A TOUR OF p-PERMUTATION MODULES AND RELATED CLASSES OF MODULES

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Abstract. This survey provides the reader with an overview of numerous results on *p-permutation modules* and the closely related classes of *endo-trivial*, *endo-permutation* and *endo-p-permutation* modules. These classes of modules play an important role in the representation theory of finite groups. For example, they are important building blocks used to understand and parametrise several kinds of categorical equivalences between blocks of finite group algebras. For this reason, there has been, since the late 1990's, much interest in classifying such modules. The aim of this manuscript is to review classical results as well as all the major recent advances in the area. The first part of this survey serves as an introduction to the topic for non-experts in modular representation theory of finite groups, outlining proof ideas of the most important results at the foundations of the theory. Simultaneously, the connections between the aforementioned classes of modules are emphasised. In this respect, results, which are dispersed in the literature, are brought together, and emphasis is put on common properties and the role played by the p-permutation modules throughout the theory. Finally, in the last part of the manuscript, lifting results from positive characteristic to characteristic zero are collected and their proofs sketched.

Contents

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1. Introduction

The class of p-permutation modules is omnipresent in the modular representation theory of finite groups. To begin with, going back to the origins of representation theory, the theory of linear representations of finite groups investigates the structural connections between groups and automorphism groups of vector spaces. One of its most basic instance comes from the consideration of a group action on a finite set X, which is extended linearly to give a *permutation representation* on X. Elementary examples are given by the natural representation of the symmetric group S_n or by the regular representation. Looking at representations of finite groups as modules over the group algebra, then a permutation representation corresponds to a permutation module, and, provided the base field has positive characteristic p, the indecomposable summands of the permutation modules are called the *p-permutation modules*.

This class of modules plays an important role in modular representation of finite groups and in the related block theory. For example, p-permutation modules are used to understand and parametrise several kinds of categorical equivalences between p-blocks of finite groups such as splendid Morita equivalences, basic equivalences, splendid Rickard equivalences, p-permutation equivalences, or the recently introduced functorial equivalences. They are also central objects to understand in the context of Alperin's weight conjecture.

The class of p-permutation modules was studied by Conlon [Con68] and Scott [Sco73]. A further fruitful approach through invariant bases, the Brauer morphism and G-algebras is due to Puig; it appears in [Bro85] by Broué. Thévenaz [Thé95, §27] also provides a detailed introduction to this theory in the language of G-algebras, and Linckelmann [Lin18a, Lin18b] has the most up-to-date collection of results on the topic with detailed proofs mixing approaches.

There are several classes of modules which are closely related and which have intensively been studied over the past 5 decades. To start with, an endo-permutation module over a p-group is one whose endomorphism ring over the base field is a permutation module. This notion was introduced in 1978 by Dade in [Dad78a, Dad78b] and studied by many authors until their classification was completed, a quarter of a century later, by Bouc in [Bou06]. However, the final results are due to the combined efforts [Alp77, CT00, CT05, CT04, Bou04, Bou00, BM04] of several authors in different combinations including Alperin, Bouc, Carlson, Dade, Mazza and Thévenaz. Crucial to this classification was the understanding of the class of endo-trivial modules, which are those (endo-permutation) modules, whose endomorphism ring is invertible in the stable module category of the group algebra. From 2006 on, once the classification of the endo-permutation had been completed, on the one hand, the focus was put on understanding and classifying endo-trivial modules over arbitrary finite groups. A complete classification has not been achieved yet, but many classes of finite groups could be treated, by many different authors (including Bessenrodt, Carlson, Grodal, Hemmer, Koshitani, Lübeck, Malle, Mazza, Nakano, Robinson, Thévenaz, and the author), using a variety of different methods, going from rank varieties to homotopy theory, passing through character theory, on top of standard module theory. On the other hand, endo-permuation modules can also be generalised to arbitrary finite groups, however, the concept of an endo-p-permutation module, i.e. one whose endomorphism ring is a p-permutation module, introduced by Urfer in [Urf06, Urf07] and developed by the author in [Las12, Las13] turns out to be more relevant. All these classes of modules are also important in block theory, as the p-permutation modules are.

Thévenaz has written a survey on endo-permutation modules [Thé07] and there are three very good surveys describing the developments towards a classification of endo-trivial modules since 2006: two brief surveys by Carlson [Car12, Car17] and a book by Mazza [Maz19]. Our aim in this survey is therefore not to provide a detailed treatment, but an introduction to all of these concepts for non-experts, emphasising their common properties and the way they interact.

Moreover, we outline the proofs of the results presented, whenever it is possible to obtain them using elementary arguments or the theory of vertices and sources, which may be thought of as the art of juggling with induction and restriction.

In Section 3 and Section 4 we give an introduction to permutation modules and p-permutation modules, outlining proof ideas of the most important results. Then, in Section 5 we review endo-permutation modules, endo-trivial modules and endo-p-permutation modules, also introducing a relative version of endo-trivial modules which has the advantage of encompassing all the latter classes of modules, through a common concept. Finally, in Section 7, we focus on relatively recent lifting results from positive characteristic to characteristic zero, which are less well-known. However, they are of great significance, as they allow for the use of ordinary character theory.

2. Notation and background material

2.1. **General notation and conventions.** Unless otherwise stated, throughout this manuscript we adopt the following notation and conventions. All groups considered are assumed to be finite, and all modules over group algebras are assumed to be finitely generated left modules. We let p denote a positive prime number, G a finite group of order divisible by p, and P a finite p-group. We denote by $\mathrm{Syl}_p(G)$ the set of all Sylow p-subgroups of G. We let $\mathcal{O}$ be a complete discrete valuation ring with field of fractions K and residue field $k := \mathcal{O}/J(\mathcal{O})$ of positive characteristic p, where $J(\mathcal{O}) =: \mathfrak{p}$ is the unique maximal ideal of $\mathcal{O}$. We assume that k is a splitting field for G and its subgroups. At first we allow the case $\mathcal{O} = k$. From Section 4 on, for simplicity, we will assume that the triple $(K, \mathcal{O}, k)$ is a splitting p-modular system for G and its subgroups. In order to state definitions or results, for which there is no difference between $\mathcal{O}$ and k, we let $R \in \{\mathcal{O}, k\}$.

If H is a subgroup of G, then we write $H \leq G$, and if $x \in G$ then we set ${}^{x}H := xHx^{-1}$ for the conjugate of H by x. The trivial RG-module is the RG-module R on which all elements of G act as the identity. Given an RG-module M, we let $M^* := \mathrm{Hom}_R(M, R)$ be the R-dual of M, and we write $L \mid M$ if L is an RG-module isomorphic to a direct summand of M. Given an RG-module M and a subgroup $H \leq G$, then the conjugate ${}^{x}M$ of M by $x \in H$ is the $R[{}^{x}H]$-module with underlying R-module M and ${}^{x}H$-action defined by $({}^{x}h) \cdot m := hm$ for any $h \in H, m \in M$. Given a subgroup $H \leq G$, an RG-module M and an RH-module U, then $\mathrm{Res}^G_H(M)$ denotes the restriction of M from G to H (i.e. $\mathrm{Res}^G_H(M) = M$ as an R-module and the given action of G is restricted to an action of H) and $\mathrm{Ind}^G_H(U) := RG \otimes_{RH} U$ the induction of U from H to G. Given a normal subgroup $N \trianglelefteq G$ and an $R[G/N]$-module V, then $\mathrm{Inf}^G_{G/N}(V)$ is the inflation of V from G/N to G, i.e. the action of $g \in G$ on $\mathrm{Inf}^G_{G/N}(V)$ is defined to be the action of the left coset gN on V. If M is an RG-module, then the first syzygy of M is by definition the kernel of a projective cover of M and is denoted by $\Omega(M)$. If P is a p-group and $Q \leq P$, then we denote by $\Omega_{P/Q}$ the relative syzygy operator with respect to Q, which by definition returns the kernel of a Q-projective relative cover of the given module. We refer to [Thé85, Car96] for details on this notion and to §2.4 for Q-projectivity. In particular $\Omega_{P/Q}(R)$ is the kernel of a relative Q-projective cover of the trivial module R and with this notation $\Omega = \Omega_{P/\{1\}}$.

Finally, we recall that an RG-lattice is an RG-module which is free as an R-module. When $R = \mathcal{O}$, we denote by $\mathcal{O}G$-**lat** the category of all $\mathcal{O}G$-lattices of finite R-rank, and when $R = k$, then we denote by kG-**mod** the category of finite-dimensional kG-modules. Further standard notation, used in this manuscript but not introduced here, is as in [Thé95, Web16, Lin18a, Lin18b].

2.2. **Reduction modulo $\mathfrak{p}$ and liftable modules.** *Reduction modulo* $\mathfrak{p}$ (sometimes simply called *reduction modulo* p) is the functor

$$k\otimes_{\mathcal{O}}- : \quad \mathcal{O}G\text{-}\mathbf{lat} \quad\longrightarrow\quad kG\text{-}\mathbf{mod}$$

mapping an $\mathcal{O}G$-lattice L to $k\otimes_{\mathcal{O}} L\cong L/\mathfrak{p}L$ and with standard action on morphisms via the universal property of the tensor product. If L_1, L_2 are $\mathcal{O}G$-lattices and $\varphi\in \mathrm{Hom}_{\mathcal{O}G}(L_1,L_2)$ is a morphism, then the reduction modulo $\mathfrak{p}$ of φ is

$$\bar{\varphi}: L_1/\mathfrak{p}L_1 \longrightarrow L_2/\mathfrak{p}L_2, x+\mathfrak{p}L_1\mapsto \varphi(x)+\mathfrak{p}L_2\,.$$

Notice that two non-isomorphic $\mathcal{O}G$-lattices L_1 and L_2 may have isomorphic reductions modulo $\mathfrak{p}$. For example, given a p-group P, any two non-isomorphic $\mathcal{O}P$-lattices L_1 and L_2 of $\mathcal{O}$-rank one are such that $L_1/\mathfrak{p}L_1\cong k\cong L_2/\mathfrak{p}L_2$. Clearly, reduction modulo $\mathfrak{p}$ is always possible. In contrast, it is not always possible to go the other way around. Thus, a kG-module M is said to be *liftable* (to $\mathcal{O}$ or to $\mathcal{O}G$) if there exists an $\mathcal{O}G$-lattice L such that $M\cong L/\mathfrak{p}L$.

2.3. **The Brauer quotient.** Let M be an $\mathcal{O}G$-lattice. Given a subgroup $H\leq G$ denote by M^H the set of H-fixed points of M. Then, given subgroups $S,Q\leq G$ such that $S\leq Q\leq G$, the relative trace map t_S^Q is defined to be the map

$$t_S^Q: M^S\longrightarrow M^Q, m\mapsto \sum_{x\in[Q/S]} xm$$

and the *Brauer quotient* of M with respect to Q is the k-vector space

$$M(Q):=M^Q\Big/\left(\sum_{S<Q} t_S^Q(M^S)+\mathfrak{p}M^Q\right).$$

By definition Q acts trivially on M^Q and the action of $N_G(Q)$ leaves $\sum_{S<Q} t_S^Q(M^S)$ and $\mathfrak{p}M^Q$ invariant. Thus $M(Q)$ is endowed with the structure of a $kN_G(Q)$-module, but also with the structure of a $k[N_G(Q)/Q]$-module, and it is common to switch between the two point of views. Notice that $M(\{1\})=M/\mathfrak{p}M$. Moreover, it is easy to see that $M(Q)$ can be non-zero only if Q is a p-subgroup of G. Indeed, if Q is properly contains a Sylow p-subgroup P of G, then $|Q:P|$ is invertible and it follows that else M^Q occurs in the sum $\sum_{S<Q} t_S^Q(M^S)$. The canonical surjection $br_Q^M: M^Q\mapsto M(Q)$ is the *Brauer homomorphism* corresponding to the subgroup Q, and is clearly a homomorphism of $k[N_G(Q)/Q]$-modules. We refer the reader to [Thé95, §27] for details.

2.4. **Vertices, sources, Green correspondence.** Standard references for detailed expositions of Green's theory of vertices and sources are for example [CR81] or [Lin18a], and we recommend [Web16, Chapter 11] to anyone starting to learn about modular representation theory.

Given a subgroup $H\leq G$, an RG-lattice M is said to be *relatively H-projective* or simply *H-projective* if M is a direct summand of kH-module induced from H to G, or equivalently if $M\mid \mathrm{Ind}_H^G\mathrm{Res}_H^G(M)$. A *vertex* of an indecomposable RG-lattice M is a subgroup $Q\leq G$ which is minimal subject to the property that M is relatively Q-projective. The set of all vertices of M is denoted by $\mathrm{vtx}(M)$. Given $Q\in\mathrm{vtx}(M)$, an *RQ-source* (or simply a *source*) of M is an

RQ-lattice T such that $M \mid \mathrm{Ind}_Q^G(T)$. Essential properties of vertices and sources, to have in mind in order to understand this text, are the following.

Properties 2.1

(a) The vertices of an indecomposable RG-lattice form a G-conjugacy class of p-subgroups of G.

(b) For a fixed vertex Q of an indecomposable RG-lattice M, a source of M is defined up to conjugacy by elements of $N_G(Q)$, and is a direct summand of $\mathrm{Res}_Q^G(M)$.

(c) The projective indecomposable RG-modules (henceforth *the PIMs of RG*) are precisely the indecomposable RG-lattices with vertex $\{1\}$.

(d) The vertices of the trivial RG-lattice R are the Sylow p-subgroups. (See Lemma 4.4(b).)

(e) If $Q \leq G$ is a p-subgroup and $H \leq G$ a subgroup containing $N_G(Q)$, then the *Green correspondence* with respect to $(G, H; Q)$ (see [Lin18a, Theorem 5.2.1]) is the following bijection:

$$\left\{\begin{matrix}\text{isomorphism classes}\\ \text{of indecomposable}\\ RG\text{-modules}\\ \text{with vertex } Q\end{matrix}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{matrix}\text{isomorphism classes}\\ \text{of indecomposable}\\ RH\text{-modules}\\ \text{with vertex } Q\end{matrix}\right\}$$

$$\begin{matrix} M & \mapsto & f(M) \\ g(N) & \leftarrow\!\shortmid & N \end{matrix}$$

where $f(M) \mid \mathrm{Res}_H^G(M)$ is the unique indecomposable summand with vertex Q and similarly $g(N) \mid \mathrm{Ind}_H^G(N)$ is the unique indecomposable summand with vertex Q. Moreover, corresponding modules have a source in common.

(f) Let M be an indecomposable RG-lattice and let $Q \leq G$ be a p-subgroup. Then the following assertions are equivalent:

 (i) Q is a vertex of M;

 (ii) Q is a maximal p-subgroup of G with the property that $(\mathrm{End}_R(M))(Q) \neq 0$.

Properties (a) to (e) are standard and to be found with a proof in any textbook on modular representation theory, although the Green correspondence requires some work to be proved. Property (f) requires Higman's criterion and we refer to [Lin18a, Theorem 5.6.9].

Finally, we note that in this context, the central objects of study in this survey, the p-permutation RG-modules, are described by the following definition. However, we will first treat them through the more intuitive approach of permutation bases.

Definition 2.2

An indecomposable RG-module M is called a **trivial source** RG-module if there exists $Q \in \mathrm{vtx}(M)$ such that the trivial RQ-module R is a source of M.

3. Permutation modules

We start with a short review of permutation RG-modules. In this view, we begin with the basic definition of a permutation representation together with fundamental examples, at the foundations of representation theory of finite groups.

3.1. **Permutation representations.** Given a finite group G and a finite G-set X, that is, a finite set X endowed with a left action $G \times X \longrightarrow X, (g, x) \mapsto g \cdot x$, we may construct the free R-module RX with basis X. More explicitly, extending the given action R-linearly yields an R-representation

$$\begin{array}{rccl} \rho_X : & G & \longrightarrow & \mathrm{Aut}_R(RX) \\ & g & \mapsto & \big(\rho_X(g) : RX \longrightarrow RX, \sum_{x \in X} \lambda_x x \mapsto \sum_{x \in X} \lambda_x (g \cdot x)\big) \end{array}$$

of G, called the *permutation representation* of G on X.

Two fundamental examples are the following.

(1) If $G = S_n$ $(n \in \mathbb{Z}_{\geq 1})$ is the symmetric group on n letters and $X = \{1, 2, \ldots, n\}$, then ρ_X is the *natural representation* of S_n.
(2) If $X = G$ and the left action of G on X is just multiplication in G, then ρ_X is the *regular representation* of G.

3.2. **Permutation modules.** The RG-module RX corresponding to the R-representation ρ_X of Subsection 3.1 is called the *permutation RG-module on X*. This leads to the following general definition.

Definition 3.1

An arbitrary RG-module is called a **permutation** RG-module if it admits an R-basis X which is invariant under the action of the group G.

It is clear that the basis X is then a finite G-set. Also, a permutation RG-lattice is R-free by definition, hence an RG-lattice.

Example 3.2

(a) An induced module of the form $\mathrm{Ind}_H^G(R)$ with $H \leq G$ is always a permutation RG-module. Indeed, as
$$\mathrm{Ind}_H^G(R) = RG \otimes_{RH} R = \bigoplus_{g \in [G/H]} g \otimes R$$
as R-lattice, it has an obvious G-invariant R-basis $\{g \otimes 1_R \mid g \in [G/H]\}$.
(b) The regular module $RG = RG \otimes_R R = \mathrm{Ind}_{\{1\}}^G(R)$ is clearly a permutation RG-module on G; see also Example (2) in Subsection 3.1.

The following elementary observation shows that an arbitrary permutation RG-module is isomorphic to a direct sum of induced modules of the form $\mathrm{Ind}_H^G(R)$ for various subgroups $H \leq G$.

Observation 3.3

If RX is a permutation RG-module on X, then a decomposition of X as a disjoint union of G-orbits, say $X = \bigsqcup_{i=1}^{n} X_i$ $(n \in \mathbb{Z}_{\geq 0})$, yields a decomposition of RX as a direct sum of RG-submodule as

$$RX = \bigoplus_{i=1}^{n} RX_i \,,$$

where each RX_i is called a *transitive* permutation module. Furthermore, for each $1 \leq i \leq n$,

$$RX_i \cong R[G/H_i] \cong \operatorname{Ind}_{H_i}^{G}(R) \,,$$

where $H_i := \operatorname{Stab}_G(x_i)$, the stabiliser in G of some $x_i \in X_i$.

Corollary 3.4

Up to isomorphism, there are only finitely many indecomposable permutation RG-modules.

Many standard operations on modules preserve permutation modules.

Properties 3.5

(a) The image of a permutation module under induction, restriction and inflation is again a permutation module.
(b) The class of permutation RG-modules is closed under conjugation, finite direct sums, taking of the R-dual and finite tensor products over R.

Proof: (a) Let $H \leq G$. If RY is a permutation RH-module on Y, then the set

$$\{g \otimes y \mid g \in [G/H], y \in Y\}$$

is G-invariant R-basis of $\operatorname{Ind}_H^G(RY)$. If RX is a permutation RG-module on X, then X is also an H-invariant R-basis of $\operatorname{Res}_H^G(RX)$. Similarly for inflation, just take the same invariant R-basis.
(b) The claim about conjugation can be proved as in Example 3.2. If M_1 and M_2 are permutation RG-modules on X_1 and X_2 respectively, then $M_1 \oplus M_2$ is a permutation module on $X_1 \sqcup X_2$ and $M_1 \otimes_R M_2$ is a permutation module on $\{x_1 \otimes x_2 \mid x_1 \in X_1, x_2 \in X_2\}$. If M is a permutation RG-module on X, then X^* is a permutation module on the dual basis to X. ■

Lemma 3.6

If P is a p-group, then for any subgroup $Q \leq P$, the permutation module $\operatorname{Ind}_Q^P(R)$ is indecomposable with vertex Q and RQ-source R.

Proof: To prove the indecomposability of $\operatorname{Ind}_Q^P(R)$, we may assume that $R = k$. Indeed, if $\operatorname{Ind}_Q^P(\mathcal{O})$ decomposes, then so does $k \otimes_{\mathcal{O}} \operatorname{Ind}_Q^P(\mathcal{O}) = \operatorname{Ind}_Q^P(k)$. Then, it suffices to prove that the socle of $\operatorname{Ind}_Q^P(k)$ is indecomposable. Now, as P is a p-group and k is a splitting field for G, up to isomorphism, the only simple kP-module is the trivial module. Hence $\operatorname{soc}(\operatorname{Ind}_Q^P(k))$ is a direct sum of trivial submodules. This together with Frobenius reciprocity yields

$$\begin{aligned}\dim_k \operatorname{soc}(\operatorname{Ind}_Q^P(k)) = \dim_k \operatorname{Hom}_{kP}(k, \operatorname{Ind}_Q^P(k)) &= \dim_k \operatorname{Hom}_{kQ}(\operatorname{Res}_Q^P(k), k) \\ &= \dim_k \operatorname{Hom}_{kQ}(k, k) = \dim_k k = 1 \,,\end{aligned}$$

which forces $\operatorname{soc}(\operatorname{Ind}_Q^P(k))$ to be indecomposable. Next, as two induced modules $\operatorname{Ind}_{Q_1}^P(R)$ and $\operatorname{Ind}_{Q_2}^P(R)$ are isomorphic if and only if the subgroups Q_1 and Q_2 of P are conjugate, clearly $Q \in \operatorname{vtx}(\operatorname{Ind}_Q^P(R))$ and R is a source by definition. ■

As a result, we have parametrised the indecomposable permutation modules over p-groups.

Corollary 3.7

If P is a p-group, then the isomorphism classes of indecomposable permutation RP-modules are parametrised by the P-conjugacy classes of subgroups of P.

4. p-PERMUTATION MODULES

4.1. **Definition and characterisations.** Coming back to Observation 3.3, we emphasise that the transitive permutation modules need not be indecomposable in general, although it is the case for p-groups. For instance, it is well-known that the PIMs of kG are the indecomposable summands of the group algebra $kG \cong \mathrm{Ind}_{\{1\}}^{G}(k)$ and each of them occurs with multiplicity equal to the dimension of its simple socle. Therefore, in general, it is necessary to investigate the indecomposable direct summands of the (transitive) permutation RG-modules. The following lemma is crucial to understand these summands.

Lemma 4.1

Let $P \in \mathrm{Syl}_p(G)$. If L is an RG-module such that $\mathrm{Res}_P^G(L)$ is a permutation RP-module, then $\mathrm{Res}_P^G(M)$ is also a permutation RP-module for any $M \mid L$.

Proof: By Observation 3.3 there exist $n \in \mathbb{Z}_{\geq 0}$ and subgroups $Q_i \leq G$ $(1 \leq i \leq n)$ such that

$$\mathrm{Res}_P^G(L) \cong \bigoplus_{i=1}^{n} \mathrm{Ind}_{Q_i}^{P}(R)\,,$$

where each $\mathrm{Ind}_{Q_i}^{P}(R)$ is indecomposable by Lemma 3.6. Thus, by the Krull–Schmidt theorem, if $M \mid L$, then $\mathrm{Res}_P^G(M)$ is isomorphic to the direct sum of some of the summands in the decomposition, hence is again a permutation RP-module. ■

This leads us to the following equivalent characterisations of the direct summands of the permutation RG-modules

Proposition-Definition 4.2 (*Characterisations of p-permutation modules*)

Let M be an RG-module and let $P \in \mathrm{Syl}_p(G)$. Then, the following conditions are equivalent:

(a) $\mathrm{Res}_Q^G(M)$ is a permutation RQ-module for each p-subgroup $Q \leq G$;

(b) $\mathrm{Res}_P^G(M)$ is a permutation RP-module;

(c) M has an R-basis which is invariant under the action of P;

(d) M is isomorphic to a direct summand of a permutation RG-module;

(e) M is isomorphic to a direct sum of trivial source RG-modules.

If M fulfils any of these equivalent conditions, then it is called a ***p*-permutation** RG-module.

At this point, we note that *p-permutation* RG-modules and *trivial source* RG-modules are essentially two different pieces of terminology for the same concept. Some authors tend to favour the use of the terminology *p-permutation module* to emphasise the existence of a P-invariant basis and reserve the terminology *trivial source module* for an indecomposable module with a trivial source (as introduced above). Other authors tend to favour the use of the terminology *trivial source module* to mean a *direct sum of RG-modules with trivial sources*, that is, our definition of a p-permutation module.

Proof:
(a)$\Leftrightarrow$(b): It is obvious that (a) implies (b). For the sufficient condition, notice that $\mathrm{Res}^G_{{}^gP}(M) \cong {}^g(\mathrm{Res}^G_P(M))$ for each $g \in G$, and recall that any p-subgroup is contained in a conjugate of P by the Sylow theorems. Thus, as by Properties 3.5 restriction and conjugation preserve permutation modules, requiring that $\mathrm{Res}^G_P(M)$ is a permutation RP-module implies that $\mathrm{Res}^G_Q(M)$ is a permutation RQ-module for each p-subgroup $Q \leq G$, because

(b)$\Leftrightarrow$(c): This is immediate from the definition of a permutation RP-module.

(b)$\Rightarrow$(e): If M satisfies (b), then by Lemma 4.1(a) we can assume that M is indecomposable. So let $Q \in \mathrm{vtx}(M)$. Then $M \mid \mathrm{Ind}^G_Q(\mathrm{Res}^G_Q(M))$ by Q-projectivity. Since $\mathrm{Res}^G_Q(M)$ is a permutation RQ-module by assumption, again by Observation 3.3 there exist $n \in \mathbb{Z}_{\geq 1}$ and subgroups $R_i \leq Q$ $(1 \leq i \leq n)$ such that
$$\mathrm{Res}^G_Q(M) \cong \bigoplus_{i=1}^{n} \mathrm{Ind}^Q_{R_i}(R)\,.$$
Inducing this module to G, we deduce that M, being indecomposable, is isomorphic to a direct summand of $\mathrm{Ind}^G_{R_i}(R)$ for some $1 \leq i \leq n$. By the minimality of vertices it follows that $R_i = Q$ and that the trivial RQ-module R must be a source of M.

(e)$\Rightarrow$(d): If L is an indecomposable trivial source module, say with vertex $Q \leq G$, then by definition of a source $L \mid \mathrm{Ind}^G_Q(R)$. This implies (d) as $\mathrm{Ind}^G_Q(R)$ is a permutation RG-module and any finite direct sum of permutation RG-modules is again a permutation RG-module.

(d)$\Rightarrow$(b): Assume that $M \mid Z$, where Z is a permutation RG-module. Then $\mathrm{Res}^G_P(M) \mid \mathrm{Res}^G_P(Z)$, where $\mathrm{Res}^G_P(Z)$ is again a permutation RP-module by Properties 3.5(a). Thus, it follows from Lemma 4.1(a) that $\mathrm{Res}^G_P(M)$ is a permutation RP-module. ■

Properties 4.3

(a) Any direct summand of a p-permutation RG-module is a p-permutation RG-module.
(b) The image of a p-permutation module under induction, restriction, inflation, conjugation and taking of the R-dual is a p-permutation module.
(c) Finite direct sums and tensor products over R of p-permutation RG-modules are p-permutation RG-modules.

Proof: Assertion (a) follows immediately from the characterisation of p-permutation modules in Proposition-Definition 4.2 and Lemma 4.1. Assertions (b) and (c) are straightforward consequences of the same assertions for permutation modules (see Properties 3.5) and the characterisation of p-permutation modules. ■

Example 4.4

(a) Projective RG-module are p-permutation RG-modules, since they are direct summands of free RG-modules and $RG = \mathrm{Ind}^G_{\{1\}}(R)$.
(b) The trivial RG-module R is a p-permutation module. Indeed, $\mathrm{Res}^G_Q(R) = R$ for any Q-subgroup $Q \leq G$, so R is a trivial source module.

Furthermore, we claim that $\mathrm{vtx}(R) = \mathrm{Syl}_p(G)$. Indeed, if $Q \in \mathrm{vtx}(R)$ and $P \in \mathrm{Syl}_p(G)$ is such that $Q \leq P$, then $R \mid \mathrm{Ind}_Q^G(R)$ and so the Mackey formula yields

$$\mathrm{Res}_P^G(R) \mid \mathrm{Res}_P^G(\mathrm{Ind}_Q^G(R)) \cong \bigoplus_{x \in [P \backslash G/Q]} \mathrm{Ind}_{P \cap {}^xQ}^P(R)$$

where all summands $\mathrm{Ind}_{P \cap {}^xQ}^P(R)$ are indecomposable by Lemma 3.6.

(c) Lemma 3.6 shows that if G is a p-group, then any p-permutation module is in fact a permutation module. Hence, the concept of a p-permutation is reduced to the concept of a permutation module, and hence the former is not need for p-groups.

4.2. **Green correspondence for p-permutation modules.** The Green correspondence provides us with a theoretical classification of all indecomposable p-permutation modules, vertex by vertex.

Lemma 4.5

Let M be an indecomposable p-permutation RG-module and let $Q \trianglelefteq G$ be a normal p-subgroup. If Q is contained in a vertex of M, then Q acts trivially on M, that is, $M^Q = M$.

Proof: Let $S \in \mathrm{vtx}(M)$ be such that $Q \leq S$. Then $M \mid \mathrm{Ind}_S^G(R) \cong R(G/S)$ (the permutation RG-module on G/S). Now, as $Q \trianglelefteq G$, we have $yxS = xS$ for all $x \in G$ and for all $y \in Q$, proving that Q acts trivially on $\mathrm{Ind}_S^G(R)$, and hence on M. ∎

Theorem 4.6 (*Green Correspondence for p-permutation modules*)

(a) If M is an indecomposable p-permutation RG-module with vertex $Q \leq G$, then Q acts trivially on the $RN_G(Q)$-Green correspondent $f(M)$ of M, and $f(M)$ can be viewed as an $R[N_G(Q)/Q]$-module. As such, $f(M)$ is indecomposable and projective.

(b) If N is a projective indecomposable $R[N_G(Q)/Q]$-module, then $\mathrm{Inf}_{N_G(Q)/Q}^{N_G(Q)}(N)$ is an indecomposable $RN_G(Q)$-module with vertex Q and trivial source. Its RG-Green correspondent is an indecomposable p-permutation RG-module.

(c) There are bijections

$$\left\{\begin{array}{c}\text{isomorphism classes}\\ \text{of indecomposable}\\ p\text{-permutation}\\ RG\text{-modules}\\ \text{with vertex } Q\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}\text{isomorphism classes}\\ \text{of indecomposable}\\ p\text{-permutation}\\ RN_G(Q)\text{-modules}\\ \text{with vertex } Q\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}\text{isomorphism classes of}\\ \text{projective indecomposable}\\ R[N_G(Q)/Q]\text{-modules}\end{array}\right\}.$$

Proof: By Proposition-Definition 4.2 any indecomposable p-permutation module has a trivial source. Thus Assertions (a) and (b) are immediate consequences of Lemma 4.5 and Properties 2.1(c),(e). The first bijection in Assertion (c) is then also given by Properties 2.1(e) applied to kG-modules with a trivial source, whereas the second bijection follows from (a) and (b). ∎

We emphasise that the characterisation of the indecomposable p-permutation RG-modules obtained via the Green correspondence is theoretically very powerful, however, does not provide us with a concrete description of such modules, because the first bijection in Assertion (c) above is not constructive. In general, the question of describing the structure of the indecomposable

p-permutation RG-modules remains a difficult question, even for small groups, or modules belonging to blocks with small defect groups. One of the main issues being that p-permutation modules are determined by the source-algebra equivalence class of the block, but not by its Morita equivalence class. As a matter of fact, the question is rather complex already for cyclic blocks. A complete solution in this case can be found in [HL21].

Finally, we mention the following result of Okuyama's showing that simple modules with a trivial source have a simple Green correspondent.

Theorem 4.7 ([Oku81, Theorem 2.3])

Let T be a simple kG-module and let $Q \in \mathrm{vtx}(T)$. Let f denote the Green correspondence with respect to $(G, N_G(Q); Q)$. If T is a p-permutation kG-module, then the Green correspondent $f(T)$ of T is a simple $kN_G(Q)$-module, and $(\dim_k(T))_p = |G : Q|_p$.

4.3. **Weight modules and Alperin's Weight Conjecture.** If $Q \leq G$ is a p-subgroup and S is a simple $kN_G(Q)$-module with vertex Q, then the pair (Q, S) is called a *weight of G with respect to Q*. In this case, Q is a *weight subgroup*, S is a *weight kG-module* and the kG-Green correspondent $g(S)$ of S is called a *weight Green correspondent.*

It is easy to see that any weight subgroup Q is p-radical subgroup, i.e. $Q = O_p(N_G(Q))$ is the largest normal p-subgroup of $N_G(Q)$. Also conjugation induces an equivalence relation on the set of all weights of G. Alperin originally stated his Weight Conjecture in [Alp87] as follows.

Conjecture 4.8 (***Alperin's Weight Conjecture***)

The number of weights of G, considered up to conjugation, is equal to the number of isomorphism classes of simple kG-modules.

Weight modules are strongly related to p-permutation modules, as we see through the following observations.

Remark 4.9

If (Q, S) is a weight of G, then S is simple and projective as a $k[N_G(Q)/Q]$-module since $Q \in \mathrm{vtx}(S)$. Conversely, the inflation from $N_G(Q)/Q$ to Q of a projective simple $k[N_G(Q)/Q]$-module is a weight module of G. Thus, in view of Theorem 4.6, it is is clear that weight modules, as well as their kG-Green correspondents, are p-permutation modules.

Lemma 4.10 ([Alp87, Lemma 1])

Let $P \in \mathrm{Syl}_p(G)$. If S is a weight kG-module, then its kG-Green correspondent $g(S)$ is isomorphic to an indecomposable direct summand of $\mathrm{Ind}_P^G(k)$.

Proof: By the Burry–Carlson–Puig theorem, it is enough to prove that $S \mid \mathrm{Res}^G_{N_G(Q)}(\mathrm{Ind}_P^G(k))$, which follows from the Mackey formula. ■

It follows that understanding the direct summands of a Sylow permutation module $\mathrm{Ind}_P^G(k)$ gives some control on the number of weights. For instance, this strategy was used by Cabanes in [Cab88] to prove Alperin's Weight Conjecture for finite groups of Lie type in their defining characteristic. More precisely, he proves that in this case the image under the Brauer morphism of each indecomposable direct summand X of $\mathrm{Ind}_P^G(k)$ with vertex Q, i.e. the Green correspondent $f(X)$ of X, is projective simple as a $k[N_G(Q)/Q]$-module and hence a weight module.

In fact, the more general context in which the endomorphism algebra $\mathrm{End}_{kG}(\mathrm{Ind}_P^G(k))$ is quasi-Frobenius was further examined by Naehrig in [Nae10]. One of the main ideas is that under this assumption,

$$\begin{aligned}|\{\text{weights of } G\}| &\leq |\{\text{indecomposable direct summands of } \mathrm{Ind}_P^G(k)\}/_{\cong}| \\ &= |\{\text{simple } \mathrm{End}_{kG}(\mathrm{Ind}_P^G(k))\text{-modules}\}| \\ &= |\{\text{simple } kG\text{-modules}\}|\end{aligned}$$

where the last equality holds by [Nae10, Theorem 3.1(d)] due to Green. This implies that in this special case Alperin's Weight Conjecture is true if and only if each indecomposable direct summand of $\mathrm{Ind}_P^G(k)$ is a weight Green correspondent.

4.4. **Approach via the Brauer quotient.** A further fruitful approach to p-permutation modules through the Brauer morphism is due to Puig and appears in [Bro85] by Broué. As a matter of fact, the Brauer construction applied to p-permutation modules is particularly well-behaved and provides us with alternative characterisations of vertices, sources and Green correspondents in this case.

Proposition 4.11

Let M be a p-permutation RG-module. Let $Q \leq G$ be a p-subgroup, let X be a Q-invariant R-basis of M and X^Q be the set of Q-fixed elements of X. Then the following assertions hold:

(a) the image $br_Q^M(X^Q)$ of X^Q in $M(Q)$ is a k-basis of $M(Q)$;

(b) $\mathrm{rk}_R(M) \equiv \dim_k(M(Q)) \pmod p$;

(c) if $X^Q = \emptyset$, then $M(Q) = \{0\}$;

(d) $M(Q)$ is a p-permutation $kN_G(Q)$-module on which Q acts trivially, i.e. a p-permutation $kN_G(Q)/Q$-module.

Proof: (a) Write $[Q\backslash X]$ for a set of representatives of Q-orbits in X, and for each $x \in [Q\backslash X]$ let Q_x be the stabiliser of x. Then a straightforward computation shows that $\{t_{Q_x}^Q(x) \mid x \in [Q\backslash X]\}$ is an R-basis of M^Q. Then, it is easy to check that when $R = \mathcal{O}$ we have

$$\sum_{S<Q} t_S^Q(M^S) + \mathfrak{p}M^Q = \left(\bigoplus_{\substack{x\in[Q\backslash X] \\ Q_x<Q}} R \cdot t_{Q_x}^Q(x) \right) \oplus \left(\bigoplus_{x\in X^Q} \mathfrak{p} \cdot x \right),$$

proving that $M(Q) = \bigoplus_{x\in X^Q} k \cdot br_Q^M(x)$ as k-vector space.

(b) The set $X \setminus X^Q$ consists of the Q-orbits of X which are not singletons. Thus, since Q is a p-group, we have $|X| \equiv |X^Q| \pmod p$ and the claim follows from (a).

(c) The claim is also immediate from (a).

(d) Let $P \in \mathrm{Syl}_p(N_G(Q))$ (which necessarily contains Q) and let Y be a P-invariant R-basis of M. Then X is Q-invariant, so (a) applies. Now, as P normalises Q, certainly X^Q is P-invariant.

Thus the k-basis $br_Q^M(X^Q)$ given by (a) is invariant under the action of the Sylow p-subgroup P/Q of $N_G(Q)/Q$. The claim follows. ■

Proposition 4.12

Let M be an indecomposable p-permutation RG-module and let $Q \le G$ be a p-subgroup. Then the following assertions hold:

(a) $M(Q) \neq \{0\}$ if and only if Q is contained in a vertex of M;

(b) Q is a vertex of M if and only if Q is maximal with the property that $M(Q) \neq \{0\}$.

Proof: By Properties 2.1(f), the p-subgroup Q is contained in a vertex of M if and only if $(\mathrm{End}_R(M))(Q) \neq \{0\}$. Thus, Assertion (a) follows from the fact that

$$(\mathrm{End}_R(M))(Q) \cong \mathrm{End}_k(M(Q))$$

(see [Lin18a, Proposition 5.6.11]). Assertion (b) follows from the maximality property in Properties 2.1(f). ■

Remark 4.13

In fact, a p-subgroup $Q \le G$ is a vertex of an indecomposable p-permutation RG-module M if and only if $M(Q) \neq \{0\}$ and a projective $k[N_G(Q)/Q]$-module. Furthermore, the Brauer construction gives us an alternative way to understand the bijections of Theorem 4.6. More precisely, the correspondence $M \mapsto M(Q)$ induces a bijection

$$\left\{\begin{array}{c}\text{isomorphism classes}\\ \text{of indecomposable}\\ p\text{-permutation}\\ RG\text{-modules}\\ \text{with vertex } Q\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}\text{isomorphism classes of}\\ \text{projective indecomposable}\\ k[N_G(Q)/Q]\text{-modules}\end{array}\right\}.$$

When working over the residue field k the Green correspondence for p-permutation modules has the following nice characterisations in terms of the Brauer quotient.

Theorem 4.14 ([Bro85, (3.4)],[Lin18a, Proposition 5.8.7],[Thé95, (27.4)])

Let M be a p-permutation kG-module and let $Q \le G$ be a p-subgroup. Then, there is an isomorphism of $kN_G(Q)$-modules

$$\mathrm{Res}^G_{N_G(Q)}(M) \cong L_1 \oplus L_2$$

such that the following properties hold:

(i) every indecomposable direct summand of L_1 has a vertex containing Q, i.e. trivial action of Q;

(ii) the vertices of the indecomposable direct summands of L_2 do not contain Q; and

(iii) $M(Q) \cong L_1(Q) \cong L_1$ as $kN_G(Q)$-modules and $L_2(Q) = \{0\}$.

Moreover, if M is indecomposable and $Q \in \mathrm{vtx}(M)$, then $M(Q) = L_1$ is indecomposable and is the $kN_G(Q)$-Green correspondent of M.

Proof: Since M is a p-permutation module, so are the indecomposable direct summands of $\mathrm{Res}^G_{N_G(Q)}(M)$ by Properties 4.3. Therefore, we may choose a decomposition of $\mathrm{Res}^G_{N_G(Q)}(M)$ into indecomposable direct summands and set L_1 to be the direct sum of all such summands with a vertex contained in Q and L_2 to be the direct sum of the remaining summands. Now,

Q acts trivially on all the summands of L_1 by Lemma 4.5. By Proposition 4.12(a), $L_2(Q) = \{0\}$, and so $M(Q) \cong L_1(Q) = L_1$. Thus (i), (ii) and (ii) hold.
Next, assume that M is indecomposable. Then M has a trivial source by Proposition-Definition 4.2, and therefore so does its $kN_G(Q)$-Green correspondent $f(M)$. By the definition of the Green correspondence,

$$\mathrm{Res}^G_{N_G(Q)}(M) = f(M) \oplus X$$

where X is a direct sum of indecomposable $kN_G(Q)$-modules having a vertex strictly contained in Q. Hence $X(Q) = 0$ by Proposition 4.12, implying that

$$M(Q) = (\mathrm{Res}^G_{N_G(Q)}(M))(Q) \cong f(M)\,.$$

■

It follows that the Brauer quotients can be used in order to determine the isomorphism type of p-permutation kG-modules.

Proposition 4.15

Let M and N be projective-free p-permutation kG-modules. Then the following assertions are equivalent:
(a) $M \cong N$;
(b) $M(Q) \cong N(Q)$ for every non-trivial p-subgroup $Q \leq G$.

Proof: It is clear that if $M \cong N$, then their Brauer quotients with respect to any non-trivial p-subgroup $Q \leq G$ are isomorphic. In order to prove the sufficient condition, assume that (b) holds. Clearly, by the Krull–Schmidt theorem, we may assume that M and N have no isomorphic direct summands. Let $Q \leq G$ be a p-subgroup which is maximal subject to $M(Q) \neq \{0\}$. As M is projective-free $Q \neq \{1\}$. Now, the maximality of Q implies that M and N have no direct summands with a vertex strictly containing Q. Choosing a direct sum decomposition $M \cong \bigoplus_{i=1}^n M_i \oplus Y$, where each M_i has vertex Q and none of the indecomposable direct summands of Y has vertex Q, then

$$M(Q) \cong \bigoplus_{i=1}^{n} f(M_i)$$

by Theorem 4.14. Similarly for N. Hence, we conclude from (b) that M and N have an indecomposable direct summand in common with vertex Q, which is a contradiction. ■

5. Endo-permutation, endo-trivial, endo-p-permutation modules

In this section we define and review several classes of RG-modules, which are closely related to the the class of p-permutation modules, because their R-endomorphism ring is by definition a p-permutation RG-module. These come in different flavours, depending on further restrictions put on the latter p-permutation module.

We will see in Section 6 that all the $\mathcal{O}G$-modules defined in this section are in fact automatically $\mathcal{O}G$-lattices, i.e. free as $\mathcal{O}$-modules. For this reason, talking about lattices or about modules does not make a difference here. Moreover, throughout this section, if not said otherwise, P denotes a p-group and G denotes a finite group of order divisible by p. Moreover, in order to understand the definitions below, we recall that the endomorphism algebra $\mathrm{End}_R(M)$ of an RG-module M is naturally endowed with the structure of an RG-module through the action of G by conjugation, that is,

$$^{g}\varphi(m) := g \cdot \varphi(g^{-1} \cdot m) \quad \forall\, g \in G, \forall\, \varphi \in \mathrm{End}_R(M) \text{ and } \forall\, m \in M\,.$$

Moreover, if M is an RG-lattice, then $\mathrm{End}_R(M) \cong M^* \otimes_R M$, where G acts diagonally on $M^* \otimes_R M$.

5.1. **Endo-permutation modules over p-groups.** Endo-permutation modules were first introduced and thoroughly studied by Dade in his celebrated two-part paper [Dad78a, Dad78b]. They play a crucial role in modular representation theory of finite groups. To give a few examples, they appear naturally as sources of simple modules for p-soluble groups (see [Thé95, §30]). They appear in Puig's characterisation of the source-algebra of nilpotent blocks in [Pui88], or in Linckelmann's classification of blocks with cyclic defect groups up to source-algebra equivalence (see [Lin18b, Chapter 11]). They also appear in the theory of basic Morita equivalences and associated Picard groups, currently under intensive investigation by several authors working towards classifications of blocks up to Morita equivalence and the verification of Donovan's conjecture for various classes of small defect groups (see e.g. [BKL20, EL20] and the references therein).

Definition 5.1

An RP-module M is an **endo-permutation** RP-module if $\mathrm{End}_R(M)$ is a permutation RP-module. Moreover, an endo-permutation RP-module is called **capped** if it has a direct summand with vertex P.

Dade proved in [Dad78b, Theorem 6.6.] that all endo-permutation modules can be described from the knowledge of the *indecomposable capped endo-permutation RP-modules*. The characterisation of p-permutation modules via the Brauer quotient we gave in Subsection 4.4 yields the following characterisation of capped endo-permuation modules, essential throughout the theory.

Lemma 5.2

Let M be an endo-permutation RP-module. The following assertions are equivalent:

(a) M is capped;

(b) the Brauer quotient $(\mathrm{End}_R(M))\,(P)$ is non-zero;

(c) there exists a fixed point in a P-invariant R-basis of $\mathrm{End}_R(M)$;

(d) there exists a P-set Y such that $\mathrm{End}_R(M) \cong R \oplus RY$ as an RP-module.

Stability properties are the following.

Properties 5.3

(a) The class of endo-permutation RP-modules contains the permutation RP-modules.

(b) The class of (capped) endo-permutation RP-modules is closed under taking direct summands, R-duals, and finite tensor products over R.

(c) The restriction, the inflation, the conjugation and the tensor induction of a (capped) endo-permutation module is again a (capped) endo-permutation module.

(d) For any $n \in \mathbb{Z}$ the relative syzygy module $\Omega^n_{P/Q}(R)$ is an endo-permutation RP-module.

Note that the induction of an endo-permutation module is not necessarily an endo-permutation module. Instead, the tensor induction is the correct operation to be used in this context.

Proof: Assertion (a) is straightforward from Properties 3.5 as $\mathrm{End}_R(M) \cong M^* \otimes_R M$. Assertions (b) and (c) are proved in [Thé95, §28], except for the claim about tensor induction follows from [BT00, Lemma 2.1]. Assertion (d) is [Alp01a, Theorem 1 and Theorem 2]. ■

A classification of the capped endo-permutation kP-modules was achieved through the work of several authors in a long series of articles, starting with Dade's initial two-part article [Dad78a, Dad78b] in 1978 and ending with the work of Bouc [Bou06] in 2006, with crucial steps achieved by Bouc and Thévenaz in [BT00] and by Carlson and Thévenaz in [CT00, CT05, CT04]. At this stage, we emphasise that Thévenaz has written a very detailed survey [Thé07] on the classification of endo-permutation modules and its chronological developments. For this reason we do not give proofs and simply refer the reader to [Thé07] for further details.

The initial idea that enabled this classification, introduced by Dade in [Dad78a, Dad78b], is the fact that the class of capped endo-permutation RP-modules subject to a certain equivalence relation can be endowed with the structure of an abelian group, known nowadays as the *Dade Group of P*. The class of capped endo-permutation RP-modules is not closed under direct sums, so the direct sum cannot be used as a group operation, but the tensor product over R can, as we describe below.

Proposition-Definition 5.4 ([Dad78a])

(a) If M is a capped endo-permutation RP-module, then any two indecomposable direct summands of M with vertex P are isomorphic. We write $\mathrm{Cap}(M)$ for the isomorphism class of such an indecomposable summand and call it the **cap** of M.

(b) Two capped endo-permutation RP-modules M and N are called **equivalent** provided $\mathrm{Cap}(M) \cong \mathrm{Cap}(N)$, or equivalently provided $R \mid M \otimes_R N^*$. This defines an equivalence relation on the class of capped endo-permutation RP-modules and the **Dade group of P**, denoted $D_R(P)$, is the resulting set of equivalence classes, endowed with the structure of an abelian group via the composition law

$$\begin{array}{rccc} +: & D_R(P) \times D_R(P) & \longrightarrow & D_R(P) \\ & ([M],[N]) & \mapsto & [M]+[N] := [M \otimes_R N] \,. \end{array}$$

The zero element is the class $[R]$ of the trivial RP-module and the opposite of a class $[M]$ is the class $[M^*]$ of the R-dual.

In Subsection 5.2 we will summerize the main milestones of this classification, which comes down to determining the structure of the Dade group. However, before we can proceed, we need to introduce several important subgroups of the Dade group.

In fact, one of the starting points of the classification was the following theorem of Lluis Puig, who, in the 1980's, introduced the notion of a *Dade P-algebra*. The connection with endo-permutation modules is the following. To start with, an RP-module M is an endo-permutation RP-module if and only if $\mathrm{End}_R(M)$ is a permutation P-algebra, i.e. a P-algebra admitting a P-invariant R-basis. A *Dade P-algebra* (over R) is defined to be an R-simple permutation P-algebra A such that $A(P) \neq 0$. Thus Lemma 5.2 yields: if M is a capped endo-permutation RP-module, then $\mathrm{End}_R(M)$ is a Dade P-algebra. Conversely, any Dade P-algebra gives rise to a capped endo-permutation RP-module, unique over $R = k$. A very clear exposition of these facts is to be found in [Thé95, §28 to §30]. The approach via Dade P-algebras lead in particular to the following fundamental result on the structure of the Dade group.

Theorem 5.5 ([Pui90, Corollary 2.4])

The Dade group $D_R(P)$ of P is finitely generated as an abelian group.

Notation 5.6

In view of the latter result, we may write $D_R^{\text{tors}}(P) \oplus D_k^{\text{free}}(P)$, where the first summand is the torsion part of $D_R(P)$ and the second summand its free part. Moreover, in view of Properties 5.3(d), we may consider the subgroup $D_R^{\Omega}(P) := \langle [\Omega^n_{P/Q}(R)] \mid Q < P \rangle$ of $D_R(P)$ generated by the relative syzygy modules of the trivial module.

The subgroups $D_R^{\text{tors}}(P)$ and $D_R^{\Omega}(P)$ are essential building blocks for the determination of the structure of the the Dade group. Yet, another important building block is given by the subgroup of *endo-trivial* RP-modules, also introduced by Dade in [Dad78a, Dad78b]. In fact this notion was introduced independently by Alperin in [Alp77], who called them *invertible* module, as they are invertible in the stable module category.

Definition 5.7 ([Dad78a])

An RP-module M is called **endo-trivial** if there exists a projective RP-module F such that

$$\mathrm{End}_R(M) \cong R \oplus F$$

as an RP-module, or equivalently if $M^* \otimes_R M \cong R$ in the stable module category of RP.

Note that if such an isomorphism exists, then F is the kernel of the trace map

$$\mathrm{Tr} : M^* \otimes_R M \longrightarrow R, \varphi \otimes m \mapsto \varphi(m)\,.$$

Moreover, we will often simply write $\mathrm{End}_R(M) \cong R \oplus (\text{proj})$ instead of specifying a projective module F.

The class of endo-trivial modules has less stability properties than the class of endo-permutation modules. Clearly, the inflation of an endo-trivial RP-module is not an endo-trivial module, as the inflation of a projective RP-module is not a projective module in general. However, the following properties hold.

Properties 5.8

(a) The trivial module R is endo-trivial.

(b) If M and N are endo-trivial RP-modules, then so are the following RP-modules:

  (i) M^*;

  (ii) $\mathrm{Res}^P_Q(M)$ for any $Q \leq P$;

  (iii) the tensor product $M \otimes_R N$;

  (iv) $\Omega^n(M)$ for any $n \in \mathbb{Z}$.

(c) If M is an endo-trivial RP-module, then there exist an indecomposable endo-trivial RP-module M_0, unique up to isomorphism, and a projective RP-module X such that $M \cong M_0 \oplus X$.

(d) Any endo-trivial RP-module is a capped endo-permutation RP-module. Furthermore, $\mathrm{Cap}(M) = M_0$.

Proof: Assertions (a), (b)(i)–(iii) are immediate from the definitions. Assertion (b)(iv) follows from elementary properties of syzygy modules:

$$\begin{aligned}\mathrm{End}_R(\Omega^n(M)) &\cong \Omega^n(M)^* \otimes_R \Omega^n(M) \\ &\cong \Omega^0(M^* \otimes M) \oplus (\mathrm{proj}) \\ &\cong \Omega^0(R \oplus (\mathrm{proj})) \oplus (\mathrm{proj}) \cong R \oplus (\mathrm{proj}).\end{aligned}$$

To obtain Assertion (c), observe that if we decompose M as $M = M_0 \oplus M_1$, then

$$R \oplus (\mathrm{proj}) \cong \mathrm{End}_R(M) \cong \mathrm{End}_R(M_0) \oplus \mathrm{Hom}_R(M_0, M_1) \oplus \mathrm{Hom}_R(M_0, M_1)^* \oplus \mathrm{End}_R(M_1)$$

as RG-modules and the claim follows from the Krull–Schmidt theorem, as R is non-projective and can then only occur once as a direct summand of $\mathrm{End}_R(M_0)$ or of $\mathrm{End}_R(M_1)$, but not of both. It is also clear from the definitions that any endo-trivial RP-module is an endo-permutation module, and it is capped by the characterisation of the capped modules in Lemma 5.2. ■

These properties allow us to define a group structure on the class of endo-trivial RP-modules, which can be identified with a subgroup of the Dade group.

Proposition-Definition 5.9 ([Dad78a])

(a) Two endo-trivial RP-modules M and N are called **equivalent** provided $M_0 \cong N_0$. This defines an equivalence relation on the class of capped endo-entriovial RP-modules and the **group of endo-trivial** RP**-modules of** P, denoted $T_R(P)$, is the resulting set of equivalence classes, endowed with the structure of an abelian group via the composition law

$$\begin{array}{cccc} +: & T_R(P) \times T_R(P) & \longrightarrow & T_R(P) \\ & ([M],[N]) & \mapsto & [M]+[N] := [M \otimes_R N]. \end{array}$$

The zero element is the class $[R]$ of the trivial RP-module and the opposite of a class $[M]$ is the class $[M^*]$ of the R-dual.

(b) There is a canonical injective group homomorphism

$$T_R(P) \hookrightarrow D_R(P), [M] \mapsto [M].$$

Notice that the equivalence classes in $D_R(P)$ are larger than in $T_R(P)$ and the class of an endo-trivial RP-module may contain modules that are not endo-trivial.

5.2. **The structure of the Dade group of a p-group.** The determination of the structure of the Dade group and of the group of endo-trivial modules was essentially realised over $R = k$. Their structure over $R = \mathcal{O}$ can then be deduced from the lifting results from k to $\mathcal{O}$ considered in Section 7. We record below the main steps which lead to the final classification. To start with, the abelian case was already understood by Dade when he started the theory.

Theorem 5.10 ([Dad78a, Dad78b])

If P is an abelian p-group, then $T_k(P)$ is cyclic generated by the class $[\Omega(k)]$ and

$$D_k(P) = \bigoplus_{Q<P} T_k(P/Q) \cong \mathbb{Z}^r \oplus (\mathbb{Z}/2\mathbb{Z})^c,$$

where each $T(P/Q)$ is identified with a subgroup of $D_k(P)$ via inflation, r is the number of non-cyclic quotients P/Q, and c is the number of cyclic quotients of P/Q of order at least 3.

The structure of the Dade group in finite and tame representation type plays a role in the final classification. If $P \cong C_{p^n}$ $(n \geq 1)$ is cyclic of order p^n (in multiplicative notation), then the structure of $D_k(P)$ is clear from Dade's theorem above, namely

$$D_k(P) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^n & \text{if } p \text{ is odd,} \\ (\mathbb{Z}/2\mathbb{Z})^{n-1} & \text{if } p = 2. \end{cases}$$

When $p = 2$, there is a missing generator coming from the fact that $T_k(C_2) = \{[k]\}$ as $\Omega(k) \cong k$ in this case. If P is a dihedral, a semi-dihedral, or a generalised quaternion 2-group, then the structure of $T_k(P)$ was obtained by Carlson and Thévenaz in [CT00, §5-§7]. In the same article, they prove the following general result about the structure of $D_k(P)$ in these cases.

Lemma 5.11 ([CT00, Lemma 10.2])

Assume P is a cyclic p-group, or, provided $p = 2$, a dihedral, a semi-dihedral, or a generalised quaternion 2-group . Then,

$$D_k(P) = T_k(P) \oplus D_k(P/Z)$$

where Z is the unique central subgroup of order p of P.

This led to the determination of $D_k(P)$ in tame representation type, using an induction argument, for which the starting point is the fact that $D_k(C_2 \times C_2) = T_k(C_2 \times C_2) \cong \mathbb{Z}$ and generated by $[\Omega(k)]$. See [CT00, Theorem 10.3] for details.

The classification of all endo-trivial kP-modules, or equivalently the determination of the structure of $T_k(P)$ was the next main step. It is mainly due to Carlson and Thévenaz again in [CT05, CT04]. In [CT05] they obtain the structure of the torsion part $T_k^{\text{tors}}(P)$ of $T_k(P)$.

Theorem 5.12 ([CT05, Corollary 12.6 and Corollary 12.7])

(a) If P is neither cyclic, nor generalised quaternion, nor semi-dihedral, then
$$T_k^{\text{tors}}(P) = \{[k]\}\,.$$

(b) If all maximal elementary abelian p-subgroups of P have rank at least 3, then $T(P) \cong \mathbb{Z}$ generated by $[\Omega(k)]$.

Therefore, it remains to consider the case in which P possesses elementary abelian p-subgroups of rank 2.

Theorem 5.13 ([CT04, Theorem 7.1])

Assume P has at least one elementary abelian subgroup of rank 2 but is not semi-dihedral. Let c be the number of P-conjugacy classes of maximal elementary abelian subgroups of P, and set $r := c$ if the p-rank of P is 2 and $r := c + 1$ if the p-rank of P is at least 3. Then, $T_k(P) \cong \mathbb{Z}^r$.

Another main step was the determination of the structure of the torsion part of $D_k(P)$ in odd characteristic, obtained earlier by Bouc and Thévenaz.

Theorem 5.14 ([BT00, Theorem A])

If p is odd, then $D_k^{\text{tors}}(P) \cong (\mathbb{Z}/2\mathbb{Z})^r$, where r is the number of P-conjugacy classes of non-trivial cyclic subgroups of P.

Using the results of [CT04], Bouc and Mazza [BM04] determined the structure of the Dade group of (almost) extra-special p-groups. Meanwhile, Bouc obtained crucial results on the tensor induction of relative syzygy modules in [Bou00], and in [Bou04] he established connections between the dual Burnside ring and the Dade group explicit. He also developed the machinary of biset functors, which, applied to the Dade group, lead to the final classification.

Theorem 5.15 ([Bou06, 7.7. Theorem and 8.4. Corollary])

Let P be a p-group. Then $D_k(P) = D_k^{\Omega}(P) + D_k^{\mathrm{tors}}(P)$. More precisely, the following hold:
(i) $D_k(P) = D_k^{\Omega}(P)$, provided p is odd; and
(ii) if $p = 2$, then $D_k(P) = D_k^{\Omega}(P) + {}_2D_k(P)$ where ${}_2D_k(P) = \{[M] \in D_k(P) \mid 2[M] = [k]\}$.

Notice that the situation in characteristic 2 is more complicated than in odd characteristic, as there exist torsion endo-permutation kP-modules whose classes do not lie in $D_k^{\Omega}(P)$. These are called *exotic*. For the precise description of these modules we refer the reader to [Bou06].

5.3. **Endo-trivial modules over arbitrary finite groups.** Endo-trivial modules over p-groups served as building blocks for Bouc's description of the Dade group and the classification of endo-permutation modules. Directly after this classification was achieved people turned their attention to endo-trivial modules over finite groups. Indeed, it is clear that Definition 5.7 does not use the fact that the group is a p-group and makes sense for an arbitrary finite group.

Definition 5.16

An RG-module M is termed **endo-trivial** iff $\mathrm{End}_R(M) \cong M^* \otimes_R M \cong R \oplus (\mathrm{proj})$, where (proj) denotes a projective RG-module.

There are many reasons for wanting to understand these modules. Equivalently, we could say that an endo-trivial RG-module is an RG-module whose R-endomorphism ring is isomorphic to the trivial module in the stable module category of RG. Moreover, tensoring over R with an endo-trivial RG-module is a self-stable equivalence of Morita type, implying that this class of modules can be identified with an important part of the Picard group of the stable module category of RG. It was also proved by Bleher and Chinburg [BC00] that they are the modules whose deformation rings are universal (as opposed to versal).

On top of Thévenaz' survey [Thé07] on endo-permutation modules already mentioned, there are three very good surveys describing the developments towards a classification of endo-trivial modules since 2006: two brief surveys by Carlson [Car12, Car17] and a book by Mazza [Maz19], to which we refer the reader for a detailed treatment of the subject. For this reason, below we are only briefly going to describe stability properties and similarities of this class of modules with the other classes we have studied that far.

Example 5.17

(a) The trivial module R and its syzygies $\Omega^n(R)$ $(n \in \mathbb{Z})$ are endo-trivial RG-modules, by the same argument we gave in the proof of Properties 5.8(b)(iv).
(b) Any RG-module Z such that $\mathrm{rk}_R(Z) = 1$ is endo-trivial, as $Z^* \otimes_R Z \cong R$. We write $X_R(G)$ for the set of all isomorphism classes of rank one RG-modules. This is a group for the tensor product $\otimes_R$ over R.

Properties 5.18

Let M and N be endo-trivial RG-modules.

(a) We have $\mathrm{rk}_R(M) \equiv \pm 1 \pmod{|G|_p}$.

(b) The RG-modules M^* and $M \otimes_k N$ are endo-trivial.

(c) If M is indecomposable, then the vertices of M are the Sylow p-subgroups of G.

(d) There is a direct sum decomposition $M \cong M_0 \oplus (\mathrm{proj})$ where M_0 is the unique indecomposable direct summand of M which is endo-trivial.

(e) If $P \in \mathrm{Syl}_p(G)$, then M is endo-trivial if and only if $\mathrm{Res}^G_P(M)$ is endo-trivial.

Proof: Assertions (b) to (e) will be proved in a more general context in Properties 5.38. Assertion (a) follows from the fact that, by definition, $\mathrm{rk}_R(\mathrm{End}_R(M)) = \mathrm{rk}_R(M)^2 \equiv 1$ modulo the R-rank of projective RG-module, which itself has R-rank divisible by $|G|_p$. ■

As for p-groups, these properties lead to a group structure, which allows us to classify the endo-trivial RG-modules in an easier way.

Proposition-Definition 5.19 ([CMN06])

Two endo-trivial RG-modules M and N are called **equivalent** provided $M_0 \cong N_0$. This defines an equivalence relation on the class of endo-entriovial RG-modules and the **group of endo-trivial RG-modules of** G, denoted $T_R(G)$, is the resulting set of equivalence classes, endowed with the structure of an abelian group via the composition law

$$\begin{array}{rccc} +: & T_R(G) \times T_R(G) & \longrightarrow & T_R(G) \\ & ([M],[N]) & \mapsto & [M]+[N] := [M \otimes_R N]\,. \end{array}$$

The zero element is the class $[R]$ of the trivial RG-module and the opposite of a class $[M]$ is the class $[M^*]$ of the R-dual.

As we will see in Section 7 any endo-trivial kG-module is liftable to an endo-trivial $\mathcal{O}G$-lattice. This fact allows us to assume that $R = k$, without loosing essential information. In fact, all articles concerned with classifications of endo-trivial modules assume that $R = k$. We sum up here some of the main results in this direction.

To begin with, the group of endo-trivial modules is also finitely generated, and the rank of its free part can be characterised in terms of the local structure of the group G.

Theorem 5.20 ([CMN06, Corollary 2.5, Theorem 3.1])

(a) The abelian group $T_k(G)$ is finitely generated, and so we can write

$$T_k(G) = T_k^{\mathrm{tors}}(G) \oplus T_k^{\mathrm{free}}(G)$$

where $T_k^{\mathrm{tors}}(G)$ is the torsion subgroup of $T_k(G)$ and $T_k^{\mathrm{free}}(G)$ is a torsion-free direct sum complement of $T_k^{\mathrm{tors}}(G)$ in $T_k(G)$. In particular $T_k^{\mathrm{tors}}(G)$ is finite.

(b) The rank of $T_k^{\mathrm{free}}(G)$ is equal to the number of conjugacy classes of maximal elementary abelian p-subgroups of rank 2 if G has p-rank 2, or this number plus one if G has p-rank greater than 2.

(c) Assume the p-rank of G is greater than p if p is odd, or greater than 4 if $p = 2$. Then, the rank of $T_k^{\mathrm{free}}(G)$ is one and $T_k^{\mathrm{free}}(G)$ is generated by the class of the syzygy module $\Omega(k)$.

As a consequence, in general, the difficult problem is to understand the torsion part $T_k^{\mathrm{tors}}(G)$ of the group $T_k(G)$. In this respect, the following characterisation through restriction to a Sylow p-subgroup and its normaliser are essential. For a subgroup $H \leq G$, we may always consider the group homomorphism

$$\mathrm{res}^G_H : T_k(G) \longrightarrow T_k(H), [M] \mapsto [\mathrm{Res}^G_H(M)]$$

and when $P \in \mathrm{Syl}_p(G)$, then we let $K(G) := \ker(\mathrm{res}^G_P)$ be its kernel. Clearly, in this way, we obtain a chain of inclusions

$$X_k(G) \leq K(G) \leq T_k^{\mathrm{tors}}(G) \leq T_k(G)\,.$$

It turns out that in many situations $T_k^{\mathrm{tors}}(G) = X_k(G)$, or $T_k^{\mathrm{tors}}(G) = K(G)$, but this is not the general case and although recent work of Grodal [Gro23] using homotopy theory brought many answers towards the structure of $T_k^{\mathrm{tors}}(G)$ its structure is still an open question in general.

Theorem 5.21 ([CMN06, Proposition 2.6])

Let $P \in \mathrm{Syl}_p(G)$ and let $H \leq G$ such that $H \geq N_G(P)$. Then the following assertions hold.

(a) The restriction homomorphism $\mathrm{res}^G_H : T_k(G) \longrightarrow T_k(H)$ is injective. More precisely, if M is an indecomposable endo-trivial kG-module such that $\mathrm{Res}^G_H(M) \cong L \oplus (\mathrm{proj})$ where L is an indecomposable kH-module, then M is the kG-Green correspondent of L and L is endo-trivial.

(b) If P is neither cyclic, nor generalised quaternion, nor semi-dihedral, then

$$T_k^{\mathrm{tors}}(N_G(P)) = X(N_G(P))\,.$$

(c) If P is neither cyclic, nor generalised quaternion, nor semi-dihedral, then

$$T_k^{\mathrm{tors}}(G) = K(G)\,.$$

Remark 5.22

The theorem tells us that in most cases, determining the torsion part of $T_k(G)$ comes down to determining which Green correspondents of the one-dimensional $kN_G(P)$-modules are endo-trivial, or in other words, determining which indecomposable p-permutation kG-modules with full vertex are endo-trivial.

This is in general a very hard question, and has generated a lot of work, by many different authors. We collect below the most important classes of finite groups for which the structure of $T_k(G)$ has been fully determined.

- Finite groups of Lie type in their defining characteristic: Carlson–Mazza–Nakano in [CMN06].
- Groups with a normal Sylow p-subgroup: Mazza in [Maz07]
- Symmetric and alternating groups: Carlson–Mazza–Nakano in [CMN09] andCarlson-Hemmer–Mazza–Nakano in [CHM10].
- p-soluble groups: Carlson–Mazza-Thévenaz in [CMT11].
- Groups with a cyclic Sylow p-subgroup: Mazza–Thévenaz in [MT07].
- Groups with a generalised quaternion Sylow 2-subgroup: Carlson–Mazza–Thévenaz in [CMT13].
- Sporadic groups and their covering groups: Lassueur–Malle–Schulte in [LMS16], Lassueur–Malle in [LM15], Lassueur–Mazza in [LM15b] and Craven in [Cra21].
- Schur covering groups of the symmetric and alternating groups: Lassueur–Mazza in [LM15a].

- Finite groups of Lie type in type A in non-defining characteristic: Carlson–Mazza–Nakano in [CMN14, CMN16].
- A reduction to p'-central extensions: Lassueur–Thévenaz in [LT17].
- Groups with a Klein-four or dihedral Sylow 2-subgroup: Koshitani–Lassueur in [KL15, KL16].
- Groups with a semi-dihedral Sylow 2-subgroup: Carlson–Mazza–Thévenaz in [CMT13] and Koshitani–Lassueur in [KL22].
- Finite groups of Lie type in non-defining characteristic, torsion-free part only: Carlson–Grodal–Mazza–Nakano in [CGMN22].

5.4. **Endo-p-permutation modules over arbitrary groups.** Considering an arbitrary finite group G, as seen in the previous subsections, the notion of a permutation module, which is good over a p-group, must be replaced by the notion of a p-permutation module in order to obtain similar behaviours and stability properties. Similarly, the notion of an endo-permutation module has to be replaced by the notion of an *endo-p-permutation module* in order to obtain a group structure similar to that of the *Dade group of a p-group*. Such modules were introduced by Urfer in his doctoral thesis [Urf06] (in French) as described below. We refer to Urfer's article [Urf07] for a published version in English, unfortunately not as complete as his thesis. Urfer defines endo-p-permutation modules over k, but the part of his work we present below holds over $\mathcal{O}$ as well.

Definition 5.23 ([Urf07, Definition 1.1])

An RG-module M is an **endo-p-permutation** RG-module if its restriction to any p-subgroup Q of G is an endo-permutation RQ-module.

Remark 5.24

(a) Clearly, an RG-module M is an endo-p-permutation module if and only if $\mathrm{End}_R(M)$ is a p-permutation module. Indeed, this follows from the fact that
$$\mathrm{Res}^G_Q(\mathrm{End}_R(M)) \cong \mathrm{End}_R(\mathrm{Res}^G_Q(M))$$
for any p-subgroup $Q \leq G$ and from the characterisation of p-permutation modules in Proposition-Definition 4.2.

(b) We also see from (a) and Proposition-Definition 4.2 that, in fact, it suffices to require that the restriction to a Sylow p-subgroup of G is an endo-permutation module.

(c) It is immediate from (a) that an RQ-source of an indecomposable endo-p-permutation RG-module with vertex $Q \leq G$ is an endo-permutation RG-module.

To begin with, we see that all the classes of kG-modules we have studied so far are subclasses of the the class of endo-p-permutation modules.

Example 5.25

(a) Any projective RG-module, any permutation RG-module, and more generally, any p-permutation RG-module M is an endo-p-permutation RG-module. Indeed, this is clear since $\mathrm{End}_R(M) \cong M^* \otimes_R M$ and by Properties 4.3 the R-dual and the tensor product of p-permutation RG-modules are again p-permutation RG-modules.

(b) If P is a p-group, then an RP-module is an endo-p-permutation RP-module if and only if it is an endo-permutation RP-module, as we have already observed in Example 4.4(c) that any p-permutation RP-module is a permutation RP-module in this case.

(c) Any endo-trivial RG-module is an endo-p-permutation RG-module, as any projective RG-module is a p-permutation RG-module.

Stability properties of the class of endo-p-permutation RG-modules are the following.

Properties 5.26

(a) The R-dual and any direct summand of an endo-p-permutation RG-module is an endo-p-permutation RG-module.

(b) The tensor product over R of two endo-p-permutation RG-modules is an endo-p-permutation RG-module.

(c) The image of an endo-p-permutation module under restriction, inflation, tensor-induction and conjugation is again an endo-p-permutation module.

(d) If M and N are two indecomposable endo-p-permutation RG-modules with a common vertex $Q \leq G$, then $M \otimes_R N$ possesses a (not necessarily unique) indecomposable direct summand $(M \otimes_R N)_\bullet$ which is an endo-p-permutation RG-module with vertex Q, and all other indecomposable direct summands have a vertex contained in Q (possibly equal to Q).

Notice that, in contrast, direct sums and standard induction do not preserve the class of endo-p-permutation modules.

Proof: For the R-dual, direct summands, tensor products over R, and conjugation the claims follow immediately from the equivalent characterisation of endo-p-permutation RG-modules in Remark 5.24(a) and the stability properties of p-permutation RG-modules under these operations from Properties 4.3. For restriction, there is nothing to do, and for tensor induction, restrict to p-subgroups, use Mackey's formula and the facts that conjugation, restriction, tensor induction and tensor product preserve endo-permutation modules. This proves Assertions (a), (b), and (c). To prove Assertion (d) observe that as M and N are relatively Q-projective, so is any direct summand of $M \otimes_R N$. Moreover, as $Q \in \mathrm{vtx}(M)$ and $Q \in \mathrm{vtx}(N)$, $\mathrm{Res}^G_Q(M)$ and $\mathrm{Res}^G_Q(N)$ are capped endo-permutation RQ-modules, and therefore so is

$$\mathrm{Res}^G_Q(M) \otimes_R \mathrm{Res}^G_Q(N) \cong \mathrm{Res}^G_Q(M \otimes_R N)\,.$$

So, there is an indecomposable direct summand $L \mid M \otimes_R N$ such that

$$\mathrm{Cap}(\mathrm{Res}^G_Q(M \otimes_R N)) \mid \mathrm{Res}^G_Q(L)\,,$$

proving that a vertex of L contains a conjugate of Q, and hence $Q \in \mathrm{vtx}(L)$, as required. ■

The main question that follows is of course, whether the class of endo-p-permutation modules can be endowed with a good equivalence relation in order to define a group structure similar to that of the Dade group of a p-group. There are in the literature two attempts to define such a group structure, the first one by Urfer in his doctoral thesis [Urf06, Urf07] and the second one by the author in [Las13], which we describe in the next subsection.

Urfer's construction is based on the following equivalence relation, generalising Dade's original approach to the definition of the Dade group in [Dad78a, Dad78b], and called *compatibility*.

Definition 5.27

Two endo-p-permutation RG-modules M and N are **compatible** if their direct sum $M \oplus N$ is an endo-p-permutation RG-module.

Observe that

$$\mathrm{End}_R(M \oplus N) \cong \mathrm{End}_R(M) \oplus \mathrm{End}_R(N) \oplus \mathrm{Hom}_R(M,N) \oplus \mathrm{Hom}_R(N,M)$$

and $\mathrm{Hom}_R(M,N) \cong \mathrm{Hom}_R(N,M)^*$ as RG-modules. Therefore, it is clear that M and N are compatible if and only if either $\mathrm{Hom}_R(M,N)$ or $\mathrm{Hom}_R(N,M)$ is a p-permutation RG-module. In particular, it is clear that isomorphic endo-p-permutation RG-modules are compatible.

Remark 5.28

If $H \leq G$, then the induction $\mathrm{Ind}_H^G(M)$ of an endo-p-permutation RH-module to G is, in general, not an endo-p-permutation RG-module. However, the compatibility relation yields the following criterion: $\mathrm{Ind}_H^G(M)$ is an endo-p-permutation RG-module if and only if the endo-p-permutation $R[{}^xH \cap H]$-modules $\mathrm{Res}^H_{{}^xH\cap H}(M)$ and $\mathrm{Res}^{{}^xH}_{{}^xH\cap H}({}^xM)$ are compatible for every $x \in G$. See [Urf07, Lemma 1.3].

This result leads us naturally to considering G-stable points of the Dade group. Indeed, as the Dade group $D_R(-)$ is in fact a Mackey functor (over $\mathbb{Z}$), we may consider its G-stable points. (See [Thé90] for an introduction to Mackey functors). In other words, if $Q \leq G$ is a p-subgroup, then an element $d \in D_R(Q)$ is called *G-stable* if

$$\mathrm{Res}^{{}^xQ}_{{}^xQ\cap Q} \circ c_x(d) = \mathrm{Res}^{Q}_{{}^xQ\cap Q}(d)$$

for every $x \in G$, where c_x is conjugation by $x \in G$. Then, $D_R(Q)^{G\text{-st}}$ denotes the subgroup consisting of the G-stable elements of $D_R(Q)$.

Theorem 5.29 ([Urf07, Theorem 1.5])

Let M be an indecomposable RG-module with vertex Q and RQ-source S. Then, M is an endo-p-permutation module if and only if S is an endo-permutation RQ-module whose class $[S]$ in the Dade group $D_R(Q)$ belongs to $D_R(Q)^{G\text{-st}}$.

Proof: Assume first that M is an endo-p-permutation module. Then, by definition $\mathrm{Res}^G_Q(M)$ is endo-permutation, and so is S as direct summand of the latter by Properties 5.26(a). Moreover, as ${}^xM \cong M$ for every $x \in G$, clearly $[\mathrm{Res}^G_Q(M)] \in D_R(Q)^{G\text{-st}}$, but $S = \mathrm{Cap}(\mathrm{Res}^G_Q(M))$, hence $[S] = [\mathrm{Res}^G_Q(M)]$. Conversely, if S is an endo-permutation RQ-module such that $[S] \in D_R(Q)^{G\text{-st}}$, then it follows from Remark 5.28 that $\mathrm{Ind}^G_Q(S)$ is an endo-p-permutation RG-module, and hence so is M as direct summand of the latter. ∎

A first consequence of this theorem is the fact that for indecomposable modules with a common vertex, compatibilitty is detected locally by the sources of the modules.

Proposition 5.30 ([Urf07, Proposition 1.6])

Two indecomposable endo-permutation RG-modules M and N with vertex $Q \leq G$ are compatible if and only if their RQ-sources are isomorphic.

Proof: If M and N are compatible, then by definition $M \oplus N$ is an endo-p-permutation RG-module and $\mathrm{Res}^G_Q(M\oplus N)$ is an endo-permutation RQ-module. Now, let S and S' be RQ-sources of M and N respectively. Then

$$S \oplus S' \mid \mathrm{Res}^G_Q(M) \oplus \mathrm{Res}^G_Q(N) \cong \mathrm{Res}^G_Q(M \oplus N)\,,$$

proving that $S\oplus S'$ is a capped endo-permutation RQ-module. Thus, it follows from Proposition-Definition 5.4(a) that $S \cong \mathrm{Cap}(S \oplus S') \cong S'$. Conversely, if $S \cong S'$, then $M, N \mid \mathrm{Ind}^G_Q(S)$. Moreover, $[S] \in D_R(Q)^{G\text{-st}}$ by Theorem 5.29, and hence $\mathrm{Ind}^G_Q(S)$ is an endo-p-permutation RG-module by Remark 5.28. If $M \cong N$, there is nothing to do, else $M \oplus N \mid \mathrm{Ind}^G_Q(S)$ and the claim follows from Properties 5.26(a). ∎

With the notion of compatibility Urfer defined the following group structure, which has never been given a name to.

Proposition-Definition 5.31 ([Urf06, Definition 2.15])

Let $Q \leq G$ be a p-subgroup. Then, compatibility is an equivalence relation on the class of indecomposable endo-p-permutation RG-modules with vertex Q, and the resulting set $D_Q(G)$ of equivalence classes endowed with the composition law

$$\begin{array}{rccc} +: & D_Q(G) \times D_Q(G) & \longrightarrow & D_Q(G) \\ & ([M],[N]) & \mapsto & [M]+[N] := [(M \otimes_R N)_\bullet] \end{array}$$

is an abelian group. The zero element is the class $[R]$ of the trivial RG-module and the opposite of a class $[M]$ is the class $[M^*]$ of the R-dual.

Proof: The addition $+$ is well-defined since the equivalence class $[(M \otimes_R N)_\bullet]$ does not depend on the choice of the summand $(M \otimes_R N)_\bullet$ of $M \otimes_R N$ given by Properties 5.26(d). Indeed, if $(M\otimes_R N)_*$ is another indecomposable direct summand of $M\otimes_R N$ with vertex Q, then $(M\otimes_R N)_\bullet$ and $(M \otimes_R N)_*$ are compatible, because $M \otimes_R N$ is endo-p-permutation by Properties 5.26(b) and therefore so is $(M \otimes_R N)_\bullet \oplus (M \otimes_R N)_*$ by Properties 5.26(a). The remaining claims are immediate. ∎

The first observation to make is that this new group structure generalises the constructions of the Dade group of a p-group to arbitrary finite groups.

Remark 5.32

(a) If P is a p-group, then $D_P(P) \cong D_R(P)$ via the map sending the equivalence class $[M]$ of an indecomposable endo-permutation RP-module M to its class $[M]$ in $D_R(P)$.

(b) As the proof above and Proposition 5.30 show, an equivalence class in $D_Q(G)$ may contain several indecomposable endo-p-permutation modules, namely all indecomposable endo-p-permutation modules with a common RQ-source.

Theorem 5.33 ([Urf06, Proposition 2.19])

If $Q \leq G$ is a p-subgroup, then $D_Q(G) \cong D_R(Q)^{G\text{-st}}$.

Proof: It is easy to verify that the map $D_Q(G) \longrightarrow D_R(Q)^{G\text{-st}}, [M] \mapsto [\mathrm{Cap}(\mathrm{Res}^G_Q M)]$ is an isomorphism, with inverse given by the map $D_R(Q)^{G\text{-st}} \longrightarrow D_Q(G), [S] \mapsto [(\mathrm{Ind}^G_Q(S))_\bullet]$, where $(\mathrm{Ind}^G_Q(S))_\bullet$ is an indecomposable direct summand of $\mathrm{Ind}^G_Q(S)$ having Q as a vertex. ∎

Finally, we emphasise that the group $D_Q(G)$ in fact classifies the sources of the indecomposable endo-p-permutation RG-modules with vertex Q, but not such modules themselves. In the next subsection, we show how to overcome this problem.

5.5. **Relative endo-trivial modules and the generalised Dade group of a finite group.** Replacing projectivity by *relative projectivity* with respect to subgroups, or more generally with respect to modules, we can define the *relative endo-trivial modules.* A detailed treatment of relative projectivity with respect to modules can be found Carlson's lecture notes [Car96] and in the author's work in [Las12, Las11, Las13]. We present below only the essential notions that allow us to define *relative endo-trivial modules.* Although adaptation to $\mathcal{O}$ are possible, some of the results we present below are specific to fields, hence we assume $R = k$.

Definition 5.34

Let V be a kG-module. Then, a kG-module M is termed **relatively V-projective**, or simply **V-projective**, if there exists a kG-module N such that $M \mid V \otimes_k N$.

It is easy to see that projectivity relative to V is equivalent to projectivity relative to V^* and it is also equivalent to projectivity relative to $V^* \otimes_k V \cong \mathrm{End}_k(V)$.

Example 5.35

(a) If $H \leq G$, then a kG-module M is H-projective if and only if M is projective relative to the induced kG-module $V =: \mathrm{Ind}_H^G(k)$. Indeed, M is H-projective if and only if

$$M \mid \mathrm{Ind}_H^G(\mathrm{Res}_H^G(M)) \cong \mathrm{Ind}_H^G(k \otimes \mathrm{Res}_H^G(M)) \cong \mathrm{Ind}_H^G(k) \otimes M\,.$$

In other words, projectivity relative to modules generalises projectivity relative to subgroups.

(b) By Properties 2.1(c) ordinary projectivity is just $\{1\}$-projectivity in terms of projectivity relative to subgroups. Thus, it is clear from (a), that ordinary projectivity can be thought of as V-projectivity for $V := \mathrm{Ind}_{\{1\}}^G(k) \cong kG$.
In addition, a projective kG-module is V-projective for any kG-module V. Indeed, if M is projective, then there exists $n \in \mathbb{N}$ such that $M \mid (kG)^n$. Thus, as $V \otimes_k kG \cong (kG)^{\dim_k V}$, we have $M \mid V \otimes_k (kG)^n$.

Remark 5.36

A well-known result of Benson and Carlson (often called the *Benson–Carlson Theorem*) [BC86, Theorem 2.1] states that if M and N are indecomposable kG-modules, then $k \mid M \otimes_k N$ if and only if $N \cong M^*$ and $p \nmid \dim_k M$. An easy consequence of this result (see [Las11, Proposition 2.2.2]) is that the following three assertions are equivalent:

(a) the trivial kG-module k is V-projective;

(b) at least one of the indecomposable direct summands of V has k-dimension coprime to p;

(c) any kG-module is V-projective.

We would like to avoid this situation, which is not interesting. Thus, we call a kG-module V **absolutely p-divisible** provided p divides the k-dimension of all the indecomposable direct summands of V.

Considering kG-modules V which are absolutely p-divisible, we can naturally generalise the definition of an endo-trivial module.

Definition 5.37 ([Las12, Definition 3.1.1])

Let V be an absolutely p-divisible kG-module. A kG-module M is termed **endo-trivial relative to the kG-module V** or simply **V-endo-trivial** iff

$$\mathrm{End}_k(M) \cong M^* \otimes_k M \cong k \oplus (V\text{-proj})\,,$$

where (V-proj) denotes a V-projective kG-module.

Remark 5.36 tells us that the ordinary endo-trivial kG-modules, introduced in §5.1 and §5.3, are V-endo-trivial modules for $V := kG$. In addition, by Remark 5.36, the assumption on the absolute p-divisibility of V ensures that a V-endo-trivial kG-module is not V-projective, and conversely.

Properties 5.38

Let V be an absolutely p-divisible kG-module. Let M and N be V-endo-trivial kG-modules. Then, the following assertions hold.

(a) We have $\dim_k(M)^2 \equiv 1 \pmod p$, and if $V = \mathrm{Ind}_Q^G(k)$ (i.e. if we consider projectivity relative to the p-subgroup $Q \leq G$), then $\dim_k(M)^2 \equiv 1 \pmod{|P:Q|}$ where $P \in \mathrm{Syl}_p(G)$ and $P \geq Q$.

(b) The kG-modules M^* and $M \otimes_k N$ are V-endo-trivial.

(c) If M is indecomposable, then the vertices of M are the Sylow p-subgroups of G. Moreover, if (P, S) is a vertex-source pair for M, then S is a $\mathrm{Res}^G_P(V)$-endo-trivial module, and S occurs with multiplicity one as a direct summand of $\mathrm{Res}^G_P(M)$.

(d) There is a direct sum decomposition $M \cong M_0 \oplus (V\text{-proj})$ where M_0 is the unique indecomposable direct summand of M which is V-endo-trivial.

(e) If $P \in \mathrm{Syl}_p(G)$, then M is V-endo-trivial if and only if $\mathrm{Res}^G_P(M)$ is $\mathrm{Res}^G_P(V)$-endo-trivial.

Proof: (a) Because V is absolutely p-divisible, so is any V-projective kG-module. Hence, the assumption that $M^* \otimes_k M \cong \mathrm{End}_k(M) \cong k \oplus (V\text{-proj})$ yields $\dim_k(M)^2 \equiv 1 \pmod p$. The second claim follows from the fact that any Q-projective kG-module has dimension divisible by $|P:Q|$ (see [CR81, (19.26) Theorem]).
(b) The claim about the dual is immediate from the definition. For the second claim, by the assumption, we have

$$\begin{aligned}(M \otimes_k N)^* \otimes_k (M \otimes_k N) &\cong (M^* \otimes_k M) \otimes_k (N^* \otimes_k N)\\ &\cong (k \oplus (V\text{-proj})) \otimes_k (k \oplus (V\text{-proj})) \cong k \oplus (V\text{-proj})\,.\end{aligned}$$

(c) It is clear that the vertices of M are the Sylow p-subgroups of G as $\dim_k(M)$ is coprime to p by (a). Moreover, we have

$$\mathrm{Res}^G_P(M)^* \otimes_k \mathrm{Res}^G_P(M) \cong \mathrm{Res}^G_P(M^* \otimes_k M) \cong \mathrm{Res}^G_P(k \oplus (V\text{-proj})) \cong k \oplus (\mathrm{Res}^G_P(V)\text{-proj})\,,$$

thus, as $S \mid \mathrm{Res}^G_P(M)$, we have $S \otimes_k S^* \mid k \oplus (\mathrm{Res}^G_P(V)\text{-proj})$. The claim follows, because if k were not a direct summand of $S \otimes_k S^*$, then M would be V-projective and therefore not V-endo-trivial.
(d) Assuming that M decomposes as $M = M_0 \oplus M_1$, then

$$k \oplus (V\text{-proj})) \cong \mathrm{End}_k(M) \cong \mathrm{End}_k(M_0) \oplus \mathrm{Hom}_k(M_0, M_1) \oplus \mathrm{Hom}_k(M_1, M_0) \oplus \mathrm{End}_k(M_1)\,.$$

As a result, the Krull–Schmidt theorem forces the trivial module k to be a direct summand of either $\mathrm{End}_k(M_0)$, or $\mathrm{End}_k(M_1)$. Indeed, if it were not the case, k would be a direct summand of $\mathrm{Hom}_k(M_0, M_1)$ or $\mathrm{Hom}_k(M_1, M_0)$. But the two latter modules being dual to each other, $k \oplus k$ would be a direct summand of $\mathrm{End}_k(M)$, which is not possible as V is absolutely p-divisible. Thus we may assume that $\mathrm{End}_k(M_0) \cong k \oplus (V\text{-proj})$ and $\mathrm{End}_k(M_1)$ is V-projective, possibly zero. It follows that M_0 is V-endo-trivial and M_1 is V-projective, and we can iterate this process until M_0 is indecomposable.

(e) The necessary condition is clear by (c). Conversely, as $P \in \mathrm{Syl}_p(G)$, M is P-projective and

$$\begin{aligned} M \mid \mathrm{Ind}_P^G \circ \mathrm{Res}_P^G(M) &\cong \mathrm{Ind}_P^G(k \oplus (\mathrm{Res}_P^G(V)\text{-proj})) \\ &\cong \mathrm{Ind}_P^G(k) \oplus \mathrm{Ind}_P^G(\mathrm{Res}_P^G(V)\text{-proj}) = \mathrm{Ind}_P^G(k) \oplus (V\text{-proj})\,. \end{aligned}$$

Moreover, M is V-projective-free by assumption, thus the Krull–Schmidt theorem yields that $M \mid \mathrm{Ind}_P^G(k)$. In consequence, P being a vertex of M, k is a source of M. ■

With these stability properties, we can now copy the procedure we have seen for endo-trivial modules over p-groups in order to obtain a group structure from the class of V-endo-trivial modules, which generalises the *group of endo-trivial kG-modules*, based on the previous proposition.

Proposition-Definition 5.39 ([Las11, Proposition 3.5.1])

Let V be an absolutely p-divisible kG-module.

(a) The relation $\sim_V$ defined on the class of V-endo-entriovial kG-modules by setting

$$M \sim N \;:\Leftrightarrow\; M_0 \cong N_0$$

is an equivalence relation and we let $T_V(G)$ denote the resulting set of equivalence classes.

(b) The set $T_V(G)$ endowed with the composition law

$$\begin{array}{rccc} +: & T_V(G) \times T_V(G) & \longrightarrow & T_V(G) \\ & ([M],[N]) & \mapsto & [M]+[N] := [M \otimes_k N]\,. \end{array}$$

is an abelian group, called the **group of V-endo-trivial modules** of G. The zero element is the class $[k]$ of the trivial kG-module and the opposite of a class $[M]$ is the class $[M^*]$ of the k-dual.

For further general properties of groups of V-endo-trivial kG-modules, we refer to [Las12, Las11]. Our aim here is to show how this construction can be used in order to define a group structure similar to that of the Dade group of a p-group for an arbitrary finite group of order divisible by p. In this view, we need to restrict our attention to endo-p-permutation modules, which are the equivalent of the capped endo-permutation modules over p-groups, and which we are going to regard as V-endo-trivial kG-modules for a well-chosen absolutely p-divisible kG-module V defined below.

Notation 5.40

Given $P \in \mathrm{Syl}_p(G)$, set $\mathcal{F}_G := \{Q \lneq P\}$ and $V(\mathcal{F}_G) := \bigoplus_{Q \in \mathcal{F}_G} \mathrm{Ind}_Q^G(k)$.

Clearly, $V(\mathcal{F}_G)$-projectivity corresponds to projectivity relative to the family of all non maximal p-subgroups of G and hence does not depend on the choice of the Sylow p-subgroup P. It is therefore an absolutely p-divisible kG-module. With this notion we can come back to endo-p-permutation kG-modules.

Proposition-Definition 5.41 ([Las13, Proposition 5.2])

Let $P \in \mathrm{Syl}_p(G)$. Let M be an endo-p-permutation kG-module. Then, M is called **strongly capped** if it fulfils the following equivalent conditions:

(a) M is $V(\mathcal{F}_G)$-endo-trivial;

(b) $\mathrm{Res}^G_P(M)$ is $V(\mathcal{F}_P)$-endo-trivial;

(c) M has a unique indecomposable direct summand with vertex P, called the *cap* of M and denoted $\mathrm{Cap}(M)$, and, in addition, if S is a kP-source for $\mathrm{Cap}(M)$, then the multiplicity of S as a direct summand of $\mathrm{Res}^G_P(M)$ is one;

(d) $\mathrm{End}_k(M) \cong k \oplus N$ where N is a p-permutation kG-module, all of whose indecomposable direct summands have a vertex strictly contained in P.

Proving the equivalence of the conditions (a) to (d) is not difficult, but requires a series of technical results on V-projectivity and V-endo-trivial modules, which we have not presented here. We refer therefore to [Las13, Proposition 5.2] for a complete proof. However, we easily see that this subclass of the class of endo-p-permutation kG-modules has all the good stability properties, we may expect it to have.

Lemma 5.42

The class of strongly capped endo-p-permutation kG-modules is closed under taking k-duals, tensor products over k and restrictions to a subgroup containing a Sylow p-subgroup.

Proof: By Properties 5.26(a) and Properties 5.38(b) taking duals and tensor products are stable operations for both the classes of endo-p-permutation modules and of $V(\mathcal{F}_G)$-endo-trivial modules, therefore they are stable for strongly capped endo-p-permutation modules. Now, if $H \leq G$ contains a Sylow p-subgroup of G, then the restriction to H of an endo-p-permutation module is an endo-p-permutation module by Properties 5.26(c) and the restriction to H of a $V(\mathcal{F}_G)$-endo-trivial module is a $V(\mathcal{F}_H)$-endo-trivial (see [Las13, Lemma 3.1]). Thus, the restriction to H of a strongly capped endo-p-permutation module is strongly capped. ■

This leads us to the definition of a *generalised Dade group of a finite group* as follows.

Proposition-Definition 5.43

(a) The relation $\sim$ defined on the class of all strongly capped endo-p-permutation kG-modules by setting
$$M \sim N :\Leftrightarrow \mathrm{Cap}(M) \cong \mathrm{Cap}(N)$$
is an equivalence relation and we let $D_k(G)$ denote the resulting set of equivalence classes.

(b) The set $D_k(G)$ endowed with the composition law
$$\begin{array}{rccc} +: & D_k(G) \times D_k(G) & \longrightarrow & D_k(G) \\ & ([M],[N]) & \mapsto & [M]+[N] := [M \otimes_k N]\,. \end{array}$$
is an abelian group, called the **generalised Dade group of** G. The zero element is the class $[k]$ of the trivial kG-module and the opposite of a class $[M]$ is the class $[M^*]$ of the k-dual.

(c) The group $D_k(G)$ can be identified with a subgroup of $T_{V(\mathcal{F}_G)}(G)$ through the natural embedding
$$\begin{array}{rccc} \imath: & D_k(G) & \longrightarrow & T_{V(\mathcal{F}_G)}(G) \\ & [M] & \longmapsto & [M]\,. \end{array}$$

Proof: Assertion (a) is clear. Lemma 5.42 and the uniqueness of the caps ensure that the assignment

$$([M],[N]) \longmapsto [M \otimes_k N]$$

is a well-defined composition law for $D_k(G)$. Moreover, the map $\imath$ is well-defined as $\sim$ is the restriction to the class of strongly capped endo-p-permutation of the equivalence relation $\sim_{V(\mathcal{F}_G)}$. It is a group homomorphism because the addition is induced by $\otimes_k$ on both sides. It is injective because $\ker(\imath) = \{[k]\}$. Indeed, if $\imath([M]) = [k]$, then $M \sim_{V(\mathcal{F}_G)} k$ which is equivalent to $M \sim k$ because both M and k are strongly capped endo-p-permutation modules, proving (b) and (c). ■

Notice that any ordinary endo-trivial module is strongly capped, and in particular, so is any one-dimensional kG-module. Therefore, up to identifications, the groups $T_k(G)$ and $X_k(G)$ can also be viewed as subgroups of $D_k(G)$ and we have a series of subgroup inclusions

$$X_k(G) \leq T_k(G) \leq D_k(G) \leq T_{V(\mathcal{F}_G)}(G)\,.$$

Remark 5.44

If G is a p-group, then certainly the *generalised Dade group* we have constructed above is isomorphic to the Dade group of G as defined in Proposition-Definition 5.4. In this case, a strongly capped endo-permutation kP-module is simply a capped endo-permutation such that its cap has multiplicity one. So certainly, this definition can be made over $R = \mathcal{O}$ and we obtain a *generalised Dade group* $D_{\mathcal{O}}(P)$.

The structure of the generalised Dade group can be linked to Urfer's characterisation of the indecomposable endo-p-permutation modules with full vertex through the G-stable points of the Dade group of a Sylow p-subgroup via the short exact sequence given by the following theorem.

Theorem 5.45 ([Las13, Theorem 7.3])

Let $P \in \mathrm{Syl}_p(G)$. Write $X := X_k(N_G(P))$ for the group of one-dimensional $kN_G(P)$-modules and let $\Gamma(X)$ be the subgroup of $D_k(G)$ consisting of the classes of the kG-Green correspondents of the modules in X. Then, restriction from G to P yields a short exact sequence

$$0 \longrightarrow \Gamma(X) \hookrightarrow D_k(G) \xrightarrow{\mathrm{Res}^G_P} D_k(P)^{G\text{-}st} \longrightarrow 0$$

of abelian groups.

Since $\Gamma(X) \cong X$ is finite and the Dade Group $D_k(P)$ is finitely generated, so is the generalised Dade group.

Corollary 5.46 ([Las13, Corollary 7.3])

The generalised Dade group $D_k(G)$ of a finite group G is finitely generated.

We refer to [Las12, Las13] for computations of the structure the generalised Dade group in some concrete examples; for instance groups with a cyclic Sylow p-subgroup, groups with a Klein-four Sylow 2-subgroup, p-nilpotent groups, or $\mathrm{GL}_3(p)$ in its defining characteristic.

Finally, we note that as in the case of p-groups, the subgroup $D_k^{\Omega}(G)$ of $D_k(G)$ generated by all the relative syzygy modules with respect to subfamilies of $\mathcal{F}_G$ plays an important role in the structure of the generalised Dade group. For example, [Las13, Corollary 12.8] tells us that when p is odd and the normaliser $N_G(P)$ of a Sylow p-subgroup of G controls fusion in G, then

$$D(G) = D_k^{\Omega}(G) + \Gamma(X)\,.$$

6. $\mathcal{O}G$-modules which are necessarily $\mathcal{O}G$-lattices

In §3, §4, and §5 we allowed ourselves to talk about $\mathcal{O}G$-modules, when we actually meant $\mathcal{O}G$-lattices. We now explain why this is not an issue. Permutation $\mathcal{O}G$-modules are clearly $\mathcal{O}$-free by definition, and so are p-permutation $\mathcal{O}G$-modules. We prove below that endo-p-permutation $\mathcal{O}G$-modules are also necessarily free when regarded as $\mathcal{O}$-modules, that is, $\mathcal{O}G$-lattices. In particular, so are endo-trivial and endo-permutation $\mathcal{O}G$-modules.

Proposition 6.1

If M is an $\mathcal{O}G$-module such that $\mathrm{End}_{\mathcal{O}}(M)$ is free when regarded as an $\mathcal{O}$-module, then so is M when regarded as an $\mathcal{O}$-module.

Proof: Since $\mathcal{O}$ is a discrete valuation ring, it is in particular a principal ideal domain. Thus, by the structure theorem for finitely generated modules over principal ideal domains, the module M, regraded as an $\mathcal{O}$-module, admits a direct sum decomposition of the form

$$M \cong \bigoplus_{i=1}^{r} \left(\bigoplus_{j_i=1}^{s_i} \mathcal{O}/\mathfrak{p}^{n_i} \right) \oplus (\mathcal{O}\text{-free summands}),$$

where r is a non-negative integer and $s_i, n_i\,(1 \le i \le r)$ are positive integers. Now, we claim that if the torsion part of M is not trivial, then neither is the torsion part of $\mathrm{End}_{\mathcal{O}}(M)$. Indeed, for every $1 \le i \le r$ we have that

$$\mathrm{End}_{\mathcal{O}}(\mathcal{O}/\mathfrak{p}^{n_i}) \cong \mathcal{O}/\mathfrak{p}^{n_i}$$

as $\mathcal{O}$-module, therefore the $\mathcal{O}$-linear endomorphisms of M have the form

$$\mathrm{End}_{\mathcal{O}}(M) \cong \bigoplus_{i=1}^{r} \left(\bigoplus_{j_i=1}^{s_i} \mathrm{End}_{\mathcal{O}}\left(\mathcal{O}/\mathfrak{p}^{n_i}\right) \right) \oplus X \cong \bigoplus_{i=1}^{r} \left(\bigoplus_{j_i=1}^{s_i} \mathcal{O}/\mathfrak{p}^{n_i} \right) \oplus X$$

(as $\mathcal{O}$-module) for some $\mathcal{O}$-module X. Since $\mathrm{End}_{\mathcal{O}}(M)$ is assumed to be $\mathcal{O}$-free, this forces M to be $\mathcal{O}$-free as well. ■

Corollary 6.2

Any endo-p-permutation $\mathcal{O}G$-module is free when regarded as an $\mathcal{O}$-module. In particular, so is any endo-trivial or any endo-permutation $\mathcal{O}G$-module.

Proof: If M is an endo-p-permutation $\mathcal{O}G$-module, then by definition $\mathrm{End}_{\mathcal{O}}(M)$ is a p-permutation $\mathcal{O}G$-module, hence $\mathcal{O}$-free and the claim follows from Proposition 6.1. Endo-trivial and endo-permutation $\mathcal{O}G$-modules are endo-p-permutation modules. The claim follows. ■

7. Lifting from positive characteristic to characteristic zero

We now turn to lifting results. It turns out that most of the classes of kG-modules, which we have introduced so far in this survey consist of modules which are liftable to characteristic zero. Being liftable is a rather rare property and proving liftability of modules defined by a common property is in general a difficult task. It is the reason why the results presented in this section were obtained one-by-one over a long period of time. We present them with their proofs, provided the approach involved do not go beyond the methods and techniques introduced so far.

In fact, amongst finitely generated kG-modules very few classes of modules are known to be liftable to $\mathcal{O}G$-lattices. The Fong–Swan theorem (see e.g. [Dor71, Theorem 72.1]) asserts that all simple modules of p-soluble groups are liftable, and Hiß [His85, His87] studies groups whose brauer-characters are liftable and gives a converse to Fong–Swan theorem result. It is also well-known that projective kG-modules lift to projective $\mathcal{O}G$-modules in a unique way. Scott proved that this remarkable property can be generalised to p-permutation kG-modules, and, in turn, the latter implies that several other related classes of modules introduced in Section 5 are liftable.

7.1. **Lifting p-permutation kG-modules.** The lifting of p-permutation modules is due to Scott, who proved that the k-endomorphism ring of a transitive permutation kG-module is liftable.

Theorem 7.1 ([Sco73, Proposition 1])

(a) If L_1 and L_2 are p-permutation $\mathcal{O}G$-modules, then the natural homomorphism

$$\mathrm{Hom}_{\mathcal{O}G}(L_1, L_2) \longrightarrow \mathrm{Hom}_{kG}(L_1/\mathfrak{p}L_1, L_2/\mathfrak{p}L_2),\ \varphi \mapsto \bar{\varphi}$$

induced by reduction modulo $\mathfrak{p}$ is surjective.

(b) If L is an indecomposable p-permutation $\mathcal{O}G$-modules with vertex Q, then $L/\mathfrak{p}L$ is an indecomposable p-permutation kG-module with vertex Q.

(c) Every p-permutation kG-module lifts to a p-permutation $\mathcal{O}G$-module, unique up to isomorphism.

Proof: (a) By the characterisation of p-permutation kG-modules in Proposition-Definition 4.2 it is enough to prove that statement (a) holds for transitive permutation $\mathcal{O}G$-modules. So assume $L_1 = \mathrm{Ind}_{Q_1}^G(\mathcal{O})$ and $L_2 = \mathrm{Ind}_{Q_2}^G(\mathcal{O})$ for some p-subgroups $Q_1, Q_2 \leq G$. Applying Frobenius' reciprocity twice and Mackey's formula we obtain

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}G}(L_1, L_2) &\cong \mathrm{Hom}_{\mathcal{O}Q_2}(\mathrm{Res}^G_{Q_2}\mathrm{Ind}^G_{Q_1}(\mathcal{O}), \mathcal{O})\\ &\cong \bigoplus_{x\in[Q_2\backslash G/Q_1]} \mathrm{Hom}_{\mathcal{O}Q_2}(\mathrm{Ind}^{Q_2}_{Q_2\cap {}^xQ_1}(\mathcal{O}), \mathcal{O})\\ &\cong \bigoplus_{x\in[Q_2\backslash G/Q_1]} \mathrm{Hom}_{\mathcal{O}(Q_2\cap {}^xQ_1)}(\mathcal{O}, \mathcal{O}) \cong \bigoplus_{x\in[Q_2\backslash G/Q_1]} \mathcal{O}\,.\end{aligned}$$

Hence the $\mathcal{O}$-rank of $\mathrm{Hom}_{\mathcal{O}G}(L_1, L_2)$ is $|Q_2\backslash G/Q_1|$. The same argument with k instead of $\mathcal{O}$ shows that the k-dimension of $\mathrm{Hom}_{kG}(L_1/\mathfrak{p}L_1, L_2/\mathfrak{p}L_2)$ is also $|Q_2\backslash G/Q_1|$ and surjectivity follows.

(b) As L is indecomposable, $\mathrm{End}_{\mathcal{O}G}(L)$ is a local ring. By (a), $\mathrm{End}_{kG}(L/\mathfrak{p}L)$ is isomorphic to a quotient of $\mathrm{End}_{\mathcal{O}G}(L)$, hence is also local (see e.g. [Lin18a, Corollary 4.4.5]), and thus $L/\mathfrak{p}L$ is indecomposable. Now, by Higman's criterion, there exists $\varphi \in \mathrm{End}_{\mathcal{O}Q}(L)$ such that $\mathrm{Id}_L = \mathrm{tr}_Q^G(\varphi)$ and Q is minimal with this property. Hence $\mathrm{Id}_{L/\mathfrak{p}L} = \mathrm{tr}_Q^G(\bar{\varphi})$ and thus Q contains a vertex of $L/\mathfrak{p}L$. On the other hand, using the fact that L has a trivial source by Proposition-Definition 4.2, certainly $\mathcal{O} \mid \mathrm{Res}_Q^G(L)$, implying that $k \mid \mathrm{Res}_Q^G(L/\mathfrak{p}L)$. As k has vertex Q, we obtain that Q is contained in a vertex of $L/\mathfrak{p}L$, proving (b).

(c) Again by the characterisation of p-permutation $\mathcal{O}G$-lattices in Proposition-Definition 4.2, it suffices to prove that the claim holds for an indecomposable p-permutation kG-module. So, let M be an indecomposable kG-module with vertex Q and trivial source. Then, $M \mid \mathrm{Ind}_Q^G(k)$ and there exists an idempotent $\iota \in \mathrm{End}_{kG}(\mathrm{Ind}_Q^G(k))$ such that $M = \iota(\mathrm{Ind}_Q^G(k))$, which is unique up to conjugacy (see e.g. [Lin18a, Corollary 4.6.10]). Now, by (a) the canonical map

$$\mathrm{End}_{\mathcal{O}G}(\mathrm{Ind}_Q^G(\mathcal{O})) \twoheadrightarrow \mathrm{End}_{kG}(\mathrm{Ind}_Q^G(k))$$

is surjective. Therefore, by the lifting theorem for idempotents, there exists an idempotent $\pi \in \mathrm{End}_{\mathcal{O}G}(\mathrm{Ind}_Q^G(\mathcal{O}))$, unique up to conjugacy, such that ι is the reduction modulo $\mathfrak{p}$ of π. Then $L := \pi(\mathrm{Ind}_Q^G(\mathcal{O}))$ is a direct summand of $\mathrm{Ind}_Q^G(\mathcal{O})$ such that $L/\mathfrak{p}L \cong M$, unique up to isomorphism, which is indecomposable with vertex Q by part (b). ■

This leads to the following character-theoretic characterisations of p-permutations modules.

Lemma 7.2 ([Sco73, Theorem 5], [Lan81, Lemma 2 and Corollary 1], [Lan83, Lemma 12.6])

Let $\widehat{M}$ be an indecomposable p-permutation $\mathcal{O}G$-module let $M := \widehat{M}/\mathfrak{p}\widehat{M}$ be its reduction modulo $\mathfrak{p}$, and let $\chi_{\widehat{M}}$ be the character afforded by $K \otimes_{\mathcal{O}} \widehat{M}$. Then the following assertions hold.

(a) If $Q \leq G$ is a p-subgroup, then $\dim_k(\mathrm{soc}(\mathrm{Res}_Q^G(M))) = \langle \chi_{\widehat{M}}, 1_Q \rangle_Q$.

(b) If $x \in G$ is a p-element, then $\chi_{\widehat{M}}(x)$ is equal to the multiplicity of the trivial module as a direct summand of $\mathrm{Res}_{\langle x \rangle}^G(M)$. In particular, $\chi_{\widehat{M}}(x)$ is a non-negative integer.

(c) If $x \in G$ is a p-element, then $\chi_{\widehat{M}}(x) \neq 0$ if and only if x belongs to a vertex of M.

Proof: (a) Let $S \leq G$ be a vertex of M. Then $M \mid \mathrm{Ind}_S^G(k)$ and by Mackey's formula and Lemma 3.6 any indecomposable direct summand of $\mathrm{Res}_Q^G(M)$ is of the form

$$\mathrm{Ind}_{Q\cap {}^gS}^Q \mathrm{Res}_{Q\cap {}^gS}^G(k) = \mathrm{Ind}_{Q\cap {}^gS}^Q(k)$$

for some $g \in G$. Moreover,

$$\dim_k(\mathrm{soc}(\mathrm{Ind}_{Q\cap {}^gS}^Q(k))) = 1 = \langle \mathrm{Ind}_{Q\cap {}^gS}^Q(1_{Q\cap {}^gS}), 1_Q \rangle_Q$$

where $\mathrm{Ind}_{Q\cap {}^gS}^Q(1_{Q\cap {}^gS})$ is the character afforded by $\mathrm{Ind}_{Q\cap {}^gS}^Q(\mathcal{O})$. Summing over the direct summands of $\mathrm{Res}_Q^G(M)$ yields the claim.

(b) Letting $Q := \langle x \rangle$ in the proof of (a), we obtain

$$\mathrm{Ind}_{Q\cap {}^gS}^Q(1_{Q\cap {}^gS})(x) = \begin{cases} 1 & \text{if } Q \cap {}^gS = Q\,, \\ 0 & \text{otherwise} \end{cases}$$

and the claim follows.

(c) Because $S \in \mathrm{vtx}(M)$, we have $\langle x \rangle \cap {}^gS = \langle x \rangle$ for some $g \in G$ if and only if $x \in {}^gS$, as required. ■

7.2. **Lifting endo-permutation kP-modules.** The question whether endo-permutation over p-groups form a class of liftable modules was open for a long time. As we will see in the next subsection, Alperin proved in 2001 that the subclass endo-trivial modules is. However, the result for endo-permutation modules was only obtained in 2006 by Bouc as a consequence of their classification.

The first observation to make is as above that the reduction modulo $\mathfrak{p}$ of endo-permutation $\mathcal{O}P$-modules is extremely well-behaved.

Lemma 7.3

Let L be an endo-permutation $\mathcal{O}P$-module and consider $L/\mathfrak{p}L$ its reduction modulo $\mathfrak{p}$. Then, the following assertions hold:

(a) $L/\mathfrak{p}L$ is an endo-permutation kP-module;

(b) L is indecomposable if and only if $L/\mathfrak{p}L$ is;

(c) if L is indecomposable, then L and $L/\mathfrak{p}L$ have the same vertices;

(d) reduction modulo $\mathfrak{p}$ induces a well-defined group homomorphism

$$\pi_{\mathfrak{p}} : D_{\mathcal{O}}(P) \longrightarrow D_k(P), [L] \mapsto [L/\mathfrak{p}L]\,.$$

Proof: See Lemma 7.16 for a more general version of Assertions (a)–(c). Assertion (d) is immediate from the definition of the Dade group and (a)–(c). ■

Theorem 7.4 ([Bou06, Corollary 8.5])

If P is a p-group, then the following assertions hold:

(a) the group homomorphism $\pi_{\mathfrak{p}} : D_{\mathcal{O}}(P) \longrightarrow D_k(P)$ is surjective;

(b) any endo-permutation kP-module lifts to an endo-permutation kG-module.

The idea of the proof is essentially that it suffices to prove that the generators of the Dade group $D_k(P)$ are liftable modules.

Proof: (Sketch.) Any relative syzygy module $\Omega_{P/Q}(k)$ lifts to a relative syzygy module over $\mathcal{O}$. In characteristic 2, when P is generalised quaternion, then straightforward calculations show that any indecomposable capped endo-permutation whose class lies in $D_k(P)^{\text{tors}}_{\text{ex}}$ is liftable as well by straightforward calculations. Thus, it follows from Theorem 5.15 that any element of $D_k(P)$ is liftable, proving (a). Since any endo-permutation kP-module can be described in terms of the indecomposable capped ones, (b) follows. ■

Going further, it is easy to describe the kernel of $\pi_{\mathfrak{p}}$, and we aim to prove that $\pi_{\mathfrak{p}}$ is in fact a split morphism.

Lemma 7.5

The kernel of $\pi_{\mathfrak{p}} : D_{\mathcal{O}}(P) \longrightarrow D_k(P)$ is isomorphic to the group $X_{\mathcal{O}}(P)$ of $\mathcal{O}P$-lattices with $\mathcal{O}$-rank equal to 1.

Proof: If L is an indecomposable endo-permutation $\mathcal{O}P$-module with $[L] \in \ker(\pi_p)$, then clearly $\dim_k(L/\mathfrak{p}L) = 1$, hence $\mathrm{rk}_{\mathcal{O}}(L) = 1$. If, conversely, $\mathrm{rk}_{\mathcal{O}}(L) = 1$, then the one-dimensional kP-module $L/\mathfrak{p}L$ must be trivial since there are no non-trivial p^n-th roots of unity in k. Therefore $[L] \in \ker(\pi_p)$. ■

Next, we explain why studying the Dade group of G over k is equivalent to Puig's approach via the *Dade group of Dade P-algebras*, but not over $\mathcal{O}$. We already explained in §5.1, that the endomorphism algebra $\mathrm{End}_R(M)$ of an endo-permutation RP-module M is naturally endowed with the structure of a so-called *Dade P-algebra* (i.e. an $\mathcal{O}$-simple permutation P-algebra whose Brauer quotient with respect to P is non-zero). Furthermore, there exists also a version of the Dade group, denoted by $D_R^{alg}(P)$, obtained by defining an equivalence relation on the class of all Dade P-algebras rather than capped endo-permutation RP-lattices, where multiplication is given by the tensor product over R. We refer to [Thé95, §28-29] for this construction. This induces a canonical homomorphism

$$d_R : D_R(P) \longrightarrow D_R^{alg}(P)\,, \quad [M] \mapsto [\mathrm{End}_R(M)]\,,$$

which is surjective by [Thé95, Proposition 28.12]. The identity element of $D_R^{alg}(P)$ being the class of the trivial P-algebra R, it follows that the kernel of d_R is isomorphic to $X_R(P)$ when $R = \mathcal{O}$, whereas it is trivial when $R = k$. Now, reduction modulo $\mathfrak{p}$ also induces a group homomorphism

$$\pi_p^{alg} : D_{\mathcal{O}}^{alg}(P) \longrightarrow D_k^{alg}(P)\,, \quad [A] \mapsto [A/\mathfrak{p}A]\,.$$

Because $\mathrm{End}_{\mathcal{O}}(M)/\mathfrak{p}\mathrm{End}_{\mathcal{O}}(M) \cong \mathrm{End}_k(M/\mathfrak{p}M)$ for any $\mathcal{O}P$-lattice, it follows that we have a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & X_{\mathcal{O}}(P) \arrow[r] \arrow[d, hook] & 1 \arrow[d, hook] \\
X_{\mathcal{O}}(P) \arrow[r, hook] \arrow[d] & D_{\mathcal{O}}(P) \arrow[r, two heads, "d_{\mathcal{O}}"] \arrow[d, "\pi_p"'] & D_{\mathcal{O}}^{alg}(P) \arrow[d, "\pi_p^{alg}"] \\
1 \arrow[r, hook] & D_k(P) \arrow[r, "\cong", "d_k"'] & D_k^{alg}(P)
\end{tikzcd}$$

The injectivity of π_p^{alg} follows from the commutativity of the bottom-right square because

$$\ker(\pi_p^{alg} d_{\mathcal{O}}) = \ker(d_k \pi_p) = \ker(\pi_p) = X(P)$$

and its image under $d_{\mathcal{O}}$ yields $\ker(\pi_p^{alg}) = d_{\mathcal{O}}(X(P)) = \{1\}$. The surjectivity of π_p implies that π_p^{alg} is also surjective, hence an isomorphism, implying that

$$D_k(P) \cong D_k^{alg}(P) \cong D_{\mathcal{O}}^{alg}(P)\,.$$

Also, finding a group-theoretic section of π_p is equivalent to finding a group-theoretic section of $d_{\mathcal{O}}$.

In odd characteristic, the explicit construction of such a group-theoretic section for $d_{\mathcal{O}}$ is due to Puig and can be found in print in [Thé95, Remark 29.6]. Below, we translate this construction from Dade P-algebras to endo-permutation modules. In characteristic 2 the question of the existence of such a section was open for a long time and eventually proved in [LT19], relying on

Bouc's classification of endo-permutation kP-modules.

Warning: For the remainder of this subsection we need to identify the Dade group $D_R(P)$ with the generalised Dade group of P over R, what we are allowed to do by Remark 5.44. So the elements of the classes in $D_R(P)$ are not just capped endo-permutation RP-modules, but *strongly capped endo-permutation RP-modules.* This is one of the key arguments used in [LT19] which makes the construction of a section in characteristic 2 possible.

Theorem 7.6 ([LT19, Theorem 1.1])

(a) The group homomorphism $\pi_p : D_{\mathcal{O}}(P) \to D_k(P)$ has a group-theoretic section.
(b) There is a group isomorphism $D_{\mathcal{O}}(P) \cong X_{\mathcal{O}}(P) \times D_k(P)$.

We prove this theorem in several steps.

Proof of Theorem 7.6(b): By Lemma 7.5, $\ker(\pi_p) \cong X_{\mathcal{O}}(P)$, hence it follows from Assertion (a) that $D_{\mathcal{O}}(P) \cong X(P) \times D_k(P)$. ■

To prove Assertion (a), we need to consider determinants. Recall that given an $\mathcal{O}P$-lattice L, we may consider the composition of the underlying representation of P with the determinant homomorphism $\det : \mathrm{GL}(L) \longrightarrow \mathcal{O}^\times$. This is a linear character of P and is called the *determinant* of L. If the determinant of L is the trivial character, then we say that L is an $\mathcal{O}P$-lattice of determinant 1. It is immediate that the $\mathcal{O}$-dual L^* of an $\mathcal{O}P$-lattice L of determinant 1 also has determinant 1 as the action of $g \in P$ on $\varphi \in L^*$ is given by $(g\cdot\varphi)(x) = \varphi(g^{-1}x)$ for all $x \in L$, and hence $\det(g, L^*) = \det(g^{-1}, L) = \det(g, L)^{-1}$. Similarly, the tensor product $L \otimes_{\mathcal{O}} N$ of two $\mathcal{O}P$-lattices L and N of determinant 1 also has determinant 1 since the determinant of a tensor product satisfies the well-known property $\det(g, L \otimes_{\mathcal{O}} N) = \det(g, L)^{\mathrm{rk}_{\mathcal{O}} N} \cdot \det(g, N)^{\mathrm{rk}_{\mathcal{O}} L}$ for any $g \in P$.

Now, amongst the lifts of a strongly capped endo-permutation kP-module M, there always exists one which has determinant 1 (see e.g. [Thé95, Lemma 28.1]), on the one hand because we assume that k is large enough, and on the other hand because $\dim_k(M)$ is prime to p as M is strongly capped (and not just capped!). This lift of determinant 1 is unique, up to isomorphism, and will be written Φ_M. It follows from the remarks above that $\Phi_M^* \cong \Phi_{M^*}$, and if N is another strongly capped endo-permutation kP-module, then $\Phi_{M\otimes_k N} \cong \Phi_M \otimes_{\mathcal{O}} \Phi_N$. The construction of a section for $\pi_{\mathfrak{p}}$ in odd characteristic then relies on the following crucial properties which fail when $p = 2$.

Lemma 7.7

Let p be an odd prime. Then the following assertions hold:
(a) any permutation $\mathcal{O}P$-module has determinant 1; and
(b) if $[L] \in D_{\mathcal{O}}(P)$ is such that L is indecomposable and has determinant 1, then any element of the class $[L]$ has determinant 1.

Proof: Let $L := \mathcal{O}X$ be a permutation $\mathcal{O}P$-module with $\mathcal{O}$-basis X permuted under the action of P. Then, for any $g \in P$, the permutation action of g on X decomposes as a product of cycles of odd length, because the order of g is odd. Any such cycle is an even permutation, so the determinant of the action of g on L is 1, proving (a).

Next, let $[L] \in D_{\mathcal{O}}(P)$ be such that L is indecomposable and has determinant 1. By the definition of the generalised Dade group, an arbitrary element of the class $[L]$ has the form $L \otimes_{\mathcal{O}} \mathcal{O}X$ where $\mathcal{O}X$ is a strongly capped permutation $\mathcal{O}P$-module. Since $\mathcal{O}X$ has determinant 1 by (a), so does the tensor product $L \otimes_{\mathcal{O}} \mathcal{O}X$ and the claim follows. ∎

Then, to obtain Theorem 7.6(a) when $p \geq 3$, it suffices to prove the following Lemma.

Lemma 7.8

Suppose that p is an odd prime. Then the map

$$\sigma_{\mathfrak{p}} : D_k(P) \longrightarrow D_{\mathcal{O}}(P)\,,\ [M] \to [\Phi_M]$$

is a well-defined group homomorphism which is a section for $\pi_{\mathfrak{p}}$.

Proof: First, we claim that if M_1 and M_2 are two strongly capped endo-permutation kP-modules and $N := \mathrm{Cap}(M_1 \otimes_k M_2)$, then $\mathrm{Cap}(\Phi_{M_1} \otimes_{\mathcal{O}} \Phi_{M_2}) = \Phi_N$. Indeed, on the one hand $\Phi_{M_1 \otimes_k M_2} \cong \Phi_{M_1} \otimes_{\mathcal{O}} \Phi_{M_2}$ and on the other hand by definition of the generalised Dade group there exists a strongly capped permutation kG-module kX such that $M_1 \otimes_k M_2 = N \otimes_k kX$ and so $\Phi_{M_1 \otimes_k M_2} \cong \Phi_N \otimes_{\mathcal{O}} \Phi_{kX} = \Phi_N \otimes_{\mathcal{O}} \mathcal{O}X$ by Lemma 7.7 as we assume that p is odd, proving that Φ_N is the cap of $\Phi_{M_1} \otimes_{\mathcal{O}} \Phi_{M_2}$.

Now, it follows from the claim that if M is a strongly capped endo-permutation module, then $[\Phi_M] = [\Phi_{\mathrm{Cap}(M)}]$, proving that $\sigma_{\mathfrak{p}}$ is well-defined, and it is clear that it is a group homomorphism again because having determinant 1 is preserved by the tensor product over $\mathcal{O}$. Finally, it is straightforward that $\pi_{\mathfrak{p}} \circ \sigma_{\mathfrak{p}}$ is the identity on $D_k(P)$. ∎

Let us now turn to characteristic 2.

Observation 7.9

Lemma 7.8 fails when $p = 2$ in general. It is clear that a (strongly capped) permutation kP-module always lifts in a unique way to a (strongly capped) permutation $\mathcal{O}P$-module. However, we emphasise that this lift may be different from the lift of determinant 1. It follows that two strongly capped endo-permutation $\mathcal{O}P$-modules in the same class in $D_{\mathcal{O}}(P)$ need not have the same determinant, and so the map $\sigma_{\mathfrak{p}}$ is not well-defined when $p = 2$. Moreover, it should be noted that for $P = C_{2^n}$, there are two natural lifts for $\Omega(k)$. One of them is $\Omega(\mathcal{O})$, but it does not have determinant 1 in this case. Indeed, if $P = \langle g \rangle$ we obtain from the short exact sequence $0 \to \Omega(\mathcal{O}) \to \mathcal{O}P \to \mathcal{O} \to 0$ that

$$\det(g, \Omega(\mathcal{O}))\det(g, \mathcal{O}) = \det(g, \mathcal{O}P) = -1$$

since the action by permutation of g on P is given by a cycle of even length, hence an odd permutation, and it follows that $\det(g, \Omega(\mathcal{O})) = -1$ as $\det(g, \mathcal{O}) = 1$. The other one is $\Phi_{\Omega(k)}$, which turns out to be isomorphic to $\mathcal{O}^- \otimes_{\mathcal{O}} \Omega(\mathcal{O})$, where $\mathcal{O}^-$ denotes the one-dimensional module with the generator of C_{2^n} acting by -1.

Proof of Theorem 7.6(a) when $p = 2$. When $p = 2$ it follows from Bouc's precise classification of endo-permutation modules in [Bou06, Section 8] that the structure of the Dade group is

$$D_k(P) \cong (\mathbb{Z}/2\mathbb{Z})^a \times (\mathbb{Z}/4\mathbb{Z})^b \times \mathbb{Z}^c$$

for some non-negative integers a, b, c. Now, choose a generator for each factor $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, or $\mathbb{Z}$. We claim that the class of each of these generator can be lifted to an element of $D_{\mathcal{O}}(P)$ of the same order. First, let $[M] \in D_k(P)$ be an element of order 2. Then $M \cong M^*$ and so

$\Phi_M \cong \Phi_{M^*} \cong \Phi_M^*$. Hence $[\Phi_M] = [\Phi_M]^{-1}$ in $D_{\mathcal{O}}(P)$, proving that $[\Phi_M]$ has order 2. Second, let $[M] \in D_k(P)$ be an element of order 4. Then, setting $N := M \otimes_k M$, $[N]$ has order 2 and so $[\Phi_N]$ has order 2 in $D_{\mathcal{O}}(P)$ by the previous argument. Therefore,

$$[\Phi_M]^4 = [(\Phi_M)^{\otimes 4}] = [\Phi_{M^{\otimes 4}}] = [\Phi_{N^{\otimes 2}}] = [\Phi_N \otimes_{\mathcal{O}} \Phi_N] = [\Phi_N]^2 = [\mathcal{O}]\,,$$

proving that $[\Phi_M]$ has order 4 in $D_{\mathcal{O}}(P)$. Finally, if $[M] \in D_k(P)$ is an element of infinite order, then $[\Phi_M]$ also has infinite order. This procedure for the generators obviously extends to a group homomorphism $D_k(P) \to D_{\mathcal{O}}(P)$ which is a group-theoretic section for $\pi_{\mathfrak{p}}$. ■

7.3. **Lifting endo-trivial kG-modules.** It was proved by Alperin in 2001 that endo-trivial modules over p-groups are liftable. The corresponding result for arbitrary groups was obtained in 2016, as described below.

Lemma 7.10

Let L be an $\mathcal{O}G$-module. Then, L is endo-trivial if and only if $L/\mathfrak{p}L$ is an endo-trivial kG-module.

Proof: The necessary condition is clear. Indeed, if L is endo-trivial, then $\mathrm{End}_{\mathcal{O}}(L) \cong \mathcal{O} \oplus X$, where X is a projective $\mathcal{O}G$-module. Therefore,

$$\mathrm{End}_k(L/\mathfrak{p}L) \cong \mathrm{End}_{\mathcal{O}}(L)/\mathfrak{p}\,\mathrm{End}_{\mathcal{O}}(L) \cong k \oplus X/\mathfrak{p}X$$

where $X/\mathfrak{p}X$ is a projective kG-module by Theorem 7.1(b). Conversely, assume that $L/\mathfrak{p}L$ is endo-trivial, i.e. $(L/\mathfrak{p}L)^* \otimes_k L/\mathfrak{p}L \cong k \oplus Y$ for some projective kG-module Y, and the $\mathrm{rk}_{\mathcal{O}} L = \dim_k L/\mathfrak{p}L$ is coprime to p. It follows that $L^* \otimes_{\mathcal{O}} L \cong \mathcal{O} \oplus X$ for some $\mathcal{O}G$-lattice X and the Krull–Schmidt theorem yields $X/\mathfrak{p}X \cong Y$, proving that X is projective, and hence L is endo-trivial. ■

Theorem 7.11 ([Alp01b, Theorem])

If P is a p-group, then any endo-trivial kP-module lifts to an endo-trivial $\mathcal{O}P$-module.

Alperin's proof is based on the key fact that the image of a representation of a p-group lies in the special linear group. We state below a slightly more general version of this result and its proof.

Theorem 7.12 ([LMS16, Proposition 2.4])

Let M be an endo-trivial kG-module of dimension n and let $\rho : G \to \mathrm{GL}_n(k)$ be a matrix representation corresponding to M. If the image of ρ lies in $\mathrm{SL}_n(k)$, then V lifts to an endo-trivial $\mathcal{O}G$-module.

Proof: For each $m \geq 1$, let $\mathrm{SL}_n(\mathcal{O}, m)$ be the normal subgroup of $\mathrm{SL}_n(\mathcal{O})$ consisting of the elements of $\mathrm{SL}_n(\mathcal{O})$ congruent to the identity matrix I_n modulo $\mathfrak{p}^m$. Then

$$\mathrm{SL}_n(\mathcal{O}, 1) \supseteq \mathrm{SL}_n(\mathcal{O}, 2) \supseteq \mathrm{SL}_n(\mathcal{O}, 3) \supseteq \cdots$$

is a central series of $\mathrm{SL}_n(\mathcal{O})$ such that for each $m \geq 1$,

$$\mathrm{SL}_n(\mathcal{O}, m)/\mathrm{SL}_n(\mathcal{O}, m+1) \cong \mathfrak{sl}_n(k)$$

as $\mathrm{GL}_n(k)$-modules, where $\mathfrak{sl}_n(k) := \ker(\mathrm{Tr})$ is the kernel of the trace operator $\mathrm{Tr} : M_n(k) \longrightarrow k$, endowed with the structure of a $k\mathrm{GL}_n(k)$-module via the conjugation action of $\mathrm{GL}_n(k)$. Now, as M is endo-trivial, $n \equiv \pm 1 \pmod p$. Thus, the trace map $\mathrm{Tr}: M_n(k) \cong \mathrm{End}_k(M) \longrightarrow k$ splits as a kG-homomorphism and

$$\mathrm{End}_k(M) \cong k \oplus \mathfrak{sl}_n(k)$$

implying that $\mathfrak{sl}_n(k)$ is projective, when regarded as a kG-module via ρ.

Next, taking a pull-back X_2 of ρ and the homomorphism induced by reduction modulo $\mathfrak{p}$ from $\mathrm{SL}_n(\mathcal{O})/\mathrm{SL}_n(\mathcal{O},2) \to \mathrm{SL}_n(k)$, which has kernel $\mathrm{SL}_n(\mathcal{O},1)/\mathrm{SL}_n(\mathcal{O},2) \cong \mathfrak{sl}_n(k)$, yields a group extension

$$1 \to \mathfrak{sl}_n(k) \to X_2 \to G \to 1\,.$$

This extension splits because $H^r(G, \mathfrak{sl}_n(k)) = \mathrm{Ext}^r_{kG}(k, \mathfrak{sl}_n(k)) = 0$ for each $r \geq 1$ as $\mathfrak{sl}_n(k)$ is an injective kG-module. As a consequence, ρ lifts to a homomorphism $\rho_2 : G \to \mathrm{SL}_n(\mathcal{O})/\mathrm{SL}_n(\mathcal{O},2)$. Inductively, for every $m > 2$, we can construct a homomorphism $\rho_m : G \to \mathrm{SL}_n(\mathcal{O})/\mathrm{SL}_n(\mathcal{O},m)$ lifting $\rho_{m-1} : G \to \mathrm{SL}_n(\mathcal{O})/\mathrm{SL}_n(\mathcal{O},m-1)$. Finally,

$$\mathrm{SL}_n(\mathcal{O}) \cong \varprojlim_{m\geq 2} \mathrm{SL}_n(\mathcal{O})/\mathrm{SL}_n(\mathcal{O},m)\,,$$

so that the universal property of the projective limit yields the desired group homomorphism $\tilde{\rho} : G \longrightarrow \mathrm{SL}_n(\mathcal{O})$ lifting ρ. Moreover, if $\widehat{M}$ is an $\mathcal{O}G$-module lifting M, it is endo-trivial by Lemma 7.10. The claim follows. ∎

This allows us to extend Alperin's result to finite groups in general.

Theorem 7.13 ([LMS16, Theorem 1.3])

Let G be a finite group. Then, any endo-trivial kG-module lifts to an endo-trivial $\mathcal{O}G$-module.

Proof: Let M be an endo-trivial kG-module, say of dimension n, and let $\rho : G \to \mathrm{GL}_n(k)$ be a matrix representation corresponding to M. To start with, we may assume that ρ is faithful. Indeed, if $\bar{\rho} : G/\ker(\rho) \longrightarrow \mathrm{GL}_n(k)$ denotes the induced representation and $\bar{\rho}$ is liftable to $\bar{\rho}_{\mathcal{O}} : G/\ker(\rho) \longrightarrow \mathrm{GL}_n(\mathcal{O})$, then we have a commutative diagram

$$\begin{tikzcd} G \arrow[r, two heads, "q"] \arrow[drr, "\rho"'] & G/\ker(\rho) \arrow[r, "\bar{\rho}_{\mathcal{O}}"] \arrow[dr, "\bar{\rho}"] & \mathrm{GL}_n(\mathcal{O}) \arrow[d, two heads, "\mathrm{mod}\ \mathfrak{p}"] \\ & & \mathrm{GL}_n(k) \end{tikzcd}$$

where q is the quotient homomorphism. So, $\bar{\rho}_{\mathcal{O}} \circ q$ lifts ρ. Therefore, we may assume that $G \leq \mathrm{GL}_n(k)$. Now, set $G_1 := GC$ with

$$C := \{aI_n \mid a^n = \det(g) \text{ for some } g \in G\}$$

and $G_0 := G_1 \cap \mathrm{SL}_n(k)$. So, the situation is as follows:

$$\begin{tikzcd}[column sep=small] & \mathrm{GL}_n(k) \arrow[d, dash] & \\ & G_1 \arrow[dl, dash, "p'"'] \arrow[dr, dash, "p'"] & \\ G & & G_0 \end{tikzcd}$$

Clearly, G_1 is a central product of G with C, and of G_0 with C. As $G \leq G_1$ has p'-index the embedding $G_1 \leq \mathrm{GL}_n(k)$ defines an endo-trivial module, and in turn restricting from G_1 to G_0 the embedding $G_0 \leq \mathrm{SL}_n(k)$ defines an endo-trivial kG_0-module as $G_0 \leq G_1$ has p'-index, too. The latter kG_0-module lifts to an endo-trivial $\mathcal{O}G_0$-module by Theorem 7.12, and denoting the corresponding representation by ψ we have $\psi(G_0) \leq \mathrm{SL}_n(\mathcal{O})$.

Now, reduction modulo $\mathfrak{p}$ induces a bijection between the group of p'-roots of unity in $\mathcal{O}$ and the group of roots of unity in k, sending $\psi(G_0) \cap Z(\mathrm{SL}_n(\mathcal{O}))$ onto $G_0 \cap Z(\mathrm{SL}_n(k))$. The inverse defines a lift of C into $\{aI_n \mid a \in \mathcal{O}^\times\} \leq \mathrm{GL}_n(\mathcal{O})$, which agrees with ψ on $G_0 \cap Z(\mathrm{SL}_n(k))$ and which we also denote by ψ. Then $G_1 = G_0C \cong \psi(G_0)\psi(C) \leq \mathrm{GL}_n(\mathcal{O})$ is a faithful representation of G_1 which lifts $G_1 \leq \mathrm{GL}_n(k)$. Again, as $|G_1 : G|$ and $|G_1 : G_0|$ are prime to p, this defines an endo-trivial $\mathcal{O}G$-module lifting the initial representation of G. ■

This lifting result opened the door to a fruitful approach to the theory of endo-trivial modules through ordinary character theory, based on the following results.

Proposition 7.14 ([LMS16, Corollary 2.3])

Let M be an endo-trivial kG-module lifting to the endo-trivial $\mathcal{O}G$-module L such that $K \otimes_{\mathcal{O}} L$ affords the character χ. Then, $|\chi(g)| = 1$ for all p-singular elements $g \in G$.

Proof: Since M is endo-trivial, $M \otimes_k M^* \cong \mathrm{End}_k(M) \cong k \oplus$ (proj). It follows that

$$\chi \cdot \bar{\chi} = 1_G + \Phi$$

where 1_G denotes the trivial character and Φ is the character of $K \otimes_{\mathcal{O}} Z$ with Z the lift to $\mathcal{O}$ of a projective kG-module. Thus, the claim follows from the fact that $\Phi(g) = 0$ for all p-singular elements $g \in G$. ■

Due to publication delays the following result, characterising endo-trivial modules which are at the same time p-permutation modules, appeared earlier than the previous one, although it is a consequence.

Theorem 7.15 ([LM15, Theorem 2.2])

Let M be an indecomposable p-permutation kG-module, let $\widehat{M}$ be its lift to a p-permutation $\mathcal{O}G$-lattice and let $\chi_{\widehat{M}}$ be the character afforded by $K \otimes_{\mathcal{O}} \widehat{M}$. Then, M is endo-trivial if and only if $\chi_{\widehat{M}}(x) = 1$ for each non-trivial p-element $x \in G$.

Proof: Since M is a p-permutation module, we know from Lemma 7.2(b) that $\chi_{\widehat{M}}(x)$ is a non-negative integer for each p-element $x \in G$. First, assuming that M is endo-trivial, by Proposition 7.14 this forces $\chi_{\widehat{M}}(x) = 1$ for each non-trivial p-element $x \in G$.
Conversely, assume that $\chi_{\widehat{M}}(x) = 1$ for each non-trivial p-element $x \in G$. Notice that we may also assume that $\dim_k(M) > 1$, as any one-dimensional kG-module is endo-trivial. Now, as $\dim_k(M) \equiv \chi_{\widehat{M}}(x) = 1 \pmod p$ for any non-trivial p-element $x \in G$ (see e.g. [JL93, Corollary 22.27]), $k \mid M \otimes_k M^*$ with multiplicity one by [BC86, Thm. 2.1]. Moreover, as M is a p-permutation module, so is $M^* \otimes_k M$ by Properties 4.3, and we may decompose

$$M^* \otimes_k M \cong k \oplus N_1 \oplus \ldots \oplus N_r\,,$$

where $N_1, \ldots, N_r$ ($r \in \mathbb{Z}_{\geq 1}$) are non-trivial indecomposable p-permutation kG-modules. At the level of characters we have

$$\bar{\chi}_{\widehat{M}} \cdot \chi_{\widehat{M}} = 1_G + \chi_{\widehat{N_1}} + \cdots + \chi_{\widehat{N_r}}$$

and it follows immediately that $\chi_{\widehat{N_i}}(x) = 0$ for each $1 \leq i \leq r$. Hence, $\mathrm{vtx}(N_i) = \{\{1\}\}$ for each $1 \leq i \leq r$ by Lemma 7.2(c). In other words, N_i is projective for each $1 \leq i \leq r$, proving that M is endo-trivial. ■

The aforementioned results led to a classification of the simple endo-trivial modules for the finite quasi-simple groups based on ordinary character theory in [LMS16, LM15, LM16, LM17], partially involving computer algebra methods and computation with GAP4, CHEVIE, or MAGMA. Character-theoretical arguments were also used to determine the structure of the group $T_k(G)$ of endo-trivial modules, and in particular of its torsion subgroup $T_k^{\mathrm{tors}}(G)$, for certain finite quasi-simple groups of Lie type in [LM15], for the sporadic groups and their covering groups in [LM15b], for the Schur covers of the alternating and symmetric groups in [LM15a], or four some groups with dihedral or semi-dihedral Sylow 2-subgroups in [KL15, KL16, KL22].

7.4. **Lifting endo-p-permutation kG-modules.** The question whether endo-p-permutation kG-modules also form a class of liftable modules is natural. It was already raised by Urfer in his PhD thesis, but only answered 12 years later in [LT18], again exploiting strongly capped endo-p-permutation modules and the generalised Dade group of a finite group rather than Urfer's original approach. The proof we give below also provides us with an alternative proof for the liftability of endo-trivial modules over arbitrary finite groups, provided it is known that endo-trivial modules over p-groups are liftable.

We start by showing that the reduction modulo $\mathfrak{p}$ of endo-p-permutation $\mathcal{O}G$-modules is extremely well-behaved.

Lemma 7.16

Let L be an endo-p-permutation $\mathcal{O}G$-module and consider $L/\mathfrak{p}L$ its reduction modulo $\mathfrak{p}$. Then, the following assertions hold:

(a) $L/\mathfrak{p}L$ is an endo-p-permutation kP-module;

(b) if $A := \mathrm{End}_{\mathcal{O}}(L)$, then the natural homomorphism $k \otimes_{\mathcal{O}} A^G \longrightarrow (k \otimes_{\mathcal{O}} A)^G$ is an isomorphism of k-algebras;

(c) L is indecomposable if and only if $L/\mathfrak{p}L$ is;

(d) if L is indecomposable, then L and $L/\mathfrak{p}L$ have the same vertices.

Proof: (a) Let $P \in \mathrm{Syl}_p(G)$. If X is an $\mathcal{O}$-basis of $\mathrm{End}_{\mathcal{O}}(L) \cong L^* \otimes_{\mathcal{O}} L$ which is P-invariant, then so is its image in $\mathrm{End}_k(L/\mathfrak{p}L) \cong (L/\mathfrak{p}L)^* \otimes_k (L/\mathfrak{p}L)$, proving (a).
(b) Observe that it follows from Theorem 7.1(a), that the canonical map $k \otimes_{\mathcal{O}} U^G \longrightarrow (k \otimes_{\mathcal{O}} U)^G$ is a kG-isomorphism for any p-permutation kG-module U. (See also [LT18, Lemma 3.1].) Therefore, writing $A = \bigoplus_{i=1}^m U_i$ as a direct sum of indecomposable p-permutation $\mathcal{O}G$-modules, we obtain that the canonical homomorphism

$$k \otimes_{\mathcal{O}} A^G \cong \bigoplus_{i=1}^m k \otimes_{\mathcal{O}} U_i^G \quad \longrightarrow \quad \bigoplus_{i=1}^m (k \otimes_{\mathcal{O}} U_i)^G \cong (k \otimes_{\mathcal{O}} A)^G$$

is an isomorphism of k-algebras.
(c) It is clear that $L/\mathfrak{p}L$ is decomposable if L is, hence it remains to prove the necessary condition. Setting $A := \mathrm{End}_{\mathcal{O}}(L)$ as in (b), it is enough to prove that $\mathrm{End}_{kG}(L/\mathfrak{p}L) = (k \otimes_{\mathcal{O}} A)^G$ is a local algebra. Write $\psi : A^G \longrightarrow A^G/\mathfrak{p}A^G$ for the canonical homomorphism given by reduction modulo $\mathfrak{p}$. By Nakayama's Lemma $\mathfrak{p}A^G \subseteq J(A^G)$, so that any maximal left ideal of A^G

contains $\mathfrak{p}A^G$. Therefore

$$\psi^{-1}(J(A^G/\mathfrak{p}A^G)) = \psi^{-1}\left(\bigcap_{\mathfrak{m}\in\mathrm{Maxl}(A^G/\mathfrak{p}A^G)} \mathfrak{m}\right) = \bigcap_{\substack{\mathfrak{a}\in\mathrm{Maxl}(A^G)\\ \mathfrak{a}\supseteq\mathfrak{p}A^G}} \mathfrak{a} = J(A^G)\,,$$

where Maxl denotes the set of maximal left ideals of the considered ring. Thus ψ induces an isomorphism $A^G/J(A^G) \cong (k\otimes_{\mathcal{O}} A^G)/J(k\otimes_{\mathcal{O}} A^G)$. Now, by (b), $k\otimes_{\mathcal{O}} A^G \cong (k\otimes_{\mathcal{O}} A)^G$ as k-algebras, thus

$$\mathrm{End}_{kG}(L/\mathfrak{p}L)/J(\mathrm{End}_{kG}(L/\mathfrak{p}L)) \cong (k\otimes_{\mathcal{O}} A)^G/J((k\otimes_{\mathcal{O}} A)^G) \cong A^G/J(A^G)\,.$$

This is a skew-field, as required, since we assume that L is indecomposable. Hence $L/\mathfrak{p}L$ is also indecomposable.

(d) Let $Q\in\mathrm{vtx}(L)$. Consider a decomposition of $\mathrm{End}_{\mathcal{O}}(L)$ into indecomposable summands

$$\mathrm{End}_{\mathcal{O}}(L) \cong L\otimes_{\mathcal{O}} L^* \cong U_1\oplus\cdots\oplus U_n\,.$$

Then there is also a decomposition

$$\mathrm{End}_k(L/\mathfrak{p}L) \cong k\otimes_{\mathcal{O}}\mathrm{End}_{\mathcal{O}}(L) \cong U_1/\mathfrak{p}U_1\oplus\cdots\oplus U_n/\mathfrak{p}U_n\,.$$

As L is an endo-p-permutation $\mathcal{O}G$-module, U_i is a p-permutation module for each $1\leq i\leq n$. Thus, by Scott's theorem (Theorem 7.1), for each $1\leq i\leq n$, the module $U_i/\mathfrak{p}U_i$ is indecomposable and the vertices of U_i and $U_i/\mathfrak{p}U_i$ are the same. Now, by Properties 5.26(d), each U_i as a vertex contained in Q and one of them has vertex Q. Therefore $U_i/\mathfrak{p}U_i$ has a vertex contained in Q for each $1\leq i\leq n$ and one of them has vertex Q, and it follows that $Q\in\mathrm{vtx}(L/\mathfrak{p}L)$. ■

Theorem 7.17 ([LT18, Theorem 4.2])

Let M be an indecomposable endo-p-permutation kG-module and let $Q\in\mathrm{vtx}(M)$. Then, there exists an indecomposable endo-p-permutation $\mathcal{O}G$-module $\widehat{M}$ with vertex Q such that $\widehat{M}/\mathfrak{p}\widehat{M}\cong M$.

Proof: Let S be a kQ-source of M. By Theorem 5.29, S is a capped endo-permutation kQ-module such that $[S]\in D_k(Q)^{G\text{-}st}$. Then [LT18, Lemma 4.1] shows that there exists an endo-permutation $\mathcal{O}Q$-module $\widehat{S}$ lifting S such that $[\widehat{S}]\in D_{\mathcal{O}}(P)^{G\text{-}st}$. Moreover, $\mathrm{Ind}_Q^G(\widehat{S})$ is an endo-p-permutation $\mathcal{O}G$-lattice by Remark 5.28 (also true over $\mathcal{O}$). Now, consider a decomposition of $\mathrm{Ind}_Q^G(\widehat{S})$ into indecomposable summands

$$\mathrm{Ind}_Q^G(\widehat{S}) = L_1\oplus\cdots\oplus L_s \quad (s\in\mathbb{Z}_{>0})\,.$$

By Properties 5.26, each L_i $(1\leq i\leq s)$ is an endo-p-permutation $\mathcal{O}G$-module. Then, by Lemma 7.16,

$$\mathrm{Ind}_Q^G(S) \cong \mathrm{Ind}_Q^G(\widehat{S})/\mathfrak{p}\,\mathrm{Ind}_Q^G(\widehat{S}) \cong L_1/\mathfrak{p}L_1\oplus\cdots\oplus L_s/\mathfrak{p}L_s$$

is a decomposition of $\mathrm{Ind}_Q^G(S)$ into indecomposable summands which preserves the vertices of the indecomposable summands. Because S is a source of M, there exists an index $1\leq i\leq s$ such that $M\cong L_i/\mathfrak{p}L_i$, poving that $\widehat{M} := L_i$ lifts M. The claim about the vertices is clear by the previous Lemma. ■

7.5. **Lifting fusion stable endo-permutation kG-modules and Brauer-friendly kG-modules.** Finally, we mention two lifting results, of the same flavour as the ones presented above, concerning two further classes of modules strongly linked to endo-permutation modules and involving certain stability conditions with respect to a group or to a block fusion system. (For the definition of a fusion system we refer to the book [AKO11] by Aschbacher–Kessar–Oliver devoted to this topic.) The first one by Kessar and Linckelmann is concerned with fusion stable endo-permutation kG-modules. To understand the statement of this theorem, we need to give the definition of *the Dade group of a fusion system on a finite p-group* introduced by Linckelmann and Mazza in [LM09].

Let P be a finite p-group and $\mathcal{F}$ be a saturated fusion system on P. If $Q \leq P$ is a subgroup, $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$ and M is an RP-module, denote by $\mathrm{Res}_{\varphi}(M)$ the RQ-module which is equal to M as an R-lattice and with $u \in Q$ acting on $m \in M$ as $\varphi(u)\cdot m$. Then, the class $[V] \in D_R(P)$ of a capped endo-permutation RP-module V is called $\mathcal{F}$-stable if the endo-permutation RQ-modules $\mathrm{Res}_{\varphi}(V)$ and $\mathrm{Res}^P_Q(V)$ have isomorphic caps, for any subgroup $Q \leq P$ and any morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$. The *Dade group of $\mathcal{F}$* can then be defined as the following subgroup of the Dade group of P:

$$D_R(P,\mathcal{F}) := \{[V] \in D_R(P) \mid [V] \text{ is } \mathcal{F}\text{-stable}\},$$

and it is easily verified that the reduction modulo $\mathfrak{p}$ of an $\mathcal{F}$-stable element of $D_{\mathcal{O}}(P)$ is an $\mathcal{F}$-stable element of $D_k(P)$. In other words, the surjective homomorphism $\pi_{\mathfrak{p}} : D_{\mathcal{O}}(P) \twoheadrightarrow D_k(P)$ restricts to a group homomorphism

$$\pi_{\mathfrak{p}} : D_{\mathcal{O}}(P,\mathcal{F}) \longrightarrow D_k(P,\mathcal{F})\,.$$

Theorem 7.18 ([KL18, Lemma 8.4])

Let P be a finite p-group and let $\mathcal{F}$ be a saturated fusion system on P. Then the group homomorphism

$$\pi_{\mathfrak{p}} : D_{\mathcal{O}}(P,\mathcal{F}) \longrightarrow D_k(P,\mathcal{F})$$

is surjective.

Assuming that G and H are two finite groups and k is a splitting field for $G \times H$, Kessar and Linckelmann used Theorem 7.18 in order to prove that a Morita equivalence (resp. a stable equivalence of Morita type) between two blocks of kG and kH induced by an indecomposable (kG,kH)-bimodule M with endo-permutation source V can be lifted to a Morita equivalence (resp. a stable equivalence of Morita type) between the corresponding blocks of $\mathcal{O}G$ and $\mathcal{O}H$ induced by an $(\mathcal{O}G,\mathcal{O}H)$-bimodule L with endo-permutation source W such that $k \otimes_{\mathcal{O}} L \cong M$ and $k \otimes_{\mathcal{O}} W \cong V$. See [KL18, Theorem 1.13].

Finally we come to the liftability of Brauer-friendly kG-modules. *Brauer-friendly RG*-modules were introduced by Biland in his doctoral thesis, also in French. See [Bil14] for a published version in English. We do not define these modules formally in this manuscript, as their definition is rather technical and goes beyond the methods and objects we have introduced so far. However, we note that any indecomposable Brauer-friendly RG-module has an endo-permutation source which is subject to a certain stability condition with respect to the fusion system of the block of RG containing the module, in the spirit of the $\mathcal{F}$-stability introduced above. Brauer-friendly RG-modules are not necessarily G-stable and provide us with natural examples of RG-modules with endo-permutation source which are not endo-p-permutation modules.

Remark 7.19

Watanabe [Wat20, Theorem 4.1] proves that an indecomposable Brauer-friendly kG-module lying in a block of kG lifts to a Brauer-friendly $\mathcal{O}G$-module belonging to the corresponding block of $\mathcal{O}G$, provided the fusion system of the block is saturated.

8. Glossary

To finish with, we summarise the different classes of modules and group structures introduced in this survey. We recall that p is a prime number, G denotes a finite group of order divisible by p, P a p-group and $R \in \{\mathcal{O}, k\}$.

Modules:

- A *permutation RG-module* is an RG-module admitting an R-basis which is (globally) invariant under the action of the group G.
  See Definition 3.1.
- A *p-permutation RG-module* is an RG-module admitting an R-basis which is (globally) invariant under the action of a Sylow p-subgroup of G.
  See Proposition-Definition 4.2.
- An *endo-permutation RP-module* is an RP-module M such that $\mathrm{End}_R(M)$ is a permutation RP-module.
  See Definition 5.1.
- A *capped* endo-permutation RP-module is an endo-permutation RP-module admitting an indecomposable direct summand with vertex P.
  See Definition 5.1.
- An *endo-trivial RG-module* is an RG-module M such that $\mathrm{End}_R(M) \cong R \oplus X$ as an RG-module and where X is a projective RG-module.
  See Definition 5.7 and Definition 5.16.
- An *endo-trivial kG-module relative to a kG-module V* (which is absolutely p-divisible) is a kG-module M such that $\mathrm{End}_k(M) \cong k \oplus X$ as a kG-module and where X is a kG-module which is projective relatively to V.
  See Definition 5.37.
- An *endo-p-permutation RG-module* is an RG-module whose restriction to any p-subgroup Q of G is an endo-permutation RQ-module.
  See Definition 5.23.
- A *strongly capped* endo-p-permutation kG-module is an endo-p-permutation kG-module which is also endo-trivial relatively to the kG-module $V(\mathcal{F}(G)) := \bigoplus_Q \mathrm{Ind}_Q^G(k)$ where the sum runs through the proper p-subgroups of G.
  See Proposition-Definition 5.41.

Group structures:

- $D_R(P)$ denotes the *Dade group* of the p-group P over R.
  See Proposition-Definition 5.4.
- $D_k(G)$ denotes the *generalised Dade group* of the arbitrary finite group G over the field k.
  See Proposition-Definition 5.43.
- $D_Q(G)$ denotes the group of compatibility classes endo-p-permutation RG-modules with vertex Q.
  See Proposition-Definition 5.31.
- $T_R(G)$ denotes the *group of endo-trivial modules* of the group G over R.
  See Proposition-Definition 5.19.
- $D_R^{\mathrm{tors}}(P)$ denotes the torsion subgroup of $D_R(P)$ and $D_R^{\mathrm{free}}(P)$ denotes the torsion-free part of $D_R(P)$.
  See Notation 5.6.
- $T_k^{\mathrm{tors}}(G)$ denotes the torsion subgroup of $T_k(G)$ and $T_k^{\mathrm{free}}(G)$ denotes the torsion-free part of $T_k(G)$.
  See Theorem 5.21.
- $X_R(G)$ denotes the group of RG-modules with R-rank equal to 1.
  See Example 5.3(b).
- $K(G)$ denotes the kernel of the restriction homomorphism $\mathrm{res}^G_P : T_k(G) \longrightarrow T_k(P)$ where $P \in \mathrm{Syl}_p(G)$.
  See Theorem 5.21.
- $T_V(G)$ denotes the *group of V-endo-trivial kG-modules*, where V is an absolutely p-divisible kG-module.
  See Proposition-Definition 5.39.

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