Chapter 5. The Mackey Formula and Clifford Theory

The results in this chapter go more deeply into the theory. We start with the so-called *Mackey decomposition formula*, which provides us with yet another relationship between induction and restriction. After that we explain Clifford's theorem, which explains what happens when a simple representation is restricted to a normal subgroup. These results are essential and have many consequences throughout representation theory of finite groups.

Notation: throughout this chapter, unless otherwise specified, we let G denote a finite group and K be a commutative ring. All modules over group algebra considered are assumed to be **finitely generated** and **free as K-modules**, hence **of finite K-rank**.

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21 Double cosets

Definition 21.1 (*Double cosets*)

Given subgroups H and L of G we define for each $g \in G$

$$HgL := \{hgk \in G \mid h \in H, k \in L\}$$

and call this subset of G the (H, L)-**double coset** of g. Moreover, we let $H\backslash G/L$ denote the set of (H, L)-double cosets of G.

First, we want to prove that the (H, L)-double cosets partition the group G.

Lemma 21.2

Let $H, L \leqslant G$.

(a) Each (H, L)-double coset is a disjoint union of right cosets of H and a disjoint union of left cosets of L.

(b) Any two (H, L)-double cosets either coincide or are disjoint. Hence, letting $[H\backslash G/L]$ denote a set of representatives for the (H, L)-double cosets of G, we have

$$G = \bigsqcup_{g\in[H\backslash G/L]} HgL\,.$$

Proof:

(a) If $hgk \in HgL$ and $k_1 \in L$, then $hgk \cdot k_1 = hg(kk_1) \in HgL$. It follows that the entire left coset of L that contains hgk is contained in HgL. This proves that HgL is a union of left cosets of L. A similar argument proves that HgL is a union of right cosets of H.

(b) Let $g_1, g_2 \in G$. If $h_1g_1k_1 = h_2g_2k_2 \in Hg_1L \cap Hg_2L$, then $g_1 = h_1^{-1}h_2g_2k_2k_1^{-1} \in Hg_2L$ so that $Hg_1L \subseteq Hg_2L$. Similarly $Hg_2L \subseteq Hg_1L$. Thus if two double cosets are not disjoint, they coincide. ∎

If X is a left G-set we use the standard notation $G\backslash X$ for the set of orbits of G on X, and denote a set of representatives for theses orbits by $[G\backslash X]$. Similarly if Y is a right G-set we write Y/G and $[Y/G]$. We shall also repeatedly use the orbit-stabiliser theorem without further mention: in other words, if X is a transitive left G-set and $x \in X$ then $X \cong G/\operatorname{Stab}_G(x)$ (i.e. the set of left cosets of the stabiliser of x in G), and similarly for right G-sets.

Exercise 21.3

(a) Let $H, L \leqslant G$. Prove that the set of (H, L)-double cosets is in bijection with the set of orbits $H\backslash(G/L)$, and also with the set of orbits $(H\backslash G)/L$ under the mappings

$$HgL \mapsto H(gL) \in H\backslash(G/L)$$

$$HgL \mapsto (Hg)L \in (H\backslash G)/L.$$

This justifies the notation $H\backslash G/L$ for the set of (H, L)-double cosets.

(b) Let $G = S_3$. Consider $H = L := S_2 = \{\mathrm{Id}, (1\ 2)\}$ as a subgroup of S_3. Prove that

$$[S_2\backslash S_3/S_2] = \{\mathrm{Id}, (1\ 2\ 3)\}$$

while

$$S_2\backslash S_3/S_2 = \{\, \{\mathrm{Id}, (1\ 2)\}, \{(1\ 2\ 3), (1\ 3\ 2), (1\ 3), (2\ 3)\}\}\,.$$

22 The Mackey formula

If H and L are subgroups of G, we wish to describe what happens if we induce a KL-module from L to G and then restrict it to H.

Remark 22.1

We need to examine KG as a (KH, KL)-bimodule, with left and right external laws by multiplication in G. Since $G = \bigsqcup_{g\in[H\backslash G/L]} HgL$, we have

$$KG = \bigoplus_{g\in[H\backslash G/L]} K\langle HgL\rangle$$

as (KH, KL)-bimodule, where $K\langle HgL\rangle$ denotes the free K-module with K-basis HgL.

Now if M is a KL-module, we will also write ${}^gM$ for $g\otimes M$, which is a left $K({}^gL)$-module with

$$(gkg^{-1})\cdot(g\otimes m) = g\otimes km$$

for each $k\in L$ and each $m\in M$. With this notation, we have

$$K\langle HgL\rangle \cong KH\otimes_{K(H\cap {}^gL)} (g\otimes KL)\,,$$

where $hgk\in HgK$ corresponds to $h\otimes g\otimes k$.

Theorem 22.2 (*Mackey formula*)

Let $H, L\leqslant G$ and let M be a KL-module. Then

$$M\!\uparrow_L^G\downarrow_H^G \cong \bigoplus_{g\in[H\backslash G/L]} ({}^gM\!\downarrow^{{}^gL}_{H\cap {}^gL})\uparrow^H_{H\cap {}^gL}$$

as KH-modules.

Proof: It follows from Remark 22.1 that as left KH-modules we have

$$\begin{aligned} M\!\uparrow_L^G\downarrow_H^G \cong (KG\otimes_{KL} M)\downarrow_H^G &\cong \bigoplus_{g\in[H\backslash G/L]} K\langle HgL\rangle\otimes_{KL} M \\ &\cong \bigoplus_{g\in[H\backslash G/L]} KH\otimes_{K(H\cap {}^gL)} (g\otimes KL)\otimes_{KL} M \\ &\cong \bigoplus_{g\in[H\backslash G/L]} KH\otimes_{K(H\cap {}^gL)} (g\otimes M)\downarrow^{{}^gL}_{H\cap {}^gL} \\ &\cong \bigoplus_{g\in[H\backslash G/L]} ({}^gM\!\downarrow^{{}^gL}_{H\cap {}^gL})\uparrow^H_{H\cap {}^gL}\,. \end{aligned}$$

■

Exercise 22.3

Let $H, L\leqslant G$, let M be a KL-module and let N be a KH-module. Use the Mackey formula to prove that:

(a) $M\!\uparrow_L^G \otimes_K N\!\uparrow_H^G \cong \bigoplus_{g\in[H\backslash G/L]} ({}^gM\!\downarrow^{{}^gL}_{H\cap {}^gL} \otimes_K N\!\downarrow^H_{H\cap {}^gL})\uparrow^G_{H\cap {}^gL}$;

(b) $\mathrm{Hom}_K(M\!\uparrow_L^G, N\!\uparrow_H^G) \cong \bigoplus_{g\in[H\backslash G/L]} (\mathrm{Hom}_K({}^gM\!\downarrow^{{}^gL}_{H\cap {}^gL}, N\!\downarrow^H_{H\cap {}^gL}))\uparrow^G_{H\cap {}^gL}$.

23 Clifford theory

We now turn to *Clifford's theorem*, which we present in a weak and a strong form. Clifford theory is a collection of results about induction and restriction of simple modules from/to normal subgroups.

Throughout this section, we assume that K is a field.

First we emphasise again, that this is no loss of generality: indeed if S were a simple KG-module with K an arbitrary commutative ring, then letting I be the annihilator in K of S, we have that I is a maximal ideal of K, so that K/I is a field and S is a $(K/I)G$-module.

Theorem 23.1 (*Clifford's Theorem, weak form*)

If $U \trianglelefteq G$ is a normal subgroup and S is a simple KG-module, then $S\downarrow_U^G$ is semisimple.

Proof: Let V be any simple KU-submodule of $S\downarrow_U^G$. Now, notice that for every $g \in G$, $gV := \{gv \mid v \in V\}$ is also a KU-submodule of $S\downarrow_U^G$, since $U \trianglelefteq G$ for any $u \in U$, we have

$$u \cdot gV = g \cdot \underbrace{(g^{-1}ug)}_{\in U} V = gV$$

Moroever, gV is also simple, since if W were a non-trivial proper KU-submodule of gV then $g^{-1}W$ would also be a a non-trivial proper submodule of $g^{-1}gV = V$. Now $\sum_{g\in G} gV$ is non-zero and it is a KG-submodule of S, which is simple, hence $\sum_{g\in G} gV = S$. Restricting to U, we obtain that

$$S\downarrow_U^G = \sum_{g\in G} gV$$

is a sum of simple KU-submodules. Hence $S\downarrow_U^G$ is semisimple. ∎

Remark 23.2

The kU-submodules gV which appear in the proof of Theorem 23.1 are isomorphic to modules we have seen before: more precisely the map

$$\begin{array}{ccc} g\otimes V & \longrightarrow & gV \\ g\otimes v & \mapsto & gv \end{array}$$

is a KU-isomorphism, since $U \trianglelefteq G$ implies that ${}^gU = U$ and hence the action of U on $g\otimes V$ (see Remark 22.1) and gV is prescribed in the same way.

Theorem 23.3 (*Clifford's Theorem, strong form*)

Let $U \trianglelefteq G$ be a normal subgroup and let S be a simple KG-module. Then we may write

$$S\downarrow_U^G = S_1^{a_1} \oplus \cdots \oplus S_r^{a_r}$$

where $r \in \mathbb{Z}_{>0}$ and $S_1, \dots, S_r$ are non-isomorphic simple KU-modules, occurring with multiplicities $a_1, \dots, a_r$ respectively. Moreover, the following statements hold:

(i) the group G permutes the homogeneous components of $S\downarrow_U^G$ transitively;

(ii) $a_1 = a_2 = \cdots = a_r$ and $\dim_K(S_1) = \cdots = \dim_K(S_r)$; and

(iii) $S \cong (S_1^{a_1})\uparrow_{H_1}^G$ as KG-modules, where $H_i = \mathrm{Stab}_G(S_1^{a_1})$.

Proof: The fact that $S\downarrow_U^G$ is semisimple and hence can be written as a direct sum as claimed follows from Theorem 23.1. Moreover, by the chapter on semisimplicity of rings and modules, we know that for each $1 \leqslant i \leqslant r$ the homogeneous component $S_i^{a_i}$ is characterised by Proposition 12.1. Now, if $g \in G$ then $g(S_i^{a_i}) = (gS_i)^{a_i}$, where gS_i is simple (see the proof of the weak form of Clifford's Theorem). Hence there exists an index $1 \leqslant j \leqslant r$ such that $gS_i = S_j$ and $g(S_i^{a_i}) \subseteq g(S_j^{a_j})$. Because $\dim_K(S_i) = \dim_K(gS_i)$, we have that $a_i \leqslant a_j$. Similarly, since $S_j = g^{-1}S_i$, we obtain $a_j \leqslant a_j$. Hence $a_i = a_j$ holds. Because

$$S = gS = g(S_1^{a_1}) \oplus \cdots \oplus g(S_r^{a_r}),$$

we actually have that G permutes the homogeneous components. Moreover, as $\sum_{g\in G} g(S_1^{a_1})$ is a non-zero KG-submodule of S, which is simple, we have that $\sum_{g\in G} g(S_1^{a_1}) = S$, and so the action on the homogeneous components is transitive. This establishes both (i) and (ii).
For (iii), we define a K-homomorphism via the map

$$\begin{array}{rccc} \Phi : & (S_1^{a_1})\uparrow_{H_1}^G = KG \otimes_{KH_1} S_1^{a_1} = \bigoplus_{g\in[G/H_1]} g \otimes S_1^{a_1} & \longrightarrow & S \\ & g \otimes m & \mapsto & gm \end{array}$$

that is, where $g \otimes m \in g \otimes S_1^{a_1}$. This is in fact a KG-homomorphism. Furthermore, the K-subspaces $g(S_1^{a_1})$ of S are in bijection with the cosets G/H_1, since G permutes them transitively by (i), and the stabiliser of one of them is H_1. Thus both $KG \otimes_{KH_1} S_1^{a_1}$ and S are the direct sum of $|G : H_1|$ K-subspaces $g \otimes S_1^{a_1}$ and $g(S_1^{a_1})$ respectively, each K-isomorphic to $S_1^{a_1}$ (via $g \otimes m \leftrightarrow m$ and $gm \leftrightarrow m$). Thus the restriction of Φ to each summand is an isomorphism, and so Φ itself must be bijective, hence a KG-isomorphism. ■

One application of Clifford's theory is for example the following Corollary:

Corollary 23.4

Assume $K = \overline{K}$ is algebraically closed of arbitrary characteristic and G is a p-group for some prime number p. Then every simple KG-module has the form $X\uparrow_H^G$, where X is a 1-dimensional KH-module for some subgroup $H \leqslant G$.

Proof: We proceed by induction on $|G|$. If $|G| = 1$ or G is a prime number, then G is abelian and all simple modules are 1-dimensional, so we are done. So assume $|G|$ is reducible, and let S be a simple KG-module and consider the subgroup

$$U := \{g \in G \mid g \cdot x = x \;\forall\, x \in S\}.$$

This is obviously a normal subgroup of G since it is the kernel of the K-representation associated to S. Hence $S = \mathrm{Inf}_{G/U}^G(T)$ for a simple $K[G/U]$-module T.
Now, if $U \neq \{1\}$, then $|G/U| < |G|$, so by the induction hypothesis there exists a subgroup $H/U \leqslant G/U$ and a $K[H/U]$-module Y such that $T = \mathrm{Ind}_{H/U}^{G/U}(Y)$. But then

$$S = \mathrm{Inf}_{G/U}^G(T) = \mathrm{Inf}_{G/U}^G \circ \mathrm{Ind}_{H/U}^{G/U}(Y) = \mathrm{Ind}_H^G \circ \mathrm{Inf}_{H/U}^H(Y),$$

so that setting $X := \mathrm{Inf}_H^{H/U}(Y)$ yields the result. Thus we may assume $U = \{1\}$.
If G is abelian, then all simple modules are 1-dimensional, so we are done. Assume now that G is not abelian. Then G has a normal abelian subgroup A that is not central. Indeed, to construct this subgroup A, let $Z_2(G)$ denote the second center of G, that is, the preimage in G of $Z(G/Z(G))$. If $x \in Z_2(G)\backslash Z(G)$, then $A := \langle Z(G), x\rangle$ is a normal abelian subgroup not contained in $Z(G)$. Now, applying Clifford's Theorem yields:

$$S\downarrow_A^G = S_1^{a_1} \oplus \cdots \oplus S_r^{a_r}$$

where $r \in \mathbb{Z}_{>0}$, $S_1, \ldots, S_r$ are non-isomorphic simple KA-modules and $S = (S_1^{a_1})\uparrow_{H_1}^G$, where $H_1 = \mathrm{Stab}_G(S_1^{a_1})$. We argue that $V := S_1^{a_1}$ must be a simple KH_1-module, since if it had a proper submodule W, then $W\uparrow_{H_1}^{G}$ would be a proper submodule of S, which is simple. If $H_1 \neq G$ then by the induction hypothesis $V = X\uparrow_{H}^{H_1}$, where $H \leqslant H_1$ and X is a 1-dimensional KH-module. Thefore, by transitivity of the induction, we have

$$S = (S_1^{a_1})\uparrow_{H_1}^{G} = (X\uparrow_{H}^{H_1})\uparrow_{H_1}^{G} = X\uparrow_{H}^{G},$$

as required.
Finally, the case $H_1 = G$ cannot happen. For if it were to happen then

$$S\downarrow_{A}^{G} = S_1^{a_1},$$

is simple, hence of dimension 1 since A is abelian. The elements of A must therefore act via scalar multiplication on S. Since such an action would commute with the action of G, which is faithful on S, we deduce that $A \subseteq Z(G)$, which contradicts the construction of A. ■