A SHORT INTRODUCTION TO THE MODULAR REPRESENTATION THEORY OF FINITE GROUPS

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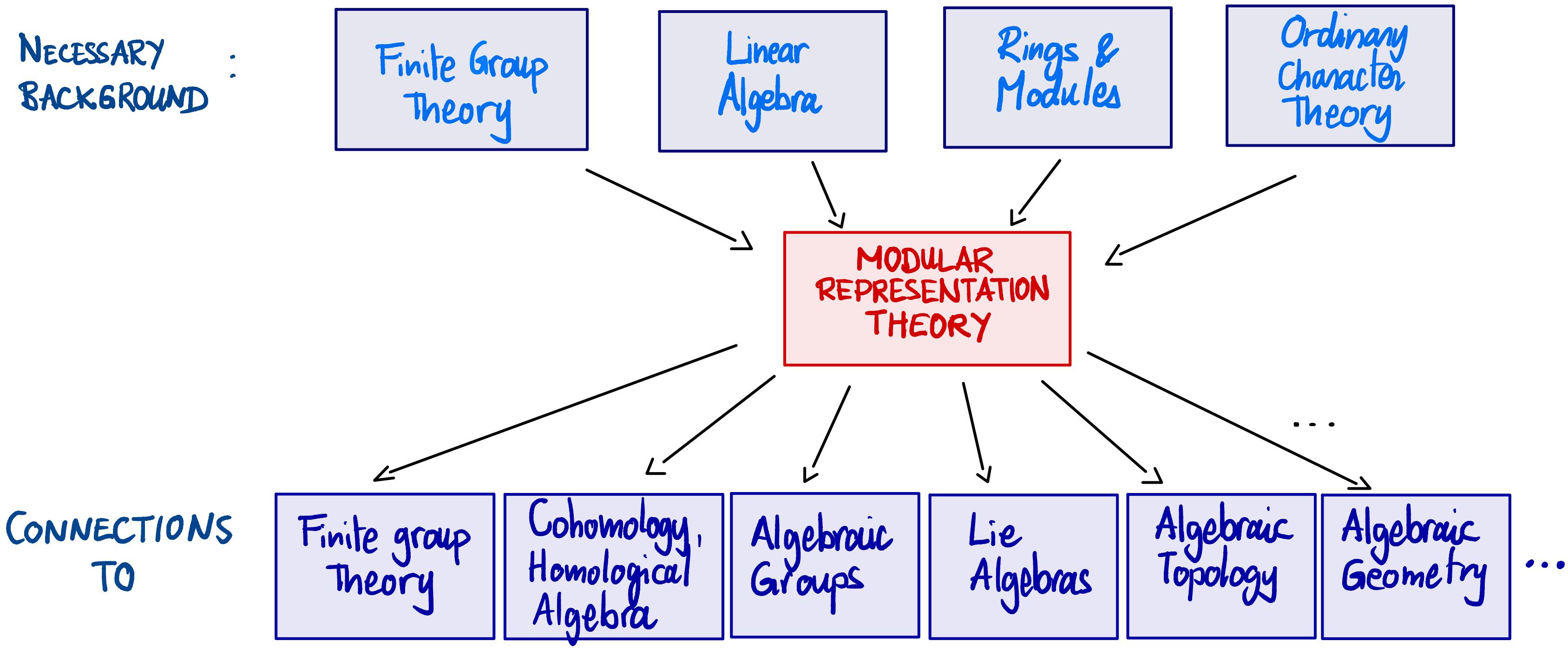
BEFORE AND AFTER



WARNING :

Modular representation theory of finite groups is a beautiful topic with deep theoretical results / questions / answers (!),

but it is

HARD (VERY HARD) !

→ Often : not much room for combinatorics ! :<

3 Main approaches to representation theory of finite groups

1. Representations

$\rho : G \to GL(V)$

and :

- their characters
  $\chi_\rho : G \longrightarrow K$
  $g \mapsto Tr(\rho(g))$

  if $char(K) = 0$

- their Brauer characters
  $\varphi_\rho : G_{p'} = \{g \in G \mid p \nmid o(g)\} \longrightarrow \mathbb{C}$

  if $char(K) > 0$

2. kG - modules

→ study of the indecomposable / simple / projective / ... modules

→ properties of the module category

3. p-Block theory / algebra approach

→ $kG = B_0 \oplus B_1 \oplus \cdots \oplus B_n$

→ study of the k-algebra structure of kG, resp. of the B_i s

! FIND GOOD ⟷ CONNECTIONS

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Class of permutation kG-modules

← "Lift" kG-modules to char. zero
⇝ associate characters

→ Parametrize many equivalences of block algebras

↓

Computer algebra !

ORDINARY REPRESENTATION THEORY → through characters

DEF^N: If $\rho : G \to GL(V)$ is a K-representation, then

$$\chi_\rho : G \longrightarrow K$$
$$g \mapsto Tr(\rho(g))$$

is the K-character of associated to ρ. Moreover χ_ρ is irreducible if ρ is.

STANDARD RESULTS FOR $K = \mathbb{C}$:

(1) $\rho_1 \sim \rho_2 \iff \chi_{\rho_1} = \chi_{\rho_2}$

(2) Characters are class functions : $\chi_\rho(hgh^{-1}) = \chi_\rho(g) \quad \forall g, h \in G$.

(3) $Irr_{\mathbb{C}}(G) := \{$ irreducible $\mathbb{C}$-characters of $G\}$ is finite. In fact,

$$|Irr_{\mathbb{C}}(G)| = \# \text{ conjugacy classes of } G$$

(4) $\chi \in Irr_{\mathbb{C}}(G) \implies \chi(1) \mid |G|$

(5) $\sum_{\chi \in Irr_{\mathbb{C}}(G)} \chi(1)^2 = |G|$

(6) G is abelian $\iff \chi(1) = 1 \quad \forall \chi \in Irr_{\mathbb{C}}(G)$

(7) The character table $(\chi(g))_{\substack{\chi \in Irr_{\mathbb{C}}(G) \\ g \in RepConjCl(G)}}$ contains a lot of information about G.

K-Representations		KG-Modules
K-representation of G	$\longleftrightarrow$	KG-module
degree	$\longleftrightarrow$	K-dimension
homomorphism of K-representations	$\longleftrightarrow$	homomorphism of KG-modules
equivalent K-representations	$\longleftrightarrow$	isomorphism of KG-modules
subrepresentation	$\longleftrightarrow$	KG-submodule
direct sum of representations $\rho_{V_1} \oplus \rho_{V_2}$	$\longleftrightarrow$	direct sum of KG-modules $V_1 \oplus V_2$
irreducible representation	$\longleftrightarrow$	simple (= irreducible) KG-module
the trivial representation	$\longleftrightarrow$	the trivial KG-module K
the regular representation of G	$\longleftrightarrow$	the regular KG-module KG°
completely reducible K-representation	$\longleftrightarrow$	semisimple KG-module (= completely reducible)
every K-representation of G is completely reducible	$\longleftrightarrow$	KG is semisimple
...		...