Chapter 1. Foundations of Representation Theory

In this chapter we review four important module-theoretic theorems, which lie at the foundations of *representation theory of finite groups*:

1. **Schur's Lemma**: about homomorphisms between simple modules.
2. **The Jordan-Hölder Theorem**: about "uniqueness" properties of composition series.
3. **Nakayama's Lemma**: about an essential property of the Jacobson radical.
4. **The Krull-Schmidt Theorem**: about direct sum decompositions into indecomposable submodules.

Notation: throughout this chapter, unless otherwise specified, we let R denote an arbitrary unital and associative ring.

Again results which intersect the *Commutative Algebra* lecture are stated without proof.

References:

[CR90] C. W. Curtis and I. Reiner, *Methods of representation theory. Vol. I*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1990.

[Dor72] L. Dornhoff, *Group representation theory. Part B: Modular representation theory*, Marcel Dekker, Inc., New York, 1972.

[NT89] H. Nagao and Y. Tsushima, *Representations of finite groups*, Translated from the Japenese. Academic Press, Inc., Boston, MA, 1989.

[Rot10] J. J. Rotman, *Advanced modern algebra. 2nd ed.*, Providence, RI: American Mathematical Society (AMS), 2010.

6 (Ir)Reducibility and (in)decomposability

Submodules and direct sums of modules allow us to introduce the two main notions that will enable us to break modules in *elementary* pieces in order to simplify their study.

Definition 6.1 (***simple/irreducible module / indecomposable module***)

(a) An R-module M is called **reducible** if it admits an R-submodule U such that $0 \lneq U \lneq M$. An R-module M is called **simple** (or **irreducible**) if it is non-zero and not reducible.

(b) An R-module M is called **decomposable** if M possesses two non-zero proper submodules M_1, M_2 such that $M = M_1 \oplus M_2$. An R-module M is called **indecomposable** if it is non-zero and not decomposable.

Remark 6.2

Clearly any simple module is also indecomposable. However, the converse does not hold in general. Exercise: find a counter-example!

Exercise 6.3

Prove that if $(R, +, \cdot)$ is a ring, then $R^\circ := R$ itself maybe seen as an R-module via left multiplication in R, i.e. where the external composition law is given by

$$R \times R^\circ \longrightarrow R^\circ, (r, m) \mapsto r \cdot m\,.$$

We call R° the **regular** R-module.

Prove that the R-submodules of R° are precisely the left ideals of R. Moreover, $I \lhd R$ is a maximal left ideal of $R \Leftrightarrow R^\circ/I$ is a simple R-module, and $I \lhd R$ is a minimal left ideal of $R \Leftrightarrow I$ is simple when regarded as an R-submodule of R°.

7 Schur's Lemma

Schur's Lemma is a basic result, which lets us understand homomorphisms between *simple* modules, and, more importantly, endomorphisms of such modules.

Theorem 7.1 (***Schur's Lemma***)

(a) Let V, W be simple R-modules. Then:

  (i) $\mathrm{End}_R(V)$ is a skew-field, and

  (ii) if $V \ncong W$, then $\mathrm{Hom}_R(V, W) = 0$.

(b) If K is an algebraically closed field, A is a K-algebra, and V is a simple A-module such that $\dim_K V < \infty$, then

$$\mathrm{End}_A(V) = \{\lambda \,\mathrm{Id}_V \mid \lambda \in K\} \cong K\,.$$

Proof :

(a) First, we claim that every $f \in \mathrm{Hom}_R(V, W)\backslash\{0\}$ admits an inverse in $\mathrm{Hom}_R(V, W)$.
Indeed, $f \neq 0 \implies \ker f \subsetneq V$ is a proper R-submodule of V and $\{0\} \neq \mathrm{Im}\, f$ is a non-zero R-submodule of W. But then, on the one hand, $\ker f = \{0\}$, because V is simple, hence f is injective, and on the other hand, $\mathrm{Im}\, f = W$ because W is simple. It follows that f is also surjective, hence

bijective. Therefore, by Example 1(d), f is invertible with inverse $f^{-1} \in \mathrm{Hom}_R(V, W)$.

Now, (ii) is straightforward from the above. For (i), by Exercise 2.2, $\mathrm{End}_R(V)$ is a ring, which is obviously non-zero as $\mathrm{End}_R(V) \ni \mathrm{Id}_V$ and $\mathrm{Id}_V \neq 0$ because $V \neq 0$ since it is simple. Thus, as any $f \in \mathrm{End}_R(V)\setminus\{0\}$ is invertible, $\mathrm{End}_R(V)$ is a skew-field.

(b) Let $f \in \mathrm{End}_A(V)$. By the assumptions on K, f has an eigenvalue $\lambda \in K$. Let $v \in V\setminus\{0\}$ be an eigenvector of f for λ. Then $(f - \lambda\,\mathrm{Id}_V)(v) = 0$. Therefore, $f - \lambda\,\mathrm{Id}_V$ is not invertible and

$$f - \lambda\,\mathrm{Id}_V \in \mathrm{End}_A(V) \quad \overset{(a)}{\Longrightarrow} \quad f - \lambda\,\mathrm{Id}_V = 0 \quad \Longrightarrow \quad f = \lambda\,\mathrm{Id}_V\,.$$

Hence $\mathrm{End}_A(V) \subseteq \{\lambda\,\mathrm{Id}_V \mid \lambda \in K\}$, but the reverse inclusion also obviously holds, so that

$$\mathrm{End}_A(V) = \{\lambda\,\mathrm{Id}_V\} \cong K\,.$$

■

8 Composition series and the Jordan-Hölder Theorem*

From Chapter 2 on, we will assume that all modules we work with can be broken into *simple* modules in the sense of the following definition.

Definition 8.1 (*Composition series / composition factors / composition length*)

Let M be an R-module.

(a) A **series** (or **filtration**) of M is a finite chain of submodules

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M \qquad (n \in \mathbb{Z}_{\geqslant 0})\,.$$

(b) A **composition series** of M is a series

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M \qquad (n \in \mathbb{Z}_{\geqslant 0})$$

where M_i/M_{i-1} is simple for each $1 \leqslant i \leqslant n$. The quotient modules M_i/M_{i-1} are called the **composition factors** (or the **constituents**) of M and the integer n is called the **composition length** of M.

Notice that, clearly, in a composition series all inclusions are in fact strict because the quotient modules are required to be simple, hence non-zero.

Next we see that the existence of a *composition series* implies that the module is *finitely generated*. However, the converse does not hold in general. This is explained through the fact that the existence of a composition series is equivalent to the fact that the module is both *Noetherian* and *Artinian*.

Definition 8.2 (*Chain conditions / Artinian and Noetherian rings and modules*)

(a) An R-module M is said to satisfy the **descending chain condition** (D.C.C.) on submodules (or to be **Artinian**) if every descending chain $M = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_r \supseteq \ldots \supseteq \{0\}$ of submodules eventually becomes stationary, i.e. $\exists\, m_0$ such that $M_m = M_{m_0}$ for every $m \geqslant m_0$.

(b) An R-module M is said to satisfy the **ascending chain condition** (A.C.C.) on submodules (or to be **Noetherian**) if every ascending chain $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r \subseteq \ldots \subseteq M$ of submodules eventually becomes stationary, i.e. $\exists\, m_0$ such that $M_m = M_{m_0}$ for every $m \geqslant m_0$.

(c) The ring R is called **left Artinian** (resp. **left Noetherian**) if the regular module R° is Artinian (resp. Noetherian).

Theorem 8.3 (*Jordan-Hölder*)

Any series of R-submodules $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r = M$ ($r \in \mathbb{Z}_{\geqslant 0}$) of an R-module M may be refined to a composition series of M. In addition, if

$$0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_n = M \quad (n \in \mathbb{Z}_{\geqslant 0})$$

and

$$0 = M_0' \subsetneq M_1' \subsetneq \ldots \subsetneq M_m' = M \quad (m \in \mathbb{Z}_{\geqslant 0})$$

are two composition series of M, then $m = n$ and there exists a permutation $\pi \in \mathfrak{S}_n$ such that $M_i'/M_{i-1}' \cong M_{\pi(i)}/M_{\pi(i)-1}$ for every $1 \leqslant i \leqslant n$. In particular, the composition length is well-defined.

Proof: See *Commutative Algebra*. ■

Corollary 8.4

If M is an R-module, then TFAE:

(a) M has a composition series;

(b) M satisfies D.C.C. and A.C.C. on submodules;

(c) M satisfies D.C.C. on submodules and every submodule of M is finitely generated.

Proof: See *Commutative Algebra*. ■

Theorem 8.5 (*Hopkins' Theorem*)

If M is a module over a left Artinian ring, then TFAE:

(a) M has a composition series;

(b) M satisfies D.C.C. on submodules;

(c) M satisfies A.C.C. on submodules;

(d) M is finitely generated.

Proof: See *Commutative Algebra*. (Or Exercise: deduce it from the properties of the Jacobson radical and semisimplicity, which we are going to develop in the next sections.) ■

9 The Jacobson radical and Nakayama's Lemma*

The Jacobson radical is one of the most important two-sided ideals of a ring. As we will see in the next sections and Chapter 2, this ideal carries a lot of information about the structure of a ring and that of its modules.

Proposition-Definition 9.1 (*Annihilator / Jacobson radical*)

(a) Let M be an R-module. Then $\mathrm{ann}_R(M) := \{r \in R \mid rm = 0 \ \ \forall\, m \in M\}$ is a two-sided ideal of R, called **annihilator** of M.

(b) The **Jacobson radical** of R is the two-sided ideal

$$J(R) := \bigcap_{\substack{V \text{ simple} \\ R\text{-module}}} \mathrm{ann}_R(V) = \{x \in R \mid 1 - axb \in R^\times \ \ \forall\, a, b \in R\}\,.$$

(c) If V is a simple R-module, then there exists a maximal left ideal $I \lhd R$ such that $V \cong R^\circ/I$ (as R-modules) and

$$J(R) = \bigcap_{\substack{I \lhd R, \\ I \text{ maximal} \\ \text{left ideal}}} I\,.$$

Proof: See *Commutative Algebra.* ■

Exercise 9.2

(a) Prove that any simple R-module may be seen as a simple $R/J(R)$-module.

(b) Conversely, prove that any simple $R/J(R)$-module may be seen as a simple R-module. [Hint: use a change of the base ring via the canonical morphism $R \longrightarrow R/J(R)$.]

(c) Deduce that R and $R/J(R)$ have the same simple modules.

Theorem 9.3 (*Nakayama's Lemma*)

If M is a finitely generated R-module and $J(R)M = M$, then $M = 0$.

Proof: See *Commutative Algebra.* ■

Remark 9.4

One often needs to apply Nakayama's Lemma to a finitely generated quotient module M/U, where U is an R-submodule of M. In that case the result may be restated as follows:

$$M = U + J(R)M \quad \Longrightarrow \quad U = M$$