

Throughout this exercise sheet G denotes a finite group, V a \mathbb{C} -vector space and all vector spaces are assumed to be finite-dimensional. Each Exercise is worth 4 points.

EXERCISE 9

Prove the following assertions:

- (a) Two equivalent (matrix) \mathbb{C} -representations of G afford the same character.
- (b) The set $\mathcal{C}l(G)$ of class functions on G is a \mathbb{C} -subspace of \mathbb{C}^G and $\dim_{\mathbb{C}}(\mathcal{C}l(G)) = |C(G)|$.

EXERCISE 10

Let $Z(G)$ denote the centre of G . Let $\rho : G \rightarrow GL(V)$ be an irreducible \mathbb{C} -representation of G of degree n affording the character χ . Prove that:

- (a) If $z \in Z(G)$, then $\chi(z) = n\zeta$, where ζ is an $o(z)$ -th root of unity.
- (b) If ρ is faithful and $g \in G$, then: $g \in Z(G)$ if and only if $|\chi(g)| = \chi(1)$.

EXERCISE 11 (The dual representation)

Let V be a G -vector space and write $V^* := \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ for the dual space. Prove that:

- (a) V^* is endowed with the structure of a G -vector space via

$$\begin{aligned} G \times V^* &\longrightarrow V^* \\ (g, f) &\mapsto g \cdot f \end{aligned}$$

where $g \cdot f(v) := f(g^{-1} \cdot v) \forall v \in V$.

- (b) If $\rho_V : G \rightarrow GL(V)$ is a \mathbb{C} -representation of G decomposing as a direct sum $\rho_{V_1} \oplus \rho_{V_2}$ of two subrepresentations, then $\rho_{V^*} = \rho_{V_1^*} \oplus \rho_{V_2^*}$.
- (c) The character of ρ_{V^*} is $\chi_{V^*} = \overline{\chi_V}$.
- (d) Compute the dual representations of the representations of S_3 and S_4 of Exercise 8.

EXERCISE 12

- (a) Let V be a G -vector space and write χ_V for the corresponding character. Write $V^G := \{v \in V \mid g \cdot v = v \forall g \in G\}$ for the subspace of fixed points of V under the action of G . Prove that

$$\dim_{\mathbb{C}} V^G = \frac{1}{|G|} \sum_{g \in G} \chi_V(g).$$

- (b) Let X be a G -set and let χ_X denote that character of the associated permutation representation.
 - (i) Prove that $\chi_X(g) = |\text{Fix}_X(g)|$ for every $g \in G$.
 - (ii) Use character theory to prove that the number of orbits of G on X is

$$\frac{1}{|G|} \sum_{g \in G} |\text{Fix}_X(g)|.$$