



## CHAPTER 4. BURNSIDE'S $p^aq^b$ -THEOREM

### Notation

$G$  finite group

$C_i = [g_i]$  ( $1 \leq i \leq r$ ) : conj. classes of  $G$

$\hat{C}_i$  ( $1 \leq i \leq r$ ) : class sums

$w_i$  ( $1 \leq i \leq r$ ) : central character of  $(G)$ .

$\text{Irr}(G) = \{\chi_1, \dots, \chi_r\}$ ,  $r = |C(G)|$ ;  $n_i := \chi_i(1)$

$m_{jkl}$  ( $1 \leq j, k, l \leq r$ ) : class mult. constants

### 1. Algebraic integers and character values.

Recap on algebraic integers (See Commutative Algebra.)

- \* An element  $b \in \mathbb{C}$  is called an algebraic integer iff it is integral over  $\mathbb{Z}$ , i.e.  $\exists f \in \mathbb{Z}[X]$  monic s.t.  $f(b) = 0$ .
- \* If  $b \in \mathbb{Q}$  is integral over  $\mathbb{Z}$ , then  $b \in \mathbb{Z}$ .
- \* If  $A \subseteq \mathbb{C}$  is a subring, then  $\{b \in \mathbb{C} \mid b \text{ integral over } A\}$  is a subring of  $\mathbb{C}$ .
- \* In particular  $\mathcal{O} := \{b \in \mathbb{C} \mid b \text{ algebraic integer}\}$  is a subring of  $\mathbb{C}$ , so that sums and products of algebraic integers are algebraic integers.

Lemma 4.1: Character values are algebraic integers.



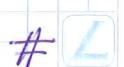
Proof: Let  $\chi$  be a character of  $G$ , and let  $g \in G$ . Set  $n := \deg(\chi)$ .

$$\stackrel{\text{Prop 2.11(b)}}{\Rightarrow} \chi(g) = \epsilon_1 + \dots + \epsilon_n \text{ with } \epsilon_i \text{ } \text{o}(g)\text{-th root of unity} \quad \forall 1 \leq i \leq n.$$

But then  $\epsilon_i$  is a zero of  $x^n - 1 \in \mathbb{Z}[x]$ , where  $n = \text{o}(g)$ .  
 $\forall 1 \leq i \leq n$ .

$\Rightarrow \epsilon_1, \dots, \epsilon_n$  are algebraic integers.

$\Rightarrow$  so is  $\chi(g)$  as a sum of alg. integers.



## Theorem 4.2 [Integrality Theorem]

The values of the central characters of  $G$  are algebraic integers.

Proof: Cor. 3.13  $\Rightarrow w_i(\hat{C}_j) w_i(\hat{C}_k) = \sum_{\ell=1}^r m_{j,k,\ell} w_i(\hat{C}_\ell)$   $\forall 1 \leq i, j, k \leq r$ .

For  $1 \leq i \leq r$  set:  $\mathbf{W}_i = (w_i(\hat{C}_1), \dots, w_i(\hat{C}_r))^t$

$$M_k := (m_{j,k,\ell})_{\substack{1 \leq j \leq r \\ 1 \leq \ell \leq r}}$$

so that  $M_k \mathbf{W}_i = w_i(\hat{C}_k) \mathbf{W}_i$

Now as  $w_i(1) = 1$  certainly  $w_i \neq 0$ , therefore  $\mathbf{W}_i$  is an eigenvector of  $M_k$  with eigenvalue  $w_i(\hat{C}_k)$

$$\Rightarrow w_i(\hat{C}_k) \text{ is a zero of } f := \det(M_k - X I_r)$$

But the class mult. csts  $m_{j,k,\ell} \in \mathbb{Z}$  by defn  $\Rightarrow f \in \mathbb{Z}[X]$

Hence  $w_i(\hat{C}_k)$  is an algebraic integer.

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Corollary 4.3:  $\frac{|G|}{n_i} \chi_i(g_j)$  is an algebraic integer.

Proof: Apply Prop. 3.15 (a). #

Corollary 4.4:  $\chi(1) \mid |G| \quad \forall \chi \in \text{Irr}(G)$ .

Proof: Let  $\chi_i \in \text{Irr}(G) = \{\chi_1, \dots, \chi_r\}$ . We have

$$\begin{aligned} \frac{|G|}{n_i} &= \frac{|G|}{n_i} \langle \chi_i, \chi_i \rangle \stackrel{\text{orthog}}{=} \frac{1}{n_i} \sum_{j=1}^r |\langle C_j | \chi_i(g_j) \rangle \chi_i(g_j)| \\ &= \sum_{j=1}^r \frac{|\langle g_j | \chi_i(g_j) \rangle \chi_i(g_j)|}{n_i} \\ &= \sum_{j=1}^r \underbrace{w_i(\hat{C}_j)}_{\substack{\text{alg. int.} \\ \text{by 4.2}}} \cdot \underbrace{\chi_i(g_j)}_{\substack{\text{alg. int.} \\ \text{by 4.1}}} \end{aligned}$$

L

$\Rightarrow \frac{|G|}{n_i} \in \mathbb{Q}$  is an algebraic integer

$\Rightarrow \frac{|G|}{n_i} \in \mathbb{Z} \Rightarrow n_i \mid |G|$ . #

Example 8:  $G = GL_3(\mathbb{F}_2)$  is a simple group

$$\begin{aligned} |G| &= (2^3-1)(2^3-2)(2^3-2^2) = 168 \\ &= 2^3 \cdot 3 \cdot 7 \end{aligned}$$

$G$  has 6 conjugacy classes (without proof here)

Question: can we compute  $\chi(1) \forall \chi \in \text{Irr}(G)$ ?

Solution: (1)  $G$  simple non-abelian  $\Rightarrow [G, G] = G$

$$\Rightarrow \chi(1) \geq 2 \quad \forall \chi \in \text{Irr}(G) \setminus \{1_G\}$$

(2) Exercise 18p Sheet 5:

$$G \text{ simple} \Rightarrow \chi(1) \neq 2 \quad \forall \chi \in \text{Irr}(G)$$

(3) Set  $n_i := \chi_i(1) \quad \forall 1 \leq i \leq 6$ .

$$\Rightarrow 168 = n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 + n_6^2$$

(1)+(2)  $\Rightarrow$  the possibilities are

1	2	4	5	6	X	} 9, 5 X 16!
1	2	3	3	8	X	
1	2	5	X	7	8	
1	X	4	7	7	7	not possible by (2)
1	3	3	6	7	8	

Important current research topic: find further arithmetic properties of the degrees of the irreducible characters.

E.g. the following conjecture is of particular interest. L

## L Conjecture 4.5 [McKay, 1972] Status: open!

Let  $G$  be a finite group,  $p \in \mathbb{P}$ , and  $P \in \text{Syl}_p(G)$ . Then

$$|\{x \in \text{Irr}(G) \mid p \nmid x(1)^2\}| = |\{x \in \text{Irr}(N_G(P)) \mid p \nmid x(1)^2\}|.$$

Note: Proven for soluble groups by Okuyama-Wajima.

Example 9:

$G = S_4$ . Then  $|\text{Irr}(G)| = 5$  by Ex. 6.

with  $\chi_1(1) = 1, \chi_2(1) = 1, \chi_3(1) = 2, \chi_4(1) = \chi_5(1) = 3$

(a)  $p=2$  yields  $P \cong D_8$  with  $N_G(P) = P$

By the Exercise,  $|\text{Irr}(D_8)| = 5$  with character degrees  
1, 1, 1, 1, 2

(b)  $p=3$  yields  $P \cong A_3$  with  $N_G(P) \cong S_3$

Ex. 5  $\Rightarrow |\text{Irr}(S_3)| = 3$  with char. degrees 1, 1, 2

$\Rightarrow$  The McKay Conjecture holds in both cases.

## 2. Burnside's vanishing theorem.

Definition 4.6: Let  $\chi$  be a character of  $G$ . Then the centre of  $\chi$  is  $Z(\chi) := \{g \in G \mid |\chi(g)| = \chi(1)\}$

Example 15:  $G = S_3$  has char. table

	1	1	1
$\chi_1$	1	1	1
$\chi_2$	1	-1	1
$\chi_3$	2	0	-1

by Ex. 5.

$$\Rightarrow Z(\chi_1) = Z(\chi_2) = G$$

$$Z(\chi_3) = \{1_G\}$$



-  Proposition 4.8 Let  $\chi$  be a character of  $G$ . Then
- (a)  $Z(\chi) \trianglelefteq G$ .
  - (b)  $\chi|_{Z(\chi)} = \chi(1)\lambda$  for some linear character  $\lambda \in \text{Irr}(Z(\chi))$ .
  - (c)  $Z(\chi)/\ker(\chi)$  is a cyclic group
  - (d)  $\chi \in \text{Irr}(G) \Rightarrow Z(\chi)/\ker(\chi) = Z(G/\ker(\chi))$

Proof: Let  $\rho$  be a representation affording  $\chi$  and set  $n := \chi(1)$ .

(a) Let  $g \in Z(\chi)$ : • Prop 2.2(b)  $\Rightarrow \chi(g) = \epsilon_1 + \dots + \epsilon_n$   
with  $\epsilon_1, \dots, \epsilon_n$   $\circ(g)$ -th roots of unity

$$\bullet \quad (\oplus): |\chi(g)| = n$$

So only possibility is that  $\epsilon_1 = \dots = \epsilon_n =: \epsilon(g)$

$$\Rightarrow \rho(g) = \epsilon(g) \text{Id}_V$$

• Now if  $g, h \in Z(\chi)$ , then  $\rho_V(gh^{-1}) = \rho_V(g)\rho_V(h)^{-1}$   
 $= \epsilon(g)\epsilon(h)^{-1} \text{Id}_V$  (scalar multiple)  
 $\Rightarrow gh^{-1} \in Z(\chi)$

$$\Rightarrow Z(\chi) \trianglelefteq G.$$

• Finally if  $g \in Z(\chi)$  and  $h \in G$ , then

$$\begin{aligned} \rho_V(hgh^{-1}) &= \rho_V(h)\epsilon(g)\text{Id}_V\rho_V(h)^{-1} \\ &= \epsilon(g)\text{Id}_V = \rho_V(g) \Rightarrow Z(\chi) \trianglelefteq G. \end{aligned}$$

(b) The map  $\lambda: Z(\chi) \rightarrow \mathbb{C}^\times$  is a grp homomorphism  
 $g \mapsto \epsilon(g)$

hence a linear character of  $Z(\chi)$ .

Moreover, clearly  $\chi|_{Z(\chi)} = \chi(1)\lambda$  as claimed.

(c) By the 1st iso theorem  $Z(\chi)/\ker(\lambda) \cong \overbrace{\text{Im}(\lambda)}^{\leq \mathbb{C}^\times}$ .



Hence  $\mathbb{Z}(x)/\ker(x)$  is cyclic since it is a finite subgroup of  $\mathbb{C}^\times$ .

(d) By (a)  $\forall g \in \mathbb{Z}(x)$ ,  $\rho_V(g) = \epsilon(g) \text{Id}_V$  scalar matrix

$$\Rightarrow \rho_V(\mathbb{Z}(x)) \leq \mathbb{Z}(\rho_V(G))$$

Again by the 1st Iso theorem:

$$\mathbb{Z}(x)/\ker(x) \cong \rho_V(\mathbb{Z}(x)) \leq \mathbb{Z}(\rho_V(G))$$

||

$$\mathbb{Z}(G/\ker(x))$$

||

$$\mathbb{Z}(G/\ker(x))$$

On the other hand, if  $g\ker(x) \in \mathbb{Z}(G/\ker(x))$ , then by

Schur's Lemma  $\rho_V(g) = \psi \text{Id}_V$  for some  $\psi \in \mathbb{C}^\times$

$$\Rightarrow g \in \mathbb{Z}(x) \Rightarrow \mathbb{Z}(G/\ker(x)) \leq \mathbb{Z}(x)/\ker(x)$$

Corollary 4.9  $\bigcap_{x \in \text{Irr}(G)} \mathbb{Z}(x) = \mathbb{Z}(G)$

Proof: Exercise See Ex. Sheet 5

Remark: one can prove that  $x(1) \mid |G : \mathbb{Z}(x)| \quad \forall x \in \text{Irr}(G)$

Remarks on field extensions: Let  $m > 1$  be an integer and let

$\zeta_m$  be a primitive  $m$ -th root of unity in  $\mathbb{C}$ . Then:

- $L := \mathbb{Q}(\zeta_m)$  is the splitting field of  $x^m - 1$ , hence  $L/\mathbb{Q}$  is a Galois extension.

- For  $\sigma \in \text{Gal}(L/\mathbb{Q})$  we must have  $\sigma(\zeta_m) = \zeta_m^k$  for some  $k \in \mathbb{Z}$  with  $\gcd(k, m) = 1$



- this defines an isomorphism

$$\text{Gal}(L/\mathbb{Q}) \xrightarrow{\cong} (\mathbb{Z}/m\mathbb{Z})^\times$$

$\Rightarrow \text{Gal}(L/\mathbb{Q})$  is an abelian group.

### Theorem 4.10 [Burnside]

Let  $\chi \in \text{Irr}(G)$  with  $\chi(1) = n$ , and let  $C$  be a conjugacy class of  $G$  with  $\gcd(n, |C|) = 1$ . Let  $g \in C$ . Then either  $\chi(g) = 0$ , or  $g \in Z(\chi)$  (i.e.  $|\chi(g)| = n$ ).

Proof:  $\gcd(n, |C|) = 1 \Rightarrow \exists u, v \in \mathbb{Z}$  s.t.  $un + v|C| = 1$

$$\text{Set } \alpha = \frac{1}{n}\chi(g) = \underbrace{u\chi(g)}_{\in \mathbb{Z} \text{ alg. int.}} + \underbrace{v\frac{|C|}{n}\chi(g)}_{\in \mathbb{Z} \text{ alg. int. by 3.1}} \Rightarrow \alpha \text{ algebraic integer.}$$

If  $g \in Z(\chi)$  we are done ✓

Else  $|\chi(g)| < n \Rightarrow |\alpha| < 1$ .

Let  $m = \alpha(g)$ . Since  $\chi(g)$  is a sum of  $m$ -th roots of unity  
 $\Rightarrow \chi(g) \in \mathbb{Q}(\mathbb{F}_m)$

Therefore if  $\sigma \in \text{Gal}(\mathbb{Q}(\mathbb{F}_m)/\mathbb{Q})$ , we have

$\sigma(\chi(g)) = \text{sum of } n \text{ } m\text{-th roots of unity}$

$$\Rightarrow |\sigma(\chi(g))| \leq n$$

$$\Rightarrow |\sigma(\alpha)| \leq 1$$

$$\Rightarrow \left| \prod_{\sigma \in \text{Gal}(\mathbb{Q}(\mathbb{F}_m)/\mathbb{Q})} \sigma(\alpha) \right| < 1 \text{ as } |\text{Id}(\alpha)| = |\alpha| < 1.$$

but this is the norm of the algebraic integer  $\alpha$

i.e. the constant term of the min. polynomial of  $\alpha$  in  $\mathbb{Q}(\alpha)/\mathbb{Q}$

$$\Rightarrow \alpha = 0 \Rightarrow \chi(g) = 0 \quad \#$$

L Corollary 4.11 : In the situation of Thm 4.10 with

1.  $\chi(1) = n > 1$
2.  $C \neq \{1\}$
3.  $G$  is simple

Then  $\chi(g) = 0$ .

Proof:  $\chi(g) \neq 0 \xrightarrow{\text{Thm 4.10}} g \in Z(\chi) \Rightarrow Z(\chi) \neq 1$

But  $Z(\chi) \trianglelefteq G$  and  $G$  is simple  $\Rightarrow Z(\chi) = G$

Prop 4.8(a)

So Prop 4.8(b)  $\Rightarrow \chi$  is a sum of linear characters  
with  $n > 1$  terms  
 $\Rightarrow \chi$  reducible  $\Downarrow$  #

Exercise 4.12: Let  $G$  be a finite cyclic group.

Let  $S := \{g \in G \mid \langle g \rangle = G\}$ . (set of generators).

If  $\chi(g) \neq 0 \forall g \in S$ , then

$$\sum_{g \in S} |\chi(g)|^2 \geq |S|.$$

Theorem 4.13 (Burnside's Vanishing Theorem).

Let  $\chi \in \text{Irr}(G)$  with  $\chi(1) > 1$ . Then  $\exists g \in G$  with  $\chi(g) = 0$ .

Proof: Define an equivalence relation on  $G$  as follows :

$$\text{for } g_1, g_2 \in G : g_1 \sim g_2 \Leftrightarrow \langle g_1 \rangle = \langle g_2 \rangle$$

(clear: reflexive, symmetric, transitive.)

Assume now that  $\chi(g) \neq 0 \forall g \in G$ .

Let  $Z \leq G$  be a cyclic subgroup and let  $S := \{g \in Z \mid \langle g \rangle \neq Z\}$

Restricting  $\chi$  to  $Z$ , Ex 4.12  $\Rightarrow \sum_{g \in S} |\chi(g)|^2 \geq |S|$

moreover:  
 $S$  is an equiv  
class of  $\sim$

Letting  $Z$  running over all cyclic subgroups of  $G$ , we get:

$\sum_{g \in G} |\chi(g)|^2 = \sum_{S \subseteq G/\langle h \rangle} \sum_{g \in S} |\chi(g)|^2 \geq |G|$  and since  $S \subseteq G/\langle h \rangle$   
 $\sum_{g \in G \setminus \{1\}} |\chi(g)|^2 \geq |G| - 1$   
 $\Rightarrow |G| = \sum_{g \in G} |\chi(g)\overline{\chi(g)}| \geq |G| - 1 + \underbrace{\chi(1)^2}_{\geq 1} > |G| \quad \#$

A stronger result whose proof requires the classification of finite simple groups:

[Malle - Navarro - Olsson, 2000] Let  $\chi \in \text{Irr}(G)$  with  $\chi(1) > 1$   
 Then  $\exists g \in G$  of prime power order  
 s.t.  $\chi(g) = 0$ .

### 3. Burnside's $p^aq^b$ -Theorem

We now see one of the most famous applications of Character Theory to the structure of finite simple groups: the proof of Burnside's  $p^aq^b$ -Thm.

Example 10:

In the Einf. in die Algebra we have seen that:

(a)  $G$  finite group with  $|G| = p^2$  for some prime  $p \in \mathbb{P} \Rightarrow G$  is abelian.

Here is another proof using character theory:

Let  $\chi \in \text{Irr}(G) \Rightarrow \chi(1) | |G| \Rightarrow \chi(1) \in \{1, p, p^2\}$

But then  $p^2 = |G| = \sum_{\chi \in \text{Irr}(G)} \chi(1)^2 = \underbrace{\chi_G(1)^2}_{=1} + \sum_{\substack{\chi \in \text{Irr}(G) \\ \chi \neq \chi_G}} \chi(1)^2$

$\Rightarrow$  any  $\chi \in \text{Irr}(G) \setminus \{\chi_G\}$  is linear

Gr. 3.6  $\Rightarrow G$  abelian  $\#$

(b)  $G$   $p$ -group ( $p \in \mathbb{P}$ )  $\Rightarrow G$  soluble group



Here is another proof using character theory :

Induction on  $|G|$  : • if  $|G|=p$ ,  $|G|=p^2$  ✓ (a)

• So assume  $p \nmid |G| \geq p^3$ .

$\chi(1) \in \{1, p, p^2, p^3, \dots\} \quad \forall \chi \in \text{Irr}(G)$ .

$$\text{As } \sum_{\chi \in \text{Irr}(G)} \chi(1)^2 = 1 + \sum_{\substack{\chi \in \text{Irr}(G) \\ \chi \neq 1_G}} \chi(1)^2 = p^n$$

$\therefore p^{n-1}$

there are at least  $p$  linear characters in  $\text{Irr}(G)$

$\stackrel{3.6}{\Rightarrow} [G, G] < G \Rightarrow [G, G]$  soluble by induction hypo.  
and so is  $G/[G, G]$

$\Rightarrow G$  is soluble (sandwich principle)

Theorem 4.14: Let  $G$  be a finite non-abelian simple group, and

(Burnside) let  $C \in C(G)$  be a conjugacy class of prime power order,  
i.e.  $|C| = p^a$  for some  $p \in \mathbb{P}$  and some  $a \in \mathbb{N}_0$ .

Then  $C = \{1\}$ .

Proof: Assume  $C \neq \{1\}$ , and let  $g \in C$  ( $g \neq 1$ )

Let  $\chi \in \text{Irr}(G) \setminus \{1_G\}$ .

As  $G$  is simple non-abelian  $[G, G] = G \stackrel{\text{or 3.6}}{\Rightarrow} \chi(1) > 1$

$\Rightarrow$  either  $p \mid \chi(1)$  or  $\underbrace{(\chi(1), |C|)}_{\text{by 4.11}} = 1$

$$\chi(g) = 0$$

2<sup>nd</sup> Orth. Relations yield

$$0 = 1 + \sum_{\chi \in \text{Irr}(G) \setminus \{1_G\}} \chi(g)\chi(1)$$

$$= 1 + \sum_{\substack{\chi \in \text{Irr}(G) \setminus \{1_G\} \\ \chi(g) \neq 0}} \underbrace{\chi(g)\chi(1)}_{p \mid \chi(1)}$$

So either the sum is 0  
and  $0 = 1$  ↯

$$\text{or } -\frac{1}{p} = \sum_{\substack{\chi \in \text{Irr}(G) \setminus \{1_G\} \\ \chi(g) \neq 0}} \frac{\chi(g)\chi(1)}{p}$$

algebraic integer by Lemma 4.1



## Theorem 4.15 [Burnside's $p^aq^b$ -Theorem, "simple" version]

Let  $p, q \in \mathbb{P}$ , and let  $a, b \in \mathbb{N}_0$  with  $a+b \geq 2$ .

If  $G$  is a group of order  $p^aq^b$ , then  $G$  is not simple.

Proof: • 1st assume that either  $a=0$ , or  $b=0$ , then  $G$  is a  $q$ -group or a  $p$ -group (resp.) with  ~~$|G| \neq q^2$~~   $q^2$  (resp.  $p^2$ )  $\mid |G|$   
 $\xrightarrow{\text{Ex 10(b)}}$   $G$  is soluble  $\Rightarrow \exists H \neq N \trianglelefteq G$  with  $|G:N|=q$ , resp.p.  
 $\Rightarrow G$  not simple.

• Now assume that  $a \geq 1, b \geq 1$ . Let  $Q \in \text{Syl}_q(G)$  ( $|Q| = p^r$ ). Because  $Q$  is a  $q$ -group,  $Z(Q) \neq \{1\}$   
 so let  $g \in Z(Q) \setminus \{1\}$

Then  $Q \leq C_G(g) \stackrel{\text{class eq.}}{\Rightarrow} |\square| = |G : C_G(g)| = p^r$   
 for some  $r \leq a$ .

If  $G$  were simple, then  $G$  would be non-abelian as  $a+b \geq 2$

$\xrightarrow{\text{Thm 4.14}} [g] = \{1\}$  ( $\Rightarrow g \in Z(G)$ )  $\nexists g \neq 1$ .

#

Corollary 4.16: Every group of order  $p^aq^b$  is soluble.

Proof:  $\xrightarrow{\text{Induction on } a+b}$

• Clean if  $a+b \leq 1$

• Assume  $a+b \geq 2$  and let  $G$  be a finite grp s.t.  $|G| = p^aq^b$

Burnside  $\Rightarrow \exists H \neq N \trianglelefteq G$

and both  $|H|, |G/H| \leqslant p^{a+b}$

$\Rightarrow H, G/H$  soluble by the induction hypothesis

$\Rightarrow G$  is soluble as well by the sandwich principle. #



Category	Sub-category	Definition	Measures		Notes
			Score	Description	
1. General Information	1.1 Personal Information	Information about the individual, such as name, age, gender, and contact details.	Score	1	
1. General Information	1.2 Professional Information	Information about the individual's professional background, such as job title, industry, and experience.	Score	1	
2. Communication Style	2.1 Directness	The extent to which the individual communicates directly and assertively.	Score	1	
2. Communication Style	2.2 Indirectness	The extent to which the individual communicates indirectly and diplomatically.	Score	1	
3. Message Content	3.1 Problem Solving	The extent to which the message focuses on problem-solving and finding solutions.	Score	1	
3. Message Content	3.2 Relationship Focus	The extent to which the message focuses on maintaining and improving relationships.	Score	1	
4. Non-verbal Cues	4.1 Body Language	The extent to which body language, such as posture and eye contact, conveys confidence or openness.	Score	1	
4. Non-verbal Cues	4.2 Tone of Voice	The extent to which tone of voice, such as pitch and volume, conveys enthusiasm or seriousness.	Score	1	
5. Cultural Sensitivity	5.1 Awareness	The extent to which the individual is aware of cultural differences and their impact on communication.	Score	1	
5. Cultural Sensitivity	5.2 Adaptation	The extent to which the individual adapts communication style to cultural context.	Score	1	
6. Self-awareness	6.1 Self-reflection	The extent to which the individual reflects on their own communication style and its impact on others.	Score	1	
6. Self-awareness	6.2 Self-improvement	The extent to which the individual actively works to improve their communication skills.	Score	1	
7. Professional Development	7.1 Training	The extent to which the individual has received formal training in communication skills.	Score	1	
7. Professional Development	7.2 Practice	The extent to which the individual practices communication skills in various contexts.	Score	1	
8. Overall Score		The final score based on the weighted average of all measures.	Score	1	