REPRESENTATION THEORY — EXERCISE SHEET 1 — **TU KAISERSLAUTERN**

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Due date: **Tuesday, 12th of November 2019, 6 p.m.** — WS 2019/20

Throughout, R denotes a ring, and, unless otherwise stated, all rings are assumed to be *associative rings with* 1, and modules are assumed to be *left* modules.

EXERCISE 1.

(a) Let $(M,+)$ be an abelian group and let $n \geq 0$ be an integer such that $0 = nm = m + \ldots + m$ (n times) for every $m \in M$. Prove that M is a $\mathbb{Z}/n\mathbb{Z}$-module for the external composition law

$$\begin{array}{rccl} *: & \mathbb{Z}/n\mathbb{Z} \times M & \longrightarrow & M \\ & (\bar{k}, m) & \mapsto & \bar{k} * m := m + \ldots + m \; (k \text{ times}) \,. \end{array}$$

(b) Let $(R, +, \cdot)$ be a ring. Prove that R is naturally endowed with the structure of a $\mathbb{Z}$-algebra.

EXERCISE 2.

(a) Let R be a ring, and let M, N be R-modules. Prove that:

(1) $\mathrm{End}_R(M)$, endowed with the pointwise addition of maps and the usual composition of maps, is a ring.

(2) The abelian group $\mathrm{Hom}_R(M, N)$ is a left R-module for the external composition law defined by

$$(rf)(m) := f(rm) = rf(m) \qquad \forall\, r \in R,\ \forall f \in \mathrm{Hom}_R(M,N),\ \forall m \in M\,.$$

(b) Let now R be a commutative ring, A be an R-algebra, and M be an A-module. Prove that $\mathrm{End}_R(M)$ and $\mathrm{End}_A(M)$ are R-algebras.

The next exercises require the content of the 2nd week of lectures.

EXERCISE 3.

(a) Assume R is a commutative ring and I is an ideal of R. Let M be a left R-module. Prove that there is an isomorphism of left R-modules $R/I \otimes_R M \cong M/IM$.

(b) Let m, n be coprime positive integers. Compute $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$, and $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$.

(c) Let K be a field and let U, V be finite-dimensional K-vector spaces. Prove that there is a natural isomorphism of K-vector spaces:

$$\mathrm{Hom}_K(U, V) \cong U^* \otimes_K V\,.$$

Exercise 4.
Prove that if $(R, +, \cdot)$ is a ring, then $R^\circ := R$ itself may be seen as an R-module via left multiplication in R, i.e. where the external composition law is given by

$$R \times R^\circ \longrightarrow R^\circ, (r, m) \mapsto r \cdot m\,.$$

We call R° the **regular** R-module.

Prove that the R-submodules of R° are prescisely the left ideals of R. Moreover, $I \triangleleft R$ is a maximal left ideal of $R \Leftrightarrow R^\circ/I$ is a simple R-module, and $I \triangleleft R$ is a minimal left ideal of R $\Leftrightarrow$ I is simple when regarded as an R-submodule of R°.

Exercise 5.

(a) Prove that any simple R-module may be seen as a simple $R/J(R)$-module.

(b) Conversely, prove that any simple $R/J(R)$-module may be seen as a simple R-module. [Hint: use a change of the base ring via the canonical morphism $R \longrightarrow R/J(R)$.]

(c) Deduce that R and $R/J(R)$ have the same simple modules.

Exercise 6.

(a) Let p is a prime number and $R := \{\frac{a}{b} \in \mathbb{Q} \mid p \nmid b\}$. Prove that $R \setminus R^\times = \{\frac{a}{b} \in R \mid p|a\}$ and deduce that R is local.

(b) Let K be a field and let $R := \left\{A = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 0 & a_1 & \dots & a_{n-1} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & a_1 \end{pmatrix} \in M_n(K)\right\}$.
Prove that $R \setminus R^\times = \{A \in R \mid a_1 = 0\}$ and deduce that R is local.

Exercise 7.

(a) Let K be a field and let A be the K-algebra $\left\{\begin{pmatrix} a_1 & a \\ 0 & a_1 \end{pmatrix} \mid a_1, a \in K\right\}$. Consider the A-module $V := K^2$, where A acts by left matrix multiplication. Prove that:

 (1) $\{\begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in K\}$ is a simple A-submodule of V; but
 (2) V is not semisimple.

(b) Prove that any submodule and any quotient of a completely reducible module is again completely reducible.