# Chapter 3. Representation Theory of Finite Groups

Representation theory of finite groups is originally concerned with the ways of writing a finite group $G$ as a group of matrices, that is using group homomorphisms from $G$ to the general linear group $\mathrm{GL}_n(K)$ of invertible $n \times n$-matrices with coefficients in a field $K$ for some positive integer $n$. Thus, we shall first define representations of groups using this approach. Our aim is then to translate such homomorphisms $G \longrightarrow \mathrm{GL}_n(K)$ into the language of module theory in order to be able to apply the theory we have developed so far.

**Notation**: throughout this chapter, unless otherwise specified, we let $G$ denote a finite group and $K$ be a commutative ring. Moroever, all modules considered are assumed to be **finitely generated**, hence **of finite rank** if they are free.

**References:**


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## 14 Linear representations of finite groups

**Definition 14.1 (*$K$-representation, matrix representation*)**

(a) A $K$-**representation** of $G$ is a group homomorphism $\rho : G \longrightarrow \mathrm{GL}(V)$, where $V \cong K^n$ ($n \in \mathbb{Z}_{>0}$) is a free $K$-module of finite rank.

(b) A **matrix representation** of $G$ is a group homomorphism $X : G \longrightarrow \mathrm{GL}_n(K)$ ($n \in \mathbb{Z}_{>0}$).

In both cases the integer $n$ is called the **degree** of the representation.

A representation is called an **ordinary representation** if $K$ is a field of characteristic zero (or more generally of characteristic not dividing $|G|$), and it is called a **modular representation** if $K$ is a field of characteristic $p$ dividing $|G|$.

**Remark 14.2**

Recall that every choice of a basis $B$ of $V$ yields a group homomorphism

$$\alpha_B : GL(V) \longrightarrow \mathrm{GL}_n(K), \varphi \mapsto (\varphi)_B$$

(where $(\varphi)_B$ denotes the matrix of $\varphi$ in the basis $B$). Therefore, a $K$-representation $\rho : G \longrightarrow \mathrm{GL}(V)$ together with the choice of a basis $B$ of $V$ gives rise to a matrix representation of $G$:

$$G \xrightarrow{\rho} \mathrm{GL}(V) \xrightarrow{\alpha_B} \mathrm{GL}_n(K)$$

Conversely, any matrix representation $X : G \longrightarrow \mathrm{GL}_n(K)$ gives rise to a $K$-representation

$$\begin{array}{rccl} \rho : & G & \longrightarrow & \mathrm{GL}(K^n) \\ & g & \mapsto & \rho(g) : K^n \longrightarrow K^n, v \mapsto X(g)v \end{array}$$

where $X(g)v$ is the standard matrix multiplication (namely we set $V = K^n$).

**Example 8**

(a) If $G$ is an arbitrary finite group, then

$$\begin{array}{rccl} \rho : & G & \longrightarrow & \mathrm{GL}(K) \cong K^\times \\ & g & \mapsto & \rho(g) := \mathrm{Id}_K \leftrightarrow 1_K \end{array}$$

is a $K$-representation of $G$, called <u>**the**</u> **trivial representation** of $G$.

(b) Let $G = S_n$ ($n \geqslant 1$) be the symmetric group on $n$ letters. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $V := K^n$. Then

$$\begin{array}{rccl} \rho : & S_n & \longrightarrow & \mathrm{GL}(K^n) \\ & \sigma & \mapsto & \rho(\sigma) : K^n \longrightarrow K^n, e_i \mapsto e_{\sigma(i)} \end{array}$$

is a $K$-representation, called **natural representation** of $S_n$.

(c) More generally, if $X$ is a finite $G$-set, i.e. a finite set endowed with a left action $\cdot : G \times X \longrightarrow X$, and $V$ is a free $K$-module with basis $\{e_x \mid x \in X\}$, then

$$\begin{array}{rccl} \rho_X : & G & \longrightarrow & \mathrm{GL}(V) \\ & g & \mapsto & \rho_X(g) : V \longrightarrow V, e_x \mapsto e_{g \cdot x} \end{array}$$

is a $K$-representation of $G$, called **permutation representation**.

Clearly (b) is a special case of (c) with $G = S_n$ and $X = \{1, 2, \ldots, n\}$.

If $X = G$ and the left action $\cdot : G \times X \longrightarrow X$ is just the multiplication in $G$, then $\rho_X =: \rho_{\mathrm{reg}}$ is called the **regular representation** of $G$.

**Definition 14.3 (*Equivalent representations*)**

Let $\rho_1 : G \longrightarrow \mathrm{GL}(V_1)$ and $\rho_2 : G \longrightarrow \mathrm{GL}(V_2)$ be two representations of $G$, where $V_1, V_2$ are two free $K$-modules of finite rank. Then $\rho_1$ and $\rho_2$ are called **equivalent** (or **similar**, or **isomorphic**) if there exists a $K$-isomorphism $\alpha : V_1 \longrightarrow V_2$ such that $\rho_2(g) = \alpha \circ \rho_1(g) \circ \alpha^{-1}$ for each $g \in G$.

$$\begin{tikzcd} V_1 \arrow[r, "\rho_1(g)"] \arrow[d, phantom, ""] & V_1 \arrow[d, "\alpha"] \\ V_2 \arrow[u, "\alpha^{-1}"] \arrow[r, "\rho_2(g)"'] \arrow[ur, phantom, "\circlearrowleft"] & V_2 \end{tikzcd}$$

In this case, we write $\rho_1 \sim \rho_2$.

Clearly $\sim$ is an equivalence relation.

## 15 The group algebra and its modules

We now want to be able to see $K$-representations of a group $G$ as *modules*, and more precisely as *modules* over a $K$-algebra depending on the group $G$, which is called the *group algebra*:

**Lemma-Definition 15.1 (*Group algebra*)**

The **group ring** $KG$ is the ring whose elements are the linear combinations $\sum_{g \in G} \lambda_g g$ with $\lambda_g \in K$, and addition and multiplication are given by

$$\sum_{g \in G} \lambda_g g + \sum_{g \in G} \mu_g g = \sum_{g \in G} (\lambda_g + \mu_g) g \quad \text{and} \quad \big(\sum_{g \in G} \lambda_g g\big) \cdot \big(\sum_{h \in G} \mu_h h\big) = \sum_{g,h \in G} (\lambda_g \mu_h) gh$$

respectively. Thus $KG$ is a $K$-algebra, which as a $K$-module is free with basis $G$. Hence we usually call $KG$ the **group algebra of $G$ over $K$** rather than simply *group ring*.

**Proof:** By definition $KG$ is a free $K$-module with basis $G$, and the multiplication in $G$ is extended by $K$-bilinearity to the given multiplication $\cdot : KG \times KG \longrightarrow KG$. It is then straightforward that $KG$ bears both the structures of a ring and of a $K$-module. Finally, axiom (A3) of $K$-algebras follows directly from the definition of the multiplication and the commutativity of $K$. ■

**Remark 15.2**

Clearly the $K$-rank of $KG$ is $|G|$ and $G \subseteq (KG)^\times$. Moreover, $KG$ is commutative if and only if $G$ is an abelian group. Also note that if $K$ is a field, then it is clear that $KG$ a left Artinian ring because we may consider $K$-dimesnions, so that by Hopkin's Theorem a $KG$-module is finitely generated if and only if it admits a composition series.

**Proposition 15.3**

(a) Any $K$-representation $\rho : G \longrightarrow \mathrm{GL}(V)$ of $G$ gives rise to a $KG$-module structure on $V$, where the external composition law is defined by the map

$$\begin{array}{rccc} \cdot : & G \times V & \longrightarrow & V \\ & (g, v) & \mapsto & g \cdot v := \rho(g)(v) \end{array}$$

extended by $K$-linearity to the whole of $KG$.

(b) Conversely, every $KG$-module $(V, +, \cdot)$ defines a $K$-representation

$$\begin{array}{rccl} \rho_V : & G & \longrightarrow & \mathrm{GL}(V) \\ & g & \mapsto & \rho_V(g) : V \longrightarrow V, v \mapsto \rho_V(g) := g \cdot v \end{array}$$

of the group $G$.

**Proof:**

(a) Since $V$ is a $K$-module, it is equipped with an internal addition $+$ such that $(V, +)$ is an abelian group. It is then straightforward to check that the given external composition law makes $(V, +)$ into a $KG$-module.

(b) Clearly, it follows from the $KG$-module axioms that $\rho_V(g) \in \mathrm{GL}(V)$ and also that $\rho_V(g_1 g_2) = \rho_V(g_1) \circ \rho_V(g_2)$ für alle $g_1, g_2 \in G$, hence $\rho_V$ is a group homomorphism. ■

Notice that, since $G$ is a group, the map $KG \longrightarrow KG$ such that $g \mapsto g^{-1}$ for each $g \in G$ is an anti-automorphism. It follows that any *left* $KG$-module $M$ may be regarded as a *right* $KG$-module via the right $G$-action $m \cdot g := g^{-1} \cdot m$. Thus the sidedness of $KG$-modules is not usually an issue.

**Example 9**

The trivial representation of Example 8(b) corresponds to the so-called **trivial** $KG$**-module**, that is the commutative ring $K$ itself seen as a $KG$-module via the $G$-action

$$\begin{aligned} \cdot\, : G \times K &\longrightarrow K \\ (g, \lambda) &\longmapsto g \cdot \lambda := \lambda \end{aligned}$$

extended by $K$-linearity to the whole of $KG$.

**Exercise 15.4**

Let $G$ be a finite group and let $K$ be a commutative ring. Prove that the regular representation $\rho_{reg}$ of $G$ defined in Exampale 8(c) corresponds to the regular $KG$-module $KG^{\circ}$ via Proposition 15.3.

**Remark 15.5**

More generally, through Proposition 15.3, we may transport terminology and properties from $KG$-modules to representations and conversely.

For instance, we say that a representation is *irreducible* (or *simple*) if the corresponding $KG$-module is irreducible (= simple). (Notice that it is tradition to use the term *simple* for modules, and the term irreducible for *representations*.)

**Lemma 15.6**

Two representations $\rho_1 : G \longrightarrow \mathrm{GL}(V_1)$ and $\rho_2 : G \longrightarrow \mathrm{GL}(V_2)$ are equivalent if and only if $V_1 \cong V_2$ as $KG$-modules.

**Proof:** If $\rho_1 \sim \rho_2$ and $\alpha : V_1 \longrightarrow V_2$ is a $K$-isomorphism such that $\rho_2(g) = \alpha \circ \rho_1(g) \circ \alpha^{-1}$ for each $g \in G$, then by Proposition 15.3 for every $v \in V_1$ and every $g \in G$ we have

$$g \cdot \alpha(v) = \rho_2(g)(\alpha(v)) = \alpha(\rho_1(g)(v)) = \alpha(g \cdot v),$$

hence $\alpha$ is a $KG$-isomorphism. Conversely, if $\alpha : V_1 \longrightarrow V_2$ is a $KG$-isomorphism, then certainly it is a $K$-homomorphism and for each $g \in G$ and by Proposition ?? for each $v \in V_2$ we have

$$\alpha \circ \rho_1(g) \circ \alpha^{-1}(v) = \alpha(\rho_1(g)(\alpha^{-1}(v)) = \alpha(g \cdot \alpha^{-1}(v)) = g \cdot \alpha(\alpha^{-1}(v)) = g \cdot v = \rho_2(g)(v)\,,$$

hence $\rho_2(g) = \alpha \circ \rho_1(g) \circ \alpha^{-1}$ for each $g \in G$. ■

Finally we introduce an ideal of $KG$ which encodes a lot of information about $KG$-modules.

**Proposition-Definition 15.7 (*The augmentation ideal*)**

The map $\varepsilon : KG \longrightarrow K, \sum_{g\in G} \lambda_g g \mapsto \sum_{g\in G} \lambda_g$ is an algebra homomorphism, called **augmentation homomorphism (or map)**. Its kernel $\ker(\varepsilon) =: I(KG)$ is an ideal and it is called the **augmentation ideal** of $KG$. The following statements hold:

(a) $I(KG) = \{\sum_{g\in G} \lambda_g g \in KG \mid \sum_{g\in G} \lambda_g = 0\} = \mathrm{ann}_{KG}(K)$ and if $K$ is a field $I(KG) \supseteq J(KG)$;

(b) $KG/I(KG) \cong K$ as $K$-algebras;

(c) $I(KG)$ is a free $K$-module of rank $|G|$-1 with $K$-basis $\{g-1 \mid g \in G\backslash\{1\}\}$;

**Proof:** Clearly, the map $\varepsilon : KG \longrightarrow K$ is the unique extension by $K$-linearity of the trivial representation $G \longrightarrow K^\times \subseteq K, g \mapsto 1_K$ to $KG$, hence is an algebra homomorphism and its kernel is an ideal of the algebra $KG$.

(a) $I(KG) = \ker(\varepsilon) = \{\sum_{g\in G} \lambda_g g \in KG \mid \sum_{g\in G} \lambda_g = 0\}$ by definition of $\varepsilon$. The second equality is obvious by definition of $\mathrm{ann}_{KG}(K)$, and the last inclusion follows from the definition of the Jacobson radical.

(b) follows from the 1st isomorphism theorem.

(c) Let $\sum_{g\in G} \lambda_g g \in I(KG)$. Then $\sum_{g\in G} \lambda_g = 0$ and hence

$$\sum_{g\in G} \lambda_g g = \sum_{g\in G} \lambda_g g - 0 = \sum_{g\in G} \lambda_g g - \sum_{g\in G} \lambda_g = \sum_{g\in G} \lambda_g (g-1) = \sum_{g\in G\backslash\{1\}} \lambda_g (g-1)\,,$$

which proves that the set $\{g-1 \mid g \in G\backslash\{1\}\}$ generates $I(KG)$ as a $K$-module. The above computations also shows that

$$\sum_{g\in G\backslash\{1\}} \lambda_g (g-1) = 0 \quad \Longrightarrow \quad \sum_{g\in G} \lambda_g g = 0$$

Hence $\lambda_g = 0\ \forall\, g \in G$, which proves that the set $\{g-1 \mid g \in G\backslash\{1\}\}$ is also $K$-linearly independent, hence a $K$-basis of $I(KG)$. ■

**Lemma 15.8**

If $K$ is a field of positive characteristic $p$ and $G$ is $p$-group, then $I(KG) = J(KG)$.

**Exercise 15.9 (*Proof of Lemma 15.8. Proceed as indicated.*)**

(a) (Facultative: you can accept this result and treat (b), (c) and (d) only.) Recall that an ideal $I$ of a ring $R$ is called a **nil ideal** if each element of $I$ is nilpotent. Prove that if $I$ is a nil left ideal in a left Artinian ring $R$ then $I$ is nilpotent.

(b) Prove that $g-1$ is a nilpotent element for each $g \in G \backslash \{1\}$ and deduce that $I(KG)$ is a nil ideal of $KG$.

(c) Deduce from (a) and (b) that $I(KG) \subseteq J(KG)$ using Exercise 10 on Exercise Sheet 2.

(d) Conclude that $I(KG) = J(KG)$ using Proposition-Definition 15.7.

## 16 Semisimplicity and Maschke's Theorem

Throughout this section, we assume that $K$ is a field.

Our first aim is to prove that the semisimplicity of the group algebra depends on both the characteristic of the field and the order of the group.

**Theorem 16.1 (*Maschke*)**

If $\mathrm{char}(K) \nmid |G|$, then $KG$ is a semisimple $K$-algebra.

**Proof:** By Proposition-Definition 11.2, we need to prove that every s.e.s. $0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ of $KG$-modules splits. However, the field $K$ is clearly semisimple (again by Proposition-Definition 11.2). Hence any such sequence regarded as a s.e.s. of $K$-vector spaces and $K$-linear maps splits. So let $\sigma : N \longrightarrow M$ be a $K$-linear section for $\psi$ and set

$$\tilde{\sigma} := \tfrac{1}{|G|}\textstyle\sum_{g\in G} g^{-1}\sigma g : \begin{array}{ccc} N & \longrightarrow & M \\ n & \mapsto & \frac{1}{|G|}\sum_{g\in G} g^{-1}\sigma(gn). \end{array}$$

We may divide by $|G|$, since $\mathrm{char}(K) \nmid |G|$ implies that $|G| \in K^\times$. Now, if $h \in G$ and $n \in N$, then

$$\tilde{\sigma}(hn) = \frac{1}{|G|}\sum_{g\in G} g^{-1}\sigma(ghn) = h\frac{1}{|G|}\sum_{g\in G}(gh)^{-1}\sigma(ghn) = h\tilde{\sigma}(n)$$

and

$$\psi\tilde{\sigma}(n) = \frac{1}{|G|}\sum_{g\in G}\psi\left(g^{-1}\sigma(gn)\right) \overset{\psi\, KG\text{-lin}}{=} \frac{1}{|G|}\sum_{g\in G} g^{-1}\psi\sigma(gn) = \frac{1}{|G|}\sum_{g\in G} g^{-1}gn = n$$

where the last-but-one equality holds because $\psi\sigma = \mathrm{Id}_N$. Thus $\tilde{\sigma}$ is a $KG$-linear section for $\psi$. ■

**Example 10**

If $K = \mathbb{C}$ is the field of complex numbers, then $\mathbb{C}G$ is a semisimple $\mathbb{C}$-algebra, since $\mathrm{char}(\mathbb{C}) = 0$.

It turns out that the converse to Maschke's theorem also holds. We obtain it using the properties of the augmentation ideal.

**Theorem 16.2 (*Converse of Maschke's Theorem*)**

If $KG$ is a semisimple $K$-algebra, then $\mathrm{char}(K) \nmid |G|$.

**Proof:** Set $\mathrm{char}(K) =: p$ and let us assume that $p \mid |G|$. In particular $p$ must be a prime number. We have to prove that then $KG$ is not semisimple.

**Claim:** If $0 \neq V \subset KG$ is a $KG$-submodule of $KG^{\circ}$, then $V \cap I(KG) \neq 0$.

Indeed: Let $v = \sum_{g\in G} \lambda_g g \in V \setminus \{0\}$. If $\varepsilon(v) = 0$ we are done. Else, set $t := \sum_{h\in G} h$. Then

$$\varepsilon(t) = \sum_{h\in G} 1 = |G| = 0$$

as $\mathrm{char}(K) \mid |G|$. Hence $t \in I(KG)$. Now consider the element $tv$. On the one hand $tv \in V$ since $V$ is a submodule of $KG^{\circ}$, and on the other hand $tv \in I(KG)\setminus\{0\}$ since

$$tv = \Big(\sum_{h\in G} h\Big)\Big(\sum_{g\in G} \lambda_g g\Big) = \sum_{h,g\in G} (1_K \cdot \lambda_g) hg = \sum_{x\in G} \Big(\sum_{g\in G} \lambda_g\Big) x = \sum_{x\in G} \varepsilon(v) x \Rightarrow \varepsilon(tv) = \sum_{x\in G} \varepsilon(v) = |G|\varepsilon(v) = 0\,.$$

The Claim implies that $I(KG)$, which is a $KG$-submodule by definition, cannot have a complement in$KG^{\circ}$. Therefore, by Proposition-Definition 11.1 $KG^{\circ}$ is not semisimple and hence $KG$ is not semisimple by Theorem-Definition 11.2. ■

In the case the filed $K$ is algebraically closed, the following Exercise offers a second proof exploiting Artin-Wedderburn.

**Exercise 16.3 (*Proof of the Converse of Maschke's Theorem for $K = \overline{K}$*)**

Assume $K = \overline{K}$ is an algebraically closed field of characteristic $p$ with $p \mid |G|$. Set $T := \langle \sum_{g\in G} g \rangle_K$.

(a) Prove that we have a series of $KG$-submodules given by $KG^{\circ} \supsetneq I(KG) \supseteq T \supsetneq 0$.

(b) Deduce that $KG^{\circ}$ has at least two composition factors isomorphic to the trivial module $K$.

(c) Deduce that $KG$ is not a semisimple $K$-algebra using Theorem 13.2.

## 17 Simple modules over algebraically closed fields

Throughout this section, we assume that $K = \overline{K}$ is an algebraically closed field.

As mentioned in Chapter 2, §13 this hypothesis may always be replaced by the weaker assumption that the field $K$ is a splitting field for the group algebra $KG$, which we simply call a *splitting filed for G*.

We state here some elementary facts about simple $KG$-modules, which we obtain as consequences of the Artin-Wedderburn structure theorem.

**Corollary 17.1**

There are only finitely many isomorphism classes of simple $KG$-modules, or equivalently, there are only finitely many irreducible $K$-representations of $G$, up to similarity.

**Proof:** Since $K = \overline{K}$, the first claim follows from Corollary 13.4 and the equivalent characterisation from Proposition 15.3. ■

**Corollary 17.2**

If $G$ is an abelian group, then any simple $KG$-module is one-dimensional, or equivalently, all irreducible $K$-representations of $G$ have degree one.

**Proof:** Since $K = \overline{K}$ and $KG$ is commutative the first claim follows from Corollary 13.5 and the equivalent characterisation from Proposition 15.3. ■

**Corollary 17.3**

Let $p$ be a prime number. If $G$ is a $p$-group and $\mathrm{char}(K) = p$, then the trivial module is the unique simple $KG$-module, up to isomorphism.

**Proof:** By Lemma 15.8 we have $J(KG) = I(KG)$. Thus $KG/J(KG) \cong K$ as $K$-algebras by Proposition-Definition 15.7. Now, as $K$ is commutative, $Z(K) = K$, and it follows from Corollary 13.4 that

$$|\mathcal{M}(KG)| = \dim_K Z(KG/J(KG)) = \dim_K K = 1\,.$$

■

**Remark 17.4**

Another standard proof for Corollary 17.3 consists in using a result of Brauer's stating that $|\mathcal{M}(KG)|$ equals the number of conjugacy classes of $G$ of order not divisible by the characteristic of the field $K$.

**Corollary 17.5**

If $\mathrm{char}(K) \nmid |G|$, then $|G| = \sum_{S\in\mathcal{M}(KG)} \dim_K(S)^2$.

**Proof:** Since $\mathrm{char}(K) \nmid |G|$, the group algebra $KG$ is semisimple by Maschke's Theorem. Thus it follows from Theorem 13.2 that

$$\sum_{S\in\mathcal{M}(KG)} \dim_K(S)^2 = \dim_K(KG) = |G|\,.$$

■