JUN.-PROF. DR. CAROLINE LASSUEUR **BERNHARD BÖHMLER**
Due date: **Thursday 04.07.2019, 12:00** SS 2019

Throughout this exercise sheet G denotes a finite group. Each Exercise is worth 4 points.

EXERCISE 21
Let G be a finite group of odd order, and let r denote the number of conjugacy classes of G. Use character theory to prove that

$$r \equiv |G| \pmod{16}\,.$$

[Hint: Label the set $\mathrm{Irr}(G)$ of irreducible characters taking dual characters into account.]

EXERCISE 22
Prove that the character table is determined by the class multiplication constants, and conversely.

EXERCISE 23
Prove the following assertions:

(a) if $\chi \in \mathrm{Irr}(G)$, then $Z(\chi)/\ker(\chi) \cong Z\left(G/\ker(\chi)\right)$;

(b) $\bigcap_{\chi\in\mathrm{Irr}(G)} Z(\chi) = Z(G)$.

EXERCISE 24 (Facultative, requires Galois Theory)
Assume G is a finite cyclic group of order $m \in \mathbb{Z}_{>0}$, and let $S := \{g \in G \mid \langle g\rangle = G\}$. Let χ be a character of G such that $\chi(g) \neq 0$ for each $g \in S$. Prove that:

(a) the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ permutes S;

(b) $\prod_{g\in S} |\chi(g)|^2 \in \mathbb{Q}$, and hence an integer;

(c) $\prod_{g\in S} |\chi(g)|^2 \geq 1$;

(d) deduce that $\sum_{g\in S} |\chi(g)|^2 \geq |S|$.