Throughout this exercise sheet K denotes a field of arbitrary characteristic and G a finite group. Each Exercise is worth 4 points.

Note: For this first Exercise Sheet, we only ask you to hand in Exercises 1 and 2. Exercises 3 and 4 require the lecture of the second week, they will be discussed as *presence* Exercises with the assistant during the Exercise Class on Friday, 26th of April.

Exercise 1
Let V be a finite-dimensional K-vector space. Prove the following assertions:

(a) If $\rho : G \longrightarrow \mathrm{GL}(V)$ is a K-representation of G, then V becomes a G-set via the left action

$$\begin{array}{ccc} G \times V & \longrightarrow & V \\ (g, v) & \mapsto & g.v := \rho(g)(v) \end{array}$$

and moreover the following conditions are satisfied $\forall\, g \in G$, $\forall\, x, y \in V$, and $\forall\, \lambda \in K$:

(i) $g.(x + y) = g.x + g.y$; and
(ii) $g.(\lambda x) = \lambda(g.x)$.

(b) Conversely a K-linear action of G on V, i.e. an action $G \times V \longrightarrow V$ satisfying (i) and (ii), gives rise to a K-representation $\rho : G \longrightarrow \mathrm{GL}(V)$ by setting $\rho(g)(v) := g.v\ \forall\, g \in G$, $\forall\, v \in V$.

Exercise 2
Assume $char(K) \neq 2$ and let $G := C_2 \times C_2$ be the Klein-four group. Find four pairwise non-equivalent (matrix) representations of G over K of degree one.

Exercise 3

(a) Prove that the trivial representation of G is a sub-representation of any permutation representation of G.

(b) Assume $K = \mathbb{C}$ and $G = C_3$. Find a decomposition into a direct sum of three irreducible representations of the regular representation of G.
[Hint: use (a) and $\omega \in \mathbb{C}$ a cubic root of unity to find three G-invariant subspaces of $V := \langle \{e_g \mid g \in G\} \rangle_K$]

Exercise 4 (Maschke's Theorem does not hold without the assumption that $char(K) \nmid |G|$.)
Let p be a prime number, let $G := C_p = \langle g \mid g^p = 1 \rangle$ and let $K := \mathbb{F}_p$. Let $B := (e_1, e_2)$ be the standard ordered basis of $V := K^2$. Consider the matrix representation

$$\begin{array}{cccc} R : & G & \longrightarrow & GL_2(K) \\ & g^b & \mapsto & \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \end{array}$$

(a) Prove that $W := Ke_1$ is G-invariant and deduce that R is reducible.

(b) Prove that there is no direct sum decomposition of V into irreducible G-invariant subspaces.