Throughout this exercise sheet $K = \mathbb{C}$ is the field of complex numbers, $(G, \cdot)$ is a finite group, and V a finite-dimensional $\mathbb{C}$-vector space.

Exercise 15
Let G and H be two finite groups. Prove that:

(a) if $\lambda, \chi \in \mathrm{Irr}(G)$ and $\lambda(1) = 1$, then $\lambda \cdot \chi \in \mathrm{Irr}(G)$;

(b) the set $\{\chi \in \mathrm{Irr}(G) \mid \chi(1) = 1\}$ of linear characters of G forms a group for the product of characters;

(c) $\mathrm{Irr}(G \times H) = \{\chi \cdot \psi \mid \chi \in \mathrm{Irr}(G), \psi \in \mathrm{Irr}(H)\}$.

[Hint: Use Corollary 9.8(d) and the degree formula.]

Exercise 16

(a) Let $N \trianglelefteq G$ and let $\rho : G/N \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation of G/N with character χ.

  (i) Prove that if ρ is irreducible, then so is $\mathrm{Inf}_{G/N}^{G}(\rho)$.

  (ii) Compute the kernel of $\mathrm{Inf}_{G/N}^{G}(\rho)$ provided that ρ is faithful.

(b) Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation of G with character χ. Prove that

$$\ker(\chi) = \ker(\rho)\,,$$

thus is a normal subgroup of G.

(c) Prove that if $N \trianglelefteq G$, then

$$N = \bigcap_{\substack{\chi \in \mathrm{Irr}(G) \\ N \subseteq \ker(\chi)}} \ker(\chi)\,.$$

(d) Prove that G is simple if and only if $\chi(g) \neq \chi(1)$ for each $g \in G \setminus \{1\}$ and each $\chi \in \mathrm{Irr}(G) \setminus \{\mathbf{1}_G\}$.

Exercise 17 (**Exercise to hand in / 8 points**)

(a) Compute the character tables of the dihedral group D_8 of order 8 and of the quaternion group Q_8.

[Hint: In each case, determine the commutator subgroup and deduce that there are 4 linear characters.]

(b) If $\rho : G \longrightarrow \mathrm{GL}(V)$ is a $\mathbb{C}$-representation of G and $\det : \mathrm{GL}(V) \longrightarrow \mathbb{C}^*$ denotes the determinant homomorphism, then we define a linear character of G through

$$\det\nolimits_{\rho} := \det \circ \rho : G \longrightarrow \mathbb{C}^*\,,$$

called the **determinant of** ρ. Prove that, although the finite groups D_8 and Q_8 have the "same" character table, they can be distinguished by considering the determinants of their irreducible $\mathbb{C}$-representations.

EXERCISE 18 (**This exercise can be handed in for bonus points / 4 points**)
Compute the complex character table of the alternating group A_4 through the following steps:

1. Determine the conjugacy classes of A_4 (there are 4 of them) and the corresponding centraliser orders. [Justify your computations / arguments.]
2. Determine the degrees of the 4 irreducible characters of A_4.
3. Determine the linear characters of A_4.
4. Determine the non-linear character of A_4 using the 2nd Orthogonality Relations.