Character Theory of Finite Groups — Exercise Sheet 2 **TU Kaiserslautern**

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Throughout this exercise sheet K denotes a field of arbitrary characteristic, G a finite group, V a K-vector space and all vector spaces are assumed to be finite-dimensional.
Each Exercise is worth 4 points.

Exercise 5 (Alternative proof of Maschke's Theorem over the field $\mathbb{C}$.)
Assume $K = \mathbb{C}$ and let V be a G-vector space over K.

(a) Prove that there exists a G-invariant scalar product $\langle -, - \rangle : V \times V \longrightarrow \mathbb{C}$, i.e. such that

$$\langle g.u, g.v \rangle = \langle u, v \rangle \quad \forall\, g \in G, \forall\, u, v \in V\,.$$

[Hint: consider a not-necessarily G-invariant scalar product on V, and use a sum on the elements of G to produce a G-invariant one.]

(b) Deduce that every G-invariant subspace W of V admits a G-invariant complement.

Exercise 6
Let G be a finite abelian group.

(a) Let V be an irreducible G-vector space over $K = \mathbb{C}$. Prove that any subspace of V is G-invariant.

(b) Deduce that any irreducible $\mathbb{C}$-representation of G has degree one.

Exercise 7
Let $N \trianglelefteq G$ and let $\pi : G \longrightarrow G/N, g \mapsto gN$ be the quotient homomorphism. Given a K-representation $\rho : G/N \longrightarrow \mathrm{GL}(V)$, we set

$$\mathrm{Inf}_{G/N}^{G}(\rho) := \rho \circ \pi : G \longrightarrow \mathrm{GL}(V)\,,$$

the **inflation of ρ from G/N to G**. This is obviously a K-representation of G.

(a) Prove that if ρ is irreducible, then so is $\mathrm{Inf}_{G/N}^{G}(\rho)$.

(b) Compute the kernel of $\mathrm{Inf}_{G/N}^{G}(\rho)$ provided that ρ is faithful.

Exercise 8
Let $G := S_n$ be the symmetric group on n letters ($n \in \mathbb{Z}_{\geq 1}$).

(a) Describe all the irreducible $\mathbb{C}$-representations of the symmetric group S_3.

(b) Exhibit two irreducible $\mathbb{C}$-representation of degree 1 and one irreducible $\mathbb{C}$-representation of degree 2 of S_4.

(c) Prove that S_n has at most two irreducible K-representations of degree 1. Find a necessary and sufficient condition on K under which S_n has exactly two irreducible K-representations of degree 1.