Chapter 2: Characters

Notation: G finite group
$K = \mathbb{C}$ the field of cplx numbers (in part alg. closed!)
V is always a finite-dimensional K-vector space

1. The character of a representation

We now introduce "characters", which are functions $G \to \mathbb{C}$, coming from representations and which encode a lot of information about the representation itself in a more compact and efficient way.

Definition 2.1:

(a) Let $\rho_V : G \to GL(V)$ be a $\mathbb{C}$-representation. The <u>character</u> of ρ_V is the function
$$\chi_V : G \longrightarrow \mathbb{C}, \quad g \longmapsto \operatorname{Tr}(\rho_V(g)) =: \chi_V(g)$$
We also say that ρ_V, or the G-vector space V, <u>affords</u> the charater χ_V.

(b) If ρ_V is irreducible, then χ_V is also called <u>irreducible</u>, else <u>reducible</u>.

Remark:
- Recall that the trace of a linear map is computed by choosing a basis of the vector space and this is independent of this choice!
  Here choose B a $\mathbb{C}$-basis of V, get $\forall g \in G$:
  $$\chi_V(g) = \operatorname{Tr}(\rho_V(g)) = \operatorname{Tr}((\rho_V(g))_B)$$
- For a matrix repres. $R : G \to GL_n(\mathbb{C})$, the character of R is then
  $$\chi_R : G \to \mathbb{C}, \quad g \mapsto \chi_R(g) = \operatorname{Tr}(R(g))$$
- Two equivalent (matrix) $\mathbb{C}$-representations afford <u>the same</u> character $\rightsquigarrow$ [Exercise Sheet 3]

Proposition 2.2 Let $\rho_V : G \to GL(V)$ be a $\mathbb{C}$-representation.

(a) $\chi_V(1_G) = \dim_{\mathbb{C}} V$ (This is called the <u>degree</u> of χ_V)

(b) $\chi_V(g)$ is a sum of $o(g)$-th roots of unity $\forall g \in G$

(c) $|\chi_V(g)| \le \chi_V(1_G)$ $\forall g \in G$.

(d) $\chi_V(g^{-1}) = \overline{\chi_V(g)}$ $\forall g \in G$

(e) If $\rho_V = \rho_{V_1} \oplus \rho_{V_2}$, a direct sum of two subrepresentations, then:

$$\chi_V = \chi_{V_1} + \chi_{V_2}.$$

Proof: (a) $\rho_V(1_G) = Id_V \Rightarrow \chi_V(1_G) = Tr(I_n) = \dim_{\mathbb{C}} V$.

Set $n := \dim_{\mathbb{C}} V$

(b) follows directly from Thm 1.18

(c) By (b) $\chi_V(g) = \varepsilon_1 + \dots + \varepsilon_n$, where ε_i $(1 \le i \le n)$ is an $o(g)$-th root of unity

$$\Rightarrow |\chi_V(g)| \le \underbrace{|\varepsilon_1|}_{1} + \dots + \underbrace{|\varepsilon_n|}_{1} = n = \chi_V(1_G).$$

(d) By Thm 1.18, $\exists$ a $\mathbb{C}$-basis B of V s.t.

$$(\rho_V(g))_B = \begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_n \end{pmatrix} \quad \text{with } \varepsilon_i \ o(g)\text{-th root of unity } \forall 1 \le i \le n.$$

$$\Rightarrow (\rho_V(g^{-1}))_B = \begin{pmatrix} \varepsilon_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \varepsilon_n^{-1} \end{pmatrix} = \begin{pmatrix} \overline{\varepsilon_1} & & \\ & \ddots & \\ & & \overline{\varepsilon_n} \end{pmatrix}$$

$$\Rightarrow \chi_V(g^{-1}) = \overline{\varepsilon_1} + \dots + \overline{\varepsilon_n} = \overline{\varepsilon_1 + \dots + \varepsilon_n} = \overline{\chi_V(g)}.$$

(e) Let B_i be a $\mathbb{C}$-basis of V_i $(i = 1,2)$

$\Rightarrow B := B_1 \sqcup B_2$ is a $\mathbb{C}$-basis of V

Then for $g \in G$, we have

$$(\rho_V(g))_B = \left(\begin{array}{c|c} (\rho_{V_1}(g))_{B_1} & 0 \\ \hline 0 & (\rho_{V_2}(g))_{B_2} \end{array}\right)$$

$$\Rightarrow \chi_V(g) = Tr((\rho_V(g))_B) = Tr((\rho_{V_1}(g))_{B_1}) + Tr((\rho_{V_2}(g))_{B_2})$$
$$= \chi_{V_1}(g) + \chi_{V_2}(g)$$

#

Recall: The group G acts on itself by conjugation via

$$G \times G \longrightarrow G$$
$$(g, x) \longmapsto gxg^{-1} = {}^g x$$

Orbits of this action are the conjugacy classes $[x] := \{{}^g x \mid g \in G\}$.
The stabiliser of $x \in G$ is its centraliser $C_G(x) := \{g \in G \mid gx = xg\}$ and $|C_G(x)| = \frac{|G|}{|[x]|}$.

Nota: we set $C(G) := \{[x] \mid x \in G\}$, the set of conjugacy classes of G.

Definition 2.3 A function $f : G \longrightarrow \mathbb{C}$ which is constant on each conjugacy class of G, i.e. s.t. $f(gxg^{-1}) = f(x)$ $\forall x, g \in G$, is called a class function.

Proposition 2.4: Characters are class functions.

Proof: Let χ_V be the character afforded by the $\mathbb{C}$-representation $\rho_V : G \rightarrow GL(V)$. We have $\forall x, g \in G$:

$$\chi_V(gxg^{-1}) = Tr(\rho_V(gxg^{-1})) \underset{\rho_V \text{ hom}}{=} Tr(\rho_V(g)\rho_V(x)\rho_V(g)^{-1})$$
$$= Tr(\rho_V(x)) \quad (Tr \text{ is an invariant of similarity})$$
$$= \chi_V(x).$$

#

Notation: Recall that $\mathbb{C}^G = \{f : G \rightarrow \mathbb{C} \mid f \text{ function}\}$ denotes the $\mathbb{C}$-vector space of $\mathbb{C}$-valued functions of G.

Set $\mathcal{Cl}(G) := \{f \in \mathbb{C}^G \mid f \text{ class function}\}$

This is a $\mathbb{C}$-subspace of $\mathbb{C}^G$ (Check it: Exercise 9, Sheet 3) called the space of class functions on G.

Moreover, $\dim_{\mathbb{C}} \mathcal{Cl}(G) = |C(G)|$ (Exercise 9, Sheet 3)

Corollary 2.5: There are at most $|C(G)|$ ~~C~~ linearly independent (irreducible) characters of G.

Proof: Prop. 2.4 $\Rightarrow$ characters are elements of $\mathcal{Cl}(G)$, whose dimension is $|C(G)|$. The claim follows. #

Proposition 2.6: The binary operation $\langle\, ,\, \rangle : \mathcal{Cl}(G) \times \mathcal{Cl}(G) \longrightarrow \mathbb{C}$

$$(f_1, f_2) \longmapsto \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}$$

is a scalar product on $\mathcal{Cl}(G)$.

Proof: • sesquilinearity: Let $\alpha \in \mathbb{C}$, $f_1, f_2 \in \mathcal{Cl}(G)$. Then

• $\langle f_1, \alpha f_2\rangle = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{(\alpha f_2)(g)} = \frac{1}{|G|}\sum_{g\in G} f_1(g)\,\overline{\alpha}\,\overline{f_2(g)}$

$= \overline{\alpha}\left(\frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}\right) = \overline{\alpha}\langle f_1, f_2\rangle$

• linearity in the 1st variable is clear: $\langle \alpha f_1, f_2\rangle = \alpha\langle f_1, f_2\rangle$ ✓

• sums in each variable are clear ✓

• $\langle\, ,\,\rangle$ is hermitian: $\langle f_1, f_2\rangle = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}$

$= \overline{\frac{1}{|G|}\sum_{g\in G} \overline{f_1(g)}\, f_2(g)} = \overline{\langle f_2, f_1\rangle}$ $\forall f_1, f_2 \in \mathcal{Cl}(G)$

• $\langle\, ,\,\rangle$ is positive-definite: $\forall f \in \mathcal{Cl}(G)$ we have

$$\langle f, f\rangle = \frac{1}{|G|}\sum_{g\in G} f(g)\overline{f(g)} = \frac{1}{|G|}\sum_{g\in G} \underbrace{|f(g)|^2}_{\in \mathbb{R}_+} \geq 0$$

moreover $\langle f, f\rangle = 0 \iff f = 0$. #

Theorem 2.7 [Orthogonality of characters / 1st Orthogonality relations]

Let χ_V, χ_W be the characters of two irreducible $\mathbb{C}$-representations ρ_V, ρ_W resp.

Then: (a) if $\rho_V \not\sim \rho_W$, then $\langle \chi_V, \chi_W\rangle = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)\chi_W(g^{-1}) = 0$

(b) $\langle \chi_V, \chi_V\rangle = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)\chi_V(g^{-1}) = 1$

(**Remark:** In other words the set of irreducible characters of G form an orthonormal system in $\mathcal{Cl}(G)$.
We want to see that they are actually a basis.)

Proof: Let $B = (e_1, \dots, e_n)$ be an ordered basis of V, and let $C = (f_1, \dots, f_m)$ be an ordered basis of W.
Let $R^V : G \to GL_n(\mathbb{C})$ and $R^W : G \to GL_m(\mathbb{C})$ be the induced matrix representations from ρ_V and ρ_W (resp.) w.r.t. to B and C (resp.). Then

$$(*) \quad \sum_{g\in G} \chi_V(g)\chi_W(g^{-1}) = \sum_{g\in G} \sum_{k=1}^{n} \sum_{\ell=1}^{m} R^V(g)_{kk} R^W(g^{-1})_{\ell\ell}$$

$$\overset{\text{Coro. 1.15}}{=} \sum_{k=1}^{n} \sum_{\ell=1}^{m} \frac{|G|}{n} \delta_{V,W} \delta_{k\ell}\delta_{k\ell} \overset{k=\ell}{=} \sum_{k=1}^{n} \frac{|G|}{n} \underbrace{\delta_{V,W}}_{\begin{cases} =1 \text{ if } V\sim W \\ =0 \text{ if } V \not\sim W\end{cases}}$$

Hence if $\rho_V \not\sim \rho_W$, we get

$$\langle \chi_V, \chi_W \rangle \overset{(*)}{=} \frac{1}{|G|} \sum_{k=1}^{n} \frac{|G|}{n} \cdot 0 = 0 \quad ; \text{ and}$$

$$\langle \chi_V, \chi_V \rangle \overset{(*)}{=} \frac{1}{|G|} \sum_{k=1}^{n} \frac{|G|}{n} \cdot 1 = 1 \qquad \#$$

Notation: We set $\mathrm{Irr}(G) := \{\text{pairwise distinct irred. characters of } G\} =: \{\chi_1, \dots, \chi_r\}$
• We will also use the notation 1_G for the character of the trivial representation.

Corollary 2.8: The set $\mathrm{Irr}(G) := \{\text{pairwise distinct irreducible characters of } G\}$ forms a linearly independent orthonormal system in $\mathcal{Cl}(G)$ w.r.t. $\langle\, ,\rangle$.

Proof: • Thm 2.7 $\Rightarrow$ $\mathrm{Irr}(G)$ orthonormal system w.r.t. $\langle\, , \rangle$. ✓
• $\mathbb{C}$-linear independence: Write $\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_r\}$, then: if $\sum_{i=1}^{r} \lambda_i \chi_i = 0$ for some $\lambda_i \in \mathbb{C}$ $(1 \le i \le r)$, then $\forall 1\le j\le r$:

$$0 = \langle \sum_{i=1}^{r} \lambda_i \chi_i, \chi_j \rangle \overset{\text{Thm 2.7}}{=} \sum_{i=1}^{r} \lambda_i \underbrace{\langle \chi_i, \chi_j \rangle}_{\delta_{ij}}$$

$$= \lambda_j \qquad /\#$$

Corollary 2.9: (Multiplicities)

Let $\rho_V : G \to GL(V)$ be a $\mathbb{C}$-representation.
Let $\rho_V = \rho_{V_1} \oplus \dots \oplus \rho_{V_s}$ a decomposition of ρ_V into irreducible subrepresentations. Then the following hold:

(a) If ρ_W is an irreducible representation of G, then the multiplicity of ρ_W in $\rho_{V_1} \oplus \dots \oplus \rho_{V_s}$ is equal to $\langle \chi_V, \chi_W \rangle$

(b) This multiplicity is independent of the chosen decomposition of ρ_V into ~~[illegible]~~ irred. summands.

Proof: (a) We may assume we have chosen the labelling s.t.

$$\rho_V = \underbrace{\rho_{V_1} \oplus \cdots \oplus \rho_{V_\ell}}_{\rho_{V_i} \sim \rho_W \ \forall 1 \le i \le \ell} \oplus \underbrace{\rho_{V_{\ell+1}} \oplus \cdots \oplus \rho_{V_s}}_{\rho_{V_i} \not\sim \rho_W \ \forall \ell+1 \le i \le s}$$

Then

$$\langle \chi_V, \chi_W \rangle = \sum_{i=1}^{\ell} \underbrace{\langle \underbrace{\chi_{V_i}}_{=\chi_W}, \chi_W \rangle}_{\overset{\text{Thm 2.7}}{=} 1} + \sum_{i=\ell+1}^{s} \underbrace{\langle \chi_{V_i}, \chi_W \rangle}_{\overset{\text{Thm 2.7}}{=} 0} = \ell$$

(b) Is obvious since $\langle \chi_V, \chi_W \rangle$ depends only on V and W, but not on the chosen decomposition. #

Corollary 2.10 If $\rho_V : G \to GL(V)$ and $\rho_{V'} : G \to GL(V')$ are two $\mathbb{C}$-representations of G, then:

$$\rho_V \sim \rho_{V'} \iff \chi_V = \chi_{V'}$$

Proof: "$\Rightarrow$" ✓ [Exercise 9, Sheet 3]

"$\Leftarrow$" Decompose $\rho_V, \rho_{V'}$ into direct sums of irreducible subrepresentations:

$$\rho_V = \underbrace{(\rho_{V_{1,1}} \oplus \cdots \oplus \rho_{V_{1,m_1}})}_{\text{all } \sim \rho_{V_1}} \oplus \cdots \oplus \underbrace{(\rho_{V_{s,1}} \oplus \cdots \oplus \rho_{V_{s,m_s}})}_{\text{all } \sim \rho_{V_s}}$$

$$\rho_{V'} = \underbrace{(\rho_{V'_{1,1}} \oplus \cdots \oplus \rho_{V'_{1,m'_1}})}_{\text{all } \sim \rho_{V_1}} \oplus \cdots \oplus \underbrace{(\rho_{V'_{s,1}} \oplus \cdots \oplus \rho_{V'_{s,m'_s}})}_{\text{all } \sim \rho_{V_s}}$$

where $m_i, m_i' \ge 0 \ \forall 1 \le i \le s$. (⚠ some of the m_i's, m_i''s may be zero) and the ρ_{V_i}'s are pairwise non-equivalent.

Then $\chi_V = \chi_{V'} \Rightarrow \underset{\overset{\| \text{Cor. 2.9}}{m_i}}{\langle \chi_V, \chi_{V_i} \rangle} = \underset{\overset{\| \text{Cor. 2.9}}{m_i'}}{\langle \chi_{V'}, \chi_{V_i} \rangle}$

hence $\rho_V = \rho_{V'}$.

Corollary 2.11 [Irreducibility Criterion]

A $\mathbb{C}$-representation $\rho_V : G \to GL(V)$ is irreducible $\Leftrightarrow \langle \chi_V, \chi_V \rangle = 1$.

Proof: "$\Rightarrow$" ✓ Thm 2.7(b).

"$\Leftarrow$" Write $\rho_V = \underbrace{(\rho_{V_{1,1}} \oplus \cdots \oplus \rho_{V_{1,m_1}})}_{\text{all} \sim \rho_{V_1}} \oplus \cdots \oplus \underbrace{(\rho_{V_{s,1}} \oplus \cdots \oplus \rho_{V_{s,m_s}})}_{\text{all} \sim \rho_{V_s}}$

where $\rho_{V_1}, \ldots, \rho_{V_s}$ are pairwise non-equivalent and $m_1, \ldots, m_s \geq 1$.

We have $1 = \langle \chi_V, \chi_V \rangle \underset{\text{thm 2.7}}{=} m_1^2 \underbrace{\langle \chi_{V_1}, \chi_{V_1} \rangle}_{1} + \cdots + m_s^2 \underbrace{\langle \chi_{V_s}, \chi_{V_s} \rangle}_{1}$

$= m_1^2 + \cdots + m_s^2$

$\Rightarrow$ w.l.o.g. $m_1 = 1$ and $m_i = 0 \ \forall 2 \leq i \leq s$.

$\Rightarrow \rho_V = \rho_{V_1}$ is irreducible.

#

2. The regular character

Recall from Ex. 1(d) that to a finite G-set X we can associate a permutation representation $\rho_X : G \to GL(V)$, $g \mapsto \rho_X(g) : V \to V$, $e_x \mapsto e_{g.x}$, where $V = \bigoplus_{x \in X} K e_x$.

We write $\mathrm{Fix}_X(g) := \{x \in X \mid g.x = x\}$, the set of fixed points of $g \in G$ on X.

Proposition 2.12: Let χ_X be the character of the permutation representation ρ_X.

Then $\chi_X(g) = |\mathrm{Fix}_X(g)| \quad \forall g \in G$.

For the action of G on itself by left multiplication, we get:

Corollary 2.13: Let χ_{reg} be the character of the regular representation of G. We have:

$$\chi_{reg}(g) = \delta_{1g} |G| = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{else} \end{cases}$$

Proof: Follows directly from Prop. 2.12 since $\mathrm{Fix}_G(1_G) = G$ and $\mathrm{Fix}_G(g) = \emptyset$ if $g \neq 1_G$. #

Proof of 2.12: The diagonal entries of the matrix of $\rho_X(g)$ in the basis $\{e_x \mid x \in X\}$ are:

$$(\rho(g))_{xx} = \begin{cases} 1 & \text{if } g.x = x \\ 0 & \text{if } g.x \neq x \end{cases} \quad \forall g \in G.$$

$$\Rightarrow \chi_X(g) = \sum_{x\in X} (\rho(g))_{xx} = |\mathrm{Fix}_X(g)| \quad \forall g \in G. \quad \#$$

Theorem 2.14: (decomposition of the regular representation)

Up to equivalence every irreducible representation ρ^i of G occurs with multiplicity $n_i = \deg(\rho^i)$ in ρ_{reg}. In other words

$$\chi_{reg} = \sum_{i=1}^{r} \chi_i(1)\chi_i \quad , \text{where } \mathrm{Irr}(G) = \{\chi_1, \dots, \chi_r\}$$

Proof: Cor. 2.9 $\Rightarrow$ $n_i = \langle \chi_{reg}, \chi_i \rangle = \frac{1}{|G|} \sum_{g\in G} \underbrace{\chi_{reg}(g)}_{\overset{2.13}{=} \delta_{1g}|G|} \chi_i(g^{-1})$

$$\overset{g=1}{=} \frac{1}{|G|} |G| \chi_i(1) = \chi_i(1) = \deg(\rho^i) \quad \#$$

Corollary 2.15: (degree formula)

$$|G| = \sum_{i=1}^{r} \chi_i(1)^2 \quad , \text{where } \mathrm{Irr}(G) = \{\chi_1, \dots, \chi_r\}.$$

Proof: Thm 2.14 yields $\chi_{reg} = \sum_{i=1}^{r} \chi_i(1)\chi_i$

$\Rightarrow$ evaluation of χ_{reg} in 1_G yields:

$$|G| = \chi_{reg}(1_G) = \sum_{i=1}^{r} \chi_i(1)\chi_i(1) . \quad \#$$

Corollary 2.16 The $\sum_{i=1}^{r} n_i^2$ functions $R^i_{k\ell} : G \to \mathbb{C}$ of Thm. 1.16 form a $\mathbb{C}$-basis of $\mathbb{C}^G$.

Proof: Thm 1.16 says that the $R^i_{k\ell}$'s are $\mathbb{C}$-lin. indep.
Cor 2.15 $\Rightarrow$ their number $= |G| = \dim_{\mathbb{C}} \mathbb{C}^G$
$\Rightarrow$ they form a generating set. $\#$

3. The 2^{nd} Orthogonality Relations

Theorem 2.17: The set $\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_r\}$ of the pairwise distinct irreducible characters of G form a $\mathbb{C}$-basis of $\mathcal{Cl}(G)$. In particular $r = |C(G)|$.

Proof:
- By Cor. 2.8., $\mathrm{Irr}(G)$ is a $\mathbb{C}$-linearly independent set. ✓
- Remains to show: $\chi_1, \dots, \chi_r$ generate $\mathcal{Cl}(G)$ as a $\mathbb{C}$-vect. space.

Indeed: By Cor. 2.16, the functions $R^i_{k\ell}$, $1 \le i \le r$, $1 \le k, \ell \le n_i$ form a basis of $\mathbb{C}^G$.

$\Rightarrow \forall f \in \mathbb{C}^G$, $\exists$ coefficients $c^i_{k\ell} \in \mathbb{C}$ s.t.

$$f = \sum_{i=1}^{r} \sum_{k,\ell=1}^{n_i} c^i_{k\ell} R^i_{k\ell}$$

Therefore $\forall g, h \in G$, we have

$$f(hgh^{-1}) = \Big(\sum_{i=1}^{r} \sum_{k,\ell=1}^{n_i} c^i_{k\ell} R^i_{k\ell}\Big)(hgh^{-1})$$

$$= \sum_{i=1}^{r} \sum_{k,\ell=1}^{n_i} c^i_{k\ell} \sum_{s,t=1}^{n_i} R^i(h)_{ks} R^i(g)_{st} R^i(h^{-1})_{t\ell}$$

$$= \sum_{i=1}^{r} \sum_{s,t=1}^{n_i} \Big(\sum_{k,\ell=1}^{n_i} c^i_{k\ell} R^i(h)_{ks} R^i(h^{-1})_{t\ell}\Big) R^i(g)_{st} \quad (\circledast)$$

Now if $f \in \mathcal{Cl}(G)$, then $f(g) = f(hgh^{-1})$ $\forall h \in G$, so that $f(g) = \frac{1}{|G|} |G| f(g) = \frac{1}{|G|} \sum_{h \in G} f(hgh^{-1})$

$$\overset{\circledast}{\Longrightarrow} f(g) = \frac{1}{|G|} \sum_{h \in G} \sum_{i=1}^{r} \sum_{s,t=1}^{n_i} \sum_{k,\ell=1}^{n_i} c^i_{k\ell} R^i(h)_{ks} R^i(h^{-1})_{t\ell} R^i(g)_{st}$$

$$= \frac{1}{|G|} \sum_{i=1}^{r} \sum_{s,t=1}^{n_i} \sum_{k,\ell=1}^{n_i} c^i_{k\ell} \underbrace{\Big(\sum_{h \in G} R^i(h)_{ks} R^i(h^{-1})_{t\ell}\Big)}_{\overset{\text{Cor 1.5}}{=} \frac{|G|}{n_i} \delta_{k\ell} \delta_{st}} R^i(g)_{st}$$

$$\overset{\substack{\ell=k\\ t=s}}{=} \frac{1}{|G|} \sum_{i=1}^{r} \sum_{s=1}^{n_i} \sum_{k=1}^{n_i} c^i_{kk} \frac{|G|}{n_i} R^i(g)_{ss}$$

$$= \sum_{i=1}^{r} \sum_{s=1}^{n_i} \underbrace{\Big(\sum_{k=1}^{n_i} c^i_{kk} \frac{1}{n_i}\Big)}_{=: c_i \in \mathbb{C}} R^i(g)_{ss}$$

$$= \sum_{i=1}^{r} c_i \underbrace{\sum_{s=1}^{n_i} R^i(g)_{ss}}_{\chi_i(g)} = \sum_{i=1}^{r} c_i \chi_i(g)$$

Hence $f = \sum_{i=1}^{r} c_i \chi_i$, $(c_i \in \mathbb{C})$, as req. #

Corollary 2.18 Let $f \in \mathcal{C}\ell(G)$. Then:

(a) $f = \sum_{i=1}^{r} \langle f, \chi_i \rangle \chi_i$

(b) $\langle f, f \rangle = \sum_{i=1}^{r} \langle f, \chi_i \rangle^2$

(c) f is a character $\Leftrightarrow$ $\langle f, \chi_i \rangle \in \mathbb{Z}_{\geq 0}$ $\forall 1 \leq i \leq r$

(d) $f \in \mathrm{Irr}(G) \Leftrightarrow$ f is a character and $\langle f, f \rangle = 1$.

Proof: (a)+(b) hold for any orthonormal basis w.r.t. $\langle \, , \rangle$ (GDM)

(c) "$\Rightarrow$" by Cor 2.9. $\langle f, \chi_i \rangle$ is a multiplicity $\rightsquigarrow \in \mathbb{Z}_{\geq 0}$.

"$\Leftarrow$" If $\langle f, \chi_i \rangle = m_i \in \mathbb{Z}_{\geq 0}$, then f is the character of $\bigoplus_{i=1}^{r} \bigoplus_{j=1}^{m_i} \rho^i$, where ρ^i is a repres. affording χ_i.

(d) follows from the properties of $\langle \, , \rangle$. #

Corollary 2.19 (2nd Orthogonality Relations)

Let $C_1, \ldots, C_r$ denote the conjugacy classes of G and let $g_i \in C_i$ $\forall 1 \leq i \leq r$. Then

$$\sum_{k=1}^{r} \chi_k(g_j)\chi_k(g_i^{-1}) = \delta_{ij} \frac{|G|}{|C_i|} = \delta_{ij} |C_G(g_i)| \quad \forall 1 \leq i, j \leq r.$$

Proof: Consider the following matrices in $M_r(\mathbb{C})$:

$$A := (\chi_k(g_j))_{\substack{1 \leq k \leq r \\ 1 \leq j \leq r}}, \quad B := \left(\frac{|C_k|}{|G|} \chi_j(g_k^{-1}) \right)_{\substack{1 \leq k \leq r \\ 1 \leq j \leq r}}$$

Claim: $B = A^{-1}$

Indeed: $(AB)_{st} = \sum_{\ell=1}^{r} A_{s\ell} B_{\ell t} = \sum_{\ell=1}^{r} \frac{|C_\ell|}{|G|} \chi_s(g_\ell)\chi_t(g_\ell^{-1})$

characters are class functions $\longrightarrow$ $= \frac{1}{|G|} \sum_{g \in G} \chi_s(g) \chi_t(g^{-1})$

$= \langle \chi_s, \chi_t \rangle = \delta_{st}$

by the 1st Orth. Relations.

This holds $\forall 1 \leq s, t \leq r \Rightarrow AB = I_r \Rightarrow BA = I_r$ and so $B = A^{-1}$

In particular $\delta_{ij} = (BA)_{ij} = \sum_{k=1}^{r} B_{ik} A_{kj}$

$$\Rightarrow \underbrace{\sum_{k=1}^{r} B_{ik} A_{kj}}_{\delta_{ij}} = \sum_{k=1}^{r} \frac{|C_i|}{|G|} \chi_k(g_i^{-1}) \chi_k(g_j) = \sum_{k=1}^{r} \frac{|C_i|}{|G|} \chi_k(g_j) \chi_k(g_i^{-1})$$

$$\Rightarrow \sum_{k=1}^{r} \chi_k(g_j) \chi_k(g_i^{-1}) = \delta_{ij} \frac{|G|}{|C_i|} \underset{\substack{\uparrow \\ \text{Orbit-stabiliser} \\ \text{Thm.}}}{=} \delta_{ij} |C_G(g_i)|$$

#

Remark: Assuming $1_G \in C_1$ and choosing $i=j=1$, $g_1 = 1_G$, we obtain again the degree formula $\sum_{i=1}^{r} \chi_i(1)^2 = |G|$ (since $|C_1| = 1$).

4. Linear characters

Definition 2.20: A character χ of G of degree 1 (i.e. $\chi(1_G) = 1_{\mathbb{C}}$) is called a linear character of G.

Example 3: In Exercise , you proved that any $\mathbb{C}$-repres. of an abelian group G has degree 1.
We find this result again via a character argument:

G abelian $\Rightarrow gxg^{-1} = x \quad \forall g, x \in G$
$\Rightarrow [x] = \{x\} \quad \forall x \in G$ (conj. classes are singletons)

$\Rightarrow \dim_{\mathbb{C}} \mathcal{C}\ell(G) = |C(G)| = |G|$
$\Rightarrow |\mathrm{Irr}(G)| = |G|$
$\Rightarrow \sum_{i=1}^{|G|} \chi_i(1)^2 = |G|$ by the degree formula

This forces $\chi_i(1) = 1 \quad \forall \chi_i \in \mathrm{Irr}(G) = \{\chi_1, \dots, \chi_{|G|}\}$

5. Tensor products

Definition 2.21: Let V, W be $\mathbb{C}$-vector spaces with bases $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$ resp. We define $V \otimes_{\mathbb{C}} W$ to be the

$n\cdot m$-dimensional $\mathbb{C}$-vector space with basis $\{v_i \otimes w_j \mid 1\leq i\leq n, 1\leq j\leq m\}$, and call this space the tensor product of V and W (balanced) over $\mathbb{C}$.

Remarks: (a) An arbitrary element of $V\otimes_{\mathbb{C}} W$ has the form

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_{ij}(v_i\otimes w_j) \quad \text{with } \{\lambda_{ij}\}_{\substack{1\leq i\leq n\\ 1\leq j\leq m}} \subseteq \mathbb{C}$$

(b)
$$-\otimes- : \{v_i \mid 1\leq i\leq n\}\times\{w_j \mid 1\leq j\leq m\} \longrightarrow \{v_i\otimes w_j \mid 1\leq i\leq n, 1\leq j\leq m\}$$
$$v_i \times w_j \longmapsto v_i\otimes w_j$$

can be extented by $\mathbb{C}$-bilinearity to

$$-\otimes- : V\times W \longrightarrow V\otimes_{\mathbb{C}} W$$
$$\Big(v=\sum_{i=1}^{n}\lambda_i v_i, w=\sum_{j=1}^{m}\mu_j w_j\Big) \longmapsto v\otimes w = \sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_i\mu_j(v_i\otimes w_j)$$

It follows that $\forall\, v\in V$, $w\in W$ and $\lambda\in\mathbb{C}$,

$$v\otimes(\lambda w) = (\lambda v)\otimes w = \lambda(v\otimes w),$$

and $\forall\, x_1,\dots,x_k\in V$ and $\forall\, y_1,\dots,y_\ell\in W$,

$$\Big(\sum_{i=1}^{k} x_i\Big)\otimes\Big(\sum_{j=1}^{\ell} y_j\Big) = \sum_{i=1}^{k}\sum_{j=1}^{\ell} x_i\otimes y_j$$

Hence arbitrary elements of $V\otimes_{\mathbb{C}} W$ are $\mathbb{C}$-linear combinations of elements $v\otimes w$ with $v\in V$, $w\in W$.

(c) Up to isomorphism $V\otimes_{\mathbb{C}} W$ is independent of the choice of the $\mathbb{C}$-bases of V and W.

Lemma 2.22: If V is a G-vector space and W is an H-vector space (for H a finite grp), then $V\otimes_{\mathbb{C}} W$ becomes a $G\times H$-vector space via the following action:

$$(G\times H)\times(V\otimes_{\mathbb{C}} W) \longrightarrow V\otimes_{\mathbb{C}} W$$
$$((g,h), v\otimes w) \longmapsto (g,h)\cdot(v\otimes w) := g.v\otimes h.w$$

Moreover if $\rho_V : G\to GL(V)$ and $\rho_W : H\to GL(W)$ denote the corresponding representations, then

$$\rho_{V\otimes_{\mathbb{C}} W} = \rho_V\otimes\rho_W : G\times H \longrightarrow GL(V\otimes_{\mathbb{C}} W)$$
$$(g,h)\longmapsto \rho_V(g)\otimes\rho_W(h) : V\otimes_{\mathbb{C}} W\longrightarrow V\otimes_{\mathbb{C}} W$$
$$v\otimes w\longmapsto \rho_V(g)(v)\otimes\rho_W(h)(w)$$

Proof: Exercise, Sheet 4.

Recall: If $A=(A_{ij})_{1\le i,j\le n}\in M_n(\mathbb{C})$ and $B:=(B_{kl})_{1\le k,l\le m}\in M_m(\mathbb{C})$ are matrices, then their <u>Kronecker product</u> (or <u>tensorproduct</u>) is the matrix

$$A\otimes B \quad \begin{pmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & & \vdots \\ A_{n1}B & \cdots & A_{nn}B \end{pmatrix} \in M_{nm}(\mathbb{C})$$

$$\left(\text{E.g. } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix} . \right)$$

Lemma 2.23: If $R: G \to GL_n(\mathbb{C})$ and $S: H \to GL_m(\mathbb{C})$ are matrix representations of the finite grps G and H, then

$$R\otimes S: G\times H \longrightarrow GL_{nm}(\mathbb{C})$$
$$(g,h) \longmapsto (R\otimes S)(g,h) := R(g)\otimes S(h)$$

is a matrix representation of $G\times H$.

Lemma 2.24 If V is a G-vector space and W an H-vector space, then $\chi_{V\otimes_{\mathbb{C}}W} = \chi_V \cdot \chi_W$

Proof: Let $B_V=(v_1,\dots,v_n)$ ordered basis of V

$B_W=(w_1,\dots,w_n)$ ordered basis of W

and consider $B_{V\otimes W}=(v_1\otimes w_1,\dots,v_1\otimes w_m,\dots,v_n\otimes w_1,\dots,v_n\otimes w_m)$ as ordered basis of $V\otimes_{\mathbb{C}} W$

Then $\chi_{V\otimes_{\mathbb{C}}W}(g,h) = Tr(\rho_{V\otimes_{\mathbb{C}}W}(g,h)) = Tr(\rho_V(g)\otimes\rho_V(h))$

$\forall g\in G$
$\forall h\in H$

$$= Tr((\rho_{V\otimes_{\mathbb{C}}W}(g,h))_{B_{V\otimes W}})$$

$$= Tr((\rho_V(g)\otimes\rho_W(h))_{B_{V\otimes W}}) \quad = \circledast$$

If $\rho_V(g)(v_p) = \sum_{i=1}^{n} a_{ip} v_i \quad \rightsquigarrow \quad (a_{ip})_{1\le i,p\le n} = \rho_V(g)_{B_V}$

$\rho_W(h)(w_q) = \sum_{j=1}^{m} b_{jq} w_j \quad \rightsquigarrow \quad (b_{jq})_{1\le j,q\le n} = \rho_W(h)_{B_W}$

$\Rightarrow (\rho_V\otimes\rho_W)(g,h)(v_p\otimes w_q) = \rho_V(g)(v_p)\otimes\rho_W(h)(w_q) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ip}b_{jq}(v_i\otimes w_j)$

$$\Rightarrow \chi_{V\otimes W}(g,h) = * = \sum_{p=1}^{n}\sum_{q=1}^{m} a_{pp} b_{qq} = \left(\sum a_{pp}\right)\left(\sum b_{qq}\right)$$

$$= \mathrm{Tr}\left(\left(\rho_V(g)\right)_{B_V}\right) \cdot \mathrm{Tr}\left(\left(\rho_W(h)\right)_{B_W}\right)$$

$$= \chi_V(g) \cdot \chi_W(h)$$

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Corollary 2.25 Let G and H be finite groups.

(a) $\mathrm{Irr}(G \times H) = \{\chi \cdot \psi \mid \chi \in \mathrm{Irr}(G), \psi \in \mathrm{Irr}(H)\}$.

(b) If χ and μ are two characters of G, then $\chi \cdot \mu$ is also a character of G.

(c) If $\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_r\}$, then

$$R(G) = \left\{\sum_{i=1}^{r} m_i \chi_i \mid m_i \in \mathbb{Z}\right\} \text{ is a subring of } \mathcal{C}\ell(G)$$

called <u>representation ring</u> of G.

(d) $\{\chi \in \mathrm{Irr}(G) \mid \chi(1) = 1\}$ is a group for the multiplication of characters

(e) If $\chi, \lambda \in \mathrm{Irr}(G)$ with λ linear, then $\chi \cdot \lambda \in \mathrm{Irr}(G)$.

<u>Proof:</u> (a) + (d) + (e) : Exercise

(b) $i: G \hookrightarrow G \times G$, $g \mapsto (g,g)$ is a grp hm with $i(G) \cong G$

and $G \xhookrightarrow{i} G \times G \xrightarrow{\chi \cdot \mu} \mathbb{C}$ becomes a character of G. By abuse of notation we still denote this character of G as $\chi \cdot \mu$ (instead of $(\chi \cdot \mu)|_{i(G)}$)

(c) Only for general knowledge
See lecture Representation theory
for uses of this ring.

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