An Introduction to the Cohomology of Groups

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0. What is group cohomology?

For each group G and representation M of G there are abelian groups $H_n(G, M)$ and $H^n(G, M)$ where $n = 0, 1, 2, 3, \ldots$, called the nth homology and cohomology of G with coefficients in M. To understand this we need to know what a *representation* of G is. It is the same thing as $\mathbb{Z}G$-module, but for this we need to know what the *group ring* $\mathbb{Z}G$ is, so some preparation is required. The homology and cohomology groups may be defined topologically and also algebraically.

We will not do much with the topological definition, but to say something about it consider the following result:

THEOREM (Hurewicz 1936). *Let X be a path-connected space with $\pi_n X = 0$ for all $n \geq 2$ (such X is called 'aspherical'). Then X is determined up to homotopy by $\pi_1(x)$.*

If $G = \pi_1(X)$ for some aspherical space X we call X an Eilenberg-MacLane space $K(G, 1)$, or (if the group is discrete) the classifying space BG. (It classifies principal G-bundles, whatever they are.)

If an aspherical space X is locally path connected the universal cover $\tilde{X}$ is contractible and $X = \tilde{X}/G$. Also $H_n(X)$ and $H^n(X)$ depend only on $\pi_1(X)$. If $G = \pi_1(X)$ we may thus define

$$H_n(G, \mathbb{Z}) = H_n(X) \quad \text{and} \quad H^n(G, \mathbb{Z}) = H^n(X)$$

and because X is determined up to homotopy equivalence the definition does not depend on X.

As an example we could take X to be d loops joined together at a point. Then $\pi_1(X) = F_d$ is free on d generators and $\pi_n(X) = 0$ for $n \geq 2$. Thus according to the above definition

$$H_n(F_d, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}^d & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Also, the universal cover of X is the tree on which F_d acts freely, and it is contractible.

The theorem of Hurewicz tells us what the group cohomology is if there happens to be an aspherical space with the right fundamental group, but it does not say that there always is such a space.

THEOREM (Eilenberg and MacLane 1953). *Given a group G there exists a connected CW complex X which is aspherical with $\pi_1(X) = G$.*

Algebraically, several of the low-dimensional homology and cohomology groups had been studied earlier than the topologically defined groups or the general definition of group cohomology. In 1904 Schur studied a group isomorphic to $H_2(G, \mathbb{Z})$, and this group is known as the *Schur multiplier* of G. In 1932 Baer studied $H^2(G, A)$ as a group of equivalence classes of extensions. It was in 1945 that Eilenberg and MacLane introduced an algebraic approach which included these groups as special cases. The definition is that

$$H_n(G, M) = \mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, M) \quad \text{and} \quad H^n(G, M) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M).$$

In order to deal with these definitions we need to know something about Ext and Tor.

Before studying these things, let us look at Baer's group of extensions. A *group extension* is a short exact sequence of groups

$$1 \to A \to E \to G \to 1$$

(so the image of A is normal in E, the quotient is isomorphic to G). If A is abelian, such an extension determines a module action of G on A via conjugation within E: given $g \in G$, $a \in A$ let $\bar{g} \in E$ be an element which maps on to g. Then $a \mapsto {}^{\bar{g}}a = \bar{g}a\bar{g}^{-1}$ is the action of g on a. We check this action is well defined, giving a homomorphism $G \to \mathrm{Aut}(A)$, i.e. A is a representation of G.

Given a representation A of G, an extension of G by A will mean an exact sequence

$$1 \to A \to E \to G \to 1$$

such that the action of G on A induced by conjugation within D is the same as the given action.

Two extensions of G by A are equivalent if and only if they can appear in a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & E_1 & \longrightarrow & G \\ \| & & \big\downarrow \phi & & \| \\ A & \longrightarrow & E_2 & \longrightarrow & G \end{array}$$

for some homomorphism $\phi : E_1 \to E_2$. Such a homomorphism is necessarily an isomorphism (use the 5-lemma, or the snake lemma, to be described). Therefore 'equivalence' is an equivalence relation on the set of extensions of G by A. As a warning, it is possible to have non-equivalent extensions whose middle groups are isomorphic.

We put $H^2(G, A) := \{$equivalence classes of extensions of G by $A\}$, and define an addition on $H^2(G, A)$ as follows. Given extensions

$$1 \to A \to E_i \xrightarrow{\pi_i} G \to 1$$

$i = 1, 2$, form

$$\begin{array}{ccccccccc}
1 & \longrightarrow & A \times A & \longrightarrow & E_1 \times E_2 & \longrightarrow & G \times G & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow \scriptstyle\text{diagonal} & & \\
1 & \longrightarrow & A \times A & \longrightarrow & X & \longrightarrow & G & \longrightarrow & 1 \\
 & & \scriptstyle\text{add} \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & A & \longrightarrow & Y & \longrightarrow & G & \longrightarrow & 1
\end{array}$$

where

$$\begin{aligned}
X &= \{(e_1, e_2) \in E_1 \times E_2 \mid \pi_1 e_1 = \pi_2 e_2\} \\
Y &= X / \{(a, -a) \mid a \in A\}
\end{aligned}$$

The bottom row is an extension of G by A called the *Baer sum* of the two extensions. We define the sum of the equivalence classes of the two extensions to be the equivalence class of their Baer sum. Under this operation $H^2(G, A)$ becomes an abelian group in which the zero element is the semidirect product. At this point these facts and the background justification that the Baer sum is well defined on equivalence classes, could be taken as an exercise. We will establish the group structure on $H^2(G, A)$ in a later section. We will also show as an example that when $G = C_2 \times C_2$ and $A = C_2$ there are eight equivalence classes of extensions: one is the direct product $E \cong C_2 \times C_2 \times C_2$, there are three equivalence classes where $E \cong C_4 \times C_2$, three where $E \cong D_8$, and one where $E \cong Q_8$.