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Throughout this exercise sheet all vector spaces are assumed to be finite-dimensional. Each Exercise is worth 4 points.

Exercise 13

Let G and H be finite groups. Let V and W be two $\mathbb{C}$-vector spaces.

(a) Assume V is a G-vector space and W is an H-vector space. Prove that $V \otimes_{\mathbb{C}} W$ is a $G \times H$-vector space for the action defined by $(g,h).(v \otimes w) = g.v \otimes h.w \;\forall\, g \in G, h \in H$, $v \in V$, $w \in W$ with corresponding representation $\rho_{V\otimes_{\mathbb{C}}W} = \rho_V \otimes \rho_W$.

(b) Let $R : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$, $S : H \longrightarrow \mathrm{GL}_m(\mathbb{C})$ be matrix representations. Prove that $R \otimes S$ is a matrix representation of $G \times H$.

Exercise 14

Let G and H be finite groups. Prove the following assertions:

(a) $\mathrm{Irr}(G \times H) = \{\chi \cdot \psi \mid \chi \in \mathrm{Irr}(G), \psi \in \mathrm{Irr}(H)\}$.

(b) $\{\chi \in \mathrm{Irr}(G) \mid \chi(1) = 1\}$ is a group for the multiplication of characters.

(c) If $\chi, \lambda \in \mathrm{Irr}(G)$ and λ is linear, then $\chi \cdot \lambda \in \mathrm{Irr}(G)$.

Exercise 15

Compute the complex character table of the alternating group $\mathfrak{A}_4$ through the following steps:

1. Determine the conjugacy classes of $\mathfrak{A}_4$ (there are 4 of them) and the corresponding centraliser orders.
2. Determine the degrees of the 4 irreducible characters of $\mathfrak{A}_4$.
3. Determine the linear characters of $\mathfrak{A}_4$.
4. Determine the non-linear character of $\mathfrak{A}_4$ using the 2nd Orthogonality Relations.

Exercise 16

Let G be a finite group. Prove the following assertions:

(a) If $N \trianglelefteq G$, then:

$$N = \bigcap_{\substack{\chi\in\mathrm{Irr}(G)\\ N\subseteq\ker(\chi)}} \ker(\chi)$$

(b) G is a simple group if and only if $\chi(g) \neq \chi(1)\ \forall \chi \in \mathrm{Irr}(G) \setminus \{1_G\}$, and $\forall\, g \in G \setminus \{1\}$.