Throughout this exercise sheet G denotes a finite group and all vector spaces are assumed to be finite-dimensional. Each Exercise is worth 4 points.

EXERCISE 17
Let G be a finite group of odd order, and let r denote the number of conjugacy classes of G. Use character theory to prove that

$$r \equiv |G| \pmod{16}.$$

[Hint: Label the set $\mathrm{Irr}(G)$ of irreducible characters taking dual characters into account.]

EXERCISE 18 (The determinant of a representation)
If $\rho : G \longrightarrow \mathrm{GL}(V)$ is a $\mathbb{C}$-representation of G and $\det : \mathrm{GL}(V) \longrightarrow \mathbb{C}^*$ denotes the determinant homomorphism, then we define

$$\det_\rho := \det \circ \rho : G \longrightarrow \mathbb{C}^*.$$

Prove the following assertions:

(a) $\det_\rho$ is a linear character of G.

(b) If G is a non-abelian simple group (or more generally if G is perfect, i.e. $G = [G, G]$), then the image $\rho(G)$ of ρ is a subgroup of $\mathrm{SL}(V)$.

(c) If G is a non-abelian simple group and $\chi \in \mathrm{Irr}(G)$, then $\chi(1) \neq 2$.

(d) The finite groups D_8 and Q_8 cannot be distinguished by their character tables, but they can be distinguished by considering the determinants of their irreducible $\mathbb{C}$-representations[1].

Recap: D_8 is the *group of isometries of the square* and admits the following presentation

$$D_8 = \langle \sigma, \rho \mid \rho^4 = \sigma^2 = 1 \text{ and } \sigma\rho\sigma^{-1} = \rho^{-1} \rangle,$$

whereas Q_8 is the *quaternion group of order* 8: as a set $Q_8 = \{\pm 1, \pm i \pm j, \pm k\}$ and is endowed with the multiplication given by

$$1 = 1_{Q_8}, \quad (-1)^2 = 1, \quad i^2 = j^2 = k^2 = -1, \quad i \cdot j = k = -j \cdot i, \quad j \cdot k = i = -k \cdot j, \quad k \cdot i = j = -i \cdot k.$$

Turn the page over!

[1] Advice: If you work in groups of two students, then one student could compute the character table of D_8 and the other the character table of Q_8.

Definition 1
Let R be an associative ring with 1. A **left R-module** is an abelian group $(M,+)$ endowed with an **external composition law**

$$*: R \times M \longrightarrow M, (r,m) \mapsto r * m$$

such that the map

$$\begin{array}{rccl} \lambda: & R & \longrightarrow & \mathrm{End}(M) \\ & r & \mapsto & \lambda(r) := \lambda_r : M \longrightarrow M, m \mapsto r * m\,, \end{array}$$

is a ring homomorphism.

(In other words, left R-modules satisfy the same axioms as vector spaces, but the field is replaced by a ring.)

Furthermore, the left R-module M is said to be **of finite type** if it admits a finite generating set $X \subseteq M$ (i.e. if for any $m \in M$, there exists $\{r_x\}_{x \in X} \subset R$ such that $m = \sum_{x \in X} r_x * x$).

Exercise 19
Check that a G-vector space over an arbitrary field K is a left KG-module, and conversely that any left KG-module is a G-vector space.

Exercise 20
Let A be a subring of a commutative ring B with 1. An element $b \in B$ is called *integral over A* iff b is the zero of a monic polynomial $f \in A[X]$. In case $B = \mathbb{C}$, then we say that $b \in \mathbb{C}$ is an *algebraic integer* if b is integral over $A = \mathbb{Z}$.

(a) If $b \in \mathbb{Q}$ is an algebraic integer, then $b \in \mathbb{Z}$.

(b) If $b \in B$, then TFAE:

  (i) b is integral over A;
  (ii) $A[b]$ is a left A-module of finite type;
  (iii) there is a subring S of B containing A and b which is of finite type as a left A-module.

(c) The set $\{b \in B \mid b \text{ is integral over A}\}$ is a subring of B.