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STEP 1: The degree formula yields

$$168 = |G| = \sum_{i=1}^{6} \chi_i(1)^2 = 1 + \sum_{i=2}^{6} \chi_i(1)^2$$

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$\implies \chi_i(1) \geq 2 \quad \forall \; 2 \leq i \leq 6$

So, we have the following 5 possibilities for the degrees:

$\chi_1(1)$	$\chi_2(1)$	$\chi_3(1)$	$\chi_4(1)$	$\chi_5(1)$	$\chi_6(1)$
1	2	4	5	6	9
1	2	3	3	8	9
1	2	5	5	7	8
1	2	4	7	7	7
1	3	3	7	7	8

STEP 3. Use integrality! In part. $\chi_i(1) \mid |G| = 168 = 2^3 \cdot 3 \cdot 7 \quad \forall\, 1 \leq i \leq 6$:

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1	2	4	5	6	$\cancel{9}$
1	2	3	3	8	$\cancel{8}$
1	2	$\cancel{5}$	$\cancel{5}$	7	8
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1	2	4	5	6	~~9~~
1	2	3	3	8	~~9~~
1	2	~~5~~	~~5~~	7	8
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remains 2 possibilities

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[Ex. 21, Sheet 6] $\Rightarrow \chi_2(1) \neq 2$ as G is simple.

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Conclusion: the degrees of the irreducible characters are

$$1 \quad 3 \quad 3 \quad 7 \quad 7 \quad 8$$