Wednesday. Chapter 4. p-Modular Systems and Brauer Characters

R. Brauer started in the late 1920's a systematic investigation of group representations over fields of positive characteristic. In order to relate group representations over fields of positive characteristic to character theory in characteristic zero, Brauer worked with a triple of rings $(F, \mathcal{O}, k)$, called a *p-modular system*, and consisting of a complete discrete valuation ring $\mathcal{O}$ with a residue field $k := \mathcal{O}/J(\mathcal{O})$ of prime characteristic p and fraction field $F := \mathrm{Frac}(\mathcal{O})$ of characteristic zero. These are used to gain information about kG and its modules (which is/are extremely complicated) from the group algebra FG, which is semisimple and therefore much better understood, via the group algebra $\mathcal{O}G$. This explains why we considered arbitrary associative rings (resp. algebras / fields) in the previous chapters rather than immediately focusing on fields of positive characteristic.

Notation. Throughout this chapter, unless otherwise specified, we let p be a prime number and let $\Lambda \in \{F, \mathcal{O}, k\}$.

30 p-modular systems

Recall that a commutative ring $\mathcal{O}$ is local iff $\mathcal{O} \setminus \mathcal{O}^\times = J(\mathcal{O})$, i.e. $J(\mathcal{O})$ is the unique maximal ideal of $\mathcal{O}$. Moreover, by the commutativity assumption this happens if and only if $\mathcal{O}/J(\mathcal{O})$ is a field. In such a situation, we write $k := \mathcal{O}/J(\mathcal{O})$ and call this field **the residue field of the local ring** $\mathcal{O}$. To ease up notation, we will often write $\mathfrak{p} := J(\mathcal{O})$. This is because our aim is a situation in which the residue field is a field of positive characteristic p. Moreover, a commutative ring $\mathcal{O}$ is called a **discrete valuation ring** if $\mathcal{O}$ is a local principal ideal domain such that $J(\mathcal{O}) \neq 0$. Such a discrete valuation ring is called **complete** if it is complete in the $J(\mathcal{O})$-adic topology.

Definition 30.1 (*p-modular systems*)

Let p be a prime number.

(a) A triple of rings $(F, \mathcal{O}, k)$ is called a **p-modular system** if:

   (1) $\mathcal{O}$ is a complete discrete valuation ring of characteristic zero,

   (2) $F = \mathrm{Frac}(\mathcal{O})$ is the field of fractions of $\mathcal{O}$ (also of characteristic zero), and

   (3) $k = \mathcal{O}/J(\mathcal{O})$ is the residue field of $\mathcal{O}$ and has characteristic p.

(b) If G is a finite group, then a p-modular system $(F, \mathcal{O}, k)$ is a called a **splitting p-modular system for** G, if both F and k are splitting fields for G.

It is often helpful to visualise p-modular systems and the condition on the characteristic of the rings involved through the following commutative diagram of rings and ring homomorphisms:

$$\begin{tikzcd} \mathbb{Q} \arrow[d, hook] & \mathbb{Z} \arrow[l, hook'] \arrow[r, two heads] \arrow[d, hook] & \mathbb{F}_p \arrow[d, hook] \\ F & \mathcal{O} \arrow[l, hook'] \arrow[r, two heads] & k \end{tikzcd}$$

where the hook arrows are the canonical inclusions and the two-head arrows the quotient morphisms. Clearly, these morphisms also extend naturally to ring homomorphisms

$$FG \hookleftarrow \mathcal{O}G \twoheadrightarrow kG$$

between the corresponding group algebras (each mapping an element $g \in G$ to itself).

Example 13

One usually works with a splitting p-modular system for all subgroups of G, because it allows us avoid problems with field extensions. By a theorem of Brauer on splitting fields such a p-modular system can always be obtained by adjoining a primitive m-th root of unity to $\mathbb{Q}_p$, where m is the exponent of G. (Notice that this extension is unique.) So we may as well assume that our situation is as given in the following commutative diagram:

$$\begin{tikzcd} \mathbb{Q}_p \arrow[d, hook] & \mathbb{Z}_p \arrow[l, hook'] \arrow[r, two heads] \arrow[d, hook] & \mathbb{F}_p \arrow[d, hook] \\ F & \mathcal{O} \arrow[l, hook'] \arrow[r, two heads] & k \end{tikzcd}$$

More generally, we have the following result, which we mention without proof. The proof can be found in [CR90, §17A].

Theorem 30.2

Let $(F, \mathcal{O}, k)$ be a p-modular system. Let G be a finite group of exponent m. Then the following assertions hold.

(a) The field F contains all m-th roots of unity if and only if F contains the cyclotomic field of m-th roots of unity;

(b) If F contains all m-th roots of unity, then so does k and F and k are splitting fields for G and all its subgroups.

Remark 30.3

If $(F, \mathcal{O}, k)$ is a p-modular system, then it is not possible to have F and k algebraically closed, while assuming $\mathcal{O}$ is complete. (Depending on your knowledge on valuation rings, you can try to prove this as an exercise!)

Let us now investigate changes of the coefficients given in the setting of a p-modular system for group algebras involved.

Definition 30.4

Let $\mathcal{O}$ be a commutative local ring. A finitely generated $\mathcal{O}G$-module L is called an **$\mathcal{O}G$-lattice** if it is free (= projective) as an $\mathcal{O}$-module.

Remark 30.5 (*Changes of the coefficients*)

Let $(F, \mathcal{O}, k)$ be a p-modular system and write $\mathfrak{p} := J(\mathcal{O})$. If L is an $\mathcal{O}G$-module, then:

- setting $L^F := F \otimes_{\mathcal{O}} L$ defines an FG-module, and
- reduction modulo $\mathfrak{p}$ of L, that is $\overline{L} := L/\mathfrak{p}L \cong k \otimes_{\mathcal{O}} L$ defines a kG-module.

We note that, when seen as an $\mathcal{O}$-module, an $\mathcal{O}G$-module L may have torsion, which is lost on passage to F. In order to avoid this issue, we usually only work with $\mathcal{O}G$-lattices. In this way, we obtain functors

$$FG\text{-}\mathbf{mod} \hookleftarrow \mathcal{O}G\text{-}\mathbf{lat} \twoheadrightarrow kG\text{-}\mathbf{mod}$$

between the corresponding categories of finitely generated $\mathcal{O}G$-lattices and finitely generated FG-, kG-modules.

A natural question to ask is: which FG-modules, respectively kG-modules, come from $\mathcal{O}G$-lattices? In the case of FG-modules we have the following answer.

Proposition-Definition 30.6

Let $\mathcal{O}$ be a complete discrete valuation ring and let $F := \mathrm{Frac}(\mathcal{O})$ be the fraction field of $\mathcal{O}$. Then, for any finitely generated FG-module V there exists an $\mathcal{O}G$-lattice L which has an $\mathcal{O}$-basis which is also an F-basis. In this situation $V \cong L^F$ and we call L an **$\mathcal{O}$-form** of V.

Proof: Choose an F-basis $\{v_1, \dots, v_n\}$ of V and set $L := \mathcal{O}Gv_1 + \cdots + \mathcal{O}Gv_n \subseteq V$. ∎

On the other hand, the question has a negative answer for kG-modules.

Definition 30.7 (*liftable kG-module*)

Let $\mathcal{O}$ be a commutative local ring with unique maximal ideal $\mathfrak{p} := J(\mathcal{O})$ and residue field $k := \mathcal{O}/\mathfrak{p}$. A kG-module M is called **liftable** if there exists an $\mathcal{O}G$-lattice $\widehat{M}$ whose reduction modulo $\mathfrak{p}$ of M is isomorphic to M, that is

$$\widehat{M}/\mathfrak{p}\widehat{M} \cong M\,.$$

(Alternatively, it is also said that M is **liftable to an $\mathcal{O}G$-lattice**, or **liftable to $\mathcal{O}$**, or **liftable to characteristic zero**.)

Even though every $\mathcal{O}G$-lattice can be reduced modulo $\mathfrak{p}$ to produce a kG-module, not every kG-module is liftable to an $\mathcal{O}G$-lattice.

Being liftable for a kG-module is a rather rare property. However, some classes of kG-modules do lift.

Remark 30.8

It follows from the *lifting of idempotents* theorem that projective indecomposable kG-modules are liftable to projective indecomposable $\mathcal{O}G$-lattices:

Any (projective) indecomposable kG-module is liftable to a (projective) indecomposable $\mathcal{O}G$-lattice. More generally, any trivial source kG-module M is liftable to an $\mathcal{O}G$-lattice. More precisely, among all lifts of M a unique one is a trivial source and we denote it by $\widetilde{M}$.

The F-character of $F \otimes_{\mathcal{O}} \widetilde{M}$ is called **the ordinary character of** M.