The aim of this preliminary chapter is to introduce (resp. recall) the basics of the theory of modules over finite dimensional algebras, which we will use throughout. We review elementary definitions and constructions such as quotients, direct sum, direct products, tensor products and exact sequences, where we emphasise the approach via universal properties.

The **main aim of this lecture** is to study the so-called *representation theory of finite groups*, which amounts to studying modules over a specific ring, called the *group ring* (or *group algebra*), which is built from the group itself as a vector space with a basis given by the group elements. Hence we already get a first feeling that "juggling with algebraic structures" will be one of the recurrent feature of this lecture.

**Notation**: throughout this chapter we let $R$ and $S$ denote rings, and unless otherwise specified, all rings are assumed to be *unital* and *associative*.

Most results are stated without proof, as they have been / will be studied in the B.Sc. lecture *Commutative Algebra*. As further reference we recommend for example:

**Reference:**
[Rot10]  J. J. ROTMAN, *Advanced modern algebra. 2nd edition*, Providence, RI: American Mathematical Society (AMS), 2010.

## 1   Modules, submodules, morphisms*

**Definition 1.1 (*Left $R$-module, right $R$-module, $(R, S)$-bimodule*)**

    (a) A **left $R$-module** is an ordered triple $(M, +, \cdot)$, where $(M, +)$ an abelian group and $\cdot : R \times M \longrightarrow M, (r, m) \mapsto r \cdot m$ is a **scalar multiplication** (or **external composition law**) such that the map

$$\begin{aligned} \lambda : \quad R &\longrightarrow \quad \mathrm{End}(M) \\ r &\mapsto \quad \lambda(r) := \lambda_r : M \longrightarrow M, m \mapsto r \cdot m \, , \end{aligned}$$

is a ring homomorphism.

    (b) A **right $R$-module** is defined analogously using a scalar multiplication $\cdot : M \times R \longrightarrow M$, $(m, r) \mapsto m \cdot r$ on the right-hand side.

(c) An $(R, S)$-**bimodule** is an abelian group $(M, +)$ which is both a left $R$-module and a right $S$-module, and which satisfies the axiom

$$r \cdot (m \cdot s) = (r \cdot m) \cdot s \qquad \forall\, r \in R, \forall\, s \in S, \forall\, m \in M\,.$$

**Convention**: Unless otherwise stated, in this lecture we always work with left modules. When no confusion is to be made, we will simply write "$R$-module" to mean "left $R$-module", denote $R$-modules by their underlying sets and write $rm$ instead of $r \cdot m$. Definitions for right modules and bimodules are similar to those for left modules, hence in the sequel we omit them.

**Definition 1.2 ($R$-submodule)**

An $R$-**submodule** of an $R$-module $M$ is a subgroup $U \leqslant M$ such that $r \cdot u \in U \,\forall\, r \in R, \forall\, u \in U$.

**Definition 1.3 (Morphisms)**

A **(homo)morphism** of $R$-modules (or an $R$-**linear map**, or an $R$-**homomorphism**) is a map of $R$-modules $\varphi : M \longrightarrow N$ such that:

(i)  $\varphi$ is a group homomorphism; and

(ii)  $\varphi(r \cdot m) = r \cdot \varphi(m) \,\forall\, r \in R, \forall\, m \in M$.

An injective (resp. surjective) morphism of $R$-modules is sometimes called a **monomorphism** (resp. an **epimorphism**) and we often denote it with a *hook arrow* "$\hookrightarrow$" (resp. a *two-head arrow* "$\twoheadrightarrow$").

A bijective morphism of $R$-modules is called an **isomorphism** (or an $R$-**isomorphism**), and we write $M \cong N$ if there exists an $R$-isomorphism between $M$ and $N$.

A morphism from an $R$-module to itself is called an **endomorphism** and a bijective endomorphism is called an **automorphism**.

**Notation**: We let $_R\mathbf{Mod}$ denote the category of left $R$-modules (with $R$-linear maps as morphisms), we let $\mathbf{Mod}_R$ denote the category of right $R$-modules (with $R$-linear maps as morphisms), and we let $_R\mathbf{Mod}_S$ denote the category of $(R, S)$-bimodules (with $(R, S)$-linear maps as morphisms). For the language of category theory, see the Appendix.

**Example 1**

(a) Exercise: Prove that Definition 1.1(a) is equivalent to requiring that $(M, +, \cdot)$ satisfies the following axioms:

(M1)  $(M, +)$ is an abelian group;

(M2)  $(r_1 + r_2) \cdot m = r_1 \cdot m + r_2 \cdot m$ for each $r_1, r_2 \in R$ and each $m \in M$;

(M3)  $r \cdot (m_1 + m_2) = r \cdot m_1 + r \cdot m_2$ for each $r \in R$ and all $m_1, m_2 \in M$;

(M4)  $(rs) \cdot m = r \cdot (s \cdot m)$ for each $r, s \in R$ and all $m \in M$.

(M5)  $1_R \cdot m = m$ for each $m \in M$.

In other words, modules over rings satisfy the same axioms as vector spaces over fields. Hence: Vector spaces over a field $K$ are $K$-modules, and conversely.

(b) Abelian groups are $\mathbb{Z}$-modules, and conversely.
Exercise: check it! What is the external composition law?

(c) If the ring $R$ is commutative, then any right module can be made into a left module, and conversely.
Exercise: check it! Where does the commutativity come into play?

(d) If $\varphi : M \longrightarrow N$ is a morphism of $R$-modules, then the kernel $\ker(\varphi) := \{m \in M \mid \varphi(m) = 0_N\}$ of $\varphi$ is an $R$-submodule of $M$ and the image $\mathrm{Im}(\varphi) := \varphi(M) = \{\varphi(m) \mid m \in M\}$ of $f$ is an $R$-submodule of $N$.
If $M = N$ and $\varphi$ is invertible, then the inverse is the usual set-theoretic *inverse map* $\varphi^{-1}$ and is also an $R$-homomorphism.
Exercise: check it!

(e) **Change of the base ring:** if $\varphi : S \longrightarrow R$ is a ring homomorphism, then every $R$-module $M$ can be endowed with the structure of an $S$-module with external composition law given by

$$\cdot : S \times M \longrightarrow M, (s, m) \mapsto s \cdot m := \varphi(s) \cdot m .$$

Exercise: check it!

**Notation 1.4**

Given $R$-modules $M$ and $N$, we set $\mathrm{Hom}_R(M, N) := \{\varphi : M \longrightarrow N \mid \varphi \text{ is an } R\text{-homomorphism}\}$.
This is an abelian group for the pointwise addition of maps:

$$+ : \quad \mathrm{Hom}_R(M, N) \times \mathrm{Hom}_R(M, N) \quad \longrightarrow \quad \mathrm{Hom}_R(M, N)$$
$$(\varphi, \psi) \quad\quad \mapsto \quad \varphi + \psi : M \longrightarrow N, m \mapsto \varphi(m) + \psi(m) .$$

In case $N = M$, we write $\mathrm{End}_R(M) := \mathrm{Hom}_R(M, M)$ for the set of endomorphisms of $M$ and $\mathrm{Aut}_R(M)$ for the set of automorphisms of $M$, i.e. the set of invertible endomorphisms of $M$.

**Lemma–Definition 1.5 (*Quotients of modules*)**

Let $U$ be an $R$-submodule of an $R$-module $M$. The quotient group $M/U$ can be endowed with the structure of an $R$-module in a natural way via the external composition law

$$R \times M/U \longrightarrow M/U$$
$$(r, m + U) \longmapsto r \cdot m + U$$

The canonical map $\pi : M \longrightarrow M/U, m \mapsto m + U$ is $R$-linear and we call it the **canonical** (or **natural**) **homomorphism**.

**Proof:** We assume known from the "Algebraische Strukturen" that $\pi$ is a group homomorphism.
Exercise: check that $\pi$ preserves the scalar multiplication.                                    ■

**Definition 1.6 (*Cokernel, coimage*)**

Let $\varphi \in \mathrm{Hom}_R(M, N)$. The **cokernel** of $\varphi$ is the quotient $R$-module $\mathrm{coker}(\varphi) := N/\mathrm{Im}\,\varphi$, and the **coimage** of $\varphi$ is the quotient $R$-module $M/\ker \varphi$.

**Theorem 1.7 (*The universal property of the quotient and the isomorphism theorems*)**

(a) **Universal property of the quotient**: Let $\varphi : M \longrightarrow N$ be a homomorphism of $R$-modules. If $U$ is an $R$-submodule of $M$ such that $U \subseteq \ker(\varphi)$, then there exists a unique $R$-module homomorphism $\overline{\varphi} : M/U \longrightarrow N$ such that $\overline{\varphi} \circ \pi = \varphi$, or in other words such that the following diagram commutes:

$$
\begin{array}{ccc}
M & \xrightarrow{\ \varphi\ } & N \\
{\scriptstyle \pi}\downarrow & \circlearrowleft \quad \nearrow & \\
M/U & {\scriptstyle \exists!\,\overline{\varphi}} &
\end{array}
$$

Concretely, $\overline{\varphi}(m + U) = \varphi(m) \ \forall \ m + U \in M/U$.

(b) **1st isomorphism theorem**: With the notation of (a), if $U = \ker(\varphi)$, then

$$\overline{\varphi} : M/\ker(\varphi) \longrightarrow \operatorname{Im}(\varphi)$$

is an isomorphism of $R$-modules.

(c) **2nd isomorphism theorem**: If $U_1, U_2$ are $R$-submodules of $M$, then so are $U_1 \cap U_2$ and $U_1 + U_2$, and there is an an isomorphism of $R$-modules

$$(U_1 + U_2)/U_2 \cong U_1/(U_1 \cap U_2).$$

(d) **3rd isomorphism theorem**: If $U_1 \subseteq U_2$ are $R$-submodules of $M$, then there is an an isomorphism of $R$-modules

$$(M/U_1)/(U_2/U_1) \cong M/U_2.$$

(e) **Correspondence theorem**: If $U$ is an $R$-submodule of $M$, then there is a bijection

$$
\begin{array}{ccc}
\{R\text{-submodules } X \text{ of } M \mid U \subseteq X\} & \longleftrightarrow & \{R\text{-submodules of } M/U\} \\
X & \mapsto & X/U \\
\pi^{-1}(Z) & \mapsfrom & Z.
\end{array}
$$

**Proof**: We assume it is known (e.g. from the "Einführung in die Algebra") that these results hold for abelian groups and morphisms of abelian groups.
Exercise: check that they carry over to the $R$-module structure. ∎

**Definition 1.8 (*Generating set / R-basis / finitely generated/free R-module*)**

Let $M$ be an $R$-module and let $X \subseteq M$ be a subset. Then:

(a) $M$ is said to be **generated by** $X$ if every element $m \in M$ may be written as an $R$-linear combination $m = \sum_{x \in X} \lambda_x x$, i.e. where $\lambda_x \in R$ is almost everywhere 0. In this case we write $M = \langle X \rangle_R$ or $M = \sum_{x \in X} Rx$.

(b) $M$ is said to be **finitely generated** if it admits a finite set of generators.

(c) $X$ is an $R$-**basis** (or simply a **basis**) if $X$ generates $M$ and if every element of $M$ can be written *in a unique way* as an $R$-linear combination $\sum_{x \in X} \lambda_x X$ (i.e. with $\lambda_x \in R$ almost everywhere 0).

(d) $M$ is called **free** if it admits an $R$-basis $X$, and $|X|$ is called $R$-**rank** of $M$.
**Notation:** In this case we write $M = \bigoplus_{x \in X} Rx$.

**Warning**: If the ring $R$ is not commutative, then it is not true in general that two different bases of a free $R$-module have the same number of elements.

**Proposition 1.9 (*Universal property of free modules*)**

Let $M$ be a free $R$-module with $R$-basis $X$. If $N$ is an $R$-module and $f : X \longrightarrow N$ is a map (of sets), then there exists a unique $R$-homomorphism $\hat{f} : M \longrightarrow N$ such that the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\;f\;} & N \\
{\scriptstyle \text{inc}}\downarrow & \circlearrowleft \nearrow & \\
M & {\scriptstyle \exists! \hat{f}} &
\end{array}
$$

We say that $\hat{f}$ is obtained by **extending $f$ by $R$-linearity**.

**Proof:** Given an $R$-linear combination $\sum_{x \in X} \lambda_x x \in M$, set $\hat{f}(\sum_{x \in X} \lambda_x x) := \sum_{x \in X} \lambda_x f(x)$. The claim follows.  ∎

# 2 Algebras

In this lecture we aim at studying modules over specific rings, which are in particular *algebras*.

**Definition 2.1 (*Algebra*)**

Let $R$ be a commutative Ring.

(a) An $R$-**algebra** is an ordered quadruple $(A, +, \cdot, *)$ such that the following axioms hold:

**(A1)** $(A, +, \cdot)$ is a Ring;

**(A2)** $(A, +, *)$ is a left $R$-module; and

**(A3)** $r * (a \cdot b) = (r * a) \cdot b = a \cdot (r * b) \; \forall \, a, b \in A, \, \forall \, r \in R$.

(b) A map $f : A \to B$ between two $R$-algebras is called an **algebra homomorphism** iff:

   (i) $f$ a is homomorphism of $R$-modules;

   (ii) $f$ is a ring homomorphism.

**Example 2 (*Algebras*)**

(a) The ring $R$ itself is an $R$-algebra.
[The internal composition law "·" and the external composition law "∗" coincide in this case.]

(b) For each $n \in \mathbb{Z}_{\geqslant 1}$ the set $M_n(R)$ of $n \times n$-matrices with coefficients in $R$ is an $R$-algebra for its usual $R$-module and Ring structures.
[Note: in particular $R$-algebras need not be commutative rings in general!]

(c) Let $K$ be a field. Then for each $n \in \mathbb{Z}_{\geqslant 1}$ the polynom ring $K[X_1, \ldots, X_n]$ is a $K$-algebra for its usual $K$-vector space and Ring structure.

(d) $\mathbb{R}$ and $\mathbb{C}$ are $\mathbb{Q}$-algebras, $\mathbb{C}$ is an $\mathbb{R}$-algebra, ...

(e) Rings are $\mathbb{Z}$-algebras.
Exercise: Check it!

**Example 3 (*Modules over algebras*)**

(a) $A = M_n(R) \Rightarrow R^n$ is an $A$-module for the external composition law given by left matrix multiplication $A \times R^n \longrightarrow R^n, (B, x) \mapsto Bx$.

(b) If $K$ is a field and $V$ a $K$-vector space, then $V$ becomes an $A$-algebra for $A := \operatorname{End}_K(V)$ together with the external composition law

$$A \times V \longrightarrow V, (\varphi, v) \mapsto \varphi(v).$$

Exercise: Check it!

(c) An arbitrary $A$-module $M$ can be seen as an $R$-module via a change of the base ring since $R \longrightarrow A, r \mapsto r * 1_A$ is a homomorphism of rings by the algebra axioms.

**Exercise 2.2**

(a) Let $R$ be a ring, and let $M, N$ be $R$-modules. Prove that:

(1) $\operatorname{End}_R(M)$, endowed with the pointwise addition of maps and the usual composition of maps, is a ring.

(2) The abelian group $\operatorname{Hom}_R(M, N)$ is a left $R$-module for the external composition law defined by

$$(rf)(m) := f(rm) = rf(m) \qquad \forall\, r \in R,\ \forall f \in \operatorname{Hom}_R(M, N),\ \forall m \in M.$$

(b) Let now $R$ be a commutative ring, $A$ be an $R$-algebra, and $M$ be an $A$-module. Prove that $\operatorname{End}_R(M)$ and $\operatorname{End}_A(M)$ are $R$-algebras.

# 3  Direct products and direct sums*

Let $\{M_i\}_{i \in I}$ be a family of $R$-modules. Then the abelian group $\prod_{i \in I} M_i$, that is the **direct product** of $\{M_i\}_{i \in I}$ seen as a family of abelian groups, becomes an $R$-module via the following external composition law:

$$R \times \prod_{i \in I} M_i \longrightarrow \prod_{i \in I} M_i$$
$$\left(r, (m_i)_{i \in I}\right) \longmapsto \left(r \cdot m_i\right)_{i \in I}$$

Exercise: check it!

**Proposition 3.1 (Universal property of the direct product)**

For each $j \in I$, we let $\pi_j : \prod_{i \in I} M_i \longrightarrow M_j$ denotes the $j$-th projection from the direct product to the module $M_j$. If $\{\varphi_i : L \longrightarrow M_i\}_{i \in I}$ is a collection of $R$-linear maps, then there exists a unique morphism of $R$-modules $\varphi : L \longrightarrow \prod_{i \in I} M_i$ such that $\pi_j \circ \varphi = \varphi_j$ for every $j \in I$.
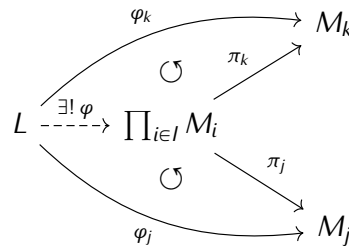


In other words, the map

$$\operatorname{Hom}_R \left( L, \prod_{i \in I} M_i \right) \longrightarrow \prod_{i \in I} \operatorname{Hom}_R(L, M_i)$$
$$\varphi \mapsto \left( \pi_i \circ \varphi \right)_i$$

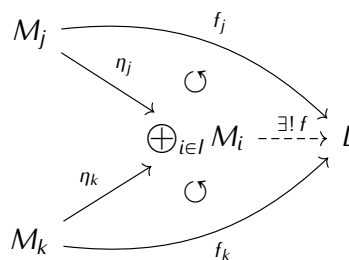is an isomorphism of abelian groups.

**Proof:** Exercise!                                                                              ∎

Let now $\bigoplus_{i \in I} M_i$ be the subgroup of $\prod_{i \in I} M_i$ consisting of the elements $(m_i)_{i \in I}$ such that $m_i = 0$ almost everywhere (i.e. $m_i = 0$ except for a finite subset of indices $i \in I$). This subgroup is called the **direct sum** of the family $\{M_i\}_{i \in I}$ and is in fact an $R$-submodule of the product. Exercise: check it!

**Proposition 3.2 (Universal property of the direct sum)**

For each $j \in I$, we let $\eta_j : M_j \longrightarrow \bigoplus_{i \in I} M_i$ denote the canonical injection of $M_j$ in the direct sum. If $\{f_i : M_i \longrightarrow L\}_{i \in I}$ is a collection of $R$-linear maps, then there exists a unique morphism of $R$-modules $\varphi : \bigoplus_{i \in I} M_i \longrightarrow L$ such that $f \circ \eta_j = f_j$ for every $j \in I$.



In other words, the map

$$\operatorname{Hom}_R \left( \bigoplus_{i \in I} M_i, L \right) \longrightarrow \prod_{i \in I} \operatorname{Hom}_R(M_i, L)$$
$$f \mapsto \left( f \circ \eta_i \right)_i$$

is an isomorphism of abelian groups.

**Proof:** Exercise!                                                                              ∎

**Remark 3.3**

It is clear that if $|I| < \infty$, then $\bigoplus_{i \in I} M_i = \prod_{i \in I} M_i$.

The direct sum as defined above is often called an *external* direct sum. This relates as follows with the usual notion of *internal* direct sum:

**Definition 3.4 (*"Internal" direct sums*)**

Let $M$ be an $R$-module and $N_1, N_2$ be two $R$-submodules of $M$. We write $M = N_1 \oplus N_2$ if every $m \in M$ can be written in a unique way as $m = n_1 + n_2$, where $n_1 \in N_1$ and $n_2 \in N_2$.

In fact $M = N_1 \oplus N_2$ (internal direct sum) if and only if $M = N_1 + N_2$ and $N_1 \cap N_2 = \{0\}$.

**Proposition 3.5**

If $N_1, N_2$ and $M$ are as in Definition 3.4 and $M = N_1 \oplus N_2$ then the map

$$\varphi : \quad M \quad \longrightarrow \quad N_1 \times N_2 = N_1 \oplus N_2 \quad \text{(external direct sum)}$$
$$m = n_1 + n_2 \quad \mapsto \quad (n_1, n_2)$$

defines an $R$-isomorphism.
Moreover, the above generalises to arbitrary internal direct sums $M = \bigoplus_{i \in I} N_i$.

**Proof:** Exercise! ∎

## 4 Exact sequences*

Exact sequences constitute a very useful tool for the study of modules. Often we obtain valuable information about modules by *plugging them* in short exact sequences, where the other terms are known.

**Definition 4.1 (*Exact sequence*)**

A sequence $L \xrightarrow{\varphi} M \xrightarrow{\psi} N$ of $R$-modules and $R$-linear maps is called **exact** (**at** $M$) if $\operatorname{Im} \varphi = \ker \psi$.

**Remark 4.2 (*Injectivity/surjectivity/short exact sequences*)**

(a) $L \xrightarrow{\varphi} M$ is injective $\Longleftrightarrow 0 \longrightarrow L \xrightarrow{\varphi} M$ is exact at $L$.

(b) $M \xrightarrow{\psi} N$ is surjective $\Longleftrightarrow M \xrightarrow{\psi} N \longrightarrow 0$ is exact at $N$.

(c) $0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ is exact (i.e. at $L$, $M$ and $N$) if and only if $\varphi$ is injective, $\psi$ is surjective and $\psi$ induces an $R$-isomorphism $\overline{\psi} : M/\operatorname{Im} \varphi \longrightarrow N, m + \operatorname{Im} \varphi \mapsto \psi(m)$.

Such a sequence is called a **short exact sequence** (**s.e.s.** for short).

(d) If $\varphi \in \operatorname{Hom}_R(L, M)$ is an injective morphism, then there is a s.e.s.

$$0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\pi} \operatorname{coker}(\varphi) \longrightarrow 0$$

where $\pi$ is the canonical projection.

(e) If $\psi \in \operatorname{Hom}_R(M, N)$ is a surjective morphism, then there is a s.e.s.

$$0 \longrightarrow \ker(\psi) \stackrel{i}{\longrightarrow} M \stackrel{\psi}{\longrightarrow} N \longrightarrow 0 \,,$$

where $i$ is the canonical injection.

## Proposition 4.3

Let $Q$ be an $R$-module. Then the following holds:

(a) $\operatorname{Hom}_R(Q, -) : {}_R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ is a *left* exact covariant functor. In other words, if $0 \longrightarrow L \stackrel{\varphi}{\longrightarrow} M \stackrel{\psi}{\longrightarrow} N \longrightarrow 0$ is a s.e.s of $R$-modules, then the induced sequence

$$0 \longrightarrow \operatorname{Hom}_R(Q, L) \stackrel{\varphi_*}{\longrightarrow} \operatorname{Hom}_R(Q, M) \stackrel{\psi_*}{\longrightarrow} \operatorname{Hom}_R(Q, N)$$

is an exact sequence of abelian groups. Here $\varphi_* := \operatorname{Hom}_R(Q, \varphi)$, that is $\varphi_*(\alpha) = \varphi \circ \alpha$ for every $\alpha \in \operatorname{Hom}_R(Q, L)$ and similarly for $\psi_*$.

(b) $\operatorname{Hom}_R(-, Q) : {}_R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ is a *left* exact contravariant functor. In other words, if $0 \longrightarrow L \stackrel{\varphi}{\longrightarrow} M \stackrel{\psi}{\longrightarrow} N \longrightarrow 0$ is a s.e.s of $R$-modules, then the induced sequence

$$0 \longrightarrow \operatorname{Hom}_R(N, Q) \stackrel{\psi^*}{\longrightarrow} \operatorname{Hom}_R(M, Q) \stackrel{\varphi^*}{\longrightarrow} \operatorname{Hom}_R(L, Q)$$

is an exact sequence of abelian groups. Here $\varphi^* := \operatorname{Hom}_R(\varphi, Q)$, that is $\varphi^*(\alpha) = \alpha \circ \varphi$ for every $\alpha \in \operatorname{Hom}_R(M, Q)$ and similarly for $\psi^*$.

Notice that $\operatorname{Hom}_R(Q, -)$ and $\operatorname{Hom}_R(-, Q)$ are not *right* exact in general. Exercise: find counter-examples!

**Proof:** One easily checks that $\operatorname{Hom}_R(Q, -)$ and $\operatorname{Hom}_R(-, Q)$ are functors. Exercise!

(a) · **Exactness at** $\operatorname{Hom}_R(Q, L)$**:** Clear.
   · **Exactness at** $\operatorname{Hom}_R(Q, M)$**:** We have

$$\beta \in \ker \psi_* \Longleftrightarrow \psi \circ \beta = 0 \Longleftrightarrow \operatorname{Im} \beta \subset \ker \psi = \operatorname{Im} \varphi$$
$$\Longleftrightarrow \forall q \in Q, \exists! \; l_q \in L \text{ such that } \beta(q) = \varphi(l_q)$$
$$\Longleftrightarrow \exists \text{ a map } \lambda : Q \longrightarrow L \text{ which sends } q \text{ to } l_q \text{ and such that } \varphi \circ \lambda = \beta$$
$$\stackrel{\varphi \text{ inj}}{\Longleftrightarrow} \exists \lambda \in \operatorname{Hom}_R(Q, L) \text{ which send } q \text{ to } l_q \text{ and such that } \varphi \circ \lambda = \beta$$
$$\Longleftrightarrow \beta \in \operatorname{Im} \varphi_* \,.$$

(b) Similar. Exercise! ∎

## Lemma–Definition 4.4 (*Split short exact sequence*)

A s.e.s. $0 \longrightarrow L \stackrel{\varphi}{\longrightarrow} M \stackrel{\psi}{\longrightarrow} N \longrightarrow 0$ of $R$-modules is called **split** if it satisfies one of the following equivalent conditions:

(a)  $\psi$ admits an $R$-linear section, i.e. if $\exists\, \sigma \in \mathrm{Hom}_R(N, M)$ such that $\psi \circ \sigma = \mathrm{Id}_N$;

(b)  $\varphi$ admits an $R$-linear retraction, i.e. if $\exists\, \rho \in \mathrm{Hom}_R(M, L)$ such that $\rho \circ \varphi = \mathrm{Id}_L$;

(c)  $\exists$ an $R$-isomorphism $\alpha : M \longrightarrow L \oplus N$ such that the following diagram commutes:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L & \overset{\varphi}{\longrightarrow} & M & \overset{\psi}{\longrightarrow} & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Id}_L} & \circlearrowleft & \downarrow{\scriptstyle \alpha} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{Id}_N} & & \\
0 & \longrightarrow & L & \overset{i}{\longrightarrow} & L \oplus N & \overset{p}{\longrightarrow} & N & \longrightarrow & 0 \,,
\end{array}
$$

where $i$, resp. $p$, are the canonical inclusion, resp. projection.

**Proof:** Exercise!                                                                              ■

**Remark 4.5**

If the sequence splits and $\sigma$ is a section, then $M = \varphi(L) \oplus \sigma(N)$. If the sequence splits and $\rho$ is a retraction, then $M = \varphi(L) \oplus \ker(\rho)$.

**Example 4**

The sequence

$$
0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \overset{\varphi}{\longrightarrow} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \overset{\pi}{\longrightarrow} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0
$$

defined by $\varphi([1]) = ([1], [0])$ and where $\pi$ is the canonical projection onto the cokernel of $\varphi$ is split but the sequence

$$
0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \overset{\varphi}{\longrightarrow} \mathbb{Z}/4\mathbb{Z} \overset{\pi}{\longrightarrow} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0
$$

defined by $\varphi([1]) = ([2])$ and $\pi$ is the canonical projection onto the cokernel of $\varphi$ is not split. Exercise: justify this fact using a straightforward argument.

## 5   Tensor products[*]

**Definition 5.1 (*Tensor product of R-modules*)**

Let $M$ be a right $R$-module and let $N$ be a left $R$-module. Let $F$ be the free abelian group (= free $\mathbb{Z}$-module) with basis $M \times N$. Let $G$ be the subgroup of $F$ generated by all the elements

$$
\begin{aligned}
&(m_1 + m_2, n) - (m_1, n) - (m_2, n), \quad \forall m_1, m_2 \in M, \forall n \in N, \\
&(m, n_1 + n_2) - (m, n_1) - (m, n_2), \quad \forall m \in M, \forall n_1, n_2 \in N, \text{ and} \\
&(mr, n) - (m, rn), \quad \forall m \in M, \forall n \in N, \forall r \in R.
\end{aligned}
$$

The **tensor product of $M$ and $N$ (balanced over $R$)**, is the abelian group $M \otimes_R N := F/G$. The class of $(m, n) \in F$ in $M \otimes_R N$ is denoted by $m \otimes n$.

**Remark 5.2**

(a) $M \otimes_R N = \langle m \otimes n \mid m \in M, n \in N \rangle_{\mathbb{Z}}$.

(b) In $M \otimes_R N$, we have the relations

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n, \quad \forall m_1, m_2 \in M, \forall n \in N,$$
$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2, \quad \forall m \in M, \forall n_1, n_2 \in N, \text{ and}$$
$$mr \otimes n = m \otimes rn, \quad \forall m \in M, \forall n \in N, \forall r \in R.$$

In particular, $m \otimes 0 = 0 = 0 \otimes n \ \forall \ m \in M, \ \forall \ n \in N$ and $(-m) \otimes n = -(m \otimes n) = m \otimes (-n)$ $\forall \ m \in M, \ \forall \ n \in N$.

**Definition 5.3 (*R-balanced map*)**

Let $M$ and $N$ be as above and let $A$ be an abelian group. A map $f : M \times N \longrightarrow A$ is called **R-balanced** if

$$f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n), \quad \forall m_1, m_2 \in M, \forall n \in N,$$
$$f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2), \quad \forall m \in M, \forall n_1, n_2 \in N,$$
$$f(mr, n) = f(m, rn), \quad \forall m \in M, \forall n \in N, \forall r \in R.$$

**Remark 5.4**

The canonical map $t : M \times N \longrightarrow M \otimes_R N, (m, n) \mapsto m \otimes n$ is $R$-balanced.

**Proposition 5.5 (*Universal property of the tensor product*)**

Let $M$ be a right $R$-module and let $N$ be a left $R$-module. For every abelian group $A$ and every $R$-balanced map $f : M \times N \longrightarrow A$ there exists a unique $\mathbb{Z}$-linear map $\overline{f} : M \otimes_R N \longrightarrow A$ such that the following diagram commutes:



**Proof:** Let $\iota : M \times N \longrightarrow F$ denote the canonical inclusion, and let $\pi : F \longrightarrow F/G$ denote the canonical projection. By the universal property of the free $\mathbb{Z}$-module, there exists a unique $\mathbb{Z}$-linear map $\tilde{f} : F \longrightarrow A$ such that $\tilde{f} \circ \iota = f$. Since $f$ is $R$-balanced, we have that $G \subseteq \ker(\tilde{f})$. Therefore, the universal property of the quotient yields the existence of a unique homomorphism of abelian groups $\overline{f} : F/G \longrightarrow A$ such that $\overline{f} \circ \pi = \tilde{f}$:



Clearly $t = \pi \circ \iota$, and hence $\overline{f} \circ t = \overline{f} \circ \pi \circ \iota = \tilde{f} \circ \iota = f$. ∎

**Remark 5.6**

Let $M$ and $N$ be as in Definition 5.1.

(a) Let $\{M_i\}_{i \in I}$ be a collection of right $R$-modules, $M$ be a right $R$-module, $N$ be a left $R$-module and $\{N_j\}_{i \in J}$ be a collection of left $R$-modules. Then, we have

$$\bigoplus_{i \in I} M_i \otimes_R N \cong \bigoplus_{i \in I} (M_i \otimes_R N)$$
$$M \otimes_R \bigoplus_{j \in J} N_j \cong \bigoplus_{j \in J} (M \otimes_R N_j).$$

(This is easily proved using both the universal property of the direct sum and of the tensor product.)

(b) There are natural isomorphisms of abelian groups given by $R \otimes_R N \cong N$ via $r \otimes n \mapsto rn$, and $M \otimes_R R \cong M$ via $m \otimes r \mapsto mr$.

(c) It follows from (b), that if $P$ is a free left $R$-module with $R$-basis $X$, then $N \otimes_R P \cong \bigoplus_{x \in X} N$, and if $P$ is a free right $R$-module with $R$-basis $X$, then $P \otimes_R M \cong \bigoplus_{x \in X} M$.

(d) Let $Q$ be a third ring. Then we obtain module structures on the tensor product as follows:

   (i) If $M$ is a $(Q, R)$-bimodule and $N$ a left $R$-module, then $M \otimes_R N$ can be endowed with the structure of a left $Q$-module via

   $$q \cdot (m \otimes n) = q \cdot m \otimes n \quad \forall q \in Q, \forall m \in M, \forall n \in N.$$

   (ii) If $M$ is a right $R$-module and $N$ an $(R, S)$-bimodule, then $M \otimes_R N$ can be endowed with the structure of a right $S$-module via

   $$(m \otimes n) \cdot s = qm \otimes n \cdot s \quad \forall s \in S, \forall m \in M, \forall n \in N.$$

   (iii) If $M$ is a $(Q, R)$-bimodule and $N$ an $(R, S)$-bimodule. Then $M \otimes_R N$ can be endowed with the structure of a $(Q, S)$-bimodule via the external composition laws defined in (i) and (ii).

(e) Assume $R$ is commutative. Then any $R$-module can be viewed as an $(R, R)$-bimodule. Then, in particular, $M \otimes_R N$ becomes an $R$-module (both on the left and on the right).

(f) For instance, it follows from (e) that if $K$ is a field and $M$ and $N$ are $K$-vector spaces with $K$-bases $\{x_i\}_{i \in I}$ and $\{y_j\}_{j \in J}$ resp., then $M \otimes_K N$ is a $K$-vector space with a $K$-basis given by $\{x_i \otimes y_j\}_{(i,j) \in I \times J}$.

(g) **Tensor product of morphisms:** Let $f : M \longrightarrow M'$ be a morphism of right $R$-modules and $g : N \longrightarrow N'$ be a morphism of left $R$-modules. Then, by the universal property of the tensor product, there exists a unique $\mathbb{Z}$-linear map $f \otimes g : M \otimes_R N \longrightarrow M' \otimes_R N'$ such that $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$.

**Proof:** Exercise! ∎

**Exercise 5.7**

(a) Assume $R$ is a commutative ring and $I$ is an ideal of $R$. Let $M$ be a left $R$-module. Prove that there is an isomorphism of left $R$-modules $R/I \otimes_R M \cong M/IM$.

(b) Let $m, n$ be coprime positive integers. Compute $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$, and $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$.

(c) Let $K$ be a field and let $U, V$ be finite-dimensional $K$-vector spaces. Prove that there is a natural isomorphism of $K$-vector spaces:

$$\operatorname{Hom}_K(U, V) \cong U^* \otimes_K V \,.$$

**Proposition 5.8 (*Right exactness of the tensor product*)**

(a) Let $N$ be a left $R$-module. Then $- \otimes_R N : \mathbf{Mod}_R \longrightarrow \mathbf{Ab}$ is a right exact covariant functor.

(b) Let $M$ be a right $R$-module. Then $M \otimes_R - :_R \mathbf{Mod} \longrightarrow \mathbf{Ab}$ is a right exact covariant functor.

**Remark 5.9**

The functors $- \otimes_R N$ and $M \otimes_R -$ are not left exact in general.