Chapter 6. Projective Modules for the Group Algebra (Part I)

We continue developing techniques to describe modules that are not semisimple and in particular indecomposable modules. The indecomposable projective modules are the indecomposable summands of the regular module. Since every module is a homomorphic image of a direct sum of copies of the regular module, by knowing the structure of the projectives we gain some insight into the structure of all modules.

Notation: throughout this chapter, unless otherwise specified, we let G denote a finite group and K is a field. All modules over group algebras considered are assumed to be **finitely generated**, hence **of finite K-dimension**. We recall that then $KG/J(KG)$ is semisimple.

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24 Radical, socle, head

Definition 24.1

Let M be a KG-module.

(a) The **radical** of M is its submodule

$$\mathrm{rad}(M) := \bigcap_{\substack{V\subset M,\\ V \text{ maximal}\\ KG\text{-submodule}}} V\,.$$

(b) The **head** (or **top**) of M is the quotient module $\mathrm{hd}(M) := M/\operatorname{rad}(M)$.

(c) The **socle** of M, denoted $\mathrm{soc}(M)$ is the sum of all simple submodules of M.

Lemma 24.2

Let M be a KG-module. Then the following KG-submodules of M are equal:

(1) $\operatorname{rad}(M)$;

(2) $J(KG)M$;

(3) the smallest KG-submodule of M with semisimple quotient.

Proof:

"(3)=(1)": Recall that if $V \subset M$ is a maximal submodule, then M/V is simple. Moreover, if $V_1, \ldots, V_r$ ($r \in \mathbb{Z}_{>0}$) are maximal submodules of M, then the map

$$\begin{array}{cccc} \varphi: & M & \longrightarrow & M/V_1 \oplus \cdots \oplus M/V_r \\ & m & \mapsto & (m + V_1, \ldots, m + V_r) \end{array}$$

is a KG-homomorphism with kernel $\ker(\varphi) = V_1 \cap \cdots \cap V_r$. Hence $M/(V_1 \cap \cdots \cap V_r) \cong \operatorname{Im}(\varphi)$ is semisimple, since it is a submodule of a semisimple module. Therefore $M/\operatorname{rad}(M)$ is a semisimple quotient. It remains to see that it is the smallest such quotient.

If $X \subseteq M$ is a KG-submodule with M/X semisimple, then by the Correspondence Theorem, there exists KG-submodules $X_1, \ldots, X_r$ of M ($r \in \mathbb{Z}_{>0}$) containing X such that

$$M/X \cong X_1/X \oplus \cdots \oplus X_r/X \qquad \text{and} \qquad X_i/X \text{ is simple } \forall\, 1 \leqslant i \leqslant r\,.$$

For each $1 \leqslant i \leqslant r$, let Y_i be be the kernel of the projection homomorphism $M \twoheadrightarrow M/X \twoheadrightarrow X_i/X$, so that Y_i is maximal (as X_i/X is simple) and $X = Y_1 \cap \ldots \cap Y_r$. Thus $X \supseteq \operatorname{rad}(M)$, as required.

"(1)⊆(2)": Observe that the quotient module $M/J(KG)M$ is a $KG/J(KG)$-module as $J(KG)\,(M/J(KG)M) = 0$. Now, as $KG/J(KG)$ is semisimple (by Proposition 11.5 and Proposition 11.7), $M/J(KG)M$ is a semisimple $KG/J(KG)$-module by definition of a semisimple ring, but then it is also semisimple as a KG-module. Since we have already proved that $\operatorname{rad}(M)$ is the largest KG-submodule of M with semisimple quotient, we must have that $\operatorname{rad}(M) \subseteq J(KG)M$.

"(2)⊆(1)": As $M/J(KG)M$ is semisimple, certainly $J(KG)\,(M/J(KG)M) = 0$, because $J(KG)$ annihilates all simple KG-module by definition. Hence $J(KG)M \subseteq \operatorname{rad}(M)$. ■

Example 13

If M is a semisimple KG-module, then $\mathrm{soc}(M) = M$ by definition, $\operatorname{rad}(M) = 0$ by the above Lemma and $\mathrm{hd}(M) = M$.

Exercise 24.3

Let M be a KG-module. Prove that the following KG-submodules of M are equal:

(1) $\mathrm{soc}(M)$;

(2) the largest semisimple KG-submodule of M;

(3) $\{m \in M \mid J(KG)m = 0\}$.

25 Projective modules

We have seen that over a semisimple ring, all simple modules appear as direct summands of the regular modules. For non-semisinple rings this is not true any more, but replacing simple modules by the so-called *projective* modules, we will obtain a similar characterisation.

Proposition-Definition 25.1 (*Projective module*)

Let R be an arbitrary ring and let P be an R-module. Then the following are equivalent:

(a) The functor $\mathrm{Hom}_R(P, -)$ is exact. In other words, the image of any s.e.s. of R-modules under $\mathrm{Hom}_R(P, -)$ is again a s.e.s.

(b) If $\psi \in \mathrm{Hom}_R(M, N)$ is a surjective morphism of R-modules, then the morphism of abelian groups $\psi_* : \mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(P, N)$ is surjective. In other words, for every pair of morphisms

$$\begin{tikzcd} & P \arrow[d, "\alpha"] \\ M \arrow[r, "\psi"'] & N \end{tikzcd}$$

where ψ is surjective, there exists a KG-morphism $\beta : P \longrightarrow M$ such that $\alpha = \psi\beta$.

(c) If $\pi : M \longrightarrow P$ is a surjective R-linear map, then π splits, i.e., there exists $\sigma \in \mathrm{Hom}_R(P, M)$ such that $\pi \circ \sigma = \mathrm{Id}_P$.

(d) P is isomorphic to a direct summand of a free R-module.

If P satisfies these equivalent conditions, then P is called **projective**. Moroever, a projective indecomposble module is called a **PIM**.

Proof: See Commutative Algebra. ■

Example 14

(a) Any free module is projective.

(b) Let e be an idempotent element of the ring R. Then, $R \cong Re \bigoplus R(1 - e)$ and Re is projective but not free if $e \neq 0, 1$.

(c) A direct sum of modules $\{P_i\}_{i \in I}$ is projective if and only if each P_i is projective.

(c) If R is semisimple, then on the one hand any projective indecomposable module is simple and conversely, since R° is semisimple. It follows that any R-module is projective.

26 Projective modules for the group algebra

We now want to prove that the PIMs of KG are in bijection with the simple KG-modules, and hence that there are a finite number of them, up to isomorphism.

Theorem 26.1

(a) If P is a projective indecomposable KG-module, then $P/\operatorname{rad}(P)$ is a simple KG-module.

(b) If M is a KG-module and $M/\operatorname{rad}(M) \cong P/\operatorname{rad}(P)$ for a projective indecomposable KG-module P, then there exists a surjective KG-homomorphism $\varphi : P \longrightarrow M$. In particular, if M is also projective indecomposable, then $M/\operatorname{rad}(M) \cong P/\operatorname{rad}(P)$ if and only if $M \cong P$.

(c) There is a bijection

$$\begin{array}{ccc} \left\{\begin{smallmatrix}\text{projective indecomposable}\\ KG\text{-modules}\end{smallmatrix}\right\}/\cong & \overset{\sim}{\longleftrightarrow} & \left\{\begin{smallmatrix}\text{simple}\\ KG\text{-modules}\end{smallmatrix}\right\}/\cong \\ P & \mapsto & P/\operatorname{rad}(P)\,. \end{array}$$

Proof:

(a) By Lemma 24.2, $P/\operatorname{rad}(P)$ is semisimple, hence it suffices to prove that it is indecomposable, or equivalently, by Proposition 10.4 that $\operatorname{End}_{KG}(P/\operatorname{rad}(P))$ is a local ring.
Now, if $\varphi \in \operatorname{End}_{KG}(P)$, then by Lemma 24.2, we have

$$\varphi(\operatorname{rad}(P)) = \varphi(J(KG)P) = J(KG)\varphi(P) \subseteq J(KG)P = \operatorname{rad}(P)\,.$$

Therefore, by the universal property of the quotient, φ induces a unique KG-homomorphism $\overline{\varphi} : P/\operatorname{rad}(P) \longrightarrow P/\operatorname{rad}(P)$ such that the following diagram commutes:

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "\pi_P"'] \arrow[dr, phantom, "\circlearrowleft"] & P \arrow[d, "\pi_P"] \\ P/\operatorname{rad}(P) \arrow[r, dashed, "\overline{\varphi}"'] & P/\operatorname{rad}(P) \end{tikzcd}$$

Then, the map

$$\begin{array}{cccc} \Phi: & \operatorname{End}_{KG}(P) & \longrightarrow & \operatorname{End}_{KG}(P/\operatorname{rad}(P)) \\ & \varphi & \mapsto & \overline{\varphi} \end{array}$$

is clearly a K-algebra homomorphism. Moreover Φ is surjectiv. Indeed, if $\psi \in \operatorname{End}_{KG}(P/\operatorname{rad}(P))$, then by the definition of a projective module there exists a KG-homomorphism $\varphi : P \longrightarrow P$ such that $\psi \circ \pi_P = \pi_P \circ \varphi$. But then ψ statisfies the diagram of the universal property of the quotient and by uniqueness $\psi = \overline{\varphi}$.
Finally, as P is indecomposable $\operatorname{End}_{KG}(P)$ is local, hence any element of $\operatorname{End}_{KG}(P)$ is either nilpotent or invertible, and by surjectivity of Φ the same holds for $\operatorname{End}_{KG}(P/\operatorname{rad}(P))$, which in turn must belocal.

(b) Consider the diagram

$$\begin{tikzcd} & & P \arrow[d, "\pi_P"] \\ M \arrow[r, "\pi_M"'] & M/\operatorname{rad}(M) \arrow[r, "\cong", "\psi"'] & P/\operatorname{rad}(P) \end{tikzcd}$$

where π_M and π_P are the quotient morphisms. As P is projective, by definition, there exists a KG-homomorphism $\varphi : P \longrightarrow M$ such that $\pi_P = \psi \circ \pi_M \circ \varphi$.
It follows that $M = \varphi(P) + \operatorname{rad}(M) = \varphi(P) + J(KG)M$, so that $\varphi(P) = M$ by Nakayama's Lemma. Finally, if M is a PIM, the surjective homomorphism φ splits by definition of a projective module, and hence $M \mid P$. But as both modules are indecomposable, we have $M \cong P$. Conversely, if $M \cong P$, then clearly $M/\operatorname{rad}(M) \cong P/\operatorname{rad}(P)$.

(c) The given map between the two sets is well-defined by (a) and (b), and it is injective by (b). It remains to prove that it is surjective. So let S be a simple KG-module. As S is finitely generated, there exists a free KG-module F and a surjective KG-homomorphism $\psi : F \longrightarrow S$. But then there is an indecomposable direct summand P of F such that $\psi|_P : P \longrightarrow S$ is non-zero, hence surjective as S is simple. Clearly $\operatorname{rad}(P) \subseteq \ker(\psi|_P)$ since it is the smallest KG-submodule with semisimple quotient by Lemma 24.2. Then the universal property of the quotient yields a surjective homomorphism $P/\operatorname{rad}(P) \longrightarrow S$ induced by $\psi|_P$. Finally, as $P/\operatorname{rad}(P)$ is simple, $P/\operatorname{rad}(P) \cong S$ by Schur's Lemma. ■

Corollary 26.2

Assume $K = \overline{K}$. Then, in the decomposition of the regular module KG° into a direct sum of indecomposable KG-submodule, each isomorphism type of projective indecomposable KG-module occurs with multiplicity $\dim_K(P/\operatorname{rad}(P))$.

Proof: Let $KG^\circ = P_1 \oplus \cdots \oplus P_r$ ($r \in \mathbb{Z}_{>0}$) be such a decomposition. In particular, $P_1, \dots P_r$ are PIMs. Then

$$J(KG) = J(KG)KG^\circ = J(KG)P_1 \oplus \cdots \oplus J(KG)P_r = \operatorname{rad}(P_1) \oplus \cdots \oplus \operatorname{rad}(P_r)$$

by Lemma 24.2. Therefore,

$$KG/J(KG) \cong P_1/\operatorname{rad}(P_1) \oplus \cdots \oplus P_r/\operatorname{rad}(P_r)\,,$$

where each summand is simple by Theorem 26.1(a). Now as $KG/J(KG)$ is semisimple, by Theorem 13.2, any simple $KG/J(KG)$-module occurs in this decomposition with multiplicity equal to its K-dimension. Thus the claim follows from the bijection of Theorem 26.1(c). ■

Lemma 26.3

If P is a projective KG-module and $H \leqslant G$, then $P\downarrow_H^G$ is a projective KH-module.

Proof: We have already seen that $KG\downarrow_H^G \cong KH \oplus \cdots \oplus KH$, where KH occurs with multiplicity $|G : H|$. Hence the restriction from G to H of a free module is again free. Now, by definition $P \mid F$ for some free KG-module F, so that $P\downarrow_H^G \mid F\downarrow_H^G$ and the claim follows. ■

Corollary 26.4

Let $K = \overline{K}$ be an algebraically closed field of characteristic $p > 0$ and let P be a projective KG-module. If a Sylow p-subgroup Q of G has order p^a with $a \in \mathbb{Z}_{>0}$, then $p^a \mid \dim_K(P)$.

Proof: By Lemma 26.3, $P\downarrow_Q^G$ is projective. Moreover, by Corollary 17.3 the trivial KQ-module is the unique simple KQ-module, hence KQ has a unique PIM, namely KQ° itself, which has dimension $|Q| = p^a$. Hence $P\downarrow_Q^G$ is a direct sum of copies of KQ° and the claim follows. ■

Exercise 26.5

Prove that

(a) If P is a projective KG-module, then so is P^*.

(b) If $H \leqslant G$ and P is a projective KH-module, then $P{\uparrow}_H^G$ is a projective KG-module.

(c) If P is a projective KG-module and M is an arbitrary KG-module, then $P \otimes_K M$ is projective.

(d) If P is a projective indecomposable KG-module, then $\mathrm{soc}(P)$ is simple. (Hint: consider duals.)

Exercise 26.6

Let S be a simple KG-module and let P_S denote the corresponding PIM (i.e. $P_S/\operatorname{rad}(P_S) \cong S$). Let M be an arbitrary KG-module. Prove that:

(a) If T is a simple KG-module then

$$\dim_K \operatorname{Hom}_{KG}(P_S, T) = \begin{cases} \dim_K \operatorname{End}_{KG}(S) & \text{if } S \cong T, \\ 0 & \text{otherwise.} \end{cases}$$

(b) The multiplicity of S as a composition factor of M is

$$\dim_K \operatorname{Hom}_{KG}(P_S, M)/\dim_K \operatorname{End}_{KG}(S)\,.$$