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# **FreeFEM Documentation**

***Release v4.2.1***

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In collaboration with:





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CHAPTER  
ONE

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## INTRODUCTION

**FreeFEM** is a partial differential equation solver for non-linear multi-physics systems in 2D and 3D.

Problems involving partial differential equations from several branches of physics, such as fluid-structure interactions, require interpolations of data on several meshes and their manipulation within one program. **FreeFEM** includes a fast interpolation algorithm and a language for the manipulation of data on multiple meshes.

**FreeFEM** is written in C++ and its language is a C++ idiom.



## 1.1 New features

### 1.1.1 Hash Matrix

A new internal management of matrix inside the FreeFEM core have been introduced in FreeFEM 4.0, for better performance.

### 1.1.2 Surface Finite Element

The release version of the surface finite element is available in FreeFEM 4.2.1. with some examples in examples/3dSurf. The documentation is being prepared.

### 1.1.3 CMake

A compilation process using CMake is under development

## 1.2 Download FreeFEM

### 1.2.1 Latest binary packages

FreeFEM v4.2.1 release runs under MacOS X, Ubuntu and Windows 64 bits.

Operating System	FreeFEM Version	Size	Date
MacOS 10.14	–	–	–
MacOS 10.10.5 up to 10.13.5	4.2.1	417 Mb	May 31, 2019
Ubuntu 16.04 or higher	4.2.1	52.6 Mb	May 31, 2019
Windows 64bits	4.2.1	135 Mb	May 31, 2019
Source_4.2.1	4.2.1	3.2 Mb	May 31, 2019
previous releases	–	–	–

The source code is available on the FreeFEM GitHub Repository.

---

**Note:** The support ended for all releases under Windows 32 bits.

---

### 1.2.2 Syntax highlighters

Lexer type	Version	Description
Emacs	0.3	<a href="#">freefem-mode.el</a>
Textmate 2	1.0	<a href="#">FreeFem.tmbundle</a>
Gedit	1.0	<a href="#">ffpp.lang</a>
Atom	0.3	<a href="#">language-freefem</a> or via the Atom package manager
Pygments	1.0	<a href="#">freefem.py</a>
Vim	0.1	<a href="#">edp.vim</a>

## 1.3 Installation guide

### 1.3.1 Easy installation

First, go to the [download page](#) and choose your platform: Linux, MacOS or Windows.

---

**Note:** Binary packages are available for Microsoft Windows, MacOS and some Linux distributions.

---

Install **FreeFEM** by double-clicking on the appropriate file. Under Linux and MacOS the install directory is one of the following /usr/local/bin, /usr/local/share/freefem++, /usr/local/lib/ff++

#### Windows binary installation

First download the windows installation executable, then double click to install **FreeFEM**.

In most cases just answer yes (or type return) to all questions.

Otherwise in the Additional Task windows, check the box “Add application directory to your system path.” This is required otherwise the program ffglut.exe will not be found.

By now you should have two new icons on your desktop:

- FreeFem++ (VERSION).exe, the freefem++ application.
- FreeFem++ (VERSION) Examples, a link to the freefem++ examples folder.

where (VERSION) is the version of the files (for example 3.59).

By default, the installed files are in C:\Programs Files\FreeFem++. In this directory, you have all the .dll files and other applications: FreeFem++-nw.exe, ffglut.exe, ... The syntax for the command-line tools are the same as those of FreeFem.exe.

#### MacOS X binary installation

Download the MacOS X binary version file, extract all the files by double clicking on the icon of the file, go to the the directory and put the FreeFem+.app application in the /Applications directory.

If you want terminal access to **FreeFEM** just copy the file FreeFem++ in a directory of your \$PATH shell environment variable.

#### Arch AUR package

An up-to-date package of **FreeFEM** for Arch is available on the Archlinux user repository.

To install it:

```
1 git clone https://aur.archlinux.org/freefem++-git.git
2 cd freefem++-git
3 makepkg -si
```

---

**Note:** Thanks to Stephan Husmann

---

### 1.3.2 Text-editor

#### Atom

In order to get the syntax highlighting in Atom, you have to install the FreeFEM language support.

You can do it directly in Atom: Edit -> Preferences -> Install, and search for language-freefem-offical.

To launch scripts directly from Atom, you have to install the atom-runner package. Once installed, modify the Atom configuration file (Edit -> Config...) to have something like that:

```
1 " * ":
2 ...
3
4   runner:
5     extensions:
6       edp: "FreeFem++"
7     scopes:
8       "Freefem++": "FreeFem++"
```

Reboot Atom, and use Alt+R to run a FreeFem++ script.

#### Gedit

In order to get the syntax highlighting in Gedit, you have to download the Gedit parser and copy it in /usr/share/gtksourceview-3.0/language-specs/.

### 1.3.3 Compilation

#### Using autotools

---

**Note:** 2 worked versions of FreeFEM are possible: minimal and full: sequential and without plugins (contains in 3rdparty) full: parallel with available plugins. ... note:: We advise you to use the package manager for macOS Homebrew to get the different packages required available [here](#)

---

#### Compilation on OSX (>=10.13)

1. Install Xcode, Xcode Command Line tools and Xcode Additional Tools from the Apple website
2. Install gcc and gfortran from Homebrew

```
1 brew install gcc
```

3. To use FreeFEM parallel version, install the openmpi source code

```
1 curl -L https://download.open-mpi.org/release/open-mpi/v4.0/openmpi-4.0.1.tar.gz
2 --output openmpi-4.0.1.tar.gz
3 tar xf openmpi-4.0.1
4 cd openmpi-4.0.1/
```

- with brew gcc gfortran compilers

```
4 ./configure CC=clang CXX=clang++ FC=gfortran-9 F77=gfortran-9 --prefix=/usr/local
```

- with LLVM gcc and brew gfortran compilers

```
4 ./configure CC=gcc-9 CXX=g++-9 FC=gfortran-9 F77=gfortran-9 --prefix=/usr/local
```

```
5 make
```

```
6 sudo make install
```

#### 4. Install the required libraries for **FreeFEM**

```
1 brew install m4 git flex bison
```

#### 5. If you want build your own configure according your system, install autoconf and automake from Homebrew (optional, see note in step 10)

```
1 brew install autoconf
```

```
2 brew install automake
```

#### 6. To use **FreeFEM** with its plugins, install rom Homebrew suitesparse, hdf5, cmake, wget

```
1 brew install suitesparse hdf5 cmake wget
```

#### 7. Install **gsl**

```
1 curl -O http://mirror.cyberbits.eu/gnu/gsl/gsl-2.5.tar.gz
```

```
2 tar zxvf gsl-2.5.tar.gz
```

```
3 cd gsl-2.5
```

```
4 ./configure
```

```
5 make
```

```
6 sudo make install
```

#### 8. Download the latest Git for Mac installer **git** and the **FreeFEM** source from the repository

```
1 git clone https://github.com/FreeFem/FreeFem-sources.git
```

#### 9. Configure your source code

```
1 cd FreeFem-sources
```

```
2 autoreconf -i
```

---

**Note:** if your autoreconf version is too old, do `tar zxvf AutoGeneratedFile.tar.gz`

---

- with LLVM gcc and brew gfortran compilers

```
3 ./configure --enable-download CC=clang CXX=clang++ F77=gfortran-9 FC=gfortran-9
```

- with brew gcc and brew gfortran compilers

```
3 ./configure --enable-download CC=gcc-9 CXX=g++-9 F77=gfortran-9 FC=gfortran-9
```

#### 10. Download the 3rd party packages to use FreeFEM plugins

```
1 ./3rdparty/getall -a
```

---

**Note:** All the third party packages have their own licence

---

### 11. Compile petsc & slepc

```
1 cd 3rdparty/ff-petsc
2 make petsc-slepc SUDO=sudo
3 cd -
```

### 12. Reconfigure with petsc and slepc

```
1 ./reconfigure
```

### 13. Build FreeFEM executable

```
1 make
2 make check
3 sudo make install
```

---

**Note:** To install FreeFEM, it is recommended to change the user ID of your installation directory instead of using SUDO.

---

## Compilation on Ubuntu

### 1. Install the following dependencies

```
1 sudo apt-get update && sudo apt-get upgrade
2 sudo apt-get install cpp freeglut3-dev g++ gcc gfortran \
3     m4 make patch pkg-config wget python unzip \
4     libopenblas-dev liblapack-dev libhdf5-dev libgsl-dev \
5     libscotch-dev libfftw3-dev libarpack2-dev libsuitesparse-dev \
6     libmumps-seq-dev libnlopt-dev coinor-libipopt-dev libgmm++-dev libtet1.5-dev \
7     gnuplot-qt autoconf automake autotools-dev bison flex gdb valgrind git cmake
8
9 # mpich is required for the FreeFem parallel computing version
10 sudo apt-get install mpich
```

**Warning:** In the oldest distribution of Ubuntu, libgsl-dev does not exists, use libgsl2-dev instead

### 2. Download **FreeFEM** source from the repository

```
1 git clone https://github.com/FreeFem/FreeFem-sources.git
```

### 3. Autoconf

```
1 cd FreeFem-sources
2 autoreconf -i
```

---

**Note:** if your autoreconf version is too old, do `tar zxvf AutoGeneratedFile.tar.gz`

---

#### 4. Configure

```
1 ./configure --enable-download --enable-optim
```

---

**Note:** To see all the options, type `./configure --help`

---

#### 5. Download the 3rd party packages

```
1 ./3rdparty/getall -a
```

---

**Note:** All the third party packages have their own licence

---

#### 6. Compile petsc & slepc

```
1 cd 3rdparty/ff-petsc
2 make petsc-slepc SUDO=sudo
3 cd -
```

#### 7. Reconfigure with petsc and slepc

```
1 ./reconfigure
```

#### 8. Build

```
1 make
```

---

**Note:** If your computer has many threads, you can run `make` in parallel using `make -j16` for 16 threads, for example.

---



---

**Note:** Optionnally, check the compilation with `make check`

---

#### 9. Install

```
1 sudo make install
```

### Compilation on Arch Linux

**Warning:** As Arch is in rolling release, the following information can be quickly outdated !

**Warning:** FreeFEM fails to compile using the newest version of gcc 8.1.0, use an older one instead.

1. Install the following dependencies:

```
1 pacman -Syu
2 pacman -S git openmpi gcc-fortran wget python
3     freeglut m4 make patch gmm
4     blas lapack hdf5 fftw arpack suitesparse
5     gnuplot autoconf automake bison flex gdb
6     valgrind cmake texlive-most
```

2. Download the **FreeFEM** source from the repository

```
1 git clone https://github.com/FreeFem/FreeFem-sources.git
```

3. Autoconf

```
1 cd FreeFem-sources
2 autoreconf -i
```

4. Configure

```
1 ./configure --enable-download --enable-optim
```

---

**Note:** To see all the options, type `./configure --help`

---

5. Download the packages

```
1 ./3rdparty/getall -a
```

---

**Note:** All the third party packages have their own licence

---

6. Compile petsc & slepc

```
1 cd 3rdparty/ff-petsc
2 make petsc-slepc SUDO=sudo
3 cd -
```

7. Reconfigure with petsc and slepc

```
1 ./reconfigure
```

8. Build

```
1 make
```

---

**Note:** If your computer has many threads, you can run `make` in parallel using `make -j16` for 16 threads, for example.

---

---

**Note:** Optionnally, check the compilation with `make check`

---

9. Install

```
1 sudo make install
```

## Compilation on Linux with Intel software tools

Follow the [guide](#)

## Compilation on Windows

1. Install [MS MPI v9](#) (msmpisdk.msi and MSMpiSetup.exe)
2. Install [Msys2](#) (x86\_64 version)
3. Start MSYS2 MSYS
4. Open MSYS2 MSYS terminal to install dependancies
  - for 64bits system:

```
1 source shell mingw64
2
3 pacman -Syu
4 pacman -S autoconf automake-wrapper bash bash-completion \
5   bison bsdcpio bsdtar bzip2 coreutils curl dash file filesystem \
6   findutils flex gawk gcc gcc-fortran gcc-libs grep gzip inetutils info less \
7   ↵lndir \
8   make man-db git mingw-w64-x86_64-freeglut mingw-w64-x86_64-gcc \
9   mingw-w64-x86_64-gcc-fortran mingw-w64-x86_64-gsl mingw-w64-x86_64-hdf5 \
10  mingw-w64-x86_64-openblas mintty msys2-keyring msys2-launcher-git \
11  msys2-runtime ncurses pacman pacman-mirrors pactoys-git patch pax-git python \
12  perl pkg-config pkgfile rebase sed tar tftp-hpa time tzcode unzip util-linux \
    ↵which \
13  mingw-w64-x86_64-libmicroutils mingw-w64-x86_64-arpack cmake
```

- for 32bits system:

```
1 source shell mingw32
2
3 pacman -Syu
4 pacman -S autoconf automake-wrapper bash bash-completion \
5   bison bsdcpio bsdtar bzip2 coreutils curl dash file filesystem \
6   findutils flex gawk gcc gcc-fortran gcc-libs grep gzip inetutils info less \
7   ↵lndir \
8   make man-db git mingw-w64-i686-freeglut mingw-w64-i686-gcc \
9   mingw-w64-i686-gcc-fortran mingw-w64-i686-gsl mingw-w64-i686-hdf5 \
10  mingw-w64-i686-openblas mintty msys2-keyring msys2-launcher-git \
11  msys2-runtime ncurses pacman pacman-mirrors pactoys-git patch pax-git \
12  perl pkg-config pkgfile rebase sed tar tftp-hpa time tzcode unzip util-linux which
```

5. Open MingW64 terminal (or MingW32) to compile **FreeFEM**

```
1 git clone https://github.com/FreeFem/FreeFem-sources
2 cd FreeFem-sources
3 autoreconf -i
4 ./configure --enable-download --disable-hips
5 ./3rdparty/getall -a
```

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```

6 make -j4
7 make check
8 make install

```

The **FreeFEM** executable (and some other like ffmedit, ...) are in C:\msys64\mingw64\bin (or C:\msys32\mingw32\bin).

### 1.3.4 Environment variables and init file

**FreeFEM** reads a user's init file named freefem++.pref to initialize global variables: verbosity, includepath, loadpath.

---

**Note:** The variable `verbosity` changes the level of internal printing (0: nothing unless there are syntax errors, 1: few, 10: lots, etc. ...), the default value is 2.

The included files are found in the `includepath` list and the load files are found in the `loadpath` list.

---

The syntax of the file is:

```

1 verbosity = 5
2 loadpath += "/Library/FreeFem++/lib"
3 loadpath += "/Users/hecht/Library/FreeFem++/lib"
4 includepath += "/Library/FreeFem++/edp"
5 includepath += "/Users/hecht/Library/FreeFem++/edp"
6 # This is a comment
7 load += "funcTemplate"
8 load += "myfunction"
9 load += "MUMPS_seq"

```

The possible paths for this file are

- under Unix and MacOs

```

1 /etc/freefem++.pref
2 $(HOME) /.freefem++.pref
3 freefem++.pref

```

- under windows

```

1 freefem++.pref

```

We can also use shell environment variables to change verbosity and the search rule before the init files.

```

1 export FF_verbosity=50
2 export FF_INCLUDEPATH="dir1;dir2"
3 export FF_LOADPATH="dir3;dir4"

```

---

**Note:** The separator between directories must be ";" and not ":" because ":" is used under Windows.

---



---

**Note:** To show the list of init of **FreeFEM**, do

```

1 export FF_VERBOSE=100;
2 ./FreeFem++-nw

```

## 1.4 Contributing

### 1.4.1 Bug report

#### Concerning the FreeFEM documentation

Open an [Issue](#) on **FreeFem-doc** repository.

#### Concerning the FreeFEM compilation or usage

Open an [Issue](#) on **FreeFem-sources** repository.

### 1.4.2 Improve content

Ask one of the contributors for Collaborator Access or make a [Pull Request](#).

## 1.5 Citation

### 1.5.1 If you use FreeFEM, please cite the following reference in your work:

#### APA

```

1 Hecht, F. (2012). New development in FreeFem++. Journal of numerical mathematics, 20(3-4), 251-266.

```

#### ISO690

```

1 HECHT, Frédéric. New development in FreeFem++. Journal of numerical mathematics, 2012,
  ↪ vol. 20, no 3-4, p. 251-266.

```

#### MLA

```

1 Hecht, Frédéric. "New development in FreeFem++." Journal of numerical mathematics 20.
  ↪ 3-4 (2012): 251-266.

```

#### BibTeX

```
1 @article{MR3043640,
2   AUTHOR = {Hecht, F.},
3   TITLE = {New development in FreeFem++},
4   JOURNAL = {J. Numer. Math.},
5   FJOURNAL = {Journal of Numerical Mathematics},
6   VOLUME = {20}, YEAR = {2012},
7   NUMBER = {3-4}, PAGES = {251--265},
8   ISSN = {1570-2820},
9   MRCLASS = {65Y15},
10  MRNUMBER = {3043640}
11 }
```

## 1.6 Authors

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Former PhD student at LJLL, optimization interface with [nlopt](#), [ipopt](#), [cmaes](#), ...

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### Jacques Morice

Former Post-Doc at LJLL, three dimensions mesh generation and coupling with [medit](#)

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<https://sites.google.com/a/comfos.org/comfos/>

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<a href="https://www.linkedin.com/in/loancannard">https://www.linkedin.com/in/loancannard</a>

And all the dedicated Github contributors



**TUTORIALS**

The **FreeFEM** language is *typed*, polymorphic and reentrant with *macro generation*.

Every variable must be typed and declared in a statement. Each statement is separated from the next by a semicolon ;.

The **FreeFEM** language is a C++ idiom with something that is more akin to LaTeX.

For the specialist, one key guideline is that **FreeFEM** rarely generates an internal finite element array, this was adopted for speed and consequently **FreeFEM** could be hard to beat in terms of execution speed, except for the time lost in the interpretation of the language (which can be reduced by a systematic usage of varf and matrix instead of problem).

The Development Cycle: Edit–Run/Visualize–Revise

Many examples and tutorials are given there after and in the *examples section*. It is better to study them and learn by example.

If you are a beginner in the finite element method, you may also have to read a book on variational formulations.

The development cycle includes the following steps:

**Modeling:** From strong forms of PDE to weak forms, one must know the variational formulation to use **FreeFEM**; one should also have an eye on the reusability of the variational formulation so as to keep the same internal matrices; a typical example is the time dependent heat equation with an implicit time scheme: the internal matrix can be factorized only once and **FreeFEM** can be taught to do so.

**Programming:** Write the code in **FreeFEM** language using a text editor such as the one provided in your integrated environment.

**Run:** Run the code (here written in file mycode.edp). That can also be done in terminal mode by :

```
1 FreeFem++ mycode.edp
```

**Visualization:** Use the keyword plot directly in mycode.edp to display functions while **FreeFEM** is running. Use the plot-parameter wait=1 to stop the program at each plot.

**Debugging:** A global variable debug (for example) can help as in wait=true to wait=false.

```
1 bool debug = true;
2
3 border a(t=0, 2.*pi){x=cos(t); y=sin(t); label=1;};
4 border b(t=0, 2.*pi){x=0.8+0.3*cos(t); y=0.3*sin(t); label=2;};
5
6 plot(a(50) + b(-30), wait=debug); //plot the borders to see the intersection
//so change 0.8 in 0.3 in b
//if debug == true, press Enter to continue
9
```

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```

10 mesh Th = buildmesh(a(50) + b(-30));
11 plot(Th, wait=debug); //plot Th then press Enter
12
13 fespace Vh(Th,P2);
14 Vh f = sin(pi*x)*cos(pi*y);
15 Vh g = sin(pi*x + cos(pi*y));
16
17 plot(f, wait=debug); //plot the function f
18 plot(g, wait=debug); //plot the function g

```

Changing debug to false will make the plots flow continuously. Watching the flow of graphs on the screen (while drinking coffee) can then become a pleasant experience.

#### Error management

Error messages are displayed in the console window. They are not always very explicit because of the template structure of the C++ code (we did our best!). Nevertheless they are displayed at the right place. For example, if you forget parenthesis as in:

```

1 bool debug = true;
2 mesh Th = square(10,10;
3 plot(Th);

```

then you will get the following message from FreeFEM:

```

1   2 : mesh Th = square(10,10;
2   Error line number 2, in file bb.edp, before token ;
3 parse error
4   current line = 2
5 Compile error : parse error
6     line number :2, ;
7 error Compile error : parse error
8     line number :2, ;
9   code = 1

```

If you use the same symbol twice as in:

```

1 real aaa = 1;
2 real aaa;

```

then you will get the message:

```

1 2 : real aaa; The identifier aaa exists
2     the existing type is <Pd>
3     the new type is <Pd>

```

If you find that the program isn't doing what you want you may also use cout to display in text format on the console window the value of variables, just as you would do in C++.

The following example works:

```

1 ...
2 fespace Vh(Th, P1);
3 Vh u;
4 cout << u;
5 matrix A = a(Vh, Vh);
6 cout << A;

```

Another trick is to *comment in and out* by using `//` as in C++. For example:

```
1 real aaa =1;
2 // real aaa;
```

## 2.1 Getting started

For a given function  $f(x, y)$ , find a function  $u(x, y)$  satisfying :

$$\begin{aligned} -\Delta u(x, y) &= f(x, y) && \text{for all } (x, y) \text{ in } \Omega \\ u(x, y) &= 0 && \text{for all } (x, y) \text{ on } \partial\Omega \end{aligned} \quad (2.1)$$

Here  $\partial\Omega$  is the boundary of the bounded open set  $\Omega \subset \mathbb{R}^2$  and  $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ .

We will compute  $u$  with  $f(x, y) = xy$  and  $\Omega$  the unit disk. The boundary  $C = \partial\Omega$  is defined as:

$$C = \{(x, y) | x = \cos(t), y = \sin(t), 0 \leq t \leq 2\pi\}$$

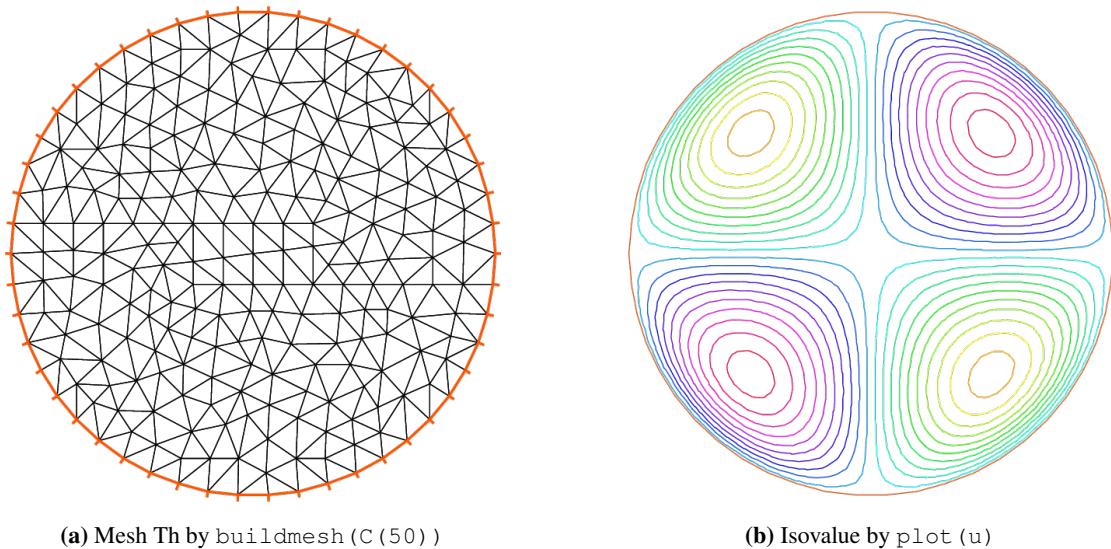
---

**Note:** In FreeFEM, the domain  $\Omega$  is assumed to be described by the left side of its boundary.

---

The following is the FreeFEM program which computes  $u$ :

```
1 // Define mesh boundary
2 border C(t=0, 2*pi){x=cos(t); y=sin(t);}
3
4 // The triangulated domain Th is on the left side of its boundary
5 mesh Th = buildmesh(C(50));
6
7 // The finite element space defined over Th is called here Vh
8 fespace Vh(Th, P1);
9 Vh u, v; // Define u and v as piecewise-P1 continuous functions
10
11 // Define a function f
12 func f= x*y;
13
14 // Get the clock in second
15 real cpu=clock();
16
17 // Define the PDE
18 solve Poisson(u, v, solver=LU)
19   = int2d(Th) ( // The bilinear part
20     dx(u)*dx(v)
21     + dy(u)*dy(v)
22   )
23   - int2d(Th) ( // The right hand side
24     f*v
25   )
26   + on(C, u=0); // The Dirichlet boundary condition
27
28 // Plot the result
29 plot(u);
30
31 // Display the total computational time
32 cout << "CPU time = " << (clock()-cpu) << endl;
```

**Fig. 2.1:** Poisson's equation

As illustrated in Fig. 2.1b, we can see the isovalue of  $u$  by using **FreeFEM** `plot` command (see line 29 above).

---

**Note:** The qualifier `solver=LU` (line 18) is not required and by default a multi-frontal `LU` is used.

The lines containing `clock` are equally not required.

---

**Tip:** Note how close to the mathematics **FreeFEM** language is.

Lines 19 to 24 correspond to the mathematical variational equation:

$$\int_{T_h} \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) dx dy = \int_{T_h} f v dx dy$$

for all  $v$  which are in the finite element space  $V_h$  and zero on the boundary  $C$ .

---

**Tip:** Change `P1` into `P2` and run the program.

---

This first example shows how **FreeFEM** executes with no effort all the usual steps required by the finite element method (FEM). Let's go through them one by one.

### On the line 2:

The boundary  $\Gamma$  is described analytically by a parametric equation for  $x$  and for  $y$ . When  $\Gamma = \sum_{j=0}^J \Gamma_j$  then each curve  $\Gamma_j$  must be specified and crossings of  $\Gamma_j$  are not allowed except at end points.

The keyword `label` can be added to define a group of boundaries for later use (boundary conditions for instance). Hence the circle could also have been described as two half circle with the same label:

```

1 border Gamma1(t=0, pi){x=cos(t); y=sin(t); label=C};
2 border Gamma2(t=pi, 2.*pi){x=cos(t); y=sin(t); label=C};
```

Boundaries can be referred to either by name (Gamma1 for example) or by label (`C` here) or even by its internal number here 1 for the first half circle and 2 for the second (more examples are in [Meshing Examples](#)).

### On the line 5

The triangulation  $\mathcal{T}_h$  of  $\Omega$  is automatically generated by `buildmesh(C(50))` using 50 points on `C` as in Fig. 2.1a.

The domain is assumed to be on the left side of the boundary which is implicitly oriented by the parametrization. So an elliptic hole can be added by typing:

```
1 border C(t=2.*pi, 0) {x=0.1+0.3*cos(t); y=0.5*sin(t);};
```

If by mistake one had written:

```
1 border C(t=0, 2.*pi) {x=0.1+0.3*cos(t); y=0.5*sin(t);};
```

then the inside of the ellipse would be triangulated as well as the outside.

---

**Note:** Automatic mesh generation is based on the Delaunay-Voronoi algorithm. Refinement of the mesh are done by increasing the number of points on  $\Gamma$ , for example `buildmesh(C(100))`, because inner vertices are determined by the density of points on the boundary.

Mesh adaptation can be performed also against a given function  $f$  by calling `adaptmesh(Th, f)`.

---

Now the name  $\mathcal{T}_h$  (`Th` in FreeFEM) refers to the family  $\{T_k\}_{k=1,\dots,n_t}$  of triangles shown in Fig. 2.1a.

Traditionally  $h$  refers to the mesh size,  $n_t$  to the number of triangles in  $\mathcal{T}_h$  and  $n_v$  to the number of vertices, but it is seldom that we will have to use them explicitly.

If  $\Omega$  is not a polygonal domain, a “skin” remains between the exact domain  $\Omega$  and its approximation  $\Omega_h = \cup_{k=1}^{n_t} T_k$ . However, we notice that all corners of  $\Gamma_h = \partial\Omega_h$  are on  $\Gamma$ .

### On line 8:

A finite element space is, usually, a space of polynomial functions on elements, triangles here only, with certain matching properties at edges, vertices etc. Here `fespace Vh(Th, P1)` defines  $V_h$  to be the space of continuous functions which are affine in  $x, y$  on each triangle of  $T_h$ .

As it is a linear vector space of finite dimension, basis can be found. The canonical basis is made of functions, called the *hat function*  $\phi_k$  which are continuous piecewise affine and are equal to 1 on one vertex and 0 on all others. A typical hat function is shown on Fig. 2.2b.

---

**Note:** The easiest way to define  $\phi_k$  is by making use of the *barycentric coordinates*  $\lambda_i(x, y)$ ,  $i = 1, 2, 3$  of a point  $q = (x, y) \in T$ , defined by  $\sum_i \lambda_i = 1$ ,  $\sum_i \lambda_i q^i = \vec{q}$  where  $q^i$ ,  $i = 1, 2, 3$  are the 3 vertices of  $T$ . Then it is easy to see that the restriction of  $\phi_k$  on  $T$  is precisely  $\lambda_k$ .

---

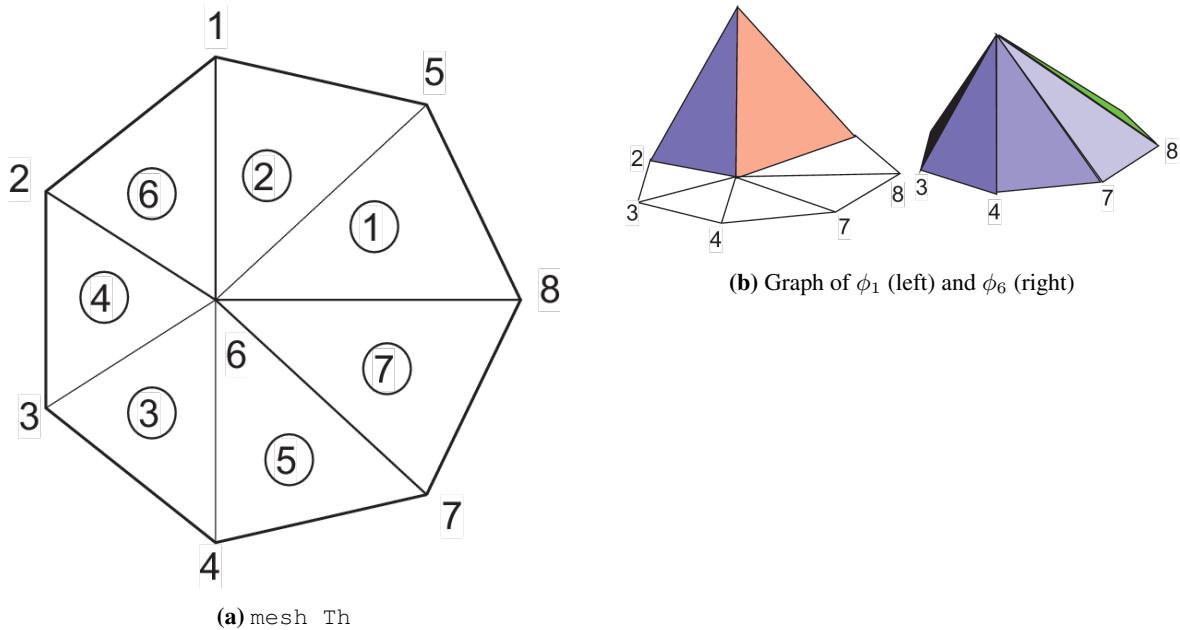
Then:

$$V_h(\mathcal{T}_h, P_1) = \left\{ w(x, y) \mid w(x, y) = \sum_{k=1}^M w_k \phi_k(x, y), w_k \text{ are real numbers} \right\} \quad (2.2)$$

where  $M$  is the dimension of  $V_h$ , i.e. the number of vertices. The  $w_k$  are called the *degree of freedom* of  $w$  and  $M$  the number of degree of freedom.

It is said also that the *nodes* of this finite element method are the vertices.

### Setting the problem



**Fig. 2.2:** Hat functions

On line 9, `Vh u, v` declares that  $u$  and  $v$  are approximated as above, namely:

$$u(x, y) \simeq u_h(x, y) = \sum_{k=0}^{M-1} u_k \phi_k(x, y) \quad (2.3)$$

On the line 12, the right hand side `f` is defined analytically using the keyword `func`.

Line 18 to 26 define the bilinear form of equation (2.1) and its Dirichlet boundary conditions.

This *variational formulation* is derived by multiplying (2.1) by  $v(x, y)$  and integrating the result over  $\Omega$ :

$$-\int_{\Omega} v \Delta u \, dx dy = \int_{\Omega} v f \, dx dy$$

Then, by Green's formula, the problem is converted into finding  $u$  such that

$$a(u, v) - \ell(f, v) = 0 \quad \forall v \text{ satisfying } v = 0 \text{ on } \partial\Omega.$$

with:

$$\begin{aligned} a(u, v) &= \int_{\Omega} \nabla u \cdot \nabla v \, dx dy \\ \ell(f, v) &= \int_{\Omega} f v \, dx dy \end{aligned} \quad (2.4)$$

In FreeFEM the **Poisson** problem can be declared only as in:

```
1 Vh u,v; problem Poisson(u,v) = ...
```

and solved later as in:

```
1 Poisson; //the problem is solved here
```

or declared and solved at the same time as in:

```
1 Vh u,v; solve Poisson(u,v) = ...
```

and (2.4) is written with  $\text{dx}(u) = \partial u / \partial x$ ,  $\text{dy}(u) = \partial u / \partial y$  and:

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx dy \longrightarrow \text{int2d}(Th) (\text{dx}(u) * \text{dx}(v) + \text{dy}(u) * \text{dy}(v))$$

$$\int_{\Omega} fv \, dx dy \longrightarrow \text{int2d}(Th) (f * v) \text{ (Notice here, } u \text{ is unused)}$$

**Warning:** In FreeFEM bilinear terms and linear terms should not be under the same integral indeed to construct the linear systems FreeFEM finds out which integral contributes to the bilinear form by checking if both terms, the unknown (here  $u$ ) and test functions (here  $v$ ) are present.

### Solution and visualization

On line 15, the current time in seconds is stored into the real-valued variable `cpu`.

Line 18, the problem is solved.

Line 29, the visualization is done as illustrated in Fig. 2.1b.

(see [Plot for zoom, postscript and other commands](#)).

Line 32, the computing time (not counting graphics) is written on the console. Notice the C++-like syntax; the user needs not study C++ for using FreeFEM, but it helps to guess what is allowed in the language.

### Access to matrices and vectors

Internally FreeFEM will solve a linear system of the type

$$\sum_{j=0}^{M-1} A_{ij} u_j - F_i = 0, \quad i = 0, \dots, M-1; \quad F_i = \int_{\Omega} f \phi_i \, dx dy \quad (2.5)$$

which is found by using (2.3) and replacing  $v$  by  $\phi_i$  in (2.4). The Dirichlet conditions are implemented by penalty, namely by setting  $A_{ii} = 10^{30}$  and  $F_i = 10^{30} * 0$  if  $i$  is a boundary degree of freedom.

---

**Note:** The number  $10^{30}$  is called `tgv` (*très grande valeur* or *very high value* in english) and it is generally possible to change this value, see the item :freefem`solve, tgv=‘

---

The matrix  $A = (A_{ij})$  is called *stiffness matrix*. If the user wants to access  $A$  directly he can do so by using (see section [Variational form, Sparse matrix, PDE data vector](#) for details).

```
1 varf a (u,v)
2   = int2d(Th) (
3     dx(u) * dx(v)
4     + dy(u) * dy(v)
5   )
6   + on (C, u=0)
7   ;
8 matrix A = a(Vh, Vh); //stiffness matrix
```

The vector  $F$  in (2.5) can also be constructed manually:

```

1 varf l(unused,v)
2   = int2d(Th) (
3     f*v
4   )
5   + on(C, unused=0)
6   ;
7 Vh F;
8 F[] = l(0,Vh); //F[] is the vector associated to the function F

```

The problem can then be solved by:

```

1 u[] = A^-1*F[]; //u[] is the vector associated to the function u

```

**Note:** Here  $u$  and  $F$  are finite element function, and  $u[]$  and  $F[]$  give the array of value associated ( $u[] \equiv (u_i)_{i=0,\dots,M-1}$  and  $F[] \equiv (F_i)_{i=0,\dots,M-1}$ ).

So we have:

$$u(x, y) = \sum_{i=0}^{M-1} u[i] \phi_i(x, y), \quad F(x, y) = \sum_{i=0}^{M-1} F[i] \phi_i(x, y)$$

where  $\phi_i, i = 0, \dots, M - 1$  are the basis functions of  $Vh$  like in equation (ref{equation3}), and  $M = Vh.ndof$  is the number of degree of freedom (i.e. the dimension of the space  $Vh$ ).

The linear system (2.5) is solved by UMFPACK unless another option is mentioned specifically as in:

```

1 Vh u, v;
2 problem Poisson(u, v, solver=CG) = int2d(... 

```

meaning that `Poisson` is declared only here and when it is called (by simply writing `Poisson;`) then (2.5) will be solved by the Conjugate Gradient method.

## 2.2 Classification of partial differential equations

**Summary :** It is usually not easy to determine the type of a system. Yet the approximations and algorithms suited to the problem depend on its type:

- Finite Elements compatible (LBB conditions) for elliptic systems
- Finite difference on the parabolic variable and a time loop on each elliptic subsystem of parabolic systems; better stability diagrams when the schemes are implicit in time.
- Upwinding, Petrov-Galerkin, Characteristics-Galerkin, Discontinuous-Galerkin, Finite Volumes for hyperbolic systems plus, possibly, a time loop.

When the system changes type, then expect difficulties (like shock discontinuities) !

### Elliptic, parabolic and hyperbolic equations

A partial differential equation (PDE) is a relation between a function of several variables and its derivatives.

$$F \left( \varphi(x), \frac{\partial \varphi}{\partial x_1}(x), \dots, \frac{\partial \varphi}{\partial x_d}(x), \frac{\partial^2 \varphi}{\partial x_1^2}(x), \dots, \frac{\partial^m \varphi}{\partial x_d^m}(x) \right) = 0, \quad \forall x \in \Omega \subset \mathbb{R}^d$$

The range of  $x$  over which the equation is taken, here  $\Omega$ , is called the *domain* of the PDE. The highest derivation index, here  $m$ , is called the *order*. If  $F$  and  $\varphi$  are vector valued functions, then the PDE is actually a *system* of PDEs.

Unless indicated otherwise, here by convention *one* PDE corresponds to one scalar valued  $F$  and  $\varphi$ . If  $F$  is linear with respect to its arguments, then the PDE is said to be *linear*.

The general form of a second order, linear scalar PDE is  $\frac{\partial^2 \varphi}{\partial x_i \partial x_j}$  and  $A : B$  means  $\sum_{i,j=1}^d a_{ij} b_{ij}$ .

$$\alpha \varphi + a \cdot \nabla \varphi + B : \nabla(\nabla \varphi) = f \quad \text{in } \Omega \subset \mathbb{R}^d,$$

where  $f(x), \alpha(x) \in \mathbb{R}$ ,  $a(x) \in \mathbb{R}^d$ ,  $B(x) \in \mathbb{R}^{d \times d}$  are the PDE *coefficients*. If the coefficients are independent of  $x$ , the PDE is said to have *constant coefficients*.

To a PDE we associate a quadratic form, by replacing  $\varphi$  by 1,  $\partial \varphi / \partial x_i$  by  $z_i$  and  $\partial^2 \varphi / \partial x_i \partial x_j$  by  $z_i z_j$ , where  $z$  is a vector in  $\mathbb{R}^d$ :

$$\alpha + A \cdot z + z^T B z = f.$$

If it is the equation of an ellipse (ellipsoid if  $d \geq 2$ ), the PDE is said to be *elliptic*; if it is the equation of a parabola or a hyperbola, the PDE is said to be *parabolic* or *hyperbolic*.

If  $A \equiv 0$ , the degree is no longer 2 but 1, and for reasons that will appear more clearly later, the PDE is still said to be *hyperbolic*.

These concepts can be generalized to systems, by studying whether or not the polynomial system  $P(z)$  associated with the PDE system has branches at infinity (ellipsoids have no branches at infinity, paraboloids have one, and hyperboloids have several).

If the PDE is not linear, it is said to be *non-linear*. Those are said to be locally elliptic, parabolic, or hyperbolic according to the type of the linearized equation.

For example, for the non-linear equation

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial \varphi}{\partial x} \frac{\partial^2 \varphi}{\partial x^2} = 1$$

we have  $d = 2$ ,  $x_1 = t$ ,  $x_2 = x$  and its linearized form is:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial x} \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial \varphi}{\partial x} \frac{\partial^2 u}{\partial x^2} = 0$$

which for the unknown  $u$  is locally elliptic if  $\frac{\partial \varphi}{\partial x} < 0$  and locally hyperbolic if  $\frac{\partial \varphi}{\partial x} > 0$ .

**Tip:** Laplace's equation is elliptic:

$$\Delta \varphi \equiv \frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} + \cdots + \frac{\partial^2 \varphi}{\partial x_d^2} = f, \quad \forall x \in \Omega \subset \mathbb{R}^d$$

**Tip:** The *heat* equation is parabolic in  $Q = \Omega \times ]0, T[ \subset \mathbb{R}^{d+1}$ :

$$\frac{\partial \varphi}{\partial t} - \mu \Delta \varphi = f \quad \forall x \in \Omega \subset \mathbb{R}^d, \quad \forall t \in ]0, T[$$

**Tip:** If  $\mu > 0$ , the *wave* equation is hyperbolic:

$$\frac{\partial^2 \varphi}{\partial t^2} - \mu \Delta \varphi = f \quad \text{in } Q.$$

---

**Tip:** The *convection diffusion* equation is parabolic if  $\mu \neq 0$  and hyperbolic otherwise:

$$\frac{\partial \varphi}{\partial t} + a \nabla \varphi - \mu \Delta \varphi = f$$


---

---

**Tip:** The *biharmonic* equation is elliptic:

$$\Delta(\Delta \varphi) = f \text{ in } \Omega.$$


---

### Boundary conditions

A relation between a function and its derivatives is not sufficient to define the function. Additional information on the boundary  $\Gamma = \partial\Omega$  of  $\Omega$ , or on part of  $\Gamma$  is necessary. Such information is called a *boundary condition*.

For example:

$$\varphi(x) \text{ given, } \forall x \in \Gamma,$$

is called a *Dirichlet boundary condition*. The *Neumann* condition is

$$\frac{\partial \varphi}{\partial \mathbf{n}}(x) \text{ given on } \Gamma \text{ (or } \mathbf{n} \cdot B \nabla \varphi, \text{ given on } \Gamma \text{ for a general second order PDE)}$$

where  $\mathbf{n}$  is the normal at  $x \in \Gamma$  directed towards the exterior of  $\Omega$  (by definition  $\frac{\partial \varphi}{\partial \mathbf{n}} = \nabla \varphi \cdot \mathbf{n}$ ).

Another classical condition, called a *Robin* (or *Fourier*) condition is written as:

$$\varphi(x) + \beta(x) \frac{\partial \varphi}{\partial n}(x) \text{ given on } \Gamma.$$

Finding a set of boundary conditions that defines a unique  $\varphi$  is a difficult art.

In general, an elliptic equation is well posed (*i.e.*  $\varphi$  is unique) with one Dirichlet, Neumann or Robin condition on the whole boundary.

Thus, Laplace's equation is well posed with a Dirichlet or Neumann condition but also with :

$$\varphi \text{ given on } \Gamma_1, \frac{\partial \varphi}{\partial n} \text{ given on } \Gamma_2, \Gamma_1 \cup \Gamma_2 = \Gamma, \Gamma_1 \cap \Gamma_2 = \emptyset.$$

Parabolic and hyperbolic equations rarely require boundary conditions on all of  $\Gamma \times ]0, T[$ . For instance, the heat equation is well posed with :

$$\varphi \text{ given at } t = 0 \text{ and Dirichlet or Neumann or mixed conditions on } \partial\Omega.$$

Here  $t$  is time so the first condition is called an initial condition. The whole set of conditions is also called Cauchy condition.

The wave equation is well posed with :

$$\varphi \text{ and } \frac{\partial \varphi}{\partial t} \text{ given at } t = 0 \text{ and Dirichlet or Neumann or mixed conditions on } \partial\Omega.$$

## 2.3 Membrane

**Summary :** Here we shall learn how to solve a Dirichlet and/or mixed Dirichlet Neumann problem for the Laplace operator with application to the equilibrium of a membrane under load. We shall also check the accuracy of the method and interface with other graphics packages

An elastic membrane  $\Omega$  is attached to a planar rigid support  $\Gamma$ , and a force  $f(x)dx$  is exerted on each surface element  $dx = dx_1 dx_2$ . The vertical membrane displacement,  $\varphi(x)$ , is obtained by solving Laplace's equation:

$$-\Delta\varphi = f \text{ in } \Omega$$

As the membrane is fixed to its planar support, one has:

$$\varphi|_{\Gamma} = 0$$

If the support wasn't planar but had an elevation  $z(x_1, x_2)$  then the boundary conditions would be of non-homogeneous Dirichlet type.

$$\varphi|_{\Gamma} = z$$

If a part  $\Gamma_2$  of the membrane border  $\Gamma$  is not fixed to the support but is left hanging, then due to the membrane's rigidity the angle with the normal vector  $n$  is zero; thus the boundary conditions are:

$$\varphi|_{\Gamma_1} = z, \quad \frac{\partial\varphi}{\partial n}|_{\Gamma_2} = 0$$

where  $\Gamma_1 = \Gamma - \Gamma_2$ ; recall that  $\frac{\partial\varphi}{\partial n} = \nabla\varphi \cdot n$ . Let us recall also that the Laplace operator  $\Delta$  is defined by:

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2}$$

**Todo:** Check references

With such “mixed boundary conditions” the problem has a unique solution (see Dautray-Lions (1988), Strang (1986) and Raviart-Thomas (1983)). The easiest proof is to notice that  $\varphi$  is the state of least energy, i.e.

$$E(\phi) = \min_{\varphi-z \in V} E(v), \quad \text{with} \quad E(v) = \int_{\Omega} \left( \frac{1}{2} |\nabla v|^2 - fv \right)$$

and where  $V$  is the subspace of the Sobolev space  $H^1(\Omega)$  of functions which have zero trace on  $\Gamma_1$ . Recall that ( $x \in \mathbb{R}^d$ ,  $d = 2$  here):

$$H^1(\Omega) = \{u \in L^2(\Omega) : \nabla u \in (L^2(\Omega))^d\}$$

Calculus of variation shows that the minimum must satisfy, what is known as the weak form of the PDE or its variational formulation (also known here as the theorem of virtual work)

$$\int_{\Omega} \nabla\varphi \cdot \nabla w = \int_{\Omega} fw \quad \forall w \in V$$

Next an integration by parts (Green's formula) will show that this is equivalent to the PDE when second derivatives exist.

**Warning:** Unlike Freefem+ which had both weak and strong forms, **FreeFEM** implements only weak formulations. It is not possible to go further in using this software if you don't know the weak form (i.e. variational formulation) of your problem: either you read a book, or ask help from a colleague or drop the matter. Now if you want to solve a system of PDE like  $A(u, v) = 0$ ,  $B(u, v) = 0$  don't close this manual, because in weak form it is

$$\int_{\Omega} (A(u, v)w_1 + B(u, v)w_2) = 0 \quad \forall w_1, w_2 \dots$$

### Example

Let an ellipse have the length of the semimajor axis  $a = 2$ , and unitary the semiminor axis. Let the surface force be  $f = 1$ . Programming this case with **FreeFEM** gives:

```

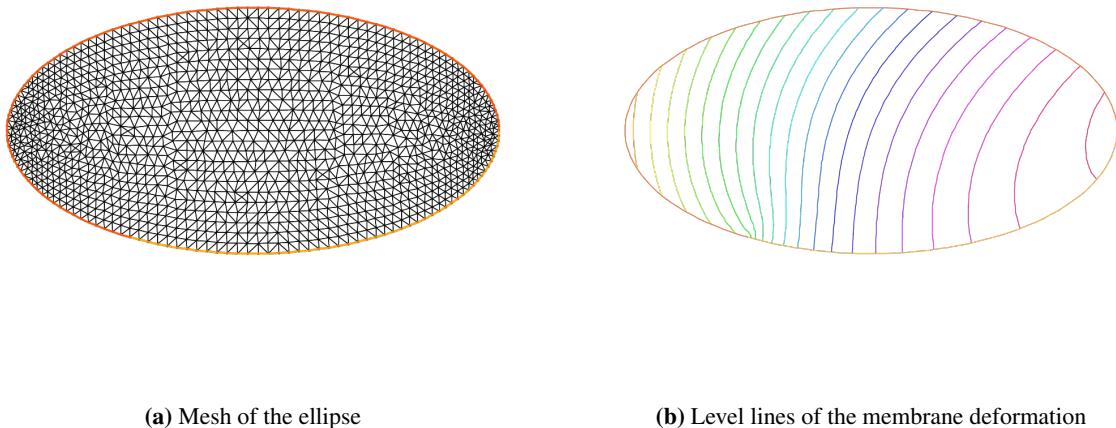
1 // Parameters
2 real theta = 4.*pi/3.;
3 real a = 2.; //The length of the semimajor axis
4 real b = 1.; //The length of the semiminor axis
5 func z = x;
6
7 // Mesh
8 border Gamma1(t=0., theta) {x=a*cos(t); y=b*sin(t);}
9 border Gamma2(t=theta, 2.*pi) {x=a*cos(t); y=b*sin(t);}
10 mesh Th = buildmesh(Gamma1(100) + Gamma2(50));
11
12 // Fespace
13 fespace Vh(Th, P2); //P2 conforming triangular FEM
14 Vh phi, w, f=1;
15
16 // Solve
17 solve Laplace(phi, w)
18   = int2d(Th) (
19     dx(phi)*dx(w)
20     + dy(phi)*dy(w)
21   )
22   - int2d(Th) (
23     f*w
24   )
25   + on(Gamma1, phi=z)
26 ;
27
28 // Plot
29 plot(phi, wait=true, ps="membrane.eps"); //Plot phi
30 plot(Th, wait=true, ps="membraneTh.eps"); //Plot Th
31
32 // Save mesh
33 savemesh(Th, "Th.msh");

```

A triangulation is built by the keyword `buildmesh`. This keyword calls a triangulation subroutine based on the Delaunay test, which first triangulates with only the boundary points, then adds internal points by subdividing the edges. How fine the triangulation becomes is controlled by the size of the closest boundary edges.

The PDE is then discretized using the triangular second order finite element method on the triangulation; as was briefly indicated in the previous chapter, a linear system is derived from the discrete formulation whose size is the number of vertices plus the number of mid-edges in the triangulation.

The system is solved by a multi-frontal Gauss LU factorization implemented in the package `UMFPACK`.



**Fig. 2.3:** Membrane

The keyword `plot` will display both  $\mathbb{T}_h$  and  $\varphi$  (remove  $\mathbb{T}_h$  if  $\varphi$  only is desired) and the qualifier `fill=true` replaces the default option (colored level lines) by a full color display.

```
plot(phi,wait=true,fill=true); //Plot phi with full color display
```

Results are on Fig. 2.3a and Fig. 2.3b.

Next we would like to check the results !

One simple way is to adjust the parameters so as to know the solutions. For instance on the unit circle  $a=1$ ,  $\varphi_e = \sin(x^2 + y^2 - 1)$  solves the problem when:

$$z = 0, f = -4(\cos(x^2 + y^2 - 1) - (x^2 + y^2)\sin(x^2 + y^2 - 1))$$

except that on  $\Gamma_2$   $\partial_n \varphi = 2$  instead of zero. So we will consider a non-homogeneous Neumann condition and solve:

$$\int_{\Omega} \nabla \varphi \cdot \nabla w = \int_{\Omega} fw + \int_{\Gamma_2} 2w \quad \forall w \in V$$

We will do that with two triangulations, compute the  $L^2$  error:

$$\epsilon = \int_{\Omega} |\varphi - \varphi_e|^2$$

and print the error in both cases as well as the log of their ratio an indication of the rate of convergence.

```

1 // Parameters
2 verbosity = 0; //to remove all default output
3 real theta = 4.*pi/3.;
4 real a=1.; //the length of the semimajor axis
5 real b=1.; //the length of the semiminor axis
6 func f = -4*(cos(x^2+y^2-1) - (x^2+y^2)*sin(x^2+y^2-1));
7 func phiexact = sin(x^2 + y^2 - 1);
8
9 // Mesh

```

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```

10 border Gamma1 (t=0., theta) {x=a*cos(t); y=b*sin(t);}
11 border Gamma2 (t=theta, 2.*pi) {x=a*cos(t); y=b*sin(t);}
12
13 // Error loop
14 real[int] L2error(2); //an array of two values
15 for(int n = 0; n < 2; n++) {
16     // Mesh
17     mesh Th = buildmesh(Gamma1(20*(n+1)) + Gamma2(10*(n+1)));
18
19     // Fespace
20     fespace Vh(Th, P2);
21     Vh phi, w;
22
23     // Solve
24     solve Laplace(phi, w)
25         = int2d(Th) (
26             dx(phi)*dx(w)
27             + dy(phi)*dy(w)
28         )
29         - int2d(Th) (
30             f*w
31         )
32         - int1d(Th, Gamma2) (
33             2*w
34         )
35         + on(Gamma1, phi=0)
36     ;
37
38     // Plot
39     plot(Th, phi, wait=true, ps="membrane.eps");
40
41     // Error
42     L2error[n] = sqrt(int2d(Th) ((phi-phiexact)^2));
43 }
44
45 // Display loop
46 for(int n = 0; n < 2; n++)
47     cout << "L2error " << n << " = " << L2error[n] << endl;
48
49 // Convergence rate
50 cout << "convergence rate = " << log(L2error[0]/L2error[1])/log(2.) << endl;

```

The output is:

```

1 L2error 0 = 0.00462991
2 L2error 1 = 0.00117128
3 convergence rate = 1.9829
4 times: compile 0.02s, execution 6.94s

```

We find a rate of 1.93591, which is not close enough to the 3 predicted by the theory.

The Geometry is always a polygon so we lose one order due to the geometry approximation in  $O(h^2)$ .

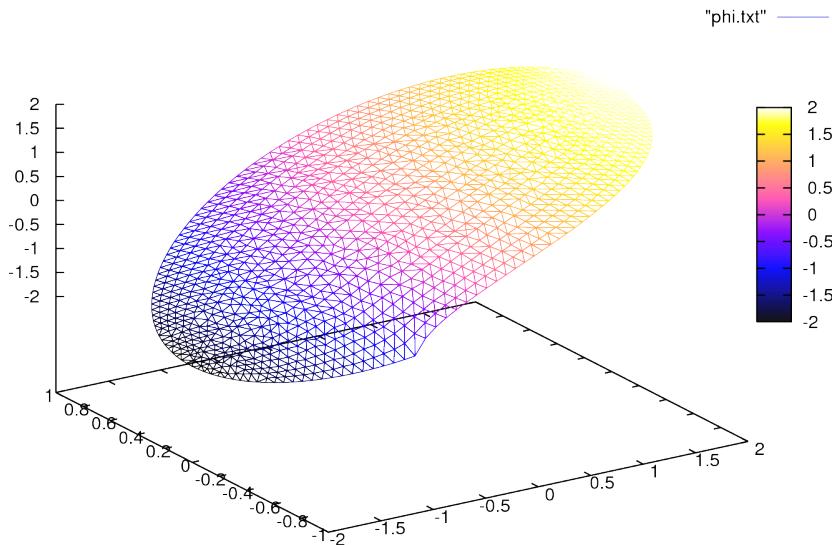
Now if you are not satisfied with the .eps plot generated by **FreeFEM** and you want to use other graphic facilities, then you must store the solution in a file very much like in C++. It will be useless if you don't save the triangulation as well, consequently you must do

```

1 {
2     ofstream ff("phi.txt");
3     ff << phi[];
4 }
5 savemesh(Th, "Th.msh");

```

For the triangulation the name is important: **the extension determines the format.**



**Fig. 2.4:** The 3D version drawn by gnuplot from a file generated by **FreeFEM**

Still that may not take you where you want. Here is an interface with gnuplot to produce the Fig. 2.4.

```

//to build a gnuplot data file
{
    ofstream ff("graph.txt");
    for (int i = 0; i < Th.nt; i++)
    {
        for (int j = 0; j < 3; j++)
            ff << Th[i][j].x << " " << Th[i][j].y << " " << phi[] [Vh(i,j)] << endl;
        ff << Th[i][0].x << " " << Th[i][0].y << " " << phi[] [Vh(i,0)] << "\n\n\n"
    }
}

```

We use the finite element numbering, where  $\text{Wh}(i, j)$  is the global index of  $j^{th}$  degrees of freedom of triangle number  $i$ .

Then open gnuplot and do:

```

1 set palette rgbformulae 30,31,32
2 splot "graph.txt" w l pal

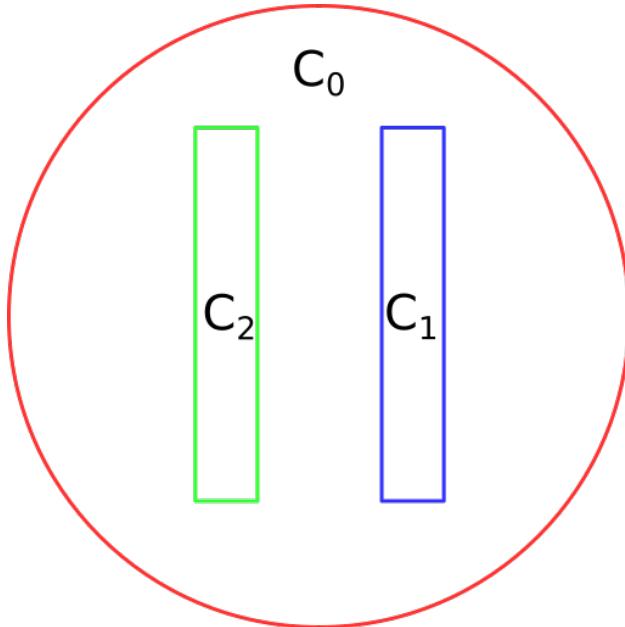
```

This works with P2 and P1, but not with P1nc because the 3 first degrees of freedom of P2 or P2 are on vertices and not with P1nc.

## 2.4 Heat Exchanger

**Summary:** Here we shall learn more about geometry input and triangulation files, as well as read and write operations.

**The problem** Let  $\{C_i\}_{1,2}$ , be 2 thermal conductors within an enclosure  $C_0$  (see Fig. 2.5).



**Fig. 2.5:** Heat exchanger geometry

The first one is held at a constant temperature  $u_1$  the other one has a given thermal conductivity  $\kappa_2$  3 times larger than the one of  $C_0$ .

We assume that the border of enclosure  $C_0$  is held at temperature  $20^\circ C$  and that we have waited long enough for thermal equilibrium.

In order to know  $u(x)$  at any point  $x$  of the domain  $\Omega$ , we must solve:

$$\nabla \cdot (\kappa \nabla u) = 0 \text{ in } \Omega, \quad u|_{\Gamma} = g$$

where  $\Omega$  is the interior of  $C_0$  minus the conductor  $C_1$  and  $\Gamma$  is the boundary of  $\Omega$ , that is  $C_0 \cup C_1$ .

Here  $g$  is any function of  $x$  equal to  $u_i$  on  $C_i$ .

The second equation is a reduced form for:

$$u = u_i \text{ on } C_i, \quad i = 0, 1.$$

The variational formulation for this problem is in the subspace  $H_0^1(\Omega) \subset H^1(\Omega)$  of functions which have zero traces on  $\Gamma$ .

$$u - g \in H_0^1(\Omega) : \int_{\Omega} \nabla u \nabla v = 0 \forall v \in H_0^1(\Omega)$$

Let us assume that  $C_0$  is a circle of radius 5 centered at the origin,  $C_i$  are rectangles,  $C_1$  being at the constant temperature  $u_1 = 60^\circ C$  (so we can only consider its boundary).

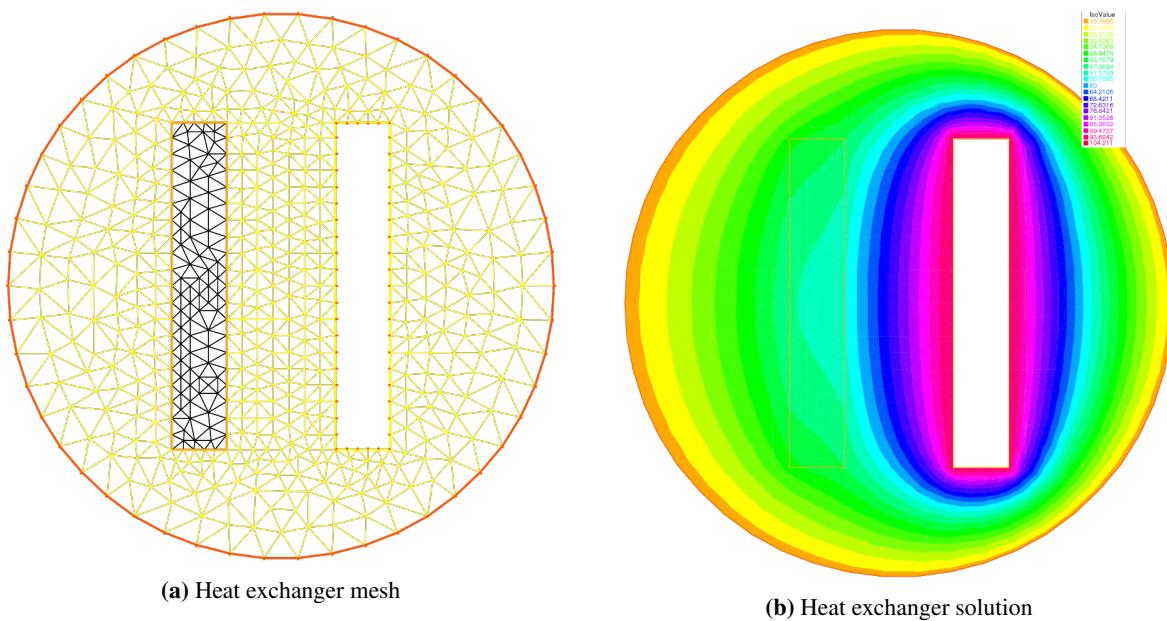
```

1 // Parameters
2 int C1=99;
3 int C2=98; //could be anything such that !=0 and C1!=C2
4
5 // Mesh
6 border C0(t=0., 2.*pi) {x=5.*cos(t); y=5.*sin(t);}
7
8 border C11(t=0., 1.) {x=1.+t; y=3.; label=C1;}
9 border C12(t=0., 1.) {x=2.; y=3.-6.*t; label=C1;}
10 border C13(t=0., 1.) {x=2.-t; y=-3.; label=C1;}
11 border C14(t=0., 1.) {x=1.; y=-3.+6.*t; label=C1;}
12
13 border C21(t=0., 1.) {x=-2.+t; y=3.; label=C2;}
14 border C22(t=0., 1.) {x=-1.; y=3.-6.*t; label=C2;}
15 border C23(t=0., 1.) {x=-1.-t; y=-3.; label=C2;}
16 border C24(t=0., 1.) {x=-2.; y=-3.+6.*t; label=C2;}
17
18 plot( C0(50) //to see the border of the domain
19     + C11(5)+C12(20)+C13(5)+C14(20)
20     + C21(-5)+C22(-20)+C23(-5)+C24(-20),
21     wait=true, ps="heatexb.eps");
22
23 mesh Th=buildmesh(C0(50)
24     + C11(5)+C12(20)+C13(5)+C14(20)
25     + C21(-5)+C22(-20)+C23(-5)+C24(-20));
26
27 plot(Th,wait=1);
28
29 // Fespace
30 fespace Vh(Th, P1);
31 Vh u, v;
32 Vh kappa=1 + 2*(x<-1)*(x>-2)*(y<3)*(y>-3);
33
34 // Solve
35 solve a(u, v)
36 = int2d(Th) (
37     kappa*
38         dx(u)*dx(v)
39         + dy(u)*dy(v)
40     )
41
42 +on(C0, u=20)
43 +on(C1, u=60)
44 ;
45
46 // Plot
47 plot(u, wait=true, value=true, fill=true, ps="HeatExchanger.eps");

```

Note the following:

- C0 is oriented counterclockwise by  $t$ , while C1 is oriented clockwise and C2 is oriented counterclockwise. This is why C1 is viewed as a hole by buildmesh.
- C1 and C2 are built by joining pieces of straight lines. To group them in the same logical unit to input the boundary conditions in a readable way we assigned a label on the boundaries. As said earlier, borders have an internal number corresponding to their order in the program (check it by adding a `cout << C22;` above). This is essential to understand how a mesh can be output to a file and re-read (see below).
- As usual the mesh density is controlled by the number of vertices assigned to each boundary. It is not possible



**Fig. 2.6:** Heat exchanger

to change the (uniform) distribution of vertices but a piece of boundary can always be cut in two or more parts, for instance  $C_{12}$  could be replaced by  $C_{121}+C_{122}$ :

```

1 // border C12(t=0.,1.) {x=2.; y=3.-6.*t; label=C1; }
2 border C121(t=0.,0.7) {x=2.; y=3.-6.*t; label=C1; }
3 border C122(t=0.7,1.) {x=2.; y=3.-6.*t; label=C1; }
4 ...
5 buildmesh(.../+* C12(20) */ + C121(12) + C122(8) + ...);

```

### **Tip: Exercise :**

Use the symmetry of the problem with respect to the axes.

Triangulate only one half of the domain, and set Dirichlet conditions on the vertical axis, and Neumann conditions on the horizontal axis.

**Writing and reading triangulation files** Suppose that at the end of the previous program we added the line

```
1 savemesh (Th, "condensor.msh");
```

and then later on we write a similar program but we wish to read the mesh from that file. Then this is how the condenser should be computed:

```

1 // Mesh
2 mesh Sh = readmesh("condensor.msh");
3
4 // Fespace
5 fespace Wh(Sh, P1);
6 Wh us, vs;
7
8 // Solve
9 solve b(us, vs)

```

---

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```

10   = int2d(Sh) (
11     dx(us)*dx(vs)
12     + dy(us)*dy(vs)
13   )
14   +on(1, us=0)
15   +on(99, us=1)
16   +on(98, us=-1)
17   ;
18
19 // Plot
20 plot(us);

```

Note that the names of the boundaries are lost but either their internal number (in the case of C0) or their label number (for C1 and C2) are kept.

## 2.5 Acoustics

**Summary :** *Here we go to grip with ill posed problems and eigenvalue problems*

Pressure variations in air at rest are governed by the wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0$$

When the solution wave is monochromatic (and that depends on the boundary and initial conditions),  $u$  is of the form  $u(x, t) = Re(v(x)e^{ikt})$  where  $v$  is a solution of Helmholtz's equation:

$$\begin{aligned} k^2 v + c^2 \Delta v &= 0 && \text{in } \Omega \\ \frac{\partial v}{\partial n}|_{\Gamma} &= g \end{aligned}$$

where  $g$  is the source.

Note the “+” sign in front of the Laplace operator and that  $k > 0$  is real. This sign may make the problem ill posed for some values of  $\frac{c}{k}$ , a phenomenon called “resonance”.

At resonance there are non-zero solutions even when  $g = 0$ . So the following program may or may not work:

```

1 // Parameters
2 real kc2 = 1.;
3 func g = y*(1.-y);
4
5 // Mesh
6 border a0(t=0., 1.) {x=5.; y=1.+2.*t;}
7 border a1(t=0., 1.) {x=5.-2.*t; y=3.;}
8 border a2(t=0., 1.) {x=3.-2.*t; y=3.-2.*t;}
9 border a3(t=0., 1.) {x=1.-t; y=1.;}
10 border a4(t=0., 1.) {x=0.; y=1.-t;}
11 border a5(t=0., 1.) {x=t; y=0.;}
12 border a6(t=0., 1.) {x=1.+4.*t; y=t;}
13
14 mesh Th = buildmesh(a0(20) + a1(20) + a2(20)
15   + a3(20) + a4(20) + a5(20) + a6(20));
16
17 // Fespace
18 fespace Vh(Th, P1);
19 Vh u, v;

```

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```

20
21 // Solve
22 solve sound(u, v)
23   = int2d(Th) (
24     u*v * kc2
25     - dx(u)*dx(v)
26     - dy(u)*dy(v)
27   )
28   - int1d(Th, a4) (
29     g * v
30   )
31 ;
32
33 // Plot
34 plot(u, wait=1, ps="Sound.eps");

```

Results are on Fig. 2.7a. But when  $kc2$  is an eigenvalue of the problem, then the solution is not unique:

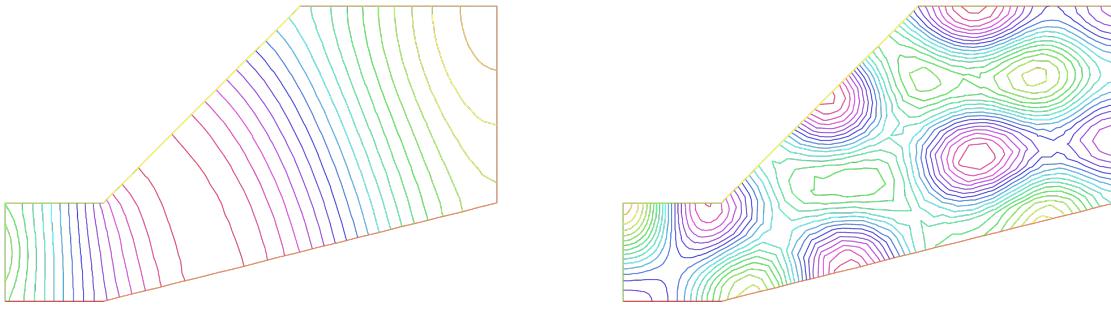
- if  $u_e \neq 0$  is an eigen state, then for any given solution  $u + u_e$  is **another** solution.

To find all the  $u_e$  one can do the following :

```

1 // Parameters
2 real sigma = 20; //value of the shift
3
4 // Problem
5 // OP = A - sigma B ; // The shifted matrix
6 varf op(u1, u2)
7   = int2d(Th) (
8     dx(u1)*dx(u2)
9     + dy(u1)*dy(u2)
10    - sigma* u1*u2
11  )
12 ;
13
14 varf b([u1], [u2])
15   = int2d(Th) (
16     u1*u2
17   )
18 ; // No Boundary condition see note \ref{note BC EV}
19
20 matrix OP = op(Vh, Vh, solver=Crout, factorize=1);
21 matrix B = b(Vh, Vh, solver=CG, eps=1e-20);
22
23 // Eigen values
24 int nev=2; // Number of requested eigenvalues near sigma
25
26 real[int] ev(nev); // To store the nev eigenvalue
27 Vh[int] eV(nev); // To store the nev eigenvector
28
29 int k=EigenValue(OP, B, sym=true, sigma=sigma, value=ev, vector=eV,
30   tol=1e-10, maxit=0, ncv=0);
31
32 cout << ev(0) << " 2 eigen values " << ev(1) << endl;
33 v = eV[0];
34 plot(v, wait=true, ps="eigen.eps");

```



(a) Amplitude of an acoustic signal coming from the left vertical wall.

(b) First eigen state ( $\lambda = (k/c)^2 = 19.4256$ ) close to 20 of eigenvalue problem:  $-\Delta\varphi = \lambda\varphi$  and  $\frac{\partial\varphi}{\partial n} = 0$  on  $\Gamma$

**Fig. 2.7:** Acoustics

## 2.6 Thermal Conduction

**Summary :** Here we shall learn how to deal with a time dependent parabolic problem. We shall also show how to treat an axisymmetric problem and show also how to deal with a nonlinear problem

### How air cools a plate

We seek the temperature distribution in a plate  $(0, Lx) \times (0, Ly) \times (0, Lz)$  of rectangular cross section  $\Omega = (0, 6) \times (0, 1)$ ; the plate is surrounded by air at temperature  $u_e$  and initially at temperature  $u = u_0 + \frac{x}{L}u_1$ . In the plane perpendicular to the plate at  $z = Lz/2$ , the temperature varies little with the coordinate  $z$ ; as a first approximation the problem is 2D.

We must solve the temperature equation in  $\Omega$  in a time interval  $(0, T)$ .

$$\begin{aligned} \partial_t u - \nabla \cdot (\kappa \nabla u) &= 0 && \text{in } \Omega \times (0, T) \\ u(x, y, 0) &= u_0 + xu_1 \\ \kappa \frac{\partial u}{\partial n} + \alpha(u - u_e) &= 0 && \text{on } \Gamma \times (0, T) \end{aligned}$$

Here the diffusion  $\kappa$  will take two values, one below the middle horizontal line and ten times less above, so as to simulate a thermostat.

The term  $\alpha(u - u_e)$  accounts for the loss of temperature by convection in air. Mathematically this boundary condition is of Fourier (or Robin, or mixed) type.

The variational formulation is in  $L^2(0, T; H^1(\Omega))$ ; in loose terms and after applying an implicit Euler finite difference approximation in time; we shall seek  $u^n(x, y)$  satisfying for all  $w \in H^1(\Omega)$ :

$$\int_{\Omega} \left( \frac{u^n - u^{n-1}}{\delta t} w + \kappa \nabla u^n \nabla w \right) + \int_{\Gamma} \alpha(u^n - u_e) w = 0$$

```

1 // Parameters
2 func u0 = 10. + 90.*x/6.;
3 func k = 1.8*(y<0.5) + 0.2;

```

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```

4   real ue = 25.;
5   real alpha=0.25;
6   real T=5.;
7   real dt=0.1 ;
8
9   // Mesh
10  mesh Th = square(30, 5, [6.*x,y]);
11
12 // Fespace
13 fespace Vh(Th, P1);
14 Vh u=u0, v, uold;
15
16 // Problem
17 problem thermic(u, v)
18   = int2d(Th) (
19     u*v/dt
20     + k*(
21       dx(u) * dx(v)
22       + dy(u) * dy(v)
23     )
24   )
25   + int1d(Th, 1, 3) (
26     alpha*u*v
27   )
28   - int1d(Th, 1, 3) (
29     alpha*ue*v
30   )
31   - int2d(Th) (
32     uold*v/dt
33   )
34   + on(2, 4, u=u0)
35 ;
36
37 // Time iterations
38 ofstream ff("thermic.dat");
39 for(real t = 0; t < T; t += dt){
40   uold = u; //equivalent to  $u^{n-1} = u^n$ 
41   thermic; //here the thermic problem is solved
42   ff << u(3., 0.5) << endl;
43   plot(u);
44 }

```

---

**Note:** We must separate by hand the bilinear part from the linear one.

---

**Note:** The way we store the temperature at point (3, 0.5) for all times in file `thermic.dat`. Should a one dimensional plot be required, the same procedure can be used. For instance to print  $x \mapsto \frac{\partial u}{\partial y}(x, 0.9)$  one would do:

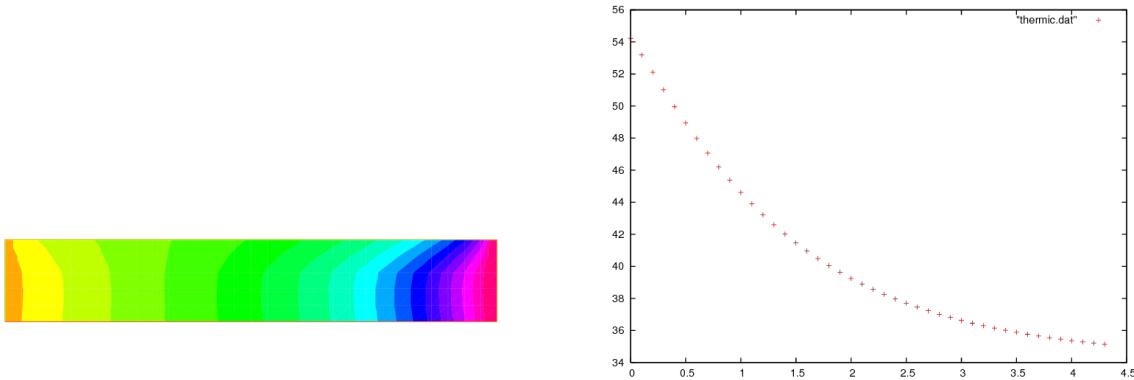
```

1 for(int i = 0; i < 20; i++)
2   cout << dy(u)(6.0*i/20.0, 0.9) << endl;

```

---

Results are shown on Fig. 2.8a and Fig. 2.8b.

(a) Temperature at  $t = 4.9$ .**Fig. 2.8:** Thermal conduction

### 2.6.1 Axisymmetry: 3D Rod with circular section

Let us now deal with a cylindrical rod instead of a flat plate. For simplicity we take  $\kappa = 1$ .

In cylindrical coordinates, the Laplace operator becomes ( $r$  is the distance to the axis,  $z$  is the distance along the axis,  $\theta$  polar angle in a fixed plane perpendicular to the axis):

$$\Delta u = \frac{1}{r} \partial_r(r \partial_r u) + \frac{1}{r^2} \partial_\theta^2 u + \partial_z^2 u.$$

Symmetry implies that we loose the dependence with respect to  $\theta$ ; so the domain  $\Omega$  is again a rectangle  $[0, R] \times [0, L]$ . We take the convention of numbering of the edges as in `square()` (1 for the bottom horizontal ...); the problem is now:

$$\begin{aligned} r \partial_t u - \partial_r(r \partial_r u) - \partial_z(r \partial_z u) &= 0 && \text{in } \Omega \\ u(t=0) &= u_0 + \frac{z}{L_z}(u_1 - u_0) \\ u|_{\Gamma_4} &= u_0 \\ u|_{\Gamma_2} &= u_1 \\ \alpha(u - u_e) + \frac{\partial u}{\partial n}|_{\Gamma_1 \cup \Gamma_3} &= 0 \end{aligned}$$

Note that the PDE has been multiplied by  $r$ .

After discretization in time with an implicit scheme, with time steps `dt`, in the **FreeFEM** syntax  $r$  becomes  $x$  and  $z$  becomes  $y$  and the problem is:

```

1 problem thermazi(u, v)
2   = int2d(Th) (
3     (u*v/dt + dx(u)*dx(v) + dy(u)*dy(v))*x
4   )
5   + int1d(Th, 3) (
6     alpha*x*u*v
7   )
8   - int1d(Th, 3) (
9     alpha*x*ue*v
10  )

```

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```

11   - int2d(Th) (
12     uold*v*x/dt
13   )
14   + on(2, 4, u=u0);

```

**Note:** The bilinear form degenerates at  $x = 0$ . Still one can prove existence and uniqueness for  $u$  and because of this degeneracy no boundary conditions need to be imposed on  $\Gamma_1$ .

## 2.6.2 A Nonlinear Problem : Radiation

Heat loss through radiation is a loss proportional to the absolute temperature to the fourth power (Stefan's Law). This adds to the loss by convection and gives the following boundary condition:

$$\kappa \frac{\partial u}{\partial n} + \alpha(u - u_e) + c[(u + 273)^4 - (u_e + 273)^4] = 0$$

The problem is nonlinear, and must be solved iteratively. If  $m$  denotes the iteration index, a semi-linearization of the radiation condition gives

$$\frac{\partial u^{m+1}}{\partial n} + \alpha(u^{m+1} - u_e) + c(u^{m+1} - u_e)(u^m + u_e + 546)((u^m + 273)^2 + (u_e + 273)^2) = 0,$$

because we have the identity  $a^4 - b^4 = (a - b)(a + b)(a^2 + b^2)$ .

The iterative process will work with  $v = u - u_e$ .

```

1 ...
2 // Parameters
3 real rad=1e-8;
4 real uek=ue+273;
5
6 // Mesh
7 fespace Vh(Th, P1);
8 Vh vold, w, v=u0-ue, b;
9
10 // Problem
11 problem thermradia(v, w)
12   = int2d(Th) (
13     v*w/dt
14     + k*(dx(v) * dx(w) + dy(v) * dy(w))
15   )
16   + int1d(Th, 1, 3) (
17     b*v*w
18   )
19   - int2d(Th) (
20     vold*w/dt
21   )
22   + on(2, 4, v=u0-ue)
23 ;
24
25 for (real t=0; t<T; t+=dt) {
26   vold = v;
27   for (int m = 0; m < 5; m++) {
28     b = alpha + rad * (v + 2*uek) * ((v+uek)^2 + uek^2);

```

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```

29     thermradia;
30 }
31 }
32 vold = v+ue;
33
34 // Plot
35 plot(vold);

```

## 2.7 Irrotational Fan Blade Flow and Thermal effects

**Summary :** Here we will learn how to deal with a multi-physics system of PDEs on a complex geometry, with multiple meshes within one problem. We also learn how to manipulate the region indicator and see how smooth is the projection operator from one mesh to another.

### Incompressible flow

Without viscosity and vorticity incompressible flows have a velocity given by:

$$u = \begin{pmatrix} \frac{\partial \psi}{\partial x_2} \\ -\frac{\partial \psi}{\partial x_1} \end{pmatrix}, \quad \text{where } \psi \text{ is solution of } \Delta \psi = 0$$

This equation expresses both incompressibility ( $\nabla \cdot u = 0$ ) and absence of vortex ( $\nabla \times u = 0$ ).

As the fluid slips along the walls, normal velocity is zero, which means that  $\psi$  satisfies:

$$\psi \text{ constant on the walls.}$$

One can also prescribe the normal velocity at an artificial boundary, and this translates into non constant Dirichlet data for  $\psi$ .

### Airfoil

Let us consider a wing profile  $S$  in a uniform flow. Infinity will be represented by a large circle  $C$  where the flow is assumed to be of uniform velocity; one way to model this problem is to write:

$$\Delta \psi = 0 \text{ in } \Omega, \quad \psi|_S = 0, \quad \psi|_C = u_\infty y$$

where  $\partial\Omega = C \cup S$

### The NACA0012 Airfoil

An equation for the upper surface of a NACA0012 (this is a classical wing profile in aerodynamics) is:

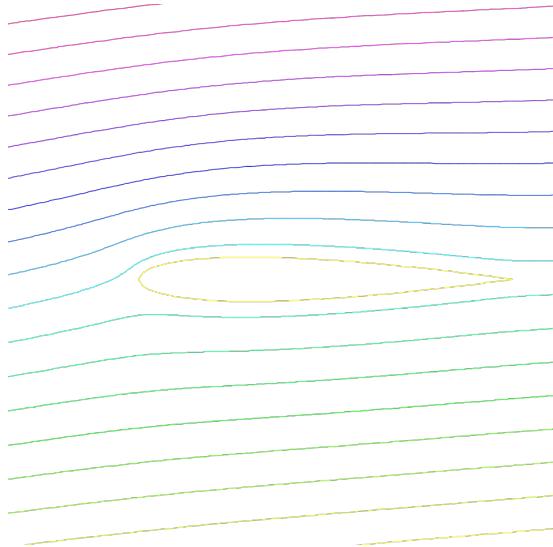
$$y = 0.17735\sqrt{x} - 0.075597x - 0.212836x^2 + 0.17363x^3 - 0.06254x^4.$$

```

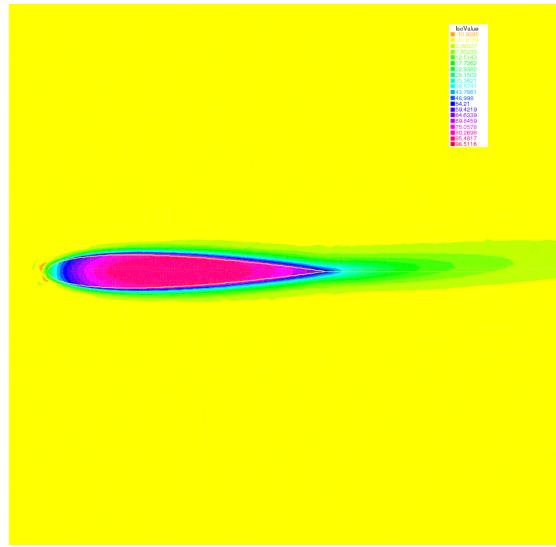
1 // Parameters
2 real S = 99;
3
4 // Mesh
5 border C(t=0., 2.*pi){x=5.*cos(t); y=5.*sin(t);}
6 border Splus(t=0., 1.){x=t; y=0.17735*sqrt(t) - 0.075597*t
- 0.212836*(t^2) + 0.17363*(t^3) - 0.06254*(t^4); label=S;}
7 border Sminus(t=1., 0.){x=t; y=-(0.17735*sqrt(t) - 0.075597*t
- 0.212836*(t^2) + 0.17363*(t^3) - 0.06254*(t^4)); label=S;}
8
9

```

(continues on next page)



(a) Zoom around the NACA0012 airfoil showing the streamlines (curve  $\psi = \text{constant}$ ). To obtain such a plot use the interactive graphic command: “+” and p.



(b) Temperature distribution at time  $T=25$  (now the maximum is at 90 instead of 120). Note that an incidence angle has been added here.

**Fig. 2.9:** The NACA0012 Airfoil

(continued from previous page)

```

10 mesh Th = buildmesh(C(50) + Splus(70) + Sminus(70));
11
12 // Fespace
13 fespace Vh(Th, P2);
14 Vh psi, w;
15
16 // Solve
17 solve potential(psi, w)
18   = int2d(Th) (
19     dx(psi)*dx(w)
20     +dy(psi)*dy(w)
21   )
22   + on(C, psi = y)
23   + on(S, psi=0)
24   ;
25
26 plot(psi, wait=1);

```

A zoom of the streamlines are shown on Fig. 2.9a.

### 2.7.1 Heat Convection around the airfoil

Now let us assume that the airfoil is hot and that air is there to cool it. Much like in the previous section the heat equation for the temperature  $v$  is

$$\partial_t v - \nabla \cdot (\kappa \nabla v) + u \cdot \nabla v = 0, \quad v(t=0) = v_0, \quad \frac{\partial v}{\partial n}|_C = 0$$

But now the domain is outside AND inside  $S$  and  $\kappa$  takes a different value in air and in steel. Furthermore there is convection of heat by the flow, hence the term  $u \cdot \nabla v$  above.

Consider the following, to be plugged at the end of the previous program:

```

1 // Parameters
2 real S = 99;
3 real dt=0.05;
4 real nbT=50;
5
6 // Mesh
7 border C(t=0., 2.*pi){x=5.*cos(t); y=5.*sin(t);}
8 border Splus(t=0., 1.){x=t; y=0.17735*sqrt(t) - 0.075597*t
- 0.212836*(t^2) + 0.17363*(t^3) - 0.06254*(t^4); label=S;}
9 border Sminus(t=1., 0.){x=t; y=-(0.17735*sqrt(t) - 0.075597*t
- 0.212836*(t^2) + 0.17363*(t^3) - 0.06254*(t^4)); label=S;}
10 border D(t=0., 2.){x=1.+t; y=0.;} // Added to have a fine mesh at trail
11 mesh Sh = buildmesh(C(25) + Splus(-90) + Sminus(-90) + D(200));
12 int steel=Sh(0.5,0).region, air=Sh(-1,0).region;
13
14
15 // Fespaces
16 fespace Vh(Sh, P2);
17 Vh psi, w;
18
19
20 fespace Wh(Sh, P1);
21 Wh v, vv;
22
23 fespace W0(Sh,P0);
24 W0 k=0.01*(region==air)+0.1*(region==steel);
25 W0 u1=dy(psi)*(region==air), u2=-dx(psi)*(region==air);
26 Wh vold = 120*(region==steel);
27
28 // Problem
29 int i;
30 problem thermic(v, vv, init=i, solver=LU)
31   = int2d(Sh)(
32     v*vv/dt
33     + k*(dx(v) * dx(vv) + dy(v) * dy(vv))
34     + 10*(u1*dx(v)+u2*dy(v)) * vv
35   )
36   - int2d(Sh)(
37     vold*vv/dt
38   )
39 ;
40
41 for(i = 0; i < nbT; i++){
42   v = vold;
43   thermic;
44   plot(v);
45 }
```

**Note:** How steel and air are identified by the mesh parameter region which is defined when buildmesh is called and takes an integer value corresponding to each connected component of  $\Omega$ ;

How the convection terms are added without upwinding. Upwinding is necessary when the Peclet number  $|u|L/\kappa$  is large (here is a typical length scale). The factor 10 in front of the convection terms is a quick way of multiplying the velocity by 10 (else it is too slow to see something).

The solver is Gauss' LU factorization and when  $\text{init} \neq 0$  the LU decomposition is reused so it is much faster after the first iteration.

## 2.8 Pure Convection : The Rotating Hill

**Summary:** Here we will present two methods for upwinding for the simplest convection problem. We will learn about Characteristics-Galerkin and Discontinuous-Galerkin Finite Element Methods.

Let  $\Omega$  be the unit disk centered at  $(0, 0)$ ; consider the rotation vector field

$$\mathbf{u} = [u_1, u_2], \quad u_1 = y, \quad u_2 = -x$$

Pure convection by  $\mathbf{u}$  is

$$\begin{aligned}\partial_t c + \mathbf{u} \cdot \nabla c &= 0 && \text{in } \Omega \times (0, T) \\ c(t=0) &= c^0 && \text{in } \Omega.\end{aligned}$$

The exact solution  $c(x_t, t)$  at time  $t$  en point  $x_t$  is given by:

$$c(x_t, t) = c^0(x, 0)$$

where  $x_t$  is the particle path in the flow starting at point  $x$  at time 0. So  $x_t$  are solutions of

$$\dot{x}_t = u(x_t), \quad x_{t=0} = x, \quad \text{where} \quad \dot{x}_t = \frac{d(t \mapsto x_t)}{dt}$$

The ODE are reversible and we want the solution at point  $x$  at time  $t$  (not at point  $x_t$ ) the initial point is  $x_{-t}$ , and we have

$$c(x, t) = c^0(x_{-t}, 0)$$

The game consists in solving the equation until  $T = 2\pi$ , that is for a full revolution and to compare the final solution with the initial one; they should be equal.

### 2.8.1 Solution by a Characteristics-Galerkin Method

In FreeFEM there is an operator called `convect([u1,u2], dt, c)` which compute  $c \circ X$  with  $X$  is the convect field defined by  $X(x) = x_{dt}$  and where  $x_\tau$  is particule path in the steady state velocity field  $\mathbf{u} = [u_1, u_2]$  starting at point  $x$  at time  $\tau = 0$ , so  $x_\tau$  is solution of the following ODE:

$$\dot{x}_\tau = u(x_\tau), \mathbf{x}_{\tau=0} = x.$$

When  $\mathbf{u}$  is piecewise constant; this is possible because  $x_\tau$  is then a polygonal curve which can be computed exactly and the solution exists always when  $\mathbf{u}$  is divergence free; `convect` returns  $c(x_{df}) = C \circ X$ .

```

1 // Parameters
2 real dt = 0.17;
3
4 // Mesh
5 border C(t=0., 2.*pi) {x=cos(t); y=sin(t);}
6 mesh Th = buildmesh(C(100));
7
8 // Fespace
9 fespace Uh(Th, P1);
10 Uh cold, c = exp(-10*((x-0.3)^2 + (y-0.3)^2));
11 Uh u1 = y, u2 = -x;
12
13 // Time loop

```

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```

14 real t = 0;
15 for (int m = 0; m < 2.*pi/dt; m++) {
16     t += dt;
17     cold = c;
18     c = convect([u1, u2], -dt, cold);
19     plot(c, cmm=" t="+t+", min="+c[].min+", max="+c[].max);
20 }

```

**Note:** 3D plots can be done by adding the qualifyer `dim=3` to the plot instruction.

The method is very powerful but has two limitations:

- it is not conservative
- it may diverge in rare cases when  $|\mathbf{u}|$  is too small due to quadrature error.

## 2.8.2 Solution by Discontinuous-Galerkin FEM

Discontinuous Galerkin methods take advantage of the discontinuities of  $c$  at the edges to build upwinding. There are many formulations possible. We shall implement here the so-called dual- $P_1^{DC}$  formulation (see [ERN2006]):

$$\int_{\Omega} \left( \frac{c^{n+1} - c^n}{\delta t} + \mathbf{u} \cdot \nabla c \right) w + \int_E (\alpha |\mathbf{n} \cdot \mathbf{u}| - \frac{1}{2} \mathbf{n} \cdot \mathbf{u}) [c] w = \int_{E_{\Gamma}^-} |\mathbf{n} \cdot \mathbf{u}| c w \quad \forall w$$

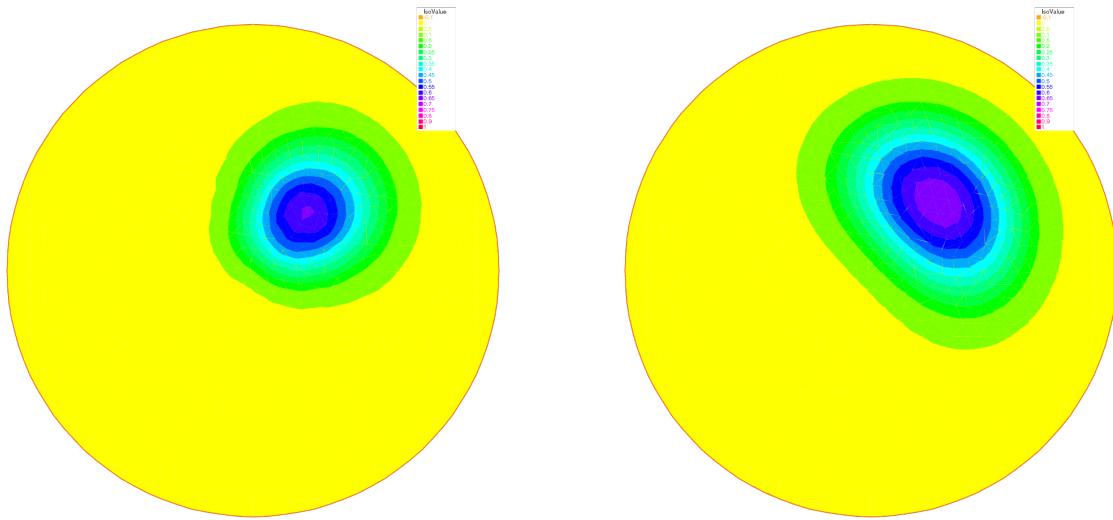
where  $E$  is the set of inner edges and  $E_{\Gamma}^-$  is the set of boundary edges where  $\mathbf{u} \cdot \mathbf{n} < 0$  (in our case there is no such edges). Finally  $[c]$  is the jump of  $c$  across an edge with the convention that  $c^+$  refers to the value on the right of the oriented edge.

```

1 // Parameters
2 real al=0.5;
3 real dt = 0.05;
4
5 // Mesh
6 border C(t=0., 2.*pi) {x=cos(t); y=sin(t);}
7 mesh Th = buildmesh(C(100));
8
9 // Fespace
10 fespace Vh(Th,P1dc);
11 Vh w, ccold, v1 = y, v2 = -x, cc = exp(-10*((x-0.3)^2 +(y-0.3)^2));
12
13 // Macro
14 macro n() (N.x*v1 + N.y*v2) // Macro without parameter
15
16 // Problem
17 problem Adual(cc, w)
18     = int2d(Th) (
19         (cc/dt + (v1*dx(cc)+v2*dy(cc))) *w
20     )
21     + intalledges(Th) (
22         (1-nTonEdge)*w*(al*abs(n)-n/2)*jump(cc)
23     )
24     - int2d(Th) (
25         ccold*w/dt
26     )

```

(continues on next page)



(a) The rotating hill after one revolution with Characteristics-Galerkin

(b) The rotating hill after one revolution with Discontinuous  $P_1$  Galerkin**Fig. 2.10:** Rotating hill

(continued from previous page)

```

27 ;
28
29 // Time iterations
30 for (real t = 0.; t < 2.*pi; t += dt) {
31   ccold = cc;
32   Adual;
33   plot(cc, fill=1, cmm="t="+t+", min="+cc[].min+", max="+ cc[].max);
34 }
35
36 // Plot
37 real [int] viso = [-0.2, -0.1, 0., 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1., 1.
38   ↪1];
39 plot(cc, wait=1, fill=1, ps="ConvectCG.eps", viso=viso);
plot(cc, wait=1, fill=1, ps="ConvectDG.eps", viso=viso);

```

**Note:** New keywords: intalledges to integrate on all edges of all triangles

$$\text{intalledges}(\text{Th}) \equiv \sum_{T \in \text{Th}} \int_{\partial T}$$

(so all internal edges are seen two times), nTonEdge which is one if the triangle has a boundary edge and two otherwise, jump to implement  $[c]$ .

Results of both methods are shown on Fig. 2.10a nad Fig. 2.10b with identical levels for the level line; this is done with the plot-modifier viso.

Notice also the macro where the parameter  $\mathbf{u}$  is not used (but the syntax needs one) and which ends with a //; it simply replaces the name  $n$  by  $(N.x*v1+N.y*v2)$ . As easily guessed  $N.x, N.y$  is the normal to the edge.

Now if you think that DG is too slow try this:

```

1 // Parameters
2 real al=0.5;
3 real dt = 0.05;
4
5 // Mesh
6 border C(t=0., 2.*pi) {x=cos(t); y=sin(t);}
7 mesh Th = buildmesh(C(100));
8
9 // Fespace
10 fespace Vh(Th,P1dc);
11 Vh w, ccold, v1 = y, v2 = -x, cc = exp(-10*((x-0.3)^2 +(y-0.3)^2));
12 Vh rhs=0;
13
14 // Macro
15 macro n() (N.x*v1 + N.y*v2) // Macro without parameter
16
17 // Problem
18 real t = 0.;
19
20 varf vAdual (cc, w)
21   = int2d(Th) (
22     (cc/dt+(v1*dx(cc)+v2*dy(cc)))*w
23   )
24   + intalldges(Th) (
25     (1-nTonEdge)*w*(al*abs(n)-n/2)*jump(cc)
26   )
27 ;
28
29 varf vBdual (cc, w)
30   = - int2d(Th) (
31     ccold*w/dt
32   )
33 ;
34
35 matrix AA = vAdual(Vh, Vh);
36 matrix BB = vBdual(Vh, Vh);
37 set (AA, init=t, solver=sparse);
38
39 // Time iterations
40 for (t = 0.; t < 2.*pi; t += dt) {
41   ccold = cc;
42   rhs[] = BB * ccold[];
43   cc[] = AA^-1 * rhs[];
44   plot(cc, fill=1, cmm="t="+t+", min="+cc[].min+", max="+cc[].max);
45 }
```

Notice the new keyword `set` to specify a solver in this framework; the modifier `init` is used to tell the solver that the matrix has not changed (`init=true`), and the name parameter are the same that in problem definition (see [Problem](#))

### 2.8.3 Finite Volume Methods can also be handled with FreeFEM but it requires programming.

For instance the  $P_0 - P_1$  Finite Volume Method of Dervieux *et al* associates to each  $P_0$  function  $c^1$  a  $P_0$  function  $c^0$  with constant value around each vertex  $q^i$  equal to  $c^1(q^i)$  on the cell  $\sigma_i$  made by all the medians of all triangles having  $q^i$  as vertex.

Then upwinding is done by taking left or right values at the median:

$$\int_{\sigma_i} \frac{1}{\delta t} (c^{1^{n+1}} - c^{1^n}) + \int_{\partial\sigma_i} u \cdot n c^- = 0, \forall i$$

It can be programmed as :

```

1 load "mat_dervieux"; //External module in C++ must be loaded
2
3 // Parameters
4 real dt = 0.025;
5
6 // Mesh
7 border a(t=0., 2.*pi){x=cos(t); y=sin(t);}
8 mesh th = buildmesh(a(100));
9
10 // Fespace
11 fespace Vh(th,P1);
12 Vh vold, vold_y, u1=-x;
13 Vh v=exp(-10*((x-0.3)^2 +(y-0.3)^2)), vWall=0, rhs=0;
14
15 // Problem
16 //qf1pTlump means mass lumping is used
17 problem FVM(v,vh) = int2d(th,qft=qf1pTlump)(v*vh/dt)
18   - int2d(th,qft=qf1pTlump)(vold*vh/dt)
19   + int1d(th,a)((u1*N.x+u2*N.y)<0)*(u1*N.x+u2*N.y)*vWall*vh
20 + rhs[];
21
22 matrix A;
23 MatUpWind0(A, th, vold, [u1, u2]);
24
25 // Time loop
26 for (int t = 0; t < 2.*pi ; t += dt) {
27   vold = v;
28   rhs[] = A * vold[];
29   FVM;
30   plot(v, wait=0);
31 }
```

the “mass lumping” parameter forces a quadrature formula with Gauss points at the vertices so as to make the mass matrix diagonal; the linear system solved by a conjugate gradient method for instance will then converge in one or two iterations.

The right hand side `rhs` is computed by an external C++ function `MatUpWind0(...)` which is programmed as :

```

1 // Computes matrix a on a triangle for the Dervieux FVM
2 int fvmP1P0(double q[3][2], // the 3 vertices of a triangle T
3             double u[2], // convection velocity on T
4             double c[3], // the P1 function on T
5             double a[3][3], // output matrix
6             double where[3]) // where>0 means we're on the boundary
{
7
8   for (int i = 0; i < 3; i++)
9     for(int j = 0; j < 3; j++) a[i][j] = 0;
10
11  for(int i = 0; i < 3; i++){
12    int ip = (i+1)%3, ipp = (ip+1)%3;
13    double unL = -(q[ip][1] + q[i][1] - 2*q[ipp][1])*u[0]
14      - (q[ip][0] + q[i][0] - 2*q[ipp][0])*u[1]);
```

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```

15     if (unL > 0) {
16         a[i][i] += unL;
17         a[ip][i] -= unL;
18     }
19     else{
20         a[i][ip] += unL;
21         a[ip][ip] -= unL;
22     }
23     if (where[i] && where[ip]) { // this is a boundary edge
24         unL = ((q[ip][1] - q[i][1])*u[0] - (q[ip][0] - q[i][0])*u[1])/2;
25         if (unL > 0){
26             a[i][i] += unL;
27             a[ip][ip] += unL;
28         }
29     }
30 }
31 return 1;
32 }
```

It must be inserted into a larger .cpp file, shown in Appendix A, which is the load module linked to **FreeFEM**.

## 2.9 The System of elasticity

### Elasticity

Solid objects deform under the action of applied forces:

a point in the solid, originally at  $(x, y, z)$  will come to  $(X, Y, Z)$  after some time; the vector  $\mathbf{u} = (u_1, u_2, u_3) = (X - x, Y - y, Z - z)$  is called the displacement. When the displacement is small and the solid is elastic, Hooke's law gives a relationship between the stress tensor  $\sigma(u) = (\sigma_{ij}(u))$  and the strain tensor  $\epsilon(u) = \epsilon_{ij}(u)$

$$\sigma_{ij}(u) = \lambda \delta_{ij} \nabla \cdot \mathbf{u} + 2\mu \epsilon_{ij}(u),$$

where the Kronecker symbol  $\delta_{ij} = 1$  if  $i = j$ , 0 otherwise, with

$$\epsilon_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

and where  $\lambda, \mu$  are two constants that describe the mechanical properties of the solid, and are themselves related to the better known constants  $E$ , Young's modulus, and  $\nu$ , Poisson's ratio:

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}.$$

### Lamé's system

Let us consider a beam with axis  $Oz$  and with perpendicular section  $\Omega$ . The components along  $x$  and  $y$  of the strain  $\mathbf{u}(x)$  in a section  $\Omega$  subject to forces  $\mathbf{f}$  perpendicular to the axis are governed by:

$$-\mu \Delta \mathbf{u} - (\mu + \lambda) \nabla(\nabla \cdot \mathbf{u}) = \mathbf{f} \text{ in } \Omega,$$

where  $\lambda, \mu$  are the Lamé coefficients introduced above.

Remark, we do not use this equation because the associated variational form does not give the right boundary condition, we simply use:

$$-\operatorname{div}(\sigma) = \mathbf{f} \text{ in } \Omega$$

where the corresponding variational form is:

$$\int_{\Omega} \sigma(u) : \epsilon(\mathbf{v}) \, dx - \int_{\Omega} \mathbf{v} f \, dx = 0;$$

where  $:$  denotes the tensor scalar product, i.e.  $a : b = \sum_{i,j} a_{ij} b_{ij}$ .

So the variational form can be written as :

$$\int_{\Omega} \lambda \nabla \cdot u \nabla \cdot v + 2\mu \epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) \, dx - \int_{\Omega} \mathbf{v} f \, dx = 0;$$

---

**Tip:** Consider an elastic plate with the undeformed rectangle shape  $[0, 20] \times [-1, 1]$ .

The body force is the gravity force  $\mathbf{f}$  and the boundary force  $\mathbf{g}$  is zero on lower, upper and right sides. The left vertical side of the beam is fixed. The boundary conditions are:

$$\begin{aligned} \sigma \cdot \mathbf{n} &= \mathbf{g} = 0 && \text{on } \Gamma_1, \Gamma_4, \Gamma_3, \\ \mathbf{u} &= \mathbf{0} && \text{on } \Gamma_2 \end{aligned}$$

---

Here  $\mathbf{u} = (u, v)$  has two components.

The above two equations are strongly coupled by their mixed derivatives, and thus any iterative solution on each of the components is risky. One should rather use FreeFEM's system approach and write:

```

1 // Parameters
2 real E = 21e5;
3 real nu = 0.28;
4
5 real f = -1;
6
7 // Mesh
8 mesh Th = square(10, 10, [20*x, 2*y-1]);
9
10 // Fespace
11 fespace Vh(Th, P2);
12 Vh u, v;
13 Vh uu, vv;
14
15 // Macro
16 real sqrt2=sqrt(2.);
17 macro epsilon(u1,u2) [dx(u1),dy(u2),(dy(u1)+dx(u2))/sqrt2] //
18 // The sqrt2 is because we want: epsilon(u1,u2)' * epsilon(v1,v2) = epsilon(u) :_
19 // epsilon(v)
20 macro div(u,v) ( dx(u)+dy(v) ) //
21
22 // Problem
23 real mu= E/(2*(1+nu));
24 real lambda = E*nu/((1+nu)*(1-2*nu));
25
26 solve lame([u, v], [uu, vv])
27 = int2d(Th)(
28     lambda * div(u, v) * div(uu, vv)
29     + 2.*mu * ( epsilon(u,v)' * epsilon(uu,vv) )
30 )
31 - int2d(Th)(
32     f*vv

```

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(a) Vector

(b) Deformation

**Fig. 2.11:** Elasticity

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```

32     )
33 + on(4, u=0, v=0)
34 ;
35
36 // Plot
37 real coef=100;
38 plot([u, v], wait=1, ps="lamevect.eps", coef=coef);
39
40 // Move mesh
41 mesh th1 = movemesh(Th, [x+u*coef, y+v*coef]);
42 plot(th1,wait=1,ps="lamedeform.eps");
43
44 // Output
45 real dxmin = u[].min;
46 real dymin = v[].min;
47
48 cout << " - dep. max x = " << dxmin << " y=" << dymin << endl;
49 cout << " dep. (20, 0) = " << u(20, 0) << " " << v(20, 0) << endl;

```

The output is:

```

1 -- square mesh : nb vertices =121 , nb triangles = 200 , nb boundary edges 40
2 -- Solve :
3           min -0.00174137 max 0.00174105
4           min -0.0263154 max 1.47016e-29
5 - dep. max x = -0.00174137 y=-0.0263154
6   dep. (20,0) = -1.8096e-07 -0.0263154
7 times: compile 0.010219s, execution 1.5827s

```

Solution of Lamé's equations for elasticity for a 2D beam deflected by its own weight and clamped by its left vertical side is shown Fig. 2.11a and Fig. 2.11b. Result are shown with a amplification factor equal to 100. The size of the arrow is automatically bound, but the color gives the real length.

## 2.10 The System of Stokes for Fluids

In the case of a flow invariant with respect to the third coordinate (two-dimensional flow), flows at low Reynolds number (for instance micro-organisms) satisfy,

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= 0 \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

where  $\mathbf{u} = (u_1, u_2)$  is the fluid velocity and  $p$  its pressure.

The driven cavity is a standard test. It is a box full of liquid with its lid moving horizontally at speed one. The pressure

and the velocity must be discretized in compatible finite element spaces for the LBB conditions to be satisfied:

$$\sup_{p \in P_h} \frac{(\mathbf{u}, \nabla p)}{|p|} \geq \beta |\mathbf{u}| \quad \forall \mathbf{u} \in U_h$$

```

1 // Parameters
2 int nn = 30;
3
4 // Mesh
5 mesh Th = square(nn, nn);
6
7 // Fespace
8 fespace Uh(Th, P1b);
9 Uh u, v;
10 Uh uu, vv;
11
12 fespace Ph(Th, P1);
13 Ph p, pp;
14
15 // Problem
16 solve stokes ([u, v, p], [uu, vv, pp])
17   = int2d(Th) (
18     dx(u)*dx(uu)
19     + dy(u)*dy(uu)
20     + dx(v)*dx(vv)
21     + dy(v)*dy(vv)
22     + dx(p)*uu
23     + dy(p)*vv
24     + pp*(dx(u) + dy(v))
25     - 1e-10*p*pp
26   )
27   + on(1, 2, 4, u=0, v=0)
28   + on(3, u=1, v=0)
29 ;
30
31 // Plot
32 plot([u, v], p, wait=1);

```

---

**Note:** We add a stabilization term  $-10e-10 * p * pp$  to fix the constant part of the pressure.

---

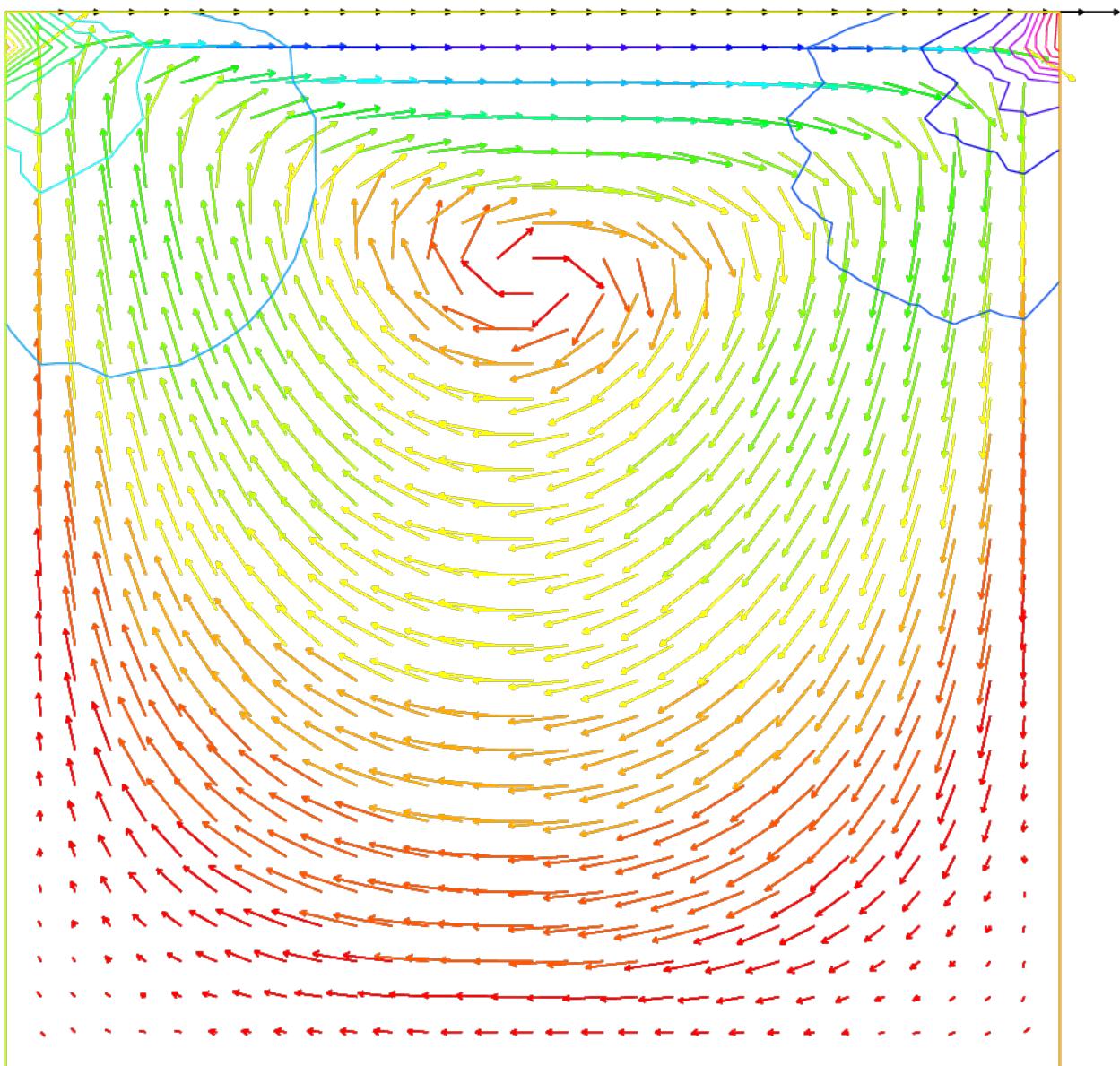
Results are shown on Fig. 2.12.

## 2.11 A projection algorithm for the Navier-Stokes equations

**Summary :** Fluid flows require good algorithms and good triangulations. We show here an example of a complex algorithm and or first example of mesh adaptation.

An incompressible viscous fluid satisfies:

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \nu \Delta \mathbf{u} &= 0 && \text{in } \Omega \times ]0, T[ \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega \times ]0, T[ \\ \mathbf{u}|_{t=0} &= \mathbf{u}^0 \\ \mathbf{u}|_{\Gamma} &= \mathbf{u}_{\Gamma} \end{aligned}$$



**Fig. 2.12:** Solution of Stokes' equations for the driven cavity problem, showing the velocity field and the pressure level lines.

A possible algorithm, proposed by Chorin, is:

$$\begin{aligned} \frac{1}{\delta t} [\mathbf{u}^{m+1} - \mathbf{u}^m o \mathbf{X}^m] + \nabla p^m - \nu \Delta \mathbf{u}^m &= 0 \\ \mathbf{u}|_{\Gamma} &= \mathbf{u}_{\Gamma} \\ \nu \partial_n \mathbf{u}|_{\Gamma_{out}} &= 0 \\ -\Delta p^{m+1} &= -\nabla \cdot \mathbf{u}^m o \mathbf{X}^m \\ \partial_n p^{m+1} &= 0 \quad \text{on } \Gamma \\ p^{m+1} &= 0 \quad \text{on } \Gamma_{out} \end{aligned}$$

where  $\mathbf{u}o\mathbf{X}(x) = \mathbf{u}(x - \mathbf{u}(x)\delta t)$  since  $\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}$  is approximated by the method of characteristics, as in the previous section.

We use the Chorin's algorithm with free boundary condition at outlet (i.e.  $p = 0, \nu \partial_n u = 0$ ), to compute a correction,  $q$ , to the pressure.

$$\begin{aligned} -\Delta q &= \nabla \cdot \mathbf{u} \\ q &= 0 \text{ on } \Gamma_{out} \end{aligned}$$

and define

$$\begin{aligned} \mathbf{u}^{m+1} &= \tilde{\mathbf{u}} + P \nabla q \delta t \\ p^{m+1} &= p^m - q \end{aligned}$$

where  $\tilde{\mathbf{u}}$  is the  $(\mathbf{u}^{m+1}, v^{m+1})$  of Chorin's algorithm, and where  $P$  is the  $L^2$  projection with mass lumping ( a sparse matrix).

### The backward facing step

The geometry is that of a channel with a backward facing step so that the inflow section is smaller than the outflow section. This geometry produces a fluid recirculation zone that must be captured correctly.

This can only be done if the triangulation is sufficiently fine, or well adapted to the flow.

---

**Note:** There is a technical difficulty in the example: the output B.C. Here we put  $p = 0$  and  $\nu \partial_n u = 0$ .

---

```

1 // Parameters
2 verbosity = 0;
3 int nn = 1;
4 real nu = 0.0025;
5 real dt = 0.2;
6 real epsv = 1e-6;
7 real epsu = 1e-6;
8 real epsp = 1e-6;
9
10 // Mesh
11 border a0 (t=1, 0) {x=-2; y=t; label=1;};
12 border a1 (t=-2, 0) {x=t; y=0; label=2;};
13 border a2 (t=0, -0.5) {x=0; y=t; label=2;};
14 border a3 (t=0, 1) {x=18*t1.2; y=-0.5; label=2;};
15 border a4 (t=-0.5, 1) {x=18; y=t; label=3;};
16 border a5 (t=1, 0) {x=-2+20*t; y=1; label=4;};
17
18 mesh Th = buildmesh(a0(3*nn) + a1(20*nn) + a2(10*nn) + a3(150*nn) + a4(5*nn) +_
19 ↵ a5(100*nn));
20 plot(Th);
21 // Fespace

```

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```

22 fespace Vh(Th, P1);
23 Vh w;
24 Vh u=0, v=0;
25 Vh p=0;
26 Vh q=0;
27
28 // Definition of Matrix dtMx and dtMy
29 matrix dtM1x, dtM1y;
30
31 // Macro
32 macro BuildMat()
33 { /* for memory managenent */
34     varf vM(unused, v) = int2d(Th)(v);
35     varf vdx(u, v) = int2d(Th)(v*dx(u)*dt);
36     varf vdy(u, v) = int2d(Th)(v*dy(u)*dt);
37
38     real[int] Mlump = vM(0, Vh);
39     real[int] one(Vh.ndof); one = 1;
40     real[int] M1 = one ./ Mlump;
41     matrix dM1 = M1;
42     matrix Mdx = vdx(Vh, Vh);
43     matrix Mdy = vdy(Vh, Vh);
44     dtM1x = dM1*Mdx;
45     dtM1y = dM1*Mdy;
46 } //
47
48 // Build matrices
49 BuildMat
50
51 // Time iterations
52 real err = 1.;
53 real outflux = 1.;
54 for(int n = 0; n < 300; n++) {
55     // Update
56     Vh uold=u, vold=v, pold=p;
57
58     // Solve
59     solve pb4u (u, w, init=n, solver=CG, eps=epsu)
60         = int2d(Th) (
61             u*w/dt
62             + nu*(dx(u)*dx(w) + dy(u)*dy(w))
63         )
64         - int2d(Th) (
65             convect([uold, vold], -dt, uold)/dt*w
66             - dx(p)*w
67         )
68         + on(1, u=4*y*(1-y))
69         + on(2, 4, u=0)
70         ;
71
72     plot(u);
73
74     solve pb4v (v, w, init=n, solver=CG, eps=epsv)
75         = int2d(Th) (
76             v*w/dt
77             + nu*(dx(v)*dx(w) + dy(v)*dy(w))
78         )

```

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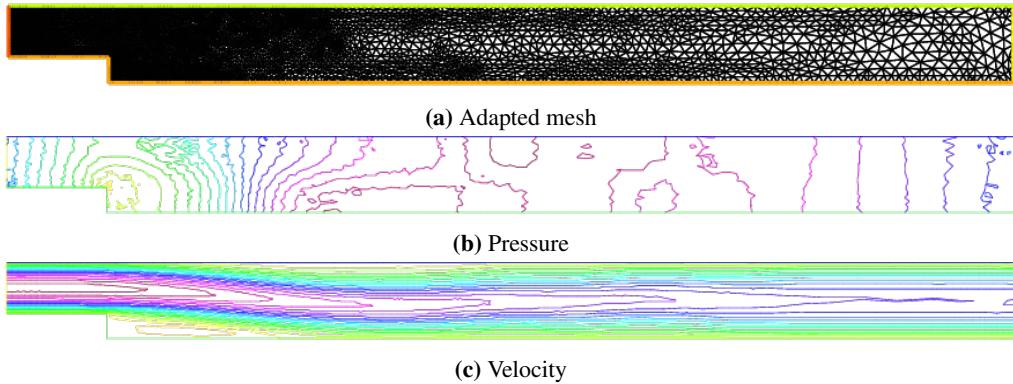
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```

79      -int2d(Th) (
80          convect ([uold,vold],-dt,vold)/dt*w
81          - dy(p)*w
82      )
83      +on(1, 2, 3, 4, v=0)
84      ;
85
86      solve pb4p (q, w, solver=CG, init=n, eps=epsp)
87      = int2d(Th) (
88          dx(q)*dx(w)+dy(q)*dy(w)
89      )
90      - int2d(Th) (
91          (dx(u)+ dy(v))*w/dt
92      )
93      + on(3, q=0)
94      ;
95
96 //to have absolute epsilon in CG algorithm.
97 epsv = -abs(epsv);
98 epsu = -abs(epsu);
99 epss = -abs(epsp);
100
101 p = pold-q;
102 u[] += dtM1x*q[];
103 v[] += dtM1y*q[];
104
105 // Mesh adaptation
106 if (n%50 == 49){
107     Th = adaptmesh(Th, [u, v], q, err=0.04, nbvx=100000);
108     plot(Th, wait=true);
109     BuildMat // Rebuild mat.
110 }
111
112 // Error & Outflux
113 err = sqrt(int2d(Th)(square(u-uold)+square(v-vold))/Th.area);
114 outflux = int1d(Th)([u,v]'*[N.x,N.y]);
115 cout << " iter " << n << " Err L2 = " << err << " outflux = " << outflux << endl;
116 if(err < 1e-3) break;
117 }
118
119 // Verification
120 assert(abs(outflux)< 2e-3);
121
122 // Plot
123 plot(p, wait=1, ps="NSprojP.eps");
124 plot(u, wait=1, ps="NSprojU.eps");

```

Rannacher's projection algorithm: result on an adapted mesh, Fig. 2.13a, showing the pressure, Fig. 2.13b, and the horizontal velocity Fig. 2.13c for a Reynolds number of 400 where mesh adaptation is done after 50 iterations on the first mesh.

**Fig. 2.13:** Navier-Stokes projection

## 2.12 Newton Method for the Steady Navier-Stokes equations

The problem is find the velocity field  $\mathbf{u} = (u_i)_{i=1}^d$  and the pressure  $p$  of a Flow satisfying in the domain  $\Omega \subset \mathbb{R}^d (d = 2, 3)$ :

$$\begin{aligned} (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p &= 0 \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

where  $\nu$  is the viscosity of the fluid,  $\nabla = (\partial_i)_{i=1}^d$ , the dot product is  $\cdot$ , and  $\Delta = \nabla \cdot \nabla$  with the same boundary conditions ( $\mathbf{u}$  is given on  $\Gamma$ ).

The weak form is find  $\mathbf{u}, p$  such that for  $\forall \mathbf{v}$  (zero on  $\Gamma$ ), and  $\forall q$ :

$$\int_{\Omega} ((\mathbf{u} \cdot \nabla) \mathbf{u}) \cdot \mathbf{v} + \nu \nabla \mathbf{u} : \nabla \mathbf{v} - p \nabla \cdot \mathbf{v} - q \nabla \cdot \mathbf{u} = 0$$

The Newton Algorithm to solve nonlinear problem is:

Find  $u \in V$  such that  $F(u) = 0$  where  $F : V \mapsto V$ .

1. choose  $u_0 \in \mathbb{R}^n$  ;
2. for ( $i = 0; i < \text{niter}; i = i + 1$ )
  1. solve  $DF(u_i)w_i = F(u_i)$ ;
  2.  $u_{i+1} = u_i - w_i$ ;

break  $\|w_i\| < \varepsilon$ .

Where  $DF(u)$  is the differential of  $F$  at point  $u$ , this is a linear application such that:

$$F(u + \delta) = F(u) + DF(u)\delta + o(\delta)$$

For Navier Stokes,  $F$  and  $DF$  are:

$$\begin{aligned} F(\mathbf{u}, p) &= \int_{\Omega} ((\mathbf{u} \cdot \nabla) \mathbf{u}) \cdot \mathbf{v} + \nu \nabla \mathbf{u} : \nabla \mathbf{v} - p \nabla \cdot \mathbf{v} - q \nabla \cdot \mathbf{u} \\ DF(\mathbf{u}, p)(\delta \mathbf{u}, \delta p) &= \int_{\Omega} ((\delta \mathbf{u} \cdot \nabla) \mathbf{u}) \cdot \mathbf{v} + ((\mathbf{u} \cdot \nabla) \delta \mathbf{u}) \cdot \mathbf{v} \\ &\quad + \nu \nabla \delta \mathbf{u} : \nabla \mathbf{v} - \delta p \nabla \cdot \mathbf{v} - q \nabla \cdot \delta \mathbf{u} \end{aligned}$$

So the Newton algorithm become:

```

1 // Parameters
2 real R = 5.;
3 real L = 15.;
4
5 real nu = 1./50.;
6 real nufinal = 1/200.;
7 real cnu = 0.5;
8
9 real eps = 1e-6;
10
11 verbosity = 0;
12
13 // Mesh
14 border cc(t=0, 2*pi){x=cos(t)/2.; y=sin(t)/2.; label=1;}
15 border ce(t=pi/2, 3*pi/2){x=cos(t)*R; y=sin(t)*R; label=1;}
16 border beb(tt=0, 1){real t=tt^1.2; x=t*L; y=-R; label=1;}
17 border beu(tt=1, 0){real t=tt^1.2; x=t*L; y=R; label=1;}
18 border beo(t=-R, R){x=L; y=t; label=0;}
19 border bei(t=-R/4, R/4){x=L/2; y=t; label=0;}
20 mesh Th = buildmesh(cc(-50) + ce(30) + beb(20) + beu(20) + beo(10) + bei(10));
21 plot(Th);
22
23 //bounding box for the plot
24 func bb = [[-1,-2],[4,2]];
25
26 // Fespace
27 fespace Xh(Th, P2);
28 Xh u1, u2;
29 Xh v1,v2;
30 Xh du1,du2;
31 Xh ulp,u2p;
32
33 fespace Mh(Th,P1);
34 Mh p;
35 Mh q;
36 Mh dp;
37 Mh pp;
38
39 // Macro
40 macro Grad(u1,u2) [dx(u1), dy(u1), dx(u2),dy(u2)] //
41 macro UgradV(u1,u2,v1,v2) [[u1,u2] *[dx(v1),dy(v1)],
42 [u1,u2] *[dx(v2),dy(v2)]] //
43 macro div(u1,u2) (dx(u1) + dy(u2)) //
44
45 // Initialization
46 u1 = (x^2+y^2) > 2;
47 u2 = 0;
48
49 // Viscosity loop
50 while(1){
51     int n;
52     real err=0;
53     // Newton loop
54     for (n = 0; n < 15; n++) {
55         // Newton
56         solve Oseen ([du1, du2, dp], [v1, v2, q])
57             = int2d(Th) (

```

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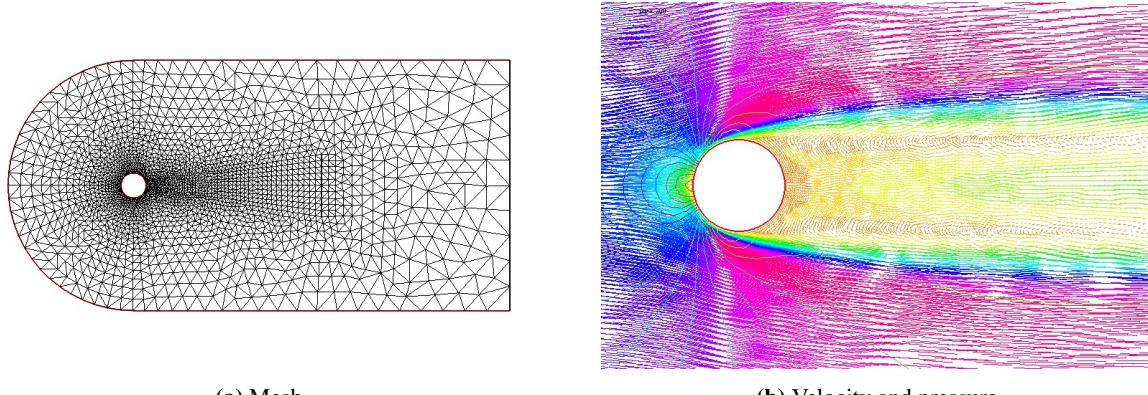
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```

58     nu * (Grad(du1,du2)' * Grad(v1,v2))
59     + UgradV(du1,du2, u1, u2)' * [v1,v2]
60     + UgradV( u1, u2,du1,du2)' * [v1,v2]
61     - div(du1,du2) * q
62     - div(v1,v2) * dp
63     - 1e-8*dp*q //stabilization term
64   )
65   - int2d(Th) (
66     nu * (Grad(u1,u2)' * Grad(v1,v2))
67     + UgradV(u1,u2, u1, u2)' * [v1,v2]
68     - div(u1,u2) * q
69     - div(v1,v2) * p
70   )
71   + on(1, du1=0, du2=0)
72 ;
73
74   u1[] -= du1[];
75   u2[] -= du2[];
76   p[] -= dp[];
77
78   real Lu1=u1[].linfty, Lu2=u2[].linfty, Lp=p[].linfty;
79   err = du1[].linfty/Lu1 + du2[].linfty/Lu2 + dp[].linfty/Lp;
80
81   cout << n << " err = " << err << " " << eps << " rey = " << 1./nu << endl;
82   if(err < eps) break; //converge
83   if( n>3 && err > 10.) break; //blowup
84 }
85
86 if(err < eps){ //converge: decrease $\\nu$ (more difficult)
87   // Plot
88   plot([u1, u2], p, wait=1, cmm=" rey = " + 1./nu , coef=0.3, bb=bb);
89
90   // Change nu
91   if( nu == nufinal) break;
92   if( n < 4) cnu = cnu^1.5; //fast converge => change faster
93   nu = max(nufinal, nu* cnu); //new viscosity
94
95   // Update
96   u1p = u1;
97   u2p = u2;
98   pp = p;
99 }
100 else{ //blowup: increase $\\nu$ (more simple)
101   assert(cnu< 0.95); //the method finally blowup
102
103   // Recover nu
104   nu = nu/cnu;
105   cnu= cnu^(1./1.5); //no conv. => change lower
106   nu = nu* cnu; //new viscosity
107   cout << " restart nu = " << nu << " Rey = " << 1./nu << " (cnu = " << cnu <<
108   ↵" ) \n";
109
110   // Recover a correct solution
111   u1 = u1p;
112   u2 = u2p;
113   p = pp;
}

```

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**Fig. 2.14:** Naver-Stokes newton

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**Note:** We use a trick to make continuation on the viscosity  $\nu$ , because the Newton method blowup owe start with the final viscosity  $\nu$ .

$\nu$  is gradually increased to the desired value.

## 2.13 A Large Fluid Problem

A friend of one of us in Auroville-India was building a ramp to access an air conditioned room. As I was visiting the construction site he told me that he expected to cool air escaping by the door to the room to slide down the ramp and refrigerate the feet of the coming visitors. I told him “no way” and decided to check numerically.

The fluid velocity and pressure are solution of the Navier-Stokes equations with varying density function of the temperature.

The geometry is trapezoidal with prescribed inflow made of cool air at the bottom and warm air above and so are the initial conditions; there is free outflow, slip velocity at the top (artificial) boundary and no-slip at the bottom. However the Navier-Stokes cum temperature equations have a RANS  $k - \epsilon$  model and a Boussinesq approximation for the buoyancy. This comes to :

$$\begin{aligned}
 \partial_t \theta + u \nabla \theta - \nabla \cdot (\kappa_T^m \nabla \theta) &= 0 \\
 \partial_t u + u \nabla u - \nabla \cdot (\mu_T \nabla u) + \nabla p + e(\theta - \theta_0) \vec{e}_2 &= 0 \\
 \nabla \cdot u &= 0 \\
 \mu_T &= c_\mu \frac{k^2}{\epsilon} \\
 \kappa_T &= \kappa \mu_T \\
 \partial_t k + u \nabla k + \epsilon - \nabla \cdot (\mu_T \nabla k) &= \frac{\mu_T}{2} |\nabla u + \nabla u^T|^2 \\
 \partial_t \epsilon + u \nabla \epsilon + c_2 \frac{\epsilon^2}{k} - \frac{c_\epsilon}{c_\mu} \nabla \cdot (\mu_T \nabla \epsilon) &= \frac{c_1}{2} k |\nabla u + \nabla u^T|^2
 \end{aligned}$$

We use a time discretization which preserves positivity and uses the method of characteristics ( $X^m(x) \approx x - u^m(x)\delta t$ )

$$\begin{aligned} \frac{1}{\delta t}(\theta^{m+1} - \theta^m \circ X^m) - \nabla \cdot (\kappa_T^m \nabla \theta^{m+1}) &= 0 \\ \frac{1}{\delta t}(u^{m+1} - u^m \circ X^m) - \nabla \cdot (\mu_T^m \nabla u^{m+1}) + \nabla p^{m+1} + e(\theta^{m+1} - \theta_0) \vec{e}_2 &= 0 \\ \nabla \cdot u^{m+1} &= 0 \\ \frac{1}{\delta t}(k^{m+1} - k^m \circ X^m) + k^{m+1} \frac{\epsilon^m}{k^m} - \nabla \cdot (\mu_T^m \nabla k^{m+1}) &= \frac{\mu_T^m}{2} |\nabla u^m + \nabla u^{mT}|^2 \\ \frac{1}{\delta t}(\epsilon^{m+1} - \epsilon^m \circ X^m) + c_2 \epsilon^{m+1} \frac{\epsilon^m}{k^m} - \frac{c_\epsilon}{c_\mu} \nabla(\mu_T^m \nabla \epsilon^{m+1}) &= \frac{c_1}{2} k^m |\nabla u^m + \nabla u^{mT}|^2 \\ \mu_T^{m+1} &= c_\mu \frac{k^{m+1}}{\epsilon^{m+1}} \\ \kappa_T^{m+1} &= \kappa \mu_T^{m+1} \end{aligned}$$

In variational form and with appropriated boundary conditions the problem is :

```

1  load "iovtk"
2
3  verbosity=0;
4
5 // Parameters
6 int nn = 15;
7 int nnPlus = 5;
8 real l = 1.;
9 real L = 15.;
10 real hSlope = 0.1;
11 real H = 6.;
12 real h = 0.5;
13
14 real reylnods =500;
15 real beta = 0.01;
16
17 real eps = 9.81/303.;
18 real nu = 1;
19 real numu = nu/sqrt(0.09);
20 real nuep = pow(nu,1.5)/4.1;
21 real dt = 0.;

22 real Penalty = 1.e-6;
23
24
25 // Mesh
26 border b1(t=0, l){x=t; y=0;};
27 border b2(t=0, L-l){x=l+t; y=-hSlope*t;};
28 border b3(t=-hSlope*(L-l), H){x=L; y=t;};
29 border b4(t=L, 0){x=t; y=H;};
30 border b5(t=H, h){x=0; y=t;};
31 border b6(t=h, 0){x=0; y=t;};

32
33 mesh Th=buildmesh(b1(nnPlus*nn*l) + b2(nn*sqrt((L-l)^2+(hSlope*(L-l))^2)) + b3(nn*(H_
34 ↪+ hSlope*(L-l))) + b4(nn*L) + b5(nn*(H-h)) + b6(nnPlus*nn*h));
35 plot(Th);

36 // Fespaces
37 fespace Vh2(Th, P1b);
38 Vh2 Ux, Uy;
39 Vh2 Vx, Vy;
40 Vh2 Upx, Upy;
41
42 fespace Vh(Th,P1);
43 Vh p=0, q;
44 Vh Tp, T=35;

```

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```

45 Vh k=0.0001, kp=k;
46 Vh ep=0.0001, epp=ep;
47
48 fespace V0h(Th,P0);
49 V0h muT=1;
50 V0h prodk, prode;
51 Vh kappa=0.25e-4, stress;
52
53 // Macro
54 macro grad(u) [dx(u), dy(u)] //
55 macro Grad(U) [grad(U#x), grad(U#y)] //
56 macro Div(U) (dx(U#x) + dy(U#y)) //
57
58 // Functions
59 func g = (x) * (1-x) * 4;
60
61 // Problem
62 real alpha = 0.;
63
64 problem Temperature(T, q)
65   = int2d(Th) (
66     alpha * T * q
67     + kappa* grad(T)' * grad(q)
68   )
69   + int2d(Th) (
70     - alpha*convect([Upx, Upy], -dt, Tp)*q
71   )
72   + on(b6, T=25)
73   + on(b1, b2, T=30)
74 ;
75
76 problem KineticTurbulence(k, q)
77   = int2d(Th) (
78     (epp/kp + alpha) * k * q
79     + muT* grad(k)' * grad(q)
80   )
81   + int2d(Th) (
82     prodk * q
83     - alpha*convect([Upx, Upy], -dt, kp)*q
84   )
85   + on(b5, b6, k=0.00001)
86   + on(b1, b2, k=beta*numu*stress)
87 ;
88
89 problem ViscosityTurbulence(ep, q)
90   = int2d(Th) (
91     (1.92*epp/kp + alpha) * ep * q
92     + muT * grad(ep)' * grad(q)
93   )
94   + int1d(Th, b1, b2) (
95     T * q * 0.001
96   )
97   + int2d(Th) (
98     prode * q
99     - alpha*convect([Upx, Upy], -dt, epp)*q
100  )
101  + on(b5, b6, ep=0.00001)

```

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```

102 + on (b1, b2, ep=beta*nuep*pow(stress,1.5))
103 ;
104
105 // Initialization with stationary solution
106 solve NavierStokes ([Ux, Uy, p], [Vx, Vy, q])
107 = int2d(Th) (
108     alpha * [Ux, Uy]' * [Vx, Vy]
109     + muT * (Grad(U) : Grad(V))
110     + p * q * Penalty
111     - p * Div(V)
112     - Div(U) * q
113 )
114 + int1d(Th, b1, b2, b4) (
115     Ux * Vx * 0.1
116 )
117 + int2d(Th) (
118     eps * (T-35) * Vx
119     - alpha*convect([Upx, Upy], -dt, Upx)*Vx
120     - alpha*convect([Upx, Upy], -dt, Upy)*Vy
121 )
122 + on (b6, Ux=3, Uy=0)
123 + on (b5, Ux=0, Uy=0)
124 + on (b1, b4, Uy=0)
125 + on (b2, Uy=-Upx*N.x/N.y)
126 + on (b3, Uy=0)
127 ;
128
129 plot([Ux, Uy], p, value=true, coef=0.2, cmm="[Ux, Uy] - p");
130
131 {
132     real[int] xx(21), yy(21), pp(21);
133     for (int i = 0 ; i < 21; i++){
134         yy[i] = i/20.;
135         xx[i] = Ux(0.5,i/20.);
136         pp[i] = p(i/20.,0.999);
137     }
138     cout << " " << yy << endl;
139     plot([xx, yy], wait=true, cmm="Ux x=0.5 cup");
140     plot([yy, pp], wait=true, cmm="p y=0.999 cup");
141 }
142
143 // Initialization
144 dt = 0.1; //probably too big
145 int nbiter = 3;
146 real coefdt = 0.25^(1./nbiter);
147 real coefcut = 0.25^(1./nbiter);
148 real cut = 0.01;
149 real tol = 0.5;
150 real coeftol = 0.5^(1./nbiter);
151 nu = 1./reynods;
152
153 T = T - 10*((x<1)*(y<0.5) + (x>=1)*(y+0.1*(x-1)<0.5));
154
155 // Convergence loop
156 real T0 = clock();
157 for (int iter = 1; iter <= nbiter; iter++) {
158     cout << "Iteration " << iter << " - dt = " << dt << endl;

```

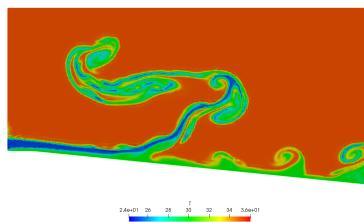
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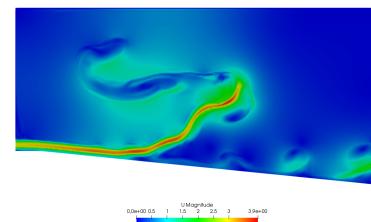
```

159 alpha = 1/dt;
160
161 // Time loop
162 real t = 0.;
163 for (int i = 0; i <= 500; i++) {
164     t += dt;
165     cout << "Time step " << i << " - t = " << t << endl;
166
167     // Update
168     Upx = Ux;
169     Upy = Uy;
170     kp = k;
171     epp = ep;
172     Tp = max(T, 25); //for beauty only should be removed
173     Tp = min(Tp, 35); //for security only should be removed
174     kp = max(k, 0.0001); epp = max(ep, 0.0001); // to be secure: should not be
→active
175     muT = 0.09*kp*kp/epp;
176
177     // Solve NS
178     NavierStokes;
179
180     // Update
181     prode = -0.126*kp*(pow(2*dx(Ux), 2)+pow(2*dy(Uy), 2)+2*pow(dx(Uy)+dy(Ux), 2))/2;
182     prodk = -prode*kp/epp*0.09/0.126;
183     kappa = muT/0.41;
184     stress = abs(dy(Ux));
185
186     // Solve k-eps-T
187     KineticTurbulence;
188     ViscosityTurbulence;
189     Temperature;
190
191     // Plot
192     plot(T, value=true, fill=true);
193     plot([Ux, Uy], p, coef=0.2, cmm=" [Ux, Uy] - p", WindowIndex=1);
194
195     // Time
196     cout << "\tTime = " << clock()-T0 << endl;
197 }
198
199 // Check
200 if (iter >= nbiter) break;
201
202 // Adaptmesh
203 Th = adaptmesh(Th, [dx(Ux), dy(Ux), dx(Ux), dy(Uy)], splitpedge=1, abserror=0,
→cutoff=cut, err=tol, inquire=0, ratio=1.5, hmin=1./1000);
204 plot(Th);
205
206 // Update
207 dt = dt * coefdt;
208 tol = tol * coeftol;
209 cut = cut * coefcut;
210 }
211 cout << "Total Time = " << clock()-T0 << endl;

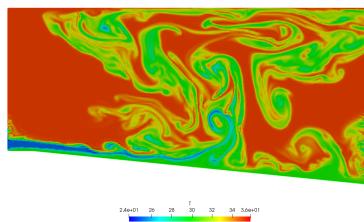
```



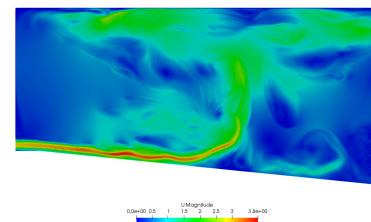
(a) Temperature at time step 100



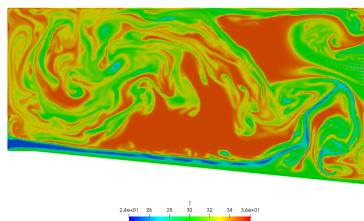
(b) Velocity at time step 100



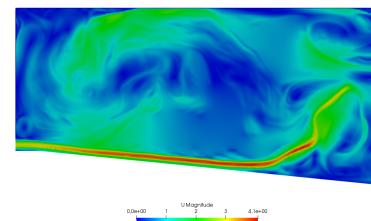
(c) Temperature at time step 200



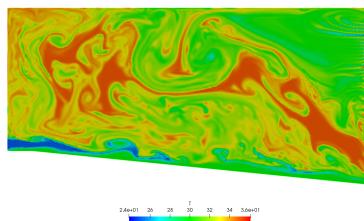
(d) Velocity at time step 200



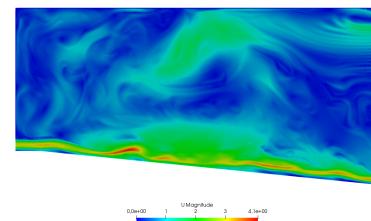
(e) Temperature at time step 300



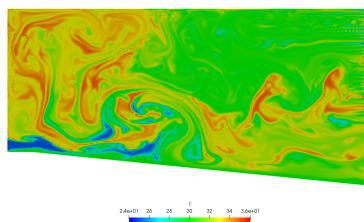
(f) Velocity at time step 300



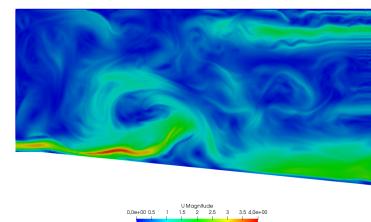
(g) Temperature at time step 400



(h) Velocity at time step 400



(i) Temperature at time step 500



(j) Velocity at time step 500

**Fig. 2.15:** A large fluid problem

## 2.14 An Example with Complex Numbers

In a microwave oven heat comes from molecular excitation by an electromagnetic field. For a plane monochromatic wave, amplitude is given by Helmholtz's equation:

$$\beta v + \Delta v = 0.$$

We consider a rectangular oven where the wave is emitted by part of the upper wall. So the boundary of the domain is made up of a part  $\Gamma_1$  where  $v = 0$  and of another part  $\Gamma_2 = [c, d]$  where for instance  $v = \sin\left(\pi \frac{y-c}{c-d}\right)$ .

Within an object to be cooked, denoted by  $B$ , the heat source is proportional to  $v^2$ . At equilibrium, one has :

$$\begin{aligned} -\Delta\theta &= v^2 I_B \\ \theta_\Gamma &= 0 \end{aligned}$$

where  $I_B$  is 1 in the object and 0 elsewhere.

In the program below  $\beta = 1/(1 - i/2)$  in the air and  $2/(1 - i/2)$  in the object ( $i = \sqrt{-1}$ ):

```

1 // Parameters
2 int nn = 2;
3 real a = 20.;
4 real b = 20.;
5 real c = 15.;
6 real d = 8.;
7 real e = 2.;
8 real l = 12.;
9 real f = 2.;
10 real g = 2.;

11
12 // Mesh
13 border a0(t=0, 1){x=a*t; y=0; label=1;}
14 border a1(t=1, 2){x=a; y=b*(t-1); label=1;}
15 border a2(t=2, 3){x=a*(3-t); y=b; label=1;}
16 border a3(t=3, 4){x=0; y=b-(b-c)*(t-3); label=1;}
17 border a4(t=4, 5){x=0; y=c-(c-d)*(t-4); label=2;}
18 border a5(t=5, 6){x=0; y=d*(6-t); label=1;}

19
20 border b0(t=0, 1){x=a-f+e*(t-1); y=g; label=3;}
21 border b1(t=1, 4){x=a-f; y=g+l*(t-1)/3; label=3;}
22 border b2(t=4, 5){x=a-f-e*(t-4); y=l+g; label=3;}
23 border b3(t=5, 8){x=a-e-f; y=l+g-l*(t-5)/3; label=3;}

24
25 mesh Th = buildmesh(a0(10*nn) + a1(10*nn) + a2(10*nn) + a3(10*nn) + a4(10*nn) +
26   ↪ a5(10*nn)
27   + b0(5*nn) + b1(10*nn) + b2(5*nn) + b3(10*nn));
28 real meat = Th(a-f-e/2, g+l/2).region;
29 real air= Th(0.01,0.01).region;
30 plot(Th, wait=1);

31 // Fespace
32 fespace Vh(Th, P1);
33 Vh R=(region-air)/(meat-air);
34 Vh<complex> v, w;
35 Vh vr, vi;
36
37 fespace Uh(Th, P1);

```

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```

38 Uh u, uu, ff;
39
40 // Problem
41 solve muwave(v, w)
42 = int2d(Th) (
43     v*w*(1+R)
44     - (dx(v)*dx(w) + dy(v)*dy(w))*(1 - 0.5i)
45 )
46 + on(1, v=0)
47 + on(2, v=sin(pi*(y-c)/(c-d)))
48 ;
49
50 vr = real(v);
51 vi = imag(v);
52
53 // Plot
54 plot(vr, wait=1, ps="rmuonde.ps", fill=true);
55 plot(vi, wait=1, ps="imuonde.ps", fill=true);
56
57 // Problem (temperature)
58 ff=1e5*(vr^2 + vi^2)*R;
59
60 solve temperature(u, uu)
61 = int2d(Th) (
62     dx(u) * dx(uu) + dy(u) * dy(uu)
63 )
64 - int2d(Th) (
65     ff*uu
66 )
67 + on(1, 2, u=0)
68 ;
69
70 // Plot
71 plot(u, wait=1, ps="tempmuonde.ps", fill=true);

```

Results are shown on Fig. 2.16a, Fig. 2.16b and Fig. 2.16c.

## 2.15 Optimal Control

Thanks to the function BFGS it is possible to solve complex nonlinear optimization problem within **FreeFEM**. For example consider the following inverse problem

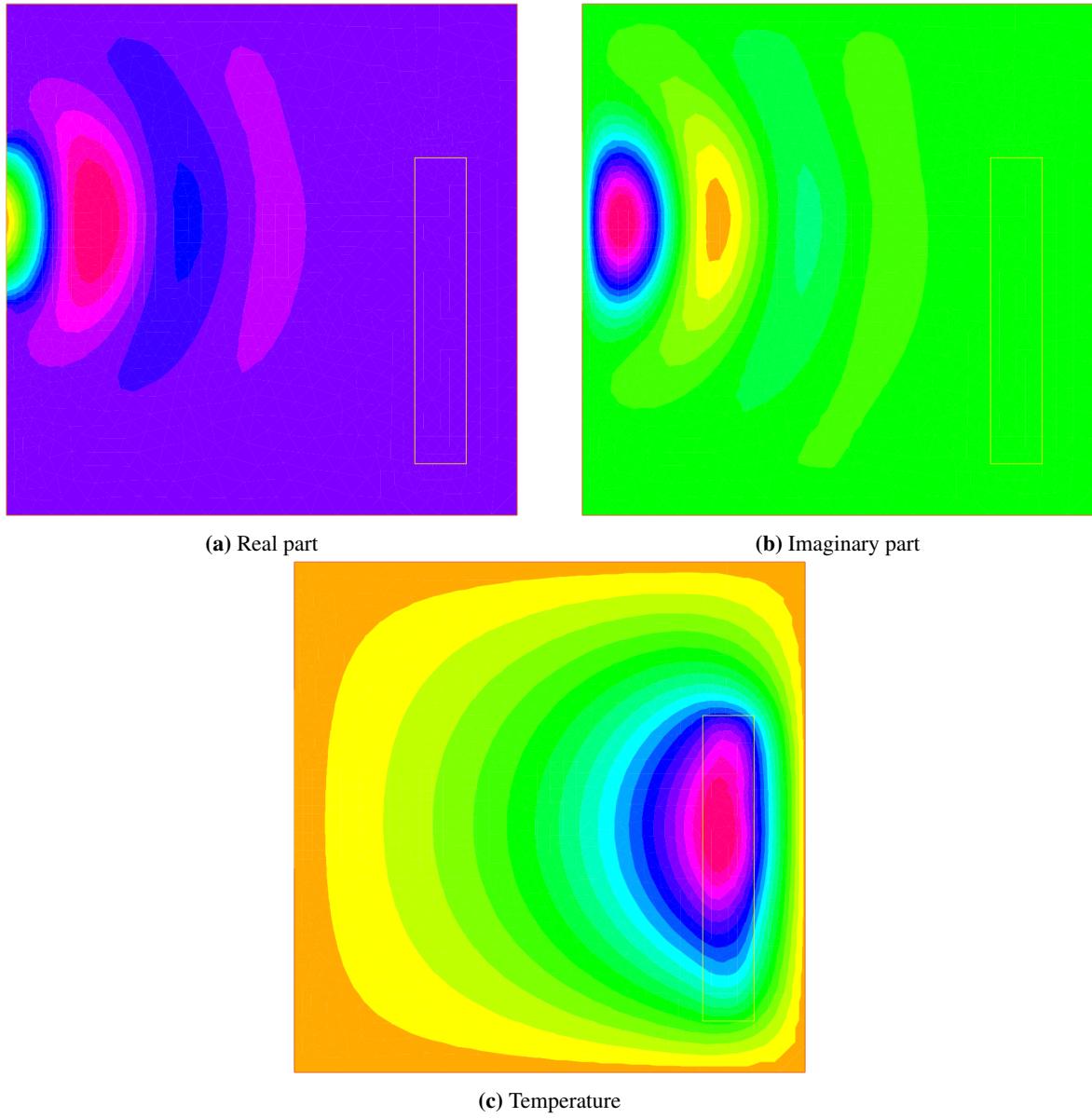
$$\begin{aligned} \min_{b,c,d \in R} J &= \int_E (u - u_d)^2 \\ -\nabla(\kappa(b,c,d) \cdot \nabla u) &= 0 \\ u|_\Gamma &= u_\Gamma \end{aligned}$$

where the desired state  $u_d$ , the boundary data  $u_\Gamma$  and the observation set  $E \subset \Omega$  are all given. Furthermore let us assume that:

$$\kappa(x) = 1 + bI_B(x) + cI_C(x) + dI_D(x) \quad \forall x \in \Omega$$

where  $B, C, D$  are separated subsets of  $\Omega$ .

To solve this problem by the quasi-Newton BFGS method we need the derivatives of  $J$  with respect to  $b, c, d$ . We self



**Fig. 2.16:** Microwave

explanatory notations, if  $\delta b, \delta c, \delta d$  are variations of  $b, c, d$  we have:

$$\begin{aligned}\delta J &\approx 2 \int_E (u - u_d) \delta u \\ -\nabla(\kappa \cdot \nabla \delta u) &\approx \nabla(\delta \kappa \cdot \nabla u) \\ \delta u|_{\Gamma} &= 0\end{aligned}$$

Obviously  $J'_b$  is equal to  $\delta J$  when  $\delta b = 1, \delta c = 0, \delta d = 0$ , and so on for  $J'_c$  and  $J'_d$ .

All this is implemented in the following program:

```

1 // Mesh
2 border aa(t=0, 2*pi){x=5*cos(t); y=5*sin(t);}
3 border bb(t=0, 2*pi){x=cos(t); y=sin(t);}
4 border cc(t=0, 2*pi){x=-3+cos(t); y=sin(t);}
5 border dd(t=0, 2*pi){x=cos(t); y=-3+sin(t);}

6
7 mesh th = buildmesh(aa(70) + bb(35) + cc(35) + dd(35));
8
9 // Fespace
10 fespace Vh(th, P1);
11 Vh Ib=((x^2+y^2)<1.0001),
12 Ic=((x+3)^2+y^2)<1.0001,
13 Id=(x^2+(y+3)^2)<1.0001,
14 Ie=((x-1)^2+y^2)<=4,
15 ud, u, uh, du;
16
17 // Problem
18 real[int] z(3);
19 problem A(u, uh)
20 = int2d(th)(
21     (1+z[0]*Ib+z[1]*Ic+z[2]*Id)*(dx(u)*dx(uh) + dy(u)*dy(uh))
22 )
23 + on(aa, u=x^3-y^3)
24 ;
25
26 // Solve
27 z[0]=2; z[1]=3; z[2]=4;
28 A;
29 ud = u;
30
31 ofstream f("J.txt");
32 func real J(real[int] & Z){
33     for (int i = 0; i < z.n; i++)
34         z[i] = Z[i];
35     A;
36     real s = int2d(th)(Ie*(u-ud)^2);
37     f << s << " ";
38     return s;
39 }
40
41 // Problem BFGS
42 real[int] dz(3), dJdz(3);
43 problem B (du, uh)
44 = int2d(th)(
45     (1+z[0]*Ib+z[1]*Ic+z[2]*Id)*(dx(du)*dx(uh) + dy(du)*dy(uh))
46 )
47 + int2d(th)(
48     (dz[0]*Ib+dz[1]*Ic+dz[2]*Id)*(dx(u)*dx(uh) + dy(u)*dy(uh))
49 )

```

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```

50    +on (aa, du=0)
51    ;
52
53 func real[int] DJ(real[int] &Z) {
54     for(int i = 0; i < z.n; i++) {
55         for(int j = 0; j < dz.n; j++)
56             dz[j] = 0;
57         dz[i] = 1;
58         B;
59         dJdz[i] = 2*int2d(th) (Ie*(u-ud)*du);
60     }
61     return dJdz;
62 }
63
64 real[int] Z(3);
65 for(int j = 0; j < z.n; j++)
66     Z[j]=1;
67
68 BFGS (J, DJ, Z, eps=1.e-6, nbiter=15, nbiterline=20);
69 cout << "BFGS: J(z) = " << J(Z) << endl;
70 for(int j = 0; j < z.n; j++)
71     cout << z[j] << endl;
72
73 // Plot
74 plot (ud, value=1, ps="u.eps");

```

In this example the sets  $B, C, D, E$  are circles of boundaries  $bb, cc, dd, ee$  and the domain  $\Omega$  is the circle of boundary  $aa$ .

The desired state  $u_d$  is the solution of the PDE for  $b = 2, c = 3, d = 4$ . The unknowns are packed into array  $z$ .

---

**Note:** It is necessary to recopy  $Z$  into  $z$  because one is a local variable while the other one is global.

---

The program found  $b = 2.00125, c = 3.00109, d = 4.00551$ .

[Fig. 2.17a](#) and [Fig. 2.17b](#) show  $u$  at convergence and the successive function evaluations of  $J$ .

Note that an *adjoint state* could have been used. Define  $p$  by:

$$\begin{aligned} -\nabla \cdot (\kappa \nabla p) &= 2I_E(u - u_d) \\ p|_{\Gamma} &= 0 \end{aligned}$$

Consequently:

$$\begin{aligned} \delta J &= -\int_{\Omega} (\nabla \cdot (\kappa \nabla p)) \delta u \\ &= \int_{\Omega} (\kappa \nabla p \cdot \nabla \delta u) \\ &= -\int_{\Omega} (\delta \kappa \nabla p \cdot \nabla u) \end{aligned}$$

Then the derivatives are found by setting  $\delta b = 1, \delta c = \delta d = 0$  and so on:

$$\begin{aligned} J'_b &= -\int_B \nabla p \cdot \nabla u \\ J'_c &= -\int_C \nabla p \cdot \nabla u \\ J'_d &= -\int_D \nabla p \cdot \nabla u \end{aligned}$$

---

**Note:** As BFGS stores an  $M \times M$  matrix where  $M$  is the number of unknowns, it is dangerously expensive to use this method when the unknown  $x$  is a Finite Element Function. One should use another optimizer such as the NonLinear Conjugate Gradient NLCG (also a key word of FreeFEM).

---

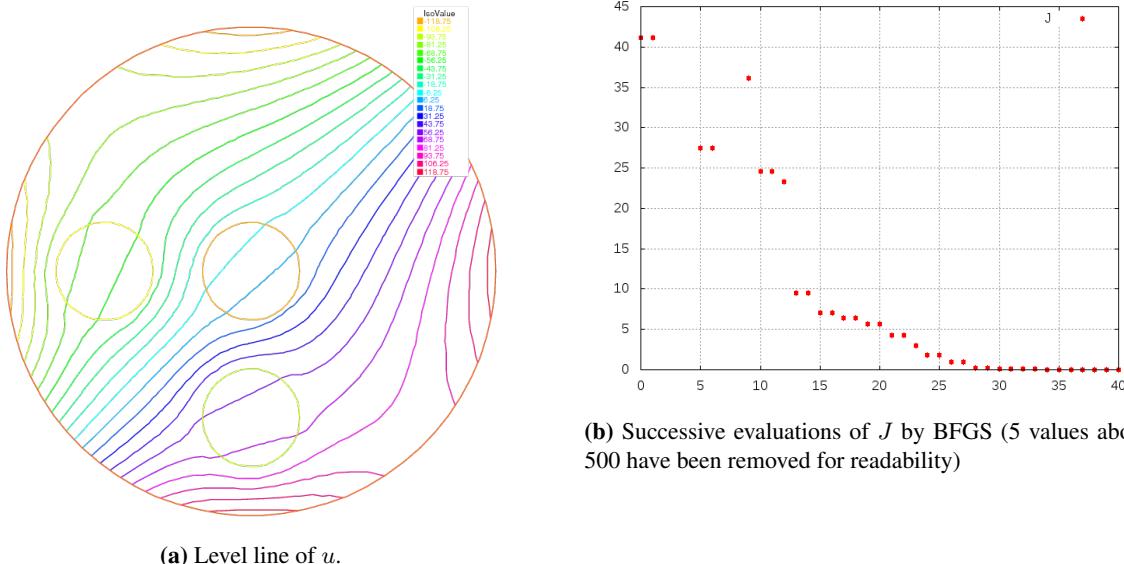


Fig. 2.17: Optimal control

## 2.16 A Flow with Shocks

Compressible Euler equations should be discretized with Finite Volumes or FEM with flux up-winding scheme but these are not implemented in **FreeFEM**. Nevertheless acceptable results can be obtained with the method of characteristics provided that the mean values  $\bar{f} = \frac{1}{2} (f^+ + f^-)$  are used at shocks in the scheme, and finally mesh adaptation.

$$\begin{aligned}\partial_t \rho + \bar{u} \nabla \rho + \bar{\rho} \nabla \cdot u &= 0 \\ \bar{\rho} (\partial_t u + \frac{\bar{\rho} u}{\bar{\rho}} \nabla u + \nabla p) &= 0 \\ \partial_t p + \bar{u} \nabla p + (\gamma - 1) \bar{\rho} \nabla \cdot u &= 0\end{aligned}$$

One possibility is to couple  $u, p$  and then update  $\rho$ , i.e.:

$$\begin{aligned}\frac{1}{(\gamma-1)\delta t \bar{p}^m} (p^{m+1} - p^m \circ X^m) + \nabla \cdot u^{m+1} &= 0 \\ \frac{\bar{\rho}^m}{\delta t} (u^{m+1} - u^m \circ \tilde{X}^m) + \nabla p^{m+1} &= 0 \\ \rho^{m+1} = \rho^m \circ X^m + \frac{\bar{\rho}^m}{(\gamma-1)\bar{p}^m} (p^{m+1} - p^m \circ X^m) &\end{aligned}$$

A numerical result is given on Fig. 2.18 and the **FreeFEM** script is

```

1 // Parameters
2 verbosity = 1;
3 int anew = 1;
4 int m = 5;
5 real x0 = 0.5;
6 real y0 = 0.;
7 real rr = 0.2;
8 real dt = 0.01;
9 real u0 = 2.;
10 real err0 = 0.00625;
11 real pena = 2.;

12
13 // Mesh
14 border ccc(t=0, 2){x=2-t; y=1;};
```

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```

15 border ddd(t=0, 1){x=0; y=1-t;};
16 border aaal(t=0, x0-rr){x=t; y=0;};
17 border cercle(t=pi, 0){x=x0+rr*cos(t); y=y0+rr*sin(t);};
18 border aaa2(t=x0+rr, 2){x=t; y=0;};
19 border bbb(t=0, 1){x=2; y=t;};
20
21 mesh Th;
22 if(anew)
23     Th = buildmesh (ccc(5*m) + ddd(3*m) + aaal(2*m) + cercle(5*m) + aaa2(5*m) +
24     ↪bbb(2*m));
25 else
26     Th = readmesh("Th_circle.mesh"); plot(Th);
27
28 // fespace
29 fespace Wh(Th, P1);
30 Wh u, v;
31 Wh ul, vl;
32 Wh uh, vh;
33
34 fespace Vh(Th, P1);
35 Vh r, rh, r1;
36
37 // Macro
38 macro dn(u) (N.x*dx(u)+N.y*dy(u)) //
39
40 // Initialization
41 if(anew){
42     ul = u0;
43     v1 = 0;
44     r1 = 1;
45 }
46 else{
47     ifstream g("u.txt"); g >> ul[];
48     ifstream gg("v.txt"); gg >> v1[];
49     ifstream ggg("r.txt"); ggg >> r1[];
50     plot(ul, ps="eta.eps", value=1, wait=1);
51     err0 = err0/10;
52     dt = dt/10;
53 }
54
55 // Problem
56 problem euler(u, v, r, uh, vh, rh)
57 = int2d(Th) (
58     (u*uh + v*vh + r*rh)/dt
59     + ((dx(r)*uh + dy(r)*vh) - (dx(rh)*u + dy(rh)*v))
60 )
61 + int2d(Th) (
62     - (
63         rh*convect([ul,v1],-dt,r1)
64         + uh*convect([ul,v1],-dt,u1)
65         + vh*convect([ul,v1],-dt,v1)
66     )/dt
67 )
68 + int1d(Th, 6) (
69     rh*u
70     )
71 + on(2, r=0)

```

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```

71 + on(2, u=u0)
72 + on(2, v=0)
73 ;
74
75 // Iterations
76 int j = 80;
77 for(int k = 0; k < 3; k++) {
78     if(k==20) {
79         err0 = err0/10;
80         dt = dt/10;
81         j = 5;
82     }
83
84     // Solve
85     for(int i = 0; i < j; i++) {
86         euler;
87         u1=u;
88         v1=v;
89         r1=abs(r);
90         cout << "k = " << k << " E = " << int2d(Th) (u^2+v^2+r) << endl;
91         plot(r, value=1);
92     }
93
94     // Mesh adaptation
95     Th = adaptmesh (Th, r, nbvx=40000, err=err0, abserror=1, nbjacobi=2, omega=1.8,_
96     ↵ratio=1.8, nbsmooth=3, splitpbedge=1, maxsubdiv=5, rescaling=1);
97     plot(Th);
98     u = u;
99     v = v;
100    r = r;
101
102    // Save
103    savemesh(Th, "Th_circle.mesh");
104    ofstream f("u.txt"); f << u[];
105    ofstream ff("v.txt"); ff << v[];
106    ofstream fff("r.txt"); fff << r[];
107    r1 = sqrt(u*u+v*v);
108    plot(r1, ps="mach.eps", value=1);
109    r1 = r;
}

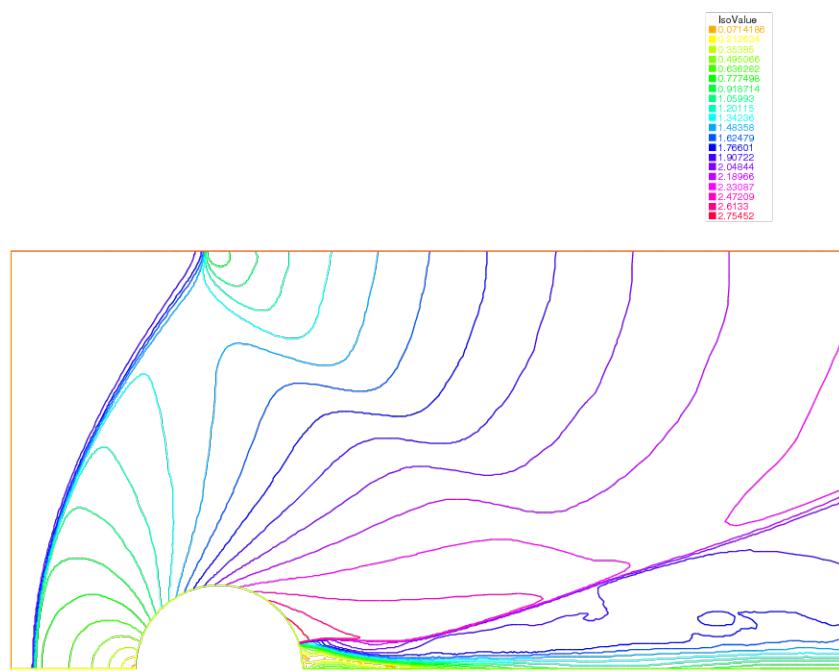
```

## 2.17 Time dependent schema optimization for heat equations

First, it is possible to define variational forms, and use this forms to build matrix and vector to make very fast script (4 times faster here).

For example solve the *ThermalConduction* problem, we must solve the temperature equation in  $\Omega$  in a time interval  $(0,T)$ .

$$\begin{aligned}
 \partial_t u - \nabla \cdot (\kappa \nabla u) &= 0 && \text{in } \Omega \times (0, T) \\
 u(x, y, 0) &= u_0 + x u_1 \\
 u &= 30 && \text{on } \Gamma_{24} \times (0, T) \\
 \kappa \frac{\partial u}{\partial n} + \alpha(u - u_e) &= 0 && \text{on } \Gamma \times (0, T)
 \end{aligned}$$



**Fig. 2.18:** Pressure for a Euler flow around a disk at Mach 2 computed by (2.6)

The variational formulation is in  $L^2(0, T; H^1(\Omega))$ ; we shall seek  $u^n$  satisfying:

$$\forall w \in V_0; \int_{\Omega} \frac{u^n - u^{n-1}}{\delta t} w + \kappa \nabla u^n \nabla w) + \int_{\Gamma} \alpha(u^n - u_{ue}) w = 0$$

where  $V_0 = \{w \in H^1(\Omega) / w|_{\Gamma_{24}} = 0\}$ .

So, to code the method with the matrices  $A = (A_{ij})$ ,  $M = (M_{ij})$ , and the vectors  $u^n, b^n, b', b'', b_{cl}$  (notation if  $w$  is a vector then  $w_i$  is a component of the vector).

$$u^n = A^{-1}b^n, \quad b' = b_0 + Mu^{n-1}, \quad b'' = \frac{1}{\varepsilon} b_{cl}, \quad b_i^n = \begin{cases} b_i'' & \text{if } i \in \Gamma_{24} \\ b_i' & \text{else} \end{cases}$$

Where with  $\frac{1}{\varepsilon} = \text{tgv} = 10^{30}$ :

$$\begin{aligned} A_{ij} &= \begin{cases} \frac{1}{\varepsilon} & \text{if } i \in \Gamma_{24}, \text{ and } j = i \\ \int_{\Omega} w_j w_i / dt + k(\nabla w_j \cdot \nabla w_i) + \int_{\Gamma_{13}} \alpha w_j w_i & \text{else} \end{cases} \\ M_{ij} &= \begin{cases} \frac{1}{\varepsilon} & \text{if } i \in \Gamma_{24}, \text{ and } j = i \\ n \int_{\Omega} w_j w_i / dt & \text{else} \end{cases} \\ b_{0,i} &= n \int_{\Gamma_{13}} \alpha u_{ue} w_i \\ b_{cl} &= u^0 \text{ the initial data} \end{aligned}$$

The Fast version script:

```
1 ...
2 Vh u0=fu0, u=u0;
```

Create three variational formulation, and build the matrices  $A, M$ .

```
1 varf vthermic (u, v)
2   = int2d(Th) (
3     u*v/dt
4     + k*(dx(u)*dx(v) + dy(u)*dy(v))
5   )
6   + int1d(Th, 1, 3) (
7     alpha*u*v
8   )
9   + on(2, 4, u=1)
10 ;
11
12 varf vthermic0 (u, v)
13   = int1d(Th, 1, 3) (
14     alpha*ue*v
15   )
16 ;
17 varf vMass (u,v)
18   = int2d(Th) (
19     u*v/dt
20   )
21   + on(2, 4, u=1)
22 ;
23
24 real tgv = 1e30;
25 matrix A = vthermic(Vh, Vh, tgv=tgv, solver=CG);
26 matrix M = vMass(Vh, Vh);
```

Now, to build the right hand size; we need 4 vectors.

```

1 real[int] b0 = vthermic0(0,Vh); //constant part of RHS
2 real[int] bcn = vthermic(0,Vh); //tgv on Dirichlet part
3 real[int] bcl = tgv*u0[]; //the Dirichlet B.C. part
4
5 // The fast loop
6 for(real t = 0; t < T; t += dt){
7     real[int] b = b0; //the RHS
8     b += M*u[]; //add the time dependent part
9     b = bcn ? bcl : b; //do $forall i$: b[i] = bcn[i] ? bcl[i] : b[i];
10    u[] = A^-1*b; //solve linear problem
11    plot(u);
12 }

```

## 2.18 Tutorial to write a transient Stokes solver in matrix form

Consider the following script to solve a time dependent Stokes problem in a cavity

```

1 // Parameters
2 real nu = 0.1;
3 real T=1.;
4 real dt = 0.1;
5
6 // Mesh
7 mesh Th = square(10, 10);
8
9 // Fespace
10 fespace Vh(Th, P2)
11 Vh u, v;
12 Vh uu, vv;
13 Vh uold=0, vold=0;
14
15 fespace Qh(Th, P1);
16 Qh p;
17 Qh pp;
18
19 // Problem
20 problem stokes (u, v, p, uu, vv, pp)
21     = int2d(Th) (
22         (u*uu+v*vv)/dt
23         + nu*(dx(u)*dx(uu) + dy(u)*dy(uu) + dx(v)*dx(vv) + dy(v)*dy(vv))
24         - p*pp*1.e-6
25         - p*(dx(uu) + dy(vv))
26         - pp*(dx(u) + dy(v))
27     )
28     - int2d(Th) (
29         (uold*uu+vold*vv)/dt
30     )
31     + on(1, 2, 4, u=0, v=0)
32     + on(3, u=1, v=0)
33 ;
34
35 // Time loop
36 int m, M = T/dt;
37 for(m = 0; m < M; m++) {
38     stokes;

```

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```

39     uold = u;
40     vold = v;
41 }
42
43 // Plot
44 plot(p, [u, v], value=true, wait=true, cmm="t=" + m*dt);

```

Every iteration is in fact of the form  $A[u, v, p] = B[uold, vold, pold] + b$  where  $A, B$  are matrices and  $b$  is a vector containing the boundary conditions.  $A, B, b$  are constructed by:

```

1 fespace Xh(Th, [P2, P2, P1]);
2 varf aa ([u, v, p], [uu, vv, pp])
3   = int2d(Th) (
4     (u*uu+v*vv)/dt
5     + nu*(dx(u)*dx(uu) + dy(u)*dy(uu) + dx(v)*dx(vv) + dy(v)*dy(vv))
6     - p*pp*1.e-6
7     - p*(dx(uu) + dy(vv))
8     - pp*(dx(u) + dy(v)))
9   )
10  + on(1, 2, 4, u=0, v=0)
11  + on(3, u=1, v=0)
12 ;
13
14 varf bb ([uold, vold, pold], [uu, vv, pp])
15   = int2d(Th) (
16     (uold*uu+vold*vv)/dt
17   )
18 //+ on(1, 2, 4, uold=0, vold=0)
19 //+ on(3, uold=1, vold=0)
20 ;
21
22 varf bcl ([uold, vold, pold], [uu, vv, pp])
23   = on(1, 2, 4, uold=0, vold=0)
24   + on(3, uold=1, vold=0)
25 ;
26
27 matrix A = aa(Xh, Xh, solver=UMFPACK);
28 matrix B = bb(Xh, Xh);
29 real[int] b = bcl(0, Xh);

```

Note that the boundary conditions are not specified in  $bb$ . Removing the comment  $//$  would cause the compiler to multiply the diagonal terms corresponding to a Dirichlet degree of freedom by a very large term ( $t_{gv}$ ); if so  $b$  would not be needed, on the condition that  $uold = 1$  on boundary 3 initially. Note also that  $b$  has a  $t_{gv}$  on the Dirichlet nodes, by construction, and so does  $A$ .

The loop will then be:

```

1 real[int] sol(Xh.ndof), aux(Xh.ndof);
2 for (m = 0; m < M; m++) {
3   aux = B*sol; aux += b;
4   sol = A^-1 * aux;
5 }

```

There is yet a difficulty with the initialization of  $sol$  and with the solution from  $sol$ . For this we need a temporary vector in  $X_h$  and here is a solution:

```

1 Xh [w1, w2, wp] = [uold, vold, pp];
2 sol = w1[]; //cause also the copy of w2 and wp
3 for (m = 0; m < M; m++) {
4     aux = B*sol; aux += b;
5     sol = A^-1 * aux;
6 }
7 w1[] = sol; u=w1; v= w2; p=wp;
8 plot(p, [u, v], value=true, wait=true, cmm="t=" + m*dt);

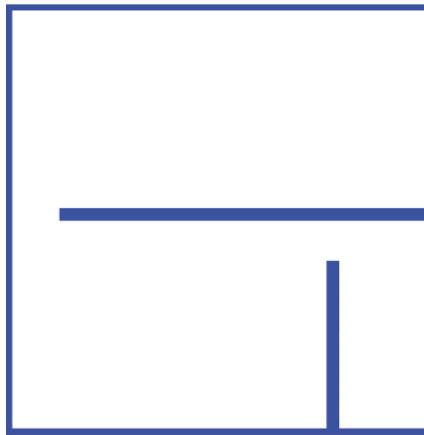
```

The freefem team agrees that the line `sol=w1[];` is mysterious as it copies also `w2` and `wp` into `sol`. Structured data such as vectors of  $X_h$  here cannot be written component by component. Hence `w1=u` is not allowed.

## 2.19 Wifi Propagation

### 2.19.1 Summary

In this tutorial, we will study the wifi signal power in a flat. An awesome flat is especially designed for the experiment, with **2** walls:



**Fig. 2.19:** Flat

Even if the flat seems small enough to be covered by wifi everywhere, it is still interesting to study where the signal's power is the lowest. We will study where to put the hotspot to get the best coverage, and as we're a bit lazy we will only put it next to the left wall.

### 2.19.2 Physics

In a nutshell, the Wifi is a electromagnetic wave that contains a signal : Internet data. Electromagnetic waves are well known by physicists and are ruled by the **4 Maxwell equations** which give you the solution for  $E$ , the electrical field, and  $B$ , the magnetic field, in space but also in time.

We don't care about the time here, because the signal period is really short so our internet quality will not change with time. Without time, we're looking for stationaries solutions, and the Maxwell equations can be simplified to one equation, the Helmholtz one :

$$\nabla^2 E + \frac{k^2}{n^2} E = 0$$

Where  $k$  is the angular wavenumber of the wifi signal, and  $n$  the refractive index of the material the wave is in.

Indeed, the main point of this study is the impact of **walls** on the signal's power, where the  $n$  is different from air (where it is 1). In walls, the refractive index is a complex number in which the two parts have a physic interpretation:

- The *real part* defines the **reflexion** of the wall (the amount of signal that doesn't pass).
- The *imaginary part* defines the **absorption** of the wall (the amount that disappears).

The wifi hotspot (simulated by a simple circle) will be the boundary condition, with a non null value for our electrical field.

### 2.19.3 Coding

#### The domain

In order to create the domain of experimentation, we need to create border objects, like this :

```

1 real a = 40, b = 40, c = 0.5;
2 border a00(t=0, 1) {x=a*t; y=0; label=1;}
3 border a10(t=0, 1) {x=a; y=b*t; label=1;}
4 border a20(t=1, 0) {x=a*t; y=b; label=1;}
5 border a30(t=1, 0) {x=0; y=b*t; label=1;}
6 border a01(t=0, 1) {x=c+(a-c*2)*t; y=c; label=1;}
7 border a11(t=0, 1) {x=a-c; y=c+(b-c*2)*t; label=1;}
8 border a21(t=1, 0) {x=c+(a-c*2)*t; y=b-c; label=1;}
9 border a31(t=1, 0) {x=c; y=c+(b-c*2)*t; label=1;}
10
11 real p = 5, q = 20, d = 34, e = 1;
12 border b00(t=0, 1) {x=p+d*t; y=q; label=3;}
13 border b10(t=0, 1) {x=p+d; y=q+e*t; label=3;}
14 border b20(t=1, 0) {x=p+d*t; y=q+e; label=3;}
15 border b30(t=1, 0) {x=p; y=q+e*t; label=3;}
16
17 real r = 30, s = 1, j = 1, u = 15;
18 border c00(t=0, 1) {x=r+j*t; y=s; label=3;}
19 border c10(t=0, 1) {x=r+j; y=s+u*t; label=3;}
20 border c20(t=1, 0) {x=r+j*t; y=s+u; label=3;}
21 border c30(t=1, 0) {x=r; y=s+u*t; label=3;}

```

#### Let's create a mesh

```

1 int n=13;
2 mesh Sh = buildmesh(a00(10*n) + a10(10*n) + a20(10*n) + a30(10*n)
3 + a01(10*n) + a11(10*n) + a21(10*n) + a31(10*n)
4 + b00(5*n) + b10(5*n) + b20(5*n) + b30(5*n)
5 + c00(5*n) + c10(5*n) + c20(5*n) + c30(5*n));
6 plot(Sh, wait=1);

```

So we are creating a mesh, and plotting it :

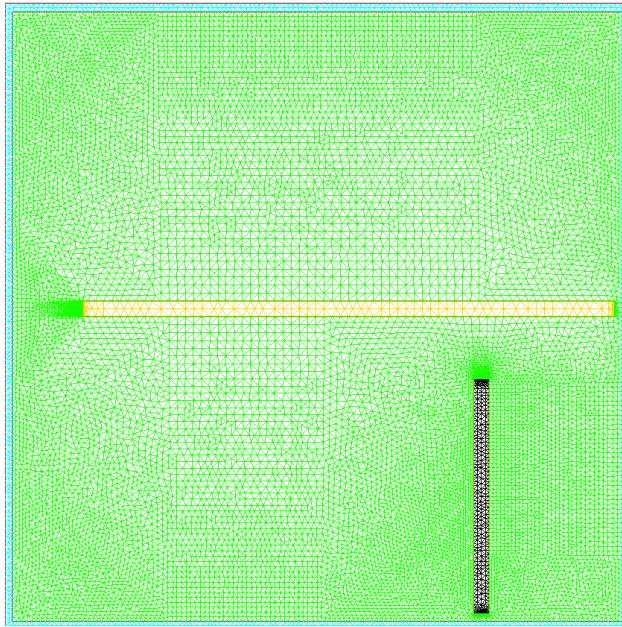
There is currently no wifi hotspot, and as we want to resolve the equation for a multiple number of position next to the left wall, let's do a `for` loop:

```

1 int bx;
2 for (bx = 1; bx <= 7; bx++) {
3     border C(t=0, 2*pi) {x=2+cos(t); y=bx*5+sin(t); label=2;}
4

```

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**Fig. 2.20:** Mesh

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```

5   mesh Th = buildmesh(a00(10*n) + a10(10*n) + a20(10*n) + a30(10*n)
6     + a01(10*n) + a11(10*n) + a21(10*n) + a31(10*n) + C(10)
7     + b00(5*n) + b10(5*n) + b20(5*n) + b30(5*n)
8     + c00(5*n) + c10(5*n) + c20(5*n) + c30(5*n));

```

The border `C` is our hotspot and as you can see a simple circle. `Th` is our final mesh, with all borders and the hotspot. Let's resolve this equation !

```

1 fespace Vh(Th, P1);
2 func real wall() {
3   if (Th(x,y).region == Th(0.5,0.5).region || Th(x,y).region == Th(7,20.5).region ||
4     Th(x,y).region == Th(30.5,2).region) { return 1; }
5   else { return 0; }
6 }
7
8 Vh<complex> v,w;
9
10 randinit(900);
11 Vh wallreflexion = randreal1();
12 Vh<complex> wallabsorption = randreal1()*0.5i;
13 Vh k = 6;
14
15 cout << "Reflexion of walls : " << wallreflexion << "\n";
16 cout << "Absorption of walls : " << wallabsorption << "\n";
17
18 problem muwave(v,w) =
19   int2d(Th) (
20     (v*w*k^2) / (1+(wallreflexion+wallabsorption)*wall())^2
21     - (dx(v)*dx(w)+dy(v)*dy(w))
22   )
23   + on(2, v=1)

```

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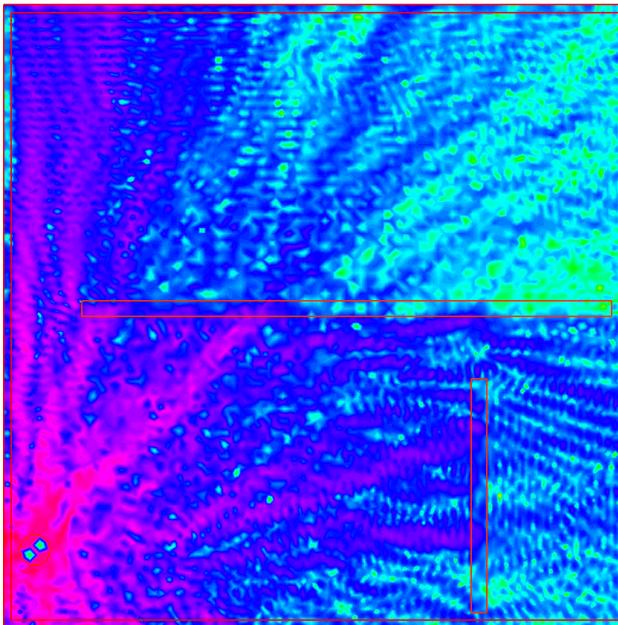
```

23 ;
24
25 muwave;
26 Vh vm = log(real(v)^2 + imag(v)^2);
27 plot(vm, wait=1, fill=true, value=0, nbiso=65);
28 }
```

A bit of understanding here :

- The `fespace` keyword defines a finite elements space, no need to know more here.
- The function `wall` return 0 if in air and 1 if in a wall (`x` and `y` are global variables).
- For this example, random numbers are used for the reflexion and the absorption.
- The problem is defined with `problem` and we solve it by calling it.

Finally, I plotted the log of the module of the solution `v` to see the signal's power, and here we are :



**Fig. 2.21:** Solution

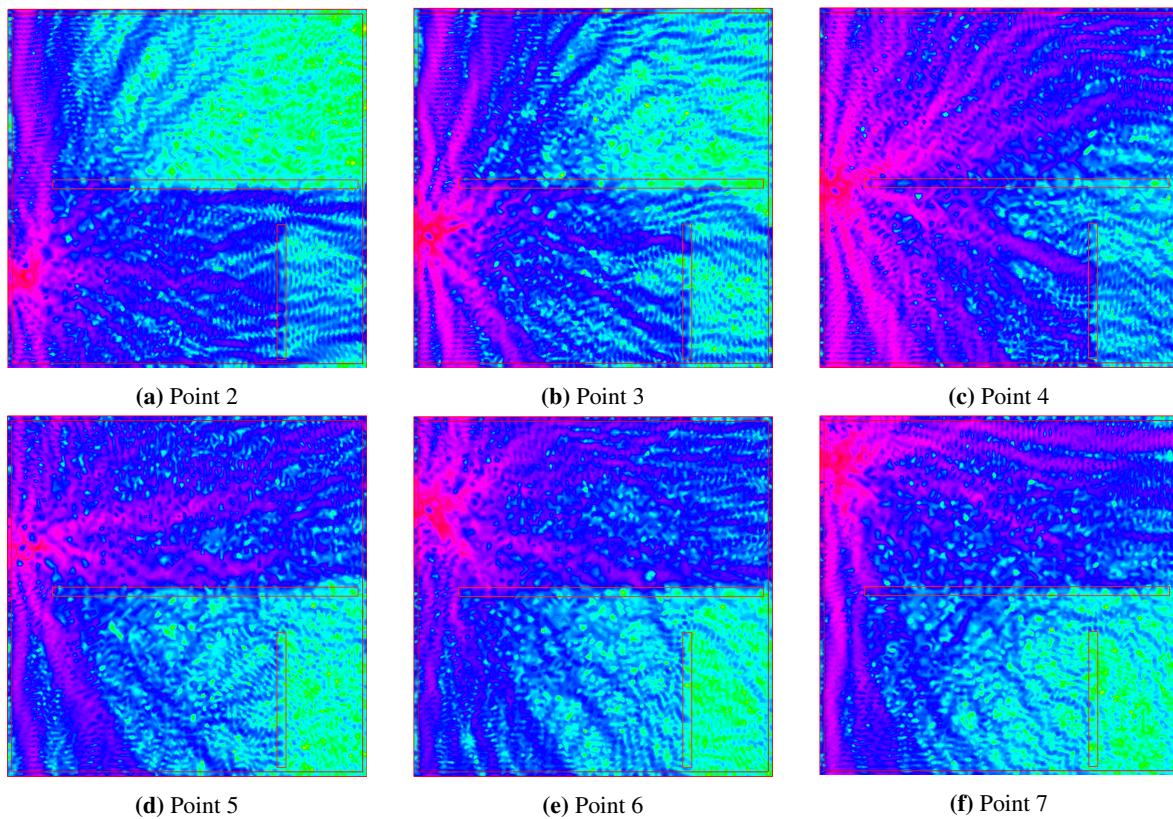
Beautiful isn't it ? This is the first position for the hotspot, but there are 6 others, and the electrical field is evolving depending on the position. You can see the other positions here :

## 2.20 Plotting in Matlab and Octave

### 2.20.1 Overview

In order to create a plot of a **FreeFEM** simulation in **Matlab**© or **Octave** two steps are necessary:

- The mesh, the finite element space connectivity and the simulation data must be exported into files
- The files must be imported into the Matlab / Octave workspace. Then the data can be visualized with the `ffmatlib` library



**Fig. 2.22:** WiFi propagation

The steps are explained in more detail below using the example of a stripline capacitor.

---

**Note:** Finite element variables must be in P1 or P2. The simulation data can be 2D or 3D.

---

## 2.20.2 2D Problem

Consider a stripline capacitor problem which is also shown in Fig. 2.23. On the two boundaries (the electrodes)  $C_A$ ,  $C_K$  a Dirichlet condition and on the enclosure  $C_B$  a Neumann condition is set. The electrostatic potential  $u$  between the two electrodes is given by the Laplace equation

$$\Delta u(x, y) = 0$$

and the electrostatic field  $\mathbf{E}$  is calculated by

$$\mathbf{E} = -\nabla u$$

```

1 int CA=3, CK=4, CB=5;
2 real w2=1.0, h=0.4, d2=0.5;
3
4 border bottomA(t=-w2,w2) { x=t; y=d2; label=CA; };
5 border rightA(t=d2,d2+h) { x=w2; y=t; label=CA; };
6 border topA(t=w2,-w2) { x=t; y=d2+h; label=CA; };
7 border leftA(t=d2+h,d2) { x=-w2; y=t; label=CA; };
8
9 border bottomK(t=-w2,w2) { x=t; y=-d2-h; label=CK; };
10 border rightK(t=-d2-h,-d2) { x=w2; y=t; label=CK; };
11 border topK(t=w2,-w2) { x=t; y=-d2; label=CK; };
12 border leftK(t=-d2,-d2-h) { x=-w2; y=t; label=CK; };
13
14 border enclosure(t=0,2*pi) {x=5*cos(t); y=5*sin(t); label=CB; }
15
16 int n=15;
17 mesh Th = buildmesh(enclosure(3*n) +
18 bottomA(-w2*n)+topA(-w2*n)+rightA(-h*n)+leftA(-h*n) +
19 bottomK(-w2*n)+topK(-w2*n)+rightK(-h*n)+leftK(-h*n));
20
21 fespace Vh(Th,P1);
22
23 Vh u,v;
24 real u0=2.0;
25
26 problem Laplace(u,v,solver=LU) =
27     int2d(Th) (dx(u)*dx(v) + dy(u)*dy(v) )
28     + on(CA,u=u0)+on(CK,u=0);
29
30 real error=0.01;
31 for (int i=0;i<1;i++) {
32     Laplace;
33     Th=adaptmesh(Th,u,err=error);
34     error=error/2.0;
35 }
36 Laplace;
37
38 Vh Ex, Ey;
```

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```

39 Ex = -dx(u);
40 Ey = -dy(u);
41
42 plot(u, [Ex,Ey],wait=true);

```

### 2.20.3 Exporting Data

The mesh is stored with the **FreeFEM** command `savemesh()`, while the connectivity of the finite element space and the simulation data are stored with the macro commands `ffSaveVh()` and `ffSaveData()`. These two commands are located in the `ffmatlib.idp` file which is included in the `ffmatlib`. Therefore, to export the stripline capacitor data the following statement sequence must be added to the **FreeFEM** code:

```

1 include "ffmatlib.idp"
2
3 //Save mesh
4 savemesh(Th,"capacitor.msh");
5 //Save finite element space connectivity
6 ffSaveVh(Th,Vh,"capacitor_vh.txt");
7 //Save some scalar data
8 ffSaveData(u,"capacitor_potential.txt");
9 //Save a 2D vector field
10 ffSaveData2(Ex,Ey,"capacitor_field.txt");

```

### 2.20.4 Importing Data

The mesh file can be loaded into the Matlab / Octave workspace using the `ffreadmesh()` command. A mesh file consists of *three main sections*:

1. The mesh points as nodal coordinates
2. A list of boundary edges including boundary labels
3. List of triangles defining the mesh in terms of connectivity

The three data sections mentioned are returned in the variables `p`, `b` and `t`. The finite element space connectivity and the simulation data can be loaded using the `ffreaddata()` command. Therefore, to load the example data the following statement sequence must be executed in Matlab / Octave:

```

1 %Add ffmatlib to the search path
2 addpath('add here the link to the ffmatlib');
3 %Load the mesh
4 [p,b,t,nv,nbe,nt,labels]=ffreadmesh('capacitor.msh');
5 %Load the finite element space connectivity
6 vh=ffreaddata('capacitor_vh.txt');
7 %Load scalar data
8 u=ffreaddata('capacitor_potential.txt');
9 %Load 2D vector field data
10 [Ex,Ey]=ffreaddata('capacitor_field.txt');

```

### 2.20.5 2D Plot Examples

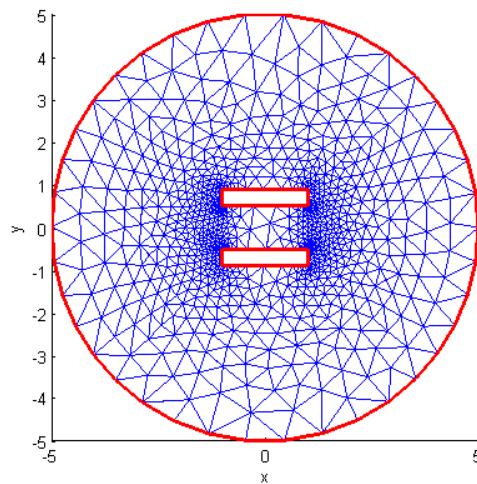
`ffpdeplot()` is a plot solution for creating patch, contour, quiver, mesh, border, and region plots of 2D geometries. The basic syntax is:

```
1 [handles,varargout] = ffpdепlot(p,b,t,varargin)
```

varargin specifies parameter name / value pairs to control the plot behaviour. A table showing all options can be found in the [ffmatlib](#) documentation. A small selection of possible plot commands is given as follows:

- Plot of the boundary and the mesh:

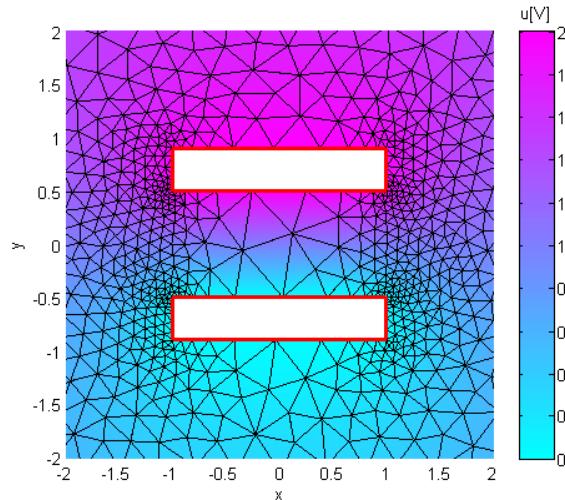
```
1 ffpdепlot(p,b,t,'Mesh','on','Boundary','on');
```



**Fig. 2.23:** Boundary and Mesh

- Patch plot (2D map or density plot) including mesh and boundary:

```
1 ffpdепlot(p,b,t,'VhSeq',vh,'XYData',u,'Mesh','on','Boundary','on',...
2 'XLim',[-2 2], 'YLim',[-2 2]);
```



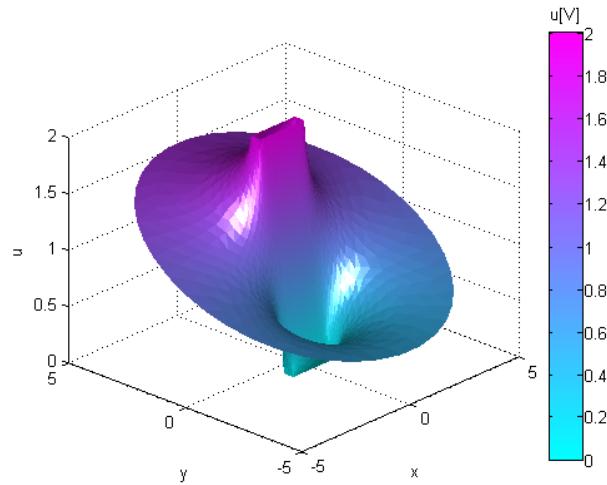
**Fig. 2.24:** Patch Plot with Mesh

- 3D surf plot:

```

1 ffpdепlot(p,b,t,'VhSeq',vh,'XYData',u,'ZStyle','continuous', ...
2           'Mesh','off');
3 lighting gouraud;
4 view([-47,24]);
5 camlight('headlight');

```



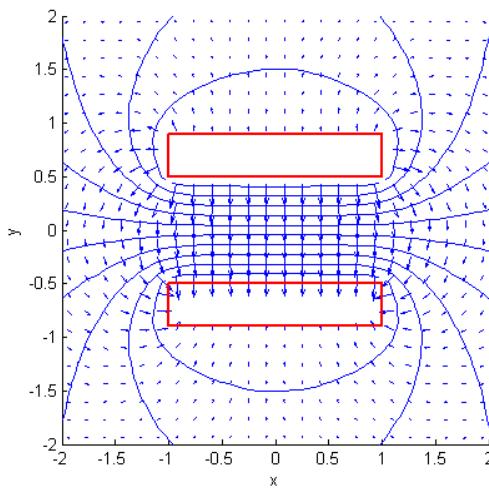
**Fig. 2.25:** 3D Surf Plot

- Contour (isovalue) and quiver (vector field) plot:

```

1 ffpdепlot(p,b,t,'VhSeq',vh,'XYData',u,'Mesh','off','Boundary','on', ...
2           'XLim',[-2 2], 'YLim',[-2 2], 'Contour','on','CCColor','b', ...
3           'XYStyle','off', 'CGridParam',[150, 150], 'ColorBar','off', ...
4           'FlowData',[Ex,Ey], 'FGridParam',[24, 24]);

```



**Fig. 2.26:** Contour and Quiver Plot

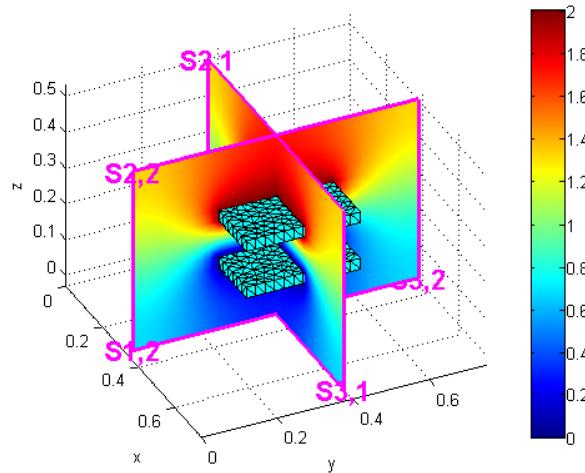
#### Download run through example:

Matlab / Octave file

FreeFEM script

## 2.20.6 3D Plot Examples

3D problems are handled by the `ffpdeplot3D()` command, which works similarly to the `ffpdeplot()` command. In particular in three-dimensions cross sections of the solution can be created. The following example shows a cross-sectional problem of a three-dimensional parallel plate capacitor.



**Fig. 2.27:** Slice on a 3D Parallel Plate Capacitor

**Download run through example:**

Matlab / Octave file

FreeFEM script

## 2.20.7 References

- Octave
- Matlab
- ffmtlib



---

CHAPTER  
THREE

---

## DOCUMENTATION

The fruit of a long maturing process, **freefem**, in its last avatar, **FreeFEM**, is a high level integrated development environment (IDE) for numerically solving partial differential equations (PDE) in dimension 2 and 3. It is the ideal tool for teaching the finite element method but it is also perfect for research to quickly test new ideas or multi-physics and complex applications.

**FreeFEM** has an advanced automatic mesh generator, capable of a posteriori mesh adaptation; it has a general purpose elliptic solver interfaced with fast algorithms, such as the multi-frontal method UMFPACK, SuperLU, MUMPS. Hyperbolic and parabolic problems are solved by iterative algorithms prescribed by the user with the high level language of **FreeFEM**. It has several triangular finite elements, including discontinuous elements. Everything is there in **FreeFEM** to prepare research quality reports with online color display, zooming and other features as well as postscript printouts.

This manual is meant for students at a Masters level, for researchers at any level, and for engineers (including financial engineering) with some understanding of variational methods for partial differential equations.

### Introduction

A partial differential equation is a relation between a function of several variables and its (partial) derivatives. Many problems in physics, engineering, mathematics and even banking are modeled by one or several partial differential equations.

**FreeFEM** is a software to solve these equations numerically. As its name implies, it is a free software (see the copyrights for full detail) based on the Finite Element Method; it is not a package, it is an integrated product with its own high level programming language. This software runs on all UNIX OS (with g++ 3.3 or later, and OpenGL), on Window XP, Vista and 7, 8, 10 and on MacOS 10 intel.

Moreover **FreeFEM** is highly adaptive. Many phenomena involve several coupled systems. Fluid-structure interactions, Lorentz forces for aluminum casting and ocean-atmosphere problems are three such systems. These require different finite element approximations and polynomial degrees, possibly on different meshes. Some algorithms like the Schwarz' domain decomposition method also requires data interpolation on multiple meshes within one program. **FreeFEM** can handle these difficulties, i.e. arbitrary finite element spaces on arbitrary unstructured and adapted bi-dimensional meshes.

The characteristics of **FreeFEM** are:

- Problem description (real or complex valued) by their variational formulations, with access to the internal vectors and matrices if needed.
- Multi-variables, multi-equations, bi-dimensional and three-dimensional static or time dependent, linear or non-linear coupled systems; however the user is required to describe the iterative procedures which reduce the problem to a set of linear problems.
- Easy geometric input by analytic description of boundaries by pieces; however this part is not a CAD system; for instance when two boundaries intersect, the user must specify the intersection points.

- Automatic mesh generator, based on the Delaunay-Voronoi algorithm; the inner point density is proportional to the density of points on the boundaries [[GEORGE1996](#)].
- Metric-based anisotropic mesh adaptation. The metric can be computed automatically from the Hessian of any **FreeFEM** function [[HECHT1998](#)].
- High level user friendly typed input language with an algebra of analytic and finite element functions.
- Multiple finite element meshes within one application with automatic interpolation of data on different meshes and possible storage of the interpolation matrices.
- A large variety of triangular finite elements: linear, quadratic Lagrangian elements and more, discontinuous P1 and Raviart-Thomas elements, elements of a non-scalar type, the mini-element,... (but no quadrangles).
- Tools to define discontinuous Galerkin finite element formulations P0, P1dc, P2dc and keywords: jump, mean, intalledges.
- A large variety of linear direct and iterative solvers (LU, Cholesky, Crout, CG, GMRES, UMFPACK, MUMPS, SuperLU, ...) and eigenvalue and eigenvector solvers (ARPARK) .
- Near optimal execution speed (compared with compiled C++ implementations programmed directly).
- Online graphics, generation of .txt,.eps,.gnu, mesh files for further manipulations of input and output data.
- Many examples and tutorials: elliptic, parabolic and hyperbolic problems, Navier-Stokes flows, elasticity, fluid structure interactions, Schwarz's domain decomposition method, eigenvalue problem, residual error indicator,  
...
- A parallel version using MPI

#### History

The project has evolved from MacFem, PCfem, written in Pascal. The first C version lead to `freefem 3.4`; it offered mesh adaptivity on a single mesh only.

A thorough rewriting in C++ led to `freefem+` (`freefem+ 1.2.10` was its last release), which included interpolation over multiple meshes (functions defined on one mesh can be used on any other mesh); this software is no longer maintained but is still in use because it handles a problem description using the strong form of the PDEs. Implementing the interpolation from one unstructured mesh to another was not easy because it had to be fast and non-diffusive; for each point, one had to find the containing triangle. This is one of the basic problems of computational geometry (see [[PREPARATA1985](#)] for example). Doing it in a minimum number of operations was the challenge. Our implementation is  $\mathcal{O}(n \log n)$  and based on a quadtree. This version also grew out of hand because of the evolution of the template syntax in C++.

We have been working for a few years now on **FreeFEM**, entirely re-written again in C++ with a thorough usage of template and generic programming for coupled systems of unknown size at compile time. Like all versions of `freefem`, it has a high level user friendly input language which is not too far from the mathematical writing of the problems.

The `freefem` language allows for a quick specification of any partial differential system of equations. The language syntax of **FreeFEM** is the result of a new design which makes use of the STL [[STROUSTRUP2000](#)], templates, and bison for its implementation; more details can be found in [[HECHT2002](#)]. The outcome is a versatile software in which any new finite elements can be included in a few hours; but a recompilation is then necessary. Therefore the library of finite elements available in **FreeFEM** will grow with the version number and with the number of users who program more new elements. So far we have discontinuous  $P_0$  elements, linear  $P_1$  and quadratic  $P_2$  Lagrangian elements, discontinuous  $P_1$  and Raviart-Thomas elements and a few others like bubble elements.

## 3.1 Notations

Here mathematical expressions and corresponding **FreeFEM** commands are explained.

### 3.1.1 Generalities

- $[\delta_{ij}]$  Kronecker delta (0 if  $i \neq j$ , 1 if  $i = j$  for integers  $i, j$ )
- $[\forall]$  for all
- $[\exists]$  there exists
- [i.e.] that is
- [PDE] partial differential equation (with boundary conditions)
- $[\emptyset]$  the empty set
- $[\mathbb{N}]$  the set of integers ( $a \in \mathbb{N} \Leftrightarrow \text{int } a$ ), int means long int inside FreeFEM
- $[\mathbb{R}]$  the set of real numbers ( $a \in \mathbb{R} \Leftrightarrow \text{real } a$ ), double inside FreeFEM
- $[\mathbb{C}]$  the set of complex numbers ( $a \in \mathbb{C} \Leftrightarrow \text{complex } a$ ), complex<double>
- $[\mathbb{R}^d]$   $d$ -dimensional Euclidean space

### 3.1.2 Sets, Mappings, Matrices, Vectors

Let  $E, F, G$  be three sets and  $A$  the subset of  $E$ .

- $[\{x \in E \mid P\}]$  the subset of  $E$  consisting of the elements possessing the property  $P$
- $[E \cup F]$  the set of elements belonging to  $E$  or  $F$
- $[E \cap F]$  the set of elements belonging to  $E$  and  $F$
- $[E \setminus A]$  the set  $\{x \in E \mid x \notin A\}$
- $[E + F]$   $E \cup F$  with  $E \cap F = \emptyset$
- $[E \times F]$  the Cartesian product of  $E$  and  $F$
- $[E^n]$  the  $n$ -th power of  $E$  ( $E^2 = E \times E, E^n = E \times E^{n-1}$ )
- $[f : E \rightarrow F]$  the mapping from  $E$  into  $F$ , i.e.,  $E \ni x \mapsto f(x) \in F$
- $[I_E \text{ or } I]$  the identity mapping in  $E$ , i.e.,  $I(x) = x \quad \forall x \in E$
- $[f \circ g]$  for  $f : F \rightarrow G$  and  $g : E \rightarrow F$ ,  $E \ni x \mapsto (f \circ g)(x) = f(g(x)) \in G$  (see [Elementary function](#))
- $[f|_A]$  the restriction of  $f : E \rightarrow F$  to the subset  $A$  of  $E$
- $[\{a_k\}]$  column vector with components  $a_k$
- $[(a_k)]$  row vector with components  $a_k$
- $[(a_k)^T]$  denotes the transpose of a matrix  $(a_k)$ , and is  $\{a_k\}$
- $[\{a_{ij}\}]$  matrix with components  $a_{ij}$ , and  $(a_{ij})^T = (a_{ji})$

### 3.1.3 Numbers

For two real numbers  $a, b$

- $[a, b]$  is the interval  $\{x \in \mathbb{R} \mid a \leq x \leq b\}$
- $]a, b]$  is the interval  $\{x \in \mathbb{R} \mid a < x \leq b\}$
- $[a, b[$  is the interval  $\{x \in \mathbb{R} \mid a \leq x < b\}$

- $]a, b[$  is the interval  $\{x \in \mathbb{R} \mid a < x < b\}$

### 3.1.4 Differential Calculus

- $[\partial f / \partial x]$  the partial derivative of  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  with respect to  $x$  (`dx(f)`)
- $[\nabla f]$  the gradient of  $f : \Omega \rightarrow \mathbb{R}$ , i.e.,  $\nabla f = (\partial f / \partial x, \partial f / \partial y)$
- $[\operatorname{div}(\mathbf{f}) \text{ or } \nabla \cdot \mathbf{f}]$  the divergence of  $\mathbf{f} : \Omega \rightarrow \mathbb{R}^d$ , i.e.,  $\operatorname{div}(\mathbf{f}) = \partial f_1 / \partial x + \partial f_2 / \partial y$
- $[\Delta f]$  the Laplacian of  $f : \Omega \rightarrow \mathbb{R}$ , i.e.,  $\Delta f = \partial^2 f / \partial x^2 + \partial^2 f / \partial y^2$

### 3.1.5 Meshes

- $[\Omega]$  usually denotes a domain on which PDE is defined
- $[\Gamma]$  denotes the boundary of  $\Omega$ , i.e.,  $\Gamma = \partial\Omega$  (keyword `border`, see [Border](#))
- $[\mathcal{T}_h]$  the triangulation of  $\Omega$ , i.e., the set of triangles  $T_k$ , where  $h$  stands for mesh size (keyword `mesh`, `buildmesh`, see [Mesh Generation](#))
- $[n_t]$  the number of triangles in  $\mathcal{T}_h$  (get by `Th.nt`)
- $[\Omega_h]$  denotes the approximated domain  $\Omega_h = \cup_{k=1}^{n_t} T_k$  of  $\Omega$ . If  $\Omega$  is polygonal domain, then it will be  $\Omega = \Omega_h$
- $[\Gamma_h]$  the boundary of  $\Omega_h$
- $[n_v]$  the number of vertices in  $\mathcal{T}_h$  (get by `Th.nv`)
- $[n_{be}]$  the number of boundary element in  $\mathcal{T}_h$  (get by `Th.nbe`)
- $[|\Omega_h|]$  the measure (area or volume) in  $\mathcal{T}_h$  (get by `Th.measure`)
- $[|\partial\Omega_h|]$  the measure of the border (length or area) in  $\mathcal{T}_h$  (get by `Th.bordermeasure`)
- $[h_{min}]$  the minimum edge size of  $\mathcal{T}_h$  (get by `Th.hmin`)
- $[h_{max}]$  the maximum edge size of  $\mathcal{T}_h$  (get by `Th.hmax`)
- $[[q^i q^j]]$  the segment connecting  $q^i$  and  $q^j$
- $[[q^{k_1}, q^{k_2}, q^{k_3}]]$  the vertices of a triangle  $T_k$  with anti-clock direction (get the coordinate of  $q^{k_j}$  by `(Th[k-1][j-1].x, Th[k-1][j-1].y)`)
- $[I_\Omega]$  the set  $\{i \in \mathbb{N} \mid q^i \notin \Gamma_h\}$

### 3.1.6 Functional Spaces

- $[L^2(\Omega)]$  the set  $\left\{ w(x, y) \mid \int_{\Omega} |w(x, y)|^2 dx dy < \infty \right\}$

$$\text{norm: } \|w\|_{0,\Omega} = \left( \int_{\Omega} |w(x, y)|^2 dx dy \right)^{1/2}$$

$$\text{scalar product: } (v, w) = \int_{\Omega} vw$$

- $[H^1(\Omega)]$  the set  $\left\{ w \in L^2(\Omega) \mid \int_{\Omega} (|\partial w / \partial x|^2 + |\partial w / \partial y|^2) dx dy < \infty \right\}$

$$\text{norm: } \|w\|_{1,\Omega} = (\|w\|_{0,\Omega}^2 + \|\nabla u\|_{0,\Omega}^2)^{1/2}$$

- $[H^m(\Omega)]$  the set  $\left\{ w \in L^2(\Omega) \mid \int_{\Omega} \frac{\partial^{|\alpha|} w}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \in L^2(\Omega) \quad \forall \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2, |\alpha| = \alpha_1 + \alpha_2 \right\}$

$$\text{scalar product: } (v, w)_{1,\Omega} = \sum_{|\alpha| \leq m} \int_{\Omega} D^\alpha v D^\alpha w$$

- $[H_0^1(\Omega)]$  the set  $\{w \in H^1(\Omega) \mid u = 0 \text{ on } \Gamma\}$
- $[L^2(\Omega)^2]$  denotes  $L^2(\Omega) \times L^2(\Omega)$ , and also  $H^1(\Omega)^2 = H^1(\Omega) \times H^1(\Omega)$

### 3.1.7 Finite Element Spaces

- $[V_h]$  denotes the finite element space created by `fespace Vh(Th, *)` in **FreeFEM** (see *Finite Elements* for `*`)
- $[\Pi_h f]$  the projection of the function  $f$  into  $V_h$  (`func f=x^2*y^3; Vh v = f;`) means  $v = \Pi_h(f)*[v]$  for FE-function  $v$  in  $V_h$  means the column vector  $(v_1, \dots, v_M)^T$  if  $v = v_1\phi_1 + \dots + v_M\phi_M$ , which is shown by `fespace Vh(Th, P2); Vh v; cout << v[] << endl;`

## 3.2 Mesh Generation

Let us begin with the two important keywords: `border` and `buildmesh`.

### 3.2.1 Generalities

#### Square

The command `square` triangulates the unit square.

The following generates a  $4 \times 5$  grid in the unit square  $[0, 1]^2$ . The labels of the boundaries are shown in Fig. 3.1.

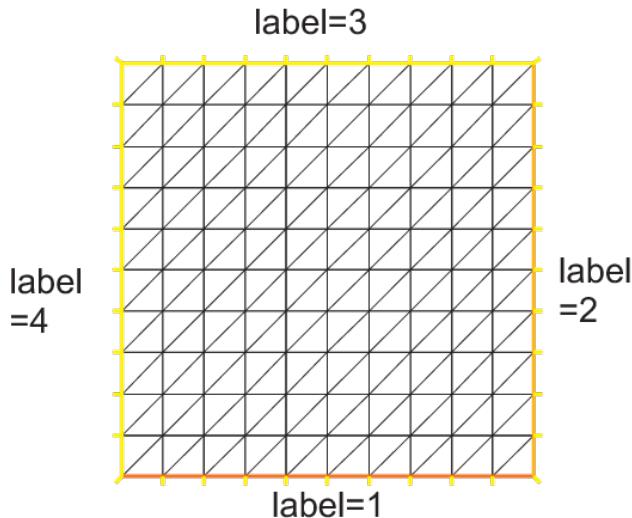
```
1 mesh Th = square(4, 5);
```

To construct a  $n \times m$  grid in the rectangle  $[x_0, x_1] \times [y_0, y_1]$ , proceed as follows:

```
1 real x0 = 1.2;
2 real x1 = 1.8;
3 real y0 = 0;
4 real y1 = 1;
5 int n = 5;
6 real m = 20;
7 mesh Th = square(n, m, [x0+(x1-x0)*x, y0+(y1-y0)*y]);
```

**Note:** Adding the named parameter `flags=icase` with `icase`:

0. will produce a mesh where all quads are split with diagonal  $x - y = constant$
1. will produce *Union Jack flag* type of mesh

**Fig. 3.1:** Boundary labels of the mesh by `square(10, 10)`

2. will produce a mesh where all quads are split with diagonal  $x + y = \text{constant}$
3. same as in case 0, except two corners where the triangles are the same as case 2 to avoid having 3 vertices on the boundary
4. same as in case 2, except two corners where the triangles are the same as case 0 to avoid having 3 vertices on the boundary

```
1 mesh Th = square(n, m, [x0+(x1-x0)*x, y0+(y1-y0)*y], flags=icase);
```

**Note:** Adding the named parameter `label=labs` will change the 4 default label numbers to `labs[i-1]`, for example `int[int] labs=[11, 12, 13, 14]`, and adding the named parameter `region=10` will change the region number to 10, for instance (v 3.8).

To see all of these flags at work, check [Square mesh example](#):

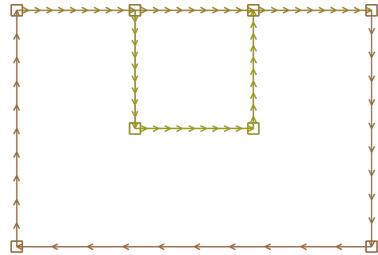
```
1 for (int i = 0; i < 5; ++i){
2     int[int] labs = [11, 12, 13, 14];
3     mesh Th = square(3, 3, flags=i, label=labs, region=10);
4     plot(Th, wait=1, cmm="square flags = "+i );
5 }
```

## Border

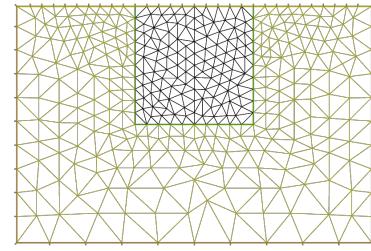
Boundaries are defined piecewise by parametrized curves. The pieces can only intersect at their endpoints, but it is possible to join more than two endpoints. This can be used to structure the mesh if an area touches a border and create new regions by dividing larger ones:

```
1 int upper = 1;
2 int others = 2;
3 int inner = 3;
```

(continues on next page)



(a) Multiple border ends intersect



(b) Generated mesh

**Fig. 3.2:** Border

(continued from previous page)

```

5  border C01(t=0, 1){x=0; y=-1+t; label=upper;}
6  border C02(t=0, 1){x=1.5-1.5*t; y=-1; label=upper;}
7  border C03(t=0, 1){x=1.5; y=-t; label=upper;}
8  border C04(t=0, 1){x=1+0.5*t; y=0; label=others;}
9  border C05(t=0, 1){x=0.5+0.5*t; y=0; label=others;}
10 border C06(t=0, 1){x=0.5*t; y=0; label=others;}
11 border C11(t=0, 1){x=0.5; y=-0.5*t; label=inner;}
12 border C12(t=0, 1){x=0.5+0.5*t; y=-0.5; label=inner;}
13 border C13(t=0, 1){x=1; y=-0.5+0.5*t; label=inner;}
14
15 int n = 10;
16 plot(C01(-n) + C02(-n) + C03(-n) + C04(-n) + C05(-n)
17     + C06(-n) + C11(n) + C12(n) + C13(n), wait=true);
18
19 mesh Th = buildmesh(C01(-n) + C02(-n) + C03(-n) + C04(-n) + C05(-n)
20     + C06(-n) + C11(n) + C12(n) + C13(n));
21
22 plot(Th, wait=true);
23
24 cout << "Part 1 has region number " << Th(0.75, -0.25).region << endl;
25 cout << "Part 2 has region number " << Th(0.25, -0.25).region << endl;
```

Borders and mesh are respectively shown in Fig. 3.2a and Fig. 3.2b.

Triangulation keywords assume that the domain is defined as being on the *left* (resp *right*) of its oriented parameterized boundary

$$\Gamma_j = \{(x, y) \mid x = \varphi_x(t), y = \varphi_y(t), a_j \leq t \leq b_j\}$$

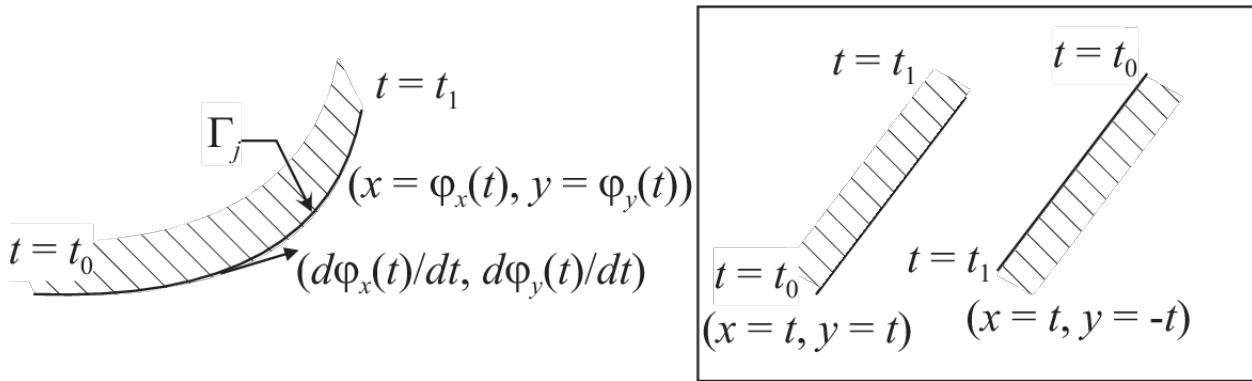
To check the orientation plot  $t \mapsto (\varphi_x(t), \varphi_y(t))$ ,  $t_0 \leq t \leq t_1$ . If it is as in Fig. 3.3, then the domain lies on the shaded area, otherwise it lies on the opposite side.

The general expression to define a triangulation with `buildmesh` is

```
1 mesh Mesh_Name = buildmesh(Gamma1(m1)+...+GammaJ(mj), OptionalParameter);
```

where  $m_j$  are positive or negative numbers to indicate how many vertices should be on  $\Gamma_j$ ,  $\Gamma = \cup_{j=1}^J \Gamma_j$ , and the optional parameter (see also [References](#)), separated with a comma, can be:

- `nbovx= int`, to set the maximum number of vertices in the mesh.



**Fig. 3.3:** Orientation of the boundary defined by  $(\phi_x(t), \phi_y(t))$

- `fixedborder= bool`, to say if the mesh generator can change the boundary mesh or not (by default the boundary mesh can change; beware that with periodic boundary conditions (see. *Finite Element*), it can be dangerous).

The orientation of boundaries can be changed by changing the sign of  $m_j$ .

The following example shows how to change the orientation. The example generates the unit disk with a small circular hole, and assigns “1” to the unit disk (“2” to the circle inside). The boundary label **must be non-zero**, but it can also be omitted.

```

1 border a(t=0, 2*pi) {x=cos(t); y=sin(t); label=1;}
2 border b(t=0, 2*pi) {x=0.3+0.3*cos(t); y=0.3*sin(t); label=2;}
3 plot(a(50) + b(30)); //to see a plot of the border mesh
4 mesh Thwithouthole = buildmesh(a(50) + b(30));
5 mesh Thwithhole = buildmesh(a(50) + b(-30));
6 plot(Thwithouthole, ps="Thwithouthole.eps");
7 plot(Thwithhole, ps="Thwithhole.eps");

```

**Note:** Notice that the orientation is changed by `b(-30)` in the 5th line. In the 7th line, `ps="fileName"` is used to generate a postscript file with identification shown on the figure.

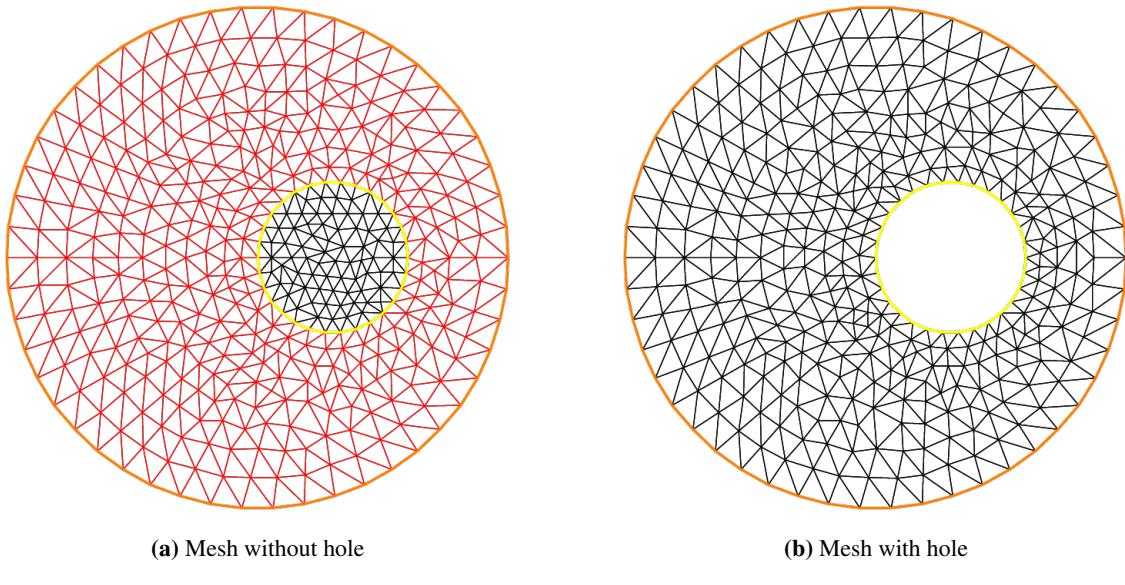
**Note:** Borders are evaluated only at the time `plot` or `buildmesh` is called so the global variables are defined at this time. In this case, since `r` is changed between the two border calls, the following code will not work because the first border will be computed with `r=0.3`:

```

1 real r=1;
2 border a(t=0, 2*pi) {x=r*cos(t); y=r*sin(t); label=1;}
3 r=0.3;
4 border b(t=0, 2*pi) {x=r*cos(t); y=r*sin(t); label=1;}
5 mesh Thwithhole = buildmesh(a(50) + b(-30)); // bug (a trap) because
6 // the two circles have the same radius = :math:`0.3`
```

## Multi-Border

Sometimes it can be useful to make an array of the border, but unfortunately it is incompatible with the **FreeFEM** syntax. To bypass this problem, if the number of segments of the discretization `n` is an array, we make an implicit loop



**Fig. 3.4:** Mesh with a hole

on all of the values of the array, and the index variable *i* of the loop is defined after the parameter definition, like in border `a(t=0, 2*pi; i)` ...

A first very small example:

```

1 border a(t=0, 2*pi; i){x=(i+1)*cos(t); y=(i+1)*sin(t); label=i; }
2 int[int] nn = [10, 20, 30];
3 plot(a(nn)); //plot 3 circles with 10, 20, 30 points

```

And a more complex example to define a square with small circles:

```

1  real[int] xx = [0, 1, 1, 0],
2      YY = [0, 0, 1, 1];
3 //radius, center of the 4 circles
4  real[int] RC = [0.1, 0.05, 0.05, 0.1],
5      XC = [0.2, 0.8, 0.2, 0.8],
6      YC = [0.2, 0.8, 0.8, 0.2];
7 int[int] NC = [-10,-11,-12,13]; //list number of :math:`\backslash pm` segments of the 4_
8   ↪circles borders
9
10 border bb(t=0, 1; i)
11 {
12     // i is the index variable of the multi border loop
13     int ii = (i+1)%4;
14     real t1 = 1-t;
15     x = xx[i]*t1 + xx[ii]*t;
16     y = yy[i]*t1 + yy[ii]*t;
17     label = 0;
18 }
19 border cc(t=0, 2*pi; i)
20 {
21     x = RC[i]*cos(t) + XC[i];
22     y = RC[i]*sin(t) + YC[i];

```

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```

23   label = i + 1;
24 }
25 int[int] nn = [4, 4, 5, 7]; //4 border, with 4, 4, 5, 7 segment respectively
26 plot(bb(nn), cc(NC), wait=1);
27 mesh th = buildmesh(bb(nn) + cc(NC));
28 plot(th, wait=1);

```

## Data Structures and Read/Write Statements for a Mesh

Users who want to read a triangulation made elsewhere should see the structure of the file generated below:

```

1 border C(t=0, 2*pi) {x=cos(t); y=sin(t);}
2 mesh Th = buildmesh(C(10));
3 savemesh(Th, "mesh.msh");

```

The mesh is shown on Fig. 3.5.

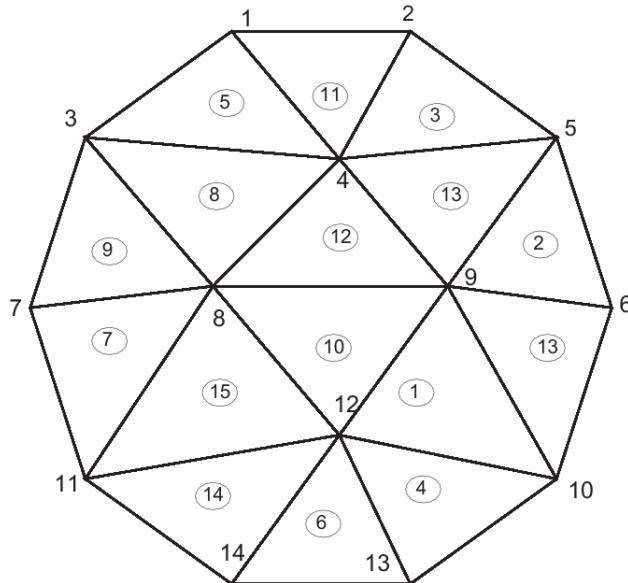
The information about Th are saved in the file `mesh.msh` whose structure is shown on Table 3.1.

There,  $n_v$  denotes the number of vertices,  $n_t$  the number of triangles and  $n_s$  the number of edges on boundary.

For each vertex  $q^i$ ,  $i = 1, \dots, n_v$ , denoted by  $(q_x^i, q_y^i)$  the  $x$ -coordinate and  $y$ -coordinate.

Each triangle  $T_k$ ,  $k = 1, \dots, n_t$  has three vertices  $q^{k_1}, q^{k_2}, q^{k_3}$  that are oriented counter-clockwise.

The boundary consists of 10 lines  $L_i$ ,  $i = 1, \dots, 10$  whose end points are  $q^{i_1}, q^{i_2}$ .



**Fig. 3.5:** Mesh by `buildmesh(C(10))`

In the Fig. 3.5, we have the following.

$$n_v = 14, n_t = 16, n_s = 10$$

$$q^1 = (-0.309016994375, 0.951056516295)$$

...

$$q^{14} = (-0.309016994375, -0.951056516295)$$

The vertices of  $T_1$  are  $q^9, q^{12}, q^{10}$ .

...

The vertices of  $T_{16}$  are  $q^9, q^{10}, q^6$ .

The edge of the 1st side  $L_1$  are  $q^6, q^5$ .

...

The edge of the 10th side  $L_{10}$  are  $q^{10}, q^6$ .

**Table 3.1:** The structure of `mesh_sample.msh`

Content of the file	Explanation
14 16 10	$n_v \quad n_t \quad n_e$
-0.309016994375 0.951056516295 1	$q_x^1 \quad q_y^1$ boundary label = 1
0.309016994375 0.951056516295 1	$q_x^2 \quad q_y^2$ boundary label = 1
...	...
-0.309016994375 -0.951056516295 1	$q_x^{14} \quad q_y^{14}$ boundary label = 1
9 12 10 0	1 <sub>1</sub> 1 <sub>2</sub> 1 <sub>3</sub> region label = 0
5 9 6 0	2 <sub>1</sub> 2 <sub>2</sub> 2 <sub>3</sub> region label = 0
...	...
9 10 6 0	16 <sub>1</sub> 16 <sub>2</sub> 16 <sub>3</sub> region label = 0
6 5 1	1 <sub>1</sub> 1 <sub>2</sub> boundary label = 1
5 2 1	2 <sub>1</sub> 2 <sub>2</sub> boundary label = 1
...	...
10 6 1	10 <sub>1</sub> 10 <sub>2</sub> boundary label = 1

In **FreeFEM** there are many mesh file formats available for communication with other tools such as `emc2`, `modulef`, ... (see [Mesh format chapter](#)).

The extension of a file implies its format. More details can be found on the file format `.msh` in the article by F. Hecht “bamg : a bidimensional anisotropic mesh generator” [[HECHT1998\\_2](#)].

A mesh file can be read into **FreeFEM** except that the names of the borders are lost and only their reference numbers are kept. So these borders have to be referenced by the number which corresponds to their order of appearance in the program, unless this number is overwritten by the keyword `label`. Here are some examples:

```

1 border floor(t=0, 1){x=t; y=0; label=1;}
2 border right(t=0, 1){x=1; y=t; label=5;}
3 border ceiling(t=1, 0){x=t; y=1; label=5;}
4 border left(t=1, 0){x=0; y=t; label=5;}

5
6 int n = 10;
7 mesh th = buildmesh(floor(n) + right(n) + ceiling(n) + left(n));
8 savemesh(th, "toto.am_fmt"); // "formatted Marrocco" format
9 savemesh(th, "toto.Th"); // "bamg"-type mesh
10 savemesh(th, "toto.msh"); // freefem format
11 savemesh(th, "toto.nopo"); // modulef format
12 mesh th2 = readmesh("toto.msh"); // read the mesh

```

```

1 // Parameters
2 int n = 10;

3
4 // Mesh
5 border floor(t=0, 1){x=t; y=0; label=1;};
6 border right(t=0, 1){x=1; y=t; label=5;};

```

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```

7  border ceiling(t=1, 0){x=t; y=1; label=5;};
8  border left(t=1, 0){x=0; y=t; label=5;};
9
10 mesh th = buildmesh(floor(n) + right(n) + ceiling(n) + left(n));
11
12 //save mesh in different formats
13 savemesh(th, "toto.am_fmt"); // format "formated Marrocco"
14 savemesh(th, "toto.Th"); // format database db mesh "bamg"
15 savemesh(th, "toto.msh"); // format freefem
16 savemesh(th, "toto.nopo"); // modulef format
17
18 // Fespace
19 fespace fempl(th, P1);
20 fempl f = sin(x)*cos(y);
21 fempl g;
22
23 //save the fespace function in a file
24 {
25     ofstream file("f.txt");
26     file << f[] << endl;
27 } //the file is automatically closed at the end of the block
28 //read a file and put it in a fespace function
29 {
30     ifstream file("f.txt");
31     file >> g[];
32 } //the file is equally automatically closed
33
34 // Plot
35 plot(g);
36
37 // Mesh 2
38 //read the mesh for freefem format saved mesh
39 mesh th2 = readmesh("toto.msh");
40
41 // Fespace 2
42 fespace Vh2(th2, P1);
43 Vh2 u, v;
44
45 // Problem
46 //solve:
47 // $u + \Delta u = g$ in $\Omega $
48 // $u=0$ on $\Gamma_1$
49 // $\frac{\partial u}{\partial n} = g$ on $\Gamma_2$
50 solve Problem(u, v)
51 = int2d(th2)(
52     u*v
53     - dx(u)*dx(v)
54     - dy(u)*dy(v)
55 )
56 + int2d(th2)(
57     - g*v
58 )
59 + int1d(th2, 5)(
60     g*v
61 )
62 + on(1, u=0)
63 ;

```

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```

64
65 // Plot
66 plot(th2, u);

```

## Mesh Connectivity and data

The following example explains methods to obtain mesh information.

```

1 // Mesh
2 mesh Th = square(2, 2);
3
4 cout << "// Get data of the mesh" << endl;
5 {
6     int NbTriangles = Th.nt;
7     real MeshArea = Th.measure;
8     real BorderLenght = Th.bordermeasure;
9
10    cout << "Number of triangle(s) = " << NbTriangles << endl;
11    cout << "Mesh area = " << MeshArea << endl;
12    cout << "Border length = " << BorderLenght << endl;
13
14    // Th(i) return the vertex i of Th
15    // Th[k] return the triangle k of Th
16    // Th[k][i] return the vertex i of the triangle k of Th
17    for (int i = 0; i < NbTriangles; i++)
18        for (int j = 0; j < 3; j++)
19            cout << i << " " << j << " - Th[i][j] = " << Th[i][j]
20            << ", x = " << Th[i][j].x
21            << ", y= " << Th[i][j].y
22            << ", label=" << Th[i][j].label << endl;
23    }
24
25    cout << "// Hack to get vertex coordinates" << endl;
26    {
27        fespace fempl(Th, P1);
28        fempl Thx=x, Thy=y;
29
30        int NbVertices = Th.nv;
31        cout << "Number of vertices = " << NbVertices << endl;
32
33        for (int i = 0; i < NbVertices; i++)
34            cout << "Th(" << i << ") : " << Th(i).x << " " << Th(i).y << " " << Th(i).
35            ↵label
36            << endl << "\told method: " << Thx[] [i] << " " << Thy[] [i] << endl;
37    }
38
39    cout << "// Method to find information of point (0.55,0.6)" << endl;
40    {
41        int TNumber = Th(0.55, 0.6).nuTriangle; //the triangle number
42        int RLabel = Th(0.55, 0.6).region; //the region label
43
44        cout << "Triangle number in point (0.55, 0.6): " << TNumber << endl;
45        cout << "Region label in point (0.55, 0.6): " << RLabel << endl;
46    }

```

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```

47 cout << " // Information of triangle" << endl;
48 {
49     int TNumber = Th(0.55, 0.6).nuTriangle;
50     real TArea = Th[TNumber].area; //triangle area
51     real TRegion = Th[TNumber].region; //triangle region
52     real TLabel = Th[TNumber].label; //triangle label, same as region for triangles
53
54     cout << "Area of triangle " << TNumber << ":" << TArea << endl;
55     cout << "Region of triangle " << TNumber << ":" << TRegion << endl;
56     cout << "Label of triangle " << TNumber << ":" << TLabel << endl;
57 }
58
59 cout << " // Hack to get a triangle containing point x, y or region number (old method)
60 ↵" << endl;
61 {
62     fespace femp0(Th, P0);
63     femp0 TNumbers; //a P0 function to get triangle numbering
64     for (int i = 0; i < Th.nt; i++)
65         TNumbers[] [i] = i;
66     femp0 RNumbers = region; //a P0 function to get the region number
67
68     int TNumber = TNumbers(0.55, 0.6); // Number of the triangle containing (0.55, 0,
69 ↵6)
70     int RNumber = RNumbers(0.55, 0.6); // Number of the region containing (0.55, 0,6)
71
72     cout << "Point (0.55,0,6) :" << endl;
73     cout << "\tTriangle number = " << TNumber << endl;
74     cout << "\tRegion number = " << RNumber << endl;
75 }
76
77 cout << " // New method to get boundary information and mesh adjacent" << endl;
78 {
79     int k = 0;
80     int l=1;
81     int e=1;
82
83     // Number of boundary elements
84     int NbBoundaryElements = Th.nbe;
85     cout << "Number of boundary element = " << NbBoundaryElements << endl;
86     // Boundary element k in {0, ..., Th.nbe}
87     int BoundaryElement = Th.be(k);
88     cout << "Boundary element " << k << " = " << BoundaryElement << endl;
89     // Vertice l in {0, 1} of boundary element k
90     int Vertex = Th.be(k)[l];
91     cout << "Vertice " << l << " of boundary element " << k << " = " << Vertex << endl;
92     // Triangle containg the boundary element k
93     int Triangle = Th.be(k).Element;
94     cout << "Triangle containing the boundary element " << k << " = " << Triangle <<
95     ↵endl;
96     // Triangle egde nubmer containing the boundary element k
97     int Edge = Th.be(k).whoinElement;
98     cout << "Triangle edge number containing the boundary element " << k << " = " <<
99     ↵Edge << endl;
100    // Adjacent triangle of the triangle k by edge e
101    int Adjacent = Th[k].adj(e); //The value of e is changed to the corresponding
102    ↵edge in the adjacent triangle
103    cout << "Adjacent triangle of the triangle " << k << " by edge " << e << " = " <<
104    ↵Adjacent << endl;

```

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```

99   cout << "\tCorresponding edge = " << e << endl;
100  // If there is no adjacent triangle by edge e, the same triangle is returned
101  // Th[k] == Th[k].adj(e)
102  // Else a different triangle is returned
103  // Th[k] != Th[k].adj(e)
104 }
105
106 cout << "// Print mesh connectivity " << endl;
107 {
108     int NbTriangles = Th.nt;
109     for (int k = 0; k < NbTriangles; k++)
110         cout << k << " : " << int(Th[k][0]) << " " << int(Th[k][1])
111         << " " << int(Th[k][2])
112         << ", label " << Th[k].label << endl;
113
114     for (int k = 0; k < NbTriangles; k++)
115         for (int e = 0, ee; e < 3; e++)
116             //set ee to e, and ee is change by method adj,
117             cout << k << " " << e << " <=> " << int(Th[k].adj((ee=e))) << " " << ee
118             << ", adj: " << (Th[k].adj((ee=e)) != Th[k]) << endl;
119
120     int NbBoundaryElements = Th.nbe;
121     for (int k = 0; k < NbBoundaryElements; k++)
122         cout << k << " : " << Th.be(k)[0] << " " << Th.be(k)[1]
123         << " , label " << Th.be(k).label
124         << " , triangle " << int(Th.be(k).Element)
125         << " " << Th.be(k).whoinElement << endl;
126
127     real[int] bb(4);
128     boundingbox(Th, bb);
129     // bb[0] = xmin, bb[1] = xmax, bb[2] = ymin, bb[3] = ymax
130     cout << "boundingbox:" << endl;
131     cout << "xmin = " << bb[0]
132         << ", xmax = " << bb[1]
133         << ", ymin = " << bb[2]
134         << ", ymax = " << bb[3] << endl;
135 }
```

The output is:

```

1 // Get data of the mesh
2 Number of triangle = 8
3 Mesh area = 1
4 Border length = 4
5 0 0 - Th[i][j] = 0, x = 0, y = 0, label=4
6 0 1 - Th[i][j] = 1, x = 0.5, y = 0, label=1
7 0 2 - Th[i][j] = 4, x = 0.5, y = 0.5, label=0
8 1 0 - Th[i][j] = 0, x = 0, y = 0, label=4
9 1 1 - Th[i][j] = 4, x = 0.5, y = 0.5, label=0
10 1 2 - Th[i][j] = 3, x = 0, y = 0.5, label=4
11 2 0 - Th[i][j] = 1, x = 0.5, y = 0, label=1
12 2 1 - Th[i][j] = 2, x = 1, y = 0, label=2
13 2 2 - Th[i][j] = 5, x = 1, y = 0.5, label=2
14 3 0 - Th[i][j] = 1, x = 0.5, y = 0, label=1
15 3 1 - Th[i][j] = 5, x = 1, y = 0.5, label=2
16 3 2 - Th[i][j] = 4, x = 0.5, y = 0.5, label=0
17 4 0 - Th[i][j] = 3, x = 0, y = 0.5, label=4
```

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```

18 4 1 - Th[i][j] = 4, x = 0.5, y= 0.5, label=0
19 4 2 - Th[i][j] = 7, x = 0.5, y= 1, label=3
20 5 0 - Th[i][j] = 3, x = 0, y= 0.5, label=4
21 5 1 - Th[i][j] = 7, x = 0.5, y= 1, label=3
22 5 2 - Th[i][j] = 6, x = 0, y= 1, label=4
23 6 0 - Th[i][j] = 4, x = 0.5, y= 0.5, label=0
24 6 1 - Th[i][j] = 5, x = 1, y= 0.5, label=2
25 6 2 - Th[i][j] = 8, x = 1, y= 1, label=3
26 7 0 - Th[i][j] = 4, x = 0.5, y= 0.5, label=0
27 7 1 - Th[i][j] = 8, x = 1, y= 1, label=3
28 7 2 - Th[i][j] = 7, x = 0.5, y= 1, label=3
29 // Hack to get vertex coordinates
30 Number of vertices = 9
31 Th(0) : 0 0 4
   old method: 0 0
32 Th(1) : 0.5 0 1
   old method: 0.5 0
33 Th(2) : 1 0 2
   old method: 1 0
34 Th(3) : 0 0.5 4
   old method: 0 0.5
35 Th(4) : 0.5 0.5 0
   old method: 0.5 0.5
36 Th(5) : 1 0.5 2
   old method: 1 0.5
37 Th(6) : 0 1 4
   old method: 0 1
38 Th(7) : 0.5 1 3
   old method: 0.5 1
39 Th(8) : 1 1 3
   old method: 1 1
40 // Method to find the information of point (0.55,0.6)
41 Triangle number in point (0.55, 0.6): 7
42 Region label in point (0.55, 0.6): 0
43 // Information of a triangle
44 Area of triangle 7: 0.125
45 Region of triangle 7: 0
46 Label of triangle 7: 0
47 // Hack to get a triangle containing point x, y or region number (old method)
48 Point (0.55,0,6) :
   Triangle number = 7
   Region number = 0
49 // New method to get boundary information and mesh adjacent
50 Number of boundary element = 8
51 Boundary element 0 = 0
52 Vertex 1 of boundary element 0 = 1
53 Triangle containing the boundary element 0 = 0
54 Triangle edge number containing the boundary element 0 = 2
55 Adjacent triangle of the triangle 0 by edge 1 = 1
   Corresponding edge = 2
56 // Print mesh connectivity
57 0 : 0 1 4, label 0
58 1 : 0 4 3, label 0
59 2 : 1 2 5, label 0
60 3 : 1 5 4, label 0
61 4 : 3 4 7, label 0
62 5 : 3 7 6, label 0

```

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```

75 6 : 4 5 8, label 0
76 7 : 4 8 7, label 0
77 0 0 <=> 3 1, adj: 1
78 0 1 <=> 1 2, adj: 1
79 0 2 <=> 0 2, adj: 0
80 1 0 <=> 4 2, adj: 1
81 1 1 <=> 1 1, adj: 0
82 1 2 <=> 0 1, adj: 1
83 2 0 <=> 2 0, adj: 0
84 2 1 <=> 3 2, adj: 1
85 2 2 <=> 2 2, adj: 0
86 3 0 <=> 6 2, adj: 1
87 3 1 <=> 0 0, adj: 1
88 3 2 <=> 2 1, adj: 1
89 4 0 <=> 7 1, adj: 1
90 4 1 <=> 5 2, adj: 1
91 4 2 <=> 1 0, adj: 1
92 5 0 <=> 5 0, adj: 0
93 5 1 <=> 5 1, adj: 0
94 5 2 <=> 4 1, adj: 1
95 6 0 <=> 6 0, adj: 0
96 6 1 <=> 7 2, adj: 1
97 6 2 <=> 3 0, adj: 1
98 7 0 <=> 7 0, adj: 0
99 7 1 <=> 4 0, adj: 1
100 7 2 <=> 6 1, adj: 1
101 0 : 0 1 , label 1, triangle 0 2
102 1 : 1 2 , label 1, triangle 2 2
103 2 : 2 5 , label 2, triangle 2 0
104 3 : 5 8 , label 2, triangle 6 0
105 4 : 6 7 , label 3, triangle 5 0
106 5 : 7 8 , label 3, triangle 7 0
107 6 : 0 3 , label 4, triangle 1 1
108 7 : 3 6 , label 4, triangle 5 1
109 boundingbox:
110 xmin = 0, xmax = 1, ymin = 0, ymax = 1

```

The real characteristic function of a mesh Th is chi (Th) in 2D and 3D where:

chi (Th) (P)=1 if  $P \in Th$

chi (Th) (P)=0 if  $P \notin Th$

### The keyword “triangulate”

**FreeFEM** is able to build a triangulation from a set of points. This triangulation is a Delaunay mesh of the convex hull of the set of points. It can be useful to build a mesh from a table function.

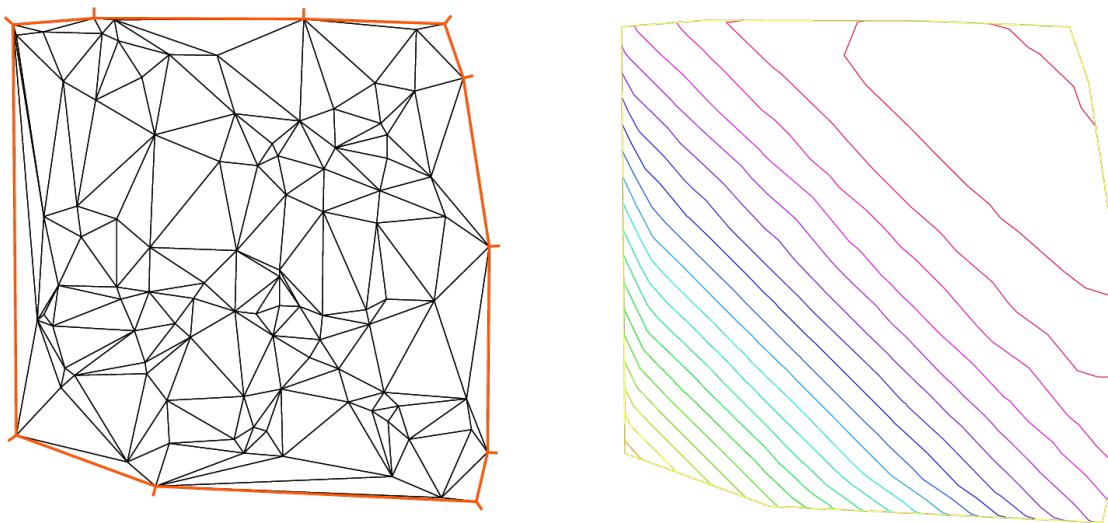
The coordinates of the points and the value of the table function are defined separately with rows of the form: x y f(x,y) in a file such as:

```

1 0.51387 0.175741 0.636237
2 0.308652 0.534534 0.746765
3 0.947628 0.171736 0.899823
4 0.702231 0.226431 0.800819
5 0.494773 0.12472 0.580623

```

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(a) Delaunay mesh of the convex hull of point set in file *xy*

(b) Isolines of table function

**Fig. 3.6:** Triangulate

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```

6 0.0838988 0.389647 0.456045
7 .....
```

The third column of each line is left untouched by the `triangulate` command. But you can use this third value to define a table function with rows of the form:  $x \ y \ f(x,y)$ .

The following example shows how to make a mesh from the file *xyf* with the format stated just above. The command `triangulate` only uses the 1st and 2nd columns.

```

1 // Build the Delaunay mesh of the convex hull
2 mesh Thxy=triangulate("xyf"); //points are defined by the first 2 columns of file
3 //xyf
4
5 // Plot the created mesh
6 plot(Thxy);
7
8 // Fespace
9 fespace Vhxy(Thxy, P1);
10 Vhxy fxy;
11
12 // Reading the 3rd column to define the function fxy
13 {
14     ifstream file("xyf");
15     real xx, yy;
16     for(int i = 0; i < fxy.n; i++)
17         file >> xx >> yy >> fxy[][i]; //to read third row only.
18                                         //xx and yy are just skipped
19
20 // Plot
21 plot(fxy);
```

One new way to build a mesh is to have two arrays: one for the  $x$  values and the other for the  $y$  values.

```

1 //set two arrays for the x's and y's
2 Vhxy xx=x, yy=y;
3 //build the mesh
4 mesh Th = triangulate(xx[], yy[]);

```

### 3.2.2 Boundary FEM Spaces Built as Empty Meshes

To define a Finite Element space on a boundary, we came up with the idea of a mesh with no internal points (called empty mesh). It can be useful to handle Lagrange multipliers in mixed and mortar methods.

So the function `emptymesh` removes all the internal points of a mesh except points on internal boundaries.

```

1 {
2     border a(t=0, 2*pi){x=cos(t); y=sin(t); label=1};
3     mesh Th = buildmesh(a(20));
4     Th = emptymesh(Th);
5     plot(Th);
6 }

```

It is also possible to build an empty mesh of a pseudo subregion with `emptymesh(Th, ssd)` using the set of edges from the mesh Th; an edge  $e$  is in this set when, with the two adjacent triangles  $e = t_1 \cap t_2$  and  $ssd[T_1] \neq ssd[T_2]$  where  $ssd$  refers to the pseudo region numbering of triangles, they are stored in the `int[int]` array of size “the number of triangles”.

```

1 {
2     mesh Th = square(10, 10);
3     int[int] ssd(Th.nt);
4     //build the pseudo region numbering
5     for(int i = 0; i < ssd.n; i++){
6         int iq = i/2; //because 2 triangles per quad
7         int ix = iq%10;
8         int iy = iq/10;
9         ssd[i] = 1 + (ix>=5) + (iy>=5)*2;
10    }
11    //build empty with all edges $e=T1 \cap T2$ and $ssd[T1] \neq ssd[T2]$
12    Th = emptymesh(Th, ssd);
13    //plot
14    plot(Th);
15    savemesh(Th, "emptymesh.msh");
16 }

```

### 3.2.3 Remeshing

#### Movemesh

Meshes can be translated, rotated, and deformed by `movemesh`; this is useful for elasticity to watch the deformation due to the displacement  $\Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y))$  of shape.

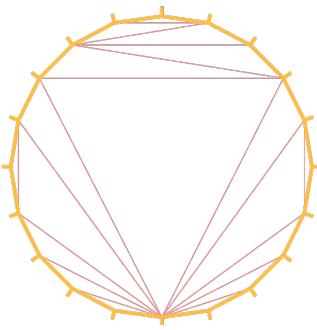
It is also useful to handle free boundary problems or optimal shape problems.

If  $\Omega$  is triangulated as  $T_h(\Omega)$ , and  $\Phi$  is a displacement vector then  $\Phi(T_h)$  is obtained by:

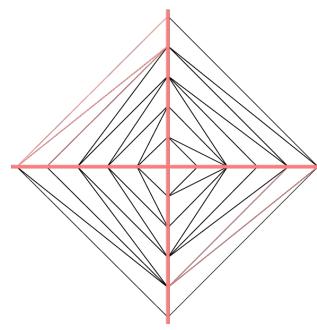
```

1 mesh Th = movemesh(Th, [Phi1, Phi2]);

```



(a) The empty mesh with boundary



(b) An empty mesh defined from a pseudo region numbering of triangle

**Fig. 3.7:** Empty mesh

Sometimes the transformed mesh is invalid because some triangles have flipped over (meaning it now has a negative area). To spot such problems, one may check the minimum triangle area in the transformed mesh with `checkmovemesh` before any real transformation.

For example:

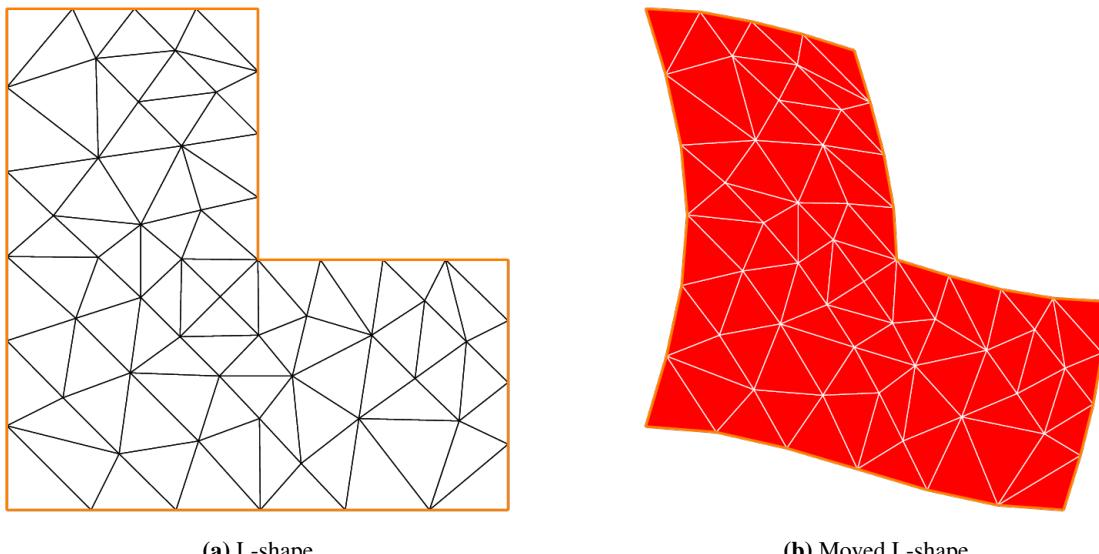
$$\begin{aligned}\Phi_1(x, y) &= x + k * \sin(y * \pi) / 10 \\ \Phi_2(x, y) &= y + k * \cos(y\pi) / 10\end{aligned}$$

for a big number  $k > 1$ .

```

1  verbosity = 4;
2
3 // Parameters
4  real coef = 1;
5
6 // Mesh
7 border a(t=0, 1){x=t; y=0; label=1;};
8 border b(t=0, 0.5){x=1; y=t; label=1;};
9 border c(t=0, 0.5){x=1-t; y=0.5; label=1;};
10 border d(t=0.5, 1){x=0.5; y=t; label=1;};
11 border e(t=0.5, 1){x=1-t; y=1; label=1;};
12 border f(t=0, 1){x=0; y=1-t; label=1;};
13 mesh Th = buildmesh(a(6) + b(4) + c(4) + d(4) + e(4) + f(6));
14 plot(Th, wait=true, fill=true, ps="Lshape.eps");
15
16 // Function
17 func uu = sin(y*pi)/10;
18 func vv = cos(x*pi)/10;
19
20 // Checkmovemesh
21 real mint0 = checkmovemesh(Th, [x, y]); //return the min triangle area
22 while(1){ // find a correct move mesh
23     real mint = checkmovemesh(Th, [x+coef*uu, y+coef*vv]);
24     if (mint > mint0/5) break; //if big enough
25     coef /= 1.5;
26 }
27
28 // Movemesh
29 Th = movemesh(Th, [x+coef*uu, y+coef*vv]);
30 plot(Th, wait=true, fill=true, ps="MovedMesh.eps");

```



**Fig. 3.8:** Move mesh

**Note:** Consider a function  $u$  defined on a mesh  $\text{Th}$ . A statement like  $\text{Th}=\text{movemesh}(\text{Th}\dots)$  does not change  $u$  and so the old mesh still exists. It will be destroyed when no function uses it. A statement like  $u = u$  redefines  $u$  on the new mesh  $\text{Th}$  with interpolation and therefore destroys the old  $\text{Th}$ , if  $u$  was the only function using it.

Now, we give an example of moving a mesh with a Lagrangian function  $u$  defined on the moving mesh.

```

1 // Parameters
2 int nn = 10;
3 real dt = 0.1;
4
5 // Mesh
6 mesh Th = square(nn, nn);
7
8 // Fespace
9 fespace Vh(Th, P1);
10 Vh u=y;
11
12 // Loop
13 real t=0;
14 for (int i = 0; i < 4; i++){
15     t = i*dt;
16     Vh f=x*t;
17     real minarea = checkmovemesh(Th, [x, y+f]);
18     if (minarea > 0) //movemesh will be ok
19     Th = movemesh(Th, [x, y+f]);
20
21     cout << " Min area = " << minarea << endl;
22
23     real[int] tmp(u[] .n);
24     tmp = u[]; //save the value
25     u = 0; //to change the FESpace and mesh associated with u
26     u[] = tmp; //set the value of u without any mesh update

```

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```

27   plot(Th, u, wait=true);
28 }
29 // In this program, since u is only defined on the last mesh, all the
30 // previous meshes are deleted from memory.

```

### 3.2.4 Regular Triangulation: hTriangle

For a set  $S$ , we define the diameter of  $S$  by

$$\text{diam}(S) = \sup\{|\mathbf{x} - \mathbf{y}|; \mathbf{x}, \mathbf{y} \in S\}$$

The sequence  $\{\mathcal{T}_h\}_{h \rightarrow 0}$  of  $\Omega$  is called *regular* if they satisfy the following:

1.  $\lim_{h \rightarrow 0} \max\{\text{diam}(T_k) | T_k \in \mathcal{T}_h\} = 0$
2. There is a number  $\sigma > 0$  independent of  $h$  such that  $\frac{\rho(T_k)}{\text{diam}(T_k)} \geq \sigma$  for all  $T_k \in \mathcal{T}_h$  where  $\rho(T_k)$  are the diameter of the inscribed circle of  $T_k$ .

We put  $h(\mathcal{T}_h) = \max\{\text{diam}(T_k) | T_k \in \mathcal{T}_h\}$ , which is obtained by

```

1 mesh Th = ....;
2 fespace Ph(Th, P0);
3 Ph h = hTriangle;
4 cout << "size of mesh = " << h[].max << endl;

```

### 3.2.5 Adaptmesh

The function:

$$f(x, y) = 10.0x^3 + y^3 + \tan^{-1}[\varepsilon / (\sin(5.0y) - 2.0x)], \varepsilon = 0.0001$$

sharply varies in value and the initial mesh given by one of the commands in the *Mesh Generation part* cannot reflect its sharp variations.

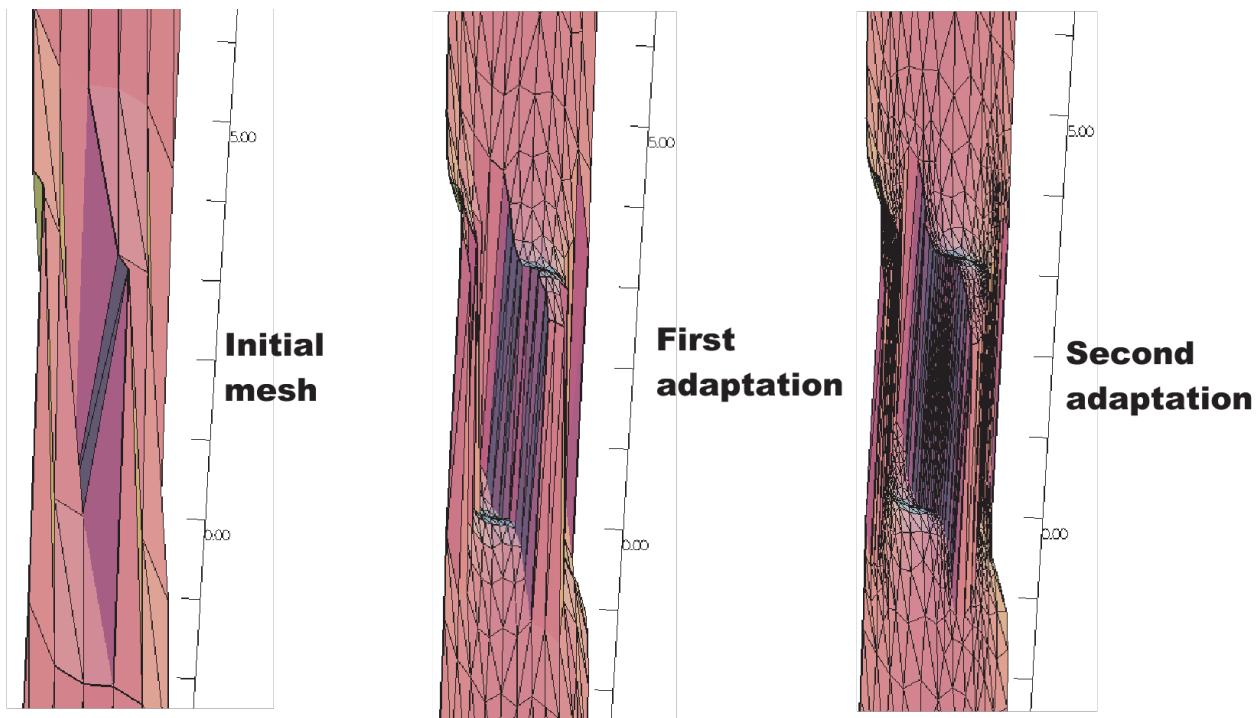
```

1 // Parameters
2 real eps = 0.0001;
3 real h = 1;
4 real hmin = 0.05;
5 func f = 10.0*x^3 + y^3 + h*atan2(eps, sin(5.0*y)-2.0*x);

6
7 // Mesh
8 mesh Th = square(5, 5, [-1+2*x, -1+2*y]);
9
10 // Fespace
11 fespace Vh(Th,P1);
12 Vh fh = f;
13 plot(fh);

14
15 // Adaptmesh
16 for (int i = 0; i < 2; i++) {
17     Th = adaptmesh(Th, fh);
18     fh = f; //old mesh is deleted
19     plot(Th, fh, wait=true);
20 }

```



**Fig. 3.9:** 3D graphs for the initial mesh and 1st and 2nd mesh adaptations

FreeFEM uses a variable metric/Delaunay automatic meshing algorithm.

The command:

```
1 mesh ATh = adaptmesh(Th, f);
```

create the new mesh ATh adapted to the Hessian

$$D^2f = (\partial^2 f / \partial x^2, \partial^2 f / \partial x \partial y, \partial^2 f / \partial y^2)$$

of a function (formula or FE-function).

Mesh adaptation is a very powerful tool when the solution of a problem varies locally and sharply.

Here we solve the *Poisson's problem*, when  $f = 1$  and  $\Omega$  is a L-shape domain.

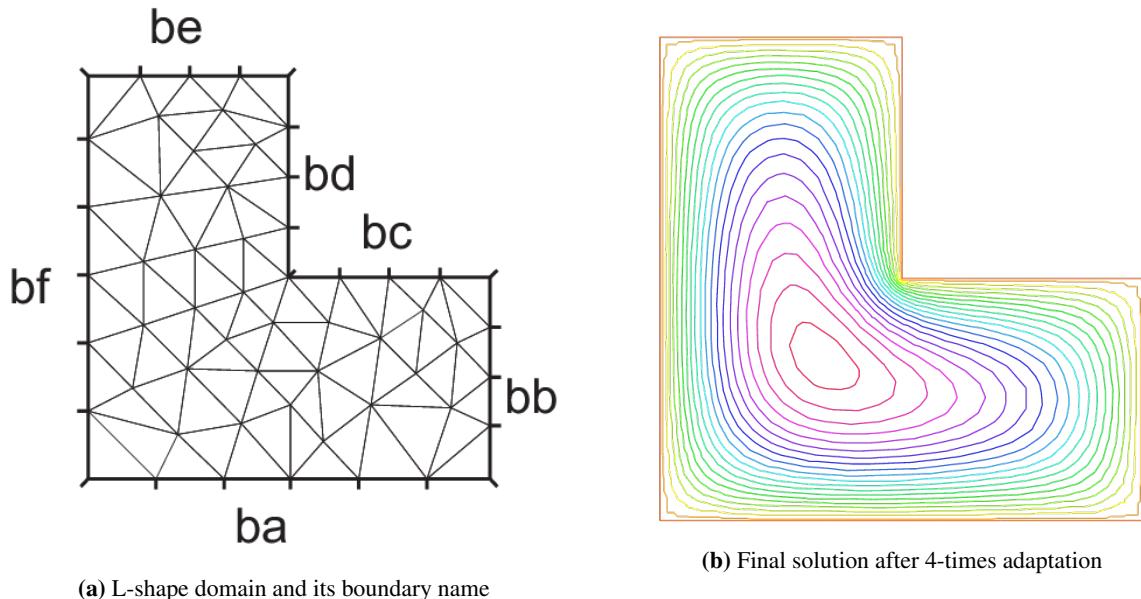
---

**Tip:** The solution has the singularity  $r^{3/2}$ ,  $r = |x - \gamma|$  at the point  $\gamma$  of the intersection of two lines  $bc$  and  $bd$  (see Fig. 3.10a).

---

```
1 // Parameters
2 real error = 0.1;
3
4 // Mesh
5 border ba(t=0, 1){x=t; y=0; label=1;};
6 border bb(t=0, 0.5){x=1; y=t; label=1;};
7 border bc(t=0, 0.5){x=1-t; y=0.5; label=1;};
8 border bd(t=0.5, 1){x=0.5; y=t; label=1;};
9 border be(t=0.5, 1){x=1-t; y=1; label=1;};
10 border bf(t=0, 1){x=0; y=1-t; label=1;};
```

(continues on next page)

**Fig. 3.10:** Mesh adaptation

(continued from previous page)

```

11 mesh Th = buildmesh (ba(6) + bb(4) + bc(4) + bd(4) + be(4) + bf(6));
12
13 // Fespace
14 fespace Vh(Th, P1);
15 Vh u, v;
16
17 // Function
18 func f = 1;
19
20 // Problem
21 problem Poisson(u, v, solver=CG, eps=1.e-6)
22   = int2d(Th) (
23     dx(u)*dx(v)
24     + dy(u)*dy(v)
25   )
26   - int2d(Th) (
27     f*v
28   )
29   + on(1, u=0);
30
31 // Adaptmesh loop
32 for (int i = 0; i < 4; i++) {
33   Poisson;
34   Th = adaptmesh(Th, u, err=error);
35   error = error/2;
36 }
37
38 // Plot
39 plot(u);

```

To speed up the adaptation, the default parameter `err` of `adaptmesh` is changed by hand; it specifies the required precision, so as to make the new mesh finer or coarser.

The problem is coercive and symmetric, so the linear system can be solved with the conjugate gradient method (parameter `solver=CG`) with the stopping criteria on the residual, here `eps=1.e-6`.

By `adaptmesh`, the slope of the final solution is correctly computed near the point of intersection of `bc` and `bd` as in Fig. 3.10b.

This method is described in detail in [HECHT1998]. It has a number of default parameters which can be modified.

If `f1, f2` are functions and `thold, Thnew` are meshes:

```

1 Thnew = adaptmesh(thold, f1 ... );
2 Thnew = adaptmesh(thold, f1,f2 ... );
3 Thnew = adaptmesh(thold, [f1,f2] ... );

```

The additional parameters of `adaptmesh` are:

See [Reference part](#) for more information

- **`hmin= Minimum edge size.`** Its default is related to the size of the domain to be meshed and the precision of the mesh generator.
- **`hmax= Maximum edge size.`** It defaults to the diameter of the domain to be meshed.
- `err=  $P_1$`  interpolation error level (0.01 is the default).
- **`errg= Relative geometrical error.`** By default this error is 0.01, and in any case it must be lower than  $1/\sqrt{2}$ . Meshes created with this option may have some edges smaller than the `-hmin` due to geometrical constraints.
- `nbx= Maximum number of vertices generated by the mesh generator` (9000 is the default).
- `nbsmooth= number of iterations of the smoothing procedure` (5 is the default).
- `nbjacobi= number of iterations in a smoothing procedure during the metric construction`, 0 means no smoothing, 6 is the default.
- **`ratio= ratio for a prescribed smoothing on the metric.`** If the value is 0 or less than 1.1 no smoothing is done on the metric. 1.8 is the default. If `ratio > 1.1`, the speed of mesh size variations is bounded by  $\log(\text{ratio})$ .

---

**Note:** As `ratio` gets closer to 1, the number of generated vertices increases. This may be useful to control the thickness of refined regions near shocks or boundary layers.

---

- `omega= relaxation parameter for the smoothing procedure`. 1.0 is the default.
- `iso= If true, forces the metric to be isotropic`. `false` is the default.
- **`abserror= If false, the metric is evaluated using the criteria of equi-repartition of relative error.`** `false` is the default. In this case the metric is defined by:

$$\mathcal{M} = \left( \frac{1}{\text{err coef}^2} \quad \frac{|\mathcal{H}|}{\max(\text{CutOff}, |\eta|)} \right)^p$$

Otherwise, the metric is evaluated using the criteria of equi-distribution of errors. In this case the metric is defined by:

$$\mathcal{M} = \left( \frac{1}{\text{err coef}^2} \quad \frac{|\mathcal{H}|}{\sup(\eta) - \inf(\eta)} \right)^p.$$

- `cutoff= lower limit for the relative error evaluation`. 1.0e-6 is the default.

- **verbosity**= informational messages level (can be chosen between 0 and  $\infty$ ). Also changes the value of the global variable `verbosity` (obsolete).
- **inquire**= To inquire graphically about the mesh. `false` is the default.
- **splitpbedge**= If true, splits all internal edges in half with two boundary vertices. `true` is the default.
- **maxsubdiv**= Changes the metric such that the maximum subdivision of a background edge is bound by `val`. Always limited by 10, and 10 is also the default.
- **rescaling**= if true, the function, with respect to which the mesh is adapted, is rescaled to be between 0 and 1. `true` is the default.
- **keepbackvertices**= if true, tries to keep as many vertices from the original mesh as possible. `true` is the default.
- **IsMetric**= if true, the metric is defined explicitly. `false` is the default. If the 3 functions  $m_{11}, m_{12}, m_{22}$  are given, they directly define a symmetric matrix field whose Hessian is computed to define a metric. If only one function is given, then it represents the isotropic mesh size at every point.  
For example, if the partial derivatives  $f_{xx} (= \partial^2 f / \partial x^2)$ ,  $f_{xy} (= \partial^2 f / \partial x \partial y)$ ,  $f_{yy} (= \partial^2 f / \partial y^2)$  are given, we can set `Th = adaptmesh(Th, fxx, fxy, fyy, IsMetric=1, nbvx=10000, hmin=hmin);`
- **power**= exponent power of the Hessian used to compute the metric. 1 is the default.
- **thetamax**= minimum corner angle in degrees. Default is  $10^\circ$  where the corner is  $ABC$  and the angle is the angle of the two vectors  $AB, BC$ , (0 imply no corner, 90 imply perpendicular corner, ...).
- **splitin2**= boolean value. If true, splits all triangles of the final mesh into 4 sub-triangles.
- **metric**= an array of 3 real arrays to set or get metric data information. The size of these three arrays must be the number of vertices. So if  $m_{11}, m_{12}, m_{22}$  are three P1 finite elements related to the mesh to adapt, you can write: `metric=[m11[], m12[], m22[]]` (see file `convect-apt.edp` for a full example)
- **nomeshgeneration**= If true, no adapted mesh is generated (useful to compute only a metric).
- **periodic**= Writing `periodic=[[4,y],[2,y],[1,x],[3,x]]`; builds an adapted periodic mesh.  
The sample builds a biperiodic mesh of a square. (see *periodic finite element spaces*, and see *the Sphere example* for a full example)

We can use the command `adaptmesh` to build a uniform mesh with a constant mesh size. To build a mesh with a constant mesh size equal to  $\frac{1}{30}$  try:

```

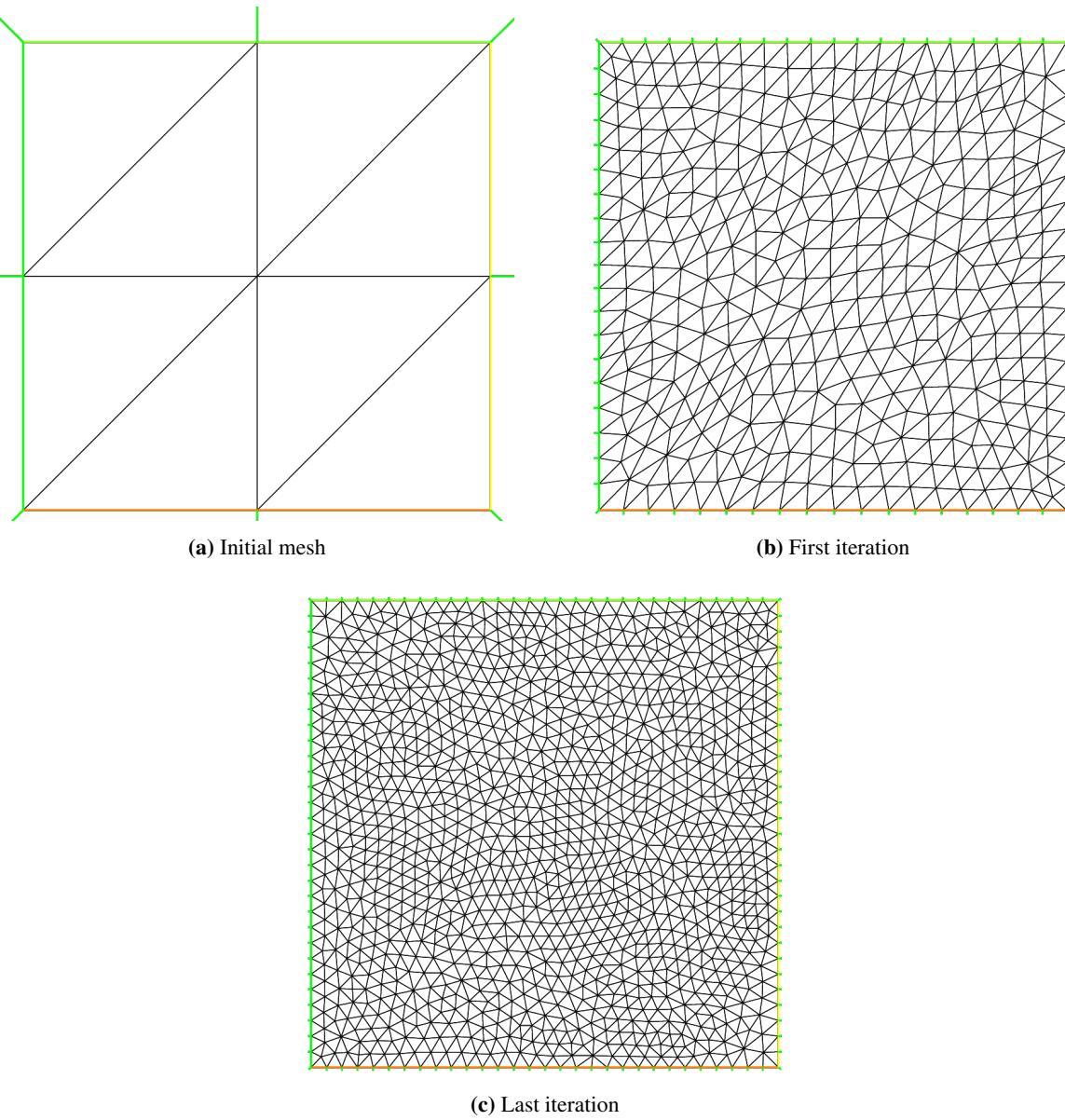
1 mesh Th=square(2, 2); //the initial mesh
2 plot(Th, wait=true, ps="square-0.eps");
3
4 Th = adaptmesh(Th, 1./30., IsMetric=1, nbvx=10000);
5 plot(Th, wait=true, ps="square-1.eps");
6
7 Th = adaptmesh(Th, 1./30., IsMetric=1, nbvx=10000); //More the one time du to
8 Th = adaptmesh(Th, 1./30., IsMetric=1, nbvx=10000); //Adaptation bound `maxsubdiv=
9 plot(Th, wait=true, ps="square-2.eps");

```

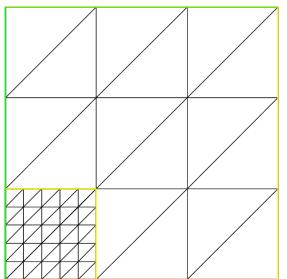
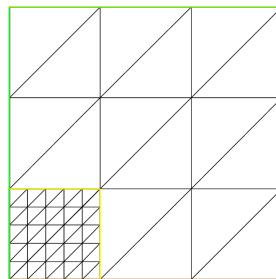
### 3.2.6 Trunc

Two operators have been introduced to remove triangles from a mesh or to divide them. Operator `trunc` has two parameters:

- **label**= sets the label number of new boundary item, one by default.



**Fig. 3.11:** Mesh adaptation

(a) Mesh of support the function P1 number 0, split in  $5 \times 5$ (b) Mesh of support the function P1 number 6, split in  $5 \times 5$ **Fig. 3.12:** Trunc

- **split=** sets the level  $n$  of triangle splitting. Each triangle is split in  $n \times n$ , one by default.

To create the mesh Th3 where all triangles of a mesh Th are split in  $3 \times 3$ , just write:

```
1 mesh Th3 = trunc(Th, 1, split=3);
```

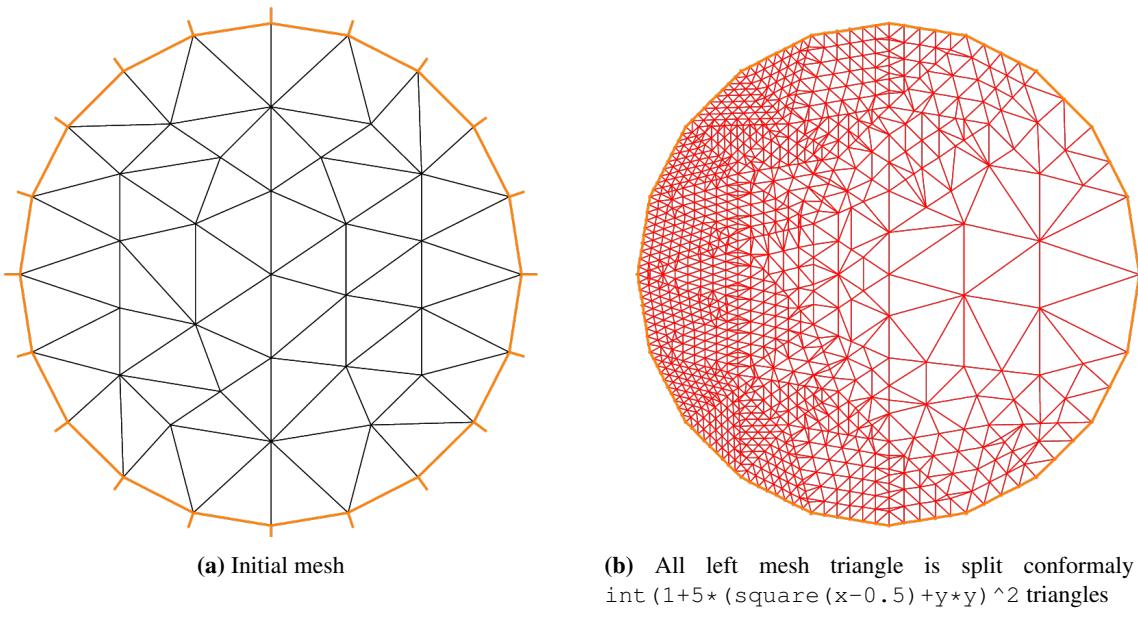
The following example construct all “truncated” meshes to the support of the basic function of the space Vh (cf.  $\text{abs}(u) > 0$ ), split all the triangles in  $5 \times 5$ , and put a label number to 2 on a new boundary.

```
1 // Mesh
2 mesh Th = square(3, 3);
3
4 // Fespace
5 fespace Vh(Th, P1);
6 Vh u=0;
7
8 // Loop on all degrees of freedom
9 int n=u.n;
10 for (int i = 0; i < n; i++){
11     u[] [i] = 1; // The basis function i
12     plot(u, wait=true);
13     mesh Sh1 = trunc(Th, abs(u)>1.e-10, split=5, label=2);
14     plot(Th, Sh1, wait=true, ps="trunc"+i+".eps");
15     u[] [i] = 0; // reset
16 }
```

### 3.2.7 Splitmesh

Another way to split mesh triangles is to use splitmesh, for example:

```
1 // Mesh
2 border a(t=0, 2*pi){x=cos(t); y=sin(t); label=1;};
3 mesh Th = buildmesh(a(20));
4 plot(Th, wait=true, ps="NotSplittedMesh.eps");
5
6 // Splitmesh
7 Th = splitmesh(Th, 1 + 5*(square(x-0.5) + y*y));
8 plot(Th, wait=true, ps="SplittedMesh.eps");
```

**Fig. 3.13:** Split mesh

### 3.2.8 Meshing Examples

**Tip:** Two rectangles touching by a side

```

1 border a(t=0, 1){x=t; y=0;};
2 border b(t=0, 1){x=1; y=t;};
3 border c(t=1, 0){x=t; y=1;};
4 border d(t=1, 0){x=0; y=t;};
5 border c1(t=0, 1){x=t; y=1;};
6 border e(t=0, 0.2){x=1; y=1+t;};
7 border f(t=1, 0){x=t; y=1.2;};
8 border g(t=0.2, 0){x=0; y=1+t;};
9 int n=1;
10 mesh th = buildmesh(a(10*n) + b(10*n) + c(10*n) + d(10*n));
11 mesh TH = buildmesh(c1(10*n) + e(5*n) + f(10*n) + g(5*n));
12 plot(th, TH, ps="TouchSide.eps");

```

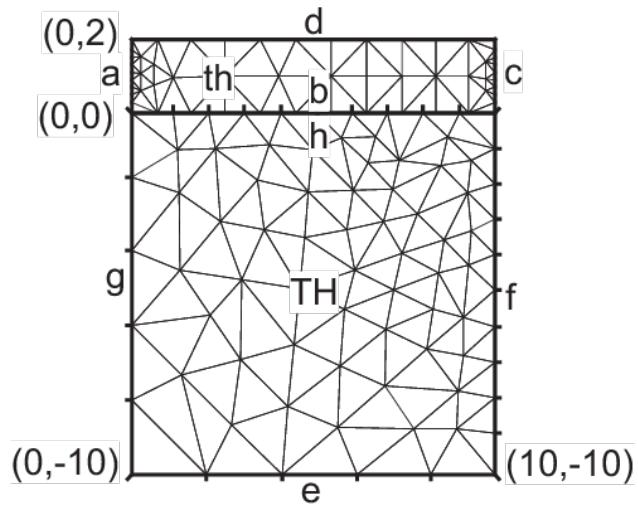
**Tip:** NACA0012 Airfoil

```

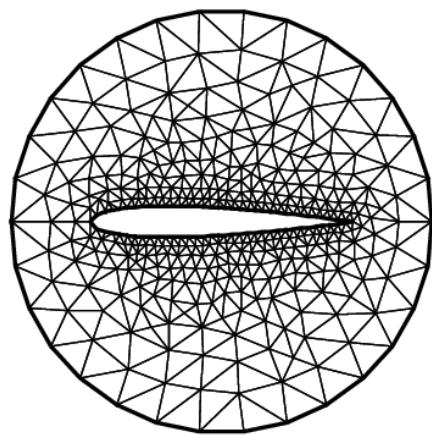
1 border upper(t=0, 1){x=t; y=0.17735*sqrt(t) - 0.075597*t - 0.212836*(t^2) + 0.
- 17363*(t^3) - 0.06254*(t^4);}
2 border lower(t=1, 0){x = t; y=-(0.17735*sqrt(t) - 0.075597*t - 0.212836*(t^2) + 0.
- 17363*(t^3) - 0.06254*(t^4));}
3 border c(t=0, 2*pi){x=0.8*cos(t) + 0.5; y=0.8*sin(t);}
4 mesh Th = buildmesh(c(30) + upper(35) + lower(35));
5 plot(Th, ps="NACA0012.eps", bw=true);

```

**Tip:** Cardioid



**Fig. 3.14:** Two rectangles touching by a side

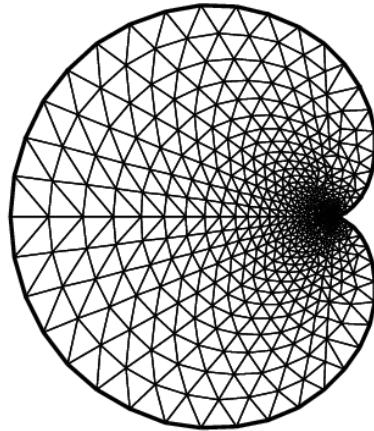


**Fig. 3.15:** NACA0012 Airfoil

```

1 real b = 1, a = b;
2 border C(t=0, 2*pi) {x=(a+b)*cos(t)-b*cos((a+b)*t/b); y=(a+b)*sin(t)-b*sin((a+b)*t/b);}
3 mesh Th = buildmesh(C(50));
4 plot(Th, ps="Cardioid.eps", bw=true);

```



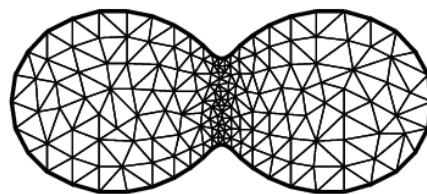
**Fig. 3.16:** Domain with Cardioid curve boundary

**Tip:** Cassini Egg

```

1 border C(t=0, 2*pi) {x=(2*cos(2*t)+3)*cos(t); y=(2*cos(2*t)+3)*sin(t);}
2 mesh Th = buildmesh(C(50));
3 plot(Th, ps="Cassini.eps", bw=true);

```



**Fig. 3.17:** Domain with Cassini egg curve boundary

**Tip:** By cubic Bezier curve

```

1 // A cubic Bezier curve connecting two points with two control points
2 func real bzi(real p0, real p1, real q1, real q2, real t){

```

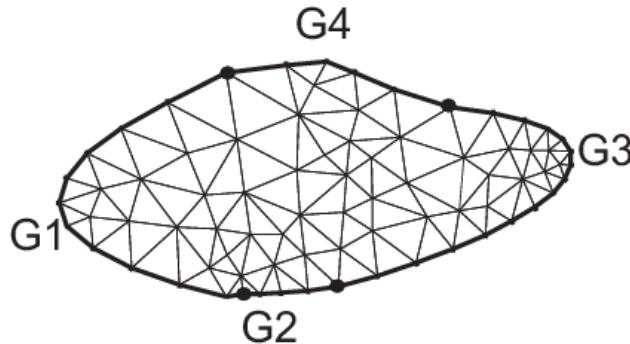
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```

3   return p0*(1-t)^3 + q1*3*(1-t)^2*t + q2*3*(1-t)*t^2 + p1*t^3;
4 }
5
6 real[int] p00 = [0, 1], p01 = [0, -1], q00 = [-2, 0.1], q01 = [-2, -0.5];
7 real[int] p11 = [1,-0.9], q10 = [0.1, -0.95], q11=[0.5, -1];
8 real[int] p21 = [2, 0.7], q20 = [3, -0.4], q21 = [4, 0.5];
9 real[int] q30 = [0.5, 1.1], q31 = [1.5, 1.2];
10 border G1(t=0, 1){
11     x=bzi(p00[0], p01[0], q00[0], q01[0], t);
12     y=bzi(p00[1], p01[1], q00[1], q01[1], t);
13 }
14 border G2(t=0, 1){
15     x=bzi(p01[0], p11[0], q10[0], q11[0], t);
16     y=bzi(p01[1], p11[1], q10[1], q11[1], t);
17 }
18 border G3(t=0, 1){
19     x=bzi(p11[0], p21[0], q20[0], q21[0], t);
20     y=bzi(p11[1], p21[1], q20[1], q21[1], t);
21 }
22 border G4(t=0, 1){
23     x=bzi(p21[0], p00[0], q30[0], q31[0], t);
24     y=bzi(p21[1], p00[1], q30[1], q31[1], t);
25 }
26 int m = 5;
27 mesh Th = buildmesh(G1(2*m) + G2(m) + G3(3*m) + G4(m));
28 plot(Th, ps="Bezier.eps", bw=true);

```

**Fig. 3.18:** Boundary drawn by Bezier curves**Tip:** Section of Engine

```

1 real a = 6., b = 1., c = 0.5;
2
3 border L1(t=0, 1){x=-a; y=1+b-2*(1+b)*t; }
4 border L2(t=0, 1){x=-a+2*a*t; y=-1-b*(x/a)*(x/a)*(3-2*abs(x)/a); }
5 border L3(t=0, 1){x=a; y=-1-b+(1+b)*t; }
6 border L4(t=0, 1){x=a-a*t; y=0; }
7 border L5(t=0, pi){x=-c*sin(t)/2; y=c/2-c*cos(t)/2; }
8 border L6(t=0, 1){x=a*t; y=c; }
9 border L7(t=0, 1){x=a; y=c+(1+b-c)*t; }
10 border L8(t=0, 1){x=a-2*a*t; y=1+b*(x/a)*(x/a)*(3-2*abs(x)/a); }

```

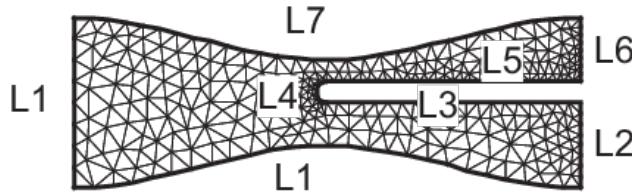
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```

11 mesh Th = buildmesh(L1(8) + L2(26) + L3(8) + L4(20) + L5(8) + L6(30) + L7(8) +
12   ↪L8(30));
plot(Th, ps="Engine.eps", bw=true);

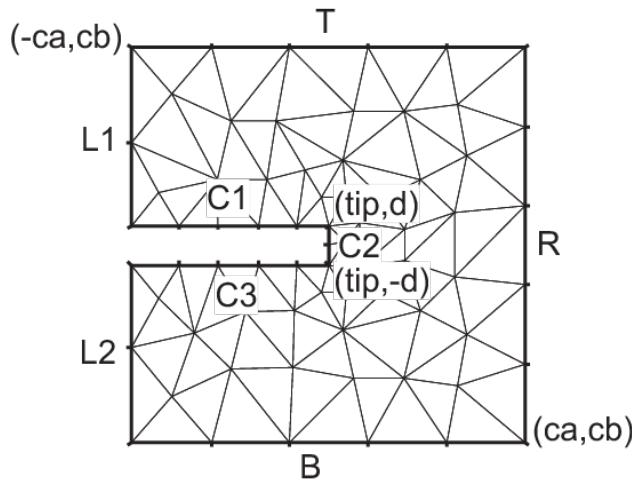
```

**Fig. 3.19:** Section of Engine**Tip:** Domain with U-shape channel

```

1 real d = 0.1; //width of U-shape
2 border L1(t=0, 1-d){x=-1; y=-d-t; }
3 border L2(t=0, 1-d){x=-1; y=1-t; }
4 border B(t=0, 2){x=-1+t; y=-1; }
5 border C1(t=0, 1){x=t-1; y=d; }
6 border C2(t=0, 2*d){x=0; y=d-t; }
7 border C3(t=0, 1){x=-t; y=-d; }
8 border R(t=0, 2){x=1; y=-1+t; }
9 border T(t=0, 2){x=1-t; y=1; }
10 int n = 5;
11 mesh Th = buildmesh(L1(n/2) + L2(n/2) + B(n) + C1(n) + C2(3) + C3(n) + R(n) + T(n));
12 plot(Th, ps="U-shape.eps", bw=true);

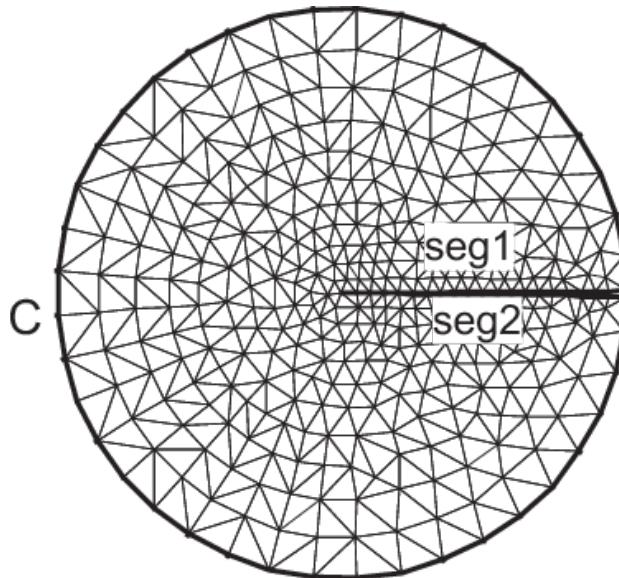
```

**Fig. 3.20:** Domain with U-shape channel changed by :freefem‘d’**Tip:** Domain with V-shape cut

```

1 real dAg = 0.02; //angle of V-shape
2 border C(t=dAg, 2*pi-dAg) {x=cos(t); y=sin(t);}
3 real[int] pa(2), pb(2), pc(2);
4 pa[0] = cos(dAg);
5 pa[1] = sin(dAg);
6 pb[0] = cos(2*pi-dAg);
7 pb[1] = sin(2*pi-dAg);
8 pc[0] = 0;
9 pc[1] = 0;
10 border seg1(t=0, 1) {x=(1-t)*pb[0]+t*pc[0]; y=(1-t)*pb[1]+t*pc[1];};
11 border seg2(t=0, 1) {x=(1-t)*pc[0]+t*pa[0]; y=(1-t)*pc[1]+t*pa[1];};
12 mesh Th = buildmesh(seg1(20) + C(40) + seg2(20));
13 plot(Th, ps="V-shape.eps", bw=true);

```



**Fig. 3.21:** Domain with V-shape cut changed by dAg

---

**Tip:** Smiling face

```

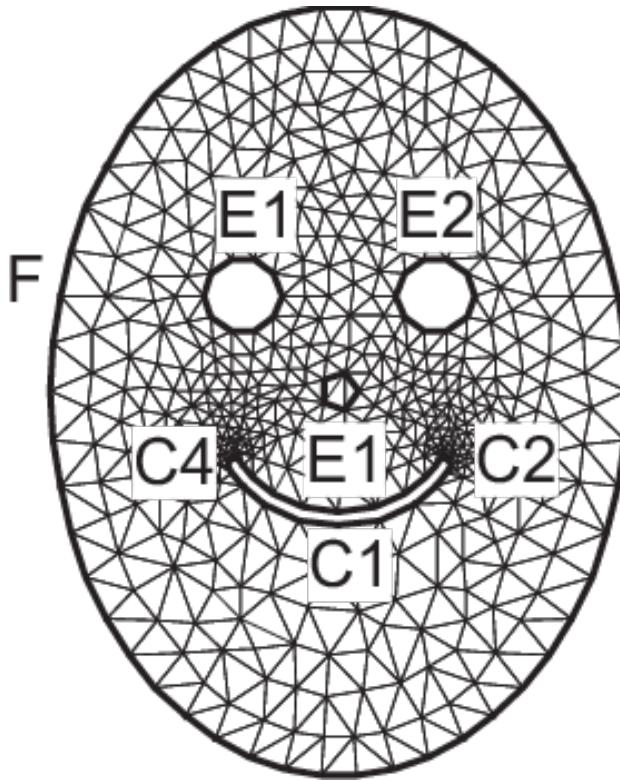
1 real d=0.1; int m = 5; real a = 1.5, b = 2, c = 0.7, e = 0.01;
2
3 border F(t=0, 2*pi) {x=a*cos(t); y=b*sin(t);}
4 border E1(t=0, 2*pi) {x=0.2*cos(t)-0.5; y=0.2*sin(t)+0.5;}
5 border E2(t=0, 2*pi) {x=0.2*cos(t)+0.5; y=0.2*sin(t)+0.5;}
6 func real st(real t){
7     return sin(pi*t) - pi/2;
8 }
9 border C1(t=-0.5, 0.5) {x=(1-d)*c*cos(st(t)); y=(1-d)*c*sin(st(t));}
10 border C2(t=0, 1) {x=((1-d)+d*t)*c*cos(st(0.5)); y=((1-d)+d*t)*c*sin(st(0.5));}
11 border C3(t=0.5, -0.5) {x=c*cos(st(t)); y=c*sin(st(t));}
12 border C4(t=0, 1) {x=(1-d*t)*c*cos(st(-0.5)); y=(1-d*t)*c*sin(st(-0.5));}
13 border C0(t=0, 2*pi) {x=0.1*cos(t); y=0.1*sin(t);}
14
15 mesh Th=buildmesh(F(10*m) + C1(2*m) + C2(3) + C3(2*m) + C4(3)
16 + C0(m) + E1(-2*m) + E2(-2*m));

```

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```
17 plot(Th, ps="SmileFace.eps", bw=true);
```



**Fig. 3.22:** Smiling face (Mouth is changeable)

. tip:: 3 points bending

```

1 // Square for Three-Point Bend Specimens fixed on Fix1, Fix2
2 // It will be loaded on Load.
3 real a = 1, b = 5, c = 0.1;
4 int n = 5, m = b*n;
5 border Left(t=0, 2*a) {x=-b; y=a-t; }
6 border Bot1(t=0, b/2-c) {x=-b+t; y=-a; }
7 border Fix1(t=0, 2*c) {x=-b/2-c+t; y=-a; }
8 border Bot2(t=0, b-2*c) {x=-b/2+c+t; y=-a; }
9 border Fix2(t=0, 2*c) {x=b/2-c+t; y=-a; }
10 border Bot3(t=0, b/2-c) {x=b/2+c+t; y=-a; }
11 border Right(t=0, 2*a) {x=b; y=-a+t; }
12 border Top1(t=0, b-c) {x=b-t; y=a; }
13 border Load(t=0, 2*c) {x=c-t; y=a; }
14 border Top2(t=0, b-c) {x=-c-t; y=a; }
15 mesh Th = buildmesh(Left(n) + Bot1(m/4) + Fix1(5) + Bot2(m/2)
16     + Fix2(5) + Bot3(m/4) + Right(n) + Top1(m/2) + Load(10) + Top2(m/2));
17 plot(Th, ps="ThreePoint.eps", bw=true);
```

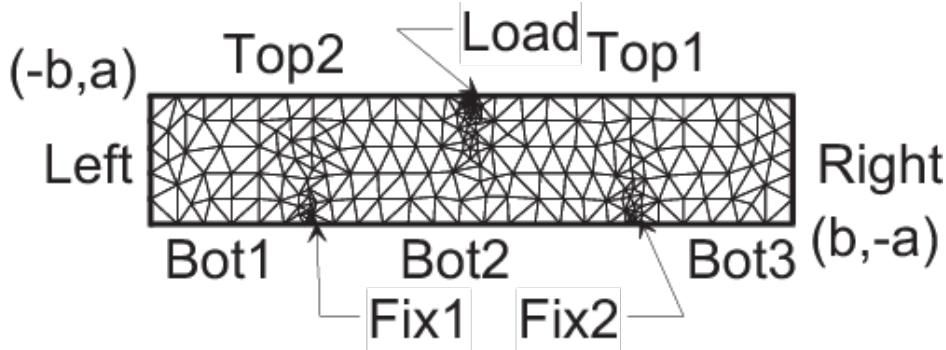


Fig. 3.23: Domain for three-point bending test

### 3.2.9 How to change the label of elements and border elements of a mesh

Changing the label of elements and border elements will be done using the keyword `change`. The parameters for this command line are for two dimensional and three dimensional cases:

- `label` = is a vector of integer that contains successive pairs of the old label number to the new label number.
- `region` = is a vector of integer that contains successive pairs of the old region number to new region number.
- `flabel` = is an integer function given the new value of the label.
- `fregion`= is an integer function given the new value of the region.

These vectors are composed of  $n_l$  successive pairs of numbers  $O, N$  where  $n_l$  is the number (label or region) that we want to change. For example, we have :

$$\begin{aligned} \text{label} &= [O_1, N_1, \dots, O_{n_l}, N_{n_l}] \\ \text{region} &= [O_1, N_1, \dots, O_{n_l}, N_{n_l}] \end{aligned} \quad (3.1)$$

An example of using this function is given here:

```

1 verbosity=3;
2
3 // Mesh
4 mesh Th1 = square(10, 10);
5 mesh Th2 = square(20, 10, [x+1, y]);
6
7 int[int] r1=[2,0];
8 plot(Th1, wait=true);
9
10 Th1 = change(Th1, label=r1); //change the label of Edges 2 in 0.
11 plot(Th1, wait=true);
12
13 int[int] r2=[4,0];
14 Th2 = change(Th2, label=r2); //change the label of Edges 4 in 0.
15 plot(Th2, wait=true);
16
17 mesh Th = Th1 + Th2; //'gluing together' of meshes Th1 and Th2
18 cout << "nb lab = " << int1d(Th1,1,3,4)(1./lenEdge)+int1d(Th2,1,2,3)(1./lenEdge)
19 << " == " << int1d(Th,1,2,3,4)(1./lenEdge) << " == " << ((10+20)+10)*2 << endl;
20 plot(Th, wait=true);
21
22 fespace Vh(Th, P1);
23 Vh u, v;
```

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```

24
25 macro Grad(u) [dx(u),dy(u)] // Definition of a macro
26
27 solve P(u, v)
28   = int2d(Th) (
29     Grad(u)'*Grad(v)
30   )
31   -int2d(Th) (
32     v
33   )
34   + on(1, 3, u=0)
35 ;
36
37 plot(u, wait=1);

```

**“gluing” different mesh** In line 17 of the previous file, the method to “gluing” different meshes of the same dimension in **FreeFEM** is using. This function is the operator “+” between meshes. The method implemented needs the point in adjacent meshes to be the same.

### 3.2.10 Mesh in three dimensions

#### Cube

A new function `cube` like the function `square` in 2d is the simple way to build cubic object, in plugin `msh3` (need `load "msh3"`).

The following code generates a  $3 \times 4 \times 5$  grid in the unit cube  $[0, 1]^3$ .

```
1 mesh3 Th = cube(3, 4, 5);
```

By default the labels are :

1. face  $y = 0$ ,
2. face  $x = 1$ ,
3. face  $y = 1$ ,
4. face  $x = 0$ ,
5. face  $z = 0$ ,
6. face  $z = 1$

and the region number is 0.

A full example of this function to build a mesh of cube  $[-1, 1]^3$  with face label given by  $(ix+4*(iy+1)+16*(iz+1))$  where  $(ix, iy, iz)$  are the coordinates of the barycenter of the current face, is given below.

```

1 load "msh3"
2
3 int[int] 16 = [37, 42, 45, 40, 25, 57];
4 int r11 = 11;
5 mesh3 Th = cube(4, 5, 6, [x*2-1, y*2-1, z*2-1], label=16, flags =3, region=r11);
6
7 cout << "Volume = " << Th.measure << ", border area = " << Th.bordermeasure << endl;
8
9 int err = 0;

```

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```

10 for(int i = 0; i < 100; ++i){
11     real s = int2d(Th,i)(1.);
12     real sx = int2d(Th,i)(x);
13     real sy = int2d(Th,i)(y);
14     real sz = int2d(Th,i)(z);
15
16     if(s) {
17         int ix = (sx/s+1.5);
18         int iy = (sy/s+1.5);
19         int iz = (sz/s+1.5);
20         int ii = (ix + 4*(iy+1) + 16*(iz+1) );
21         //value of ix,iy,iz => face min 0, face max 2, no face 1
22         cout << "Label = " << i << ", s = " << s << " " << ix << iy << iz << " : " <<
23         ii << endl;
24         if( i != ii ) err++;
25     }
26
27     real volr11 = int3d(Th,r11)(1.);
28     cout << "Volume region = " << 11 << ":" << volr11 << endl;
29     if((volr11 - Th.measure )>1e-8) err++;
30     plot(Th, fill=false);
31     cout << "Nb err = " << err << endl;
32     assert(err=0);

```

The output of this script is:

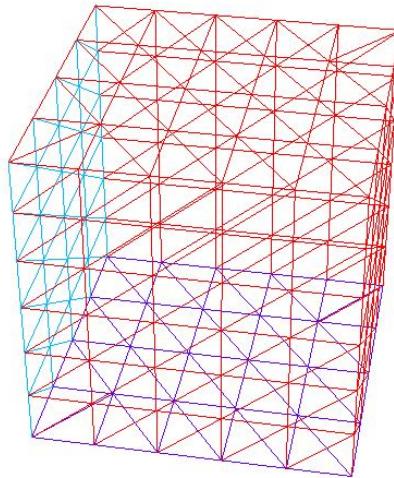
```

1 Enter: BuildCube: 3
2     kind = 3 n tet Cube = 6 / n slip 6 19
3 Cube nv=210 nt=720 nbe=296
4 Out: BuildCube
5 Volume = 8, border area = 24
6 Label = 25, s = 4 110 : 25
7 Label = 37, s = 4 101 : 37
8 Label = 40, s = 4 011 : 40
9 Label = 42, s = 4 211 : 42
10 Label = 45, s = 4 121 : 45
11 Label = 57, s = 4 112 : 57
12 Volume region = 11: 8
13 Nb err = 0

```

## Read/Write Statements for a Mesh in 3D

In three dimensions, the file mesh format supported for input and output files by **FreeFEM** are the extension .msh and .mesh. These formats are described in the *Mesh Format section*.



**Fig. 3.24:** The mesh 3d of function `cube (4, 5, 6, flags=3)`

**Extension file .msh** The structure of the files with extension .msh in 3D is given by:

$n_v$	$n_{tet}$	$n_{tri}$		
$q_x^1$	$q_y^1$	$q_z^1$	<i>Vertexlabel</i>	
$q_x^2$	$q_y^2$	$q_z^2$	<i>Vertexlabel</i>	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$q_x^{n_v}$	$q_y^{n_v}$	$q_z^{n_v}$	<i>Vertexlabel</i>	
$1_1$	$1_2$	$1_3$	$1_4$	<i>regionlabel</i>
$2_1$	$2_2$	$2_3$	$2_4$	<i>regionlabel</i>
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(n_{tet})_1$	$(n_{tet})_2$	$(n_{tet})_3$	$(n_{tet})_4$	<i>regionlabel</i>
$1_1$	$1_2$	$1_3$	<i>boundarylabel</i>	
$2_1$	$2_2$	$2_3$	<i>boundarylabel</i>	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$(n_{tri})_1$	$(n_{tri})_2$	$(n_{tri})_3$	<i>boundarylabel</i>	

In this structure,  $n_v$  denotes the number of vertices,  $n_{tet}$  the number of tetrahedra and  $n_{tri}$  the number of triangles.

For each vertex  $q^i$ ,  $i = 1, \dots, n_v$ , we denote by  $(q_x^i, q_y^i, q_z^i)$  the  $x$ -coordinate, the  $y$ -coordinate and the  $z$ -coordinate.

Each tetrahedra  $T_k$ ,  $k = 1, \dots, n_{tet}$  has four vertices  $q^{k_1}, q^{k_2}, q^{k_3}, q^{k_4}$ .

The boundary consists of a union of triangles. Each triangle  $be_j$ ,  $j = 1, \dots, n_{tri}$  has three vertices  $q^{j_1}, q^{j_2}, q^{j_3}$ .

**extension file .mesh** The data structure for a three dimensional mesh is composed of the data structure presented in [Mesh Format section](#) and a data structure for the tetrahedra. The tetrahedra of a three dimensional mesh are referred using the following field:

```

1 Tetrahedra
2 NbTetrahedra
3 Vertex1 Vertex2 Vertex3 Vertex4 Label
4 ...
5 Vertex1 Vertex2 Vertex3 Vertex4 Label

```

This field is express with the notation of [Mesh Format section](#).

## TetGen: A tetrahedral mesh generator

### TetGen

TetGen is a software developed by Dr. Hang Si of Weierstrass Institute for Applied Analysis and Stochastics in Berlin, Germany [HANG2006]. TetGen is free for research and non-commercial use. For any commercial license utilization, a commercial license is available upon request to Hang Si.

This software is a tetrahedral mesh generator of a three dimensional domain defined by its boundary. The input domain takes into account a polyhedral or a piecewise linear complex. This tetrahedralization is a constrained Delaunay tetrahedralization.

The method used in TetGen to control the quality of the mesh is a Delaunay refinement due to Shewchuk [SHEWCHUK1998]. The quality measure of this algorithm is the Radius-Edge Ratio (see Section 1.3.1 [HANG2006] for more details). A theoretical bound of this ratio of the Shewchuk algorithm is obtained for a given complex of vertices, constrained segments and facets of surface mesh, with no input angle less than 90 degrees. This theoretical bound is 2.0.

The launch of TetGen is done with the keyword `tetg`. The parameters of this command line is:

- `reftet`= sets the label of tetrahedra.
- **label=** is a vector of integers that contains the old labels number at index  $2i$  and the new labels number at index  $2i + 1$ .  
This parameter is initialized as a label for the keyword `change`.
- **switch=** A string expression. This string corresponds to the command line switch of TetGen see Section 3.2 of [HANG2006].
- `nbofholes`= Number of holes (default value: “size of `holelist / 3`”).
- **holelist=** This array corresponds to `holelist` of TetGenio data structure [HANG2006]. A real vector of size  $3 * nbofholes$ . In TetGen, each hole is associated with a point inside this domain. This vector is  $x_1^h, y_1^h, z_1^h, x_2^h, y_2^h, z_2^h, \dots$ , where  $x_i^h, y_i^h, z_i^h$  is the associated point with the  $i^{\text{th}}$  hole.
- `nbofregions`= Number of regions (default value: “size of `regionlist / 5`”).
- `regionlist`= This array corresponds to `regionlist` of TetGenio data structure [HANG2006].

The attribute and the volume constraint of region are given in this real vector of size  $5 * nbofregions$ . The  $i^{\text{th}}$  region is described by five elements:  $x$ -coordinate,  $y$ -coordinate and  $z$ -coordinate of a point inside this domain ( $x_i, y_i, z_i$ ); the attribute ( $at_i$ ) and the maximum volume for tetrahedra ( $mvol_i$ ) for this region.

The `regionlist` vector is:  $x_1, y_1, z_1, at_1, mvol_1, x_2, y_2, z_2, at_2, mvol_2, \dots$ .

- `nboffacetcl`= Number of facets constraints “size of `facetcl / 2`”).
- `facetcl`= This array corresponds to `facetconstraintlist` of TetGenio data structure [HANG2006].  
The  $i^{\text{th}}$  facet constraint is defined by the facet marker  $Ref_i^{fc}$  and the maximum area for faces  $marea_i^{fc}$ . The `facetcl` array is:  $Ref_1^{fc}, marea_1^{fc}, Ref_2^{fc}, marea_2^{fc}, \dots$ .

This parameters has no effect if switch `q` is not selected.

Principal switch parameters in TetGen:

- **p** Tetrahedralization of boundary.
- **q Quality mesh generation.** The bound of Radius-Edge Ratio will be given after the option `q`. By default, this value is 2.0.
- **a Constructs with the volume constraints on tetrahedra.** These volumes constraints are defined with the bound of the previous switch `q` or in the parameter `regionlist`.

- **A Attributes reference to region given in the `regionlist`.** The other regions have label 0.

The option `AA` gives a different label at each region. This switch works with the option `p`. If option `:freefem'r'` is used, this switch has no effect.

- **r Reconstructs and Refines a previously generated mesh.** This character is only used with the command line `tetgreconstruction`.

- **Y** This switch preserves the mesh on the exterior boundary.

This switch must be used to ensure a conformal mesh between two adjacent meshes.

- **YY** This switch preserves the mesh on the exterior and interior boundary.

- **C** The consistency of the result's mesh is testing by TetGen.

- **CC** The consistency of the result's mesh is testing by TetGen and also constrained checks of Delaunay mesh (if `p` switch is selected) or the consistency of Conformal Delaunay (if `q` switch is selected).

- **v Give information of the work of TetGen.** More information can be obtained in specified `VV` or `VVV`.

- **Q Quiet:** No terminal output except errors

- **M** The coplanar facets are not merging.

- **T Sets a tolerance for coplanar test.** The default value is  $1e - 8$ .

- **d** Intersections of facets are detected.

To obtain a tetrahedral mesh with TetGen, we need the surface mesh of a three dimensional domain. We now give the command line in **FreeFEM** to construct these meshes.

#### keyword: movemesh23

A simple method to construct a surface is to place a two dimensional domain in a three dimensional space. This corresponds to moving the domain by a displacement vector of this form  $\Phi(\mathbf{x}, \mathbf{y}) = (\Phi_1(x, y), \Phi_2(x, y), \Phi_3(x, y))$ .

The result of moving a two dimensional mesh `Th2` by this three dimensional displacement is obtained using:

```
mesh3 Th3 = movemesh23(Th2, transfo=[Phi(1), Phi(2), Phi(3)]);
```

The parameters of this command line are:

- **transfo=** `[Φ1, Φ2, Φ3]` sets the displacement vector of transformation  $\Phi(\mathbf{x}, \mathbf{y}) = [\Phi_1(x, y), \Phi_2(x, y), \Phi_3(x, y)]$ .
- **label=** sets an integer label of triangles.
- **orientation=** sets an integer orientation of mesh.
- **ptmerge= A real expression.** When you transform a mesh, some points can be merged. This parameter is the criteria to define two merging points. By default, we use

$$ptmerge = 1e - 7 \text{ Vol}(B),$$

where  $B$  is the smallest axis, parallel boxes containing the discretized domain of  $\Omega$  and  $\text{Vol}(B)$  is the volume of this box.

We can do a “gluing” of surface meshes using the process given in [Change section](#). An example to obtain a three dimensional mesh using the command line `tetg` and `movemesh23` is given below.

```

1  load "msh3"
2  load "tetgen"
3
4 // Parameters
5  real x10 = 1.;
6  real x11 = 2.;
7  real y10 = 0.;
8  real y11 = 2.*pi;
9
10 func ZZ1min = 0;
11 func ZZ1max = 1.5;
12 func XX1 = x;
13 func YY1 = y;
14
15 real x20 = 1.;
16 real x21 = 2.;
17 real y20=0.;
18 real y21=1.5;
19
20 func ZZ2 = y;
21 func XX2 = x;
22 func YY2min = 0.;
23 func YY2max = 2*pi;
24
25 real x30=0.;
26 real x31=2*pi;
27 real y30=0.;
28 real y31=1.5;
29
30 func XX3min = 1.;
31 func XX3max = 2.;
32 func YY3 = x;
33 func ZZ3 = y;
34
35 // Mesh
36 mesh Thsq1 = square(5, 35, [x10+(x11-x10)*x, y10+(y11-y10)*y]);
37 mesh Thsq2 = square(5, 8, [x20+(x21-x20)*x, y20+(y21-y20)*y]);
38 mesh Thsq3 = square(35, 8, [x30+(x31-x30)*x, y30+(y31-y30)*y]);
39
40 // Mesh 2D to 3D surface
41 mesh3 Th31h = movemesh23(Thsq1, transfo=[XX1, YY1, ZZ1max]);
42 mesh3 Th31b = movemesh23(Thsq1, transfo=[XX1, YY1, ZZ1min]);
43
44 mesh3 Th32h = movemesh23(Thsq2, transfo=[XX2, YY2max, ZZ2]);
45 mesh3 Th32b = movemesh23(Thsq2, transfo=[XX2, YY2min, ZZ2]);
46
47 mesh3 Th33h = movemesh23(Thsq3, transfo=[XX3max, YY3, ZZ3]);
48 mesh3 Th33b = movemesh23(Thsq3, transfo=[XX3min, YY3, ZZ3]);
49
50 // Gluing surfaces
51 mesh3 Th33 = Th31h + Th31b + Th32h + Th32b + Th33h + Th33b;
52 plot(Th33, cmm="Th33");
53
54 // Tetrahelize the interior of the cube with TetGen
55 real[int] domain =[1.5, pi, 0.75, 145, 0.0025];
56 mesh3 Thfinal = tetg(Th33, switch="paAAQY", regionlist=domain);
57 plot(Thfinal, cmm="Thfinal");

```

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```

58
59 // Build a mesh of a half cylindrical shell of interior radius 1, and exterior radius ↩
60 // 2 and a height of 1.5
61 func mv2x = x*cos(y);
62 func mv2y = x*sin(y);
63 func mv2z = z;
64 mesh3 Thmv2 = movemesh3(Thfinal, transfo=[mv2x, mv2y, mv2z]);
plot(Thmv2, cmm="Thmv2");

```

The command `movemesh3` is described in the following section.

### The keyword `tetgtransfo`

This keyword corresponds to a composition of command line `tetg` and `movemesh23`.

```

1 tetgtransfo(Th2, transfo=[Phi(1), Phi(2), Phi(3)], ...) = tetg(Th3surf, ...),

```

where `Th3surf = movemesh23(Th2, transfo=[Phi(1), Phi(2), Phi(3)])` and `Th2` is the input two dimensional mesh of `tetgtransfo`.

The parameters of this command line are, on one hand, the parameters `label`, `switch`, `regionlist`, `nboffacetcl`, `facetcl` of keyword `tetg` and on the other hand, the parameter `ptmerge` of keyword `movemesh23`.

---

**Note:** To use `tetgtransfo`, the result's mesh of `movemesh23` must be a closed surface and define one region only. Therefore, the parameter `regionlist` is defined for one region.

An example of this keyword can be found in line 61 of the [Build layer mesh example](#).

---

### The keyword `tetgconvexhull`

**FreeFEM**, using TetGen, is able to build a tetrahedralization from a set of points. This tetrahedralization is a Delaunay mesh of the convex hull of the set of points.

The coordinates of the points can be initialized in two ways. The first is a file that contains the coordinate of points  $X_i = (x_i, y_i, z_i)$ . This file is organized as follows:

$$\begin{array}{c}
 n_v \\
 x_1 & y_1 & z_1 \\
 x_2 & y_2 & z_2 \\
 \vdots & \vdots & \vdots \\
 x_{n_v} & y_{n_v} & z_{n_v}
 \end{array}$$

The second way is to give three arrays that correspond respectively to the  $x$ -coordinates,  $y$ -coordinates and  $z$ -coordinates.

The parameters of this command line are :

- **switch= A string expression.** This string corresponds to the command line `switch` of TetGen see Section 3.2 of [[HANG2006](#)].
- **reftet= An integer expression.** Set the label of tetrahedra.
- **label= An integer expression.** Set the label of triangles.

In the string `switch`, we can't used the option `p` and `q` of TetGen.

## Reconstruct/Refine a three dimensional mesh with TetGen

Meshes in three dimension can be refined using TetGen with the command line `tetgreconstruction`.

The parameter of this keyword are

- **region=** an integer array that changes the region number of tetrahedra. This array is defined as the parameter `reftet` in the keyword `change`.
- **label=** an integer array that changes the label of boundary triangles. This array is defined as the parameter `label` in the keyword `change`.
- **sizeofvolume=** a reel function. This function constraints the volume size of the tetrahedra in the domain (see *Isotropic mesh adaption section* to build a 3d adapted mesh).

The parameters `switch`, `nbofregions`, `regionlist`, `nboffacetcl` and `facetcl` of the command line which call TetGen (`tetg`) is used for `tetgrefine`.

In the parameter `switch=`, the character `r` should be used without the character `p`.

For instance, see the manual of TetGen [HANG2006] for effect of `r` to other character.

The parameter `regionlist` defines a new volume constraint in the region. The label in the `regionlist` will be the previous label of region.

This parameter and `nbofregions` can't be used with the parameter `sizeofvolume`.

### Example refinesphere.edp

```

1  load "msh3"
2  load "TetGen"
3  load "medit"
4
5  mesh Th = square(10, 20, [x*pi-pi/2, 2*y*pi]); // $]-pi/2, pi/2[X]0, 2pi[ $
6
7  // A parametrization of a sphere
8  func f1 = cos(x)*cos(y);
9  func f2 = cos(x)*sin(y);
10 func f3 = sin(x);
11 // Partial derivative of the parametrization DF
12 func f1x = sin(x)*cos(y);
13 func f1y = -cos(x)*sin(y);
14 func f2x = -sin(x)*sin(y);
15 func f2y = cos(x)*cos(y);
16 func f3x = cos(x);
17 func f3y = 0;
18 // M = DF^t DF
19 func m11 = f1x^2 + f2x^2 + f3x^2;
20 func m21 = f1x*f1y + f2x*f2y + f3x*f3y;
21 func m22 = f1y^2 + f2y^2 + f3y^2;
22
23 // Mesh adaptation
24 func perio = [[4, y], [2, y], [1, x], [3, x]];
25 real hh = 0.1;
26 real vv = 1/square(hh);
27 verbosity = 2;
28 Th = adaptmesh(Th, m11*vv, m21*vv, m22*vv, IsMetric=1, periodic=perio);
29 Th = adaptmesh(Th, m11*vv, m21*vv, m22*vv, IsMetric=1, periodic=perio);
30 plot(Th, wait=true);
31
32 // Construction of the surface of spheres

```

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```

33 real Rmin = 1.;
34 func f1min = Rmin*f1;
35 func f2min = Rmin*f2;
36 func f3min = Rmin*f3;
37
38 mesh3 Th3 = movemesh23(Th, transfo=[f1min, f2min, f3min]);
39
40 // Construct the volume
41 real[int] domain = [0., 0., 0., 145, 0.01];
42 mesh3 Th3sph = tetg(Th3, switch="paAAQYY", nbregions=1, regionlist=domain);
43
44 // Refine
45 int[int] newlabel = [145, 18];
46 real[int] domainrefine = [0., 0., 0., 145, 0.0001];
47 mesh3 Th3sphrefine = tetgreconstruction(Th3sph, switch="raAQ", reftet=newlabel,
48 nbregions=1, regionlist=domain, sizeofvolume=0.0001);
49
50 // Re-Refine
51 int[int] newlabel2 = [145, 53];
52 func fsize = 0.01/((1 + 5*sqrt((x-0.5)^2+(y-0.5)^2+(z-0.5)^2))^3);
53 mesh3 Th3sphrefine2 = tetgreconstruction(Th3sph, switch="raAQ", reftet=newlabel2,
54 sizeofvolume=fsize);
55
56 // Medit
57 medit("sphere", Th3sph);
58 medit("isotroperefine", Th3sphrefine);
59 medit("anisotroperefine", Th3sphrefine2);

```

## Moving mesh in three dimensions

Meshes in three dimensions can be translated, rotated, and deformed using the command line `movemesh` as in the 2D case (see [section movemesh](#)). If  $\Omega$  is tetrahedrized as  $T_h(\Omega)$ , and  $\Phi(x, y) = (\Phi_1(x, y, z), \Phi_2(x, y, z), \Phi_3(x, y, z))$  is a displacement vector then  $\Phi(T_h)$  is obtained by:

```
1 mesh3 Th = movemesh(Th, [Phi1, Phi2, Phi3], ...);
```

The parameters of `movemesh` in three dimensions are:

- **region=** sets the integer labels of the tetrahedra. 0 by default.
- **label=** sets the labels of the border faces. This parameter is initialized as the label for the keyword `change`.
- **facemerge= An integer expression.** When you transform a mesh, some faces can be merged. This parameter equals to one if the merges' faces is considered. Otherwise it equals to zero. By default, this parameter is equal to 1.
- **ptmerge = A real expression.** When you transform a mesh, some points can be merged. This parameter is the criteria to define two merging points. By default, we use

$$ptmerge = 1e-7 Vol(B),$$

where  $B$  is the smallest axis parallel boxes containing the discretion domain of  $\Omega$  and  $Vol(B)$  is the volume of this box.

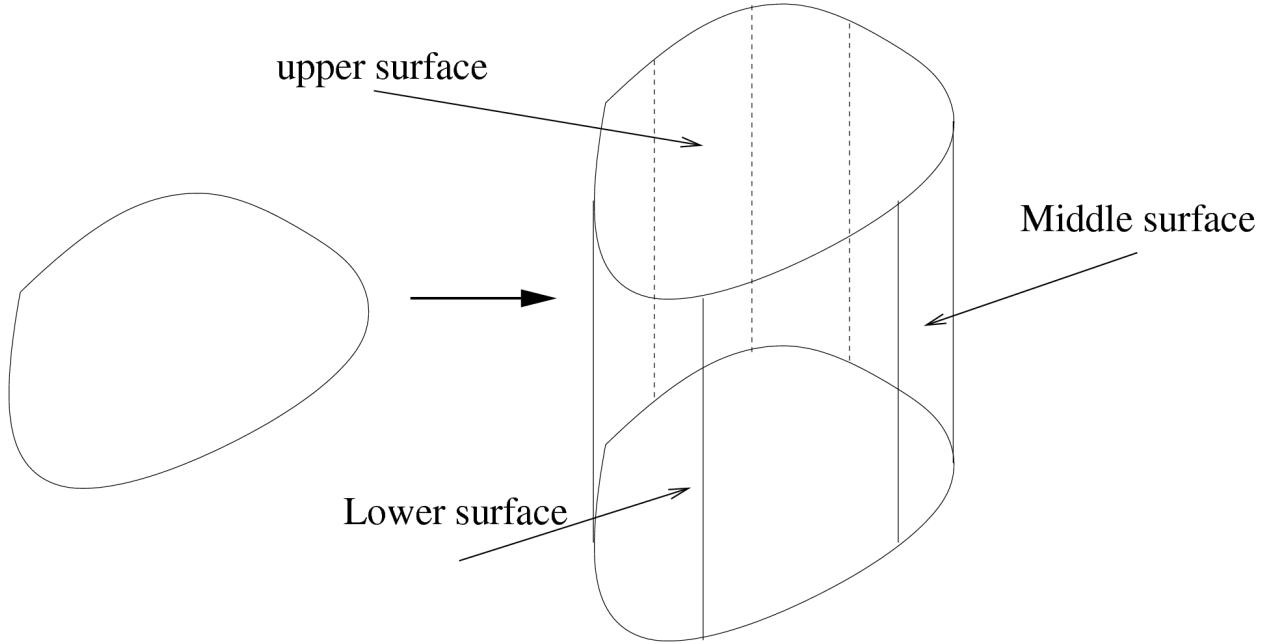
- **orientation =** An integer expression (1 by default), to reverse or not to reverse the orientation of the tetrahedra if it is not positive.

An example of this command can be found in the [Poisson's equation 3D example](#).

## Layer mesh

In this section, we present the command line to obtain a Layer mesh: `buildlayers`. This mesh is obtained by extending a two dimensional mesh in the  $z$ -axis.

The domain  $\Omega_{3d}$  defined by the layer mesh is equal to  $\Omega_{3d} = \Omega_{2d} \times [zmin, zmax]$  where  $\Omega_{2d}$  is the domain defined by the two dimensional meshes.  $zmin$  and  $zmax$  are functions of  $\Omega_{2d}$  in  $\mathbb{R}$  that defines respectively the lower surface and upper surface of  $\Omega_{3d}$ .



**Fig. 3.25:** Example of Layer mesh in three dimensions.

For a vertex of a two dimensional mesh  $V_i^{2d} = (x_i, y_i)$ , we introduce the number of associated vertices in the  $z$ -axis  $M_i + 1$ .

We denote by  $M$  the maximum of  $M_i$  over the vertices of the two dimensional mesh. This value is called the number of layers (if  $\forall i$ ,  $M_i = M$  then there are  $M$  layers in the mesh of  $\Omega_{3d}$ ).  $V_i^{2d}$  generated  $M + 1$  vertices which are defined by:

$$\forall j = 0, \dots, M, \quad V_{i,j}^{3d} = (x_i, y_i, \theta_i(z_{i,j})),$$

where  $(z_{i,j})_{j=0,\dots,M}$  are the  $M + 1$  equidistant points on the interval  $[zmin(V_i^{2d}), zmax(V_i^{2d})]$ :

$$z_{i,j} = j \delta\alpha + zmin(V_i^{2d}), \quad \delta\alpha = \frac{zmax(V_i^{2d}) - zmin(V_i^{2d})}{M}.$$

The function  $\theta_i$ , defined on  $[zmin(V_i^{2d}), zmax(V_i^{2d})]$ , is given by:

$$\theta_i(z) = \begin{cases} \theta_{i,0} & \text{if } z = zmin(V_i^{2d}), \\ \theta_{i,j} & \text{if } z \in ]\theta_{i,j-1}, \theta_{i,j}], \end{cases}$$

with  $(\theta_{i,j})_{j=0,\dots,M_i}$  are the  $M_i + 1$  equidistant points on the interval  $[zmin(V_i^{2d}), zmax(V_i^{2d})]$ .

Set a triangle  $K = (V_{i1}^{2d}, V_{i2}^{2d}, V_{i3}^{2d})$  of the two dimensional mesh.  $K$  is associated with a triangle on the upper surface (resp. on the lower surface) of layer mesh:

$$(V_{i1,M}^{3d}, V_{i2,M}^{3d}, V_{i3,M}^{3d}) \text{ (resp. } (V_{i1,0}^{3d}, V_{i2,0}^{3d}, V_{i3,0}^{3d})).$$

Also  $K$  is associated with  $M$  volume prismatic elements which are defined by:

$$\forall j = 0, \dots, M, \quad H_j = (V_{i1,j}^{3d}, V_{i2,j}^{3d}, V_{i3,j}^{3d}, V_{i1,j+1}^{3d}, V_{i2,j+1}^{3d}, V_{i3,j+1}^{3d}).$$

These volume elements can have some merged point:

- 0 merged point : prism
- 1 merged points : pyramid
- 2 merged points : tetrahedra
- 3 merged points : no elements

The elements with merged points are called degenerate elements. To obtain a mesh with tetrahedra, we decompose the pyramid into two tetrahedra and the prism into three tetrahedra. These tetrahedra are obtained by cutting the quadrilateral face of pyramid and prism with the diagonal which have the vertex with the maximum index (see [HECHT1992] for the reason of this choice).

The triangles on the middle surface obtained with the decomposition of the volume prismatic elements are the triangles generated by the edges on the border of the two dimensional mesh. The label of triangles on the border elements and tetrahedra are defined with the label of these associated elements.

The arguments of `buildlayers` is a two dimensional mesh and the number of layers  $M$ .

The parameters of this command are:

- **`zbound= [zmin, zmax]`** where `zmin` and `zmax` are functions expression. These functions define the lower surface mesh and upper mesh of surface mesh.
- **`coef= A function expression between [0,1]`**. This parameter is used to introduce degenerate element in mesh. The number of associated points or vertex  $V_i^{2d}$  is the integer part of  $\text{coef}(V_i^{2d})M$ .
- **`region=`** This vector is used to initialize the region of tetrahedra.

This vector contains successive pairs of the 2d region number at index  $2i$  and the corresponding 3d region number at index  $2i + 1$ , like `change`.

- **`labelmid=`** This vector is used to initialize the 3d labels number of the vertical face or mid face from the 2d label number.

This vector contains successive pairs of the 2d label number at index  $2i$  and the corresponding 3d label number at index  $2i + 1$ , like `change`.

- **`labelup=`** This vector is used to initialize the 3d label numbers of the upper/top face from the 2d region number.

This vector contains successive pairs of the 2d region number at index  $2i$  and the corresponding 3d label number at index  $2i + 1$ , like `change`.

- **`labeldown=`** Same as the previous case but for the lower/down face label.

Moreover, we also add post processing parameters that allow to moving the mesh. These parameters correspond to parameters `transfo`, `facemerge` and `ptmerge` of the command line `movemesh`.

The vector `region`, `labelmid`, `labelup` and `labeldown` These vectors are composed of  $n_l$  successive pairs of number  $O_i, N_l$  where  $n_l$  is the number (label or region) that we want to get.

An example of this command is given in the [Build layer mesh example](#).

**Tip:** Cube

```

1 //Cube.idp
2 load "medit"
3 load "msh3"
4
5 func mesh3 Cube (int[int] &NN, real[int, int] &BB, int[int, int] &L) {
6     real x0 = BB(0,0), x1 = BB(0,1);
7     real y0 = BB(1,0), y1 = BB(1,1);
8     real z0 = BB(2,0), z1 = BB(2,1);
9
10    int nx = NN[0], ny = NN[1], nz = NN[2];
11
12    // 2D mesh
13    mesh Thx = square(nx, ny, [x0+(x1-x0)*x, y0+(y1-y0)*y]);
14
15    // 3D mesh
16    int[int] rup = [0, L(2,1)], rdown=[0, L(2,0)];
17    int[int] rmid=[1, L(1,0), 2, L(0,1), 3, L(1,1), 4, L(0,0)];
18    mesh3 Th = buildlayers(Thx, nz, zbound=[z0,z1],
19    labelmid=rmid, labelup = rup, labeledown = rdown);
20
21    return Th;
22}

```

**Tip:** Unit cube

```

1 include "Cube.idp"
2
3 int[int] NN = [10,10,10]; //the number of step in each direction
4 real [int, int] BB = [[0,1],[0,1],[0,1]]; //the bounding box
5 int [int, int] L = [[1,2],[3,4],[5,6]]; //the label of the 6 face left,right, front,_
//back, down, right
6 mesh3 Th = Cube(NN, BB, L);
7 medit ("Th", Th);

```

**Tip:** Cone

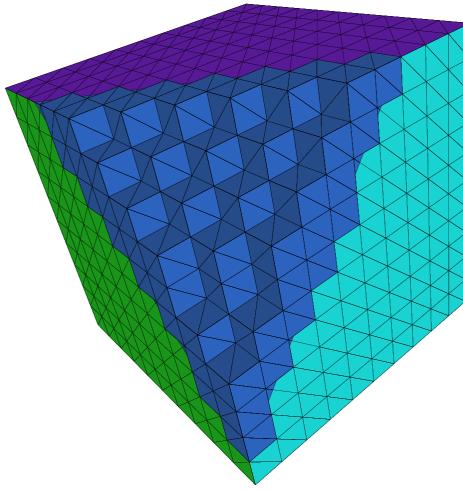
An axisymtric mesh on a triangle with degenerateness

```

1 load "msh3"
2 load "medit"
3
4 // Parameters
5 real RR = 1;
6 real HH = 1;
7
8 int nn=10;
9
10 // 2D mesh
11 border Taxe(t=0, HH){x=t; y=0; label=0;}
12 border Hypo(t=1, 0){x=HH*t; y=RR*t; label=1;}
13 border Vert(t=0, RR){x=HH; y=t; label=2;}
14 mesh Th2 = buildmesh(Taxe(HH*nn) + Hypo(sqrt(HH*HH+RR*RR)*nn) + Vert(RR*nn));
15 plot(Th2, wait=true);

```

(continues on next page)



**Fig. 3.26:** The mesh of a cube made with `cube.edp`

(continued from previous page)

```

16 // 3D mesh
17 real h = 1./nn;
18 int MaxLayersT = (int(2*pi*RR/h)/4)*4; //number of layers
19 real zminT = 0;
20 real zmaxT = 2*pi; //height 2*pi
21 func fx = y*cos(z);
22 func fy = y*sin(z);
23 func fz = x;
24 int[int] r1T = [0,0], r2T = [0,0,2,2], r4T = [0,2];
25 //trick function:
26 //The function defined the proportion
27 //of number layer close to axis with reference MaxLayersT
28 func deg = max(.01, y/max(x/HH, 0.4)/RR);
29 mesh3 Th3T = buildlayers(Th2, coef=deg, MaxLayersT,
30 zbound=[zminT, zmaxT], transfo=[fx, fy, fz],
31 facemerge=0, region=r1T, labelmid=r2T);
32 medit("cone", Th3T);
33

```

---

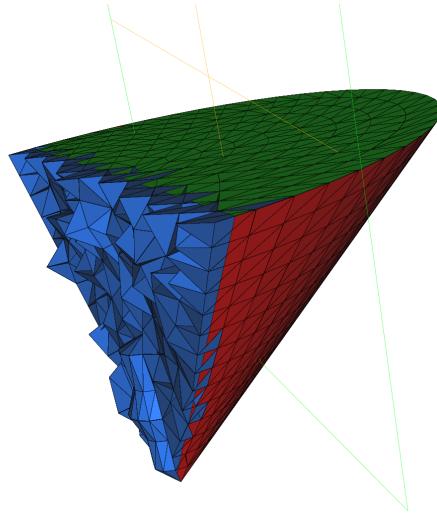
**Tip:** Buildlayer mesh

```

1 load "msh3"
2 load "TetGen"
3 load "medit"
4
5 // Parameters
6 int C1 = 99;
7 int C2 = 98;
8

```

(continues on next page)

**Fig. 3.27:** The mesh of a cone made with `cone.edp`

(continued from previous page)

```

9 // 2D mesh
10 border C01(t=0, pi){x=t; y=0; label=1;}
11 border C02(t=0, 2*pi){ x=pi; y=t; label=1;}
12 border C03(t=0, pi){ x=pi-t; y=2*pi; label=1;}
13 border C04(t=0, 2*pi){ x=0; y=2*pi-t; label=1;}
14
15 border C11(t=0, 0.7){x=0.5+t; y=2.5; label=C1;}
16 border C12(t=0, 2){x=1.2; y=2.5+t; label=C1;}
17 border C13(t=0, 0.7){x=1.2-t; y=4.5; label=C1;}
18 border C14(t=0, 2){x=0.5; y=4.5-t; label=C1;}
19
20 border C21(t=0, 0.7){x=2.3+t; y=2.5; label=C2;}
21 border C22(t=0, 2){x=3; y=2.5+t; label=C2;}
22 border C23(t=0, 0.7){x=3-t; y=4.5; label=C2;}
23 border C24(t=0, 2){x=2.3; y=4.5-t; label=C2;}
24
25 mesh Th = buildmesh(C01(10) + C02(10) + C03(10) + C04(10)
26     + C11(5) + C12(5) + C13(5) + C14(5)
27     + C21(-5) + C22(-5) + C23(-5) + C24(-5));
28
29 mesh Ths = buildmesh(C01(10) + C02(10) + C03(10) + C04(10)
30     + C11(5) + C12(5) + C13(5) + C14(5));
31
32 // Construction of a box with one hole and two regions
33 func zmin = 0.;
34 func zmax = 1.;
35 int MaxLayer = 10;
36
37 func XX = x*cos(y);
38 func YY = x*sin(y);
39 func ZZ = z;
40
41 int[int] r1 = [0, 41], r2 = [98, 98, 99, 99, 1, 56];
42 int[int] r3 = [4, 12]; //the triangles of upper surface mesh
43 //generated by the triangle in the 2D region

```

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```

44 //of mesh Th of label 4 as label 12
45 int[int] r4 = [4, 45]; //the triangles of lower surface mesh
46 //generated by the triangle in the 2D region
47 //of mesh Th of label 4 as label 45.
48
49 mesh3 Th3 = buildlayers(Th, MaxLayer, zbound=[zmin, zmax], region=r1,
50   labelmid=r2, labelup=r3, labeledown=r4);
51 medit("box 2 regions 1 hole", Th3);
52
53 // Construction of a sphere with TetGen
54 func XX1 = cos(y)*sin(x);
55 func YY1 = sin(y)*sin(x);
56 func ZZ1 = cos(x);
57
58 real[int] domain = [0., 0., 0., 0, 0.001];
59 string test = "paACQ";
60 cout << "test = " << test << endl;
61 mesh3 Th3sph = tetgtransfo(Ths, transfo=[XX1, YY1, ZZ1],
62   switch=test, nbregions=1, regionlist=domain);
63 medit("sphere 2 regions", Th3sph);

```

### 3.2.11 Meshing examples

**Tip:** Lake

```

1 load "msh3"
2 load "medit"
3
4 // Parameters
5 int nn = 5;
6
7 // 2D mesh
8 border cc(t=0, 2*pi){x=cos(t); y=sin(t); label=1;};
9 mesh Th2 = buildmesh(cc(100));
10
11 // 3D mesh
12 int[int] rup = [0, 2], rlow = [0, 1];
13 int[int] rmid = [1, 1, 2, 1, 3, 1, 4, 1];
14 func zmin = 2-sqrt(4-(x*x+y*y));
15 func zmax = 2-sqrt(3.);
16
17 mesh3 Th = buildlayers(Th2, nn,
18   coef=max((zmax-zmin)/zmax, 1./nn),
19   zbound=[zmin,zmax],
20   labelmid=rmid,
21   labelup=rup,
22   labeledown=rlow);
23
24 medit ("Th", Th);

```

**Tip:** Hole region

```

1  load "msh3"
2  load "TetGen"
3  load "medit"
4
5 // 2D mesh
6 mesh Th = square(10, 20, [x*pi-pi/2, 2*y*pi]); // ]-pi/2, pi/2[X]0,2pi[
7
8 // 3D mesh
9 //parametrization of a sphere
10 func f1 = cos(x)*cos(y);
11 func f2 = cos(x)*sin(y);
12 func f3 = sin(x);
13 //partial derivative of the parametrization
14 func f1x = sin(x)*cos(y);
15 func f1y = -cos(x)*sin(y);
16 func f2x = -sin(x)*sin(y);
17 func f2y = cos(x)*cos(y);
18 func f3x = cos(x);
19 func f3y = 0;
20 //M = DF^t DF
21 func m11 = f1x^2 + f2x^2 + f3x^2;
22 func m21 = f1x*f1y + f2x*f2y + f3x*f3y;
23 func m22 = f1y^2 + f2y^2 + f3y^2;
24
25 func perio = [[4, y], [2, y], [1, x], [3, x]];
26 real hh = 0.1;
27 real vv = 1/square(hh);
28 verbosity = 2;
29 Th = adaptmesh(Th, m11*vv, m21*vv, m22*vv, IsMetric=1, periodic=perio);
30 Th = adaptmesh(Th, m11*vv, m21*vv, m22*vv, IsMetric=1, periodic=perio);
31 plot(Th, wait=true);
32
33 //construction of the surface of spheres
34 real Rmin = 1.;
35 func f1min = Rmin*f1;
36 func f2min = Rmin*f2;
37 func f3min = Rmin*f3;
38
39 mesh3 Th3sph = movemesh23(Th, transfo=[f1min, f2min, f3min]);
40
41 real Rmax = 2.;
42 func f1max = Rmax*f1;
43 func f2max = Rmax*f2;
44 func f3max = Rmax*f3;
45
46 mesh3 Th3sph2 = movemesh23(Th, transfo=[f1max, f2max, f3max]);
47
48 //gluing meshse
49 mesh3 Th3 = Th3sph + Th3sph2;
50
51 cout << " TetGen call without hole " << endl;
52 real[int] domain2 = [1.5, 0., 0., 145, 0.001, 0.5, 0., 0., 18, 0.001];
53 mesh3 Th3fin = tetcg(Th3, switch="paAAQYY", nbofregions=2, regionlist=domain2);
54 medit("Sphere with two regions", Th3fin);
55
56 cout << " TetGen call with hole " << endl;
57 real[int] hole = [0.,0.,0.];

```

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```

58 real[int] domain = [1.5, 0., 0., 53, 0.001];
59 mesh3 Th3finhole = tetg(Th3, switch="paAAQYY",
60   nbofholes=1, holelist=hole, nbofregions=1, regionlist=domain);
61 medit("Sphere with a hole", Th3finhole);

```

## Build a 3d mesh of a cube with a balloon

First the MeshSurface.idp file to build boundary mesh of a Hexaedra and of a Sphere:

```

1 func mesh3 SurfaceHex (int[int] &N, real[int, int] &B, int[int, int] &L, int_
→orientation) {
2     real x0 = B(0, 0), x1 = B(0, 1);
3     real y0 = B(1, 0), y1 = B(1, 1);
4     real z0 = B(2, 0), z1 = B(2, 1);
5
6     int nx = N[0], ny = N[1], nz = N[2];
7
8     mesh Thx = square(ny, nz, [y0+(y1-y0)*x, z0+(z1-z0)*y]);
9     mesh Thy = square(nx, nz, [x0+(x1-x0)*x, z0+(z1-z0)*y]);
10    mesh Thz = square(nx, ny, [x0+(x1-x0)*x, y0+(y1-y0)*y]);
11
12    int[int] refx = [0, L(0,0)], refX = [0, L(0,1)]; //Xmin, Ymax faces labels_
→renumbering
13    int[int] refy = [0, L(1,0)], refY = [0, L(1,1)]; //Ymin, Ymax faces labels_
→renumbering
14    int[int] refz = [0, L(2,0)], refZ = [0, L(2,1)]; //Zmin, Zmax faces labels_
→renumbering
15
16    mesh3 Thx0 = movemesh23(Thx, transfo=[x0, x, y], orientation=-orientation,_
→label=refx);
17    mesh3 Thx1 = movemesh23(Thx, transfo=[x1, x, y], orientation=+orientation,_
→label=refX);
18    mesh3 Thy0 = movemesh23(Thy, transfo=[x, y0, y], orientation=+orientation,_
→label=refy);
19    mesh3 Thy1 = movemesh23(Thy, transfo=[x, y1, y], orientation=-orientation,_
→label=refY);
20    mesh3 Thz0 = movemesh23(Thz, transfo=[x, y, z0], orientation=-orientation,_
→label=refz);
21    mesh3 Thz1 = movemesh23(Thz, transfo=[x, y, z1], orientation=+orientation,_
→label=refZ);
22    mesh3 Th = Thx0 + Thx1 + Thy0 + Thy1 + Thz0 + Thz1;
23
24    return Th;
25}
26
27 func mesh3 Sphere (real R, real h, int L, int orientation){
28     mesh Th=square(10, 20, [x*pi-pi/2, 2*y*pi]); //]-pi/2, pi/2[X]0,2pi[
29
30     func f1 = cos(x)*cos(y);
31     func f2 = cos(x)*sin(y);
32     func f3 = sin(x);
33
34     func f1x = sin(x)*cos(y);
35     func f1y = -cos(x)*sin(y);

```

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```

36   func f2x = -sin(x)*sin(y);
37   func f2y = cos(x)*cos(y);
38   func f3x = cos(x);
39   func f3y = 0;
40
41   func m11 = f1x^2 + f2x^2 + f3x^2;
42   func m21 = f1x*f1y + f2x*f2y + f3x*f3y;
43   func m22 = f1y^2 + f2y^2 + f3y^2;
44
45   func perio = [[4, y], [2, y], [1, x], [3, x]]; //to store the periodic condition
46
47   real hh = h/R; //hh mesh size on unite sphere
48   real vv = 1/square(hh);
49   Th = adaptmesh(Th, m11*vv, m21*vv, m22*vv, IsMetric=1, periodic=perio);
50   Th = adaptmesh(Th, m11*vv, m21*vv, m22*vv, IsMetric=1, periodic=perio);
51   Th = adaptmesh(Th, m11*vv, m21*vv, m22*vv, IsMetric=1, periodic=perio);
52   Th = adaptmesh(Th, m11*vv, m21*vv, m22*vv, IsMetric=1, periodic=perio);
53   int[int] ref = [0, L];
54
55   mesh3 ThS = movemesh23(Th, transfo=[f1*R, f2*R, f3*R], orientation=orientation,
56   ↪refface=ref);
57
58   return ThS;
}

```

The test of the two functions and the call to TetGen mesh generator:

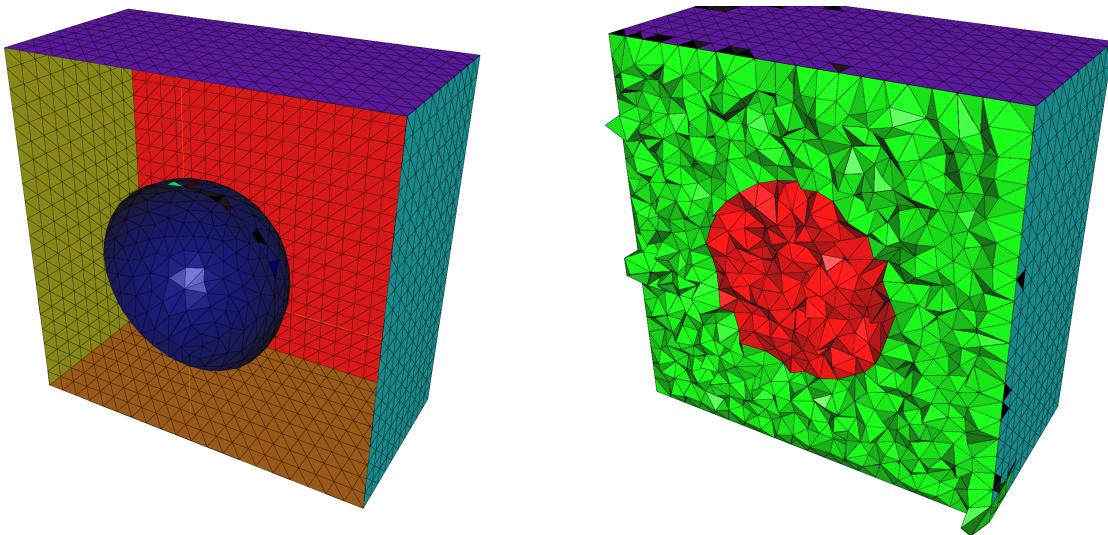
```

1  load "msh3"
2  load "TetGen"
3  load "medit"
4  include "MeshSurface.idp"
5
6  // Parameters
7  real hs = 0.1; //mesh size on sphere
8  int[int] N = [20, 20, 20];
9  real [int,int] B = [[-1, 1], [-1, 1], [-1, 1]];
10 int [int,int] L = [[1, 2], [3, 4], [5, 6]];
11
12 // Mesh
13 mesh3 ThH = SurfaceHex(N, B, L, 1);
14 mesh3 ThS = Sphere(0.5, hs, 7, 1);
15
16 mesh3 ThHS = ThH + ThS;
17 medit("Hex-Sphere", ThHS);
18
19 real voltet = (hs^3)/6.;
20 cout << "voltet = " << voltet << endl;
21 real[int] domain = [0, 0, 0, 1, voltet, 0, 0, 0.7, 2, voltet];
22 mesh3 Th = tetg(ThHS, switch="pqaAYYQ", nbofregions=2, regionlist=domain);
23 medit("Cube with ball", Th);

```

### 3.2.12 Medit

The keyword `medit` allows to display a mesh alone or a mesh and one or several functions defined on the mesh using the Pascal Frey's freeware `medit`. `medit` opens its own window and uses OpenGL extensively. Naturally to use this command `medit` must be installed.



(a) The surface mesh of the hex with internal sphere

(b) The tetrahedral mesh of the cube with internal ball

**Fig. 3.28:** Cube sphere

A visualization with `medit` of scalar solutions  $f_1$  and  $f_2$  continuous, piecewise linear and known at the vertices of the mesh  $\text{Th}$  is obtained using:

```
1 medit("sol1 sol2", Th, f1, f2, order=1);
```

The first plot named `sol1` displays  $f_1$ . The second plot named `sol2` displays  $f_2$ .

The arguments of the function `medit` are the name of the different scenes (separated by a space) of `medit`, a mesh and solutions.

Each solution is associated with one scene. The scalar, vector and symmetric tensor solutions are specified in the format described in the section dealing with the keyword `savesol`.

The parameters of this command line are :

- **`order= 0 if the solution is given at the center of gravity of elements.`** 1 is the solution is given at the vertices of elements.
- **`meditff= set the name of execute command of medit.`** By default, this string is `medit`.
- **`save= set the name of a file .sol or .solb to save solutions.`**

This command line allows also to represent two different meshes and solutions on them in the same windows. The nature of solutions must be the same. Hence, we can visualize in the same window the different domains in a domain decomposition method for instance. A visualization with `medit` of scalar solutions  $h_1$  and  $h_2$  at vertices of the mesh  $\text{Th}_1$  and  $\text{Th}_2$  respectively are obtained using:

```
1 medit("sol2domain", Th1, h1, Th2, h2, order=1);
```

---

#### Tip: Medit

```
1 load "medit"
2
3 // Initial Problem:
4 // Resolution of the following EDP:
```

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```

5 // -Delta u_s = f on \Omega = { (x,y) | 1 <= sqrt(x^2+y^2) <= 2 }
6 // -Delta u_1 = f1 on \Omega_1 = { (x,y) | 0.5 <= sqrt(x^2+y^2) <= 1. }
7 // u = 1 on Gamma
8 // Null Neumann condition on Gamma_1 and on Gamma_2
9 // We find the solution u by solving two EDP defined on domain Omega and Omega_1
10 // This solution is visualize with medit
11
12 verbosity=3;
13
14 // Mesh
15 border Gamma(t=0, 2*pi){x=cos(t); y=sin(t); label=1;};
16 border Gamma1(t=0, 2*pi){x=2*cos(t); y=2*sin(t); label=2;};
17 border Gamma2(t=0, 2*pi){x=0.5*cos(t); y=0.5*sin(t); label=3;};
18
19 mesh Th = buildmesh(Gamma1(40) + Gamma(-40)); //Omega
20 mesh Th1 = buildmesh(Gamma(40) + Gamma2(-40)); //Omega_1
21
22 // Fespace
23 fespace Vh(Th, P2);
24 func f = sqrt(x*x + y*y);
25 Vh us, v;
26
27 fespace Vh1(Th1, P2);
28 func f1 = 10*sqrt(x*x+y*y);
29 Vh1 u1, v1;
30
31 // Macro
32 macro Grad2(us) [dx(us), dy(us)] // EOM
33
34 // Problem
35 problem Lap2dOmega (us, v, init=false)
36     = int2d(Th) (
37         Grad2(v)' * Grad2(us)
38     )
39     - int2d(Th) (
40         f*v
41     )
42     +on(1, us=1)
43 ;
44
45 problem Lap2dOmega1 (u1, v1, init=false)
46     = int2d(Th1) (
47         Grad2(v1)' * Grad2(u1)
48     )
49     - int2d(Th1) (
50         f1*v1
51     )
52     + on(1, u1=1)
53 ;
54
55 // Solve
56 Lap2dOmega;
57 Lap2dOmega1;
58
59 // Plot with medit
60 medit("solution", Th, us, Th1, u1, order=1, save="testsavemedit.solb");

```

### 3.2.13 Mshmet

Mshmet is a software developed by P. Frey that allows to compute an anisotropic metric based on solutions (i.e. Hessian-based). This software can return also an isotropic metric. Moreover, `mshmet` can also construct a metric suitable for levelset interface capturing. The solution can be defined on 2D or 3D structured/unstructured meshes. For example, the solution can be an error estimate of a FE solution.

Solutions for `mshmet` are given as an argument. The solution can be a `func`, a vector `func`, a symmetric tensor, a `fespace` function, a `fespace` vector function and a `fespace` symmetric tensor. The symmetric tensor argument is defined as this type of data for `datasol` argument. This software accepts more than one solution.

For example, the metric  $M$  computed with `mshmet` for the solution  $u$  defined on the mesh  $Th$  is obtained by writing:

```
1 fespace Vh(Th, P1);
2 Vh u; //a scalar fespace function
3 real[int] M = mshmet(Th, u);
```

The parameters of the keyword `mshmet` are :

- `normalization` = (b) do a normalization of all solution in [0, 1].
- `aniso` = (b) build anisotropic metric if 1 (default 0: isotropic)
- `levelset` = (b) build metric for levelset method (default: `false`)
- `verbosity` = (l) level of verbosity
- `nbregul` = (l) number of regularization's iteration of solutions given (default 0).
- `hmin` = (d)
- `hmax` = (d)
- `err` = (d) level of error.
- `width` = (d) the width
- **`metric` = a vector of double.** This vector contains an initial metric given to `mshmet`. The structure of the metric vector is described in the next paragraph.
- **`options` = a vector of integer of size 7.** This vector contains the integer parameters of `mshmet` (for expert only).
  - `loptions(0)`: normalization (default 1).
  - **`loptions(1): isotropic parameters (default 0)`.** 1 for isotropic metric results otherwise 0.
  - **`loptions(2): level set parameters (default 0)`.** 1 for building level set metric otherwise 0.
  - **`loptions(3): debug parameters (default 0)`.** 1 for turning on debug mode otherwise 0.
  - `loptions(4)`: level of verbosity (default 10).
  - `loptions(5)`: number of regularization's iteration of solutions given (default 0).
  - **`loptions(6): previously metric parameter (default 0)`.** 1 for using previous metric otherwise 0.
- **`options= a vector of double of size 4.`** This vector contains the real parameters of `mshmet` (for expert only).
  - `doptions(0)`: `hmin` : min size parameters (default 0.01).
  - `doptions(1)`: `hmax` : max size parameters (default 1.0).
  - `doptions(2)`: `eps` : tolerance parameters (default 0.01).
  - `doptions(2)`: `width` : relative width for Level Set ( $0 < w < 1$ ) (default 0.05).

The result of the keyword `mshmet` is a `real [int]` which contains the metric computed by `mshmet` at the different vertices  $V_i$  of the mesh.

With  $nv$  is the number of vertices, the structure of this vector is:

$$M_{iso} = (m(V_0), m(V_1), \dots, m(V_{nv}))^t$$

for a isotropic metric  $m$ . For a symmetric tensor metric  $h = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$ , the parameters `metric` is:

$$M_{aniso} = (H(V_0), \dots, H(V_{nv}))^t$$

where  $H(V_i)$  is the vector of size 6 defined by  $[m_{11}, m_{21}, m_{22}, m_{31}, m_{32}, m_{33}]$

---

**Tip:** `mshmet`

```

1  load "mshmet"
2  load "medit"
3  load "msh3"
4
5 // Parameters
6 real error = 0.01;
7 func zmin = 0;
8 func zmax = 1;
9 int MaxLayer = 10;
10
11 // Mesh
12 border a(t=0, 1.0){x=t; y=0; label=1;};
13 border b(t=0, 0.5){x=1; y=t; label=2;};
14 border c(t=0, 0.5){x=1-t; y=0.5; label=3;};
15 border d(t=0.5, 1){x=0.5; y=t; label=4;};
16 border e(t=0.5, 1){x=1-t; y=1; label=5;};
17 border f(t=0.0, 1){x=0; y=1-t; label=6;};
18 mesh Th = buildmesh(a(6) + b(4) + c(4) + d(4) + e(4) + f(6));
19 mesh3 Th3 = buildlayers(Th, MaxLayer, zbound=[zmin, zmax]);
20
21 // Fespace
22 fespace Vh3(Th3, P2);
Vh3 u3, v3;
24
25 fespace Vh3P1(Th3, P1);
Vh3P1 usol;
27
28 // Problem
29 problem Problem2(u3, v3, solver=sparse)
30   = int3d(Th3) (
31     u3*v3*1.0e-10
32     + dx(u3)*dx(v3)
33     + dy(u3)*dy(v3)
34     + dz(u3)*dz(v3)
35   )
36   - int3d(Th3) (
37     v3
38   )
39   +on(0, 1, 2, 3, 4, 5, 6, u3=0)
40 ;

```

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```

41
42 // Solve
43 Problem2;
44 cout << u3[] .min << " " << u3[] .max << endl;
45
46 medit ("Sol", Th3, u3);
47
48 real[int] bb = mshmet(Th3,u3);
49 cout << "Metric:" << bb << endl;
50 for (int ii = 0; ii < Th3.nv; ii++)
51     usol[] [ii] = bb[ii];
52
53 medit ("Metric", Th3, usol);

```

### 3.2.14 FreeYams

FreeYams is a surface mesh adaptation software which is developed by P. Frey. This software is a new version of yams. The adapted surface mesh is constructed with a geometric metric tensor field. This field is based on the intrinsic properties of the discrete surface.

Also, this software allows to construct a simplification of a mesh. This decimation is based on the Hausdorff distance between the initial and the current triangulation. Compared to the software yams, FreeYams can be used also to produce anisotropic triangulations adapted to levelset simulations. A technical report on freeYams documentation is available [here](#).

To call FreeYams in **FreeFEM**, we used the keyword `freeyams`. The arguments of this function are the initial mesh and/or metric. The metric with `freeyams` are a `func`, a `fespace` function, a symmetric tensor function, a symmetric tensor `fespace` function or a vector of double (`real[int]`). If the metric is a vector of double, this data must be given in `metric` parameter. Otherwise, the metric is given in the argument.

For example, the adapted mesh of `Thinit` defined by the metric `u` defined as `fespace` function is obtained by writing:

```

1 fespace Vh(Thinit, P1);
2 Vh u;
3 mesh3 Th = freeyams(Thinit, u);

```

The symmetric tensor argument for `freeyams` keyword is defined as this type of data for `datasol` argument.

- `aniso= (b)` aniso or iso metric (default 0, iso)
- `mem= (l)` memory of for `freeyams` in Mb (default -1, `freeyams` choose)
- `hmin= (d)`
- `hmax= (d)`
- `gradation= (d)`
- `option= (l)`
  - 0 : mesh optimization (smoothing+swapping)
  - 1 : decimation+enrichment adaptated to a metric map. (default)
  - -1 : decimation adaptated to a metric map.
  - 2 : decimation+enrichment with a Hausdorff-like method

- -2 : decimation with a Hausdorff-like method
  - 4 : split triangles recursively.
  - 9 : No-Shrinkage Vertex Smoothing
- ridgeangle= (d)
  - absolute= (b)
  - verbosity= (i)
- **metric= vector expression.** This parameters contains the metric at the different vertices on the initial mesh.  
With  $nv$  is the number of vertices, this vector is:

$$M_{iso} = (m(V_0), m(V_1), \dots, m(V_{nv}))^t$$

for a scalar metric  $m$ . For a symmetric tensor metric  $h = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$ , the parameters **metric** is:

$$M_{aniso} = (H(V_0), \dots, H(V_{nv}))^t$$

where  $H(V_i)$  is the vector of size 6 defined by  $[m_{11}, m_{21}, m_{22}, m_{31}, m_{32}, m_{33}]$

- **options= a vector of integer of size 13.** This vectors contains the integer options of FreeYams. (just for the expert)
  - **loptions(0): anisotropic parameter (default 0).** If you give an anisotropic metric 1 otherwise 0.
  - **loptions(1): Finite Element correction parameter (default 0).** 1 for no Finite Element correction otherwise 0.
  - **loptions(2): Split multiple connected points parameter (default 1).** 1 for splitting multiple connected points otherwise 0.
  - loptions(3): maximum value of memory size in Mbytes (default -1: the size is given by freeyams).
  - **loptions(4): set the value of the connected component which we want to obtain.** (Remark: freeyams give an automatic value at each connected component).
  - loptions(5): level of verbosity
  - **loptions(6): Create point on straight edge (no mapping) parameter (default 0).** 1 for creating point on straight edge otherwise 0.
  - **loptions(7): validity check during smoothing parameter.** This parameter is only used with No-Shrinkage Vertex Smoothing optimization (optimization option parameter 9). 1 for No validity checking during smoothing otherwise 0.
  - loptions(8): number of desired's vertices (default -1).
  - loptions(9): number of iteration of optimizations (default 30).
  - **loptions(10): no detection parameter (default 0).** 1 for detecting the ridge on the mesh otherwise 0. The ridge definition is given in the parameter doptions(12).
  - **loptions(11): no vertex smoothing parameter (default 0).** 1 for smoothing the vertices otherwise 0.
  - loptions(12): Optimization level parameter (default 0).
  - 0 : mesh optimization (smoothing+swapping)

- 1 : decimation+enrichment adapted to a metric map.
  - -1: decimation adapted to a metric map.
  - 2 : decimation+enrichment with a Hausdorff-like method
  - -2: decimation with a Hausdorff-like method
  - 4 : split triangles recursively.
  - 9 : No-Shrinkage Vertex Smoothing
- **doptions= a vector of double of size 11.** This vectors contains the real options of freeyams.
- doptions(0): Set the geometric approximation (Tangent plane deviation) (default 0.01).
  - doptions(1): Set the lamda parameter (default -1).
  - doptions(2): Set the mu parameter (default -1).
  - doptions(3): Set the gradation value (Mesh density control) (default 1.3).
  - doptions(4): Set the minimal size(hmin) (default -2.0: the size is automatically computed).
  - doptions(5): Set the maximal size(hmax) (default -2.0: the size is automatically computed).
  - doptions(6): Set the tolerance of the control of Chordal deviation (default -2.0).
  - doptions(7): Set the quality of degradation (default 0.599).
  - doptions(8): Set the declic parameter (default 2.0).
  - doptions(9): Set the angular walton limitation parameter (default 45 degree).
  - doptions(10): Set the angular ridge detection (default 45 degree).

**Tip:** freeyams

```

1 load "msh3"
2 load "medit"
3 load "freeyams"
4
5 // Parameters
6 int nn = 20;
7 real zmin = 0;
8 real zmax = 1;
9
10 // Mesh
11 mesh Th2 = square(nn, nn);
12 int[int] rup = [0, 2], rdown = [0, 1];
13 int[int] rmid = [1, 1, 2, 1, 3, 1, 4, 1];
14 mesh3 Th = buildlayers(Th2, nn, zbound=[zmin, zmax], reffacemid=rmid, reffaceup=rup, ↵
15 ↵ reffacelow=rdown);
16 mesh3 Th3 = freeyams(Th);
17 medit ("SurfaceMesh", Th3);

```

### 3.2.15 mmg3d

---

**Todo:** mmg3d-v4.0

---

Mmg3d is a 3D remeshing software developed by C. Dobrzynski and P. Frey.

This software allows to remesh an initial mesh made of tetrahedra. This initial mesh is adapted to a geometric metric tensor field or to a displacement vector (moving rigid body). The metric can be obtained with [mshmet](#).

---

**Note:**

- If no metric is given, an isotropic metric is computed by analyzing the size of the edges in the initial mesh.
  - If a displacement is given, the vertices of the surface triangles are moved without verifying the geometrical structure of the new surface mesh.
- 

The parameters of mmg3d are :

- **options= vector expression.** This vector contains the option parameters of mmg3d. It is a vector of 6 values, with the following meaning:
  - Optimization parameters : (default 1)
    - 0 : mesh optimization.
    - 1 : adaptation with metric (deletion and insertion vertices) and optimization.
    - 1 : adaptation with metric (deletion and insertion vertices) without optimization.
    - 4 : split tetrahedra (be careful modify the surface).
    - 9 : moving mesh with optimization.
    - 9 : moving mesh without optimization.
  - Debug mode : (default 0)
    - 1 : turn on debug mode.
    - 0 : otherwise.
  - Specify the size of bucket per dimension (default 64)
  - Swapping mode : (default 0)
    - 1 : no edge or face flipping.
    - 0 : otherwise.
  - Insert points mode : (default 0)
    - 1 : no edge splitting or collapsing and no insert points.
    - 0 : otherwise.
- 5. Verbosity level (default 3)
- **memory= integer expression.** Set the maximum memory size of new mesh in Mbytes. By default the number of maximum vertices, tetrahedra and triangles are respectively 500 000, 3000 000, 100000 which represent approximately a memory of 100 Mo.
- **metric= vector expression.** This vector contains the metric given at mmg3d. It is a vector of size  $nv$  or  $6 nv$  respectively for an isotropic and anisotropic metric where  $nv$  is the number of vertices in the initial mesh. The structure of metric vector is described in the [mshmet](#).

- `displacement= [Φ1, Φ2, Φ3]` set the displacement vector of the initial mesh  $\Phi(\mathbf{x}, \mathbf{y}) = [\Phi_1(x, y), \Phi_2(x, y), \Phi_3(x, y)]$ .
- **displVect= sets the vector displacement in a vector expression.** This vector contains the displacement at each point of the initial mesh. It is a vector of size 3  $nv$ .

**Tip:** mmg3d

```

1 load "msh3"
2 load "medit"
3 load "mmg3d"
4 include "Cube.idp"
5
6 // Parameters
7 int n = 6;
8 int[int] Nxyz = [12, 12, 12];
9 real [int, int] Bxyz = [[0., 1.], [0., 1.], [0., 1.]];
10 int [int, int] Lxyz = [[1, 1], [2, 2], [2, 2]];
11
12 // Mesh
13 mesh3 Th = Cube(Nxyz, Bxyz, Lxyz);
14
15 real[int] isometric(Th.nv);
16 for (int ii = 0; ii < Th.nv; ii++)
17     isometric[ii] = 0.17;
18
19 mesh3 Th3 = mmg3d(Th, memory=100, metric=isometric);
20
21 // Plot
22 medit("Initial", Th);
23 medit("Isometric", Th3);

```

**Tip:** Falling spheres

```

1 load "msh3"
2 load "TetGen"
3 load "medit"
4 load "mmg3d"
5 include "MeshSurface.idp"
6
7 // Parameters
8 real hs = 0.8;
9 int[int] N = [4/hs, 8/hs, 11.5/hs];
10 real [int, int] B = [[-2, 2], [-2, 6], [-10, 1.5]];
11 int [int, int] L = [[311, 311], [311, 311], [311, 311]];
12
13 int[int] opt = [9, 0, 64, 0, 0, 3];
14 real[int] vit=[0, 0, -0.3];
15 func zero = 0.;
16 func dep = vit[2];
17
18 // Mesh
19 mesh3 ThH = SurfaceHex(N, B, L, 1);
20 mesh3 ThSg = Sphere(1, hs, 300, -1);
21 mesh3 ThSd = Sphere(1, hs, 310, -1);

```

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```

22 ThSd = movemesh3(ThSd, transfo=[x, 4+y, z]);
23 mesh3 ThHS = ThH + ThSg + ThSd; //gluing surface meshes
24 medit("ThHS", ThHS);
25
26 real voltet = (hs^3)/6.;
27 real[int] domain = [0, 0, -4, 1, voltet];
28 real [int] holes = [0, 0, 0, 0, 4, 0];
29 mesh3 Th = tetg(ThHS, switch="pqaAAYYQ", nbofregions=1, regionlist=domaine,
   ↵nbofholes=2, holelist=holes);
30 medit("Box-With-two-Ball", Th);
31
32 // Fespace
33 fespace Vh(Th, P1);
34 Vh uh,vh;
35
36 // Macro
37 macro Grad(u) [dx(u),dy(u),dz(u)]
38
39 // Problem
40 problem Lap (uh, vh, solver=CG)
41   = int3d(Th) (
42     Grad(uh)' * Grad(vh)
43   )
44   + on(310, 300, uh=dep)
45   + on(311, uh=0.)
46 ;
47
48 // Falling loop
49 for(int it = 0; it < 29; it++){
50   cout << " ITERATION " << it << endl;
51
52   // Solve
53   Lap;
54
55   // Plot
56   plot(Th, uh);
57
58   // Sphere falling
59   Th = mmg3d(Th, options=opt, displacement=[zero, zero, uh], memory=1000);
60 }
```

### 3.2.16 A first 3d isotropic mesh adaptation process

**Tip:** Adaptation 3D

```

1 load "msh3"
2 load "TetGen"
3 load "mshmet"
4 load "medit"
5
6 // Parameters
7 int nn = 6;
8 int[int] l1111 = [1, 1, 1, 1]; //labels
```

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```

9 int[int] l01 = [0, 1];
10 int[int] l11 = [1, 1];
11
12 real errm = 1e-2; //level of error
13
14 // Mesh
15 mesh3 Th3 = buildlayers(square(nn, nn, region=0, label=l1111),
16 nn, zbound=[0, 1], labelmid=l11, labelup=l01, labeledown=l01);
17
18 Th3 = trunc(Th3, (x<0.5) | (y < 0.5) | (z < 0.5), label=1); //remove the ]0.5,1[^3_
19 //cube
20
21 // Fespace
22 fespace Vh(Th3, P1);
23 Vh u, v, usol, h;
24
25 // Macro
26 macro Grad(u) [dx(u), dy(u), dz(u)] // EOM
27
28 // Problem
29 problem Poisson (u, v, solver=CG)
30   = int3d(Th3)(
31     Grad(u)' * Grad(v)
32   )
33   - int3d(Th3)(
34     1*v
35   )
36   + on(1, u=0)
37 ;
38
39 // Loop
40 for (int ii = 0; ii < 5; ii++){
41   // Solve
42   Poisson;
43   cout << "u min, max = " << u[].min << " " << u[].max << endl;
44
45   h=0.; //for resizing h[] because the mesh change
46   h[] = mshmet(Th3, u, normalization=1, aniso=0, nbregul=1, hmin=1e-3, hmax=0.3,
47   ↪err=errm);
48   cout << "h min, max = " << h[].min << " " << h[].max << " " << h[].n << " " << Th3.
49   ↪nv << endl;
50   plot(u, wait=true);
51
52   errm *= 0.8; //change the level of error
53   cout << "Th3 " << Th3.nv < " " << Th3.nt << endl;
54   Th3 = tetgreconstruction(Th3, switch="raAQ", sizeofvolume=h*h*h/6.); //rebuild_
55   ↪mesh
56   medit("U-adap-iso-"+ii, Th3, u, wait=true);
57 }
```

### 3.2.17 Build a 2d mesh from an isoline

The idea is to get the discretization of an isoline of fluid meshes, this tool can be useful to construct meshes from image. First, we give an example of the isovalue meshes 0.2 of analytical function  $\sqrt{(x - 1/2)^2 + (y - 1/2)^2}$ , on

unit square.

```

1 load "isoline"
2
3 real[int,int] xy(3, 1); //to store the isoline points
4 int[int] be(1); //to store the begin, end couple of lines
5 {
6     mesh Th = square(10, 10);
7     fespace Vh(Th, P1);
8     Vh u = sqrt(square(x-0.5) + square(y-0.5));
9     real iso = 0.2 ;
10    real[int] viso = [iso];
11    plot(u, viso=viso, Th); //to see the iso line
12
13    int nbc = isoline(Th, u, xy, close=1, iso=iso, beginend=be, smoothing=0.1);

```

The `isoline` parameters are `Th` the mesh, the expression `u`, the bidimentionnal array `xy` to store the list coordinate of the points. The list of named parameter are :

- `iso`= value of the isoline to compute (0 is the default value)
- `close`= close the isoline with the border (default `true`), we add the part of the mesh border such the value is less than the isoline value
- `smoothing`= number of smoothing process is the  $l^r s$  where  $l$  is the length of the current line component,  $r$  the ratio,  $s$  is smoothing value. The smoothing default value is 0.
- `ratio`= the ratio (1 by default).
- `eps`= relative  $\varepsilon$  (default  $1e-10$ )
- `beginend`= array to get begin, end couple of each of sub line (resize automatically)
- `file`= to save the data curve in data file for gnuplot

In the array `xy` you get the list of vertices of the isoline, each connex line go from  $i = i_0^c, \dots, i_1^c - 1$  with  $i_0^c = be(2*c)$   $i_1^c = be(2*c + 1)$ , and where  $x_i = xy(0, i)$ ,  $y_i = xy(1, i)$ ,  $l_i = xy(2, i)$ .

Here  $l_i$  is the length of the line (the origin of the line is point  $i_0^c$ ).

The sense of the isoline is such that the upper part is at the left size of the isoline. So here : the minimum is a point 0.5, 0.5 so the curve turn in the clockwise sense, the order of each component are sort such that the number of point by component is decreasing.

```

1 cout << "Number of the line component = " << nbc << endl;
2 cout << "Number of points = " << xy.m << endl;
3 cout << "be = " << be << endl;
4
5 // shows the lines component
6 for (int c = 0; c < nbc; ++c){
7     int i0 = be[2*c], i1 = be[2*c+1]-1;
8     cout << "Curve " << c << endl;
9     for(int i = i0; i <= i1; ++i)
10        cout << "x= " << xy(0, i) << " y= " << xy(1, i) << " s= " << xy(2, i) <<_
11    endl;
12    plot([xy(0, i0:i1), xy(1, i0:i1)], wait=true, viso=viso, cmm=" curve "+c);
13 }
14
15 cout << "length of last curve = " << xy(2, xy.m-1) << endl;

```

We also have a new function to easily parametrize a discrete curve defined by the couple `be, xy`.

```

1 border Curve0(t=0, 1) {
2     int c=0; //component 0
3     int i0=be[2*c], i1=be[2*c+1]-1;
4     P=Curve(xy, i0, i1, t); //Curve 0
5     label=1;
6 }
7
8 border Curve1(t=0, 1) {
9     int c=1; //component 1
10    int i0=be[2*c], i1=be[2*c+1]-1;
11    P=Curve(xy, i0, i1, t); //Curve 1
12    label=1;
13 }
14
15 plot(Curve1(100)); //show curve
16 mesh Th = buildmesh(Curve1(-100));
17 plot(Th, wait=true);

```

Secondly, we use this idea to build meshes from an image, we use the plugins ppm2rnm to read pgm a gray scale image and then we extract the gray contour at level 0.25.

---

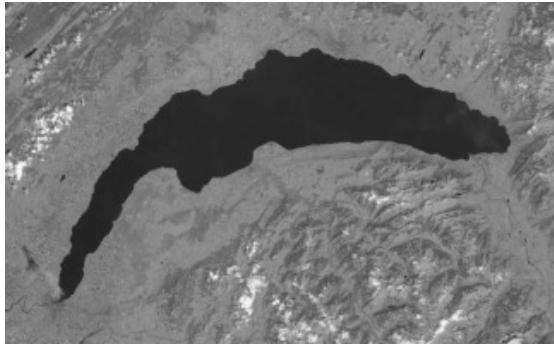
**Tip:** Leman lake

```

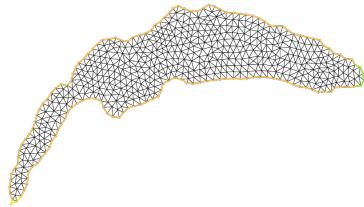
1 load "ppm2rnm"
2 load "isoline"
3
4 // Parameters
5 string leman = "LemanLake.pgm";
6 real AreaLac = 580.03; //in km^2
7 real hsize = 5;
8 real[int, int] Curves(3, 1);
9 int[int] be(1);
10 int nc; //nb of curve
11 {
12     real[int, int] ff1(leman); //read image
13     //and set it in a rect. array
14     int nx = ff1.n, ny = ff1.m;
15     //build a Cartesian mesh such that the origin is in the right place.
16     mesh Th = square(nx-1, ny-1, [(nx-1)*(x), (ny-1)*(1-y)]);
17     //warning the numbering of the vertices (x,y) is
18     //given by $i = x/nx + nx* y/ny $
19     fespace Vh(Th, P1);
20     Vh f1;
21     f1[] = ff1; //transform array in finite element functions.
22     nc = isoline(Th, f1, iso=0.25, close=1, Curves, beginend=be, smoothing=.1,
23     ↵ratio=0.5);
23 }
24
25 //The longest isoline: the lake
26 int ic0 = be(0), ic1 = be(1)-1;
27 plot([Curves(0, ic0:ic1), Curves(1, ic0:ic1)], wait=true);
28
29 int NC = Curves(2, ic1)/hsize;
30 real xl = Curves(0, ic0:ic1).max - 5;
31 real yl = Curves(1, ic0:ic1).min + 5;
32 border G(t=0, 1){P=Curve(Curves, ic0, ic1, t); label=1+(x>xl)*2+(y>yl);}

```

(continues on next page)



(a) The image of the Leman lake meshes



(b) The mesh of the lake

**Fig. 3.29:** Isoline

(continued from previous page)

```

33 plot (G(-NC), wait=true);
34
35 mesh Th = buildmesh (G(-NC));
36 plot (Th, wait=true);
37
38 real scale = sqrt (AreaLac/Th.area);
39 Th = movemesh (Th, [x*scale, y*scale]);
40 cout << "Th.area = " << Th.area << " Km^2 " << " == " << AreaLac << " Km^2 " << endl;
41 plot (Th, wait=true, ps="leman.eps");

```

### 3.3 Finite element

As stated in *tutorials*, FEM approximates all functions  $w$  as:

$$w(x, y) \simeq w_0\phi_0(x, y) + w_1\phi_1(x, y) + \cdots + w_{M-1}\phi_{M-1}(x, y)$$

with finite element basis functions  $\phi_k(x, y)$  and numbers  $w_k$  ( $k = 0, \dots, M - 1$ ). The functions  $\phi_k(x, y)$  are constructed from the triangle  $T_{i_k}$ , and called *shape functions*.

In **FreeFEM**, the finite element space:

$$V_h = \{w \mid w_0\phi_0 + w_1\phi_1 + \cdots + w_{M-1}\phi_{M-1}, w_i \in \mathbb{R}\}$$

is easily created by:

```
1 fespace IDspace (IDmesh,<IDFE>);
```

or with  $\ell$  pairs of periodic boundary conditions in 2D:

```

1 fespace IDspace (IDmesh,<IDFE>,
2   periodic=[[la1, sa1], [lb1, sb1],
3     ...
4       [lak, sak], [lbk, sbk]]);
```

and in 3D:

```

1 fespace IDspace(IDmesh,<IDFE>,
2   periodic=[[la1, sa1, ta1], [lb1, sb1, tb1],
3   ...
4   [lak, sak, tak], [lbk, sbk, tbk]]);
```

where `IDspace` is the name of the space (e.g. `Vh`), `IDmesh` is the name of the associated mesh and `<IDFE>` is an identifier of finite element type.

In 2D we have a pair of periodic boundary conditions, if  $[la_i, sa_i]$ ,  $[lb_i, sb_i]$  is a pair of `int`, and the 2 labels  $la_i$  and  $lb_i$  refer to 2 pieces of boundary to be in equivalence.

If  $[la_i, sa_i]$ ,  $[lb_i, sb_i]$  is a pair of `real`, then  $sa_i$  and  $sb_i$  give two common abscissa on the two boundary curves, and two points are identified as one if the two abscissa are equal.

In 2D, we have a pair of periodic boundary conditions, if  $[la_i, sa_i, ta_i]$ ,  $[lb_i, sb_i, tb_i]$  is a pair of `int`, the 2 labels  $la_i$  and  $lb_i$  define the 2 pieces of boundary to be in equivalence.

If  $[la_i, sa_i, ta_i]$ ,  $[lb_i, sb_i, tb_i]$  is a pair of `real`, then  $sa_i$ ,  $ta_i$  and  $sb_i$ ,  $tb_i$  give two common parameters on the two boundary surfaces, and two points are identified as one if the two parameters are equal.

---

**Note:** The 2D mesh of the two identified borders must be the same, so to be sure, use the parameter `fixedborder=true` in `buildmesh` command (see [fixedborder](#)).

---

As of today, the known types of finite elements are:

- `[P0, P03d]` piecewise constant discontinuous finite element (2d, 3d), the degrees of freedom are the barycenter element value.

$$\mathbb{P}_h^0 = \{v \in L^2(\Omega) \mid \text{for all } K \in \mathcal{T}_h \text{ there is } \alpha_K \in \mathbb{R} : v|_K = \alpha_K\} \quad (3.2)$$

- `[P1, P13d]` piecewise linear continuous finite element (2d, 3d), the degrees of freedom are the vertices values.

$$\mathbb{P}_h^1 = \{v \in H^1(\Omega) \mid \forall K \in \mathcal{T}_h, v|_K \in P_1\} \quad (3.3)$$

- `[P1dc]` piecewise linear discontinuous finite element

$$\mathbb{P}_{dc|h}^1 = \{v \in L^2(\Omega) \mid \forall K \in \mathcal{T}_h, v|_K \in P_1\} \quad (3.4)$$

**Warning:** Due to an interpolation problem, the degree of freedom is not the vertices but three vertices which move inside  $T(X) = G + .99(X - G)$  where  $G$  is the barycenter.

- `[P1b, P1b3d]` piecewise linear continuous finite element plus bubble (2d, 3d)

### The 2D Case:

$$\mathbb{P}_{b|h}^1 = \{v \in H^1(\Omega) \mid \forall K \in \mathcal{T}_h, v|_K \in P_1 \oplus \text{Span}\{\lambda_0^K \lambda_1^K \lambda_2^K\}\} \quad (3.5)$$

**The 3D Case:**

$$\mathbb{P}_{b|h}^1 = \{v \in H^1(\Omega) \mid \forall K \in \mathcal{T}_h, v|_K \in P_1 \oplus \text{Span}\{\lambda_0^K \lambda_1^K \lambda_2^K \lambda_3^K\}\}$$
 (3.6)

where  $\lambda_i^K, i = 0, \dots, d$  are the  $d + 1$  barycentric coordinate functions of the element  $K$  (triangle or tetrahedron).

- P1bl, P1bl3d piecewise linear continuous finite element plus linear bubble (2d, 3d).

The bubble is built by splitting the  $K$ , a barycenter in  $d + 1$  sub element. (need `load "Element_P1bl"`)

- [P2, P23d] piecewise  $P_2$  continuous finite element (2d, 3d)

$$\mathbb{P}_h^2 = \{v \in H^1(\Omega) \mid \forall K \in \mathcal{T}_h, v|_K \in P_2\}$$

where  $P_2$  is the set of polynomials of  $\mathbb{R}^2$  of degrees  $\leq 2$ .

- [P2b] piecewise  $P_2$  continuous finite element plus bubble

$$\mathbb{P}_h^2 = \{v \in H^1(\Omega) \mid \forall K \in \mathcal{T}_h, v|_K \in P_2 \oplus \text{Span}\{\lambda_0^K \lambda_1^K \lambda_2^K\}\}$$

- [P2dc] piecewise  $P_2$  discontinuous finite element

$$\mathbb{P}_{dc|h}^2 = \{v \in L^2(\Omega) \mid \forall K \in \mathcal{T}_h, v|_K \in P_2\}$$

**Warning:** Due to an interpolation problem, the degree of freedom is not the six  $P_2$  nodes but six nodes which move inside  $T(X) = G + .99(X - G)$  where  $G$  is the barycenter.

- [P2h] quadratic homogeneous continuous (without P1).
- [P3] piecewise  $P_3$  continuous finite element (2d) (needs `load "Element_P3"`)

$$\mathbb{P}_h^3 = \{v \in H^1(\Omega) \mid \forall K \in \mathcal{T}_h, v|_K \in P_3\}$$

where  $P_3$  is the set of polynomials of  $\mathbb{R}^2$  of degrees  $\leq 3$ .

- [P3dc] piecewise  $P_3$  discontinuous finite element (2d) (needs `load "Element_P3dc"`)

$$\mathbb{P}_{dc|h}^3 = \{v \in L^2(\Omega) \mid \forall K \in \mathcal{T}_h, v|_K \in P_3\}$$

where  $P_3$  is the set of polynomials of  $\mathbb{R}^2$  of degrees  $\leq 3$ .

- [P4] piecewise  $P_4$  continuous finite element (2d) (needs `load "Element_P4"`)

$$\mathbb{P}_h^4 = \{v \in H^1(\Omega) \mid \forall K \in \mathcal{T}_h, v|_K \in P_4\}$$

where  $P_4$  is the set of polynomials of  $\mathbb{R}^2$  of degrees  $\leq 4$ .

- [P4dc] piecewise  $P_4$  discontinuous finite element (2d) (needs `load "Element_P4dc"`)

$$\mathbb{P}_{dc|h}^4 = \{v \in L^2(\Omega) \mid \forall K \in \mathcal{T}_h, v|_K \in P_3\}$$

where  $P_4$  is the set of polynomials of  $\mathbb{R}^2$  of degrees  $\leq 3$ .

- [P0Edge] piecewise  $P_0$  discontinuous finite element (2d) contained on each edge of the mesh.
- [P1Edge] piecewise  $P_1$  discontinuous finite element (2d) (needs `load "Element_PkEdge"`)  $P_1$  on each edge of the mesh.
- [P2Edge] piecewise  $P_2$  discontinuous finite element (2d) (needs `load "Element_PkEdge"`)  $P_2$  on each edge of the mesh.
- [P3Edge] piecewise  $P_3$  discontinuous finite element (2d) (needs `load "Element_PkEdge"`)  $P_3$  on each edge of the mesh.
- [P4Edge] piecewise  $P_4$  discontinuous finite element (2d) (needs `load "Element_PkEdge"`)  $P_4$  on each edge of the mesh.
- [P5Edge] piecewise  $P_5$  discontinuous finite element (2d) (needs `load "Element_PkEdge"`)  $P_5$  on each edge of the mesh.
- [P2Morley] piecewise  $P_2$  non conform finite element (2d) (needs `load "Morley"`)

$$\mathbb{P}_h^2 = \left\{ v \in L^2(\Omega) \mid \forall K \in \mathcal{T}_h, v|_K \in P_3, \begin{cases} v \text{ continuous at vertices,} \\ \partial_n v \text{ continuous at middle of edge,} \end{cases} \right\}$$

where  $P_2$  is the set of polynomials of  $\mathbb{R}^2$  of degrees  $\leq 2$ .

**Warning:** To build the interplant of a function  $u$  (scalar) for this finite element, we need the function and 2 partial derivatives  $(u, u_x, u_y)$ , creating this vectorial finite element with 3 components  $(u, u_x, u_y)$ .

See our example for solving the BiLaplacien problem:

```

1  load "Morley"
2
3  // Parameters
4  int nn = 10;
5  real h = 0.01;
6
7  real f = 1;
8
9  // Mesh
10 mesh Th = square(nn, nn);
11 Th = adaptmesh(Th, h, IsMetric=1);
12
13 // Fespace
14 fespace Vh(Th, P2Morley); //The Morley finite element space
15 Vh [u, ux, uy], [v, vx, vy];
16
17 // Macro
18 macro bilaplaci(u, v) (dxx(u)*dxx(v) + dyy(u)*dyy(v) + 2.
    ↵*dxy(u)*dxy(v)) //
```

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```

19 // Problem
20 solve bilap ([u, ux, uy], [v, vx, vy])
21   = int2d(Th) (
22     bilaplacien(u, v)
23   )
24   - int2d(Th) (
25     f*v
26   )
27   + on(1, 2, 3, 4, u=0, ux=0, uy=0)
28   ;
29
30 // Plot
31 plot(u, cmm="u");
32

```

- [HCT]  $P_3$   $C^1$  conforms finite element (2d) (needs `load "Element_HCT"`) one 3 sub triangles.

Lets call  $\mathcal{T}_h^\Delta$  the sub mesh of  $\mathcal{T}_h$  where all triangles are split in 3 at the barycenter.

$$\mathbb{P}_h^{HCT} = \left\{ v \in C^1(\Omega) \mid \forall K \in \mathcal{T}_h^\Delta, v|_K \in P_3 \right\}$$

where  $P_3$  is the set of polynomials of  $\mathbb{R}^2$  of degrees  $\leq 3$ .

The degrees of freedom are the values of the normal derivative at the mid-point of each edge [BERNADOU1980].

**Warning:** To build the interplant of a function  $u$  (scalar) for this finite element, we need the function and 2 partial derivatives  $(u, u_x, u_y)$ , creating this vectorial finite element with 3 components  $(u, u_x, u_y)$  like in previous finite element.

- [P2BR] (needs `load "BernadiRaugel"`) the Bernadi Raugel Finite Element is a Vectorial element (2d) with 2 components, see [BERNARDI1985].

It is a 2D coupled Finite Element, where the Polynomial space is  $P_1^2$  with 3 normal bubble edge functions ( $P_2$ ). There are 9 degrees of freedom:

- 2 components at each of the 3 vertices and
- the 3 flux on the 3 edges.

- [RT0, RT03d] Raviart-Thomas finite element of degree 0.

#### The 2D Case:

$$RT0_h = \left\{ \mathbf{v} \in H(\text{div}) \mid \forall K \in \mathcal{T}_h, \mathbf{v}|_K(x, y) = \begin{pmatrix} \alpha_K^1 \\ \alpha_K^2 \\ \alpha_K^3 \end{pmatrix} + \beta_K \begin{pmatrix} x \\ y \end{pmatrix} \right\} \quad (3.7)$$

#### The 3D Case:

$$RT0_h = \left\{ \mathbf{v} \in H(\text{div}) \mid \forall K \in \mathcal{T}_h, \mathbf{v}|_K(x, y, z) = \begin{pmatrix} \alpha_K^1 \\ \alpha_K^2 \\ \alpha_K^3 \end{pmatrix} + \beta_K \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\} \quad (3.8)$$

where by writing  $\text{div } \mathbf{w} = \sum_{i=1}^d \partial w_i / \partial x_i$  with  $\mathbf{w} = (w_i)_{i=1}^d$ :

$$H(\text{div}) = \left\{ \mathbf{w} \in L^2(\Omega)^d \mid \text{div } \mathbf{w} \in L^2(\Omega) \right\}$$

and where  $\alpha_K^1, \alpha_K^2, \alpha_K^3, \beta_K$  are real numbers.

- [RT0Ortho] Raviart-Thomas Orthogonal, or Nedelec finite element type I of degree 0 in dimension 2

$$RT0Orthoh = \left\{ \mathbf{v} \in H(\text{curl}) \mid \forall K \in \mathcal{T}_h, \mathbf{v}|_K(x, y) = \begin{vmatrix} \alpha_K^1 & +\beta_K |^{-y} \\ \alpha_K^2 & x \end{vmatrix} \right\} \quad (3.9)$$

- [Edge03d] 3d Nedelec finite element or Edge Element of degree 0.

$$Edge0_h = \left\{ \mathbf{v} \in H(\text{Curl}) \mid \forall K \in \mathcal{T}_h, \mathbf{v}|_K(x, y, z) = \begin{vmatrix} \alpha_K^1 & +\beta_K^1 \\ \alpha_K^2 & \beta_K^2 \\ \alpha_K^3 & \beta_K^3 \end{vmatrix} \times \begin{vmatrix} x \\ y \\ z \end{vmatrix} \right\} : label : eq : Edge03d$$

where by writing  $\text{curl}\mathbf{w} = \begin{vmatrix} \partial w_2 / \partial x_3 - \partial w_3 / \partial x_2 \\ \partial w_3 / \partial x_1 - \partial w_1 / \partial x_3 \\ \partial w_1 / \partial x_2 - \partial w_2 / \partial x_1 \end{vmatrix}$  with  $\mathbf{w} = (w_i)_{i=1}^d$ :

$$H(\text{curl}) = \left\{ \mathbf{w} \in L^2(\Omega)^d \mid \text{curl } \mathbf{w} \in L^2(\Omega)^d \right\}$$

and  $\alpha_K^1, \alpha_K^2, \alpha_K^3, \beta_K^1, \beta_K^2, \beta_K^3$  are real numbers.

- [Edge13d] (needs load "Element\_Mixte3d") 3d Nedelec finite element or Edge Element of degree 1.
- [Edge23d] (needs load "Element\_Mixte3d") 3d Nedelec finite element or Edge Element of degree 2.
- [P1nc] piecewise linear element continuous at the mid-point of the edge only in 2D (Crouzeix-Raviart Finite Element 2D).
- [P2pnc] piecewise quadratic plus a P3 bubble element with the continuity of the 2 moments on each edge (needs load "Element\_P2pnc")
- [RT1] (needs load "Element\_Mixte")

$$RT1_h = \left\{ \mathbf{v} \in H(\text{div}) \mid \forall K \in \mathcal{T}_h, \alpha_K^1, \alpha_K^2, \beta_K \in P_1^2, P_0, \mathbf{v}|_K(x, y) = \begin{vmatrix} \alpha_K^1 & +\beta_K |^x \\ \alpha_K^2 & y \end{vmatrix} \right\} \quad (3.10)$$

- [RT1Ortho] (needs load "Element\_Mixte")

$$RT1_h = \left\{ \mathbf{v} \in H(\text{curl}) \mid \forall K \in \mathcal{T}_h, \alpha_K^1, \alpha_K^2, \beta_K \in P_1^2, P_0, \mathbf{v}|_K(x, y) = \begin{vmatrix} \alpha_K^1 & +\beta_K |^{-y} \\ \alpha_K^2 & x \end{vmatrix} \right\} \quad (3.11)$$

- [RT2] (needs load "Element\_Mixte")

$$RT2_h = \left\{ \mathbf{v} \in H(\text{div}) \mid \forall K \in \mathcal{T}_h, \alpha_K^1, \alpha_K^2, \beta_K \in P_2^2, P_1, \mathbf{v}|_K(x, y) = \begin{vmatrix} \alpha_K^1 & +\beta_K |^x \\ \alpha_K^2 & y \end{vmatrix} \right\} \quad (3.12)$$

- [RT2Ortho] (needs load "Element\_Mixte")

$$RT2_h = \left\{ \mathbf{v} \in H(\text{curl}) \mid \forall K \in \mathcal{T}_h, \alpha_K^1, \alpha_K^2, \beta_K \in P_2^2, P_1, \mathbf{v}|_K(x, y) = \begin{vmatrix} \alpha_K^1 & +\beta_K |^{-y} \\ \alpha_K^2 & x \end{vmatrix} \right\} \quad (3.13)$$

- [BDM1] (needs load "Element\_Mixte") the Brezzi-Douglas-Marini finite element:

$$BDM1_h = \{ \mathbf{v} \in H(\text{div}) \mid \forall K \in \mathcal{T}_h, \mathbf{v}|_K \in P_1^2 \} \quad (3.14)$$

- [BDM1Ortho] (needs `load "Element_Mixte"`) the Brezzi-Douglas-Marini Orthogonal also call Ned-  
elec of type II , finite element

$$BDM1Ortho_h = \{ \mathbf{v} \in H(\text{curl}) \mid \forall K \in \mathcal{T}_h, \mathbf{v}|_K \in P_1^2 \} \quad (3.15)$$

- [FEQF] (needs `load "Element_QF"`) the finite element to store functions at default quadrature points (so the quadrature is qf5pT in 2D and is qfV5 in 3d).

For over quadrature you have the following corresponding finite element's quadrature formula.

- FEQF1  $\mapsto$  qf1pT,
- FEQF2  $\mapsto$  qf2pT,
- FEQF5  $\mapsto$  qf5pT,
- FEQF7  $\mapsto$  qf7pT,
- FEQF9  $\mapsto$  qf9pT,
- FEQF13d  $\mapsto$  qfV1,
- FEQF23d  $\mapsto$  qfV2,
- FEQF53d  $\mapsto$  qfV5

You can use this element to optimize the storage and reuse of functions with a long formula inside an integral for non linear processes.

### 3.3.1 Use of freefem fespace in 2D

With the 2D finite element spaces

$$X_h = \{ v \in H^1([0, 1]^2) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_1 \}$$

$$X_{ph} = \{ v \in X_h \mid v(0, \cdot) = v(1, \cdot), v(0, 0) = v(1, 0) \}$$

$$M_h = \{ v \in H^1([0, 1]^2) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_2 \}$$

$$R_h = \{ \mathbf{v} \in H^1([0, 1]^2)^2 \mid \forall K \in \mathcal{T}_h \quad \mathbf{v}|_K(x, y) = \begin{pmatrix} \alpha_K \\ \beta_K \end{pmatrix} + \gamma_K \begin{pmatrix} x \\ y \end{pmatrix} \}$$

when  $\mathcal{T}_h$  is a mesh  $10 \times 10$  of the unit square  $[0, 1]^2$ , we only write in FreeFEM:

```

1 mesh Th = square(10, 10);
2 fespace Xh(Th, P1); //scalar FE
3 fespace Xph(Th,P1,
4   periodic=[[2, y], [4, y], [1, x], [3, x]]); //bi-periodic FE
5 fespace Mh(Th, P2); //scalar FE
6 fespace Rh(Th, RT0); //vectorial FE

```

where  $X_h$ ,  $M_h$ ,  $R_h$  expresses finite element spaces (called FE spaces)  $X_h$ ,  $M_h$ ,  $R_h$ , respectively.

To use FE-functions  $u_h, v_h \in X_h, p_h, q_h \in M_h$  and  $U_h, V_h \in R_h$ , we write:

```

1 Xh uh, vh;
2 Xph uph, vph;
3 Mh ph, qh;
4 Rh [Uxh, Uyh], [Vxh, Vyh];
5 Xh[int] Uh(10); //array of 10 functions in Xh
6 Rh[int] [Wxh, Wyh](10); //array of 10 functions in Rh
7 Wxh[5](0.5,0.5); //the 6th function at point (0.5, 0.5)
8 Wxh[5][]; //the array of the degree of freedom of the 6th function

```

The functions  $U_h, V_h$  have two components so we have

$$U_h = \begin{vmatrix} U_{xh} \\ U_{yh} \end{vmatrix} \quad \text{and} \quad V_h = \begin{vmatrix} V_{xh} \\ V_{yh} \end{vmatrix}$$

### 3.3.2 Use of fespace in 3D

With the 3D finite element spaces

$$\begin{aligned} X_h &= \{v \in H^1([0, 1]^3) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_1\} \\ X_{ph} &= \{v \in X_h \mid v(\cdot^0) = v(\cdot^1), v(\cdot^0) = v(\cdot^1)\} \\ M_h &= \{v \in H^1([0, 1]^2) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_2\} \\ R_h &= \{\mathbf{v} \in H^1([0, 1]^2)^2 \mid \forall K \in \mathcal{T}_h \quad \mathbf{v}|_K(x, y) = \begin{pmatrix} \alpha_K \\ \beta_K \end{pmatrix} + \gamma_K \begin{pmatrix} x \\ y \end{pmatrix}\} \end{aligned}$$

when  $\mathcal{T}_h$  is a mesh  $10 \times 10 \times 10$  of the unit cubic  $[0, 1]^3$ , we write in FreeFEM:

```

1 mesh3 Th = buildlayers(square(10, 10), 10, zbound=[0,1]);
2 //label: 0 up, 1 down, 2 front, 3 left, 4 back, 5 right
3 fespace Xh(Th, P1); //scalar FE
4 fespace Xph(Th, P1,
5   periodic=[[0, x, y], [1, x, y],
6   [2, x, z], [4, x, z],
7   [3, y, z], [5, y, z]]); //three-periodic FE
8 fespace Mh(Th, P2); //scalar FE
9 fespace Rh(Th, RT03d); //vectorial FE

```

where  $X_h, M_h, Rh$  expresses finite element spaces (called FE spaces)  $X_h, M_h, R_h$ , respectively.

To define and use FE-functions  $u_h, v_h \in X_h, p_h, q_h \in M_h$  and  $U_h, V_h \in R_h$ , we write:

```

1 Xh uh, vh;
2 Xph uph, vph;
3 Mh ph, qh;
4 Rh [Uxh, Uyh, Uydh], [Vxh, Vyh, Vydh];
5 Xh[int] Uh(10); //array of 10 functions in Xh
6 Rh[int] [Wxh, Wyh, Wzh](10); //array of 10 functions in Rh
7 Wxh[5](0.5,0.5,0.5); //the 6th function at point (0.5, 0.5, 0.5)
8 Wxh[5][]; //the array of the degree of freedom of the 6th function

```

The functions  $U_h, V_h$  have three components, so we have:

$$U_h = \begin{vmatrix} U_{xh} \\ U_{yh} \\ U_{zh} \end{vmatrix} \quad \text{and} \quad V_h = \begin{vmatrix} V_{xh} \\ V_{yh} \\ V_{zh} \end{vmatrix}$$

---

**Note:** One challenge of the periodic boundary condition is that the mesh must have equivalent faces.

The buildlayers mesh generator splits each quadrilateral face with the diagonal passing through the vertex with maximum number, so to be sure to have the same mesh one both face periodic the 2D numbering in corresponding edges must be compatible (for example the same variation).

By Default, the numbering of square vertex is correct.

To change the mesh numbering you can use the `change` function like:

```

1 {
2     int[int] old2new(0:Th.nv-1); //array set on 0, 1, ..., nv-1
3     fespace Vh2(Th, P1);
4     Vh2 sorder = x+y; //choose an order increasing on 4 square borders with x or y
5     sort(sorder[], old2new); //build the inverse permutation
6     int[int] new2old = old2new^-1; //inverse the permutation
7     Th = change(Th, renumv=new2old);
8 }
```

The full example is in [examples](#).

### 3.3.3 Lagrangian Finite Elements

#### P0-element

For each triangle ( $d=2$ ) or tetrahedron ( $d=3$ )  $T_k$ , the basis function  $\phi_k$  in  $Vh(Th, P0)$  is given by:

$$\phi_k(\mathbf{x}) = \begin{cases} 1 & \text{if } (\mathbf{x}) \in T_k \\ 0 & \text{if } (\mathbf{x}) \notin T_k \end{cases}$$

If we write:

```

1 Vh(Th, P0);
2 Vh fh = f(x,y);
```

then for vertices  $q^{k_i}$ ,  $i = 1, 2, \dots, d + 1$  in Fig. 3.30,  $f_h$  is built as  $f_h = f_h(x, y) = \sum_k f\left(\frac{\sum_i q^{k_i}}{d+1}\right) \phi_k$

See Fig. 3.31b for the projection of  $f(x, y) = \sin(\pi x) \cos(\pi y)$  on  $Vh(Th, P0)$  when the mesh  $Th$  is a  $4 \times 4$ -grid of  $[-1, 1]^2$  as in Fig. 3.31a.

#### P1-element

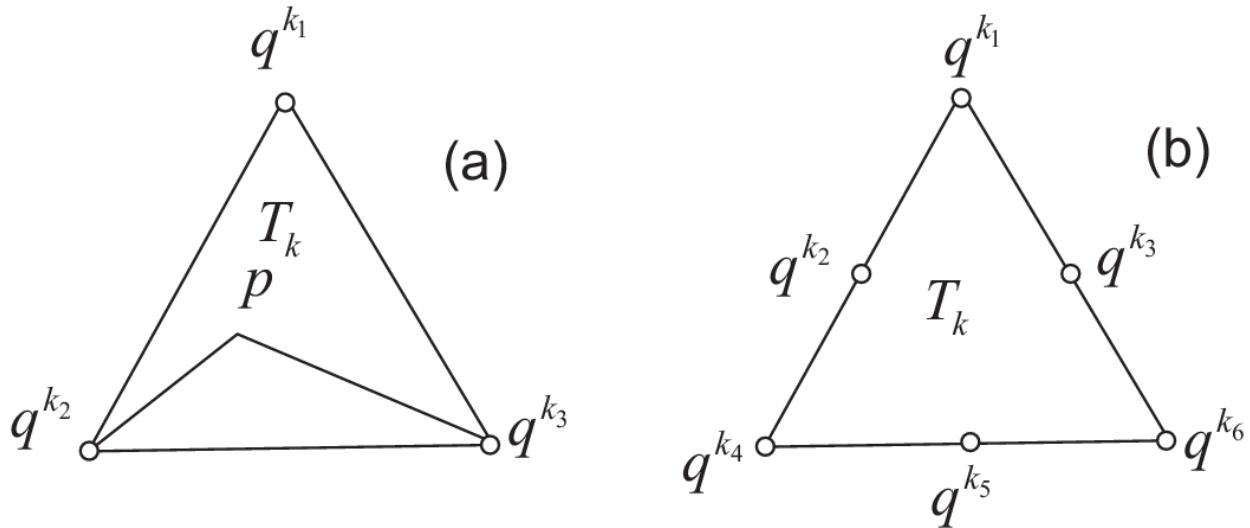
For each vertex  $q^i$ , the basis function  $\phi_i$  in  $Vh(Th, P1)$  is given by:

$$\begin{aligned} \phi_i(x, y) &= a_i^k + b_i^k x + c_i^k y \quad \text{for } (x, y) \in T_k, \\ \phi_i(q^i) &= 1, \quad \phi_i(q^j) = 0 \text{ if } i \neq j \end{aligned}$$

The basis function  $\phi_{k_1}(x, y)$  with the vertex  $q^{k_1}$  in Fig. 3.30 at point  $p = (x, y)$  in triangle  $T_k$  simply coincide with the *barycentric coordinates*  $\lambda_1^k$  (*area coordinates*):

$$\phi_{k_1}(x, y) = \lambda_1^k(x, y) = \frac{\text{area of triangle}(p, q^{k_2}, q^{k_3})}{\text{area of triangle}(q^{k_1}, q^{k_2}, q^{k_3})}$$

If we write:

**Fig. 3.30:**  $P_1$  and  $P_2$  degrees of freedom on triangle  $T_k$ 

```

1 Vh(Th, P1);
2 Vh fh = g(x,y);
```

then:

$$fh = f_h(x, y) = \sum_{i=1}^{n_v} f(q^i) \phi_i(x, y)$$

See Fig. 3.32a for the projection of  $f(x, y) = \sin(\pi x) \cos(\pi y)$  into  $Vh(Th, P1)$ .

### P2-element

For each vertex or mid-point  $q^i$ . The basis function  $\phi_i$  in  $Vh(Th, P2)$  is given by:

$$\begin{aligned}\phi_i(x, y) &= a_i^k + b_i^k x + c_i^k y + d_i^k x^2 + e_i^k xy + f_i^k y^2 \text{ for } (x, y) \in T_k, \\ \phi_i(q^i) &= 1, \quad \phi_i(q^j) = 0 \text{ if } i \neq j\end{aligned}$$

The basis function  $\phi_{k_1}(x, y)$  with the vertex  $q^{k_1}$  in Fig. 3.30 is defined by the *barycentric coordinates*:

$$\phi_{k_1}(x, y) = \lambda_1^k(x, y)(2\lambda_1^k(x, y) - 1)$$

and for the mid-point  $q^{k_2}$ :

$$\phi_{k_2}(x, y) = 4\lambda_1^k(x, y)\lambda_4^k(x, y)$$

If we write:

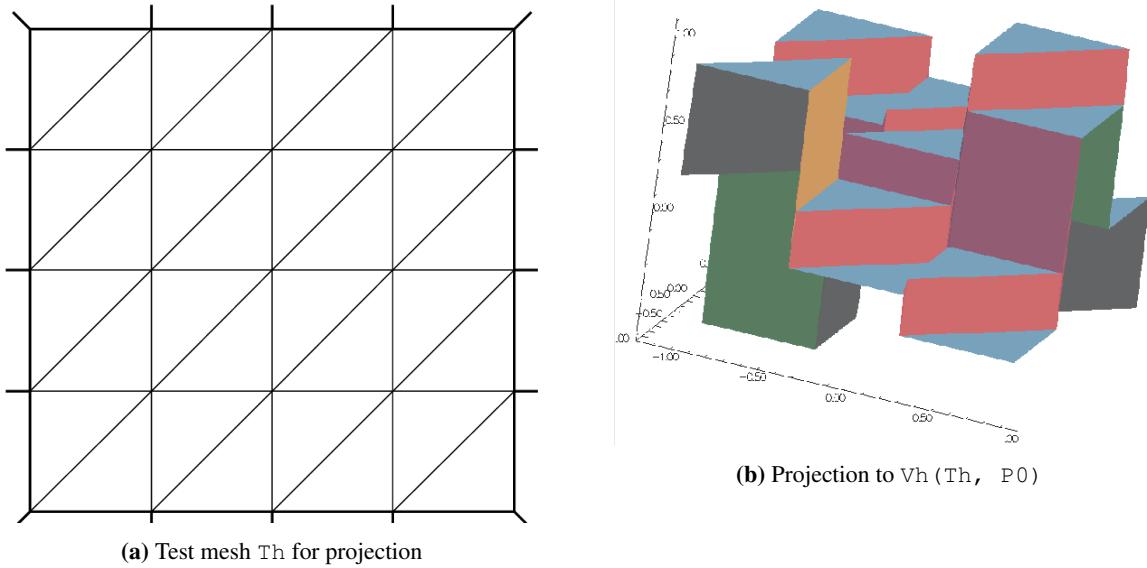
```

1 Vh(Th, P2);
2 Vh fh = f(x,y);
```

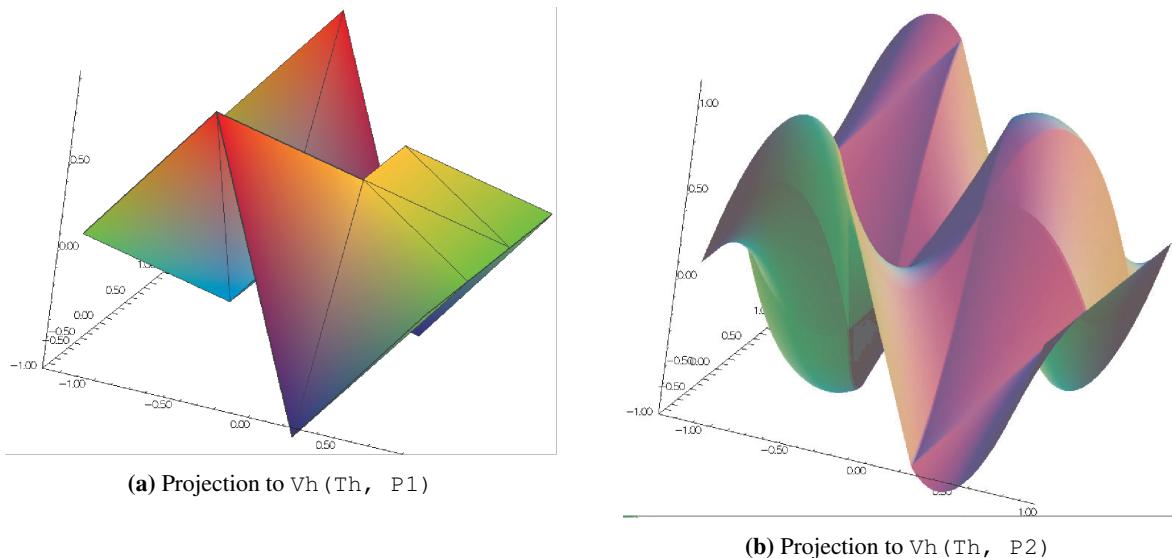
then:

$$fh = f_h(x, y) = \sum_{i=1}^M f(q^i) \phi_i(x, y) \quad (\text{summation over all vertex or mid-point})$$

See *Projection to Vh(Th, P2)* for the projection of  $f(x, y) = \sin(\pi x) \cos(\pi y)$  into  $Vh(Th, P2)$ .



**Fig. 3.31:** Finite element  $\text{P}0$



**Fig. 3.32:** Finite elements  $\text{P}1, \text{P}2$

### 3.3.4 P1 Nonconforming Element

Refer to [THOMASSET2012] for details; briefly, we now consider non-continuous approximations so we will lose the property:

$$w_h \in V_h \subset H^1(\Omega)$$

If we write:

```

1 Vh (Th, P1nc) ;
2 Vh fh = f (x.y) ;

```

then:

$$fh = f_h(x, y) = \sum_{i=1}^{n_v} f(m^i) \phi_i(x, y) \quad (\text{summation over all midpoint})$$

Here the basis function  $\phi_i$  associated with the mid-point  $m^i = (q^{k_i} + q^{k_{i+1}})/2$  where  $q^{k_i}$  is the  $i$ -th point in  $T_k$ , and we assume that  $j+1=0$  if  $j=3$ :

$$\begin{aligned} \phi_i(x, y) &= a_i^k + b_i^k x + c_i^k y \text{ for } (x, y) \in T_k, \\ \phi_i(m^i) &= 1, \quad \phi_i(m^j) = 0 \text{ if } i \neq j \end{aligned}$$

Strictly speaking  $\partial\phi_i/\partial x, \partial\phi_i/\partial y$  contain Dirac distribution  $\rho\delta_{\partial T_k}$ .

The numerical calculations will automatically *ignore* them. In [THOMASSET2012], there is a proof of the estimation

$$\left( \sum_{k=1}^{n_v} \int_{T_k} |\nabla w - \nabla w_h|^2 \text{d}x \text{d}y \right)^{1/2} = O(h)$$

The basis functions  $\phi_k$  have the following properties.

1. For the bilinear form  $a$  defined in Fig. 3.33a satisfy:

$$\begin{aligned} a(\phi_i, \phi_i) &> 0, & a(\phi_i, \phi_j) &\leq 0 \quad \text{if } i \neq j \\ \sum_{k=1}^{n_v} a(\phi_i, \phi_k) &\geq 0 \end{aligned}$$

2.  $f \geq 0 \Rightarrow u_h \geq 0$
3. If  $i \neq j$ , the basis function  $\phi_i$  and  $\phi_j$  are  $L^2$ -orthogonal:

$$\int_{\Omega} \phi_i \phi_j \text{d}x \text{d}y = 0 \quad \text{if } i \neq j$$

which is false for  $P_1$ -element.

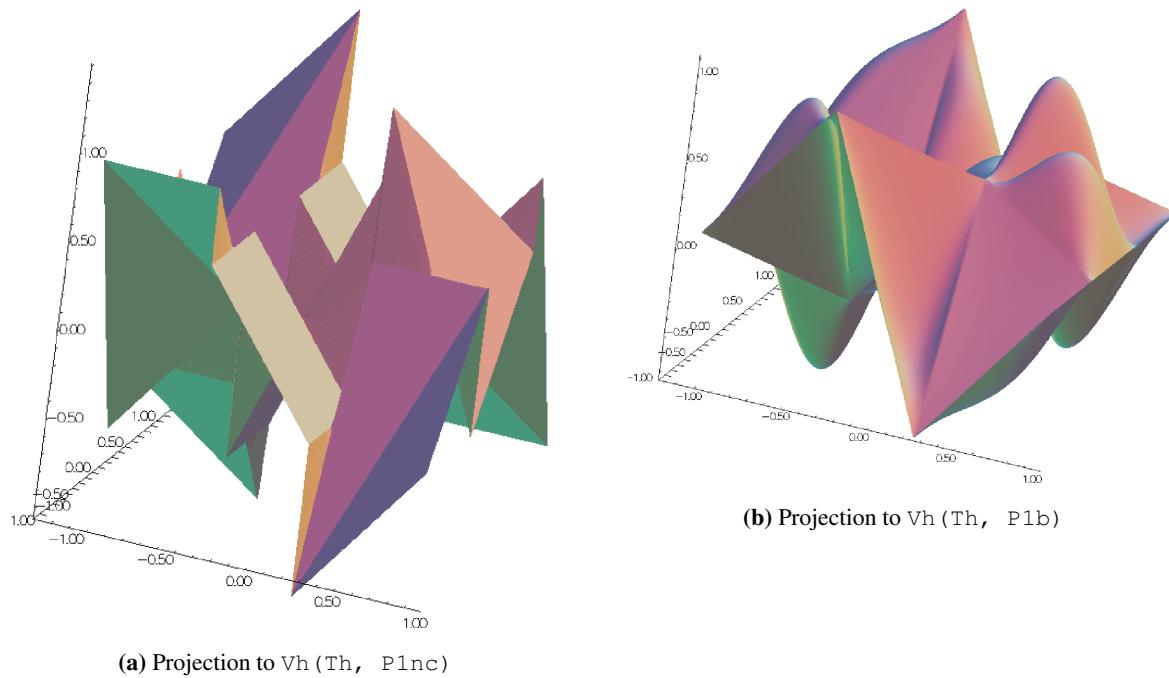
See Fig. 3.33a for the projection of  $f(x, y) = \sin(\pi x) \cos(\pi y)$  into  $Vh (Th, P1nc)$ .

### 3.3.5 Other FE-space

For each triangle  $T_k \in \mathcal{T}_h$ , let  $\lambda_{k_1}(x, y), \lambda_{k_2}(x, y), \lambda_{k_3}(x, y)$  be the area coordinate of the triangle (see Fig. 3.30), and put:

$$\beta_k(x, y) = 27\lambda_{k_1}(x, y)\lambda_{k_2}(x, y)\lambda_{k_3}(x, y)$$

called *bubble* function on  $T_k$ . The bubble function has the feature: 1.  $\beta_k(x, y) = 0$  if  $(x, y) \in \partial T_k$ .

**Fig. 3.33:** Finite elements  $P_{1nc}$ ,  $P_{1b}$ 

2.  $\beta_k(q^{k_b}) = 1$  where  $q^{k_b}$  is the barycenter  $\frac{q^{k_1} + q^{k_2} + q^{k_3}}{3}$ .

If we write:

```
1 Vh (Th, P1b);
2 Vh fh = f (x.y);
```

then:

$$f_h(x, y) = \sum_{i=1}^{n_v} f(q^i) \phi_i(x, y) + \sum_{k=1}^{n_t} f(q^{k_b}) \beta_k(x, y)$$

See Fig. 3.33b for the projection of  $f(x, y) = \sin(\pi x) \cos(\pi y)$  into  $V_h$  ( $\text{Th}$ ,  $P_{1b}$ ).

### 3.3.6 Vector Valued FE-function

Functions from  $\mathbb{R}^2$  to  $\mathbb{R}^N$  with  $N = 1$  are called scalar functions and called *vector valued* when  $N > 1$ . When  $N = 2$

```
1 fespace Vh (Th, [P0, P1]);
```

makes the space

$$V_h = \{\mathbf{w} = (w_1, w_2) \mid w_1 \in V_h(\mathcal{T}_h, P_0), w_2 \in V_h(\mathcal{T}_h, P_1)\}$$

#### Raviart-Thomas Element

In the Raviart-Thomas finite element  $RT0_h$ , the degrees of freedom are the fluxes across edges  $e$  of the mesh, where the flux of the function  $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is  $\int_e \mathbf{f} \cdot \mathbf{n}_e$ ,  $\mathbf{n}_e$  is the unit normal of edge  $e$ .

This implies an orientation of all the edges of the mesh, for example we can use the global numbering of the edge vertices and we just go from small to large numbers.

To compute the flux, we use a quadrature with one Gauss point, the mid-point of the edge.

Consider a triangle  $T_k$  with three vertices  $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ .

Lets denote the vertices numbers by  $i_a, i_b, i_c$ , and define the three edge vectors  $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$  by  $sgn(i_b - i_c)(\mathbf{b} - \mathbf{c})$ ,  $sgn(i_c - i_a)(\mathbf{c} - \mathbf{a})$ ,  $sgn(i_a - i_b)(\mathbf{a} - \mathbf{b})$ .

We get three basis functions:

$$\phi_1^k = \frac{sgn(i_b - i_c)}{2|T_k|}(\mathbf{x} - \mathbf{a}), \quad \phi_2^k = \frac{sgn(i_c - i_a)}{2|T_k|}(\mathbf{x} - \mathbf{b}), \quad \phi_3^k = \frac{sgn(i_a - i_b)}{2|T_k|}(\mathbf{x} - \mathbf{c}),$$

where  $|T_k|$  is the area of the triangle  $T_k$ . If we write:

```
1 Vh(Th, RT0);
2 Vh [f1h, f2h] = [f1(x, y), f2(x, y)];
```

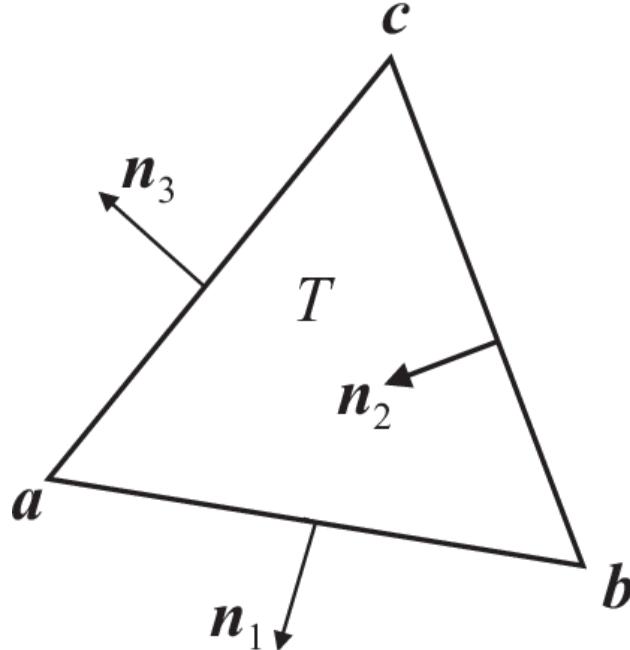
then:

$$\mathbf{f}_h = \mathbf{f}_h(x, y) = \sum_{k=1}^{n_t} \sum_{l=1}^6 n_{i_l j_l} |\mathbf{e}^{i_l}| f_{j_l}(m^{i_l}) \phi_{i_l j_l}$$

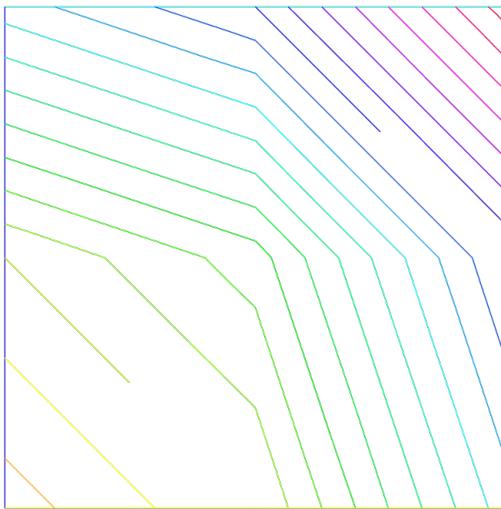
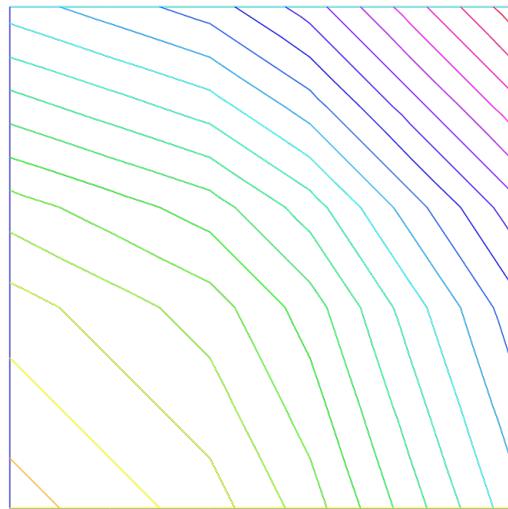
where  $n_{i_l j_l}$  is the  $j_l$ -th component of the normal vector  $\mathbf{n}_{i_l}$ ,

$$\{m_1, m_2, m_3\} = \left\{ \frac{\mathbf{b} + \mathbf{c}}{2}, \frac{\mathbf{a} + \mathbf{c}}{2}, \frac{\mathbf{b} + \mathbf{a}}{2} \right\}$$

and  $i_l = \{1, 1, 2, 2, 3, 3\}$ ,  $j_l = \{1, 2, 1, 2, 1, 2\}$  with the order of  $l$ .



**Fig. 3.34:** Normal vectors of each edge

(a) vh Iso on mesh  $2 \times 2$ (b) vh Iso on mesh  $5 \times 5$ 

```

1 // Mesh
2 mesh Th = square(2, 2);
3
4 // Fespace
5 fespace Xh(Th, P1);
6 Xh uh = x2 + y2, vh;
7
8 fespace Vh(Th, RT0);
9 Vh [Uxh, Uyh] = [sin(x), cos(y)]; //vectorial FE function
10
11 // Change the mesh
12 Th = square(5,5);
13
14 //Xh is unchanged
15 //Uxh = x; //error: impossible to set only 1 component
16 //of a vector FE function
17 vh = Uxh; //ok
18 //and now vh use the 5x5 mesh
19 //but the fespace of vh is always the 2x2 mesh
20
21 // Plot
22 plot(uh);
23 uh = uh; //do a interpolation of uh (old) of 5x5 mesh
24 //to get the new uh on 10x10 mesh
25 plot(uh);
26
27 vh([x-1/2, y]) = x2 + y2; //interpolate vh = ((x-1/2)2 + y2)

```

To get the value at a point  $x = 1, y = 2$  of the FE function uh, or  $[Uxh, Uyh]$ , one writes:

```

1 real value;
2 value = uh(2,4); //get value = uh(2, 4)
3 value = Uxh(2, 4); //get value = Uxh(2, 4)
4 //OR
5 x = 1; y = 2;

```

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```

6 value = uh; //get value = uh(1, 2)
7 value = Uxh; //get value = Uxh(1, 2)
8 value = Uyh; //get value = Uyh(1, 2)

```

To get the value of the array associated to the FE function uh, one writes

```

1 real value = uh[] [0]; //get the value of degree of freedom 0
2 real maxdf = uh[] .max; //maximum value of degree of freedom
3 int size = uh.n; //the number of degree of freedom
4 real[int] array(uh.n) = uh[]; //copy the array of the function uh

```

**Warning:** For a non-scalar finite element function [Uxh, Uyh] the two arrays Uxh [] and Uyh [] are the same array, because the degree of freedom can touch more than one component.

### 3.3.7 A Fast Finite Element Interpolator

In practice, one may discretize the variational equations by the Finite Element method. Then there will be one mesh for  $\Omega_1$  and another one for  $\Omega_2$ . The computation of integrals of products of functions defined on different meshes is difficult.

Quadrature formula and interpolations from one mesh to another at quadrature points are needed. We present below the interpolation operator which we have used and which is new, to the best of our knowledge.

Let  $\mathcal{T}_h^0 = \cup_k T_k^0$ ,  $\mathcal{T}_h^1 = \cup_k T_k^1$  be two triangulations of a domain  $\Omega$ . Let:

$$V(T_h^i) = \{C^0(\Omega_h^i) : f|_{T_k^i} \in P_0\}, \quad i = 0, 1$$

be the spaces of continuous piecewise affine functions on each triangulation.

Let  $f \in V(\mathcal{T}_h^0)$ . The problem is to find  $g \in V(\mathcal{T}_h^1)$  such that:

$$g(q) = f(q) \quad \forall q \text{ vertex of } \mathcal{T}_h^1$$

Although this is a seemingly simple problem, it is difficult to find an efficient algorithm in practice.

We propose an algorithm which is of complexity  $N^1 \log N^0$ , where  $N^i$  is the number of vertices of  $\mathcal{T}_h^i$ , and which is very fast for most practical 2D applications.

#### Algorithm

The method has 5 steps.

First a quadtree is built containing all the vertices of the mesh  $\mathcal{T}_h^0$  such that in each terminal cell there are at least one, and at most 4, vertices of  $\mathcal{T}_h^0$ .

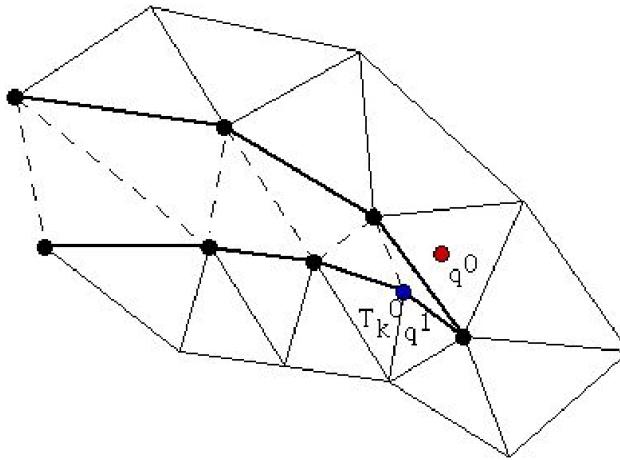
For each  $q^1$ , vertex of  $\mathcal{T}_h^1$  do:

1. Find the terminal cell of the quadtree containing  $q^1$ .
2. Find the nearest vertex  $q_j^0$  to  $q^1$  in that cell.
3. Choose one triangle  $T_k^0 \in \mathcal{T}_h^0$  which has  $q_j^0$  for vertex.
4. Compute the barycentric coordinates  $\{\lambda_j\}_{j=1,2,3}$  of  $q^1$  in  $T_k^0$ .
  - if all barycentric coordinates are positive, go to Step 5

- otherwise, if one barycentric coordinate  $\lambda_i$  is negative, replace  $T_k^0$  by the adjacent triangle opposite  $q_i^0$  and go to Step 4.
- otherwise, if two barycentric coordinates are negative, take one of the two randomly and replace  $T_k^0$  by the adjacent triangle as above.

5. Calculate  $g(q^1)$  on  $T_k^0$  by linear interpolation of  $f$ :

$$g(q^1) = \sum_{j=1,2,3} \lambda_j f(q_j^0)$$



**Fig. 3.36:** To interpolate a function at  $q^0$ , the knowledge of the triangle which contains  $q^0$  is needed. The algorithm may start at  $q^1 \in T_k^0$  and stall on the boundary (thick line) because the line  $q^0q^1$  is not inside  $\Omega$ . But if the holes are triangulated too (dotted line) then the problem does not arise.

Two problems need to be solved:

- What if  $q^1$  is not in  $\Omega_h^0$ ? Then Step 5 will stop with a boundary triangle.

So we add a step which tests the distance of  $q^1$  with the two adjacent boundary edges and selects the nearest, and so on till the distance grows.

- What if  $\Omega_h^0$  is not convex and the marching process of Step 4 locks on a boundary? By construction Delaunay-Voronoi's mesh generators always triangulate the convex hull of the vertices of the domain.

Therefore, we make sure that this information is not lost when  $\mathcal{T}_h^0, \mathcal{T}_h^1$  are constructed and we keep the triangles which are outside the domain on a special list.

That way, in step 5 we can use that list to step over holes if needed.

---

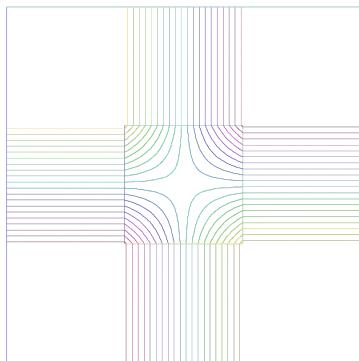
**Note:** Sometimes, in rare cases, the interpolation process misses some points, we can change the search algorithm through a global variable `searchMethod`

```

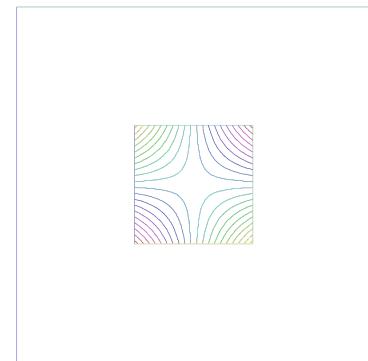
1 searchMethod = 0; // default value for fast search algorithm
2 searchMethod = 1; // safe search algorithm, uses brute force in case of missing point
3 // (warning: can be very expensive in cases where a lot of points are outside of the
4 // domain)
4 searchMethod = 2; // always uses brute force. It is very computationally expensive.

```

---



(a) Extension of a continuous FE-function



(b) Extension of discontinuous FE-function

**Fig. 3.37:** Extension of FE-function

---

**Note:** Step 3 requires an array of pointers such that each vertex points to one triangle of the triangulation.

---

**Note:** The operator `=` is the interpolation operator of **FreeFEM**, the continuous finite functions are extended by continuity to the outside of the domain.

Try the following example :

```

1 // Mesh
2 mesh Ths = square(10, 10);
3 mesh Thg = square(30, 30, [x*3-1, y*3-1]);
4 plot(Ths, Thg, wait=true);
5
6 // Fespace
7 fespace Ch(Ths, P2);
8 Ch us = (x-0.5)*(y-0.5);
9
10 fespace Dh(Ths, P2dc);
11 Dh vs = (x-0.5)*(y-0.5);
12
13 fespace Fh(Thg, P2dc);
14 Fh ug=us, vg=vs;
15
16 // Plot
17 plot(us, ug, wait=true);
18 plot(vs, vg, wait=true);

```

---

### 3.3.8 Keywords: Problem and Solve

For **FreeFEM**, a problem must be given in variational form, so we need a bilinear form  $a(u, v)$ , a linear form  $\ell(f, v)$ , and possibly a boundary condition form must be added.

```

1 problem P (u, v)
2   = a(u, v) - l(f, v)

```

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```

3   + (boundary condition)
4   ;

```

**Note:** When you want to formulate the problem and solve it in the same time, you can use the keyword `solve`.

## Weak Form and Boundary Condition

To present the principles of Variational Formulations, also called weak form, for the Partial Differential Equations, let's take a model problem: a Poisson equation with Dirichlet and Robin Boundary condition.

The problem: Find  $u$  a real function defined on a domain  $\Omega$  of  $\mathbb{R}^d$  ( $d = 2, 3$ ) such that:

$$\begin{aligned} -\nabla \cdot (\kappa \nabla u) &= f && \text{in } \Omega \\ au + \kappa \frac{\partial u}{\partial n} &= b && \text{on } \Gamma_r \\ u &= g && \text{on } \Gamma_d \end{aligned}$$

where:

- if  $d = 2$  then  $\nabla \cdot (\kappa \nabla u) = \partial_x(\kappa \partial_x u) + \partial_y(\kappa \partial_y u)$  with  $\partial_x u = \frac{\partial u}{\partial x}$  and  $\partial_y u = \frac{\partial u}{\partial y}$
- if  $d = 3$  then  $\nabla \cdot (\kappa \nabla u) = \partial_x(\kappa \partial_x u) + \partial_y(\kappa \partial_y u) + \partial_z(\kappa \partial_z u)$  with  $\partial_x u = \frac{\partial u}{\partial x}$ ,  $\partial_y u = \frac{\partial u}{\partial y}$  and  $\partial_z u = \frac{\partial u}{\partial z}$
- The border  $\Gamma = \partial\Omega$  is split in  $\Gamma_d$  and  $\Gamma_n$  such that  $\Gamma_d \cap \Gamma_n = \emptyset$  and  $\Gamma_d \cup \Gamma_n = \partial\Omega$ ,
- $\kappa$  is a given positive function, such that  $\exists \kappa_0 \in \mathbb{R}$ ,  $0 < \kappa_0 \leq \kappa$ .
- $a$  a given non negative function,
- $b$  a given function.

**Note:** This is the well known Neumann boundary condition if  $a = 0$ , and if  $\Gamma_d$  is empty.

In this case the function appears in the problem just by its derivatives, so it is defined only up to a constant (if  $u$  is a solution then  $u + c$  is also a solution).

Let  $v$ , a regular test function, null on  $\Gamma_d$ , by integration by parts we get:

$$-\int_{\Omega} \nabla \cdot (\kappa \nabla u) v \, d\omega = \int_{\Omega} \kappa \nabla v \cdot \nabla u \, d\omega - \int_{\Gamma} v \kappa \frac{\partial u}{\partial n} \, d\gamma, = \int_{\Omega} fv \, d\omega$$

where if  $d = 2$  the  $\nabla v \cdot \nabla u = (\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y})$ ,

where if  $d = 3$  the  $\nabla v \cdot \nabla u = (\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial z})$ ,

and where  $\mathbf{n}$  is the unitary outer-pointing normal of the  $\Gamma$ .

Now we note that  $\kappa \frac{\partial u}{\partial n} = -au + b$  on  $\Gamma_r$  and  $v = 0$  on  $\Gamma_d$  and  $\Gamma = \Gamma_d \cup \Gamma_n$  thus:

$$-\int_{\Gamma} v \kappa \frac{\partial u}{\partial n} = \int_{\Gamma_r} auv - \int_{\Gamma_r} bv$$

The problem becomes:

Find  $u \in V_g = \{w \in H^1(\Omega) / w = g \text{ on } \Gamma_d\}$  such that:

$$\int_{\Omega} \kappa \nabla v \cdot \nabla u \, d\omega + \int_{\Gamma_r} auv \, d\gamma = \int_{\Omega} fv \, d\omega + \int_{\Gamma_r} bv \, d\gamma, \quad \forall v \in V_0 \quad (3.16)$$

where  $V_0 = \{v \in H^1(\Omega) / v = 0 \text{ on } \Gamma_d\}$

Except in the case of Neumann conditions everywhere, the problem (3.16) is well posed when  $\kappa \geq \kappa_0 > 0$ .

---

**Note:** If we have only the Neumann boundary condition, linear algebra tells us that the right hand side must be orthogonal to the kernel of the operator for the solution to exist.

One way of writing the compatibility condition is:

$$\int_{\Omega} f d\omega + \int_{\Gamma} b d\gamma = 0$$

and a way to fix the constant is to solve for  $u \in H^1(\Omega)$  such that:

$$\int_{\Omega} (\varepsilon uv + \kappa \nabla v \cdot \nabla u) d\omega = \int_{\Omega} fv d\omega + \int_{\Gamma_r} bv d\gamma, \quad \forall v \in H^1(\Omega)$$

where  $\varepsilon$  is a small parameter ( $\sim \kappa 10^{-10} |\Omega|^{\frac{2}{d}}$ ).

Remark that if the solution is of order  $\frac{1}{\varepsilon}$  then the compatibility condition is unsatisfied, otherwise we get the solution such that  $\int_{\Omega} u = 0$ , you can also add a Lagrange multiplier to solve the real mathematical problem like in the [Lagrange multipliers example](#).

---

In FreeFEM, the bidimensional problem (3.16) becomes:

```

1 problem Pw (u, v)
2   = int2d(Th) ( //int_{Omega} kappa nabla v . nabla u
3     kappa*(dx(u)*dx(v) + dy(u)*dy(v))
4   )
5   + int1d(Th, gn) ( //int_{Gamma_r} a u v
6     a * u*v
7   )
8   - int2d(Th) ( //int_{Omega} f v
9     f*v
10  )
11  - int1d(Th, gn) ( //int_{Gamma_r} b v
12    b * v
13  )
14  + on(gd, u=g) //u = g on Gamma_d
15  ;

```

where Th is a mesh of the bi-dimensional domain  $\Omega$ , and gd and gn are respectively the boundary labels of boundary  $\Gamma_d$  and  $\Gamma_n$ .

And the three dimensional problem (3.16) becomes

```

1 macro Grad(u) [dx(u), dy(u), dz(u)] //
2 problem Pw (u, v)
3   = int3d(Th) ( //int_{Omega} kappa nabla v . nabla u
4     kappa*(Grad(u)'*Grad(v))
5   )
6   + int2d(Th, gn) ( //int_{Gamma_r} a u v
7     a * u*v
8   )
9   - int3d(Th) ( //int_{Omega} f v
10    f*v
11  )
12  - int2d(Th, gn) ( //int_{Gamma_r} b v

```

(continues on next page)

(continued from previous page)

```

13     b * v
14   )
15   + on(gd, u=g) //u = g on Gamma_d
16 ;

```

where Th is a mesh of the three dimensional domain  $\Omega$ , and gd and gn are respectively the boundary labels of boundary  $\Gamma_d$  and  $\Gamma_n$ .

### 3.3.9 Parameters affecting solve and problem

The parameters are FE functions real or complex, the number  $n$  of parameters is even ( $n = 2 * k$ ), the  $k$  first function parameters are unknown, and the  $k$  last are test functions.

---

**Note:** If the functions are a part of vectorial FE then you must give all the functions of the vectorial FE in the same order (see [Poisson problem with mixed finite element](#) for example).

---

**Note:** Don't mix complex and real parameters FE function.

---

**Warning: Bug:**

The mixing of multiple fespace with different periodic boundary conditions are not implemented.

So all the finite element spaces used for tests or unknown functions in a problem, must have the same type of periodic boundary conditions or no periodic boundary conditions.

No clean message is given and the result is unpredictable.

The parameters are:

- **solver=** LU, CG, Crout, Cholesky, GMRES, sparsesolver, UMFPACK ...

The default solver is sparsesolver (it is equal to UMFPACK if no other sparse solver is defined) or is set to LU if no direct sparse solver is available.

The storage mode of the matrix of the underlying linear system depends on the type of solver chosen; for LU the matrix is sky-line non symmetric, for Crout the matrix is sky-line symmetric, for Cholesky the matrix is sky-line symmetric positive definite, for CG the matrix is sparse symmetric positive, and for GMRES, sparsesolver or UMFPACK the matrix is just sparse.

- **eps=** a real expression.

$\varepsilon$  sets the stopping test for the iterative methods like CG.

Note that if  $\varepsilon$  is negative then the stopping test is:

$$\|Ax - b\| < |\varepsilon|$$

if it is positive, then the stopping test is:

$$\|Ax - b\| < \frac{|\varepsilon|}{\|Ax_0 - b\|}$$

- **init=** boolean expression, if it is false or 0 the matrix is reconstructed.

Note that if the mesh changes the matrix is reconstructed too.

- **precon=** name of a function (for example `P`) to set the preconditioner.

The prototype for the function `P` must be:

```
1 func real[int] P(real[int] & xx);
```

- **tgv=** Huge value ( $10^{30}$ ) used to implement Dirichlet boundary conditions.
- **tolpivot=** sets the tolerance of the pivot in UMFPACK ( $10^{-1}$ ) and, LU, Crout, Cholesky factorisation ( $10^{-20}$ ).
- **tolpivot sym=** sets the tolerance of the pivot sym in UMFPACK
- **strategy=** sets the integer UMFPACK strategy (0 by default).

### 3.3.10 Problem definition

Below `v` is the unknown function and `w` is the test function.

After the “=” sign, one may find sums of:

- Identifier(s); this is the name given earlier to the variational form(s) (type `varf`) for possible reuse.

Remark, that the name in the `varf` of the unknown test function is forgotten, we use the order in the argument list to recall names as in a C++ function,

- The terms of the bilinear form itself: if `K` is a given function,
- Bilinear part for 3D meshes `Th`

$$\begin{aligned}
 - \text{int3d}(\text{Th}) (\text{K} * \text{v} * \text{w}) &= \sum_{T \in \text{Th}} \int_T K v w \\
 - \text{int3d}(\text{Th}, 1) (\text{K} * \text{v} * \text{w}) &= \sum_{T \in \text{Th}, T \subset \Omega_1} \int_T K v w \\
 - \text{int3d}(\text{Th}, \text{levelset}=\text{phi}) (\text{K} * \text{v} * \text{w}) &= \sum_{T \in \text{Th}} \int_{T, \phi < 0} K v w \\
 - \text{int3d}(\text{Th}, 1, \text{levelset}=\text{phi}) (\text{K} * \text{v} * \text{w}) &= \sum_{T \in \text{Th}, T \subset \Omega_l} \int_{T, \phi < 0} K v w \\
 - \text{int2d}(\text{Th}, 2, 5) (\text{K} * \text{v} * \text{w}) &= \sum_{T \in \text{Th}} \int_{(\partial T \cup \Gamma) \cap (\Gamma_2 \cup \Gamma_5)} K v w \\
 - \text{int2d}(\text{Th}, 1) (\text{K} * \text{v} * \text{w}) &= \sum_{T \in \text{Th}, T \subset \Omega_1} \int_T K v w \\
 - \text{int2d}(\text{Th}, 2, 5) (\text{K} * \text{v} * \text{w}) &= \sum_{T \in \text{Th}} \int_{(\partial T \cup \Gamma) \cap (\Gamma_2 \cup \Gamma_5)} K v w \\
 - \text{int2d}(\text{Th}, \text{levelset}=\text{phi}) (\text{K} * \text{v} * \text{w}) &= \sum_{T \in \text{Th}} \int_{T, \phi=0} K v w \\
 - \text{int2d}(\text{Th}, 1, \text{levelset}=\text{phi}) (\text{K} * \text{v} * \text{w}) &= \sum_{T \in \text{Th}, T \subset \Omega_l} \int_{T, \phi=0} K v w \\
 - \text{intallfaces}(\text{Th}) (\text{K} * \text{v} * \text{w}) &= \sum_{T \in \text{Th}} \int_{\partial T} K v w
 \end{aligned}$$

- $\text{intallfaces}(\text{Th}, 1) (\mathbf{K} \star \mathbf{v} \star \mathbf{w}) = \sum_{T \in \text{Th}, T \subset \Omega_1} \int_{\partial T} K v w$

- They contribute to the sparse matrix of type `matrix` which, whether declared explicitly or not, is constructed by **FreeFEM**.

- Bilinear part for 2D meshes `Th`

- $\text{int2d}(\text{Th}) (\mathbf{K} \star \mathbf{v} \star \mathbf{w}) = \sum_{T \in \text{Th}} \int_T K v w$

- $\text{int2d}(\text{Th}, 1) (\mathbf{K} \star \mathbf{v} \star \mathbf{w}) = \sum_{T \in \text{Th}, T \subset \Omega_1} \int_T K v w$

- $\text{int2d}(\text{Th}, \text{levelset}=\text{phi}) (\mathbf{K} \star \mathbf{v} \star \mathbf{w}) = \sum_{T \in \text{Th}} \int_{T, \phi < 0} K v w$

- $\text{int2d}(\text{Th}, 1, \text{levelset}=\text{phi}) (\mathbf{K} \star \mathbf{v} \star \mathbf{w}) = \sum_{T \in \text{Th}, T \subset \Omega_l} \int_{T, \phi < 0} K v w$

- $\text{int1d}(\text{Th}, 2, 5) (\mathbf{K} \star \mathbf{v} \star \mathbf{w}) = \sum_{T \in \text{Th}} \int_{(\partial T \cup \Gamma) \cap (\Gamma_2 \cup \Gamma_5)} K v w$

- $\text{int1d}(\text{Th}, 1) (\mathbf{K} \star \mathbf{v} \star \mathbf{w}) = \sum_{T \in \text{Th}, T \subset \Omega_1} \int_T K v w$

- $\text{int1d}(\text{Th}, 2, 5) (\mathbf{K} \star \mathbf{v} \star \mathbf{w}) = \sum_{T \in \text{Th}} \int_{(\partial T \cup \Gamma) \cap (\Gamma_2 \cup \Gamma_5)} K v w$

- $\text{int1d}(\text{Th}, \text{levelset}=\text{phi}) (\mathbf{K} \star \mathbf{v} \star \mathbf{w}) = \sum_{T \in \text{Th}} \int_{T, \phi=0} K v w$

- $\text{int1d}(\text{Th}, 1, \text{levelset}=\text{phi}) (\mathbf{K} \star \mathbf{v} \star \mathbf{w}) = \sum_{T \in \text{Th}, T \subset \Omega_l} \int_{T, \phi=0} K v w$

- $\text{intalledges}(\text{Th}) (\mathbf{K} \star \mathbf{v} \star \mathbf{w}) = \sum_{T \in \text{Th}} \int_{\partial T} K v w$

- $\text{intalledges}(\text{Th}, 1) (\mathbf{K} \star \mathbf{v} \star \mathbf{w}) = \sum_{T \in \text{Th}, T \subset \Omega_1} \int_{\partial T} K v w$

- They contribute to the sparse matrix of type `matrix` which, whether declared explicitly or not, is constructed by **FreeFEM**.

- The right hand-side of the Partial Differential Equation in 3D, the terms of the linear form: for given functions  $K, f$ :

- $\text{int3d}(\text{Th}) (\mathbf{K} \star \mathbf{w}) = \sum_{T \in \text{Th}} \int_T K w$

- $\text{int3d}(\text{Th}, 1) (\mathbf{K} \star \mathbf{w}) = \sum_{T \in \text{Th}, T \in \Omega_l} \int_T K w$

- $\text{int3d}(\text{Th}, \text{levelset}=\text{phi}) (\mathbf{K} \star \mathbf{w}) = \sum_{T \in \text{Th}} \int_{T, \phi < 0} K w$

- $\text{int3d}(\text{Th}, 1, \text{levelset}=\text{phi}) (\mathbf{K} \star \mathbf{w}) = \sum_{T \in \text{Th}, T \subset \Omega_l} \int_{T, \phi < 0} K w$

- `int2d(Th, 2, 5) (K*w)` =  $\sum_{T \in \text{Th}} \int_{(\partial T \cup \Gamma) \cap (\Gamma_2 \cup \Gamma_5)} K w$
- `int2d(Th, levelset=phi) (K*w)` =  $\sum_{T \in \text{Th}} \int_{T, \phi=0} K w$
- `int2d(Th, 1, levelset=phi) (K*w)` =  $\sum_{T \in \text{Th}, T \subset \Omega_l} \int_{T, \phi=0} K w$
- `intallfaces(Th) (f*w)` =  $\sum_{T \in \text{Th}} \int_{\partial T} f w$
- A vector of type `real[int]`

- The right hand-side of the Partial Differential Equation in 2D, the terms of the linear form: for given functions  $K, f$ :

- `int2d(Th) (K*w)` =  $\sum_{T \in \text{Th}} \int_T K w$
- `int2d(Th, 1) (K*w)` =  $\sum_{T \in \text{Th}, T \subset \Omega_l} \int_T K w$
- `int2d(Th, levelset=phi) (K*w)` =  $\sum_{T \in \text{Th}} \int_{T, \phi<0} K w$
- `int2d(Th, 1, levelset=phi) (K*w)` =  $\sum_{T \in \text{Th}, T \subset \Omega_l} \int_{T, \phi<0} K w$
- `int1d(Th, 2, 5) (K*w)` =  $\sum_{T \in \text{Th}} \int_{(\partial T \cup \Gamma) \cap (\Gamma_2 \cup \Gamma_5)} K w$
- `int1d(Th, levelset=phi) (K*w)` =  $\sum_{T \in \text{Th}} \int_{T, \phi=0} K w$
- `int1d(Th, 1, levelset=phi) (K*w)` =  $\sum_{T \in \text{Th}, T \subset \Omega_l} \int_{T, \phi=0} K w$
- `intalledges(Th) (f*w)` =  $\sum_{T \in \text{Th}} \int_{\partial T} f w$
- a vector of type `real[int]`

- The boundary condition terms:

- An “on” scalar form (for Dirichlet): `on (1, u=g)`

Used for all degrees of freedom  $i$  of the boundary referred by “1”, the diagonal term of the matrix  $a_{ii} = tgv$  with the *terrible giant value* `tgv` ( $= 10^{30}$  by default), and the right hand side  $b[i] = ”(\Pi_h g)[i]” \times tgv$ , where the  $(\Pi_h g)[i]$  is the boundary node value given by the interpolation of  $g$ .

---

**Note:** if  $tgv < 0$  then we put to 0 all term of the line  $i$  in the matrix, except diagonal term  $a_{ii} = 1$ , and  $b[i] = ”(\Pi_h g)[i]”$ .

---

- An “on” vectorial form (for Dirichlet): `on (1, u1=g1, u2=g2)`

If you have vectorial finite element like RT0, the 2 components are coupled, and so you have :  
 $b[i] = "(\Pi_h(g1, g2))[i]" \times tgv$ , where  $\Pi_h$  is the vectorial finite element interpolant.

- A linear form on  $\Gamma$  (for Neumann in 2d) -int1d(Th) (f\*w) or -int1d(Th, 3) (f\*w)
- A bilinear form on  $\Gamma$  or  $\Gamma_2$  (for Robin in 2d) int1d(Th) (K\*v\*w) or int1d(Th, 2) (K\*v\*w)
- A linear form on  $\Gamma$  (for Neumann in 3d) -int2d(Th) (f\*w) or -int2d(Th, 3) (f\*w)
- A bilinear form on  $\Gamma$  or  $\Gamma_2$  (for Robin in 3d) int2d(Th) (K\*v\*w) or int2d(Th, 2) (K\*v\*w)

---

**Note:**

- If needed, the different kind of terms in the sum can appear more than once.
  - The integral mesh and the mesh associated to test functions or unknown functions can be different in the case of linear form.
  - N.x, N.y and N.z are the normal's components.
- 

**Warning:** It is not possible to write in the same integral the linear part and the bilinear part such as in  
`int1d(Th) (K*v*w - f*w)`.

### 3.3.11 Numerical Integration

Let  $D$  be a  $N$ -dimensional bounded domain.

For an arbitrary polynomial  $f$  of degree  $r$ , if we can find particular (quadrature) points  $\xi_j$ ,  $j = 1, \dots, J$  in  $D$  and (quadrature) constants  $\omega_j$  such that

$$\int_D f(\mathbf{x}) = \sum_{\ell=1}^L c_\ell f(\xi_\ell)$$

then we have an error estimate (see [CROUZEIX1984]), and then there exists a constant  $C > 0$  such that

$$\left| \int_D f(\mathbf{x}) - \sum_{\ell=1}^L \omega_\ell f(\xi_\ell) \right| \leq C |D| h^{r+1}$$

for any function  $r + 1$  times continuously differentiable  $f$  in  $D$ , where  $h$  is the diameter of  $D$  and  $|D|$  its measure (a point in the segment  $[q^i q^j]$  is given as

$$\{(x, y) | x = (1-t)q_x^i + tq_x^j, y = (1-t)q_y^i + tq_y^j, 0 \leq t \leq 1\}$$

For a domain  $\Omega_h = \sum_{k=1}^{n_t} T_k$ ,  $\mathcal{T}_h = \{T_k\}$ , we can calculate the integral over  $\Gamma_h = \partial\Omega_h$  by:

`int1d(Th) (f) = int1d(Th, qfe=*) (f) = int1d(Th, qforder=*) (f)`

where \* stands for the name of the quadrature formula or the precision (order) of the Gauss formula.

Quadrature formula on an edge					
L	qfe	qforder	Point in $[q^i, q^j]$	$\omega_\ell$	Exact on $P_k$ , $k =$
1	qf1pE	2	1/2	$\ q^i q^j\ $	1
2	qf2pE	3	$(1 \pm \sqrt{1/3})/2$	$\ q^i q^j\ /2$	3
3	qf3pE	6	$(1 \pm \sqrt{3/5})/2$ 1/2	$(5/18)\ q^i q^j\ $ $(8/18)\ q^i q^j\ $	5
4	qf4pE	8	$(1 \pm \frac{525+70\sqrt{30}}{35})/2$ $(1 \pm \frac{525-70\sqrt{30}}{35})/2$	$\frac{18-\sqrt{30}}{72}\ q^i q^j\ $ $\frac{18+\sqrt{30}}{72}\ q^i q^j\ $	7
5	qf5pE	10	$(1 \pm \frac{245+14\sqrt{70}}{21})/2$ 1/2 $(1 \pm \frac{245-14\sqrt{70}}{21})/2$	$\frac{322-13\sqrt{70}}{1800}\ q^i q^j\ $ $\frac{64}{225}\ q^i q^j\ $ $\frac{322+13\sqrt{70}}{1800}\ q^i q^j\ $	9
2	qf1pElump2	0 1		$\ q^i q^j\ /2$ $\ q^i q^j\ /2$	1

where  $|q^i q^j|$  is the length of segment  $\overline{q^i q^j}$ .

For a part  $\Gamma_1$  of  $\Gamma_h$  with the label “1”, we can calculate the integral over  $\Gamma_1$  by:

$\int_{\Gamma_1} f(x, y) ds = \text{int1d}(\text{Th}, 1) (f) = \text{int1d}(\text{Th}, 1, \text{qfe}=qf2pE) (f)$

The integrals over  $\Gamma_1$ ,  $\Gamma_3$  are given by:

$\int_{\Gamma_1 \cup \Gamma_3} f(x, y) ds$

For each triangle  $T_k = [q^{k_1} q^{k_2} q^{k_3}]$ , the point  $P(x, y)$  in  $T_k$  is expressed by the *area coordinate* as  $P(\xi, \eta)$ :

$$|T_k| = \frac{1}{2} \begin{vmatrix} 1 & q_x^{k_1} & q_y^{k_1} \\ 1 & q_x^{k_2} & q_y^{k_2} \\ 1 & q_x^{k_3} & q_y^{k_3} \end{vmatrix} \quad D_1 = \begin{vmatrix} 1 & x & y \\ 1 & q_x^{k_2} & q_y^{k_2} \\ 1 & q_x^{k_3} & q_y^{k_3} \end{vmatrix} \quad D_2 = \begin{vmatrix} 1 & q_x^{k_1} & q_y^{k_1} \\ 1 & x & y \\ 1 & q_x^{k_3} & q_y^{k_3} \end{vmatrix} \quad D_3 = \begin{vmatrix} 1 & q_x^{k_1} & q_y^{k_1} \\ 1 & q_x^{k_2} & q_y^{k_2} \\ 1 & x & y \end{vmatrix}$$

$$\xi = \frac{1}{2} D_1 / |T_k| \quad \eta = \frac{1}{2} D_2 / |T_k| \quad \text{then } 1 - \xi - \eta = \frac{1}{2} D_3 / |T_k|$$

For a two dimensional domain or a border of three dimensional domain  $\Omega_h = \sum_{k=1}^{n_t} T_k$ ,  $\mathcal{T}_h = \{T_k\}$ , we can calculate the integral over  $\Omega_h$  by:

$\int_{\Omega_h} f(x, y) ds = \text{int2d}(\text{Th}) (f) = \text{int2d}(\text{Th}, \text{qft}=\star) (f) = \text{int2d}(\text{Th}, \text{qforder}=\star) (f)$

where  $\star$  stands for the name of quadrature formula or the order of the Gauss formula.

Quadrature formula on a triangle					
$L$	qft	qforder	Point in $T_k$	$\omega_\ell$	Exact on $P_k$ , $k =$
1	qf1p	2	$(\frac{1}{3}, \frac{1}{3})$	$ T_k $	1
3	qf2p	3	$(\frac{1}{2}, \frac{1}{2})$ $(\frac{1}{2}, 0)$ $(0, \frac{1}{2})$	$ T_k /3$ $ T_k /3$ $ T_k /3$	2
7	qf5p	6	$(\frac{1}{3}, \frac{1}{3})$ $(\frac{6-\sqrt{15}}{21}, \frac{6-\sqrt{15}}{21})$ $(\frac{6-\sqrt{15}}{21}, \frac{9+2\sqrt{15}}{21})$ $(\frac{9+2\sqrt{15}}{21}, \frac{6-\sqrt{15}}{21})$ $(\frac{6+\sqrt{15}}{21}, \frac{6+\sqrt{15}}{21})$ $(\frac{6+\sqrt{15}}{21}, \frac{9-2\sqrt{15}}{21})$ $(\frac{9-2\sqrt{15}}{21}, \frac{6+\sqrt{15}}{21})$	$0.225 T_k $ $\frac{(155-\sqrt{15}) T_k }{1200}$ $\frac{(155-\sqrt{15}) T_k }{1200}$ $\frac{(155-\sqrt{15}) T_k }{1200}$ $\frac{(155+\sqrt{15}) T_k }{1200}$ $\frac{(155+\sqrt{15}) T_k }{1200}$ $\frac{(155+\sqrt{15}) T_k }{1200}$	5
3	qf1p	lump	$(0, 0)$ $(1, 0)$ $(0, 1)$	$ T_k /3$ $ T_k /3$ $ T_k /3$	1
9	qf2p	4P1	$(\frac{1}{4}, \frac{3}{4})$ $(\frac{3}{4}, \frac{1}{4})$ $(0, \frac{1}{4})$ $(0, \frac{3}{4})$ $(\frac{1}{4}, 0)$ $(\frac{3}{4}, 0)$ $(\frac{1}{4}, \frac{1}{4})$ $(\frac{1}{4}, \frac{1}{2})$ $(\frac{1}{2}, \frac{1}{4})$	$ T_k /12$ $ T_k /12$ $ T_k /12$ $ T_k /12$ $ T_k /12$ $ T_k /12$ $ T_k /6$ $ T_k /6$ $ T_k /6$	1
15	qf7p	8	See [TAYLOR2005] for detail		7
21	qf9p	10	See [TAYLOR2005] for detail		9

For a three dimensional domain  $\Omega_h = \sum_{k=1}^{n_t} T_k$ ,  $\mathcal{T}_h = \{T_k\}$ , we can calculate the integral over  $\Omega_h$  by:

$\int_{\Omega_h} f(x, y) = \text{int3d}(\text{Th}) (f) = \text{int3d}(\text{Th}, \text{qfV}=\star) (f) = \text{int3D}(\text{Th}, \text{qforder}=\star) (f)$

where  $\star$  stands for the name of quadrature formula or the order of the Gauss formula.

Quadrature formula on a tetrahedron					
$L$	qfV	qforder	Point in $T_k \in \mathbb{R}^3$	$\omega_\ell$	Exact on $P_k$ , $k =$
1	qfV1	2	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$ T_k $	1
4	qfV2	3	$G4(0.58\dots, 0.13\dots, 0.13\dots)$	$ T_k /4$	2
14	qfV5	6	$G4(0.72\dots, 0.092\dots, 0.092\dots)$ $G4(0.067\dots, 0.31\dots, 0.31\dots)$ $G6(0.45\dots, 0.045\dots, 0.45\dots)$	$0.073\dots  T_k $ $0.11\dots  T_k $ $0.042\dots  T_k $	5
4	qfV11	lump	$G4(1, 0, 0)$	$ T_k /4$	1

Where  $G4(a, b, b)$  such that  $a + 3b = 1$  is the set of the four point in barycentric coordinate:

$$\{(a, b, b, b), (b, a, b, b), (b, b, a, b), (b, b, b, a)\}$$

and where  $G6(a, b, b)$  such that  $2a + 2b = 1$  is the set of the six points in barycentric coordinate:

$$\{(a, a, b, b), (a, b, a, b), (a, b, b, a), (b, b, a, a), (b, a, b, a), (b, a, a, b)\}$$

---

**Note:** These tetrahedral quadrature formulae come from <http://nines.cs.kuleuven.be/research/ecf/mtables.html>

---

**Note:** By default, we use the formula which is exact for polynomials of degree 5 on triangles or edges (in bold in three tables).

---

It is possible to add an own quadrature formulae with using plugin `qf11to25` on segment, triangle or Tetrahedron.

The quadrature formulae in  $D$  dimension is a bidimensional array of size  $N_q \times (D+1)$  such that the  $D+1$  value of on row  $i = 0, \dots, N_p - 1$  are  $w^i, \hat{x}_1^i, \dots, \hat{x}_D^i$  where  $w^i$  is the weight of the quadrature point, and  $1 - \sum_{k=1}^D \hat{x}_k^i, \hat{x}_1^i, \dots, \hat{x}_D^i$  is the barycentric coordinate the quadrature point.

```

1 load "qf11to25"
2
3 // Quadrature on segment
4 real[int, int] qq1 = [
5     [0.5, 0],
6     [0.5, 1]
7 ];
8
9 QF1 qf1(1, qq1); //def of quadrature formulae qf1 on segment
10 //remark:
11 //1 is the order of the quadrature exact for polynome of degree < 1
12
13 //Quadrature on triangle
14 real[int, int] qq2 = [
15     [1./3., 0, 0],
16     [1./3., 1, 0],
17     [1./3., 0, 1]
18 ];
19
20 QF2 qf2(1, qq2); //def of quadrature formulae qf2 on triangle
21 //remark:
22 //1 is the order of the quadrature exact for polynome of degree < 1
23 //so must have sum w^i = 1
24
25 // Quadrature on tetrahedron
26 real[int, int] qq3 = [
27     [1./4., 0, 0, 0],
28     [1./4., 1, 0, 0],
29     [1./4., 0, 1, 0],
30     [1./4., 0, 0, 1]
31 ];
32
33 QF3 qf3(1, qq3); //def of quadrature formulae qf3 on get
34 //remark:
35 //1 is the order of the quadrature exact for polynome of degree < 1)
36
37 // Verification in 1d and 2d
38 mesh Th = square(10, 10);
39

```

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```

40 real I1 = int1d(Th, qfe=qf1) (x^2);
41 real I11 = int1d(Th, qfe=qf1pElump) (x^2);
42
43 real I2 = int2d(Th, qft=qf2) (x^2);
44 real I21 = int2d(Th, qft=qf1pTlump) (x^2);
45
46 cout << I1 << " == " << I11 << endl;
47 cout << I2 << " == " << I21 << endl;
48 assert( abs(I1-I11) < 1e-10 );
49 assert( abs(I2-I21) < 1e-10 );

```

The output is

```

1 1.67 == 1.67
2 0.335 == 0.335

```

### 3.3.12 Variational Form, Sparse Matrix, PDE Data Vector

In **FreeFEM** it is possible to define variational forms, and use them to build matrices and vectors, and store them to speed-up the script (4 times faster here).

For example let us solve the *Thermal Conduction problem*.

The variational formulation is in  $L^2(0, T; H^1(\Omega))$ ; we shall seek  $u^n$  satisfying:

$$\forall w \in V_0; \quad \int_{\Omega} \frac{u^n - u^{n-1}}{\delta t} w + \kappa \nabla u^n \nabla w + \int_{\Gamma} \alpha(u^n - u_{ue}) w = 0$$

where  $V_0 = \{w \in H^1(\Omega) / w|_{\Gamma_{24}} = 0\}$ .

So to code the method with the matrices  $A = (A_{ij})$ ,  $M = (M_{ij})$ , and the vectors  $u^n, b^n, b', b'', b_{cl}$  (notation if  $w$  is a vector then  $w_i$  is a component of the vector).

$$u^n = A^{-1}b^n, \quad b' = b_0 + Mu^{n-1}, \quad b'' = \frac{1}{\varepsilon} b_{cl}, \quad b_i^n = \begin{cases} b''_i & \text{if } i \in \Gamma_{24} \\ b'_i & \text{else if } i \notin \Gamma_{24} \end{cases}$$

Where with  $\frac{1}{\varepsilon} = \text{tgv} = 10^{30}$ :

$$\begin{aligned} A_{ij} &= \begin{cases} \frac{1}{\varepsilon} & \text{if } i \in \Gamma_{24}, \text{ and } j = i \\ \int_{\Omega} w_j w_i / dt + k(\nabla w_j \cdot \nabla w_i) + \int_{\Gamma_{13}} \alpha w_j w_i & \text{else if } i \notin \Gamma_{24}, \text{ or } j \neq i \end{cases} \\ M_{ij} &= \begin{cases} \frac{1}{\varepsilon} & \text{if } i \in \Gamma_{24}, \text{ and } j = i \\ \int_{\Omega} w_j w_i / dt & \text{else if } i \notin \Gamma_{24}, \text{ or } j \neq i \end{cases} \\ b_{0,i} &= \int_{\Gamma_{13}} \alpha u_{ue} w_i \\ b_{cl} &= u^0 \text{ the initial data} \end{aligned}$$

```

1 // Parameters
2 func fu0 = 10 + 90*x/6;
3 func k = 1.8*(y<0.5) + 0.2;
4 real ue = 25.;
5 real alpha = 0.25;
6 real T = 5;
7 real dt = 0.1 ;

```

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```

8 // Mesh
9 mesh Th = square(30, 5, [6*x, y]);
10
11 // Fespace
12 fespace Vh(Th, P1);
13 Vh u0 = fu0, u = u0;

```

Create three variational formulation, and build the matrices  $A, M$ .

```

// Problem
varf vthermic (u, v)
= int2d(Th) (
    u*v/dt
    + k*(dx(u)*dx(v) + dy(u)*dy(v))
)
+ int1d(Th, 1, 3) (
    alpha*u*v
)
+ on(2, 4, u=1)
;

varf vthermic0 (u, v)
= int1d(Th, 1, 3) (
    alpha*ue*v
)
;

varf vMass (u, v)
= int2d(Th) (
    u*v/dt
)
+ on(2, 4, u=1)
;

real tgv = 1e30;
matrix A = vthermic(Vh, Vh, tgv=tgv, solver=CG);
matrix M = vMass(Vh, Vh);

```

Now, to build the right hand size we need 4 vectors.

```

1 real[int] b0 = vthermic0(0, Vh); //constant part of the RHS
2 real[int] bcn = vthermic(0, Vh); //tgv on Dirichlet boundary node ( !=0 )
3 //we have for the node i : i in Gamma_24 -> bcn[i] != 0
4 real[int] bcl = tgv*u0[]; //the Dirichlet boundary condition part

```

---

**Note:** The boundary condition is implemented by penalization and vector  $bcn$  contains the contribution of the boundary condition  $u = 1$ , so to change the boundary condition, we have just to multiply the vector  $bcn[]$  by the current value  $f$  of the new boundary condition term by term with the operator  $.*$ .

*Uzawa model* gives a real example of using all this features.

---

And the new version of the algorithm is now:

```

1 // Time loop
2 ofstream ff("thermic.dat");
3 for(real t = 0; t < T; t += dt) {
4     // Update
5     real[int] b = b0; //for the RHS
6     b += M*u[]; //add the time dependent part
7     //lock boundary part:
8     b = bcn ? bcl : b; //do forall i: b[i] = bcn[i] ? bcl[i] : b[i]
9
10    // Solve
11    u[] = A^-1*b;
12
13    // Save
14    ff << t << " " << u(3, 0.5) << endl;
15
16    // Plot
17    plot(u);
18}
19
20 // Display
21 for(int i = 0; i < 20; i++)
22     cout << dy(u)(6.0*i/20.0, 0.9) << endl;
23
24 // Plot
25 plot(u, fill=true, wait=true);

```

**Note:** The functions appearing in the variational form are formal and local to the varf definition, the only important thing is the order in the parameter list, like in:

```

1 varf vb1([u1, u2], q) = int2d(Th)((dy(u1) + dy(u2))*q) + int2d(Th)(1*q);
2 varf vb2([v1, v2], p) = int2d(Th)((dy(v1) + dy(v2))*p) + int2d(Th)(1*p);

```

To build matrix  $A$  from the bilinear part the variational form  $a$  of type varf simply write:

```

1 A = a(Vh, Wh, [...] );
2 // where
3 //Vh is "fespace" for the unknown fields with a correct number of component
4 //Wh is "fespace" for the test fields with a correct number of component

```

Possible named parameters in , [ . . . ] are

- solver= LU, CG, Crout, Cholesky, GMRES, sparsesolver, UMFPACK ...  
The default solver is GMRES.  
The storage mode of the matrix of the underlying linear system depends on the type of solver chosen; for LU the matrix is sky-line non symmetric, for Crout the matrix is sky-line symmetric, for Cholesky the matrix is sky-line symmetric positive definite, for CG the matrix is sparse symmetric positive, and for GMRES, sparsesolver or UMFPACK the matrix is just sparse.
- factorize = If true then do the matrix factorization for LU, Cholesky or Crout, the default value is false.
- eps= A real expression.  
 $\varepsilon$  sets the stopping test for the iterative methods like CG.

Note that if  $\varepsilon$  is negative then the stopping test is:

$$\|Ax - b\| < |\varepsilon|$$

if it is positive then the stopping test is

$$\|Ax - b\| < \frac{|\varepsilon|}{\|Ax_0 - b\|}$$

- precon= Name of a function (for example `P`) to set the preconditioner.

The prototype for the function `P` must be:

```
1 func real[int] P(real[int] & xx) ;
```

- `tgv`= Huge value ( $10^{30}$ ) used to implement Dirichlet boundary conditions.
- `tolpivot`= Set the tolerance of the pivot in UMFPACK ( $10^{-1}$ ) and, LU, Crout, Cholesky factorization ( $10^{-20}$ ).
- `tolpivot sym`= Set the tolerance of the pivot sym in UMFPACK
- `strategy`= Set the integer UMFPACK strategy (0 by default).

---

**Note:** The line of the matrix corresponding to the space `Wh` and the column of the matrix corresponding to the space `Vh`.

---

To build the dual vector `b` (of type `real [int]`) from the linear part of the variational form `a` do simply:

```
1 real b(Vh.ndof);
2 b = a(0, Vh);
```

A first example to compute the area of each triangle  $K$  of mesh  $Th$ , just do:

```
1 fespace Nh(Th, P0); //the space function constant / triangle
2 Nh areaK;
3 varf varea (unused, chiK) = int2d(Th)(chiK);
4 etaK[] = varea(0, Ph);
```

Effectively, the basic functions of space  $Nh$ , are the characteristic function of the element of  $Th$ , and the numbering is the numeration of the element, so by construction:

$$\text{etaK}[i] = \int 1|_{K_i} = \int_{K_i} 1;$$

Now, we can use this to compute error indicators like in example *Adaptation using residual error indicator*.

First to compute a continuous approximation to the function  $h$  “density mesh size” of the mesh  $Th$ .

```
1 fespace Vh(Th, P1);
2 Vh h ;
3 real[int] count(Th.nv);
4 varf vmeshsizen (u, v) = intalledges(Th, qfnbpE=1)(v);
5 varf vedgecount (u, v) = intalledges(Th, qfnbpE=1)(v/lenEdge);
6
7 // Computation of the mesh size
8 count = vedgecount(0, Vh); //number of edge / vertex
9 h[] = vmeshsizen(0, Vh); //sum length edge / vertex
10 h[] = h[]./count; //mean length edge / vertex
```

To compute error indicator for Poisson equation:

$$\eta_K = \int_K h_K^2 |(f + \Delta u_h)|^2 + \int_{\partial K} h_e \left| \left[ \frac{\partial u_h}{\partial n} \right] \right|^2$$

where  $h_K$  is size of the longest edge (`hTriangle`),  $h_e$  is the size of the current edge (`lenEdge`),  $n$  the normal.

```

1 fespace Nh(Th, P0); // the space function constant / triangle
2 Nh etak;
3 varf vetaK (unused, chiK)
4   = intalldges(Th) (
5     chiK*lenEdge*square(jump(N.x*dx(u) + N.y*dy(u)))
6   )
7   + int2d(Th) (
8     chiK*square(hTriangle*(f + dxx(u) + dyy(u)))
9   )
10 ;
11
12 etak[] = vetaK(0, Ph);

```

We add automatic expression optimization by default, if this optimization creates problems, it can be removed with the keyword `optimize` as in the following example:

```

1 varf a (u1, u2)
2   = int2d(Th, optimize=0) (
3     dx(u1)*dx(u2)
4     + dy(u1)*dy(u2)
5   )
6   + on(1, 2, 4, u1=0)
7   + on(3, u1=1)
8 ;

```

or you can also do optimization and remove the check by setting `optimize=2`.

Remark, it is all possible to build interpolation matrix, like in the following example:

```

1 mesh TH = square(3, 4);
2 mesh th = square(2, 3);
3 mesh Th = square(4, 4);
4
5 fespace VH(TH, P1);
6 fespace Vh(th, P1);
7 fespace Wh(Th, P1);
8
9 matrix B = interpolate(VH, Vh); //build interpolation matrix Vh->VH
10 matrix BB = interpolate(Wh, Vh); //build interpolation matrix Wh->Vh

```

and after some operations on sparse matrices are available for example:

```

1 int N = 10;
2 real [int, int] A(N, N); //a full matrix
3 real [int] a(N), b(N);
4 A = 0;
5 for (int i = 0; i < N; i++) {
6   A(i, i) = 1+i;
7   if (i+1 < N) A(i, i+1) = -i;
8   a[i] = i;
9 }

```

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```

10 b = A*b;
11 matrix sparseA = A;
12 cout << sparseA << endl;
13 sparseA = 2*sparseA + sparseA';
14 sparseA = 4*sparseA + sparseA*5;
15 matrix sparseB = sparseA + sparseA + sparseA; ;
16 cout << "sparseB = " << sparseB(0,0) << endl;

```

### 3.3.13 Interpolation matrix

It is also possible to store the matrix of a linear interpolation operator from a finite element space  $V_h$  to another  $W_h$  to interpolate( $W_h, V_h, \dots$ ) a function.

Note that the continuous finite functions are extended by continuity outside of the domain.

The named parameters of function `interpolate` are:

- `inside`= set true to create zero-extension.
- `t`= set true to get the transposed matrix
- `op`= set an integer written below
  - 0 the default value and interpolate of the function
  - 1 interpolate the  $\partial_x$
  - 2 interpolate the  $\partial_y$
  - 3 interpolate the  $\partial_z$
- `U2Vc`= set the which is the component of  $W_h$  come in  $V_h$  in interpolate process in a int array so the size of the array is number of component of  $W_h$ , if the put `-1` then component is set to 0, like in the following example: (by default the component number is unchanged).

```

1 fespace V4h(Th4, [P1, P1, P1, P1]);
2 fespace V3h(Th, [P1, P1, P1]);
3 int[int] u2vc = [1, 3, -1]; // -1 -> put zero on the component
4 matrix IV34 = interpolate(V3h, V4h, inside=0, U2Vc=u2vc); // V3h <- V4h
5 V4h [a1, a2, a3, a4] = [1, 2, 3, 4];
6 V3h [b1, b2, b3] = [10, 20, 30];
7 b1[] = IV34*a1[];

```

So here we have: `freefem b1 == 2, b2 == 4, b3 == 0 ...`

---

**Tip:** Matrix interpolation

```

1 // Mesh
2 mesh Th = square(4, 4);
3 mesh Th4 = square(2, 2, [x*0.5, y*0.5]);
4 plot(Th, Th4, wait=true);
5
6 // Fespace
7 fespace Vh(Th, P1);
8 Vh v, vv;
9 fespace Vh4(Th4, P1);
10 Vh4 v4=x*y;

```

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```

11
12 fespace Wh(Th, P0);
13 fespace Wh4(Th4, P0);
14
15 // Interpolation
16 matrix IV = interpolate(Vh, Vh4); //here the function is exended by continuity
17 cout << "IV Vh<-Vh4 " << IV << endl;
18
19 V=v4;
20 vv[] = IV*v4[]; //here v == vv
21
22 real[int] diff= vv[] - v[];
23 cout << "|| v - vv || = " << diff.linfty << endl;
24 assert( diff.linfty<= 1e-6);
25
26 matrix IV0 = interpolate(Vh, Vh4, inside=1); //here the function is exended by zero
27 cout << "IV Vh<-Vh4 (inside=1) " << IV0 << endl;
28
29 matrix IVt0 = interpolate(Vh, Vh4, inside=1, t=1);
30 cout << "IV Vh<-Vh4^t (inside=1) " << IVt0 << endl;
31
32 matrix IV4t0 = interpolate(Vh4, Vh);
33 cout << "IV Vh4<-Vh^t " << IV4t0 << endl;
34
35 matrix IW4 = interpolate(Wh4, Wh);
36 cout << "IV Wh4<-Wh " << IW4 << endl;
37
38 matrix IW4V = interpolate(Wh4, Vh);
39 cout << "IV Wh4<-Vh " << IW4 << endl;

```

Build interpolation matrix  $A$  at a array of points  $(xx[j], yy[j])$ ,  $i = 0, 2$  here:

$$a_{ij} = dop(w_c^i(xx[j], yy[j]))$$

where  $w_i$  is the basic finite element function,  $c$  the component number,  $dop$  the type of diff operator like in op def.

```

1 real[int] xx = [.3, .4], yy = [.1, .4];
2 int c = 0, dop = 0;
3 matrix Ixx = interpolate(Vh, xx, yy, op=dop, composante=c);
4 cout << Ixx << endl;
5 Vh ww;
6 real[int] dd = [1, 2];
7 ww[] = Ixx*dd;

```

### Tip: Schwarz

The following shows how to implement with an interpolation matrix a domain decomposition algorithm based on Schwarz method with Robin conditions.

Given a non-overlapping partition  $\bar{\Omega} = \bar{\Omega}_0 \cup \bar{\Omega}_1$  with  $\Omega_0 \cap \Omega_1 = \emptyset$ ,  $\Sigma := \bar{\Omega}_0 \cap \bar{\Omega}_1$  the algorithm is:

$$\begin{aligned} -\Delta u_i &= f \text{ in } \Omega_i, \quad i = 0, 1, \\ \frac{\partial(u_1 - u_0)}{\partial n} + \alpha(u_1 - u_0) &= 0 \text{ on } \Sigma. \end{aligned}$$

The same in variational form is:

$$\int_{\Omega_i} \nabla u_i \cdot \nabla v + \int_{\Sigma} \alpha u_i v = \int_{\Omega_i} f v \\ - \int_{\Omega_j} (\nabla u_j \cdot \nabla v - fv) + \int_{\Sigma} \alpha u_j v, \quad \forall v \in H_0^1(\Omega), i, j = [0, 1] \cup [1, 0]$$

To discretized with the  $P^1$  triangular Lagrangian finite element space  $V_h$  simply replace  $H_0^1(\Omega)$  by  $V_h(\Omega_0) \cup V_h(\Omega_1)$ .

Then difficulty is to compute  $\int_{\Omega_j} \nabla u_j \cdot \nabla v$  when  $v$  is a basis function of  $V_h(\Omega_i)$ ,  $i \neq j$ .

It is done as follows (with  $\Gamma = \partial\Omega$ ):

```

1 // Parameters
2 int n = 30;
3 int Gamma = 1;
4 int Sigma = 2;
5
6 func f = 1.;
7 real alpha = 1.;
8
9 int Niter = 50;
10
11 // Mesh
12 mesh[int] Th(2);
13 int[int] reg(2);
14
15 border a0(t=0, 1){x=t; y=0; label=Gamma;};
16 border a1(t=1, 2){x=t; y=0; label=Gamma;};
17 border b1(t=0, 1){x=2; y=t; label=Gamma;};
18 border c1(t=2, 1){x=t; y=1; label=Gamma;};
19 border c0(t=1, 0){x=t; y=1; label=Gamma;};
20 border b0(t=1, 0){x=0; y=t; label=Gamma;};
21 border d(t=0, 1){x=1; y=t; label=Sigma;};
22 plot(a0(n) + a1(n) + b1(n) + c1(n) + c0(n) + b0(n) + d(n));
23 mesh TH = buildmesh(a0(n) + a1(n) + b1(n) + c1(n) + c0(n) + b0(n) + d(n));
24
25 reg(0) = TH(0.5, 0.5).region;
26 reg(1) = TH(1.5, 0.5).region;
27
28 for(int i = 0; i < 2; i++) Th[i] = trunc(TH, region==reg(i));
29
30 // Fespace
31 fespace Vh0(Th[0], P1);
32 Vh0 u0 = 0;
33
34 fespace Vh1(Th[1], P1);
35 Vh1 u1 = 0;
36
37 // Macro
38 macro grad(u) [dx(u), dy(u)] //
39
40 // Problem
41 int i;
42 varf a (u, v)
43   = int2d(Th[i])(
44     grad(u)'*grad(v)
45   )
46   + int1d(Th[i], Sigma)(
47     alpha*u*v
48   )

```

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```

49     + on(Gamma, u=0)
50 ;
51
52 varf b (u, v)
53     = int2d(Th[i])(
54         f*v
55     )
56     + on(Gamma, u=0)
57 ;
58
59 varf duldn (u, v)
60     =-int2d(Th[1])(
61         grad(u1)'*grad(v)
62         - f*v
63     )
64     + int1d(Th[1], Sigma)(
65         alpha*u1*v
66     )
67     +on(Gamma, u=0)
68 ;
69
70 varf du0dn (u, v)
71     =-int2d(Th[0])(
72         grad(u0)'*grad(v)
73         - f*v
74     )
75     + int1d(Th[0], Sigma)(
76         alpha*u0*v
77     )
78     +on(Gamma, u=0)
79 ;
80
81 matrix I01 = interpolate(Vh1, Vh0);
82 matrix I10 = interpolate(Vh0, Vh1);
83
84 matrix[int] A(2);
85 i = 0; A[i] = a(Vh0, Vh0);
86 i = 1; A[i] = a(Vh1, Vh1);
87
88 // Solving loop
89 for(int iter = 0; iter < Niter; iter++){
90     // Solve on Th[0]
91     {
92         i = 0;
93         real[int] b0 = b(0, Vh0);
94         real[int] Duldn = duldn(0, Vh1);
95         real[int] Tduldn(Vh0.ndof); Tduldn = I01'*Duldn;
96         b0 += Tduldn;
97         u0[] = A[0]^_1*b0;
98     }
99     // Solve on Th[1]
100    {
101        i = 1;
102        real[int] b1 = b(0, Vh1);
103        real[int] Du0dn = du0dn(0, Vh0);
104        real[int] Tdu0dn(Vh1.ndof); Tdu0dn = I10'*Du0dn;
105        b1 += Tdu0dn;

```

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```

106     u1[] = A[1]^_1*b1;
107 }
108 plot(u0, u1, cmm="iter="+iter);
109 }
```

### 3.3.14 Finite elements connectivity

Here, we show how to get informations on a finite element space  $W_h(\mathcal{T}_n, *)$ , where “\*” may be P1, P2, P1nc, etc.

- Wh.nt gives the number of element of  $W_h$
- Wh.ndof gives the number of degrees of freedom or unknown
- Wh.ndofK gives the number of degrees of freedom on one element
- Wh(k, i) gives the number of  $i$ th degrees of freedom of element  $k$ .

See the following example:

---

**Tip:** Finite element connectivity

```

1 // Mesh
2 mesh Th = square(5, 5);
3
4 // Fespace
5 fespace Wh(Th, P2);
6
7 cout << "Number of degree of freedom = " << Wh.ndof << endl;
8 cout << "Number of degree of freedom / ELEMENT = " << Wh.ndofK << endl;
9
10 int k = 2, kdf = Wh.ndofK; //element 2
11 cout << "Degree of freedom of element " << k << ":" << endl;
12 for (int i = 0; i < kdf; i++)
13   cout << Wh(k,i) << " ";
14 cout << endl;
```

The output is:

```

1 Number of degree of freedom = 121
2 Number of degree of freedom / ELEMENT = 6
3 Degree of freedom of element 2:
4 78 95 83 87 79 92
```

---

## 3.4 Visualization

Results created by the finite element method can be a huge set of data, so it is very important to render them easy to grasp.

There are two ways of visualization in FreeFEM:

- One, the default view, which supports the drawing of meshes, isovalues of real FE-functions, and of vector fields, all by the command `plot` (see *Plot section* below). For publishing purpose, **FreeFEM** can store these plots as postscript files.
- Another method is to use external tools, for example, `gnuplot` (see *Gnuplot section*, *medit section*, *Paraview section*, *Matlab/Octave section*) using the command `system` to launch them and/or to save the data in text files.

### 3.4.1 Plot

With the command `plot`, meshes, isovalues of scalar functions, and vector fields can be displayed.

The parameters of the `plot` command can be meshes, real FE functions, arrays of 2 real FE functions, arrays of two double arrays, to plot respectively a mesh, a function, a vector field, or a curve defined by the two double arrays.

---

**Note:** The length of an arrow is always bound to be in [5%, 5%] of the screen size in order to see something.

---

The `plot` command parameters are listed in the *Reference part*.

The keyboard shortcuts are:

- **enter** tries to show plot
- **p** previous plot (10 plots saved)
- **?** shows this help
- **+-** zooms in/out around the cursor 3/2 times
- **=** resets the view
- **r** refreshes plot
- **up, down, left, right** special keys to translate
- **3** switches 3d/2d plot keys :
  - **z,Z** focal zoom and zoom out
  - **H,h** increases or decreases the Z scale of the plot
- **mouse motion:**
  - **left button** rotates
  - **right button** zooms (ctrl+button on mac)
  - **right button +alt** translates (alt+ctrl+button on mac)
- **a,A** increases or decreases the arrow size
- **B** switches between showing the border meshes or not
- **i,I** updates or not: the min/max bound of the functions to the window
- **n,N** decreases or increases the number of iso value arrays
- **b** switches between black and white or color plotting
- **g** switches between grey or color plotting
- **f** switches between filling iso or iso line
- **l** switches between lighting or not

- **v** switches between show or not showing the numerical value of colors
- **m** switches between show or not showing the meshes
- **w** window dump in file ffglutXXXX.ppm
- \* keep/drop viewpoint for next plot
- **k** complex data / change view type
- **ESC** closes the graphics process before version 3.22, after no way to close
- **otherwise** does nothing

For example:

```

1 real[int] xx(10), yy(10);

2 mesh Th = square(5,5);

3 fespace Vh(Th, P1);

4 //plot scalar and vectorial FE function
5 Vh uh=x*x+y*y, vh=-y^2+x^2;
6 plot(Th, uh, [uh, vh], value=true, ps="three.eps", wait=true);

7 //zoom on box defined by the two corner points [0.1,0.2] and [0.5,0.6]
8 plot(uh, [uh, vh], bb=[[0.1, 0.2], [0.5, 0.6]],
9      wait=true, grey=true, fill=true, value=true, ps="threeg.eps");

10 //compute a cut
11 for (int i = 0; i < 10; i++) {
12     x = i/10.;
13     y = i/10.;
14     xx[i] = i;
15     yy[i] = uh; //value of uh at point (i/10., i/10.)
16 }
17 plot([xx, yy], ps="likegnu.eps", wait=true);

```

To change the color table and to choose the value of iso line you can do:

```

1 // from: \url{http://en.wikipedia.org/wiki/HSV_color_space}
2 // The HSV (Hue, Saturation, Value) model defines a color space
3 // in terms of three constituent components:
4 // HSV color space as a color wheel
5 // Hue, the color type (such as red, blue, or yellow):
6 // Ranges from 0-360 (but normalized to 0-100% in some applications, like here)
7 // Saturation, the "vibrancy" of the color: Ranges from 0-100%
8 // The lower the saturation of a color, the more "grayness" is present
9 // and the more faded the color will appear.
10 // Value, the brightness of the color: Ranges from 0-100%

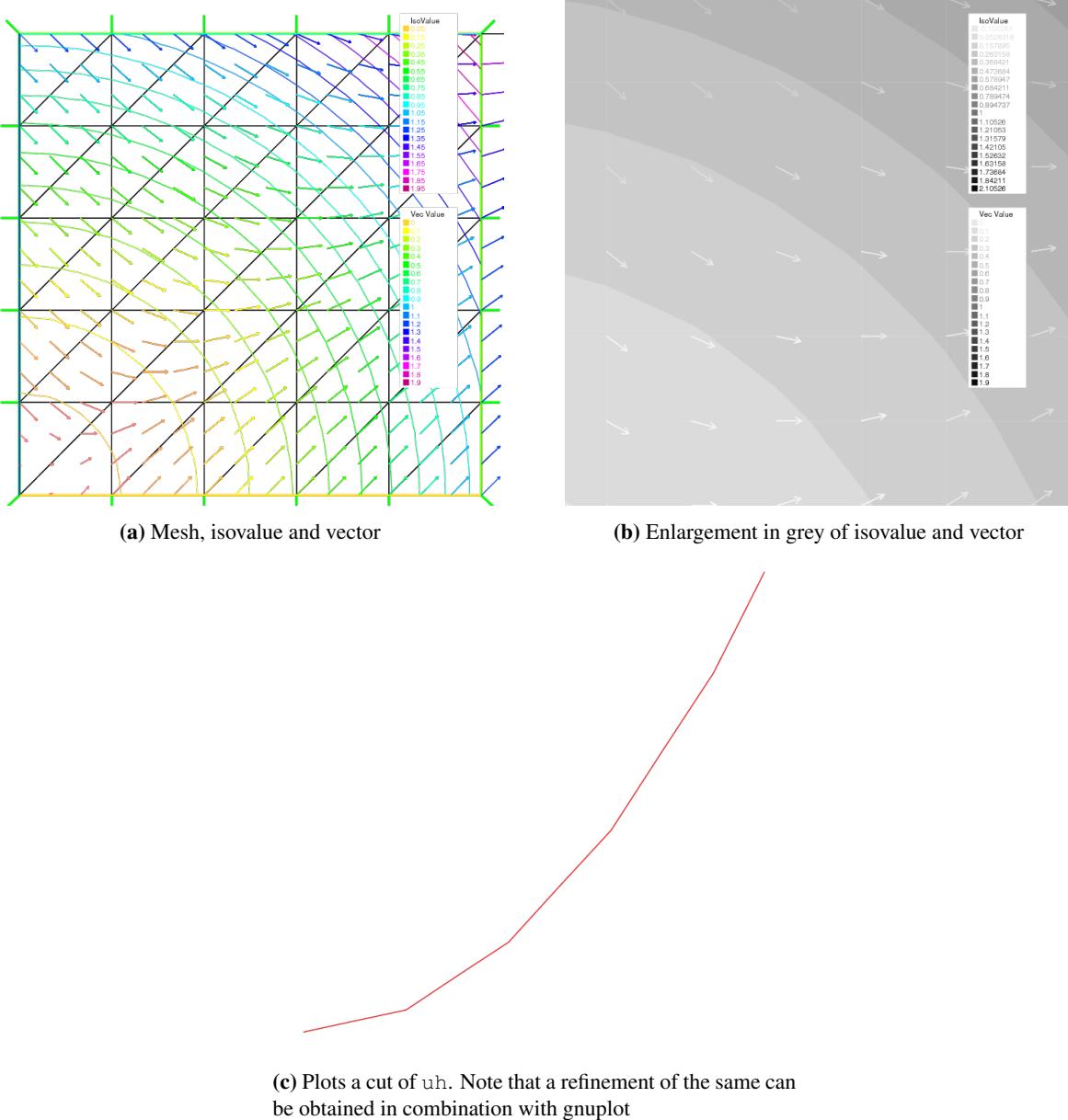
11 mesh Th = square(10, 10, [2*x-1, 2*y-1]);

12 fespace Vh(Th, P1);
13 Vh uh=2-x*x-y*y;

14 real[int] colorhsv=[ // color hsv model
15   4./6., 1, 0.5, // dark blue
16   4./6., 1, 1, // blue

```

(continues on next page)



**Fig. 3.38:** Plot

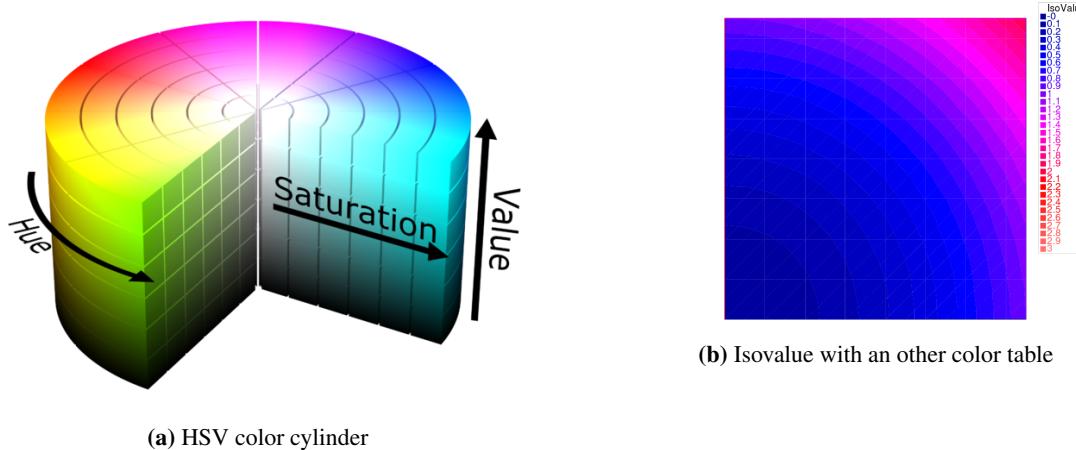


Fig. 3.39: HSV

(continued from previous page)

```

20   5./6., 1 , 1, // magenta
21   1, 1. , 1, // red
22   1, 0.5 , 1 // light red
23   ];
24   real[int] viso(31);
25
26   for (int i = 0; i < viso.n; i++)
27     viso[i] = i*0.1;
28
29   plot(uh, viso=viso(0:viso.n-1), value=true, fill=true, wait=true, hsv=colorHSV);

```

**Note:** See [HSV example](#) for the complete script.

### 3.4.2 Link with gnuplot

Example [Membrane](#) shows how to generate a gnuplot from a **FreeFEM** file. Here is another technique which has the advantage of being online, i.e. one doesn't need to quit **FreeFEM** to generate a gnuplot.

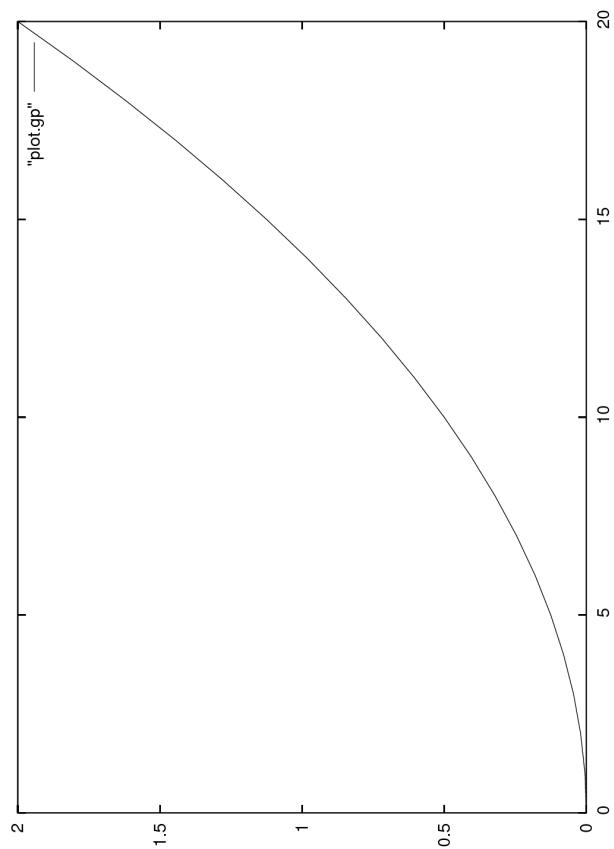
However, this works only if `gnuplot` is installed, and only on an Unix-like computer.

Add to the previous example:

```

1  // file for gnuplot
2  ofstream gnu("plot.gp");
3  for (int i = 0; i < n; i++)
4    gnu << xx[i] << " " << yy[i] << endl;
5
6
7  // to call gnuplot command and wait 5 second (due to the Unix command)
8  // and make postscript plot
9  exec("echo 'plot \"plot.gp\" w 1 \n pause 5 \n set term postscript \n set output \
  ↵\"gnuplot.eps\" \n replot \n quit' | gnuplot");

```



**Fig. 3.40:** Plots a cut of uh with gnuplot

---

**Note:** See [Plot example](#) for the complete script.

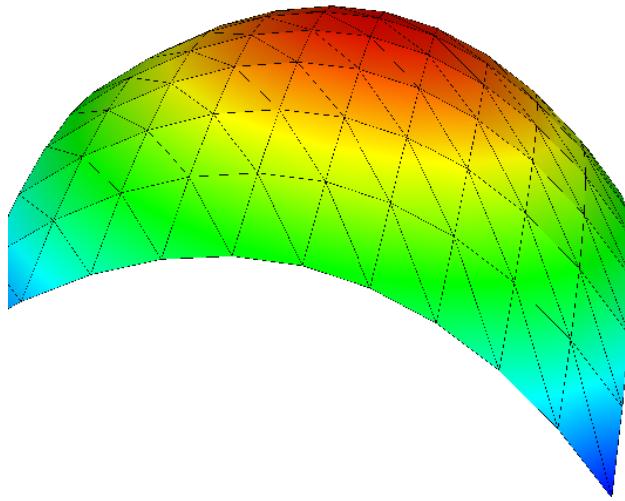
---

### 3.4.3 Link with medit

As said above, medit is a freeware display package by Pascal Frey using OpenGL. Then you may run the following example.

Now medit software is included in **FreeFEM** under `ffmedit` name.

The `medit` command parameters are listed in the [Reference part](#).



**Fig. 3.41:** `:freefem:medit` plot

With version 3.2 or later

```

1 load "medit"
2
3 mesh Th = square(10, 10, [2*x-1, 2*y-1]);
4
5 fespace Vh(Th, P1);
6 Vh u=2-x*x-y*y;
7
8 medit("u", Th, u);
```

Before:

```

1 mesh Th = square(10, 10, [2*x-1, 2*y-1]);
2
3 fespace Vh(Th, P1);
4 Vh u=2-x*x-y*y;
5
6 savemesh(Th, "u", [x, y, u*.5]); //save u.points and u.faces file
7 // build a u.bb file for medit
8 {
9   ofstream file("u.bb");
10  file << "2 1 1 " << u[].n << " 2 \n";
11  for (int j = 0; j < u[].n; j++)
```

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```

12     file << u[] [j] << endl;
13 }
14 //call medit command
15 exec("ffmedit u");
16 //clean files on unix-like OS
17 exec("rm u.bb u.faces u.points");

```

**Note:** See [Medit example](#) for the complete script.

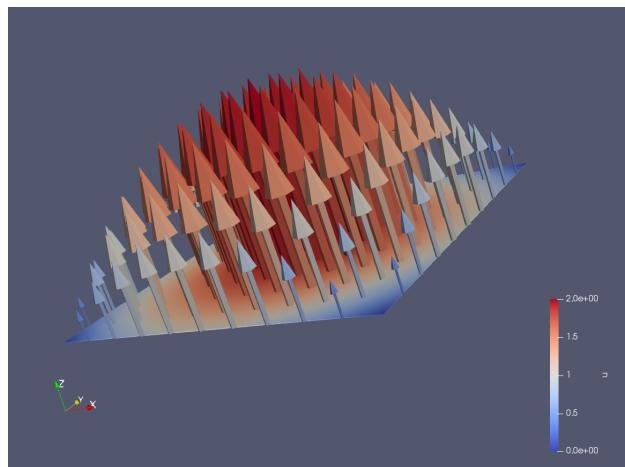
### 3.4.4 Link with Paraview

One can also export mesh or results in the .vtk format in order to post-process data using Paraview.

```

1 load "iovtk"
2
3 mesh Th = square(10, 10, [2*x-1, 2*y-1]);
4
5 fespace Vh(Th, P1);
6 Vh u=2*x*x-y*y;
7
8 int[int] Order = [1];
9 string DataName = "u";
10 savevtk("u.vtu", Th, u, dataName=DataName, order=Order);

```



**Fig. 3.42:** Paraview plot

**Warning:** Finite element variables saved using paraview **must be in P0 or P1**

**Note:** See [Paraview example](#) for the complete script.

### 3.4.5 Link with Matlab© and Octave

In order to create a plot from a **FreeFEM** simulation in **Octave** and **Matlab** the mesh, the finite element space connectivity and the simulation data must be written to files:

```

1 include "ffmatlib.idp"
2
3 mesh Th = square(10, 10, [2*x-1, 2*y-1]);
4 fespace Vh(Th, P1);
5 Vh u=2-x*x-y*y;
6
7 savemesh(Th,"export_mesh.msh");
8 ffSaveVh(Th,Vh,"export_vh.txt");
9 ffSaveData(u,"export_data.txt");

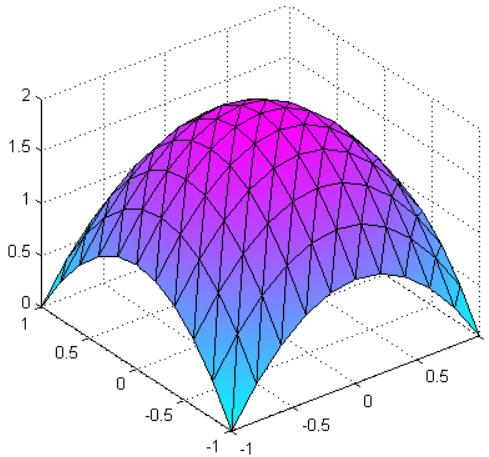
```

Within Matlab or Octave the files can be plot with the **ffmatlib** library:

```

1 addpath('path to ffmatlib');
2 [p,b,t]=ffreadmesh('export_mesh.msh');
3 vh=ffreaddata('export_vh.txt');
4 u=ffreaddata('export_data.txt');
5 ffpdeplot(p,b,t,'VhSeq',vh,'XYData',u,'ZStyle','continuous','Mesh','on');
6 grid;

```



**Fig. 3.43:** Matlab / Octave plot

---

**Note:** For more Matlab / Octave plot examples have a look at the tutorial section [Matlab / Octave Examples](#) or visit the [ffmatlib library](#) on github.

## 3.5 Algorithms & Optimization

### 3.5.1 Conjugate Gradient/GMRES

Suppose we want to solve the Euler problem (here  $x$  has nothing to do with the reserved variable for the first coordinate in **FreeFEM**):

find  $x \in \mathbb{R}^n$  such that

$$\nabla J(x) = \left( \frac{\partial J}{\partial x_i}(\mathbf{x}) \right) = 0 \quad (3.17)$$

where  $J$  is a function (to minimize for example) from  $\mathbb{R}^n$  to  $\mathbb{R}$ .

If the function is convex we can use the conjugate gradient algorithm to solve the problem, and we just need the function (named `dJ` for example) which computes  $\nabla J$ , so the parameters are the name of that function with prototype  
`func real[int] dJ(real[int] &xx);` which computes  $\nabla J$ , and a vector  $\mathbf{x}$  of type (of course the number 20 can be changed) `real[int] x(20);` to initialize the process and get the result.

Given an initial value  $\mathbf{x}^{(0)}$ , a maximum number  $i_{\max}$  of iterations, and an error tolerance  $0 < \epsilon < 1$ :

Put  $\mathbf{x} = \mathbf{x}^{(0)}$  and write

```
1 NLCG (dJ, x, precon=M, nbiter=imax, eps=epsilon, stop=stopfunc);
```

will give the solution of  $\mathbf{x}$  of  $\nabla J(\mathbf{x}) = 0$ . We can omit parameters `precon`, `nbiter`, `eps`, `stop`. Here  $M$  is the preconditioner whose default is the identity matrix.

The stopping test is

$$\|\nabla J(\mathbf{x})\|_P \leq \epsilon \|\nabla J(\mathbf{x}^{(0)})\|_P$$

Writing the minus value in `eps=`, i.e.,

```
1 NLCG (dJ, x, precon=M, nbiter=imax, eps=-epsilon);
```

We can use the stopping test:

$$\|\nabla J(\mathbf{x})\|_P^2 \leq \epsilon$$

The parameters of these three functions are:

- `nbiter`= set the number of iteration (by default 100)
- `precon`= set the preconditioner function (`P` for example) by default it is the identity, note the prototype is  
`func real[int] P(real[int] &x).`
- `eps`= set the value of the stop test  $\epsilon$  ( $= 10^{-6}$  by default) if positive then relative test  $\|\nabla J(\mathbf{x})\|_P \leq \epsilon \|\nabla J(\mathbf{x}_0)\|_P$ , otherwise the absolute test is  $\|\nabla J(\mathbf{x})\|_P^2 \leq |\epsilon|$ .
- `veps`= set and return the value of the stop test, if positive, then relative test is  $\|\nabla J(\mathbf{x})\|_P \leq \epsilon \|\nabla J(\mathbf{x}_0)\|_P$ , otherwise the absolute test is  $\|\nabla J(\mathbf{x})\|_P^2 \leq |\epsilon|$ . The return value is minus the real stop test (remark: it is useful in loop).
- `stop`= `stopfunc` add your test function to stop before the `eps` criterion. The prototype for the function `stopfunc` is

```
func bool stopfunc(int iter, real[int] u, real[int] g)
```

where  $\mathbf{u}$  is the current solution, and  $\mathbf{g}$ , the current gradient, is not preconditioned.

---

**Tip:** *Algorithms.edp*

For a given function  $b$ , let us find the minimizer  $u$  of the function

$$\begin{aligned} J(u) &= \frac{1}{2} \int_{\Omega} f(|\nabla u|^2) - \int_{\Omega} ub \\ f(x) &= ax + x - \ln(1+x), \quad f'(x) = a + \frac{x}{1+x}, \quad f''(x) = \frac{1}{(1+x)^2} \end{aligned}$$

under the boundary condition  $u = 0$  on  $\partial\Omega$ .

```

1 fespace Ph(Th, P0);
2 Ph alpha; //store df(/nabla u)^2
3
4 // The functionn J
5 //J(u) = 1/2 int_Omega f(/nabla u)^2 - int_Omega u b
6 func real J (real[int] & u) {
7   Vh w;
8   w[] = u;
9   real r = int2d(Th) (0.5*f(dx(w)*dx(w) + dy(w)*dy(w)) - b*w);
10  cout << "J(u) = " << r << " " << u.min << " " << u.max << endl;
11  return r;
12 }
13
14 // The gradiant of J
15 func real[int] dJ (real[int] & u) {
16   Vh w;
17   w[] = u;
18   alpha = df(dx(w)*dx(w) + dy(w)*dy(w));
19   varf au (uh, vh)
20   = int2d(Th) (
21     alpha*(dx(w)*dx(vh) + dy(w)*dy(vh))
22     - b*vh
23   )
24   + on(1, 2, 3, 4, uh=0)
25 ;
26
27 u = au(0, Vh);
28 return u; //warning: no return of local array
29 }
```

We also want to construct a preconditioner  $C$  with solving the problem:

find  $u_h \in V_{0h}$  such that:

$$\forall v_h \in V_{0h}, \quad \int_{\Omega} \alpha \nabla u_h \cdot \nabla v_h = \int_{\Omega} b v_h$$

where  $\alpha = f'(|\nabla u|^2)$ .

```

1 alpha = df(dx(u)*dx(u) + dy(u)*dy(u));
2 varf alap (uh, vh)
3   = int2d(Th) (
4     alpha*(dx(uh)*dx(vh) + dy(uh)*dy(vh))
5   )
6   + on(1, 2, 3, 4, uh=0)
7 ;
8
9 varf amass(uh, vh)
10  = int2d(Th) (
11    uh*vh
12  )
13  + on(1, 2, 3, 4, uh=0)
14 ;
15
16 matrix Amass = amass(Vh, Vh, solver=CG);
17 matrix Alap= alap(Vh, Vh, solver=Cholesky, factorize=1);
18
19 // Preconditionner
```

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```

20 func real[int] C(real[int] & u) {
21   real[int] w = u;
22   u = Alap^-1*w;
23   return u; //warning: no return of local array variable
24 }
```

To solve the problem, we make 10 iterations of the conjugate gradient, recompute the preconditioner and restart the conjugate gradient:

```

1 int conv=0;
2 for(int i = 0; i < 20; i++){
3   conv = NLCG(dJ, u[], nbiter=10, precon=C, veps=eps, verbosity=5);
4   if (conv) break;
5
6   alpha = df(dx(u)*dx(u) + dy(u)*dy(u));
7   Alap = alap(Vh, Vh, solver=Cholesky, factorize=1);
8   cout << "Restart with new preconditioner " << conv << ", eps =" << eps << endl;
9 }
10
11 // Plot
12 plot (u, wait=true, cmm="solution with NLCG");
```

For a given symmetric positive matrix  $A$ , consider the quadratic form

$$J(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A \mathbf{x} - \mathbf{b}^T \mathbf{x}$$

then  $J(\mathbf{x})$  is minimized by the solution  $\mathbf{x}$  of  $A\mathbf{x} = \mathbf{b}$ . In this case, we can use the function `AffineCG`

```
1 AffineCG(A, x, precon=M, nbiter=imax, eps=epsilon, stop=stp);
```

If  $A$  is not symmetric, we can use GMRES(Generalized Minimum Residual) algorithm by

```
1 AffineGMRES(A, x, precon=M, nbiter=imax, eps=epsilon);
```

Also, we can use the non-linear version of GMRES algorithm (the function  $J$  is just convex)

```
1 AffineGMRES(dJ, x, precon=M, nbiter=imax, eps=epsilon);
```

For the details of these algorithms, refer to [PIRONNEAU1998], Chapter IV, 1.3.

### 3.5.2 Algorithms for Unconstrained Optimization

Two algorithms of COOOL package are interfaced with the Newton Raphson method (called `Newton`) and the BFGS method. These two are directly available in **FreeFEM** (no dynamical link to load). Be careful with these algorithms, because their implementation uses full matrices. We also provide several optimization algorithms from the `NLOpt` library as well as an interface for Hansen's implementation of CMAES (a MPI version of this one is also available).

#### Example of usage for BFGS or CMAES

---

**Tip:** BFGS

```

1 real[int] b(10), u(10);
2
3 //J
4 func real J (real[int] & u) {
5     real s = 0;
6     for (int i = 0; i < u.n; i++)
7         s += (i+1)*u[i]*u[i]*0.5 - b[i]*u[i];
8     if (debugJ)
9         cout << "J = " << s << ", u = " << u[0] << " " << u[1] << endl;
10    return s;
11 }
12
13 //the gradiant of J (this is a affine version (the RHS is in)
14 func real[int] dJ (real[int] &u) {
15     for (int i = 0; i < u.n; i++)
16         u[i] = (i+1)*u[i];
17     if (debugdJ)
18         cout << "dJ: u = " << u[0] << " " << u[1] << " " << u[2] << endl;
19     u -= b;
20     if (debugdJ)
21         cout << "dJ-b: u = " << u[0] << " " << u[1] << " " << u[2] << endl;
22     return u; //return of global variable ok
23 }
24
25 b=1;
26 u=2;
27 BFGS(J, DJ, u, eps=1.e-6, nbiter=20, nbiterline=20);
28 cout << "BFGS: J(u) = " << J(u) << ", err = " << error(u, b) << endl;

```

It is almost the same as using the CMA evolution strategy except, that since it is a derivative free optimizer, the `dJ` argument is omitted and there are some other named parameters to control the behavior of the algorithm. With the same objective function as above, an example of utilization would be (see [CMAES Variational inequality](#) for a complete example):

```

1 load "ff-cmaes"
2 //define J, u, ...
3 real min = cmaes(J, u, stopTolFun=1e-6, stopMaxIter=3000);
4 cout << "minimum value is " << min << " for u = " << u << endl;

```

This algorithm works with a normal multivariate distribution in the parameters space and tries to adapt its covariance matrix using the information provided by the successive function evaluations (see [NLopt documentation](#) for more details). Therefore, some specific parameters can be passed to control the starting distribution, size of the sample generations, etc... Named parameters for this are the following:

- `seed`= Seed for random number generator (`val` is an integer). No specified value will lead to a clock based seed initialization.
- `initialStdDev`= Value for the standard deviations of the initial covariance matrix (`val` is a `real`). If the value  $\sigma$  is passed, the initial covariance matrix will be set to  $\sigma I$ . The expected initial distance between initial  $X$  and the `argmin` should be roughly `initialStdDev`. Default is 0.3.
- `initialStdDevs`= Same as above except that the argument is an array allowing to set a value of the initial standard deviation for each parameter. Entries differing by several orders of magnitude should be avoided (if it can't be, try rescaling the problem).
- `stopTolFun`= Stops the algorithm if function value differences are smaller than the passed one, default is  $10^{-12}$ .

- `stopTolFunHist`= Stops the algorithm if function value differences from the best values are smaller than the passed one, default is 0 (unused).
- `stopTolX`= Stopping criteria is triggered if step sizes in the parameters space are smaller than this real value, default is 0.
- `stopTolXFactor`= Stopping criteria is triggered when the standard deviation increases more than this value. The default value is  $10^3$ .
- `stopMaxFunEval`= Stops the algorithm when `stopMaxFunEval` function evaluations have been done. Set to  $900(n + 3)^2$  by default, where  $n$  is the parameters space dimension.
- `stopMaxIter`= Integer stopping the search when `stopMaxIter` generations have been sampled. Unused by default.
- `popsizes`= Integer value used to change the sample size. The default value is  $4 + \lfloor 3 \ln(n) \rfloor$ . Increasing the population size usually improves the global search capabilities at the cost of, at most, a linear reduction of the convergence speed with respect to `popsizes`.
- `paramFile`= This string type parameter allows the user to pass all the parameters using an extern file, as in Hansen's original code. More parameters related to the CMA-ES algorithm can be changed with this file. Note that the parameters passed to the CMAES function in the **FreeFEM** script will be ignored if an input parameters file is given.

### 3.5.3 IPOPT

The `ff-Ipopt` package is an interface for the **IPOPT** [WÄCHTER2006] optimizer. IPOPT is a software library for large scale, non-linear, constrained optimization. It implements a primal-dual interior point method along with filter method based line searches.

IPOPT needs a direct sparse symmetric linear solver. If your version of **FreeFEM** has been compiled with the `--enable-downlad` tag, it will automatically be linked with a sequential version of MUMPS. An alternative to MUMPS would be to download the HSL subroutines (see [Compiling and Installing the Java Interface JIOPPT](#)) and place them in the `/ipopt/Ipopt-3.10.2/ThirdParty/HSL` directory of the **FreeFEM** downloads folder before compiling.

#### Short description of the algorithm

In this section, we give a very brief glimpse at the underlying mathematics of IPOPT. For a deeper introduction on interior methods for nonlinear smooth optimization, one may consult [FORSGREN2002], or [WÄCHTER2006] for more IPOPT specific elements. IPOPT is designed to perform optimization for both equality and inequality constrained problems. However, nonlinear inequalities are rearranged before the beginning of the optimization process in order to restrict the panel of nonlinear constraints to those of the equality kind. Each nonlinear inequality is transformed into a pair of simple bound inequalities and nonlinear equality constraints by the introduction of as many slack variables as is needed :  $c_i(x) \leq 0$  becomes  $c_i(x) + s_i = 0$  and  $s_i \leq 0$ , where  $s_i$  is added to the initial variables of the problems  $x_i$ . Thus, for convenience, we will assume that the minimization problem does not contain any nonlinear inequality constraint. It means that, given a function  $f : \mathbb{R}^n \mapsto \mathbb{R}$ , we want to find:

$$x_0 = \underset{x \in V}{\operatorname{argmin}} f(x) \quad (3.18)$$

with  $V = \{x \in \mathbb{R}^n \mid c(x) = 0 \text{ and } x_l \leq x \leq x_u\}$

Where  $c : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $x_l, x_u \in \mathbb{R}^n$  and inequalities hold componentwise. The  $f$  function as well as the constraints  $c$  should be twice-continuously differentiable.

As a barrier method, interior points algorithms try to find a Karush-Kuhn-Tucker point for (3.18) by solving a sequence of problems, unconstrained with respect to the inequality constraints, of the form:

$$\text{for a given } \mu > 0, \text{ find } x_\mu = \underset{x \in \mathbb{R}^n \mid c(x)=0}{\operatorname{argmin}} B(x, \mu) \quad (3.19)$$

Where  $\mu$  is a positive real number and

$$B(x, \mu) = f(x) - \mu \sum_{i=1}^n \ln(x_{u,i} - x_i) - \mu \sum_{i=1}^m \ln(x_i - x_{l,i})$$

The remaining equality constraints are handled with the usual Lagrange multipliers method. If the sequence of barrier parameters  $\mu$  converge to 0, intuition suggests that the sequence of minimizers of (3.19) converge to a local constrained minimizer of (3.18). For a given  $\mu$ , (3.19) is solved by finding  $(x_\mu, \lambda_\mu) \in \mathbb{R}^n \times \mathbb{R}^m$  such that:

$$\begin{aligned} \nabla B(x_\mu, \mu) + \sum_{i=1}^m \lambda_{\mu,i} \nabla c_i(x_\mu) &= \nabla B(x_\mu, \mu) + J_c(x_\mu)^T \lambda_\mu = 0 \\ c(x_\mu) &= 0 \end{aligned} \quad (3.20)$$

The derivations for  $\nabla B$  only holds for the  $x$  variables, so that:

$$\nabla B(x, \mu) = \nabla f(x) + \begin{pmatrix} \mu/(x_{u,1} - x_1) \\ \vdots \\ \mu/(x_{u,n} - x_n) \end{pmatrix} - \begin{pmatrix} \mu/(x_1 - x_{l,1}) \\ \vdots \\ \mu/(x_n - x_{l,n}) \end{pmatrix}$$

If we respectively call  $z_u(x, \mu) = (\mu/(x_{u,1} - x_1), \dots, \mu/(x_{u,n} - x_n))$  and  $z_l(x, \mu)$  the other vector appearing in the above equation, then the optimum  $(x_\mu, \lambda_\mu)$  satisfies:

$$\nabla f(x_\mu) + J_c(x_\mu)^T \lambda_\mu + z_u(x_\mu, \mu) - z_l(x_\mu, \mu) = 0 \quad \text{and} \quad c(x_\mu) = 0 \quad (3.21)$$

In this equation, the  $z_l$  and  $z_u$  vectors seem to play the role of Lagrange multipliers for the simple bound inequalities, and indeed, when  $\mu \rightarrow 0$ , they converge toward some suitable Lagrange multipliers for the KKT conditions, provided some technical assumptions are fulfilled (see [FORSGREN2002]).

Equation (3.21) is solved by performing a Newton method in order to find a solution of (3.20) for each of the decreasing values of  $\mu$ . Some order 2 conditions are also taken into account to avoid convergence to local maximizers, see [FORSGREN2002] for details about them. In the most classic IP algorithms, the Newton method is directly applied to (3.20). This is in most case inefficient due to frequent computation of infeasible points. These difficulties are avoided in Primal-Dual interior point methods where (3.20) is transformed into an extended system where  $z_u$  and  $z_l$  are treated as unknowns and the barrier problems are finding  $(x, \lambda, z_u, z_l) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n$  such that:

$$\left\{ \begin{array}{rcl} \nabla f(x) + J_c(x)^T \lambda + z_u - z_l & = & 0 \\ c(x) & = & 0 \\ (X_u - X)z_u - \mu e & = & 0 \\ (X - X_l)z_l - \mu e & = & 0 \end{array} \right. \quad (3.22)$$

Where if  $a$  is a vector of  $\mathbb{R}^n$ ,  $A$  denotes the diagonal matrix  $A = (a_i \delta_{ij})_{1 \leq i,j \leq n}$  and  $e \in \mathbb{R}^n = (1, 1, \dots, 1)$ . Solving this nonlinear system by the Newton method is known as being the *primal-dual* interior point method. Here again, more details are available in [FORSGREN2002]. Most actual implementations introduce features in order to globalize the convergence capability of the method, essentially by adding some line-search steps to the Newton algorithm, or by using trust regions. For the purpose of IPOPT, this is achieved by a *filter line search* methods, the details of which can be found in [WÄCHTER2006].

More IPOPT specific features or implementation details can be found in [WÄCHTER2006]. We will just retain that IPOPT is a smart Newton method for solving constrained optimization problems, with global convergence capabilities due to a robust line search method (in the sense that the algorithm will converge no matter the initializer). Due to the underlying Newton method, the optimization process requires expressions of all derivatives up to the order 2 of the fitness function as well as those of the constraints. For problems whose Hessian matrices are difficult to compute or lead to high dimensional dense matrices, it is possible to use a BFGS approximation of these objects at the cost of a much slower convergence rate.

## IPOPT in FreeFEM

Calling the IPOPT optimizer in a **FreeFEM** script is done with the `IPOPT` function included in the `ff-Ipopt` dynamic library. IPOPT is designed to solve constrained minimization problems in the form:

$$\begin{aligned} \text{find } & x_0 = \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} f(x) \\ \text{s.t. } & \begin{cases} \forall i \leq n, x_i^{\text{lb}} \leq x_i \leq x_i^{\text{ub}} & (\text{simple bounds}) \\ \forall i \leq m, c_i^{\text{lb}} \leq c_i(x) \leq c_i^{\text{ub}} & (\text{constraints functions}) \end{cases} \end{aligned}$$

Where `ub` and `lb` stand for “upper bound” and “lower bound”. If for some  $i, 1 \leq i \leq m$  we have  $c_i^{\text{lb}} = c_i^{\text{ub}}$ , it means that  $c_i$  is an equality constraint, and an inequality one if  $c_i^{\text{lb}} < c_i^{\text{ub}}$ .

There are different ways to pass the fitness function and constraints. The more general one is to define the functions using the keyword `func`. Any returned matrix must be a sparse one (type `matrix`, not a `real[int, int]`):

```

1 func real J (real[int] &X) {...} //Fitness Function, returns a scalar
2 func real[int] gradJ (real[int] &X) {...} //Gradient is a vector
3
4 func real[int] C (real[int] &X) {...} //Constraints
5 func matrix jacC (real[int] &X) {...} //Constraints Jacobian

```

**Warning:** In the current version of **FreeFEM**, returning a `matrix` object that is local to a function block leads to undefined results. For each sparse matrix returning function you define, an `extern` matrix object has to be declared, whose associated function will overwrite and return on each call. Here is an example for `jacC`:

```

1 matrix jacCBuffer; //just declare, no need to define yet
2 func matrix jacC (real[int] &X) {
3     ...//fill jacCBuffer
4     return jacCBuffer;
5 }

```

**Warning:** IPOPT requires the structure of each matrix at the initialization of the algorithm. Some errors may occur if the matrices are not constant and are built with the `matrix A = [I, J, C]` syntax, or with an intermediary full matrix (`real[int, int]`), because any null coefficient is discarded during the construction of the sparse matrix. It is also the case when making matrices linear combinations, for which any zero coefficient will result in the suppression of the matrix from the combination. Some controls are available to avoid such problems. Check the named parameter descriptions (`checkindex`, `structhess` and `struct jac` can help). We strongly advice to use `varf` as much as possible for the matrix forging.

The Hessian returning function is somewhat different because it has to be the Hessian of the Lagrangian function:

$$(x, \sigma_f, \lambda) \mapsto \sigma_f \nabla^2 f(x) + \sum_{i=1}^m \lambda_i \nabla^2 c_i(x) \text{ where } \lambda \in \mathbb{R}^m \text{ and } \sigma \in \mathbb{R}$$

Your Hessian function should then have the following prototype:

```

1 matrix hessianLBuffer; //Just to keep it in mind
2 func matrix hessianL (real[int] &X, real sigma, real[int] &lambda) {...}

```

If the constraints functions are all affine, or if there are only simple bound constraints, or no constraint at all, the Lagrangian Hessian is equal to the fitness function Hessian, one can then omit the `sigma` and `lambda` parameters:

```

1 matrix hessianJBuffer;
2 func matrix hessianJ (real[int] &X) {...} //Hessian prototype when constraints are
   ↪affine

```

When these functions are defined, IPOPT is called this way:

```

1 real[int] Xi = ... ; //starting point
2 IPOPT(J, gradJ, hessianL, C, jacC, Xi, /*some named parameters*/);

```

If the Hessian is omitted, the interface will tell IPOPT to use the (L)BFGS approximation (it can also be enabled with a named parameter, see further). Simple bound or unconstrained problems do not require the constraints part, so the following expressions are valid:

```

1 IPOPT(J, gradJ, C, jacC, Xi, ... ); //IPOPT with BFGS
2 IPOPT(J, gradJ, hessianJ, Xi, ... ); //Newton IPOPT without constraints
3 IPOPT(J, gradJ, Xi, ... ); //BFGS, no constraints

```

Simple bounds are passed using the `lb` and `ub` named parameters, while constraint bounds are passed with the `clb` and `cub` ones. Unboundedness in some directions can be achieved by using the  $1e^{19}$  and  $-1e^{19}$  values that IPOPT recognizes as  $+\infty$  and  $-\infty$ :

```

1 real[int] xlB(n), xub(n), clb(m), cub(m);
2 //fill the arrays...
3 IPOPT(J, gradJ, hessianL, C, jacC, Xi, lb=xlB, ub=xub, clb=clb, cub=cub, /*some other
   ↪named parameters*/);

```

**P2 fitness function and affine constraints function :** In the case where the fitness function or constraints function can be expressed respectively in the following forms:

$$\forall x \in \mathbb{R}^n, f(x) = \frac{1}{2} \langle Ax, x \rangle + \langle b, x \rangle \quad (A, b) \in \mathcal{M}_{n,n}(\mathbb{R}) \times \mathbb{R}^n \\ \text{or, } C(x) = Ax + b \quad (A, b) \in \mathcal{M}_{m,n}(\mathbb{R}) \times \mathbb{R}^m$$

where  $A$  and  $b$  are constant, it is possible to directly pass the  $(A, b)$  pair instead of defining 3 (or 2) functions. It also indicates to IPOPT that some objects are constant and that they have to be evaluated only once, thus avoiding multiple copies of the same matrix. The syntax is:

```

1 // Affine constraints with "standard" fitness function
2 matrix A = ... ; //linear part of the constraints
3 real[int] b = ... ; //constant part of constraints
4 IPOPT(J, gradJ, hessianJ, [A, b], Xi, /*bounds and named parameters*/);
5 // [b, A] would work as well.

```

Note that if you define the constraints in this way, they don't contribute to the Hessian, so the Hessian should only take one `real[int]` as an argument.

```

1 // Affine constraints and P2 fitness func
2 matrix A = ... ; //bilinear form matrix
3 real[int] b = ... ; //linear contribution to f
4 matrix Ac = ... ; //linear part of the constraints
5 real[int] bc = ... ; //constant part of constraints
6 IPOPT([A, b], [Ac, bc], Xi, /*bounds and named parameters*/);

```

If both objective and constraint functions are given this way, it automatically activates the IPOPT `mehrotra_algorithm` option (better for linear and quadratic programming according to the documentation). Otherwise, this option can only be set through the option file (see the named parameters section).

A false case is the one of defining  $f$  in this manner while using standard functions for the constraints:

```

1 matrix A = ... ; //bilinear form matrix
2 real[int] b = ... ; //linear contribution to f
3 func real[int] C(real[int] &X) {...} //constraints
4 func matrix jacC(real[int] &X) {...} //constraints Jacobian
5 IPOPT([A, b], C, jacC, Xi, /*bounds and named parameters*/);

```

Indeed, when passing  $[A, b]$  in order to define  $f$ , the Lagrangian Hessian is automatically built and has the constant  $x \mapsto A$  function, with no way to add possible constraint contributions, leading to incorrect second order derivatives. So, a problem should be defined like that in only two cases:

1. constraints are nonlinear but you want to use the BFGS mode (then add `bfgs=1` to the named parameter),
2. constraints are affine, but in this case, compatible to pass in the same way

Here are some other valid definitions of the problem (cases when  $f$  is a pure quadratic or linear form, or  $C$  a pure linear function, etc...):

```

1 // Pure quadratic f - A is a matrix
2 IPOPT(A, /*constraints arguments*/, Xi, /*bound and named parameters*/);
3 // Pure linear f - b is a real[int]
4 IPOPT(b, /*constraints arguments*/, Xi, /*bound and named parameters*/);
5 // Linear constraints - Ac is a matrix
6 IPOPT(/*fitness function arguments*/, Ac, Xi, /*bound and named parameters*/);

```

**Returned Value :** The IPOPT function returns an error code of type `int`. A zero value is obtained when the algorithm succeeds and positive values reflect the fact that IPOPT encounters minor troubles. Negative values reveal more problematic cases. The associated IPOPT return tags are listed in the table below. The [IPOPT pdf documentation](#) provides a more accurate description of these return statuses:

Success	Failures
0 Solve_Succeeded	
1 Solved_To_Acceptable_Level	-1 Maximum_Iterations_Exceeded
2 Infeasible_Problem_Detected	-2 Restoration_Failed
3 Search_Direction_Becomes_Too_Small	-3 Error_In_Step_Computation
4 Diverging_Iterates	-4 Maximum_CpuTime_Exceeded
5 User_Requested_Stop	
6 Feasible_Point_Found	

Problem definition issues	Critical errors
-10 NotEnoughDegreesOfFreedom	-100 Unrecoverable_Exception
-11 Invalid_Problem_Definition	-101 NonIpopt_Exception_Thrown
-12 Invalid_Option	-102 Insufficient_Memory
-13 Invalid_Number_Detected	-199 Internal_Error

**Named Parameters :** The available named parameters in this interface are those we thought to be the most subject to variations from one optimization to another, plus a few that are interface specific. Though, as one could see at [IPOPT Linear solver](#), there are many parameters that can be changed within IPOPT, affecting the algorithm behavior. These parameters can still be controlled by placing an option file in the execution directory. Note that [IPOPT's pdf documentation](#) may provides more information than the previously mentioned online version for certain parameters. The in-script available parameters are:

- `lb, ub : real[int]` for lower and upper simple bounds upon the search variables must be of size  $n$  (search space dimension). If two components of the same index in these arrays are equal then the corresponding search variable is fixed. By default IPOPT will remove any fixed variable from the optimization process and always use the fixed value when calling functions. It can be changed using the `fixedvar` parameter.

- `clb, cub : real[int]` of size  $m$  (number of constraints) for lower and upper constraints bounds. Equality between two components of the same index  $i$  in `clb` and `cub` reflect an equality constraint.
- `struct jacc` : To pass the greatest possible structure (indexes of non null coefficients) of the constraint Jacobians under the form `[I, J]` where `I` and `J` are two integer arrays. If not defined, the structure of the constraint Jacobians, evaluated in `Xi`, is used (no issue if the Jacobian is constant or always defined with the same `varf`, hazardous if it is with a triplet array or if a full matrix is involved).
- `structhess` : Same as above but for the Hessian function (unused if  $f$  is P2 or less and constraints are affine). Here again, keep in mind that it is the Hessian of the Lagrangian function (which is equal to the Hessian of  $f$  only if constraints are affine). If no structure is given with this parameter, the Lagrangian Hessian is evaluated on the starting point, with  $\sigma = 1$  and  $\lambda = (1, 1, \dots, 1)$  (it is safe if all the constraints and fitness function Hessians are constant or build with `varf`, and here again it is less reliable if built with a triplet array or a full matrix).
- `checkindex` : A `bool` that triggers a dichotomic index search when matrices are copied from **FreeFEM** functions to IPOPT arrays. It is used to avoid wrong index matching when some null coefficients are removed from the matrices by **FreeFEM**. It will not solve the problems arising when a too small structure has been given at the initialization of the algorithm. Enabled by default (except in cases where all matrices are obviously constant).
- `warmstart` : If set to `true`, the constraints dual variables  $\lambda$ , and simple bound dual variables are initialized with the values of the arrays passed to `lm`, `lz` and `uz` named parameters (see below).
- `lm : real[int]` of size  $m$ , which is used to get the final values of the constraints dual variables  $\lambda$  and/or initialize them in case of a warm start (the passed array is also updated to the last dual variables values at the end of the algorithm).
- `lz, uz : real[int]` of size  $n$  to get the final values and/or initialize (in case of a warm start) the dual variables associated to simple bounds.
- `tol : real`, convergence tolerance for the algorithm, the default value is  $10^{-8}$ .
- `maxiter : int`, maximum number of iterations with 3000 as default value.
- `maxctime : real` value, maximum runtime duration. Default is  $10^6$  (almost 11 and a halfdays).
- `bfgs : bool` enabling or not the (low-storage) BFGS approximation of the Lagrangian Hessian. It is set to false by default, unless there is no way to compute the Hessian with the functions that have been passed to IPOPT.
- `derivativetest` : Used to perform a comparison of the derivatives given to IPOPT with finite differences computation. The possible string values are : "none" (default), "first-order", "second-order" and "only-second-order". The associated derivative error tolerance can be changed via the option file. One should not care about any error given by it before having tried, and failed, to perform a first optimization.
- `dth` : Perturbation parameter for the derivative test computations with finite differences. Set by default to  $10^{-8}$ .
- `dttol` : Tolerance value for the derivative test error detection (default value unknown yet, maybe  $10^{-5}$ ).
- `optfile : string` parameter to specify the IPOPT option file name. IPOPT will look for a `ipopt.opt` file by default. Options set in the file will overwrite those defined in the **FreeFEM** script.
- `printlevel : An int` to control IPOPT output print level, set to 5 by default, the possible values are from 0 to 12. A description of the output information is available in the [PDF documentation](#) of IPOPT.
- `fixedvar : string` for the definition of simple bound equality constraints treatment : use "make\_parameter" (default value) to simply remove them from the optimization process (the functions will always be evaluated with the fixed value for those variables), "make\_constraint" to treat them as any other constraint or "relax\_bounds" to relax fixing bound constraints.
- `mustrategy : a string` to choose the update strategy for the barrier parameter  $\mu$ . The two possible tags are "monotone", to use the monotone (Fiacco-McCormick) strategy, or "adaptive" (default setting).

- `muinit` : real positive value for the barrier parameter initialization. It is only relevant when `mustrategy` has been set to `monotone`.
- `pivtol` : real value to set the pivot tolerance for the linear solver. A smaller number pivots for sparsity, a larger number pivots for stability. The value has to be in the  $[0, 1]$  interval and is set to  $10^{-6}$  by default.
- `brf` : Bound relax factor: before starting the optimization, the bounds given by the user are relaxed. This option sets the factor for this relaxation. If it is set to zero, then the bound relaxation is disabled. This `real` has to be positive and its default value is  $10^{-8}$ .
- `objvalue` : An identifier to a `real` type variable to get the last value of the objective function (best value in case of success).
- `mumin` : minimum value for the barrier parameter  $\mu$ , a `real` with  $10^{-11}$  as default value.
- `linesearch` : A boolean which disables the line search when set to `false`. The line search is activated by default. When disabled, the method becomes a standard Newton algorithm instead of a primal-dual system. The global convergence is then no longer assured, meaning that many initializers could lead to diverging iterates. But on the other hand, it can be useful when trying to catch a precise local minimum without having some out of control process making the iterate caught by some other near optimum.

### 3.5.4 Some short examples using IPOPT

**Tip:** Ipopt variational inequality A very simple example consisting of, given two functions  $f$  and  $g$  (defined on  $\Omega \subset \mathbb{R}^2$ ), minimizing  $J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \int_{\Omega} fu$ , with  $u \leq g$  almost everywhere:

```

1 // Solve
2 // Delta u = f
3 //u < g
4 //u = 0 on Gamma
5 load "ff-Ipopt";
6
7 // Parameters
8 int nn = 20;
9 func f = 1.; //rhs function
10 real r = 0.03, s = 0.1;
11 func g = r - r/2*exp(-0.5*(square(x)-0.5) + square(y)-0.5))/square(s);
12
13 // Mesh
14 mesh Th = square(nn, nn);
15
16 // Fespace
17 fespace Vh(Th, P2);
18 Vh u = 0;
19 Vh lb = -1.e19;
20 Vh ub = g;
21
22 // Macro
23 macro Grad(u) [dx(u), dy(u)] //
24
25 // Problem
26 varf vP (u, v)
27   = int2d(Th) (
28     Grad(u) *Grad(v)
29   )
30   - int2d(Th) (

```

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```

31     f*v
32   )
33 ;

```

Here we build the matrix and second member associated to the function to fully and finally minimize it. The  $[A, b]$  syntax for the fitness function is then used to pass it to IPOPT.

```

1 matrix A = vP(Vh, Vh, solver=CG);
2 real[int] b = vP(0, Vh);

```

We use simple bounds to impose the boundary condition  $u = 0$  on  $\partial\Omega$ , as well as the  $u \leq g$  condition.

```

1 varf vGamma (u, v) = on(1, 2, 3, 4, u=1);
2 real[int] onGamma = vGamma(0, Vh);
3
4 //warning: the boundary conditions are given with lb and ub on border
5 ub[] = onGamma ? 0. : ub[];
6 lb[] = onGamma ? 0. : lb[];
7
8 // Solve
9 IPOPT([A, b], u[], lb=lb[], ub=ub[]);
10
11 // Plot
12 plot(u);

```

### Tip: Ipopt variational inequality 2

Let  $\Omega$  be a domain of  $\mathbb{R}^2$ .  $f_1, f_2 \in L^2(\Omega)$  and  $g_1, g_2 \in L^2(\partial\Omega)$  four given functions with  $g_1 \leq g_2$  almost everywhere. We define the space:

$$V = \{(v_1, v_2) \in H^1(\Omega)^2; v_1|_{\partial\Omega} = g_1, v_2|_{\partial\Omega} = g_2, v_1 \leq v_2 \text{ a.e.}\}$$

as well as the function  $J : H^1(\Omega)^2 \rightarrow \mathbb{R}$ :

$$J(v_1, v_2) = \frac{1}{2} \int_{\Omega} |\nabla v_1|^2 - \int_{\Omega} f_1 v_1 + \frac{1}{2} \int_{\Omega} |\nabla v_2|^2 - \int_{\Omega} f_2 v_2$$

The problem entails finding (numerically) two functions  $(u_1, u_2) = \underset{(v_1, v_2) \in V}{\operatorname{argmin}} J(v_1, v_2)$ .

```

1 load "ff-Ipopt";
2
3 // Parameters
4 int nn = 10;
5 func f1 = 10; //right hand side
6 func f2 = -15;
7 func g1 = -0.1; //Boundary condition functions
8 func g2 = 0.1;
9
10 // Mesh
11 mesh Th = square(nn, nn);
12
13 // Fespace
14 fespace Vh(Th, [P1, P1]);
15 Vh [uz, uz2] = [1, 1];

```

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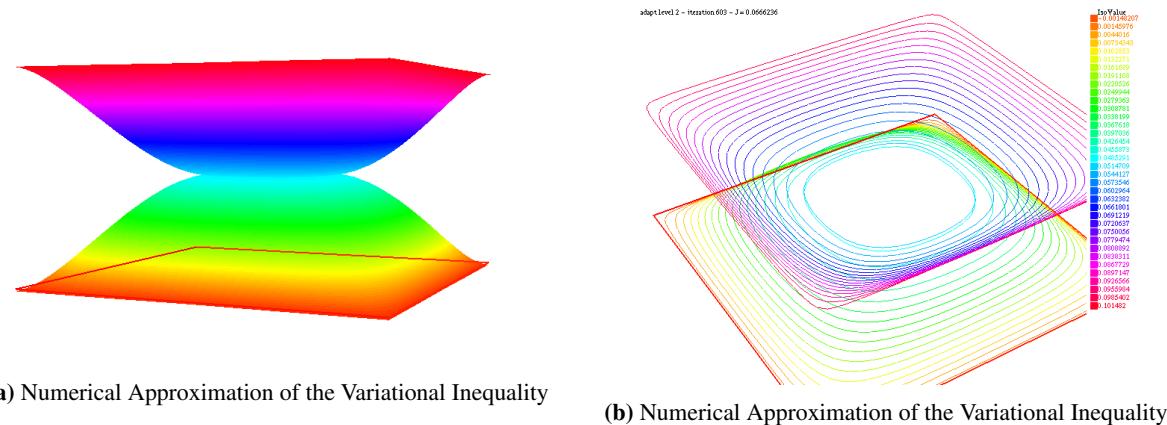
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```

16 Vh [lz, lz2] = [1, 1];
17 Vh [u1, u2] = [0, 0]; //starting point
18
19 fespace Wh(Th, [P1]);
20 Wh lm=1.;
21
22 // Macro
23 macro Grad(u) [dx(u), dy(u)] //
24
25 // Loop
26 int iter=0;
27 while (++iter){
28     // Problem
29     varf vP ([u1, u2], [v1, v2])
30         = int2d(Th) (
31             Grad(u1)'*Grad(v1)
32             + Grad(u2)'*Grad(v2)
33         )
34         - int2d(Th) (
35             f1*v1
36             + f2*v2
37         )
38     ;
39
40     matrix A = vP(Vh, Vh); //fitness function matrix
41     real[int] b = vP(0, Vh); //and linear form
42
43     int[int] II1 = [0], II2 = [1]; //Constraints matrix
44     matrix C1 = interpolate (Wh, Vh, U2Vc=II1);
45     matrix C2 = interpolate (Wh, Vh, U2Vc=II2);
46     matrix CC = -1*C1 + C2; // u2 - u1 > 0
47     Wh cl = 0; //constraints lower bounds (no upper bounds)
48
49 //Boundary conditions
50 varf vGamma ([u1, u2], [v1, v2]) = on(1, 2, 3, 4, u1=1, u2=1);
51 real[int] onGamma = vGamma(0, Vh);
52 Vh [ub1, ub2] = [g1, g2];
53 Vh [lb1, lb2] = [g1, g2];
54 ub1[] = onGamma ? ub1[] : 1e19; //Unbounded in interior
55 lb1[] = onGamma ? lb1[] : -1e19;
56
57 Vh [uzi, uzi2] = [uz, uz2], [lzi, lzi2] = [lz, lz2];
58 Wh lmi = lm;
59 Vh [uil, ui2] = [u1, u2];
60
61 // Solve
62 IPOPT([b, A], CC, uil[], lb=lb1[], clb=cl[], ub=ub1[], warmstart=iter>1, uz=uzi[],
63 ↴lz=lzi[], lm=lmi[]);
64
65 // Plot
66 plot(uil, ui2, wait=true, nbiso=60, dim=3);
67
68 if(iter > 1) break;
69
70 // Mesh adpatation
71 Th = adaptmesh(Th, [uil, ui2], err=0.004, nbvx=100000);
72 [uz, uz2] = [uzi, uzi2];

```

(continues on next page)

**Fig. 3.44:** Variational inequality

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```

72 [lz, lz2] = [lzi, lzi2];
73 [u1, u2] = [ui1, ui2];
74 lm = lmi;
75 }
```

### 3.5.5 3D constrained minimum surface with IPOPT

#### Area and volume expressions

This example is aimed at numerically solving some constrained minimum surface problems with the IPOPT algorithm. We restrain to  $C^k$  ( $k \geq 1$ ), closed, spherically parametrizable surfaces, i.e. surfaces  $S$  such that:

$$\exists \rho \in C^k([0, 2\pi] \times [0, \pi]) | S = \left\{ X = \begin{pmatrix} \rho(\theta, \phi) \\ 0 \\ 0 \end{pmatrix}, (\theta, \phi) \in [0, 2\pi] \times [0, \pi] \right\}$$

Where the components are expressed in the spherical coordinate system. Let's call  $\Omega$  the  $[0, 2\pi] \times [0, \pi]$  angular parameters set. In order to exclude self crossing and opened shapes, the following assumptions upon  $\rho$  are made:

$$\rho \geq 0 \text{ and } \forall \phi, \rho(0, \phi) = \rho(2\pi, \phi)$$

For a given function  $\rho$  the first fundamental form (the metric) of the defined surface has the following matrix representation:

$$G = \begin{pmatrix} \rho^2 \sin^2(\phi) + (\partial_\theta \rho)^2 & \partial_\theta \rho \partial_\phi \rho \\ \partial_\theta \rho \partial_\phi \rho & \rho^2 + (\partial_\phi \rho)^2 \end{pmatrix} \quad (3.23)$$

This metric is used to express the area of the surface. Let  $g = \det(G)$ , then we have:

$$\begin{aligned} \mathcal{A}(\rho) &= \int \Omega \|\partial_\theta X \wedge \partial_\phi X\| = \int \Omega \sqrt{g} \\ &= \int \Omega \sqrt{\rho^2(\partial_\theta \rho)^2 + \rho^4 \sin^2(\phi) + \rho^2(\partial_\phi \rho)^2 \sin^2(\phi)} d\theta d\phi \end{aligned} \quad (3.24)$$

The volume of the space enclosed within the shape is easier to express:

$$\mathcal{V}(\rho) = \int \Omega \int_0^{\rho(\theta, \phi)} r^2 \sin(\phi) dr d\theta d\phi = \frac{1}{3} \int \Omega \rho^3 \sin(\phi) d\theta d\phi \quad (3.25)$$

## Derivatives

In order to use a Newton based interior point optimization algorithm, one must be able to evaluate the derivatives of  $\mathcal{A}$  and  $\mathcal{V}$  with respect to  $\rho$ . Concerning the area, we have the following result:

$$\forall v \in C^1(\Omega), \langle d\mathcal{A}(\rho), v \rangle = \int \Omega \frac{1}{2} \frac{d\bar{g}(\rho)(v)}{\sqrt{g}} d\theta d\phi$$

Where  $\bar{g}$  is the application mapping the  $(\theta, \phi) \mapsto g(\theta, \phi)$  scalar field to  $\rho$ . This leads to the following expression, easy to transpose in a freefem script using:

$$\begin{aligned} \forall v \in C^1(\Omega) \\ \langle d\mathcal{A}(\rho), v \rangle &= \int \Omega (2\rho^3 \sin^2(\phi) + \rho(\partial_\theta \rho)^2 + \rho(\partial_\phi \rho)^2 \sin^2(\phi)) v \\ &\quad + \int \Omega \rho^2 \partial_\theta \rho \partial_\theta v + \rho^2 \partial_\phi \rho \sin^2(\phi) \partial_\phi v \end{aligned} \quad (3.26)$$

With a similar approach, one can derive an expression for second order derivatives. However, comporting no specific difficulties, the detailed calculus are tedious, the result is that these derivatives can be written using a  $3 \times 3$  matrix  $\mathbf{B}$  whose coefficients are expressed in term of  $\rho$  and its derivatives with respect to  $\theta$  and  $\phi$ , such that:

$$\forall (w, v) \in C^1(\Omega), d^2\mathcal{A}(\rho)(w, v) = \int \Omega \begin{pmatrix} w & \partial_\theta w & \partial_\phi w \end{pmatrix} \mathbf{B} \begin{pmatrix} v \\ \partial_\theta v \\ \partial_\phi v \end{pmatrix} d\theta d\phi \quad (3.27)$$

Deriving the volume function derivatives is again an easier task. We immediately get the following expressions:

$$\begin{aligned} \forall v, \langle d\mathcal{V}(\rho), v \rangle &= \int \Omega \rho^2 \sin(\phi) v d\theta d\phi \\ \forall w, v, d^2\mathcal{V}(\rho)(w, v) &= \int \Omega 2\rho \sin(\phi) w v d\theta d\phi \end{aligned} \quad (3.28)$$

## The problem and its script

The whole code is available in [IPOPT minimal surface & volume example](#). We propose to solve the following problem:

**Tip:** Given a positive function  $\rho_{\text{object}}$  piecewise continuous, and a scalar  $\mathcal{V}_{\max} > \mathcal{V}(\rho_{\text{object}})$ , find  $\rho_0$  such that:

$$\rho_0 = \underset{\rho \in C^1(\Omega)}{\operatorname{argmin}} \mathcal{A}(\rho), \text{ s.t. } \rho_0 \geq \rho_{\text{object}} \text{ and } \mathcal{V}(\rho_0) \leq \mathcal{V}_{\max}$$

If  $\rho_{\text{object}}$  is the spherical parametrization of the surface of a 3-dimensional object (domain)  $\mathcal{O}$ , it can be interpreted as finding the surface with minimum area enclosing the object with a given maximum volume. If  $\mathcal{V}_{\max}$  is close to  $\mathcal{V}(\rho_{\text{object}})$ , so should be  $\rho_0$  and  $\rho_{\text{object}}$ . With higher values of  $\mathcal{V}_{\max}$ ,  $\rho$  should be closer to the unconstrained minimum surface surrounding  $\mathcal{O}$  which is obtained as soon as  $\mathcal{V}_{\max} \geq \frac{4}{3}\pi \|\rho_{\text{object}}\|_\infty^3$  (sufficient but not necessary).

It also could be interesting to solve the same problem with the constraint  $\mathcal{V}(\rho_0) \geq \mathcal{V}_{\min}$  which leads to a sphere when  $\mathcal{V}_{\min} \geq \frac{1}{6}\pi \operatorname{diam}(\mathcal{O})^3$  and moves toward the solution of the unconstrained problem as  $\mathcal{V}_{\min}$  decreases.

We start by meshing the domain  $[0, 2\pi] \times [0, \pi]$ , then a periodic P1 finite elements space is defined.

```

1 load "msh3";
2 load "medit";
3 load "ff-Ipopt";
4
5 // Parameters
6 int nadapt = 3;
7 real alpha = 0.9;
8 int np = 30;
9 real regtest;
```

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```

10 int shapeswitch = 1;
11 real sigma = 2*pi/40.;
12 real treshold = 0.1;
13 real e = 0.1;
14 real r0 = 0.25;
15 real rr = 2-r0;
16 real E = 1./(e*e);
17 real RR = 1./(rr*rr);

18
19 // Mesh
20 mesh Th = square(2*np, np, [2*pi*x, pi*y]);
21
22 // Fespace
23 fespace Vh(Th, P1, periodic=[[2, y], [4, y]]);
24 //Initial shape definition
25 //outside of the mesh adaptation loop to initialize with the previous optimial shape
//→found on further iterations
26 Vh startshape = 5;

```

We create some finite element functions whose underlying arrays will be used to store the values of dual variables associated to all the constraints in order to reinitialize the algorithm with it in the case where we use mesh adaptation. Doing so, the algorithm will almost restart at the accuracy level it reached before mesh adaptation, thus saving many iterations.

```

1 Vh uz = 1., lz = 1.;
2 rreal[int] lm = [1];

```

Then, follows the mesh adaptation loop, and a rendering function, Plot3D, using 3D mesh to display the shape it is passed with medit (the movemesh23 procedure often crashes when called with ragged shapes).

```

1 for(int kkk = 0; kkk < nadapt; ++kkk) {
2     int iter=0;
3     func sin2 = square(sin(y));
4
5     // A function which transform Th in 3d mesh (r=rho)
6     //a point (theta,phi) of Th becomes ( r(theta,phi)*cos(theta)*sin(phi) , r(theta,
//→phi)*sin(theta)*sin(phi) , r(theta,phi)*cos(phi) )
7     //then displays the resulting mesh with medit
8     func int Plot3D (real[int] &rho, string cmm, bool ffplot) {
9         Vh rho;
10        rhoo[] = rho;
11        //mesh sTh = square(np, np/2, [2*pi*x, pi*y]);
12        //fespace svh(sTh, P1);
13        //Vh rhoplot = rho;
14        try{
15            mesh3 Sphere = movemesh23(Th, transfo=[rhoo(x,y)*cos(x)*sin(y), rhoo(x,
//→y)*sin(x)*sin(y), rhoo(x,y)*cos(y)]);
16            if(ffplot)
17                plot(Sphere);
18            else
19                medit(cmm, Sphere);
20        }
21        catch(...){
22            cout << "PLOT ERROR" << endl;
23        }
24        return 1;

```

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```
25     }
26 }
```

Here are the functions related to the area computation and its shape derivative, according to equations (3.24) and (3.26):

```
1 // Surface computation
2 //Maybe is it possible to use movemesh23 to have the surface function less complicated
3 //However, it would not simplify the gradient and the hessian
4 func real Area (real[int] &X) {
5     Vh rho;
6     rho[] = X;
7     Vh rho2 = square(rho);
8     Vh rho4 = square(rho2);
9     real res = int2d(Th) (sqrt(rho4*sin2 + rho2*square(dx(rho)) +_
10    rho2*sin2*square(dy(rho)))) ;
11    ++iter;
12    if(1)
13        plot(rho, value=true, fill=true, cmm="rho(theta,phi) on [0,2pi]*[0,pi] - S="_
14    "+res, dim=3);
15    else
16        Plot3D(rho[], "shape_evolution", 1);
17    return res;
18 }
19
20 func real[int] GradArea (real[int] &X) {
21     Vh rho, rho2;
22     rho[] = X;
23     rho2[] = square(X);
24     Vh sqrtPsi, alpha;
25     {
26         Vh dxrho2 = dx(rho)*dx(rho), dyrho2 = dy(rho)*dy(rho);
27         sqrtPsi = sqrt(rho2*rho2*sin2 + rho2*dxrho2 + rho2*dyrho2*sin2);
28         alpha = 2.*rho2*rho*sin2 + rho*dxrho2 + rho*dyrho2*sin2;
29     }
30     varf dArea (u, v)
31     = int2d(Th) (
32         1./sqrtPsi * (alpha*v + rho2*dx(rho)*dx(v) + rho2*dy(rho)*sin2*dy(v))
33     )
34     ;
35
36     real[int] grad = dArea(0, Vh);
37     return grad;
38 }
```

The function returning the hessian of the area for a given shape is a bit blurry, thus we won't show here all of equation (3.27) coefficients definition, they can be found in the edp file.

```
1 matrix hessianA;
2 func matrix HessianArea (real[int] &X) {
3     Vh rho, rho2;
4     rho[] = X;
5     rho2 = square(rho);
6     Vh sqrtPsi, sqrtPsi3, C00, C01, C02, C11, C12, C22, A;
7     {
8         Vh C0, C1, C2;
```

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```

9   Vh dxrho2 = dx(rho)*dx(rho), dyrho2 = dy(rho)*dy(rho);
10  sqrtPsi = sqrt( rho2*rho2*sin2 + rho2*dxrho2 + rho2*dyrho2*sin2 );
11  sqrtPsi3 = (rho2*rho2*sin2 + rho2*dxrho2 + rho2*dyrho2*sin2)*sqrtPsi;
12  C0 = 2.*rho2*rho*sin2 + rho*dxrho2 + rho*dyrho2*sin2;
13  C1 = rho2*dx(rho);
14  C2 = rho2*sin2*dy(rho);
15  C00 = square(C0);
16  C01 = C0*C1;
17  C02 = C0*C2;
18  C11 = square(C1);
19  C12 = C1*C2;
20  C22 = square(C2);
21  A = 6.*rho2*sin2 + dxrho2 + dyrho2*sin2;
22 }
23 varf d2Area (w, v)
24 =int2d(Th) (
25   1./sqrtPsi * (
26     A*w*v
27     + 2.*rho*dx(rho)*dx(w)*v
28     + 2.*rho*dx(rho)*w*dx(v)
29     + 2.*rho*dy(rho)*sin2*dy(w)*v
30     + 2.*rho*dy(rho)*sin2*w*dy(v)
31     + rho2*dx(w)*dx(v)
32     + rho2*sin2*dy(w)*dy(v)
33   )
34   + 1./sqrtPsi3 * (
35     C00*w*v
36     + C01*dx(w)*v
37     + C01*w*dx(v)
38     + C02*dy(w)*v
39     + C02*w*dy(v)
40     + C11*dx(w)*dx(v)
41     + C12*dx(w)*dy(v)
42     + C12*dy(w)*dx(v)
43     + C22*dy(w)*dy(v)
44   )
45 )
46 ;
47 hessianA = d2Area(Vh, Vh);
48 return hessianA;
49 }
```

And the volume related functions:

```

1 // Volume computation
2 func real Volume (real[int] &X) {
3   Vh rho;
4   rho[] = X;
5   Vh rho3 = rho*rho*rho;
6   real res = 1./3.*int2d(Th)(rho3*sin(y));
7   return res;
8 }
9
10 func real[int] GradVolume (real[int] &X) {
11   Vh rho;
12   rho[] = X;
13   varf dVolume(u, v) = int2d(Th)(rho*rho*sin(y)*v);
```

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```

14   real[int] grad = dVolume(0, Vh);
15   return grad;
16 }
17
18 matrix hessianV;
19 func matrix HessianVolume(real[int] &X) {
20   Vh rho;
21   rho[] = X;
22   varf d2Volume(w, v) = int2d(Th)(2*rho*sin(y)*v*w);
23   hessianV = d2Volume(Vh, Vh);
24   return hessianV;
25 }
```

If we want to use the volume as a constraint function we must wrap it and its derivatives in some **FreeFEM** functions returning the appropriate types. It is not done in the above functions in cases where one wants to use it as a fitness function. The lagrangian hessian also has to be wrapped since the Volume is not linear with respect to  $\rho$ , it has some non-null second order derivatives.

```

1 func real[int] ipVolume (real[int] &X){ real[int] vol = [Volume(X)]; return vol; }
2 matrix mdV;
3 func matrix ipGradVolume (real[int] &X) { real[int,int] dvol(1,Vh.ndof); dvol(0,:)=_
→GradVolume(X); mdV = dvol; return mdV; }
4 matrix HLagrangian;
5 func matrix ipHessianLag (real[int] &X, real objfact, real[int] &lambda) {
6   HLagrangian = objfact*HessianArea(X) + lambda[0]*HessianVolume(X);
7   return HLagrangian;
8 }
```

The `ipGradVolume` function could pose some troubles during the optimization process because the gradient vector is transformed in a sparse matrix, so any null coefficient will be discarded. Here we create the IPOPT structure manually and use the `checkindex` named-parameter to avoid bad indexing during copies. This gradient is actually dense, there is no reason for some components to be constantly zero:

```

1 int[int] gvi(Vh.ndof), gvj=0:Vh.ndof-1;
2 gvi = 0;
```

These two arrays will be passed to IPOPT with `struct jacc=[gvi,gvj]`. The last remaining things are the bound definitions. The simple lower bound must be equal to the components of the P1 projection of  $\rho_{object}$ . And we choose  $\alpha \in [0, 1]$  to set  $\mathcal{V}_{\max}$  to  $(1 - \alpha)\mathcal{V}(\rho_{object}) + \alpha\frac{4}{3}\pi\|\rho_{object}\|_\infty^3$ :

```

1 func disc1 = sqrt(1./ (RR+ (E-RR) *cos(y) *cos(y)))*(1+0.1*cos(7*x));
2 func disc2 = sqrt(1./ (RR+ (E-RR) *cos(x) *cos(x) *sin2));
3
4 if(1){
5   lb = r0;
6   for (int q = 0; q < 5; ++q) {
7     func f = rr*Gaussian(x, y, 2*q*pi/5., pi/3.);
8     func g = rr*Gaussian(x, y, 2*q*pi/5.+pi/5., 2.*pi/3.);
9     lb = max(max(lb, f), g);
10  }
11  lb = max(lb, rr*Gaussian(x, y, 2*pi, pi/3));
12 }
13 lb = max(lb, max(disc1, disc2));
14 real Vobj = Volume(lb[]);
15 real Vnvc = 4./3.*pi*pow(lb[].linfty, 3);
16
```

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```

17 if(1)
18   Plot3D(lb[], "object_inside", 1);
19   real[int] clb = 0., cub = [(1-alpha)*Vobj + alpha*Vnvc];

```

Calling IPOPT:

```

1 int res = IPOPT(Area, GradArea, ipHessianLag, ipVolume, ipGradVolume,
2   rc[], ub=ub[], lb=lb[], clb=clb, cub=cub, checkindex=1, maxiter=kkk<nadapt-1 ?_
3   ↪40:150,
4   warmstart=kkk, lm=lm, uz=uz[], lz=lz[], tol=0.00001, struct jacc=[gvi,gvj]);
5   cout << "IPOPT: res =" << res << endl ;
6
7 // Plot
8 Plot3D(rc[], "Shape_at_"+kkk, 1);
9 Plot3D(GradArea(rc[]), "ShapeGradient", 1);

```

Finally, before closing the mesh adaptation loop, we have to perform the said adaptation. The mesh is adaptated with respect to the  $X = (\rho, 0, 0)$  (in spherical coordinates) vector field, not directly with respect to  $\rho$ , otherwise the true curvature of the 3D-shape would not be well taken into account.

```

1 if (kkk < nadapt-1){
2   Th = adaptmesh(Th, rc*cos(x)*sin(y), rc*sin(x)*sin(y), rc*cos(y),
3   ↪nbvx=50000, periodic=[[2, y], [4, y]]);
4   plot(Th, wait=true);
5   startshape = rc;
6   uz = uz;
7   lz = lz;
8 }

```

Here are some pictures of the resulting surfaces obtained for decreasing values of  $\alpha$  (and a slightly more complicated object than two orthogonal discs). We return to the enclosed object when  $\alpha = 0$ :

### 3.5.6 The nlopt optimizers

The ff-Nlopt package provides a **FreeFEM** interface to the free/open-source library for nonlinear optimization, easing the use of several different free optimization (constrained or not) routines available online along with the PDE solver. All the algorithms are well documented in [Nlopt documentation](#), therefore no exhaustive information concerning their mathematical specificities will be found here and we will focus on the way they are used in a **FreeFEM** script. If needing detailed information about these algorithms, visit the website where a description of each of them is given, as well as many bibliographical links.

Most of the gradient based algorithms of Nlopt uses a full matrix approximation of the Hessian, so if you're planning to solve a large scale problem, use the IPOPT optimizer which definitely surpass them.

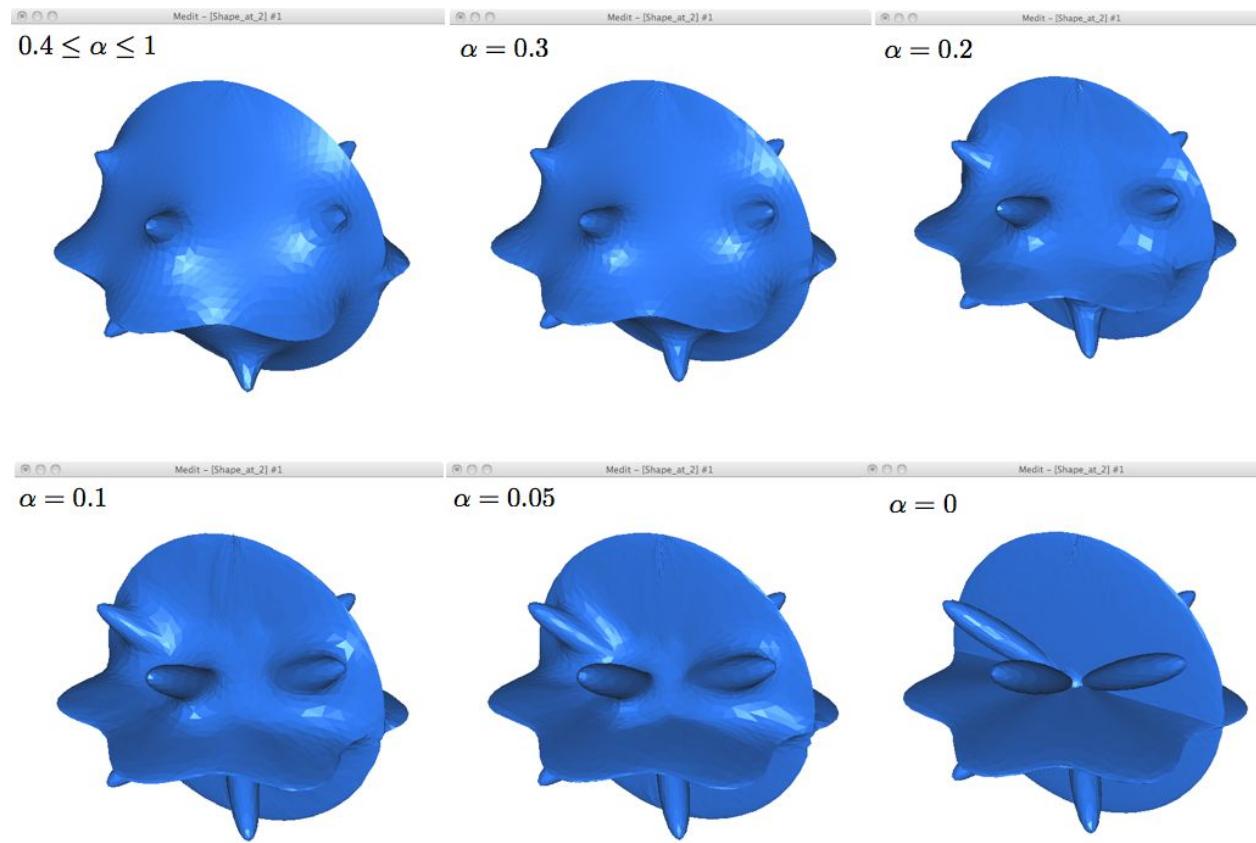
All the Nlopt features are identified that way:

```

1 load "ff-Nlopt"
2 //define J, u, and maybe grad(J), some constraints etc...
3 real min = nloptXXXXXX(J, u, //Unavoidable part
4   grad=<name of grad(J)>, //if needed
5   lb= //Lower bounds array
6   ub= //Upper bounds array
7   ... //Some optional arguments:
8   //Constraints functions names,

```

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```

9 //Stopping criteria,
10 //Algorithm specific parameters,
11 //Etc...
12 );

```

XXXXXX refers to the algorithm tag (not necessarily 6 characters long).  $\mathbf{u}$  is the starting position (a `real[int]` type array) which will be overwritten by the algorithm, the value at the end being the found *argmin*. And as usual,  $J$  is a function taking a `real[int]` type array as argument and returning a `real`. `grad`, `lb` and `ub` are “half-optional” arguments, in the sense that they are obligatory for some routines but not all.

The possible optionally named parameters are the following, note that they are not used by all algorithms (some do not support constraints, or a type of constraints, some are gradient-based and others are derivative free, etc...). One can refer to the table after the parameters description to check which are the named parameters supported by a specific algorithm. Using an unsupported parameter will not stop the compiler work, seldom breaks runtime, and will just be ignored. When it is obvious you are missing a routine, you will get a warning message at runtime (for example if you pass a gradient to a derivative free algorithm, or set the population of a non-genetic one, etc...). In the following description,  $n$  stands for the dimension of the search space.

#### Half-optional parameters :

- `grad`= The name of the function which computes the gradient of the cost function (prototype should be `real[int] → real[int]`, both argument and result should have the size  $n$ ). This is needed as soon as a gradient-based method is involved, which is ignored if defined in a derivative free context.
- `lb/ub` = Lower and upper bounds arrays (`real[int]` type) of size  $n$ . Used to define the bounds within which the search variable is allowed to move. Needed for some algorithms, optional, or unsupported for others.
- `subOpt` : Only enabled for the Augmented Lagrangian and MSL methods who need a sub-optimizer in order to work. Just pass the tag of the desired local algorithm with a `string`.

#### Constraints related parameters (optional - unused if not specified):

- `IConst/EConst` : Allows to pass the name of a function implementing some inequality (resp. equality) constraints on the search space. The function type must be `real[int] → real[int]` where the size of the returned array is equal to the number of constraints (of the same type - it means that all of the constraints are computed in one vectorial function). In order to mix inequality and equality constraints in a same minimization attempt, two vectorial functions have to be defined and passed. See example (3.29) for more details about how these constraints have to be implemented.
- `gradIConst/gradEConst` : Use to provide the inequality (resp. equality) constraints gradient. These are `real[int] → real[int,int]` type functions. Assuming we have defined a constraint function (either inequality or equality) with  $p$  constraints, the size of the matrix returned by its associated gradient must be  $p \times n$  (the  $i$ -th line of the matrix is the gradient of the  $i$ -th constraint). It is needed in a gradient-based context as soon as an inequality or equality constraint function is passed to the optimizer and ignored in all other cases.
- `tolIConst/tolEConst` : Tolerance values for each constraint. This is an array of size equal to the number of inequality (resp. equality) constraints. Default value is set to  $10^{-12}$  for each constraint of any type.

#### Stopping criteria :

- `stopFuncValue` : Makes the algorithm end when the objective function reaches this `real` value.
- `stopRelXTol` : Stops the algorithm when the relative moves in each direction of the search space is smaller than this `real` value.
- `stopAbsXTol` : Stops the algorithm when the moves in each direction of the search space is smaller than the corresponding value in this `real[int]` array.
- `stopRelFTol` : Stops the algorithm when the relative variation of the objective function is smaller than this `real` value.

- `stopAbsFTol` : Stops the algorithm when the variation of the objective function is smaller than this `real` value.
- `stopMaxFEval` : Stops the algorithm when the number of fitness evaluations reaches this `integer` value.
- `stopTime` : Stops the algorithm when the optimization time in seconds exceeds this `real` value. This is not a strict maximum: the time may exceed it slightly, depending upon the algorithm and on how slow your function evaluation is.

Note that when an AUGLAG or MLSL method is used, the meta-algorithm and the sub-algorithm may have different termination criteria. Thus, for algorithms of this kind, the following named parameters has been defined (just adding the SO prefix - for Sub-Optimizer) to set the ending condition of the sub-algorithm (the meta one uses the ones above): `SOSTopFuncValue`, `SOSTopRelXTol`, and so on... If these are not used, the sub-optimizer will use those of the master routine.

#### Other named parameters :

- `popSize` : `integer` used to change the size of the sample for stochastic search methods. Default value is a peculiar heuristic to the chosen algorithm.
- `SOPopSize` : Same as above, but when the stochastic search is passed to a meta-algorithm.
- `nGradStored` : The number (`integer` type) of gradients to “remember” from previous optimization steps: increasing this increases the memory requirements but may speed convergence. It is set to a heuristic value by default. If used with AUGLAG or MLSL, it will only affect the given subsidiary algorithm.

The following table sums up the main characteristics of each algorithm, providing the more important information about which features are supported by which algorithm and what are the unavoidable arguments they need. More details can be found in [NLOpt documentation](#).

#### Tip: Variational inequality

Let  $\Omega$  be a domain of  $\mathbb{R}^2$ ,  $f_1, f_2 \in L^2(\Omega)$  and  $g_1, g_2 \in L^2(\partial\Omega)$  four given functions with  $g_1 \leq g_2$  almost everywhere.

We define the space:

$$V = \{(v_1, v_2) \in H^1(\Omega)^2; v_1|_{\partial\Omega} = g_1, v_2|_{\partial\Omega} = g_2, v_1 \leq v_2 \text{ a.e.}\}$$

as well as the function  $J : H^1(\Omega)^2 \rightarrow \mathbb{R}$ :

$$J(v_1, v_2) = \frac{1}{2} \int_{\Omega} |\nabla v_1|^2 - \int_{\Omega} f_1 v_1 + \frac{1}{2} \int_{\Omega} |\nabla v_2|^2 - \int_{\Omega} f_2 v_2 \quad (3.29)$$

The problem consists in finding (numerically) two functions  $(u_1, u_2) = \underset{(v_1, v_2) \in V}{\operatorname{argmin}} J(v_1, v_2)$ .

This can be interpreted as finding  $u_1, u_2$  as close as possible (in a certain sense) to the solutions of the Laplace equation with respectively  $f_1, f_2$  second members and  $g_1, g_2$  Dirichlet boundary conditions with the  $u_1 \leq u_2$  almost everywhere constraint.

Here is the corresponding script to treat this variational inequality problem with one of the NLOpt algorithms.

```

1 //A brief script to demonstrate how to use the freefemm interfaced nlopt routines
2 //The problem consist in solving a simple variational inequality using one of the
3 //optimization algorithm of nlopt. We restart the algorithlm a few times after
4 //performing some mesh adaptation to get a more precise output
5
6 load "ff-NLOpt"
7
8 // Parameters
9 int kas = 3; //choose of the algorithm

```

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Id Tag	Full Name	Bounds	Gradient-Based	Stochastic	Constraints	Sub-Opt
					Equality	Inequality
DIRECT	Dividing rectangles	●				
DIRECTL	Locally biased dividing rectangles	●				
DIRECTLRand	Randomized locally biased dividing rectangles	●				
DIRECTNoScal	Dividing rectangles - no scaling	●				
DIRECTLNoScal	Locally biased dividing rectangles - no scaling	●				
DIRECTLRandNoScal	Randomized locally biased dividing rectangles - no scaling	●				
OrigDIRECT	Original Glabonsky's dividing rectangles	●			✓	
OrigDIRECTL	Original Glabonsky's locally biased dividing rectangles	●			✓	
StoGO	Stochastic(?) Global Optimization	●	●			
StoGORand	Randomized Stochastic(?) Global Optimization	●	●			
LBFGS	Low-storage BFGS		●			
PRAxis	Principal AXIS	✓				
Var1	Rank-1 shifted limited-memory variable-metric		●			
Var2	Rank-2 shifted limited-memory variable-metric		●			
TNewton	Truncated Newton	●				
TNewtonRestart	Steepest descent restarting truncated Newton	●				
TNewtonPrecond	BFGS preconditionned truncated Newton	●				
TNewtonRestartPrecond	BFGS preconditionned truncated Newton with steepest descent restarting	●				
CRS2	Controlled random search with local mutation	✓		●		
MMA	Method of moving asymptots	✓	●		✓	
COBYLA	Constrained optimization by linear approximations	✓			✓	✓
NEWUOA	NEWUOA					
NEWUOABound	NEWUOA for bounded optimization	✓				
NelderMead	Nelder-Mead simplex	✓				
Sbplx	Subplex	✓				
BOBYQA	BOBYQA	✓				
ISRES	Improved stochastic ranking evolution strategy	✓		●	✓	✓
SLSQP	Sequential least-square quadratic programming	✓	●		✓	✓
MLSL	Multi-level single-linkage	✓	●	●		●
MLSLDS	Low discrepancy multi-level single-linkage	✓	●	●		●
AUGLAG	Constraints augmented lagrangian	✓	●		✓	✓
AUGLAGEQ	Equality constraints augmented lagrangian	✓	●		✓	●

Legend :

- ✓ Supported and optional
- ✗ Should be supported and optional, may lead to weird behaviour though.
- ✗ Intrinsic characteristic of the algorithm which then need one or more unavoidable parameter to work (for stochastic algorithm, the population size always have a default value, they will then work if it is omitted)
- For routines with subsidiary algorithms only, indicates that the corresponding feature will depend on the chosen sub-optimizer.
- /✓ For routines with subsidiary algorithms only, indicates that the corresponding feature will depend on the chosen sub-optimizer.

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```

10 int NN = 10;
11 func f1 = 1.;
12 func f2 = -1.;
13 func g1 = 0.;
14 func g2 = 0.1;
15 int iter = 0;
16 int nadapt = 2;
17 real starttol = 1e-6;
18 real bctol = 6.e-12;
19
20 // Mesh
21 mesh Th = square(NN, NN);
22
23 // Fespace
24 fespace Vh(Th, P1);
25 Vh oldu1, oldu2;
26
27 // Adaptation loop
28 for (int al = 0; al < nadapt; ++al){
29     varf BVF (v, w) = int2d(Th) (0.5*dx(v)*dx(w) + 0.5*dy(v)*dy(w));
30     varf LVF1 (v, w) = int2d(Th) (f1*w);
31     varf LVF2 (v, w) = int2d(Th) (f2*w);
32     matrix A = BVF(Vh, Vh);
33     real[int] b1 = LVF1(0, Vh), b2 = LVF2(0, Vh);
34
35     varf Vbord (v, w) = on(1, 2, 3, 4, v=1);
36
37     Vh In, Bord;
38     Bord[] = Vbord(0, Vh, tgv=1);
39     In[] = Bord[] ? 0:1;
40     Vh gh1 = Bord*g1, gh2 = Bord*g2;
41
42     func real J (real[int] &X) {
43         Vh u1, u2;
44         u1[] = X(0:Vh.ndof-1);
45         u2[] = X(Vh.ndof:2*Vh.ndof-1);
46         iter++;
47         real[int] Au1 = A*u1[], Au2 = A*u2[];
48         Au1 -= b1;
49         Au2 -= b2;
50         real val = u1[]'*Au1 + u2[]'*Au2;
51         if (iter%10 == 9)
52             plot(u1, u2, nbiso=30, fill=1, dim=3, cmm="adapt level "+al+" - iteration
53             "+iter+" - J = "+val, value=1);
54         return val;
55     }
56
57     varf dBVF (v, w) = int2d(Th) (dx(v)*dx(w)+dy(v)*dy(w));
58     matrix dA = dBVF(Vh, Vh);
59     func real[int] dJ (real[int] &X) {
60         Vh u1, u2;
61         u1[] = X(0:Vh.ndof-1);
62         u2[] = X(Vh.ndof:2*Vh.ndof-1);
63
64         real[int] grad1 = dA*u1[], grad2 = dA*u2[];
65         grad1 -= b1;
66         grad2 -= b2;

```

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```

66   real[int] Grad(X.n);
67   Grad(0:Vh.ndof-1) = grad1;
68   Grad(Vh.ndof:2*Vh.ndof-1) = grad2;
69   return Grad;
70 }
71
72 func real[int] IneqC (real[int] &X) {
73   real[int] constraints(Vh.ndof);
74   for (int i = 0; i < Vh.ndof; ++i) constraints[i] = X[i] - X[i+Vh.ndof];
75   return constraints;
76 }
77
78 func real[int,int] dIneqC (real[int] &X) {
79   real[int, int] dconst(Vh.ndof, 2*Vh.ndof);
80   dconst = 0;
81   for(int i = 0; i < Vh.ndof; ++i){
82     dconst(i, i) = 1.;
83     dconst(i, i+Vh.ndof) = -1.;
84   }
85   return dconst;
86 }
87
88 real[int] BordIndex(Th.nbe); //Indexes of border d.f.
89 {
90   int k = 0;
91   for (int i = 0; i < Bord.n; ++i) if (Bord[][i]) { BordIndex[k] = i; ++k; }
92 }
93
94 func real[int] BC (real[int] &X) {
95   real[int] bc(2*Th.nbe);
96   for (int i = 0; i < Th.nbe; ++i){
97     int I = BordIndex[i];
98     bc[i] = X[I] - ghl[] [I];
99     bc[i+Th.nbe] = X[I+Th.nv] - gh2[] [I];
100   }
101   return bc;
102 }
103
104 func real[int, int] dBC(real[int] &X) {
105   real[int, int] dbc(2*Th.nbe, 2*Th.nv);
106   dbc = 0.;
107   for (int i = 0; i < Th.nbe; ++i){
108     int I = BordIndex[i];
109     dbc(i, I) = 1.;
110     dbc(i+Th.nbe, I+Th.nv) = 1.;
111   }
112   return dbc;
113 }
114
115 real[int] start(2*Vh.ndof), up(2*Vh.ndof), lo(2*Vh.ndof);
116
117 if (al == 0){
118   start(0:Vh.ndof-1) = 0.;
119   start(Vh.ndof:2*Vh.ndof-1) = 0.01;
120 }
121 else{
122   start(0:Vh.ndof-1) = oldul[];

```

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```

123     start (Vh.ndof:2*Vh.ndof-1) = oldu2[];
124 }
125
126 up = 1000000;
127 lo = -1000000;
128 for (int i = 0; i < Vh.ndof; ++i){
129     if (Bord[][][i]){
130         up[i] = gh1[][i] + bctol;
131         lo[i] = gh1[][i] - bctol;
132         up[i+Vh.ndof] = gh2[][i] + bctol;
133         lo[i+Vh.ndof] = gh2[][i] - bctol;
134     }
135 }
136
137 real mini = 1e100;
138 if (kas == 1)
139     mini = nloptAUGLAG(J, start, grad=dJ, lb=lo,
140                         ub=up, IConst=IneqC, gradIConst=dIneqC,
141                         subOpt="LBFGS", stopMaxFEval=10000, stopAbsFTol=starttol);
142 else if (kas == 2)
143     mini = nloptMMA(J, start, grad=dJ, lb=lo, ub=up, stopMaxFEval=10000, ↴
144     stopAbsFTol=starttol);
145 else if (kas == 3)
146     mini = nloptAUGLAG(J, start, grad=dJ, IConst=IneqC,
147                         gradIConst=dIneqC, EConst=BC, gradEConst=dBC,
148                         subOpt="LBFGS", stopMaxFEval=200, stopRelXTol=1e-2);
149 else if (kas == 4)
150     mini = nloptSQP(J, start, grad=dJ, IConst=IneqC,
151                     gradIConst=dIneqC, EConst=BC, gradEConst=dBC,
152                     stopMaxFEval=10000, stopAbsFTol=starttol);
153 Vh best1, best2;
154 best1[] = start(0:Vh.ndof-1);
155 best2[] = start(Vh.ndof:2*Vh.ndof-1);
156
157 Th = adaptmesh(Th, best1, best2);
158 oldu1 = best1;
159 oldu2 = best2;
}

```

### 3.5.7 Optimization with MPI

The only quick way to use the previously presented algorithms on a parallel architecture lies in parallelizing the used cost function (which is in most real life cases, the expensive part of the algorithm). Somehow, we provide a parallel version of the CMA-ES algorithm. The parallelization principle is the trivial one of evolving/genetic algorithms: at each iteration the cost function has to be evaluated  $N$  times without any dependence at all, these  $N$  calculus are then equally distributed to each process. Calling the MPI version of CMA-ES is nearly the same as calling its sequential version (a complete example of use can be found in the [CMAES MPI variational inequality example](#)):

```

1 load "mpi-cmaes"
2 ... // Define J, u and all here
3 real min = cmaesMPI(J, u, stopTolFun=1e-6, stopMaxIter=3000);
4 cout << "minimum value is " << min << " for u = " << u << endl;

```

If the population size is not changed using the `popsize` parameter, it will use the heuristic value slightly changed to be equal to the closest greatest multiple of the size of the communicator used by the optimizer. The `FreeFEM mpicommworld` is used by default. The user can specify his own MPI communicator with the named parameter `comm=`, see the MPI section of this manual for more information about communicators in `FreeFEM`.

## 3.6 Parallelization

A first attempt of parallelization of `FreeFEM` is made here with **MPI**. An extended interface with MPI has been added to `FreeFEM` version 3.5, (see the [MPI documentation](#) for the functionality of the language).

### 3.6.1 MPI

#### MPI Keywords

The following keywords and concepts are used:

- `mpiComm` to defined a *communication world*
- `mpiGroup` to defined a group of *processors* in the communication world
- `mpiRequest` to defined a request to wait for the end of the communication

#### MPI Constants

- `mpisize` The total number of *processes*,
- `mpirank` the id-number of my current process in  $\{0, \dots, \text{mpisize}-1\}$ ,
- `mpiUndefined` The `MPI_UNDEFINED` constant,
- `mpiAnySource` The `MPI_ANY_SOURCE` constant,
- `mpiCommWorld` The `MPI_COMM_WORLD` constant,
- [...] and all the keywords of `MPI_Op` for the *reduce* operator: `mpiMAX`, `mpiMIN`, `mpiSUM`, `mpiPROD`, `mpiLAND`, `mpiLOR`, `mpiLXOR`, `mpiBAND`, `mpiBXOR`.

#### MPI Constructor

```

1 // Parameters
2 int[int] proc1 = [1, 2], proc2 = [0, 3];
3 int color = 1;
4 int key = 1;
5
6 // MPI ranks
7 cout << "MPI rank = " << mpirank << endl;
8
9 // MPI
10 mpiComm comm(mpicommWorld, 0, 0); //set a MPI_Comm to MPI_COMM_WORLD
11
12 mpiGroup grp(proc1); //set MPI_Group to proc 1,2 in MPI_COMM_WORLD
13 mpiGroup grp1(comm, proc1); //set MPI_Group to proc 1,2 in comm
14
15 mpiComm ncomm1(mpicommWorld, grp); //set the MPI_Comm form grp

```

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```

16
17 mpiComm ncomm2(comm, color, key); //MPI_Comm_split(MPI_Comm comm, int color, int key, ↵
18   ↵MPI_Comm *ncomm)
19
20 mpiRequest rq; //defined an MPI_Request
21 mpiRequest[int] arr(10); //defined an array of 10 MPI_Request

```

## MPI Functions

```

1 mpiComm Comm(mpiCommWorld, 0, 0);
2
3 int MPICommSize = mpiSize(Comm);
4 int MPIRank = mpiRank(Comm);
5
6 if (MPIRank == 0) cout << "MPI Comm size = " << MPICommSize << endl;
7 cout << "MPI rank in Comm = " << mpiRank(Comm) << endl;
8
9 mpiRequest Req;
10 mpiRequest[int] ReqArray(10);
11
12 for (int i = 0; i < MPICommSize; i++) {
13   //return processor i with no Resquest in MPI_COMM_WORLD
14   processor(i);
15   //return processor any source with no Resquest in MPI_COMM_WORLD
16   processor(mpiAnySource);
17   //return processor i with no Resquest in Comm
18   processor(i, Comm);
19   //return processor i with no Resquest in Comm
20   processor(Comm, i);
21   //return processor i with Request rq in Comm
22   /* processor(i, Req, Comm);
23   //return processor i with Request rq in MPI_COMM_WORLD
24   processor(i, Req); */
25   //return processor i in MPI_COMM_WORLD in block mode for synchronously ↵
26   //communication
27   processorblock(i);
28   //return processor any source in MPI_COMM_WORLD in block mode for synchronously ↵
29   //communication
30   processorblock(mpiAnySource);
31   //return processor i in in Comm in block mode
32   processorblock(i, Comm);
33 }
34
35 mpiBarrier(Comm); //do a MPI_Barrier on communicator Comm
36 mpiWaitAny(ReqArray); //wait add of Request array,
37 mpiWait(Req); //wait on a Request
38 real t = mpiWtime(); //return MPIWtime in second
39 real tick = mpiWtick(); //return MPIWTick in second

```

where a processor is just a integer rank, pointer to a MPI\_comm and pointer to a MPI\_Request, and processorblock with a special MPI\_Request.

## MPI Communicator operator

```

1 int status; //to get the MPI status of send / recv
2 real a, b;
3
4 mpiComm comm(mpiCommWorld, 0, 0);
5 mpiRequest req;
6
7 //send a,b asynchronously to the process 1
8 processor(1) << a << b;
9 //receive a,b synchronously from the process 10
10 processor(10) >> a >> b;
11
12 //broadcast from processor of comm to other comm processor
13 // broadcast(processor(10, comm), a);
14 //send synchronously to the process 10 the data a
15 status = Send(processor(10, comm), a);
16 //receive synchronously from the process 10 the data a
17 status = Recv(processor(10, comm), a);
18
19 //send asynchronously to the process 10 the data a without request
20 status = Isend(processor(10, comm), a);
21 //send asynchronously to the process 10 the data a with request
22 status = Isend(processor(10, comm, req), a);
23 //receive asynchronously from the process 10 the data a
24 status = Irecv(processor(10, req), a);
25 //Error asynchronously without request.
26 // status = Irecv(processor(10), a);

```

where the data type of a can be of type of int, real, complex, int[int], real[int], complex[int], int[int,int], double[int,int], complex[int,int], mesh, mesh3, mesh[int], mesh3[int], matrix, matrix<complex>

```

1 //send asynchronously to the process 10 the data a with request
2 processor(10, req) << a ;
3 //receive asynchronously from the process 10 the data a with request
4 processor(10, req) >> a ;

```

If a, b are arrays or full matrices of int, real, or complex, we can use the following MPI functions:

```

1 mpiAlltoall(a, b, [comm]);
2 mpiAllgather(a, b, [comm]);
3 mpiGather(a, b, processor(..));
4 mpiScatter(a, b, processor(..));
5 mpiReduce(a, b, processor(..), mpiMAX);
6 mpiAllReduce(a, b, comm, mpiMAX);

```

Thank you to Guy-Antoine Atenekeng Kahou for his help to code this interface.

## Schwarz example in parallel

This example is a rewriting of example *Schwarz overlapping*.

```

1 ff-mpirun -np 2 SchwarzParallel.edp
2 # OR
3 mpirun -np 2 FreeFem++-mpi SchwarzParallel.edp

```

```

1  if (mpisize != 2) {
2      cout << " sorry, number of processors !=2 " << endl;
3      exit(1);
4  }
5
6 // Parameters
7 verbosity = 0;
8 int interior = 2;
9 int exterior = 1;
10 int n = 4;
11
12 // Mesh
13 border a(t=1, 2){x=t; y=0; label=exterior;}
14 border b(t=0, 1){x=2; y=t; label=exterior;}
15 border c(t=2, 0){x=t; y=1; label=exterior;}
16 border d(t=1, 0){x=1-t; y=t; label=interior;}
17 border e(t=0, pi/2){x=cos(t); y=sin(t); label=interior;}
18 border e1(t=pi/2, 2*pi){x=cos(t); y=sin(t); label=exterior;}
19 mesh[int] Th(mpisize);
20 if (mpirank == 0)
21     Th[0] = buildmesh(a(5*n) + b(5*n) + c(10*n) + d(5*n));
22 else
23     Th[1] = buildmesh(e(5*n) + e1(25*n));
24
25 broadcast(processor(0), Th[0]);
26 broadcast(processor(1), Th[1]);
27
28 // Fespace
29 fespace Vh(Th[mpirank], P1);
30 Vh u = 0, v;
31
32 fespace Vhother(Th[1-mpirank], P1);
33 Vhother U = 0;
34
35 //Problem
36 int i = 0;
37 problem pb (u, v, init=i, solver=Cholesky)
38     = int2d(Th[mpirank])(
39         dx(u)*dx(v)
40         + dy(u)*dy(v)
41     )
42     - int2d(Th[mpirank])(
43         v
44     )
45     + on(interior, u=U)
46     + on(exterior, u= 0 )
47 ;
48
49 // Loop
50 for (i = 0; i < 20; i++){
51     cout << mpirank << " - Loop " << i << endl;
52
53     // Solve
54     pb;
55     //send u to the other proc, receive in U
56     processor(1-mpirank) << u[];
57     processor(1-mpirank) >> U[];

```

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```

58
59 // Error
60 real err0, err1;
61 err0 = int1d(Th[mpirank],interior) (square(U - u));
62 // send err0 to the other proc, receive in err1
63 processor(1-mpirank) << err0;
64 processor(1-mpirank) >> err1;
65 real err = sqrt(err0 + err1);
66 cout << " err = " << err << " - err0 = " << err0 << " - err1 = " << err1 << endl;
67 if (err < 1e-3) break;
68 }
69 if (mpirank == 0)
70 plot(u, U);

```

**Todo:** script freeze in the loop

## True parallel Schwarz example

Thank you to F. Nataf

This is a explanation of the two examples [MPI-GMRES 2D](#) and [MPI-GMRES 3D](#), a Schwarz parallel with a complexity almost independent of the number of process (with a coarse grid preconditioner).

To solve the following Poisson problem on domain  $\Omega$  with boundary  $\Gamma$  in  $L^2(\Omega)$  :

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= g && \text{on } \Gamma \end{aligned}$$

where  $f$  and  $g$  are two given functions of  $L^2(\Omega)$  and of  $H^{\frac{1}{2}}(\Gamma)$ ,

Lets introduce  $(\pi_i)_{i=1,\dots,N_p}$  a regular partition of the unity of  $\Omega$ , q-e-d:

$$\pi_i \in \mathcal{C}^0(\Omega) : \quad \pi_i \geq 0 \text{ and } \sum_{i=1}^{N_p} \pi_i = 1.$$

Denote  $\Omega_i$  the sub domain which is the support of  $\pi_i$  function and also denote  $\Gamma_i$  the boundary of  $\Omega_i$ .

The parallel Schwarz method is:

Let  $\ell = 0$  the iterator and a initial guest  $u^0$  respecting the boundary condition (i.e.  $u^0|_\Gamma = g$ ).

$$\begin{aligned} \forall i = 1.., N_p : \\ -\Delta u_i^\ell &= f && \text{in } \Omega_i \\ u_i^\ell &= u^\ell && \text{on } \Gamma_i \setminus \Gamma \\ u_i^\ell &= g && \text{on } \Gamma_i \cap \Gamma \\ \\ u^{\ell+1} &= \sum_{i=1}^{N_p} \pi_i u_i^\ell \end{aligned} \tag{3.30}$$

After discretization with the Lagrange finite element method, with a compatible mesh  $\mathcal{T}_{h_i}$  of  $\Omega_i$ , i. e., the exist a global mesh  $\mathcal{T}_h$  such that  $\mathcal{T}_{h_i}$  is include in  $\mathcal{T}_h$ .

Let us denote:

- $V_{h_i}$  the finite element space corresponding to domain  $\Omega_i$ ,

- $\mathcal{N}_{hi}$  is the set of the degree of freedom  $\sigma_i^k$ ,
- $\mathcal{N}_{hi}^{\Gamma_i}$  is the set of the degree of freedom of  $V_{hi}$  on the boundary  $\Gamma_i$  of  $\Omega_i$ ,
- $\sigma_i^k(v_h)$  is the value the degree of freedom  $k$ ,
- $V_{0hi} = \{v_h \in V_{hi} : \forall k \in \mathcal{N}_{hi}^{\Gamma_i}, \sigma_i^k(v_h) = 0\}$ ,
- The conditional expression  $a ? b : c$  is defined like in :c‘C‘ of C++ language by

$$a?b:c \equiv \begin{cases} \text{if } a \text{ is true then return } b \\ \text{else return } c \end{cases}.$$

---

**Note:** We never use finite element space associated to the full domain  $\Omega$  because it is too expensive.

---

We have to defined to operator to build the previous algorithm:

We denote  $u_{h|i}^\ell$  the restriction of  $u_h^\ell$  on  $V_{hi}$ , so the discrete problem on  $\Omega_i$  of problem (3.30) is find  $u_{hi}^\ell \in V_{hi}$  such that:

$$\forall v_{hi} \in V_{0i} : \int_{\Omega_i} \nabla v_{hi} \cdot \nabla u_{hi}^\ell = \int_{\Omega_i} f v_{hi}, \quad \forall k \in \mathcal{N}_{hi}^{\Gamma_i} : \sigma_i^k(u_{hi}^\ell) = (k \in \Gamma) ? g_i^k : \sigma_i^k(u_{h|i}^\ell)$$

where  $g_i^k$  is the value of  $g$  associated to the degree of freedom  $k \in \mathcal{N}_{hi}^{\Gamma_i}$ .

In FreeFEM, it can be written has with  $\mathbf{U}$  is the vector corresponding to  $u_{h|i}^\ell$  and the vector  $\mathbf{U1}$  is the vector corresponding to  $u_{hi}^\ell$  is the solution of:

```

1 real[int] U1(Ui.n);
2 real[int] b = onG .* U;
3 b = onG ? b : Bi ;
4 U1 = Ai^-1*b;
```

where  $\text{onG}[i] = (i \in \Gamma_i \setminus \Gamma) ? 1 : 0$ , and  $\mathbf{Bi}$  the right of side of the problem, are defined by

```

1 // Fespace
2 fespace Whi(Thi, P2);
3
4 // Problem
5 varf vPb (U, V)
6   = int3d(Thi)(
7     grad(U) * grad(V)
8   )
9   + int3d(Thi)(
10    F*V
11   )
12   + on(1, U=g)
13   + on(10, U=G)
14 ;
15
16 varf vPbon (U, V) = on(10, U=1) + on(1, U=0);
17
18 matrix Ai = vPb(Whi, Whi, solver=sparsesolver);
19 real[int] onG = vPbon(0, Whi);
20 real[int] Bi=vPb(0, Whi);
```

where the FreeFEM label of  $\Gamma$  is 1 and the label of  $\Gamma_i \setminus \Gamma$  is 10.

To build the transfer/update part corresponding to (3.30) equation on process  $i$ , let us call  $n_{jpart}$  the number the neighborhood of domain of  $\Omega_i$  (i.e:  $\pi_j$  is none 0 of  $\Omega_i$ ), we store in an array  $jpart$  of size  $n_{jpart}$  all this neighborhood.

Let us introduce two array of matrix,  $Smj[j]$  to defined the vector to send from  $i$  to  $j$  a neighborhood process, and the matrix  $rMj[j]$  to after to reduce owith neighborhood  $j$  domain.

So the tranfert and update part compute  $v_i = \pi_i u_i + \sum_{j \in J_i} \pi_j u_j$  and can be write the FreeFEM function Update:

```

1 func bool Update (real[int] &ui, real[int] &vi) {
2     int n = jpart.n;
3     for (int j = 0; j < njpart; ++j) Usend[j][] = smj[j]*ui;
4     mpiRequest[int] rq(n*2);
5     for (int j = 0; j < n; ++j) Irecv(processor(jpart[j], comm, rq[j]), Ri[j][]);
6     for (int j = 0; j < n; ++j) Isend(processor(jpart[j], comm, rq[j+n]), Si[j][]);
7     for (int j = 0; j < n*2; ++j) int k = mpiWaitAny(rq);
8     // apply the unity local partition
9     vi = Pii*ui; //set to pi_i u_i
10    for (int j = 0; j < njpart; ++j) vi += rMj[j]*Vrecv[j][]; //add pi_j u_j
11    return true;
12 }
```

where the buffer are defined by:

```

1 InitU(njpart, Whij, Thij, aThij, Usend) //defined the send buffer
2 InitU(njpart, Whij, Thij, aThij, Vrecv) //defined the recv buffer
```

with the following macro definition:

```

1 macro InitU(n, Vh, Th, aTh, U) Vh[int] U(n); for (int j = 0; j < n; ++j){Th = aTh[j]; U[j] = 0;}
```

*First GMRES algorithm:* you can easily accelerate the fixed point algorithm by using a parallel GMRES algorithm after the introduction the following affine  $\mathcal{A}_i$  operator sub domain  $\Omega_i$ .

```

1 func real[int] DJ0 (real[int] & U) {
2     real[int] V(U.n), b = onG .* U;
3     b = onG ? b : Bi ;
4     V = Ai^-1*b;
5     Update(V, U);
6     V -= U;
7     return V;
8 }
```

Where the parallel MPIGMRES or MPICG algorithm is just a simple way to solve in parallel the following  $A_i x_i = b_i, i = 1,.., N_p$  by just changing the dot product by reduce the local dot product of all process with the following MPI code:

```

1 template<class R> R ReduceSum1(R s, MPI_Comm *comm) {
2     R r = 0;
3     MPI_Allreduce(&s, &r, 1, MPI_TYPE<R>::TYPE(), MPI_SUM, *comm );
4     return r;
5 }
```

This is done in MPIGC dynamics library tool.

*Second GMRES algorithm:* Use scharwz algorithm as a preconditioner of basic GMRES method to solving the parallel problem.

```

1 func real[int] DJ (real[int]& U) { //the original problem
2     ++kiter;
3     real[int] V(U.n);
4     V = Ai*U;
5     V = onGi ? 0. : V; //remove boundary term
6     return V;
7 }
8
9 func real[int] PDJ (real[int]& U) { //the preconditioner
10    real[int] V(U.n);
11    real[int] b = onG ? 0. : U;
12    V = Ai^-1*b;
13    Update(V, U);
14    return U;
15 }
```

*Third GMRES algorithm:* Add a coarse solver to the previous algorithm

First build a coarse grid on processor 0, and the

```

1 matrix AC, Rci, Pci;
2 if (mpiRank(comm) == 0)
3     AC = vPbC(VhC, VhC, solver=sparseSolver); //the coarse problem
4
5 Pci = interpolate(Whi, VhC); //the projection on coarse grid
6 Rci = Pci'*Pii; //the restriction on Process i grid with the partition pi_i
7
8 func bool CoarseSolve (real[int]& V, real[int]& U, mpiComm& comm){
9     // solving the coarse problem
10    real[int] Uc(Rci.n), Bc(Uc.n);
11    Uc = Rci*U;
12    mpiReduce(Uc, Bc, processor(0, comm), mpiSUM);
13    if (mpiRank(comm) == 0)
14        Uc = AC^-1*Bc;
15    broadcast(processor(0, comm), Uc);
16    V = Pci*Uc;
17 }
```

The New preconditionner

```

1 func real[int] PDJC (real[int]& U) {
2     // Idea: F. Nataf.
3     //  $0 \sim (I - C1A)(I - C2A) \Rightarrow I \sim -C1AC2A + C1A + C2A$ 
4     // New Prec  $P = C1 + C2 - C1AC2 = C1(I - A C2) + C2$ 
5     //  $(C1(I - A C2) + C2) U_o$ 
6     //  $V = -C2 * U_o$ 
7     // ....
8     real[int] V(U.n);
9     CoarseSolve(V, U, comm);
10    V = -V; // -C2 * U_o
11    U += Ai*V; //  $U = (I - A C2) U_o$ 
12    real[int] b = onG ? 0. : U;
13    U = Ai^-1*b; //  $C1(I - A C2) U_o$ 
14    V = U - V;
15    Update(V, U);
16    return U;
17 }
```

The code of the 4 algorithms:

```

1  real epss = 1e-6;
2  int rgmres = 0;
3  if (gmres == 1) {
4      rgmres = MPIAffineGMRES(DJ0, u[], veps=epss, nbiter=300,
5          comm=comm, dimKrylov=100, verbosity=ipart?0: 50);
6      real[int] b = onG .* u[];
7      b = onG ? b : Bi ;
8      v[] = Ai^-1*b;
9      Update(v[], u[]);
10 }
11 else if (gmres == 2)
12     rgmres = MPILinearGMRES(DJ, precon=PDJ, u[], Bi, veps=epss,
13         nbiter=300, comm=comm, dimKrylov=100, verbosity=ipart?0: 50);
14 else if (gmres == 3)
15     rgmres = MPILinearGMRES(DJ, precon=PDJC, u[], Bi, veps=epss,
16         nbiter=300, comm=comm, dimKrylov=100, verbosity=ipart?0: 50);
17 else //algo Schwarz for demo
18     for(int iter = 0; iter < 10; ++iter)
19         ...

```

We have all ingredient to solve in parallel if we have et the partitions of the unity. To build this partition we do:

The initial step on process 1 to build a coarse mesh,  $\mathcal{T}_h^*$  of the full domain, and build the partition  $\pi$  function constant equal to  $i$  on each sub domain  $\mathcal{O}_i, i = 1,.., N_p$ , of the grid with the metis graph partitioner [KARYPIS1995] and on each process  $i$  in  $1.., N_p$  do

1. Broadcast from process 1, the mesh  $\mathcal{T}_h^*$  (call `Thi.i` in FreeFEM script), and  $\pi$  function,
2. remark that the characteristic function  $\mathbf{I}_{\mathcal{O}_i}$  of domain  $\mathcal{O}_i$ , is defined by  $(\pi = i)?1:0$ ,
3. Let us call  $\Pi_P^2$  (resp.  $\Pi_V^2$ ) the  $L^2$  on  $P_h^*$  the space of the constant finite element function per element on  $\mathcal{T}_h^*$  (resp.  $V_h^*$  the space of the affine continuous finite element per element on  $\mathcal{T}_h^*$ ) and build in parallel the  $\pi_i$  and  $\Omega_i$ , such that  $\mathcal{O}_i \subset \Omega_i$  where  $\mathcal{O}_i = \text{supp}((\Pi_V^2 \Pi_C^2)^m \mathbf{I}_{\mathcal{O}_i})$ , and  $m$  is a the overlaps size on the coarse mesh (generally one), (this is done in function `AddLayers(Thi,suppli[],nlayer,phii[])`; We choose a function  $\pi_i^* = (\Pi_1^2 \Pi_0^2)^m \mathbf{I}_{\mathcal{O}_i}$  so the partition of the unity is simply defined by

$$\pi_i = \frac{\pi_i^*}{\sum_{j=1}^{N_p} \pi_j^*}$$

The set  $J_i$  of neighborhood of the domain  $\Omega_i$ , and the local version on  $V_{hi}$  can be defined the array `jpart` and `njpart` with:

```

1 Vhi pii = piistar;
2 Vhi[int] pij(nprij); //local partition of 1 = pii + sum_j pij[j]
3 int[int] jpart(npart);
4 int njpart = 0;
5 Vhi sumphi = piistar;
6 for (int i = 0; i < npart; ++i)
7     if (i != ipart){
8         if (int3d(Thi)(pijstar,j) > 0) {
9             pij[njpart] = pijstar;
10            sumphi[] += pij[njpart]++;
11            jpart[njpart++] = i;
12        }
13    }

```

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```

14 pii[] = pii[] ./ sumphi[];
15 for (int j = 0; j < njpart; ++j)
16 pij[j][] = pij[j][] ./ sumphi[];
17 jpart.resize(njpart);

```

4. We call  $\mathcal{T}_{hij}^*$  the sub mesh part of  $\mathcal{T}_{hi}$  where  $\pi_j$  are none zero. And thanks to the function `trunc` to build this array,

```

1 for (int jp = 0; jp < njpart; ++jp)
2     aThij[jp] = trunc(Thi, pij[jp] > 1e-10, label=10);

```

5. At this step we have all on the coarse mesh, so we can build the fine final mesh by splitting all meshes: `Thi`, `Thij[j]`, `Thij[j]` with **FreeFEM** `trunc` mesh function which do restriction and slipping.
6. The construction of the send/recv matrices `sMj` and *freefem*: '`rMj`' can done with this code:

```

1 mesh3 Thij = Thi;
2 fespace Whij(Thij, Pk);
3 matrix Pii; Whi wpii = pii; Pii = wpii[]; //Diagonal matrix corresponding to X pi_i
4 matrix[int] sMj(njpart), rMj(njpart); //M send/receive case
5 for (int jp = 0; jp < njpart; ++jp){
6     int j = jpart[jp];
7     Thij = aThij[jp]; //change mesh to change Whij, Whij
8     matrix I = interpolate(Whij, Whi); //Whij <- Whi
9     sMj[jp] = I * Pii; //Whi -> s Whij
10    rMj[jp] = interpolate(Whij, Whi, t=1); //Whij -> Whi
11 }

```

To buil a not too bad application, all variables come from parameters value with the following code

```

1 include "getARGV.idp"
2 verbosity = getARGV("-vv", 0);
3 int vdebug = getARGV("-d", 1);
4 int ksplit = getARGV("-k", 10);
5 int nloc = getARGV("-n", 25);
6 string sff = getARGV("-p", "", "");
7 int gmres = getARGV("-gmres", 3);
8 bool dplot = getARGV("-dp", 0);
9 int nC = getARGV("-N", max(nloc/10, 4));

```

And small include to make graphic in parallel of distribute solution of vector  $u$  on mesh  $T_h$  with the following interface:

```

1 include "MPIplot.idp"
2 func bool plotMPIall(mesh &Th, real[int] &u, string cm) {
3     PLOTMPIALL(mesh, Pk, Th, u, {cmm=cm, nbiso=20, fill=1, dim=3, value=1});
4     return 1;
5 }

```

---

**Note:** The `cmm=cm`, ... in the macro argument is a way to quote macro argument so the argument is `cmm=cm`, ...

---

### 3.6.2 Parallel sparse solvers

Parallel sparse solvers use several processors to solve linear systems of equation. Like sequential, parallel linear solvers can be direct or iterative. In **FreeFEM** both are available.

#### Using parallel sparse solvers in FreeFEM

We recall that the `solver` parameters are defined in the following commands: `solve`, `problem`, `set` (setting parameter of a matrix) and in the construction of the matrix corresponding to a bilinear form. In these commands, the parameter `solver` must be set to `sparsesolver` for parallel sparse solver. We have added specify parameters to these command lines for parallel sparse solvers. These are:

- `lparams` : vector of integer parameters (`l` is for the C++ type `long`)
- `dparams` : vector of real parameters
- `sparams` : string parameters
- `datafilename` : name of the file which contains solver parameters

The following four parameters are only for direct solvers and are vectors. These parameters allow the user to preprocess the matrix (see the section on [sparse direct solver](#) for more information).

- `permr` : row permutation (integer vector)
- `permC` : column permutation or inverse row permutation (integer vector)
- `scaler` : row scaling (real vector)
- `scalec` : column scaling (real vector)

There are two possibilities to control solver parameters. The first method defines parameters with `lparams`, `dparams` and `sparams` in `.edp` file.

The second one reads the solver parameters from a data file. The name of this file is specified by `datafilename`. If `lparams`, `dparams`, `sparams` or `datafilename` is not provided by the user, the solver's default values are used.

To use parallel solver in **FreeFEM**, we need to load the dynamic library corresponding to this solver. For example to use **MUMPS** solver as parallel solver in **FreeFEM**, write in the `.edp` file `load "MUMPS_FreeFem"`.

If the libraries are not loaded, the default sparse solver will be loaded (default sparse solver is **UMFPACK**). The [Table 3.2](#) gives this new value for the different libraries.

**Table 3.2:** Default sparse solver for real and complex arithmetics when we load a parallel sparse solver library

Libraries	Default sparse solver	
	real	complex
MUMPS_FreeFem	mumps	mumps
real_SuperLU_DIST_FreeFem	SuperLU_DIST	previous solver
complex_SuperLU_DIST_FreeFem	previous solver	SuperLU_DIST
real_pastix_FreeFem	PaStiX	previous solver
complex_pastix_FreeFem	previous solver	PaStiX
hips_FreeFem	hips	previous solver
hypre_FreeFem	hypre	previous solver
parms_FreeFem	parms	previous solver

We also add functions (see [Table 3.3](#)) with no parameter to change the default sparse solver in the `.edp` file. To use these functions, we need to load the library corresponding to the solver. An example of using different parallel sparse solvers for the same problem is given in [Direct solvers example](#).

**Table 3.3:** Functions that allow to change the default sparse solver for real and complex arithmetics and the result of these functions

Function	default sparse solver	
	real	complex
defaulttoMUMPS()	mumps	mumps
realdefaulttoSuperLUdist()	SuperLU_DIST	previous solver
complexdefaulttoSuperLUdist()	previous solver	SuperLU_DIST
realdefaulttopastix()	pastix	previous solver
complexdefaulttopastix()	previous solver	pastix
defaulttohips()	hips	previous solver
defaulttohypre()	hypre	previous solver
defaulttoparms()	parms	previous solver

**Tip:** Test direct solvers

```

1  load "MUMPS_FreeFem"
2 //default solver: real-> MUMPS, complex -> MUMPS
3  load "real_SuperLU_DIST_FreeFem"
4 //default solver: real-> SuperLU_DIST,
5 complex -> MUMPS load "real_pastix_FreeFem"
6 //default solver: real-> pastix, complex -> MUMPS
7
8 // Solving with pastix
9 {
10    matrix A =
11      [[1, 2, 2, 1, 1],
12       [2, 12, 0, 10, 10],
13       [2, 0, 1, 0, 2],
14       [1, 10, 0, 22, 0.],
15       [1, 10, 2, 0., 22]];
16
17    real[int] xx = [1, 32, 45, 7, 2], x(5), b(5), di(5);
18    b = A*xx;
19    cout << "b =" << b << endl; cout << "xx =" << xx << endl;
20
21    set(A, solver=sparse solver, datafilename="ffpastix_iparm_dparm.txt");
22    cout << "solve" << endl;
23    x = A^-1*b;
24    cout << "b =" << b << endl;
25    cout << "x =" << endl;
26    cout << x << endl;
27    di = xx - x;
28    if (mpirank == 0) {
29      cout << "x-xx =" << endl;
30      cout << "Linf =" << di.linfy << ", L2 =" << di.l2 << endl;
31    }
32
33
34 // Solving with SuperLU_DIST
35 realdefaulttoSuperLUdist();
36 //default solver: real-> SuperLU_DIST, complex -> MUMPS
37 {
38    matrix A =
39      [[1, 2, 2, 1, 1],

```

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```

40      [ 2, 12, 0, 10, 10],
41      [ 2, 0, 1, 0, 2],
42      [ 1, 10, 0, 22, 0.],
43      [ 1, 10, 2, 0., 22]];
44
45  real[int] xx = [1, 32, 45, 7, 2], x(5), b(5), di(5);
46  b = A*xx;
47  cout << "b =" << b << endl;
48  cout << "xx =" << xx << endl;
49
50  set(A, solver=sparse solver, datafilename="ffsuperlu_dist_fileparam.txt");
51  cout << "solve" << endl;
52  x = A^-1*b;
53  cout << "b =" << b << endl;
54  cout << "x =" << endl;
55  cout << x << endl;
56  di = xx - x;
57  if (mpirank == 0) {
58    cout << "x-xx =" << endl;
59    cout << "Linf =" << di.linfty << ", L2 =" << di.L2 << endl;
60  }
61 }
62
63 // Solving with MUMPS
64 defaulttoMUMPS();
65 //default solver: real-> MUMPS, complex -> MUMPS
66 {
67  matrix A =
68    [[1, 2, 2, 1, 1],
69     [ 2, 12, 0, 10, 10],
70     [ 2, 0, 1, 0, 2],
71     [ 1, 10, 0, 22, 0.],
72     [ 1, 10, 2, 0., 22]];
73
74  real[int] xx = [1, 32, 45, 7, 2], x(5), b(5), di(5);
75  b = A*xx;
76  cout << "b =" << b << endl;
77  cout << "xx =" << xx << endl;
78
79  set(A, solver=sparse solver, datafilename="ffmumps_fileparam.txt");
80  cout << "solving solution" << endl;
81  x = A^-1*b;
82  cout << "b =" << b << endl;
83  cout << "x =" << endl;
84  cout << x << endl;
85  di = xx - x;
86  if (mpirank == 0) {
87    cout << "x-xx =" << endl;
88    cout << "Linf =" << di.linfty << ", L2" << di.L2 << endl;
89  }
90 }
```

## Sparse direct solver

In this section, we present the sparse direct solvers interfaced with FreeFEM.

## MUMPS solver

MULTifrontal Massively Parallel Solver ([MUMPS](#)) is an open-source library.

This package solves linear system of the form  $A x = b$  where  $A$  is a square sparse matrix with a direct method. The square matrix considered in MUMPS can be either unsymmetric, symmetric positive definite or general symmetric.

The method implemented in MUMPS is a direct method based on a multifrontal approach. It constructs a direct factorization  $A = L U$ ,  $A = L^t D L$  depending of the symmetry of the matrix  $A$ .

**MUMPS uses the following libraries :**

- [BLAS](#),
- [BLACS](#),
- [ScaLAPACK](#).

**Warning:** MUMPS does not solve linear system with a rectangular matrix.

### MUMPS parameters:

There are four input parameters in [MUMPS](#). Two integers `SYM` and `PAR`, a vector of integer of size 40 `INCTL` and a vector of real of size 15 `CNTL`.

The first parameter gives the type of the matrix: 0 for unsymmetric matrix, 1 for symmetric positive matrix and 2 for general symmetric.

The second parameter defined if the host processor work during the factorization and solves steps : `PAR=1` host processor working and `PAR=0` host processor not working.

The parameter `INCTL` and `CNTL` is the control parameter of MUMPS. The vectors `ICNTL` and `CNTL` in MUMPS becomes with index 1 like vector in Fortran. For more details see the [MUMPS user's guide](#).

We describe now some elements of the main parameters of `INCTL` for MUMPS.

- **Input matrix parameter** The input matrix is controlled by parameters `ICNTL(5)` and `ICNTL(18)`.  
The matrix format (resp. matrix pattern and matrix entries) are controlled by `INCTL(5)` (resp. `INCTL(18)`).  
The different values of `ICNTL(5)` are 0 for assembled format and 1 for element format. In the current release of [FreeFEM](#), we consider that FE matrix or matrix is storage in assembled format. Therefore, `INCTL(5)` is treated as 0 value.  
The main option for `ICNTL(18)`: `INCLTL(18)=0` centrally on the host processor, `ICNTL(18)=3` distributed the input matrix pattern and the entries (recommended option for distributed matrix by developer of MUMPS). For other values of `ICNTL(18)` see the [MUMPS user's guide](#). These values can be used also in [FreeFEM](#).  
The default option implemented in [FreeFEM](#) are `ICNTL(5)=0` and `ICNTL(18)=0`.
- **Preprocessing parameter** The preprocessed matrix  $A_p$  that will be effectively factored is defined by

$$A_p = P D_r A Q_c D_c P^t$$

where  $P$  is the permutation matrix,  $Q_c$  is the column permutation,  $D_r$  and  $D_c$  are diagonal matrix for respectively row and column scaling.

The ordering strategy to obtain  $P$  is controlled by parameter `ICNTL(7)`. The permutation of zero free diagonal  $Q_c$  is controlled by parameter `ICNTL(6)`. The row and column scaling is controlled

by parameter `ICNTL(18)`. These option are connected and also strongly related with `ICNTL(12)` (see the [MUMPS user's guide](#) for more details).

The parameters `permr`, `scaler`, and `scalec` in **FreeFEM** allow to give permutation matrix( $P$ ), row scaling ( $D_r$ ) and column scaling ( $D_c$ ) of the user respectively.

### Calling MUMPS in FreeFEM

To call MUMPS in **FreeFEM**, we need to load the dynamic library `MUMPS_freefem.dylib` (MacOSX), `MUMPS_freefem.so` (Unix) or `MUMPS_freefem.dll` (Windows).

This is done in typing `load "MUMPS_FreeFem"` in the `.edp` file. We give now the two methods to give the option of MUMPS solver in **FreeFEM**.

- **Solver parameters is defined in .edp file: In this method, we need to give the parameters `lparams` and `dparams`.**

These parameters are defined for MUMPS by :

- `lparams[0] = SYM, lparams[1] = PAR,`
- $\forall i = 1, \dots, 40, lparams[i+1] = ICNTL(i)$
- $\forall i = 1, \dots, 15, dparams[i-1] = CNTL(i)$

- **Reading solver parameters on a file:**

The structure of data file for MUMPS in **FreeFEM** is : first line parameter `SYM` and second line parameter `PAR` and in the following line the different value of vectors `ICNTL` and `CNTL`. An example of this parameter file is given in `ffmumpsfileparam.txt`.

```

1 0 /* SYM :: 0 for non symmetric matrix, 1 for symmetric definite positive_
2   ↵matrix and 2 general symmetric matrix*/
3 1 /* PAR :: 0 host not working during factorization and solves steps, 1_
4   ↵host working during factorization and solves steps*/
5 -1 /* ICNTL(1) :: output stream for error message */
6 -1 /* ICNTL(2) :: output for diagnostic printing, statics and warning_
7   ↵message */
8 -1 /* ICNTL(3) :: for global information */
9 0 /* ICNTL(4) :: Level of printing for error, warning and diagnostic_
10  ↵message */
11 0 /* ICNTL(5) :: matrix format : 0 assembled format, 1 elemental format.._
12   ↵*/
13 7 /* ICNTL(6) :: control option for permuting and/or scaling the matrix_
14   ↵in analysis phase */
15 3 /* ICNTL(7) :: pivot order strategy : AMD, AMF, metis, pord scotch*/
16 77 /* ICNTL(8) :: Row and Column scaling strategy */
17 1 /* ICNTL(9) :: 0 solve Ax = b, 1 solve the transposed system A^t x = b_
18   ↵: parameter is not considered in the current release of FreeFEM*/
19 0 /* ICNTL(10) :: number of steps of iterative refinement */
20 0 /* ICNTL(11) :: statics related to linear system depending on ICNTL(9)_
21   ↵*/
22 1 /* ICNTL(12) :: constrained ordering strategy for general symmetric_
23   ↵matrix */
24 0 /* ICNTL(13) :: method to control splitting of the root frontal matrix_
25   ↵*/
26 20 /* ICNTL(14) :: percentage increase in the estimated working space_
27   ↵(default 20\%)*/
28 0 /* ICNTL(15) :: not used in this release of MUMPS */
29 0 /* ICNTL(16) :: not used in this release of MUMPS */
30 0 /* ICNTL(17) :: not used in this release of MUMPS */
31 3 /* ICNTL(18) :: method for given : matrix pattern and matrix entries :_
32   ↵*/

```

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```

21 0 /* ICNTL(19) :: method to return the Schur complement matrix */
22 0 /* ICNTL(20) :: right hand side form ( 0 dense form, 1 sparse form) :_
→parameter will be set to 0 for FreeFEM */
23 0 /* ICNTL(21) :: 0, 1 kept distributed solution : parameter is not_
→considered in the current release of FreeFEM */
24 0 /* ICNTL(22) :: controls the in-core/out-of-core (OOC) facility */
25 0 /* ICNTL(23) :: maximum size of the working memory in Megabyte than_
→MUMPS can allocate per working processor */
26 0 /* ICNTL(24) :: control the detection of null pivot */
27 0 /* ICNTL(25) :: control the computation of a null space basis */
28 0 /* ICNTL(26) :: This parameter is only significant with Schur option_
→(ICNTL(19) not zero). : parameter is not considered in the current_
→release of FreeFEM */
29 -8 /* ICNTL(27) (Experimental parameter subject to change in next release_
→of MUMPS) :: control the blocking factor for multiple righthand side_
→during the solution phase : parameter is not considered in the current_
→release of FreeFEM */
30 0 /* ICNTL(28) :: not used in this release of MUMPS*/
31 0 /* ICNTL(29) :: not used in this release of MUMPS*/
32 0 /* ICNTL(30) :: not used in this release of MUMPS*/
33 0 /* ICNTL(31) :: not used in this release of MUMPS*/
34 0 /* ICNTL(32) :: not used in this release of MUMPS*/
35 0 /* ICNTL(33) :: not used in this release of MUMPS*/
36 0 /* ICNTL(34) :: not used in this release of MUMPS*/
37 0 /* ICNTL(35) :: not used in this release of MUMPS*/
38 0 /* ICNTL(36) :: not used in this release of MUMPS*/
39 0 /* ICNTL(37) :: not used in this release of MUMPS*/
40 0 /* ICNTL(38) :: not used in this release of MUMPS*/
41 1 /* ICNTL(39) :: not used in this release of MUMPS*/
42 0 /* ICNTL(40) :: not used in this release of MUMPS*/
43 0.01 /* CNTL(1) :: relative threshold for numerical pivoting */
44 1e-8 /* CNTL(2) :: stopping criteria for iterative refinement */
45 -1 /* CNTL(3) :: threshold for null pivot detection */
46 -1 /* CNTL(4) :: determine the threshold for partial pivoting */
47 0.0 /* CNTL(5) :: fixation for null pivots */
48 0 /* CNTL(6) :: not used in this release of MUMPS */
49 0 /* CNTL(7) :: not used in this release of MUMPS */
50 0 /* CNTL(8) :: not used in this release of MUMPS */
51 0 /* CNTL(9) :: not used in this release of MUMPS */
52 0 /* CNTL(10) :: not used in this release of MUMPS */
53 0 /* CNTL(11) :: not used in this release of MUMPS */
54 0 /* CNTL(12) :: not used in this release of MUMPS */
55 0 /* CNTL(13) :: not used in this release of MUMPS */
56 0 /* CNTL(14) :: not used in this release of MUMPS */
57 0 /* CNTL(15) :: not used in this release of MUMPS */

```

If no solver parameter is given, we used default option of MUMPS solver.

---

**Tip:** MUMPS example

A simple example of calling MUMPS in **FreeFEM** with this two methods is given in the *Test solver MUMPS example*.

---

## SuperLU distributed solver

The package SuperLU\_DIST solves linear systems using LU factorization. It is a free scientific library

This library provides functions to handle square or rectangular matrix in real and complex arithmetics. The method implemented in SuperLU\_DIST is a supernodal method. New release of this package includes a parallel symbolic factorization. This scientific library is written in C and MPI for communications.

### SuperLU\_DIST parameters:

We describe now some parameters of SuperLU\_DIST. The SuperLU\_DIST library use a 2D-logical process group. This process grid is specified by *nprow* (process row) and *npcol* (process column) such that  $N_p = nprow \cdot npcol$  where  $N_p$  is the number of all process allocated for SuperLU\_DIST.

The input matrix parameters is controlled by “matrix=” in `sparams` for internal parameter or in the third line of parameters file. The different value are

- `matrix=assembled` global matrix are available on all process
- `matrix=distributedglobal` The global matrix is distributed among all the process
- `matrix=distributed` The input matrix is distributed (not yet implemented)

The option arguments of SuperLU\_DIST are described in the section Users-callable routine of the [SuperLU users' guide](#).

The parameter `Fact` and `TRANS` are specified in **FreeFEM** interfaces to SuperLU\_DIST during the different steps. For this reason, the value given by the user for this option is not considered.

The factorization LU is calculated in SuperLU\_DIST on the matrix  $A_p$ .

$$A_p = P_c \ P_r \ D_r \ A \ D_c \ P_c^t$$

where  $P_c$  and  $P_r$  is the row and column permutation matrix respectively,  $D_r$  and  $D_c$  are diagonal matrix for respectively row and column scaling.

The option argument `RowPerm` (resp. `ColPerm`) control the row (resp. column) permutation matrix.  $D_r$  and  $D_c$  is controlled by the parameter `DiagScale`.

The parameter `permr`, `permC`, `scaler`, and `scaleC` in **FreeFEM** is provided to give row permutation, column permutation, row scaling and column scaling of the user respectively.

The other parameters for LU factorization are `ParSymFact` and `ReplaceTinyPivot`. The parallel symbolic factorization works only on a power of two processes and need the `ParMetis` ordering. The default option argument of SuperLU\_DIST are given in the file `ffsuperlu_dist_fileparam.txt`.

### Calling SuperLU\_DIST in FreeFEM

To call SuperLU\_DIST in **FreeFEM**, we need to load the library dynamic correspond to interface. This done by the following line `load "real_superlu_DIST_FreeFem"` (resp. `load "complex_superlu_DIST_FreeFem"`) for real (resp. complex) arithmetics in the file `.edp`.

### Solver parameters is defined in .edp file:

To call SuperLU\_DIST with internal parameter, we used the parameters `sparams`. The value of parameters of SuperLU\_DIST in `sparams` are defined by :

- `nprow=1,`
- `npcol=1,`
- `matrix= distributedglobal,`
- `Fact= DOFACT,`

- Equil=NO,
- ParSymbFact=NO,
- ColPerm= MMD\_AT\_PLUS\_A,
- RowPerm= LargeDiag,
- DiagPivotThresh=1.0,
- IterRefine=DOUBLE,
- Trans=NOTRANS,
- ReplaceTinyPivot=NO,
- SolveInitialized=NO,
- PrintStat=NO,
- DiagScale=NOEQUIL

This value correspond to the parameter in the file `ffsuperlu_dist_fileparam.txt`. If one parameter is not specified by the user, we take the default value of SuperLU\_DIST.

**Reading solver parameters on a file:** The structure of data file for SuperLU\_DIST in **FreeFEM** is given in the file `ffsuperlu_dist_fileparam.txt` (default value of the **FreeFEM** interface).

```
1 /* nprow : integer value */
2 /* npcol : integer value */
3 distributedglobal /* matrix input : assembled, distributedglobal, distributed */
4 DOFACT /* Fact : DOFACT, SamePattern, SamePattern_SameRowPerm, FACTORED */
5 NO /* Equil : NO, YES */
6 NO /* ParSymbFact : NO, YES */
7 MMD_AT_PLUS_A /* ColPerm : NATURAL, MMD_AT_PLUS_A, MMD_ATA, METIS_AT_PLUS_A, PARMETIS,
   ↪ MY_PERMC */
8 LargeDiag /* RowPerm : NOROWPERM, LargeDiag, MY_PERMR */
9 1.0 /* DiagPivotThresh : real value */
10 DOUBLE /* IterRefine : NOREFINE, SINGLE, DOUBLE, EXTRA */
11 NOTRANS /* Trans : NOTRANS, TRANS, CONJ*/
12 NO /* ReplaceTinyPivot : NO, YES*/
13 NO /* SolveInitialized : NO, YES*/
14 NO /* RefineInitialized : NO, YES*/
15 NO /* PrintStat : NO, YES*/
16 NOEQUIL /* DiagScale : NOEQUIL, ROW, COL, BOTH*/
```

If no solver parameter is given, we used default option of SuperLU\_DIST solver.

---

**Tip:** A simple example of calling SuperLU\_DIST in **FreeFEM** with this two methods is given in the [Solver superLU\\_DIST example](#).

---

## PaStiX solver

**PaStiX** (Parallel Sparse matrix package) is a free scientific library under CECILL-C license. This package solves sparse linear system with a direct and block ILU(k) iterative methods. his solver can be applied to a real or complex matrix with a symmetric pattern.

### PaStiX parameters:

The input `matrix` parameter of **FreeFEM** depend on PaStiX interface. `matrix = assembled` for non distributed matrix. It is the same parameter for `SuperLU_DIST`.

There are four parameters in PaStiX : `iparm`, `dparm`, `perm` and `invp`. These parameters are respectively the integer parameters (vector of size 64), real parameters (vector of size 64), permutation matrix and inverse permutation matrix respectively. `iparm` and `dparm` vectors are described in [PaStiX RefCard](#).

The parameters `permr` and `permC` in **FreeFEM** are provided to give permutation matrix and inverse permutation matrix of the user respectively.

#### Solver parameters defined in .edp file:

To call PaStiX in **FreeFEM** in this case, we need to specify the parameters `lparams` and `dparams`. These parameters are defined by :

```
 $\forall i = 0, \dots, 63, lparams[i] = iparm[i].$ 
 $\forall i = 0, \dots, 63, dparams[i] = dparm[i].$ 
```

#### Reading solver parameters on a file:

The structure of data file for PaStiX parameters in **FreeFEM** is: first line structure parameters of the matrix and in the following line the value of vectors `iparm` and `dparm` in this order.

```
1 assembled /* matrix input :: assembled, distributed global and distributed */
2 iparm[0]
3 iparm[1]
4 ...
5 ...
6 iparm[63]
7 dparm[0]
8 dparm[1]
9 ...
10 ...
11 dparm[63]
```

An example of this file parameter is given in `ffpastix_iparm_dparm.txt` with a description of these parameters. This file is obtained with the example file `iparm.txt` and `dparm.txt` including in the PaStiX package.

If no solver parameter is given, we use the default option of PaStiX solver.

---

**Tip:** A simple example of calling PaStiX in **FreeFEM** with this two methods is given in the [Solver PaStiX example](#).

---

In Table 3.4, we recall the different matrix considering in the different direct solvers.

**Table 3.4:** Type of matrix used by the different direct sparse solver

direct solver	square matrix			rectangular matrix		
	sym	sym pattern	unsym	sym	sym pattern	unsym
SuperLU_DIST	yes	yes	yes	yes	yes	yes
MUMPS	yes	yes	yes	no	no	no
Pastix	yes	yes	no	no	no	no

#### Parallel sparse iterative solver

Concerning iterative solvers, we have chosen `pARMS`, `HIPS` and `Hypre`.

Each software implements a different type of parallel preconditioner.

So, pARMS implements algebraic domain decomposition preconditioner type such as additive Schwarz [CAI1989] and interface method; while HIPS implement hierarchical incomplete factorization and finally HYPRE implements multilevel preconditioner are AMG(Algebraic MultiGrid) and parallel approximated inverse.

To use one of these programs in **FreeFEM**, you have to install it independently of **FreeFEM**. It is also necessary to install the MPI communication library which is essential for communication between the processors and, in some cases, software partitioning graphs like **METIS** or **Scotch**.

All this preconditioners are used with Krylov subspace methods accelerators.

Krylov subspace methods are iterative methods which consist in finding a solution  $x$  of linear system  $Ax = b$  inside the affine space  $x_0 + K_m$  by imposing that  $b - Ax \perp \mathcal{L}_m$ , where  $K_m$  is Krylov subspace of dimension  $m$  defined by  $K_m = \{r_0, Ar_0, A^2r_0, \dots, A^{m-1}r_0\}$  and  $\mathcal{L}_m$  is another subspace of dimension  $m$  which depends on type of Krylov subspace. For example in GMRES,  $\mathcal{L}_m = AK_m$ .

We realized an interface which is easy to use, so that the call of these different softwares in **FreeFEM** is done in the same way. You just have to load the solver and then specify the parameters to apply to the specific solvers. In the rest of this chapter, when we talk about Krylov subspace methods we mean one among GMRES, CG and BICGSTAB.

## pARMS solver

**pARMS** (parallel Algebraic Multilevel Solver) is a software developed by Youssef Saad and al at University of Minnesota.

This software is specialized in the resolution of large sparse non symmetric linear systems of equation. Solvers developed in pARMS are of type “Krylov’s subspace”.

It consists of variants of GMRES like FGMRES (Flexible GMRES), DGMRES (Deflated GMRES) [SAAD2003] and BICGSTAB. pARMS also implements parallel preconditioner like RAS (Restricted Additive Schwarz) [CAI1989] and Schur Complement type preconditioner.

All these parallel preconditioners are based on the principle of domain decomposition. Thus, the matrix  $A$  is partitioned into sub matrices  $A_i (i = 1, \dots, p)$  where  $p$  represents the number of partitions one needs. The union of  $A_i$  forms the original matrix. The solution of the overall system is obtained by solving the local systems on  $A_i$  (see [SMITH1996]). Therefore, a distinction is made between iterations on  $A$  and the local iterations on  $A_i$ .

To solve the local problem on  $A_i$  there are several preconditioners as **ilut** (Incomplete LU with threshold), **iluk** (Incomplete LU with level of fill in) and **ARMS** (Algebraic Recursive Multilevel Solver).

---

### Tip: Default parameters

```

1  load "parms_FreeFem" //Tell FreeFem that you will use pARMS
2
3  // Mesh
4  border C(t=0, 2*pi) {x=cos(t); y=sin(t); label=1;}
5  mesh Th = buildmesh (C(50));
6
7  // Fespace
8  fespace Vh(Th, P2); Vh u, v;
9
10 // Function
11 func f= x*y;
12
13 // Problem
14 problem Poisson (u, v, solver=sparsesolver)
15   = int2d(Th) (
16     dx(u)*dx(v)

```

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```

17     + dy(u)*dy(v) )
18 + int2d(Th) (
19     - f*v
20 )
21 + on(1, u=0) ;
22
23 // Solve
24 real cpu = clock();
25 Poisson;
26 cout << " CPU time = " << clock()-cpu << endl;
27
28 // Plot
29 plot(u);

```

In line 1, the pARMS dynamic library is loaded with interface **FreeFEM**. After this, in line 15 we specify that the bilinear form will be solved by the last sparse linear solver load in memory which, in this case, is pARMS.

The parameters used in pARMS in this case are the default one since the user does not have to provide any parameter.

---

**Note:** In order to see the plot of a parallel script, run the command `FreeFem++-mpi -glut ff glut script.edp`

---

Here are some default parameters:

- solver=FGMRES,
- Krylov dimension=30,
- Maximum of Krylov=1000,
- Tolerance for convergence=1e-08 (see book [SAAD2003] to understand all this parameters),
- preconditionner=Restricted Additif Schwarz [CAI1989],
- Inner Krylov dimension=5,
- Maximum of inner Krylov dimension=5,
- Inner preconditionner=ILUK.

To specify the parameters to apply to the solver, the user can either give an integer vector for **integer parameters** and real vectors for **real parameters** or provide a **file** which contains those parameters.

---

**Tip:** User specifies parameters inside two vectors

Lets us consider Navier-Stokes example. In this example we solve linear systems coming from discretization of Navier-Stokes equations with pARMS. Parameters of solver is specified by user.

```

1 load "parms_FreeFem"
2
3 // Parameters
4 real nu = 1.;
5 int[int] iparm(16);
6 real[int] dparm(6);
7 for (int ii = 0; ii < 16; ii++)
8   iparm[ii] = -1;
9 for (int ii = 0; ii < 6; ii++)

```

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```

10  dparm[ii] = -1.0; iparm[0]=0;
11
12 // Mesh
13 mesh Th = square(10, 10);
14 int[int] wall = [1, 3];
15 int inlet = 4;
16
17 // Fespace
18 fespace Vh(Th, [P2, P2, P1]);
19
20 // Function
21 func uc = 1.;
22
23 // Problem
24 varf Stokes ([u, v, p], [ush, vsh, psh], solver=sparse solver)
25   = int2d(Th) (
26     nu*(
27       dx(u)*dx(ush)
28       + dy(u)*dy(ush)
29       + dx(v)*dx(vsh)
30       + dy(v)*dy(vsh)
31     )
32     - p*psh*(1.e-6)
33     - p*(dx(ush) + dy(vsh))
34     - (dx(u) + dy(v))*psh
35   )
36   + on(wall, wall, u=0., v=0.)
37   + on(inlet, u=uc, v=0) ;
38
39 matrix AA = Stokes(Vh, Vh);
40 set(AA, solver=sparse solver, lparams=iparm, dparams=dparm); //set pARMS as linear
41 //solver
41 real[int] bb = Stokes(0, Vh);
42 real[int] sol(AA.n);
43 SOL = AA^-1 * bb;

```

We need two vectors to specify the parameters of the linear solver. In line 5-6 of the example, we have declared these vectors(`int[int] iparm(16); real[int] dparm(6);`). In line 7-10 we have initialized these vectors by negative values.

We do this because all parameters values in pARMS are positive and if you do not change the negative values of one entry of this vector, the default value will be set.

In [Table 3.7](#) and [Table 3.8](#), we have the meaning of different entries of these vectors.

We run this example on a cluster paradent of Grid5000 and report results in [Table 3.5](#).

**Table 3.5:** Convergence and time for solving linear system

$n = 471281$ $nnz = 13 \times 10^6$ $T_e = 571.29$				
np	add(iluk)		shur(iluk)	
	nit	time	nit	time
4	230	637.57	21	557.8
8	240	364.12	22	302.25
16	247	212.07	24	167.5
32	261	111.16	25	81.5

**Table 3.6:** Legend of Table 3.5

n	matrix size
nnz	number of non null entries inside matrix
nit	number of iteration for convergence
time	Time for convergence
Te	Time for constructing finite element matrix
np	number of processor

In this example, we fix the matrix size (in term of finite element, we fix the mesh) and increase the number of processors used to solve the linear system. We saw that, when the number of processors increases, the time for solving the linear equation decreases, even if the number of iteration increases. This proves that, using pARMS as solver of linear systems coming from discretization of partial differential equation in FreeFEM can decrease drastically the total time of simulation.

**Table 3.7:** Meaning of lparams corresponding variables

Entries of iparm	Significations of each entries
iparm[0]	Krylov subspace methods Different values for this parameters are specify on <a href="#">Table 3.9</a>
iparm[1]	Preconditionner Different preconditionners for this parameters are specify on <a href="#">Table 3.10</a>
iparm[2]	Krylov subspace dimension in outer iteration: default value 30
iparm[3]	Maximum of iterations in outer iteration: default value 1000
iparm[4]	Number of level in arms when used
iparm[5]	Krylov subspace dimension in inner iteration: default value 3
iparm[6]	Maximum of iterations in inner iteration: default value 3
iparm[7]	Symmetric(=1 for symmetric) or unsymmetric matrix: default value 0(unsymmetric matrix)
iparm[8]	Overlap size between different subdomain: default value 0(no overlap)
iparm[9]	Scale the input matrix or not: Default value 1 (Matrix should be scaled)
iparm[10]	Block size in arms when used: default value 20
iparm[11]	lfil0 (ilut, iluk, and arms) : default value 20
iparm[12]	lfil for Schur complement const : default value 20
iparm[13]	lfil for Schur complement const : default value 20
iparm[14]	Multicoloring or not in ILU when used : default value 1
iparm[15]	Inner iteration : default value 0
iparm[16]	Print message when solving: default 0 (no message print) <ul style="list-style-type: none"> <li>• 0: no message is print,</li> <li>• 1: Convergence informations like number of iteration and residual,</li> <li>• 2: Timing for a different step like preconditioner,</li> <li>• 3 : Print all informations</li> </ul>

**Table 3.8:** Significations of dparams corresponding variables

Entries of dparm	Significations of each entries
dparm[0]	precision for outer iteration : default value 1e-08
dparm[1]	precision for inner iteration: default value 1e-2
dparm[2]	tolerance used for diagonal domain: : default value 0.1
dparm[3]	drop tolerance droptol0 (ilut, iluk, and arms) : default value 1e-2
dparm[4]	droptol for Schur complement const: default value 1e-2
dparm[5]	droptol for Schur complement const: default value 1e-2

**Table 3.9:** Krylov Solvers in pARMS

Values of iparm[0]	Krylov subspace methods
0	FGMRES (Flexible GMRES)
1	DGMRES (Deflated GMRES)
2	BICGSTAB

**Table 3.10:** Preconditionners in pARMS

Values of iparm[1]	Preconditionners type
0	additive Schwartz preconditioner with ilu0 as local preconditioner
1	additive Schwartz preconditioner with iluk as local preconditioner
2	additive Schwartz preconditioner with ilut as local preconditioner
3	additive Schwartz preconditioner with arms as local preconditioner
4	Left Schur complement preconditioner with ilu0 as local preconditioner
5	Left Schur complement preconditioner with ilut as local preconditioner
6	Left Schur complement preconditioner with iluk as local preconditioner
7	Left Schur complement preconditioner with arms as local preconditioner
8	Right Schur complement preconditioner with ilu0 as local preconditioner
9	Right Schur complement preconditioner with ilut as local preconditioner
10	Right Schur complement preconditioner with iluk as local preconditioner
11	Right Schur complement preconditioner with arms as local preconditioner
12	sch_ilu0, Schur complement preconditioner with global ilu0
13	SchurSymmetric GS preconditioner

## Interfacing with HIPS

HIPS (*Hierarchical Iterative Parallel Solver*) is a scientific library that provides an efficient parallel iterative solver for very large sparse linear systems. HIPS is available as free software under the CeCILL-C licence.

HIPS implements two solver classes which are the iteratives class (GMRES, PCG) and the Direct class. Concerning preconditioners, HIPS implements a type of multilevel ILU. For further informations on those preconditioners see the [HIPS documentation](#).

---

### Tip: Laplacian 3D solved with HIPS

Let us consider the 3D Laplacian example inside **FreeFEM** package where after discretization we want to solve the linear equation with HIPS.

The following example is a Laplacian 3D using Hips as linear solver. We first load Hips solver at line 2. From line 7 to 18 we specify the parameters for the Hips solver and in line 82 we set these parameters in the linear solver.

In Table 3.11 results of running on Cluster Paradent of Grid5000 are reported. We can see in this running example the efficiency of parallelism.

```

1  load "msh3"
2  load "hips_FreeFem" //load Hips library
3
4  // Parameters
5  int nn = 10;
6  real zmin = 0, zmax = 1;
7  int[int] iparm(14);
8  real[int] dparm(6);
9  for (int iiii = 0; iiii < 14; iiii++)
10    iparm[iiii] = -1;
11  for (int iiii = 0; iiii < 6; iiii++)
12    dparm[iiii] = -1;
13  iparm[0] = 0; //use iterative solver
14  iparm[1] = 1; //PCG as Krylov method
15  iparm[4] = 0; //Matrix are symmetric
16  iparm[5] = 1; //Pattern are also symmetric
17  iparm[9] = 1; //Scale matrix
18  dparm[0] = 1e-13; //Tolerance to convergence
19  dparm[1] = 5e-4; //Threshold in ILUT
20  dparm[2] = 5e-4; //Threshold for Schur preconditionner
21
22 // Functions
23 func ue = 2*x*x + 3*y*y + 4*z*z + 5*x*y + 6*x*z + 1;
24 func uex = 4*x + 5*y + 6*z;
25 func uey = 6*y + 5*x;
26 func uez = 8*z + 6*x;
27 func f = -18.;
28
29 // Mesh
30 mesh Th2 = square(nn, nn);
31
32 int[int] rup = [0,2], rdown=[0, 1];
33 int[int] rmid=[1, 1, 2, 1, 3, 1, 4, 1];
34
35 mesh3 Th=buildlayers(Th2, nn, zbound=[zmin, zmax], reffacemid=rmid,
36   reffaceup = rup, reffacelow = rdown);
37
38 // Fespace
39 fespace Vh2(Th2, P2);
40 Vh2 ux, uz, p2;
41
42 fespace Vh(Th, P2);
43 Vh uhe = ue;
44 cout << "uhe min =" << uhe[].min << ", max =" << uhe[].max << endl;
45 Vh u, v;
46 Vh F;
47
48 // Macro
49 macro Grad3(u) [dx(u), dy(u), dz(u)] //
50
51 // Problem
52 varf va (u, v)
53   = int3d(Th) (
54     Grad3(v)' * Grad3(u)
55   )

```

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```

56     + int2d(Th, 2) (
57         u*v
58     )
59     - int3d(Th) (
60         f*v
61     )
62     - int2d(Th, 2) (
63         ue*x + (uex*N.x + uey*N.y + uez*N.z)*v
64     )
65     + on(1, u=ue);
66
67 varf l (unused, v) = int3d(Th) (f*v);
68
69 real cpu=clock();
70 matrix Aa = va(Vh, Vh);
71
72 F[] = va(0, Vh);
73
74 if (mpirank == 0) {
75     cout << "Size of A =" << Aa.n << endl;
76     cout << "Non zero coefficients =" << Aa.nbcoef << endl;
77     cout << "CPU TIME FOR FORMING MATRIX =" << clock()-cpu << endl;
78 }
79
80 set(Aa, solver=sparse solver, dparams=dparm, lparams=iparm); //Set hips as linear_
81 //solver
82
83 // Solve
84 u[] = Aa^-1*F[];
85
86 // Plot
plot(u);

```

**Table 3.11:** Legend of this table are give in Table 3.6

$n = 4 \times 10^6$ $nnz = 118 \times 10^6$ $Te = 221.34$		
np	nit	time
8	190	120.34
16	189	61.08
32	186	31.70
64	183	23.44

**Tip:****Table 3.12:** Significations of lparams corresponding to HIPS interface

Entries of iparm	Significations of each entries
iparm[0]	Strategy use for solving (Iterative=0 or Hybrid=1 or Direct=2). Defaults values are : Iterative
iparm[1]	Krylov methods. If iparm[0]=0, give type of Krylov methods: 0 for GMRES, 1 for PCG
iparm[2]	Maximum of iterations in outer iteration: default value 1000
iparm[3]	Krylov subspace dimension in outer iteration: default value 40
iparm[4]	Symmetric(=0 for symmetric) and 1 for unsymmetricmatrix: default value 1 (unsymmetric matrix)
iparm[5]	Pattern of matrix are symmetric or not: default value 0
iparm[6]	Partition type of input matrix: default value 0
iparm[7]	Number of level that use the HIPS locally consistentfill-in: Default value 2
iparm[8]	Numbering in indices array will start at 0 or 1: Default value 0
iparm[9]	Scale matrix. Default value 1
iparm[10]	Reordering use inside subdomains for reducingfill-in: Only use for iterative. Default value 1
iparm[11]	Number of unknowns per node in the matrix non-zeropattern graph: Default value 1
iparm[12]	This value is used to set the number of time the normalization is applied to the matrix: Default 2.
iparm[13]	Level of informations printed during solving: Default 5.
iparm[14]	HIPS_DOMSIZE Subdomain size

**Table 3.13:** Significations of dparams corresponding to HIPS interface

dparm[0]	HIPS_PREC: Relative residual norm: Default=1e-9
dparm[1]	HIPS_DROPTOL0: Numerical threshold in ILUT for interior domain (important : set 0.0 in HYBRID: Default=0.005)
dparm[2]	HIPS_DROPTOL1 : Numerical threshold in ILUT for Schur preconditioner: Default=0.005
dparm[3]	HIPS_DROPTOLE : Numerical threshold for coupling between the interior level and Schur: Default 0.005
dparm[4]	HIPS_AMALG : Numerical threshold for coupling between the interior level and Schur: Default=0.005
dparm[5]	HIPS_DROPSCHUR : Numerical threshold for coupling between the interior level and Schur: Default=0.005

## Interfacing with HYPRE

[Hypre](#) (High Level Preconditioner) is a suite of parallel preconditioner developed at Lawrence Livermore National Lab.

There are two main classes of preconditioners developed in HYPRE: AMG (Algebraic MultiGrid) and Parasails (Parallel Sparse Approximate Inverse).

Now, suppose we want to solve  $Ax = b$ .

At the heart of AMG there is a series of progressively coarser (smaller) representations of the matrix  $A$ . Given an approximation  $\hat{x}$  to the solution  $x$ , consider solving the residual equation  $Ae = r$  to find the error  $e$ , where  $r = b - A\hat{x}$ . A fundamental principle of AMG is that it is an algebraically smooth error. To reduce the algebraically smooth errors

further, they need to be represented by a smaller defect equation (coarse grid residual equation)  $A_c e_c = r_c$ , which is cheaper to solve. After solving this coarse equation, the solution is then interpolated in fine grid represented here by matrix  $A$ . The quality of AMG depends on the choice of coarsening and interpolating operators.

The *sparse approximate inverse* approximates the inverse of a matrix  $A$  by a sparse matrix  $M$ . A technical idea to construct matrix  $M$  is to minimize the Frobenius norm of the residual matrix  $I - MA$ . For more details on this preconditioner technics see [CHOW1997].

HYPRE implement three Krylov subspace solvers: GMRES, PCG and BiCGStab.

---

**Tip:** Laplacian 3D solved with HYPRE

Let us consider again the 3D Laplacian example inside **FreeFEM** package where after discretization we want to solve the linear equation with Hypre. The following example is a Laplacian 3D using Hypre as linear solver. This is the same example as Hips one, so we just show here the lines where we set some Hypre parameters.

We first load the Hypre solver at line 2. From line 6 to 18 we specifies the parameters to set to Hypre solver and in line 22 we set parameters to Hypre solver.

It should be noted that the meaning of the entries of these vectors is different from those of Hips. In the case of HYPRE, the meaning of differents entries of vectors `iparm` and `dparm` are given in Table 3.14 to Table 3.18.

In Table 3.19 the results of running on Cluster Paradent of Grid5000 are reported. We can see in this running example the efficiency of parallelism, in particular when AMG are use as preconditioner.

```

1  load "msh3"
2  load "hipre_FreeFem" //Load Hipre librairy
3
4  // Parameters
5  int nn = 10;
6  int[int] iparm(20);
7  real[int] dparm(6);
8  for (int iiii = 0; iiii < 20; iiii++)
9    iparm[iiii] = -1;
10 for (int iiii = 0; iiii < 6; iiii++)
11   dparm[iiii] = -1;
12 iparm[0] = 2; //PCG as krylov method
13 iparm[1] = 0; //AMG as preconditionner 2: if ParaSails
14 iparm[7] = 7; //Interpolation
15 iparm[9] = 6; //AMG Coarsen type
16 iparm[10] = 1; //Measure type
17 iparm[16] = 2; //Additive schwarz as smoother
18 dparm[0] = 1e-13; //Tolerance to convergence
19 dparm[1] = 5e-4; //Threshold
20 dparm[2] = 5e-4; //Truncation factor
21
22 ...
23
24 set (Aa, solver=sparse solver, dparams=dparm, lparams=iparm);
```

---

**Table 3.14:** Definitions of common entries of `iparms` and `dparms` vectors for every preconditioner in HYPRE

iparms [0]	Solver identification: 0: BiCGStab, 1: GMRES, 2: PCG. Default=1
iparms [1]	Preconditioner identification: 0: BOOMER AMG, 1: PILUT, 2: Parasails, 3: Schwartz Default=0
iparms [2]	Maximum of iteration: Default=1000
iparms [3]	Krylov subspace dim: Default= 40
iparms [4]	Solver print info level: Default=2
iparms [5]	Solver log: Default=1
iparms [6]	Solver stopping criteria only for BiCGStab : Default=1
dparms [0]	Tolerance for convergence: Default=:math:1.0e-11

**Table 3.15:** Definitions of other entries of iparms and dparms if preconditioner is BOOMER AMG

iparm [AMG]	AMG interpolation type: Default=6
iparm [GSMG]	Specifies the use of GSMG - geometrically smooth coarsening and interpolation: Default=1
iparm [AMG]coarsen	AMG coarsen type: Default=6
iparm [AMG]local	Defines whether local or global measures are used: Default=1
iparm [AMG]cycle	AMG cycle type: Default=1
iparm [AMG]smoother	AMG Smoother type: Default=1
iparm [AMG]levels	AMG number of levels for smoothers: Default=3
iparm [AMG]sweeps	AMG number of sweeps for smoothers: Default=2
iparm [AMG]multilevel	AMG maximum number of multigrid levels: Default=25
iparm [Schwartz]	Defines which variant of the Schwartz method is used: 0: hybrid multiplicative Schwartz method (no overlap across processor boundaries) 1: hybrid additive Schwartz method (no overlap across processor boundaries) 2: additive Schwartz method 3: hybrid multiplicative Schwartz method (with overlap across processor boundaries) Default=1
iparm [Size]	Size of the system of PDEs: Default=1
iparm [Overlap]	Overlap for the Schwarz method: Default=1
iparm [Type]	Type of domain used for the Schwarz method 0: each point is a domain 1: each node is a domain (only of interest in “systems” AMG) 2: each domain is generated by agglomeration (default)
dparm [AMG]	AMG strength threshold: Default=0.25
dparm [Trunc]	Truncation factor for the interpolation: Default=1e-2
dparm [SetS]	SetS a parameter to modify the definition of strength for diagonal dominant portions of the matrix: Default=0.9
dparm [Additive]	Defines a smoothing parameter for the additive Schwartz method. Default=1

**Table 3.16:** Definitions of other entries of iparms and dparms if preconditioner is PILUT

iparms [7]	Row size in Parallel ILUT: Default=1000
iparms [8]	Set maximum number of iterations: Default=30
dparms [1]	Drop tolerance in Parallel ILUT: Default=1e-5

**Table 3.17:** Definitions of other entries of iparms and dparms if preconditioner is ParaSails

iparms	Number of levels in Parallel Sparse Approximate inverse: Default=1
iparms	Symmetric parameter for the ParaSails preconditioner: 0: nonsymmetric and/or indefinite problem, and nonsymmetric preconditioner 1: SPD problem, and SPD (factored) preconditioner 2: nonsymmetric, definite problem, and SPD (factored) preconditioner Default=0
dparms	Filters parameters. The filter parameter is used to drop small nonzeros in the preconditioner, to reduce the cost of applying the preconditioner: Default=0.1
dparms	Threshold parameter: Default=0.1

**Table 3.18:** Definitions of other entries of iparms and dparms if preconditioner is Schwartz

iparm	Defines which variant of the Schwartz method is used: 0: hybrid multiplicative Schwartz method (no overlap across processor boundaries) 1: hybrid additive Schwartz method (no overlap across processor boundaries) 2: additive Schwartz method 3: hybrid multiplicative Schwartz method (with overlap across processor boundaries) Default=1
iparm	Overlap for the Schwartz method: Default=1
iparm	Type of domain used for the Schwartz method 0: each point is a domain 1: each node is a domain (only of interest in “systems” AMG) 2: each domain is generated by agglomeration (default)

**Table 3.19:** Convergence and time for solving linear system

$n = 4 \times 10^6$	$nnz = 13 \times 10^6$	$Te = 571.29$
np	AMG	
	<i>nit</i>	<i>time</i>
8	6	1491.83
16	5	708.49
32	4	296.22
64	4	145.64

## Conclusion

With the different runs presented here, we wanted to illustrate the gain in time when we increase the number of processors used for the simulations. We saw that in every case the time for the construction of the finite element matrix is constant. This is normal because until now this phase is sequential in **FreeFEM**. In contrast, phases for solving the linear system are parallel. We saw on several examples presented here that when we increase the number of processors, in general we decrease the time used for solving the linear systems. But this is not true in every case. In several cases, when we increase the number of processors the time to convergence also increases. There are two main reasons for this. First, the increase of processors can lead to the increase of volume of exchanged data across processors consequently increasing the time for solving the linear systems.

Furthermore, in decomposition domain type preconditioners, the number of processors generally corresponds to the number of sub domains. In subdomain methods, generally when we increase the number of subdomains we decrease convergence quality of the preconditioner. This can increase the time used for solving linear equations.

To end this, we should note that good use of the preconditioners interfaced in **FreeFEM** is empiric, because it is difficult to know what is a good preconditioner for some type of problems. Although, the efficiency of preconditioners sometimes depends on how its parameters are set. For this reason we advise the user to pay attention to the meaning of the parameters in the user guide of the iterative solvers interfaced in **FreeFEM**.

## Domain decomposition

In the previous section, we saw that the phases to construct a matrix are sequential. One strategy to construct the matrix in parallel is to divide geometrically the domain into subdomains. In every subdomain we construct a local submatrix and after that we assemble every submatrix to form the global matrix.

We can use this technique to solve PDE directly in domain  $\Omega$ . In this case, in every subdomains you have to define artificial boundary conditions to form consistent equations in every subdomains. After this, you solve equation in every subdomains and define a strategy to obtain the global solution.

In terms of parallel programming for **FreeFEM**, with MPI, this means that the user must be able to divide processors available for computation into subgroups of processors and also must be able to realize different type of communications in **FreeFEM** script. Here is a wrapper of some MPI functions.

## Communicators and groups

### Groups

**mpiGroup grpe(mpiGroup gp, KN\_<long>):** Create MPI\_Group from existing group **gp** by given vector.

### Communicators

Communicators is an abstract MPI object which allows MPI user to communicate across group of processors. Communicators can be Intra-communicators(involves a single group) or Inter-communicators (involves two groups). When we not specify type of communicator it will be Intra-communicators

**mpiComm cc(mpiComm comm, mpiGroup gp):** Creates a new communicator.

**comm** communicator(handle), **gp** group which is a subset of the group of **comm** (handle). Return new communicator

**mpiComm cc(mpiGroup gp):** Same as previous constructor but default **comm** here is MPI\_COMM\_WORLD.

**mpiComm cc(mpiComm comm, int color, int key):** Creates new communicators based on **colors** and **key**. This constructor is based on MPI\_Comm\_split routine of MPI.

**mpiComm cc(MPIrank p, int key):** Same constructor than the last one.

Here **colors** and **comm** is defined in MPIrank. This constructor is based on MPI\_Comm\_split routine of MPI.

---

### Tip: Split communicator

```

1 mpiComm comm(mpiCommWorld, 0, 0);
2 int color = mpiRank(comm)%2;
3 mpiComm ccc(processor(color, comm), 0);
4 mpiComm qpp(comm, 0, 0);
5 mpiComm cp(ccc, color, 0);

```

---

**mpiComm cc(mpiComm comm, int high):** Creates an intracomunicator from an intercommunicator. **comm** inter-communicator, **high**.

Used to order the groups within **comm** (logical) when creating the new communicator. This constructor is based on MPI\_Intercomm\_merge routine of MPI.

**mpiComm cc(MPIrank p1, MPIrank p2, int tag):** This constructor creates an intercommunicator from two intra-communicators. **p1** defined local (intra)communicator and rank in **local\_comm** of leader (often 0) while **p2** defined remote communicator and rank in **peer\_comm** of remote leader (often 0). **tag** Message tag to use in constructing intercommunicator. This constructor is based on MPI\_Intercomm\_create.

---

**Tip:** Merge

```
1 mpiComm comm, cc;
2 int color = mpiRank(comm)%2;
3 int rk = mpiRank(comm);
4 int size = mpiSize(comm);
5 cout << "Color values: " << color << endl;
6 mpiComm ccc(processor((rk<size/2), comm), rk);
7 mpiComm cp(cc, color, 0);
8 int rleader;
9 if (rk == 0){ rleader = size/2; }
10 else if (rk == size/2){ rleader = 0; }
11 else{ rleader = 3; }
12 mpiComm qqp(processor(0, ccc), processor(rlider, comm), 12345);
13 int aaa = mpiSize(ccc);
14 cout << "Number of processor: " << aaa << endl;
```

---

**Process**

In **FreeFEM** we wrap MPI process by function call `processor` which create internal **FreeFEM** object call `MPIrank`. This mean that do not use `MPIrank` in **FreeFEM** script.

`processor(int rk)`: Keep process rank inside object `MPIrank`. Rank is inside `MPI_COMM_WORLD`.

`processor(int rk, mpiComm cc)` and `processor(mpiComm cc, int rk)` process rank inside communicator `cc`.

`processor(int rk, mpiComm cc)` and `processor(mpiComm cc, int rk)` process rank inside communicator `cc`.

`processorblock(int rk)`: This function is exactly the same than `processor(int rk)` but is use in case of blocking communication.

`processorblock(int rk, mpiComm cc)`: This function is exactly the same as `processor(int rk, mpiComm cc)` but uses a synchronization point.

**Points to Points communicators**

In **FreeFEM** you can call MPI points to points communications functions.

`Send(processor(int rk, mpiComm cc), Data D)` : Blocking send of Data `D` to processor of rank `rk` inside communicator `cc`. Note that Data `D` can be: `int`, `real`, `complex`, `int[int]`, `real[int]`, `complex[int]`, `Mesh`, `Mesh3`, `Matrix`.

`Recv(processor(int rk, mpiComm cc), Data D)`: Receive Data `D` from process of rank `rk` in communicator `cc`.

Note that Data `D` can be: `int`, `real`, `complex`, `int[int]`, `real[int]`, `complex[int]`, `Mesh`, `Mesh3`, `Matrix` and should be the same type than corresponding send.

`Irecv(processor(int rk, mpiComm cc), Data D)` : Non blocking send of Data `D` to processor of rank `rk` inside communicator `cc`.

Note that Data `D` can be: `int`, `real`, `complex`, `int[int]`, `real[int]`, `complex[int]`, `mesh`, `mesh3`, `matrix`.

Recv(processor(int rk, mpiComm cc), Data D): Receive corresponding to send.

## Global operations

In **FreeFEM** you can call MPI global communication functions.

broadcast(processor(int rk, mpiComm cc), Data D): Process rk Broadcast Data D to all process inside communicator cc. Note that Data D can be: int, real, complex, int[int], real[int], complex[int], Mesh, Mesh3, Matrix.

broadcast(processor(int rk), Data D): Process rk Broadcast Data D to all process inside MPI\_COMM\_WORLD. Note that Data D can be: int, real, complex, int[int], real[int], complex[int], Mesh, Mesh3, Matrix.

mpiAlltoall(Data a, Data b): Sends data a from all to all processes. Receive buffer is Data b. This is done inside communicator MPI\_COMM\_WORLD.

mpiAlltoall(Data a, Data b, mpiComm cc): Sends data a from all to all processes. Receive buffer is Data b. This is done inside communicator cc.

mpiGather(Data a, Data b, processor(mpiComm, int rk)): Gathers together values Data a from a group of processes. Process of rank rk get data on communicator rk. This function is like MPI\_Gather.

mpiAllgather(Data a, Data b): Gathers Data a from all processes and distribute it to all in Data b. This is done inside communicator MPI\_COMM\_WORLD. This function is like MPI\_Allgather.

mpiAllgather(Data a, Data b, mpiComm cc): Gathers Data a from all processes and distribute it to all in Data b. This is done inside communicator cc. This function is like MPI\_Allgather.

mpiScatter(Data a, Data b, processor(int rk, mpiComm cc)): Sends Data a from one process with rank rk to all other processes in group represented by communicator mpiComm cc.

mpiReduce(Data a, Data b, processor(int rk, mpiComm cc), MPI\_Op op) Reduces values Data a on all processes to a single value Data b on process of rank rk and communicator cc.

Operation use in reduce is: MPI\_Op op which can be: mpiMAX, mpiMIN, mpiSUM, mpiPROD, mpiLAND, mpiLOR, mpiLXOR, mpiBAND, mpiBXOR, mpiMAXLOC, mpiMINLOC.

Note that, for all global operations, only int[int] and real[int] are data type take in account in **FreeFEM**.

## HPDDM solvers

Real valued problems (diffusion, heat, elasticity and Stokes) and complex valued problems (Maxwell and Helmholtz) are given in both 2D and 3D. We detail here the 3D elasticity problem and the 3D time-dependent heat problem.

---

### Tip: Elasticity 3D

A three dimensional elasticity problem is defined. The solver is a domain decomposition method. Domain decomposition methods are a natural framework for parallel computers. The scripts run on multicores computers (from 2 to tens of thousands of cores). Recall that like in any MPI code the number of MPI processes, mpisize, is given in the command line via the option -np. We focus on the script `Elasticity3D.edp` but the other scripts have the same structure. The command line to run the example on four processes with ffglut visualization is: ff-mpirun -np 4 `Elasticity3D.edp -glut ffglut`

```

1 load "hpddm" //load HPDDM plugin
2 macro partitioner()metis//metis, scotch, or parmetis
3 macro dimension()3//2D or 3D

```

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```

4  macro vectorialfe()P1//  

5  include "macro_ddm.idp" //additional DDM functions  

6  

7 // Macro  

8 macro def(i)[i, i#B, i#C] //vector field definition  

9 macro init(i)[i, i, i] //vector field initialization  

10  

11 real Sqrt = sqrt(2.0);  

12 macro epsilon(u) [dx(u), dy(u#B), dz(u#C),  

13   (dz(u#B) + dy(u#C)) / Sqrt,  

14   (dz(u) + dx(u#C)) / Sqrt,  

15   (dy(u) + dx(u#B)) / Sqrt] //  

16 macro div(u) (dx(u) + dy(u#B) + dz(u#C)) //  

17  

18 // Parameters  

19 real f = -9000.0;  

20 real strain = 100.0;  

21 real Young = 2.0e11; // steel  

22 real poisson = 0.35;  

23  

24 func Pk = [vectorialfe, vectorialfe, vectorialfe];  

25  

26 string deflation = getARGV("-deflation", "geneo"); //coarse space construction  

27 int overlap = getARGV("-overlap", 1); //geometric overlap between subdomains  

28 int fakeInterface = getARGV("-interface", 10); //interface between subdomains  

29 int s = getARGV("-split", 1); //refinement factor  

30 int p = getARGV("-hpddm_master_p", 1);  

31  

32 mpiComm comm;  

33 bool excluded = splitComm(mpisize, p, comm, topology = getARGV("-hpddm_master_"
  ↪ topology", 0), exclude = (usedARGV("-hpddm_master_exclude") != -1));  

34  

35 // Display  

36 if (verbosity > 0 && mpirank == 0){  

37   cout << " --- " << mpirank << "/" << mpisize;  

38   cout << " - Elasticity3D.edp - input parameters: refinement factor = " << s << " - "
  ↪ overlap = " << overlap << endl;  

39 }  

40  

41 // Mesh  

42 int[int] LL = [2, 3, 2, 1, 2, 2];  

43 meshN ThBorder, Th = cube(1, 1, 1, [x, y, z]);  

44 fespace Wh(Th, Pk); //local finite element space  

45  

46 int[int] arrayIntersection; //ranks of neighboring subdomains  

47 int[int][int] restrictionIntersection(0); //local-to-neighbors renumbering  

48 real[int] D; //partition of unity  

49 {  

50   meshN ThGlobal = cube(10*getARGV("-global", 5), getARGV("-global", 5), getARGV("-"
    ↪ global", 5), [10*x, y, z], label=LL); //global mesh  

51   build(Th, ThBorder, ThGlobal, fakeInterface, s, overlap, D, arrayIntersection, "
    ↪ restrictionIntersection, Wh, Pk, comm, excluded, 3)  

52 }  

53  

54 // Problem  

55 real tmp = 1.0 + poisson;  

56 real mu = Young / (2.0 * tmp);

```

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```

57 real lambda = Young * poisson / (tmp * (1.0 - 2.0 * poisson));
58 real[int] rhs; //local right-hand side
59 matrix<real> Mat; //local operator
60 { //local weak form
61     meshN ThAugmented = Th + ThBorder;
62     varf vPb (def(u), def(v))
63         = intN(ThAugmented) (
64             lambda * div(u) * div(v)
65             + 2.0 * mu * (epsilon(u)' * epsilon(v))
66         )
67         + intN(ThAugmented) (
68             f * vC
69         )
70         + on(1, u=0.0, uB=0.0, uC=0.0)
71     ;
72
73     fespace WhAugmented(ThAugmented, Pk);
74     Mat = vPb(WhAugmented, WhAugmented, tgv=-1);
75     real[int] rhsFull = vPb(0, WhAugmented, tgv=-1);
76     matrix R = interpolate(Wh, WhAugmented);
77     renumbering(Mat, R, rhsFull, rhs);
78 }
79 ThBorder = cube(1, 1, 1, [x, y, z] );
80
81 dschwarz A(Mat, arrayIntersection, restrictionIntersection, scaling = D);
82
83 set(A, sparams = "-hpddm_schwarz_method ras -hpddm_schwarz_coarse_correction balanced"
84     ↪-hpddm_variant right -hpddm_verbose 1 -hpddm_geneo_nu 10");
85
86 matrix<real> Opt; //local operator with optimized boundary conditions
87 dpair ret;
88 {
89     int solver = getopt("schwarz_method");
90     if (solver == 1 || solver == 2 || solver == 4){ //optimized Schwarz methods
91         fespace Ph(Th, P0);
92         real kZero = getARGV("-kZero", 10.0);
93         Ph transmission = 2 * kZero * mu * (2 * mu + lambda) / (lambda + 3 * mu);
94         varf vOptimized (def(u), def(v))
95             = intN(Th) (
96                 lambda * div(u) * div(v)
97                 + 2.0 * mu * (epsilon(u)' * epsilon(v))
98             )
99             + intN1(Th, fakeInterface) (
100                 transmission * (def(u)' * def(v))
101             )
102             + on(1, u=0.0, uB=0.0, uC=0.0)
103             ;
104         Opt = vOptimized(Wh, Wh, tgv=-1);
105     }
106     if (mpisize > 1 && isSetOption("schwarz_coarse_correction")){ //two-level Schwarz
107     ↪methods
108         if(excluded)
109             attachCoarseOperator(mpiCommWorld, A);
110         else {
111             varf vPbNoPen (def(u), def(v))
112                 = intN(Th) (
113                     lambda * div(u) * div(v)

```

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```

112         + 2.0 * mu * (epsilon(u)' * epsilon(v))
113     )
114     + on(1, u=0.0, uB=0.0, uC=0.0)
115 ;
116     matrix<real> noPen = vPbNoPen(Wh, Wh, solver=CG);
117     if(deflation == "geneo") //standard GenEO, no need for RHS -> deduced
118     ↪from LHS (Neumann matrix)
119     attachCoarseOperator(MPICommWorld, A, A=noPen, ret=ret);
120     else if(deflation == "dtn"){
121         varf vMass (def(u), def(v)) = intN1(Th, fakeInterface)(u * v);
122         matrix<real> massMatrix = vMass(Wh, Wh, solver=CG);
123         attachCoarseOperator(MPICommWorld, A, A=noPen, B=massMatrix,
124     ↪pattern=Opt, ret=ret);
125     }
126     else if(deflation == "geneo-2") //GenEO-2 for optimized Schwarz methods,
127     ↪need for RHS (LHS is still Neumann matrix)
128     attachCoarseOperator(MPICommWorld, A, A=noPen, B=Opt, pattern=Opt,
129     ↪ret=ret);
130 }
131
132 // Solve
133 Wh<real> def(u); //local solution
134
135 if(Opt.n > 0) //optimized Schwarz methods
136     DDM(A, u[], rhs, excluded=excluded, ret=ret, O=Opt);
137 else
138     u[] = A^-1 * rhs;
139
140 // Error
141 real[int] err(u[].n);
142 err = A * u[]; //global matrix-vector product
143 err -= rhs;
144
145 // Plot
146 plotMPI(Th, u[], "Global solution", Pk, def, real, 3, 1)
147 plotMPI(Th, err, "Global residual", Pk, def, real, 3, 1)
148 real alpha = 2000.0;
149 meshN ThMoved = movemesh3(Th, transfo = [x + alpha*u, y + alpha*uB, z + alpha*uC]);
150 u[] = mpirank;
151 plotMPI(ThMoved, u[], "Global moved solution", Pk, def, real, 3, 1)

```

The macro build is of particular interest since it handles the data distribution among the `mpisize` MPI processes with the following steps:

- The initial mesh `ThGlobal` is partitioned by process 0 into `mpisize` submeshes
- **The partition is broadcasted to every process  $i$  for  $0 < i < \text{mpisize}$ .** From then on, all tasks are parallel.
- **Each process creates the local submesh `Th` (if the refinement factor `s` defined via the option `-split` is larger than 1, each**  
The number of extra layers added to the initial partition is monitored by the option `overlap`.
- Connectivity structures are created
- `D` is the diagonal of the local partition of unity (see [Distributed vectors in HPDDM](#))
- `arrayIntersection` is the list of neighbors of the current subdomain

- For  $j$  in `arrayIntersection`, `restrictionIntersection[j]` is the list of the degrees of freedom that belong to the intersection of the current subdomain with its neighbor  $j$ .

Then, the variational formulation `vPb` of a three dimensional elasticity problem is used to assemble a local matrix `Mat`. This matrix along with `D`, `arrayIntersection` and `restrictionIntersection` are arguments for the constructor of the distributed matrix `A`. This is enough to solve the problem with a one-level additive Schwarz method which can be either ASM or RAS.

For some problems it is interesting to use optimized interface conditions. When there are many subdomains, it is usually profitable to add a second level to the solver. Options are set in the sequel of the script:

```
1 set (A, sparams="-hpddm_schwarz_method ras -hpddm_schwarz_coarse_correction balanced -  
-hpddm_variant right -hpddm_verbose 1 -hpddm_geneo_nu 10");
```

In the above line, the first option selects the one-level preconditioner `ras` (possible choices are `ras`, `oras`, `soras`, `asm`, `osm` or `none`), the second option selects the correction formula for the second level here `balanced` (possible options are `deflated`, `additive` or `balanced`), the third option selects right preconditioning, the fourth one is verbosity level of HPDDM (different from the one of FreeFEM), the fifth one prints all possible options of HPPDM and the last one specifies the number of coarse degrees of freedom per subdomain of the GENEO coarse space. All other options of `HPDDM library` can be selected via the `FreeFEM` function `set`.

In the last part of the script, the global linear system is solved by the domain decomposition method defined above.

```
1 // Solve  
2 Wh<real> def(u); //local solution  
3  
4 if(Opt.n > 0) //optimized Schwarz methods  
5 DDM(A, u[], rhs, excluded=excluded, ret=ret, O=Opt);  
6 else  
7 u[] = A^-1 * rhs;
```

## Time dependent problem

### Tip: Heat 3D

A three dimensional heat problem

$$\frac{\partial u}{\partial t} - \Delta u = 1, \quad u(0, \cdot) := 0 \text{ in } \Omega.$$

is discretized by an implicit Euler scheme. At each time step  $n$ , we shall seek  $u^n(x, y, z)$  satisfying for all  $w \in H^1(\Omega)$ :

$$\int_{\Omega} \frac{u^n - u^{n-1}}{\delta t} w + \nabla u^n \nabla w = \int_{\Omega} w, \quad u^0 := 0 \text{ in } \Omega.$$

so that at each time step a linear system:

$$(M + dt * K)u^n[] = M * u^{n-1}[] + \delta t * F$$

is solved by a domain decomposition method where  $M$  is the mass matrix and  $K$  is the rigidity matrix. In order to save computational efforts, the domain decomposition method preconditioner is built only once and then reused for all subsequent solves with matrix  $A := M + dt * K$ . The distributed matrix vector product with matrix  $M$  is made through the call to the function `dmv` using the partition of unity associated to matrix  $A$ .

```

1 load "hpddm" //load HPDDM plugin
2 macro partitioner()metis//metis, scotch, or parmetis
3 macro dimension()3//2D or 3D
4 include "macro_ddm.idp" //additional DDM functions
5
6 // Macro
7 macro def(i)i //scalar field definition
8 macro init(i)i //scalar field initialization
9 macro grad(u) [dx(u), dy(u), dz(u)] //three-dimensional gradient
10
11 // Parameters
12 func Pk = P2; //finite element space
13
14 string deflation = getARGV("-deflation", "geneo"); //coarse space construction
15 int overlap = getARGV("-overlap", 1); //geometric overlap between subdomains
16 int fakeInterface = getARGV("-interface", 10); //interface between subdomains
17 int s = getARGV("-split", 1); //refinement factor
18 real dt = getARGV("-dt", 0.01); //time step
19 int iMax = getARGV("-iMax", 10); //number of iterations
20
21 mpiComm comm;
22 int p = getARGV("-hpddm_master_p", 1);
23 bool excluded = splitComm(mpiCommWorld, p, comm, topology = getARGV("-hpddm_master_"
    ↪topology", 0), exclude = (usedARGV("-hpddm_master_exclude") != -1));
24
25 // Display
26 if (verbosity > 0 && mpirank == 0){
27     cout << " --- " << mpirank << "/" << mpisize;
28     cout << " - Heat3D.edp - input parameters: refinement factor = " << s << " -"
    ↪overlap = " << overlap << endl;
29 }
30
31 // Mesh
32 int[int] LL = [1, 2, 1, 1, 1, 1];
33 meshN ThBorder, Th = cube(1, 1, 1, [x, y, z]);
34 fespace Wh(Th, Pk); //local finite element space
35 int[int] arrayIntersection; //ranks of neighboring subdomains
36 int[int][int] restrictionIntersection(0); //local-to-neighbors renumbering
37 real[int] D; //partition of unity
38 {
39     meshN ThGlobal = cube(getARGV("-global", 10), getARGV("-global", 10), getARGV("-"
    ↪global", 10), [x, y, z], label=LL); //global mesh
40     build(Th, ThBorder, ThGlobal, fakeInterface, s, overlap, D, arrayIntersection,
    ↪restrictionIntersection, Wh, Pk, comm, excluded)
41 }
42
43 // Problem
44 real[int] rhs; // local right-hand side
45 matrix<real> Mat; //local operator
46 matrix<real> M; //local mass matrix
47 { //local weak form
48     meshN ThAugmented = Th + ThBorder;
49     varf vPb (u, v)
50         = intN(ThAugmented) (
51             u * v
52             + dt * (grad(u)' * grad(v))
53         )

```

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```

54     + intN(ThAugmented) (
55         dt * v
56     )
57     + on(1, u=0.0)
58 ;
59 fespace WhAugmented(ThAugmented, Pk);
60 Mat = vPb(WhAugmented, WhAugmented, tgv=-1);
61 real[int] rhsFull = vPb(0, WhAugmented, tgv=-1);
62 matrix R = interpolate(Wh, WhAugmented);
63 varf vPbM (u, v) = intN(ThAugmented) (u * v);
64 M = vPbM(WhAugmented, WhAugmented);
65 renumbering(M, R, rhsFull, rhs);
66 renumbering(Mat, R, rhsFull, rhs);
67 }
68 ThBorder = cube(1, 1, 1, [x, y, z]);
69
70 dschwarz A(Mat, arrayIntersection, restrictionIntersection, scaling=D);
71
72 matrix<real> Opt; //local operator with optimized boundary conditions
73 dpair ret;
74 {
75     int solver = getopt("schwarz_method");
76     if (solver == 1 || solver == 2 || solver == 4){ //optimized Schwarz methods
77         fespace Ph(Th, P0);
78         real kZero = getARGV("-kZero", 10.0);
79         Ph transmission = kZero;
80         varf vOptimized (u, v)
81             = intN(Th) (
82                 u * v
83                 + dt * (grad(u)' * grad(v))
84             )
85             + intN1(Th, fakeInterface) (
86                 transmission * (u * v)
87             )
88             + on(1, u=0.0)
89 ;
90         Opt = vOptimized(Wh, Wh, tgv=-1);
91     }
92     if (mpisize > 1 && isSetOption("schwarz_coarse_correction")){ //two-level Schwarz
93         ↪methods
94         if(excluded)
95             attachCoarseOperator(MPICommWorld, A);
96         else {
97             varf vPbNoPen (u, v)
98                 = intN(Th) (
99                     u * v
100                    + dt * (grad(u)' * grad(v))
101                )
102                + on(1, u=0.0)
103 ;
104             matrix<real> noPen = vPbNoPen(Wh, Wh, solver=CG);
105             if(deflation == "geneo") //standard GenEO, no need for RHS -> deduced
106             ↪from LHS (Neumann matrix)
107                 attachCoarseOperator(MPICommWorld, A, A=noPen, ret = ret);
108             else if(deflation == "dtn") {
109                 varf vMass (def(u), def(v)) = intN1(Th, fakeInterface) (u * v);
110                 matrix<real> massMatrix = vMass(Wh, Wh, solver=CG);

```

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```

109         attachCoarseOperator(mpiCommWorld, A, A=noPen, B=massMatrix, pattern=Opt, ret=ret);
110     }
111     else if(deflation == "geneo-2") //GenEO-2 for optimized Schwarz methods, need for RHS (LHS is still Neumann matrix)
112     attachCoarseOperator(mpiCommWorld, A, A=noPen, B=Opt, pattern=Opt, ret=ret);
113   }
114 }
115
116
117 // Solve
118 set(A, sparams="-hpddm_reuse_preconditioner=1");
119 Wh<real> def(u) = init(0.0); //local solution
120 for (int i = 0; i < iMax; ++i){
121   real[int] newRhs(rhs.n);
122   dmv(A, M, u[], newRhs); //newRhs = M * u[]
123   newRhs += rhs;
124
125   if (Opt.n > 0) //optimized Schwarz methods
126   DDM(A, u[], newRhs, excluded=excluded, ret=ret, O=Opt);
127   else
128     u[] = A-1 * newRhs;
129
130   plotMPI(Th, u[], "Global solution", Pk, def, real, 3, 0)
131 }
```

## Distributed vectors in HPDDM

We give here some hints on the way vectors are distributed among  $np$  processes when using **FreeFEM** interfaced with HPDDM. The set of degrees of freedom  $\mathcal{N}$  is decomposed into  $np$  overlapping sets  $(\mathcal{N}_i)_{1 \leq i \leq np}$ .

A MPI-process is in charge of each subset. Let  $n := \#\mathcal{N}$  be the number of degrees of freedom of the global finite element space. Let  $R_i$  denote the restriction operator from  $\mathbb{R}^n$  onto  $\mathbb{R}^{\#\mathcal{N}_i}$ . We have also defined local diagonal matrices  $D_i \in \mathbb{R}^{\#\mathcal{N}_i} \times \mathbb{R}^{\#\mathcal{N}_i}$  so that we have a partition of unity at the algebraic level:

$$\mathbf{U} = \sum_{i=1}^{np} R_i^T D_i R_i \mathbf{U} \quad \forall \mathbf{U} \in \mathbb{R}^n. \quad (3.31)$$

A global vector  $\mathbf{U} \in \mathbb{R}^n$  is actually not stored. Rather, it is stored in a distributed way. Each process  $i$ ,  $1 \leq i \leq N$ , stores the local vector  $\mathbf{U}_i := R_i \mathbf{U} \in \mathbb{R}^{\#\mathcal{N}_i}$ .

It is important to ensure that the result of all linear algebra operators applied to this representation are coherent.

As an example, consider the scalar product of two distributed vectors  $\mathbf{U}, \mathbf{V} \in \mathbb{R}^n$ . Using the partition of unity (3.31), we have:

$$\begin{aligned} (\mathbf{U}, \mathbf{V}) &= \left( \mathbf{U}, \sum_{i=1}^{np} R_i^T D_i R_i \mathbf{V} \right) = \sum_{i=1}^{np} (R_i \mathbf{U}, D_i R_i \mathbf{V}) \\ &= \sum_{i=1}^{np} (\mathbf{U}_i, D_i \mathbf{V}_i). \end{aligned}$$

Thus, the formula for the scalar product is:

$$(\mathbf{U}, \mathbf{V}) = \sum_{i=1}^{np} (R_i \mathbf{U}, D_i R_i \mathbf{V}).$$

Local scalar products are performed concurrently. Thus, the implementation is parallel except for the sum which corresponds to a MPI\_Reduce call across the  $np$  MPI processes.

Note also that the implementation relies on the knowledge of a partition of unity so that the **FreeFEM** syntax is `dscalprod(D, u, v)`.

A axpy procedure  $y \leftarrow \alpha x + y$  for  $x, y \in \mathbb{R}^n$  and  $\alpha \in \mathbb{R}$  is easily implemented concurrently for distributed vectors in the form:

$$y_i \leftarrow \alpha x_i + y_i, \forall 1 \leq i \leq np.$$

The matrix vector product is more involved and details are given in the SIAM book [An Introduction to Domain Decomposition Methods: algorithms, theory and parallel implementation](#) and even more details are given in P. Jolivet's PhD manuscript.

## 3.7 Plugins

### 3.7.1 gsl

The interface with `gsl` spline is available in **FreeFEM**, the seven kind of spline are

0. `gslinterpcpline`: default type of spline
1. `gslinterpakima`
2. `gslinterpsteffen`
3. `gslinterplinear`
4. `gslinterppolynomial`
5. `gslinterpcplineperiodic`
6. `gslinterpakimaperiodic`

A brief wing example given all the syntax:

```

1 load "gsl"
2
3 // Parameters
4 int n = 10;
5 real[int, int] dspline(2,n+1); //data points to define the spline
6 for(int i = 0; i <= n; ++i){ //set data points
7     real xx = square(real(i)/n);
8     real yy = sin(xx*pi*2);
9     dspline(0, i) = xx;
10    dspline(1, i) = yy;
11 }
12
13 // GSL splines
14 gspline spline1(gslinterpcpline, dspline); //define the spline1
15 gspline spline11(dspline); //define the spline11
16 gspline spline2(gslinterpsteffen, dspline); //define the spline2

```

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```

17 gslspline spline3(gslinterpcsplines, dspline(0, :), dspline(1, :));
18 gslspline spline33(dspline(0, :), dspline(1, :)); //define the spline3
19 spline1 = spline2; //copy spline2 in spline1
20
21 real t = 1.;
22 real s1 = spline1(t); //evaluate the function spline1 at t
23 cout << "spline1(t) = " << s1 << endl;
24 real ds1 = spline1.d(t); //evaluate the derivative of function spline1 at t
25 cout << "spline1.d(t) = " << ds1 << endl;
26 real dds1 = spline1.dd(t); //evaluate the second derivative of function spline1 at t
27 cout << "spline1.dd(t) = " << dds1 << endl;

```

This can be usefull to build function from data value.

The list of all `gsl` functions and the **FreeFEM** equivalent is available in the *Language references* (same names without `_`).

### 3.7.2 ffrandom

Plugin to linux random functions.

The range of the random generator is from 0 to  $(2^{31}) - 1$ .

```

1 load "ffrandom"
2
3 srandomdev(); //set a true random seed
4 //warning: under window this command
5 //change the seed by randinit(random())) so all
6 //FreeFEM random function are changed
7
8 int maxrang = 2^31 - 1;
9 cout << " max range " << maxrang << endl;
10
11 cout << random() << endl;
12 cout << random() << endl;
13 cout << random() << endl;
14
15 srandom(10);
16 cout << random() << endl;
17 cout << random() << endl;
18 cout << random() << endl;

```

### 3.7.3 mmap / semaphore

The idea is just try to use Interprocess communication using POSIX Shared Memory in Linux.

We build a small library `libff-mmap-semaphore.c` and `libff-mmap-semaphore.h` to easily interface.

- mmap - allocate memory, or map files or devices into memory
- semaphore - allow processes and threads to synchronize their actions

A semaphore is an integer whose value is never allowed to fall below zero. Two operations can be performed on semaphores: increment the semaphore value by one (`sem_post`); and decrement the semaphore value by one (`sem_wait`).

If the value of a semaphore is currently zero, then a `sem_wait` operation will block until the value becomes greater than zero.

### The functions of library

First the `semaphore` interface to make synchronization:

- `typedef struct FF_P_sem *ff_Psem;` the pointer to data structure
- `ff_Psem ffsem_malloc();` malloc an empty data structure
- `void ffsem_del(ff_Psem sem);` clean and free the pointer
- `void ffsem_destroy(ff_Psem sem);` clean, close the data structure
- `void ffsem_init0(ff_Psem sem);` make a correct empty of the data structure
- `void ffsem_init(ff_Psem sem, const char *nmm, int crea);` create or use a new semaphore
- `long ffsem_post(ff_Psem sem);` nlocked, the value of the semaphore is incremented, and all threads which are waiting on the semaphore are awakened
- `long ffsem_wait(ff_Psem sem);` the semaphore referenced by `sem` is locked. When calling `sem_wait()`, if the semaphore's value is zero, the calling thread will block until the lock is acquired or until the call is interrupted by a signal.

Alternatively, the `sem_trywait()` function will fail if the semaphore is already locked, rather than blocking on the semaphore

- `long ffsem_trywait(ff_Psem p);`

Secondly, the `mmap` functions:

- `typedef struct FF_P_mmap *ff_Pmmap;` the pointer to data structure
- `ff_Psem ffmmap_malloc();` malloc an empty data structure
- `void ffmmap_del(ff_Pmmap p);` clean and free the pointer
- `void ffmmap_destroy(ff_Pmmap p);` clean, close the data structure
- `void ffmmap_init0(ff_Pmmap p);` make a correct empty of the data structure
- `long ffmmap_msync(ff_Pmmap p, long off, long ln);` call writes modified whole pages back to the filesystem and updates the file modification time. Only those pages containing `addr` and `len-1` succeeding locations will be examined.
- `void ffmmap_init(ff_Pmmap p, const char *nmm, long len);` allocate memory, or map files or devices into memory.
- `long ffmmap_read(ff_Pmmap p, void *t, size_t n, size_t off);` read `n` bytes from the mmap at memory `off` in pointer `t`.
- `long ffmmap_write(ff_Pmmap p, void *t, size_t n, size_t off);` write `n` bytes to the mmap at memory `off` in pointer `t`.

The FreeFEM corresponding functions:

- `Pmmap sharedata(filename, 1024);` new type to store the `mmap` informations of name store in string `filename` with 1024 is the size the `sharedata` zone and file.
- `Psemaphore smff("ff-slave", creat);` new type to store the semaphore of name `ff-slave` where `creat` is a boolean to create or use a existing semaphore.

- `Wait(sem)` the semaphore referenced by `sem` is locked. When calling `Wait(sem)`, if the semaphore's value is zero, the calling thread will block until the lock is acquired or until the call is interrupted by a signal. Alternatively, the `trywait(sem)` function will fail if the semaphore is already locked, rather than blocking on the semaphore.
- `Post(sem)` the semaphore referenced by `sem` is unlocked, the value of the semaphore is incremented, and all threads which are waiting on the semaphore are awakened.
- `Read(sharedata , offset, data);` read the variable `data` from the place `offset` in `sharedata` `mmap`.
- `Write(sharedata, offset, data);` write the variable `data` at the place `offset` in `sharedata` `mmap`.

The full example:

The FFMaster.c file:

```

1 #include "libff-mmap-semaphore.h"
2 #include <unistd.h>
3 #include <stdlib.h>
4 #include <stdio.h>
5 ff_Psem sem_ff, sem_c; //the semaphore for mutex
6
7 int main(int argc, const char ** argv)
8 {
9     int debug = 0;
10    ff_Pmmap shd;
11    double cff, rff;
12    long status;
13    int i;
14    if (argc > 1) debug = atoi(argv[1]);
15    ff_mmap_sem_verb = debug;
16
17    sem_ff = ffsem_malloc();
18    sem_c = ffsem_malloc();
19    shd = ff mmap_malloc();
20
21    ffsem_init(sem_ff, "ff-slave1", 1);
22    ffsem_init(sem_c, "ff-master1", 1);
23    ff mmap_init(shd, "shared-data", 1024);
24
25    status = 1;
26    ff mmap_write(shd, &status, sizeof(status), 8);
27    ff mmap_msync(shd, 0, 32);
28
29    char ff[1024];
30    sprintf(ff, "FreeFem++ FFSlave.edp -nw -ns -v %d&", debug);
31    system(ff); //launch FF++ in batch no graphics
32    if(debug) printf("cc: before wait\n");
33
34    if(debug) printf("cc: before wait 0 ff\n");
35    ffsem_wait(sem_ff);
36
37    for (i = 0; i < 10; ++i){
38        printf(" iter : %d \n", i);
39        cff = 10+i;
40        ff mmap_write(shd, &cff, sizeof(cff), 0);
41        ffsem_post(sem_c);

```

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```

42
43     if(debug) printf(" cc: before wait 2\n");
44     ffsem_wait(sem_ff);
45     ff mmap_read(shd, &rff, sizeof(rff), 16);
46     printf(" iter = %d rff= %f\n", i, rff);
47 }
48
49 status = 0; //end
50 ff mmap_write(shd, &status, sizeof(status), 8);
51 ffsem_post(sem_c);
52 printf("End Master \n");
53 ffsem_wait(sem_ff);
54 ffsem_del(sem_ff);
55 ffsem_del(sem_c);
56 ff mmap_del(shd);
57 return 0;
58 }
```

The FFSlave.edp file:

```

1 load "ff-mmap-semaphore"
2
3 Psemaphore smff("ff-slave1", 0);
4 Psemaphore smc("ff-master1", 0);
5 Pmmap sharedata("shared-data", 1024);
6 if (verbosity < 4) verbosity = 0;
7
8 // Mesh
9 mesh Th = square(10, 10);
10 int[int] Lab = [1, 2, 3, 4];
11
12 // Fespace
13 fespace Vh(Th, P1);
14 Vh u, v;
15
16 // Macro
17 macro grad(u) [dx(u), dy(u)] //
18
19 int status = 1;
20 cout << " FF status = " << status << endl;
21 real cff, rff;
22
23 // Problem
24 problem Pb (u, v)
25   = int2d(Th) (
26     grad(u)' * grad(v)
27   )
28   - int2d(Th) (
29     cff * v
30   )
31   + on(Lab, u=0)
32 ;
33
34 if (verbosity > 9) cout << " FF: before FF post\n";
35 Post(smff); //unlock master end init
36
37 while (1) {
```

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```

38 if (verbosity > 9) cout << " FF: before FF wait \n";
39 Wait(smc); //wait from cint write ok
40 Read(sharedata, 0, cff);
41 Read(sharedata, 8, status);
42
43 cout << " After wait .. FF " << cff << " " << status << endl;
44 if(status <= 0) break;
45
46 // Solve
Pb;
rff = int2d(Th)(u*u);
cout << " ** FF " << cff << " " << rff << endl;
47
48 // Write
Write(sharedata, 16, rff);
Post(smff); //unlock cc
49
50
51 Post(smff); //wait from cint
52 cout << " End FreeFEM " << endl;
53
54
55
56
57

```

To test this example of coupling C program and **FreeFEM** script:

```

1 cc -c libff-mmap-semaphore.c
2 cc FFMaster.c -o FFMaster libff-mmap-semaphore.o -g -pthread
3 ff-c++ -auto ff-mmap-semaphore.cpp
4 ./FFMaster

```

The output:

```

1 len 1024 size 0
2 len 1024 size 1024
3 FF status = 1
4 iter : 0
5 After wait .. FF 10 1
6 ** FF 10 0.161797
7 iter = 0 rff= 0.161797
8 iter : 1
9 After wait .. FF 11 1
10 ** FF 11 0.195774
11 iter = 1 rff= 0.195774
12 iter : 2
13 After wait .. FF 12 1
14 ** FF 12 0.232987
15 iter = 2 rff= 0.232987
16 iter : 3
17 After wait .. FF 13 1
18 ** FF 13 0.273436
19 iter = 3 rff= 0.273436
20 iter : 4
21 After wait .. FF 14 1
22 ** FF 14 0.317121
23 iter = 4 rff= 0.317121
24 iter : 5
25 After wait .. FF 15 1
26 ** FF 15 0.364042
27 iter = 5 rff= 0.364042

```

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```

28 iter : 6
29 After wait .. FF 16 1
30 ** FF 16 0.414199
31 iter = 6 rff= 0.414199
32 iter : 7
33 After wait .. FF 17 1
34 ** FF 17 0.467592
35 iter = 7 rff= 0.467592
36 iter : 8
37 After wait .. FF 18 1
38 ** FF 18 0.524221
39 iter = 8 rff= 0.524221
40 iter : 9
41 After wait .. FF 19 1
42 ** FF 19 0.584086
43 iter = 9 rff= 0.584086
44 End Master
45 After wait .. FF 19 0

```

## 3.8 Developers

### 3.8.1 File formats

#### Mesh file data structure

The mesh data structure, output of a mesh generation algorithm, refers to the geometric data structure and in some case to another mesh data structure.

In this case, the fields are

```

1 MeshVersionFormatted 0
2
3 Dimension [DIM] (int)
4
5 Vertices
6 [Number of vertices] (int)
7 X_1(double) Y_1(double) (Z_1(double)) Ref_1(int)
8 ...
9 X_nv(double) Y_nv(double) (Z_nv(double)) Ref_nv(int)
10
11 Edges
12 [Number of edges] (int)
13 Vertex1_1(int) Vertex2_1(int) Ref_1(int)
14 ...
15 Vertex1_ne(int) Vertex2_ne(int) Ref_ne(int)
16
17 Triangles
18 [Number of triangles] (int)
19 Vertex1_1(int) Vertex2_1(int) Vertex3_1(int) Ref_1(int)
20 ...
21 Vertex1_nt(int) Vertex2_nt(int) Vertex3_nt(int) Ref_nt(int)
22
23 Quadrilaterals

```

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```

24 [Number of Quadrilaterals] (int)
25 Vertex1_1(int) Vertex2_1(int) Vertex3_1(int) Vertex4_1(int) Ref_1(int)
26 ...
27 Vertex1_nq(int) Vertex2_nq(int) Vertex3_nq(int) Vertex4_nq(int) Ref_nq(int)
28
29 Geometry
30 [File name of geometric support] (char*)
31
32     VertexOnGeometricVertex
33         [Number of vertex on geometric vertex] (int)
34         Vertex_1(int) VertexGeometry_1(int)
35         ...
36         Vertex_nvg(int) VertexGeometry_nvg(int)
37
38     EdgeOnGeometricEdge
39         [Number of geometric edge] (int)
40         Edge_1(int) EdgeGeometry_1(int)
41         ...
42         Edge_neg(int) EdgeGeometry_neg(int)
43
44 CrackedEdges
45 [Number of cracked edges] (int)
46 Edge1_1(int) Edge2_1(int)
47 ...
48 Edge1_nce(int) Edge2_nce(int)

```

When the current mesh refers to a previous mesh, we have in addition

```

1 MeshSupportOfVertices
2 [File name of mesh support] (char*)
3
4     VertexOnSupportVertex
5         [Number of vertex on support vertex] (int)
6         Vertex_1(int) VertexSupport_1(int)
7         ...
8         Vertex_nvsv(int) VertexSupport_nvsv(int)
9
10    VertexOnSupportEdge
11        [Number of vertex on support edge] (int)
12        Vertex_1(int) EdgeSupport_1(int) USupport_1(double)
13        ...
14        Vertex_nvse(int) EdgeSupport_nvse(int) USupport_nvse(double)
15
16    VertexOnSupportTriangle
17        [Number of vertex on support triangle] (int)
18        Vertex_1(int) TriangleSupport_1(int) USupport_1(double) VSupport_1(double)
19        ...
20        Vertex_nvst(int) TriangleSupport_nvst(int) USupport_nvst(double) VSupport_nvst(double)
21
22    VertexOnSupportQuadrilaterals
23        [Number of vertex on support quadrilaterals]
24        Vertex_1(int) TriangleSupport_1(int) USupport_1(double) VSupport_1(double)
25        ...
26        Vertex_nvsq(int) TriangleSupport_nvsq(int) USupport_nvsq(double) VSupport_nvsq(double)

```

- nv means the number of vertices
- ne means the number of edges
- nt means the number of triangles
- nq means the number of quadrilaterals
- nvg means the number of vertex on geometric vertex
- neg means the number of edges on geometric edge
- nce means the number of cracked edges

## bb file type to Store Solutions

The file is formatted such that:

```

1 2 [Number of solutions] (int) [Number of vertices] (int) 2
2
3 U_1_1 (double) ... U_ns_1 (double)
4 ...
5 U_1_nv (double) ... U_ns_nv (double)
```

- ns means the number of solutions
- nv means the number of vertices
- $U_{i,j}$  is the solution component  $i$  at the vertex  $j$  on the associated mesh.

## BB file type to store solutions

The file is formatted such that:

```

1 2 [Number of solutions] (int) [Type 1] (int) ... [Type ns] (int) [Number of_
2 <vertices] (int) 2
3
4 U_1_1_1 (double) ... U_(type_k)_1_1 (double)
5 ...
6 U_1_1_1 (double) ... U_(type_k)_nbv_1 (double)
7 ...
8
9 U_1_1_ns (double) ... U_(type_k)_1_ns (double)
10 ...
11 U_1_nbv_ns (double) ... U_(type_k)_nbv_ns (double)
```

- ns means the number of solutions
- $type_k$  mean the type of solution  $k$ :
  - 1: the solution is scalar (1 value per vertex)
  - 2: the solution is vectorial (2 values per vertex)
  - 3: the solution is a  $2 \times 2$  symmetric matrix (3 values per vertex)
  - 4: the solution is a  $2 \times 2$  matrix (4 values per vertex)
- nbv means the number of vertices
- $U_{i,j,k}$  is the value of the component  $i$  of the solution  $k$  at vertex  $j$  on the associated mesh

## Metric file

A metric file can be of two types, isotropic or anisotropic.

The isotropic file is such that

```

1 [Number of vertices] (int) 1
2 h_0 (double)
3 ...
4 h_nv (double)
```

- nv is the number of vertices
- $h_i$  is the wanted mesh size near the vertex  $i$  on associated mesh.

The metric is  $\mathcal{M}_i = h_i^{-2} I$  where  $I$  is the identity matrix.

The anisotropic file is such that

```

1 [Number of vertices] (int) 3
2 a11_0 (double) a21_0 (double) a22_0 (double)
3 ...
4 a11_nv (double) a21_nv (double) a22_nv (double)
```

- nv is the number of vertices
- $a_{11,i}$ ,  $a_{21,i}$  and  $a_{22,i}$  represent metric  $\mathcal{M}_i = \begin{pmatrix} a_{11,i} & a_{12,i} \\ a_{12,i} & a_{22,i} \end{pmatrix}$  which define the wanted size in a vicinity of the vertex  $i$  such that  $h$  in direction  $u \in \mathbb{R}^2$  is equal to  $|u|/\sqrt{u \cdot \mathcal{M}_i u}$ , where  $\cdot$  is the dot product in  $\mathbb{R}^2$ , and  $|\cdot|$  is the classical norm.

## List of AM\_FMT, AMDBA Meshes

The mesh is only composed of triangles and can be defined with the help of the following two integers and four arrays:

- nbt the number of triangles
- nbv the number of vertices
- nu(1:3, 1:nbt) an integer array giving the three vertex numbers counterclockwise for each triangle
- c(1:2, 1:nbv) a real array giving the two coordinates of each vertex
- refs(1:nbv) an integer array giving the reference numbers of the vertices
- reft(1:nbt) an integer array giving the reference numbers of the triangles

## AM\_FMT Files

In Fortran the am\_fmt files are read as follows:

```

1 open (1, file='xxx.am_fmt', form='formatted', status='old')
2 read (1, *) nbv, nbt
3 read (1, *) ((nu(i, j), i=1, 3), j=1, nbt)
4 read (1, *) ((c(i, j), i=1, 2), j=1, nbv)
5 read (1, *) (reft(i), i=1, nbt)
6 read (1, *) (refs(i), i=1, nbv)
7 close(1)
```

## AM Files

In Fortran the am files are read as follows:

```

1 open (1, file='xxx.am', form='unformatted', status='old')
2 read (1, *) nbv, nbt
3 read (1, ((nu(i, j), i=1, 3), j=1, nbt),
4 & ((c(i, j), i=1, 2), j=1, nbv),
5 & (reft(i), i=1, nbt),
6 & (refs(i), i=1, nbv)
7 close (1)

```

## AMDBA Files

In Fortran the amdba files are read as follows:

```

1 open (1, file='xxx.amdba', form='formatted', status='old')
2 read (1, *) nbv, nbt
3 read (1, *) (k, (c(i, k), i=1, 2), refs(k), j=1, nbv)
4 read (1, *) (k, (nu(i, k), i=1, 3), reft(k), j=1, nbt)
5 close (1)

```

## msh Files

First, we add the notions of boundary edges

- nbbe the number of boundary edge
- nube (1:2, 1:nbbe) an integer array giving the two vertex numbers of boundary edges
- refbe (1:nbbe) an integer array giving the reference numbers of boundary edges

In Fortran the msh files are read as follows:

```

1 open (1, file='xxx.msh', form='formatted', status='old')
2 read (1, *) nbv, nbt, nbbe
3 read (1, *) ((c(i, k), i=1, 2), refs(k), j=1, nbv)
4 read (1, *) ((nu(i, k), i=1, 3), reft(k), j=1, nbt)
5 read (1, *) ((ne(i, k), i=1, 2), refbe(k), j=1, nbbe)
6 close (1)

```

## ftq Files

In Fortran the ftq files are read as follows:

```

1 open (1, file='xxx.ftq', form='formatted', status='old')
2 read (1, *) nbv, nbe, nbt, nbq
3 read (1, *) (k(j), (nu(i, j), i=1, k(j)), reft(j), j=1, nbe)
4 read (1, *) ((c(i, k), i=1, 2), refs(k), j=1, nbv)
5 close (1)

```

where if  $k(j) = 3$  when the element  $j$  is a triangle and  $k(j) = 4$  when the element  $j$  is a quadrilateral.

## sol and solb files

With the keyword `savesol`, we can store a scalar functions, a scalar finite element functions, a vector fields, a vector finite element fields, a symmetric tensor and a symmetric finite element tensor.

Such format is used in medit.

### Extension file .sol

The first two lines of the file are :

- MeshVersionFormatted 0

- Dimension [DIM] (int)

The following fields begin with one of the following keyword: SolAtVertices, SolAtEdges, SolAtTriangles, SolAtQuadrilaterals, SolAtTetrahedra, SolAtPentahedra, SolAtHexahedra.

In each field, we give then in the next line the number of elements in the solutions (SolAtVertices: number of vertices, SolAtTriangles: number of triangles, ...). In other lines, we give the number of solutions, the type of solution (1: scalar, 2: vector, 3: symmetric tensor). And finally, we give the values of the solutions on the elements.

The file must be ended with the keyword End.

The real element of symmetric tensor :

$$ST^{3d} = \begin{pmatrix} ST_{xx}^{3d} & ST_{xy}^{3d} & ST_{xz}^{3d} \\ ST_{yx}^{3d} & ST_{yy}^{3d} & ST_{yz}^{3d} \\ ST_{zx}^{3d} & ST_{zy}^{3d} & ST_{zz}^{3d} \end{pmatrix} \quad ST^{2d} = \begin{pmatrix} ST_{xx}^{2d} & ST_{xy}^{2d} \\ ST_{yx}^{2d} & ST_{yy}^{2d} \end{pmatrix} \quad (3.32)$$

stored in the extension .sol are respectively  $ST_{xx}^{3d}, ST_{xy}^{3d}, ST_{yy}^{3d}, ST_{xz}^{3d}, ST_{yz}^{3d}, ST_{zz}^{3d}$  and  $ST_{xx}^{2d}, ST_{xy}^{2d}, ST_{yy}^{2d}$

An example of field with the keyword SolAtTetrahedra:

```

1 SolAtTetrahedra
2 [Number of tetrahedra] (int)
3 [Number of solutions] (int) [Type of solution 1] (int) ... [Type of soution nt] (int)
4
5 U_1_1_1 (double) ... U_nrs_1_1 (double)
6 ...
7 U_1_ns_1 (double) ... U_(nrs_k)_ns_1 (double)
8 ...
9 ...
10
11 U_1_1_nt (double) ... U_nrs_1_nt (double)
12 ...
13 U_1_ns_nt (double) ... U_(nrs_k)_ns_nt (double)

```

- ns is the number of solutions
- typesol\_k, type of the solution number k
  - typesol\_k = 1 the solution k is scalar
  - typesol\_k = 2 the solution k is vectorial
  - typesol\_k = 3 the solution k is a symmetric tensor or symmetric matrix
- nrs\_k is the number of real to describe solution k
  - nrs\_k = 1 if the solution k is scalar
  - nrs\_k = dim if the solution k is vectorial (dim is the dimension of the solution)
  - nrs\_k = dim\*(dim+1)/2 if the solution k is a symmetric tensor or symmetric matrix
- U\_i\_j^k is a real equal to the value of the component i of the solution k at tetrahedron j on the associated mesh

The format .solb is the same as format .sol but in binary (read/write is faster, storage is less).

A real scalar functions f1, a vector fields  $\Phi = [\Phi_1, \Phi_2, \Phi_3]$  and a symmetric tensor  $ST^{3d}$  (3.32) at the vertices of the three dimensional mesh Th3 is stored in the file f1PhiTh3.sol using :

```

1 savesol("f1PhiST3dTh3.sol", Th3, f1, [Phi(1), Phi(2), Phi(3)], VV3, order=1);

```

where  $VV3 = [ST_{xx}^{3d}, ST_{yx}^{3d}, ST_{yy}^{3d}, ST_{zx}^{3d}, ST_{zy}^{3d}, ST_{zz}^{3d}]$ .

For a two dimensional mesh Th, A real scalar functions  $f2$ , a vector fields  $\Psi = [\Psi1, \Psi2]$  and a symmetric tensor  $ST^{2d}$  (3.32) at triangles is stored in the file f2PsiST2dTh3.solb using :

```
savesol("f2PsiST2dTh3.solb", Th, f2, [Psi(1), Psi(2)], VV2, order=0);
```

where  $VV2 = [ST_{xx}^{2d}, ST_{yx}^{2d}, ST_{yy}^{2d}]$

The arguments of savesol functions are the name of a file, a mesh and solutions. These arguments must be given in this order.

The parameters of this keyword are :

- `order = 0` is the solution is given at the center of gravity of elements. 1 is the solution is given at the vertices of elements.

In the file, solutions are stored in this order : scalar solutions, vector solutions and finally symmetric tensor solutions.

### 3.8.2 Adding a new finite element

#### Some notations

For a function  $f$  taking value in  $\mathbb{R}^N$ ,  $N = 1, 2, \dots$ , we define the finite element approximation  $\Pi_h f$  of  $f$ .

Let us denote the number of the degrees of freedom of the finite element by  $NbDoF$ . Then the  $i$ -th base  $\omega_i^K$  ( $i = 0, \dots, NbDoF - 1$ ) of the finite element space has the  $j$ -th component  $\omega_{ij}^K$  for  $j = 0, \dots, N - 1$ .

The operator  $\Pi_h$  is called the interpolator of the finite element.

We have the identity  $\omega_i^K = \Pi_h \omega_i^K$ .

Formally, the interpolator  $\Pi_h$  is constructed by the following formula:

$$\Pi_h f = \sum_{k=0}^{kPi-1} \alpha_k f_{j_k}(P_{p_k}) \omega_{i_k}^K \quad (3.33)$$

where  $P_p$  is a set of  $npPi$  points,

In the formula (3.33), the list  $p_k$ ,  $j_k$ ,  $i_k$  depend just on the type of finite element (not on the element), but the coefficient  $\alpha_k$  can be depending on the element.

#### Tip: Classical scalar Lagrange finite element

With the classical scalar Lagrange finite element, we have  $kPi = npPi = NbOfNode$  and

- $P_p$  is the point of the nodal points.
- the  $\alpha_k = 1$ , because we take the value of the function at the point  $P_k$ .
- $p_k = k$ ,  $j_k = k$  because we have one node per function.
- $j_k = 0$  because  $N = 1$ .

#### Tip: The Raviart-Thomas finite element

$$RT0_h = \{\mathbf{v} \in H(div) / \forall K \in \mathcal{T}_h \quad \mathbf{v}|_K(x, y) = \left| \begin{array}{c} \alpha_K \\ \beta_K \\ \gamma_K \end{array} \right| \begin{array}{c} x \\ y \end{array} \} \quad (3.34)$$

The degrees of freedom are the flux through an edge  $e$  of the mesh, where the flux of the function  $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is  $\int_e \mathbf{f} \cdot \mathbf{n}_e$ ,  $n_e$  is the unit normal of edge  $e$  (this implies a orientation of all the edges of the mesh, for example we can use the global numbering of the edge vertices and we just go to small to large number).

To compute this flux, we use a quadrature formula with one point, the middle point of the edge. Consider a triangle  $T$  with three vertices ( $\mathbf{a}, \mathbf{b}, \mathbf{c}$ ).

Let denote the vertices numbers by  $i_a, i_b, i_c$ , and define the three edge vectors  $\mathbf{e}^0, \mathbf{e}^1, \mathbf{e}^2$  by  $sgn(i_b - i_c)(\mathbf{b} - \mathbf{c})$ ,  $sgn(i_c - i_a)(\mathbf{c} - \mathbf{a})$ ,  $sgn(i_a - i_b)(\mathbf{a} - \mathbf{b})$ .

The three basis functions are:

$$\omega_0^K = \frac{sgn(i_b - i_c)}{2|T|}(x - a), \quad \omega_1^K = \frac{sgn(i_c - i_a)}{2|T|}(x - b), \quad \omega_2^K = \frac{sgn(i_a - i_b)}{2|T|}(x - c),$$

where  $|T|$  is the area of the triangle  $T$ .

So we have  $N = 2$ ,  $kPi = 6$ ;  $npPi = 3$ ; and:

- $P_p = \left\{ \frac{\mathbf{b}+\mathbf{c}}{2}, \frac{\mathbf{a}+\mathbf{c}}{2}, \frac{\mathbf{b}+\mathbf{a}}{2} \right\}$
  - $\alpha_0 = -\mathbf{e}_2^0, \alpha_1 = \mathbf{e}_1^0, \alpha_2 = -\mathbf{e}_2^1, \alpha_3 = \mathbf{e}_1^1, \alpha_4 = -\mathbf{e}_2^2, \alpha_5 = \mathbf{e}_1^2$  (effectively, the vector  $(-\mathbf{e}_2^m, \mathbf{e}_1^m)$  is orthogonal to the edge  $\mathbf{e}^m = (e_1^m, e_2^m)$  with a length equal to the side of the edge or equal to  $\int_{e^m} 1$ ).
  - $i_k = \{0, 0, 1, 1, 2, 2\}$ ,
  - $p_k = \{0, 0, 1, 1, 2, 2\}$ ,  $j_k = \{0, 1, 0, 1, 0, 1, 0, 1\}$ .
- 

## Which class to add?

Add file FE\_ADD.cpp in directory FreeFem-sources/src/femlib for example first to initialize :

```

1 #include "error.hpp"
2 #include "rgraph.hpp"
3 using namespace std;
4 #include "RNM.hpp"
5 #include "fem.hpp"
6 #include "FESpace.hpp"
7 #include "AddNewFE.h"
8
9 namespace Fem2D { ... }
```

Then add a class which derive for public TypeOfFE like:

```

1 class TypeOfFE_RTortho : public TypeOfFE { public:
2     static int Data[]; //some numbers
3     TypeOfFE_RTortho():
4         TypeOfFE(
5             0+3+0, //nb degree of freedom on element
6             2, //dimension N of vectorial FE (1 if scalar FE)
7             Data, //the array data
8             1, //nb of subdivision for plotting
9             1, //nb of sub finite element (generally 1)
10            6, //number kPi of coef to build the interpolator
11            3, //number npPi of integration point to build interpolator
12            0 //an array to store the coef \alpha_k to build interpolator
13            //here this array is no constant so we have
14            //to rebuilt for each element
15        )
16 }
```

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```

16 {
17     const R2 Pt[] = {R2(0.5, 0.5), R2(0.0, 0.5), R2(0.5, 0.0) };
18     // the set of Point in hat{K}
19     for (int p = 0, kk = 0; p < 3; p++) {
20         P_Pi_h[p] = Pt[p];
21         for (int j = 0; j < 2; j++)
22             pij_alpha[kk++] = IPJ(p, p, j);
23     }
24 } //definition of i_k, p_k, j_k in interpolator
25
26 void FB(const bool *watdd, const Mesh &Th, const Triangle &K,
27 const R2 &PHat, RNMK_ &val) const;
28
29 void Pi_h_alpha(const baseFEElement &K, KN_<double> &v) const;
30 ;

```

where the array data is formed with the concatenation of five array of size NbDoF and one array of size N.

This array is:

```

1 int TypeOfFE_RTortho::Data[] = {
2     //for each df 0, 1, 3:
3     3, 4, 5, //the support of the node of the df
4     0, 0, 0, //the number of the df on the node
5     0, 1, 2, //the node of the df
6     0, 0, 0, //the df come from which FE (generally 0)
7     0, 1, 2, //which are the df on sub FE
8     0, 0
9 }; //for each component j=0, N-1 it give the sub FE associated

```

where the support is a number 0, 1, 2 for vertex support, 3, 4, 5 for edge support, and finally 6 for element support.

The function to defined the function  $\omega_i^K$ , this function return the value of all the basics function or this derivatives in array val, computed at point Phat on the reference triangle corresponding to point R2 P=K(Phat) ; on the current triangle K.

The index  $i, j, k$  of the array val( $i, j, k$ ) correspond to:

- $i$  is the basic function number on finite element  $i \in [0, NoF[$
- $j$  is the value of component  $j \in [0, N[$
- $k$  is the type of computed value  $f(P), dx(f)(P), dy(f)(P), \dots i \in [0, \text{last\_operatortype}[$ .

**Note:** For optimization, this value is computed only if whatd[k] is true, and the numbering is defined with

```

1 enum operatortype {
2     op_id = 0,
3     op_dx = 1, op_dy = 2,
4     op_dxx = 3, op_dyy = 4,
5     op_dyx = 5, op_dxy = 5,
6     op_dz = 6,
7     op_dzz = 7,
8     op_dzx = 8, op_dxz = 8,
9     op_dzy = 9, op_dyz = 9
10    };
11 const int last_operatortype = 10;

```

The shape function:

```

1 void TypeOfFE_RTortho::FB(const bool *whatd, const Mesh &Th, const Triangle & K,
2   const R2 &PHat, RNMK_ &val) const
3 {
4   R2 P(K(PHat));
5   R2 A(K[0]), B(K[1]), C(K[2]);
6   R l0 = 1 - P.x-P.y;
7   R l1 = P.x, l2 = P.y;
8   assert(val.N() >= 3);
9   assert(val.M() == 2);
10  val = 0;
11  R a = 1./(2*K.area);
12  R a0 = K.EdgeOrientation(0) * a;
13  R a1 = K.EdgeOrientation(1) * a;
14  R a2 = K.EdgeOrientation(2) * a;
15
16  if (whatd[op_id]) { //value of the function
17    assert(val.K() > op_id);
18    RN_ f0(val('. ', 0, 0)); //value first component
19    RN_ f1(val('. ', 1, 0)); //value second component
20    f1[0] = (P.x - A.x)*a0;
21    f0[0] = -(P.y - A.y)*a0;
22
23    f1[1] = (P.x - B.x)*a1;
24    f0[1] = -(P.y - B.y)*a1;
25
26    f1[2] = (P.x - C.x)*a2;
27    f0[2] = -(P.y - C.y)*a2;
28  }
29
30  if (whatd[op_dx]) { //value of the dx of function
31    assert(val.K() > op_dx);
32    val(0,1,op_dx) = a0;
33    val(1,1,op_dx) = a1;
34    val(2,1,op_dx) = a2;
35  }
36  if (whatd[op_dy]) {
37    assert(val.K() > op_dy);
38    val(0,0,op_dy) = -a0;
39    val(1,0,op_dy) = -a1;
40    val(2,0,op_dy) = -a2;
41  }
42
43  for (int i = op_dy; i < last_operatortype; i++)
44    if (whatd[op_dx])
45      assert(op_dy);
46}

```

The function to defined the coefficient  $\alpha_k$ :

```

1 void TypeOfFE_RT::Pi_h_alpha(const baseFElement &K, KN_<double> &v) const
2 {
3   const Triangle &T(K.T);
4
5   for (int i = 0, k = 0; i < 3; i++) {
6     R2 E(T.Edge(i));
7     R signe = T.EdgeOrientation(i) ;

```

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```

8     v[k++] = signe*E.y;
9     v[k++] = -signe*E.x;
10    }
11 }
```

Now , we just need to add a new key work in **FreeFEM**.

Two way, with static or dynamic link so at the end of the file, we add:

**With dynamic link** it is very simple (see section *Dynamical link*), just add before the end of FEM2d namespace:

```

1 static TypeOfFE_RTOrtho The_TypeOfFE_RTOrtho;
2 static AddNewFE("RT0Ortho", The_TypeOfFE_RTOrtho);
3 } //FEM2d namespace
```

Try with `./load.link` command in `examples++-load/` and see `BernardiRaugel.cpp` or `Morley.cpp` new finite element examples.

**Otherwise with static link** (for expert only), add

```

1 //let the 2 globals variables
2 static TypeOfFE_RTOrtho The_TypeOfFE_RTOrtho;
3 //the name in freefem
4 static ListOfTFE typefemRTOrtho("RT0Ortho", &The_TypeOfFE_RTOrtho);
5
6 //link with FreeFEM do not work with static library .a
7 //so add a extern name to call in init_static_FE
8 // (see end of FESpace.cpp)
9 void init_FE_ADD() { };
10 //end
11 } //FEM2d namespace
```

To inforce in loading of this new finite element, we have to add the two new lines close to the end of files `src/femlib/FESpace.cpp` like:

```

1 //correct problem of static library link with new make file
2 void init_static_FE()
3 { //list of other FE file.
4     extern void init_FE_P2h();
5     init_FE_P2h();
6     extern void init_FE_ADD(); //new line 1
7     init_FE_ADD(); //new line 2
8 }
```

and now you have to change the makefile.

First, create a file `FE_ADD.cpp` containing all this code, like in file `src/femlib/Element_P2h.cpp`, after modify the `Makefile.am` by adding the name of your file to the variable `EXTRA_DIST` like:

```

1 # Makefile using Automake + Autoconf
2 # -----
3 # Id
4
5 # This is not compiled as a separate library because its
6 # interconnections with other libraries have not been solved.
7
8 EXTRA_DIST=BamgFreeFem.cpp BamgFreeFem.hpp CGNL.hpp CheckPtr.cpp \
9 ConjuguedGradientNL.cpp DOperator.hpp Drawing.cpp Element_P2h.cpp \\\
```

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```

10 Element_P3.cpp Element_RT.cpp fem3.hpp fem.cpp fem.hpp FESpace.cpp      \
11 FESpace.hpp FESpace-v0.cpp FQuadTree.cpp FQuadTree.hpp gibbs.cpp          \
12 glutdraw.cpp gmres.hpp MatriceCreuse.hpp MatriceCreuse_tpl.hpp           \
13 MeshPoint.hpp mortar.cpp mshptg.cpp QuadratureFormular.cpp                \
14 QuadratureFormular.hpp RefCounter.hpp RNM.hpp RNM_opc.hpp RNM_op.hpp     \
15 RNM_tpl.hpp      FE_ADD.cpp

```

and do in the **FreeFEM** root directory

```

1 autoreconf
2 ./reconfigure
3 make

```

For codewarrior compilation add the file in the project an remove the flag in panal PPC linker FreeFm++ Setting Dead-strip Static Initialization Code Flag.

### 3.8.3 Dynamical link

Now, it's possible to add built-in functionnalites in **FreeFEM** under the three environnents Linux, Windows and MacOS X 10.3 or newer.

It is a good idea to first try the example `load.edp` in directory `example++-load`.

You will need to install a compiler (generally `g++/gcc` compiler) to compile your function.

- Windows Install the `cygwin` environnement or the `mingw` one
- MacOs Install the developer tools `Xcode` on the apple DVD
- Linux/Unix Install the correct compiler (`gcc` for instance)

Now, assume that you are in a shell window (a `cygwin` window under Windows) in the directory `example++-load`.

---

**Note:** In the sub directory `include`, they are all the **FreeFEM** include file to make the link with **FreeFEM**.

---

**Note:** If you try to load dynamically a file with command `load "xxx"` - Under Unix (Linux or MacOs), the file `xxx.so` will be loaded so it must be either in the search directory of routine `dlopen` (see the environment variable `$LD_LIBRARY_PATH`) or in the current directory, and the suffix "`.so`" or the prefix "`./`" is automatically added.

- Under Windows, the file `xxx.dll` will be loaded so it must be in the `loadLibrary` search directory which includes the directory of the application,

#### Compilation of your module:

The script `ff-c++` compiles and makes the link with **FreeFEM**, but be careful, the script has no way to known if you try to compile for a pure Windows environment or for a cygwin environment so to build the load module under cygwin you must add the `-cygwin` parameter.

#### A first example `myfunction.cpp`

The following defines a new function call `myfunction` with no parameter, but using the  $x, y$  current value.

```

1 #include <iostream>
2 #include <cffloat>
3 using namespace std;
4 #include "error.hpp"
5 #include "AFunction.hpp"
6 #include "rgraph.hpp"
7 #include "RNM.hpp"
8 #include "fem.hpp"
9 #include "FESpace.hpp"
10 #include "MeshPoint.hpp"
11
12 using namespace Fem2D;
13 double myfunction(Stack stack) {
14     //to get FreeFEM data
15     MeshPoint &mp = *MeshPointStack(stack); //the struct to get x, y, normal, value
16     double x = mp.P.x; //get the current x value
17     double y = mp.P.y; //get the current y value
18     //cout << "x = " << x << " y=" << y << endl;
19     return sin(x)*cos(y);
20 }

```

Now the Problem is to build the link with **FreeFEM**, to do that we need two classes, one to call the function `myfunction`.

All **FreeFEM** evaluable expression must be a C++ struct/class which derivate from `E_F0`. By default this expression does not depend of the mesh position, but if they derivate from `E_F0mps` the expression depends of the mesh position, and for more details see [HECHT2002].

```

1 //A class build the link with FreeFEM
2 //generaly this class are already in AFunction.hpp
3 //but unfortunatly, I have no simple function with no parameter
4 //in FreeFEM depending of the mesh
5 template<class R>
6 class OneOperator0s : public OneOperator {
7     //the class to define and evaluate a new function
8     //It must devive from E_F0 if it is mesh independent
9     //or from E_F0mps if it is mesh dependent
10    class E_F0_F :public E_F0mps {
11        public:
12            typedef R (*func)(Stack stack);
13            func f; //the pointeur to the fnction myfunction
14            E_F0_F(func ff) : f(ff) {}
15            //the operator evaluation in FreeFEM
16            AnyType operator()(Stack stack) const {return SetAny<R>(f(stack));}
17        };
18        typedef R (*func)(Stack);
19        func f;
20        public:
21            //the function which build the FreeFEM byte code
22            E_F0 *code(const basicAC_F0 &) const { return new E_F0_F(f); }
23            //the constructor to say ff is a function without parameter
24            //and returning a R
25            OneOperator0s(func ff) : OneOperator(map_type[typeid(R).name()]), f(ff) {}
26    };

```

To finish we must add this new function in **FreeFEM** table, to do that include :

```

1 void init() {
2     Global.Add("myfunction", "( ", new OneOperator0s<double>(myfunction));
3 }
LOADFUNC(init);

```

It will be called automatically at load module time.

To compile and link, use the `ff-c++` script :

```

1 ff-c++ myfunction.cpp
2 g++ -c -g -Iinclude myfunction.cpp
3 g++ -bundle -undefined dynamic_lookup -g myfunction.o -o ./myfunction.dylib

```

To try the simple example under Linux or MacOS, do `FreeFem++-nw load.edp`

The output must be:

```

-- FreeFem++ v *.*.*.*.* (date *** * * * * * (UTC+0*00) )
Load: lg_fem lg_mesh lg_mesh3 eigenvalue
  1 : // Example of dynamic function load
  2 : //
  3 : // $Id$ 
  4 :
  5 : load "myfunction"
  6 : // dumptable(cout);
  7 : mesh Th=square(5,5);
  8 : fespace Vh(Th,P1);
  9 : Vh uh= myfunction(); // warning do not forget ()
10 : cout << uh[].min << " " << uh[].max << endl;
11 : cout << " test io ( " << endl;
12 : testio();
13 : cout << " ) end test io .. " << endl; sizestack + 1024 =1416 ( 392 )
14 :
15 -- Square mesh : nb vertices =36 , nb triangles = 50 , nb boundary edges 20
0 0.841471
16 test io (
17 test cout 3.14159
18 test cout 512
19 test cerr 3.14159
20 test cerr 512
21 ) end test io ..
22 times: compile 0.012854s, execution 0.000313s, mpirank:0
23 CodeAlloc : nb ptr 2715, size :371104 mpirank: 0
24 Ok: Normal End

```

Under Windows, launch **FreeFEM** with the mouse (or **ctrl O**) on the example.

### Example: Discrete Fast Fourier Transform

This will add FFT to **FreeFEM**, taken from **FFTW**. To download and install under `download/include` just go in `download/fftw` and try `make`.

The 1D dfft (fast discret fourier transform) for a simple array  $f$  of size  $n$  is defined by the following formula:

$$\text{dfft}(f, \varepsilon)_k = \sum_{j=0}^{n-1} f_i e^{\varepsilon 2\pi i k j / n}$$

The 2D DFFT for an array of size  $N = n \times m$  is:

$$\text{dffft}(f, m, \varepsilon)_{k+nl} = \sum_{j'=0}^{m-1} \sum_{j=0}^{n-1} f_{i+nj} e^{\varepsilon 2\pi i (kj/n + lj'/m)}$$

---

**Note:** The value  $n$  is given by  $\text{size}(f)/m$ , and the numbering is row-major order.

---

So the classical discrete DFFT is  $\hat{f} = \text{dffft}(f, -1)/\sqrt{n}$  and the reverse dFFT  $f = \text{dffft}(\hat{f}, 1)/\sqrt{n}$

---

**Note:** The 2D Laplace operator is

$$f(x, y) = 1/\sqrt{N} \sum_{j'=0}^{m-1} \sum_{j=0}^{n-1} \hat{f}_{i+nj} e^{\varepsilon 2\pi i (xj + yj')}$$

and we have

$$f_{k+nl} = f(k/n, l/m)$$

So

$$\widehat{\Delta f_{kl}} = -((2\pi)^2 ((\tilde{k})^2 + (\tilde{l})^2)) \widehat{f_{kl}}$$

where  $\tilde{k} = k$  if  $k \leq n/2$  else  $\tilde{k} = k - n$  and  $\tilde{l} = l$  if  $l \leq m/2$  else  $\tilde{l} = l - m$ .

---

And to have a real function we need all modes to be symmetric around zero, so  $n$  and  $m$  must be odd.

---

### Compile to build a new library

```

1 ff-c++ dfft.cpp .. /download/install/lib/libfftw3.a -I .. /download/install/include
2 export MACOSX_DEPLOYMENT_TARGET=10.3
3 g++ -c -I include -I .. /download/install/include dfft.cpp
4 g++ -bundle -undefined dynamic_lookup dfft.o -o ./dfft.dylib .. /download/install/lib/
  ↵ libfftw3.a

```

To test, try [FFT example](#).

### Load Module for Dervieux P0-P1 Finite Volume Method

The associed edp file is examples++-load/convect\_dervieux.edp.

See mat\_dervieux.cpp.

### More on Adding a new finite element

First read the [Adding a new finite element section](#), we add two new finite elements examples in the directory examples++-load.

## The Bernardi-Raugel Element

The Bernardi-Raugel finite element is meant to solve the Navier Stokes equations in  $u, p$  formulation; the velocity space  $P_K^{br}$  is minimal to prove the inf-sup condition with piecewise constant pressure by triangle.

The finite element space  $V_h$  is

$$V_h = \{u \in H^1(\Omega)^2; \quad \forall K \in T_h, u|_K \in P_K^{br}\}$$

where

$$P_K^{br} = \text{span}\{\lambda_i^K e_k\}_{i=1,2,3, k=1,2} \cup \{\lambda_i^K \lambda_{i+1}^K n_{i+2}^K\}_{i=1,2,3}$$

with notation  $4 = 1, 5 = 2$  and where  $\lambda_i^K$  are the barycentric coordinates of the triangle  $K$ ,  $(e_k)_{k=1,2}$  the canonical basis of  $\mathbb{R}^2$  and  $n_k^K$  the outer normal of triangle  $K$  opposite to vertex  $k$ .

See [BernardiRaugel.cpp](#).

A way to check the finite element

```

1  load "BernardiRaugel"
2
3  // Macro
4  // a macro to compute numerical derivative
5  macro DD(f, hx, hy) ( (f(x1+hx, y1+hy) - f(x1-hx, y1-hy)) / (2*(hx+hy)) ) //
6
7  // Mesh
8  mesh Th = square(1, 1, [10*(x+y/3), 10*(y-x/3)]);
9
10 // Parameters
11 real x1 = 0.7, y1 = 0.9, h = 1e-7;
12 int it1 = Th(x1, y1).nuTriangle;
13
14 // Fespace
15 fespace Vh(Th, P2BR);
16 Vh [a1, a2], [b1, b2], [c1, c2];
17
18
19 for (int i = 0; i < Vh.ndofK; ++i)
20   cout << i << " " << Vh(0,i) << endl;
21
22 for (int i = 0; i < Vh.ndofK; ++i)
23 {
24   a1[] = 0;
25   int j = Vh(it1, i);
26   a1[][j] = 1;
27   plot([a1, a2], wait=1);
28   [b1, b2] = [a1, a2]; // do the interpolation
29
30   c1[] = a1[] - b1[];
31   cout << " -----" << i << " " << c1[].max << " " << c1[].min << endl;
32   cout << " a = " << a1[] << endl;
33   cout << " b = " << b1[] << endl;
34   assert(c1[].max < 1e-9 && c1[].min > -1e-9); // check if the interpolation is
35   // correct
36
37   // check the derivative and numerical derivative
38   cout << " dx(a1)(x1, y1) = " << dx(a1)(x1, y1) << " == " << DD(a1, h, 0) << endl;

```

(continues on next page)

(continued from previous page)

```

38 assert( abs(dx(a1)(x1, y1) - DD(a1, h, 0) ) < 1e-5);
39 assert( abs(dx(a2)(x1, y1) - DD(a2, h, 0) ) < 1e-5);
40 assert( abs(dy(a1)(x1, y1) - DD(a1, 0, h) ) < 1e-5);
41 assert( abs(dy(a2)(x1, y1) - DD(a2, 0, h) ) < 1e-5);
42 }
```

A real example using this finite element, just a small modification of the Navier-Stokes P2-P1 example, just the beginning is change to

```

1 load "BernardiRaugel"
2
3 real s0 = clock();
4 mesh Th = square(10, 10);
5 fespace Vh2(Th, P2BR);
6 fespace Vh(Th, P0);
7 Vh2 [u1, u2], [up1, up2];
8 Vh2 [v1, v2];
```

And the plot instruction is also changed because the pressure is constant, and we cannot plot isovalues of piecewise constant functions.

## The Morley Element

See the example `bilapMorley.edp`.

## 3.9 ffddm

In the acronym `ffddm`, `ff` stands for FreeFEM and `ddm` for domain decomposition methods. The idea behind `ffddm` is to simplify the use of parallel solvers in FreeFEM: distributed direct methods and domain decomposition methods.

Parallelism is an important issue because, since about 2004, the clock speed of cores stagnates at 2-3 GHz. The increase in performance is almost entirely due to the increase in the number of cores per processor. All major processor vendors are producing multicore chips and now every machine is a parallel machine. Waiting for the next generation machine does not guarantee anymore a better performance of a software. To keep doubling performance parallelism must double. It implies a huge effort in algorithmic development.

Thanks to `ffddm`, FreeFEM users have access to high-level functionalities for specifying and solving their finite element problems in parallel. The first task handled by `ffddm` is the data distribution among the processors. This is done via an overlapping domain decomposition and a related distributed linear algebra. Then, solving a linear system is possible either via an interface to the parallel `MUMPS` solver or by using domain decomposition methods as preconditioners to the GMRES Krylov method. The `ffddm` framework makes it easy to use scalable Schwarz methods enhanced by a coarse space correction built either from a coarse mesh or a `GenEO` (Generalized Eigenvalue in the Overlap) coarse space, see also the book [An Introduction to Domain Decomposition Methods: algorithms, theory, and parallel implementation](#). State-of-the-art three level methods are also implemented in `ffddm`.

The `ffddm` framework is entirely written in the FreeFEM language and the ‘`idp`’ scripts can be found [here](#) (‘`ffddm*.idp`’ files). It makes it also a very good tool for learning and prototyping domain decomposition methods without compromising efficiency.

`ffddm` can also act as a wrapper for the `HPDDM` library. HPDDM is an efficient implementation of various domain decomposition methods and a variety of Krylov subspace algorithms, with advanced block and recycling methods for solving sequences of linear systems with multiple right-hand sides: GMRES and Block GMRES, CG, Block CG,

and Breakdown-Free Block CG, GCRO-DR and Block GCRO-DR. For more details on how to use HPDDM within ffddm, see [the ffddm documentation](#).

## Getting Started

```

1 macro dimension 2 // EOM           // 2D or 3D
2 include "ffddm.idp"
3 mesh Th = square(50,50);      // global mesh
4 // Step 1: Decompose the mesh
5 ffddmbuildDmesh( P , Th , mpiCommWorld )
6 // Step 2: Define your finite element
7 macro def(u) u // EOM
8 macro init(u) u // EOM
9 ffddmbuildDfespace( P , P , real , def , init , P2 )
10 // Step 3: Define your problem
11 macro grad(u) [dx(u), dy(u)] // EOM
12 macro Varf(varfName, meshName, VhName)
13     varf varfName(u,v) = int2d(meshName)(grad(u)' * grad(v)) + int2d(meshName)(1*v)
14         + on(1, u = 0); // EOM
15 ffddmsetupOperator( P , P , Varf )
16 PVhi ui, bi;
17 ffddmbuildrhs( P , Varf , bi[] )
18 // Step 4: Define the one level DD preconditioner
19 ffddmsetupPrecond( P , Varf )
20 // Step 5: Define the two-level GenEO Coarse Space
21 ffddmgeneosetup( P , Varf )
22 // Step 6: Solve the linear system with GMRES
23 PVhi x0i = 0;
24 ui[] = PfGMRES(x0i[], bi[], 1.e-6, 200, "right");
25 ffddmplot(P, ui, "u")
26 Pwritessummary

```

This example solves a Laplace problem in 2D in parallel with a two-level GenEO domain decomposition method. To try this example, just copy and paste the script above in a file ‘test.edp’ and run it on 2 cores with

```
ff-mpirun -np 2 test.edp -glut ff	glut
```

### 3.9.1 Domain Decomposition (DD)

When the size of a three dimensional problem is large (whatever it means), it is necessary to distribute data among several processors especially for solving linear systems. A natural way is to do it via domain decomposition.

#### Mesh Decomposition

The starting point is a collection of  $N$  sub-meshes  $(Th_i)_{i=1}^N$  that together form a global mesh

$$Th := \bigcup_{i=1}^N Th_i.$$

These meshes may be overlapping or not. This decomposition induces a natural decomposition of the global finite element space  $Vh$  on  $Th$  into  $N$  local finite element spaces  $(Vh_i)_{i=1}^N$  each of them defined on  $Th_i$ .

**Note** By global, we mean that the corresponding structure can be referred to in the code (most often only) by its local values. In computer science term, it corresponds to a distributed data where each piece of data is stored by a MPI process.

## Distributed Linear Algebra

The domain decomposition induces a natural decomposition of the set of the global degrees of freedom (d.o.f.)  $\mathcal{N}$  of the finite element space  $Vh$  into the  $N$  subsets of d.o.f.'s  $(\mathcal{N}_i)_{i=1}^N$  each associated with the local finite element space  $Vh_i$ . We have thus

$$\mathcal{N} = \cup_{i=1}^N \mathcal{N}_i,$$

but with duplications of some of the d.o.f.'s.

Associated with this decomposition of the set of d.o.f.'s  $\mathcal{N}$ , a *distributed vector* is a collection of local vectors  $(\mathbf{V}_i)_{1 \leq i \leq N}$  so that the values on the duplicated d.o.f.'s are the same.

---

**Note:** In mathematical terms, it can be described as follows for a real valued problem. Let  $R_i$  be the restriction operator from  $\mathbb{R}^{\#\mathcal{N}}$  to  $\mathbb{R}^{\#\mathcal{N}_i}$ , where  $\#\mathcal{N}_i$  denotes the number of elements of  $\mathcal{N}_i$ . A collection of local vectors  $(\mathbf{V}_i)_{1 \leq i \leq N} \in \Pi_{i=1}^N \mathbb{R}^{\#\mathcal{N}_i}$  is a distributed vector iff there exists a global vector  $\mathbf{V} \in \mathbb{R}^{\#\mathcal{N}}$  such that for all subset  $1 \leq i \leq N$ , we have:

$$\mathbf{V}_i = R_i \mathbf{V}.$$

We will also say that the collection of local vectors  $(\mathbf{V}_i)_{1 \leq i \leq N}$  is consistent. For a complex valued problem, simply replace  $\mathbb{R}$  with  $\mathbb{C}$ .

---

## Partition of Unity Matrices (POUM)

Let  $(D_i)_{1 \leq i \leq N}$  be square diagonal matrices of size  $\#\mathcal{N}_i$  which form a partition of unity in the sense that:

$$Id = \sum_{i=1}^N R_i^T D_i R_i \text{ in } \mathbb{R}^{\#\mathcal{N} \times \#\mathcal{N}}.$$

For instance if a degree of freedom is shared by  $k$  subdomains defining the corresponding entry of the diagonal matrix  $D$  to be  $1/k$  yields partition of unity matrices. The matrices  $R_i$  and  $D_i$  are the heart of distributed linear algebra.

## Distributed scalar product

For two global vectors  $\mathbf{U}$  and  $\mathbf{V}$  of size  $\#\mathcal{N}$ , the formula for the scalar product  $\mathbf{V}^T \mathbf{U} = (\mathbf{U}, \mathbf{V})$  in terms of their distributed vector counterparts is:

$$(\mathbf{U}, \mathbf{V}) = \left( \mathbf{U}, \sum_{i=1}^N R_i^T D_i R_i \mathbf{V} \right) = \sum_{i=1}^N (R_i \mathbf{U}, D_i R_i \mathbf{V}) = \sum_{i=1}^N (\mathbf{U}_i, D_i \mathbf{V}_i).$$

Local scalar products are performed concurrently. Thus, the implementation is parallel except for the sum which corresponds to a MPI\_Reduce call across the  $N$  MPI processes. Note also that the implementation relies on the knowledge of a partition of unity so that the FreeFEM syntax is `dscalprod(Di,u,v)` or equivalently `pr#scalprod(u,v)` where `pr` is a user defined prefix that refers to the domain decomposition and thus implicitly also to the partition of unity.

## Update

From a collection of local vectors  $(\mathbf{U}_i)_{1 \leq i \leq N}$ , it is possible ensure consistency of the duplicated data and thus creating a distributed vector  $(\mathbf{V}_i)_{1 \leq i \leq N}$  by calling the function `pr#update(Ui, TRUE)` where `pr` is a user defined prefix

that refers to the domain decomposition. This function performs the following operation for all  $1 \leq i \leq N$ :

$$\mathbf{V}_i \leftarrow R_i \sum_{j=1}^N R_j^T D_j \mathbf{U}_j$$

---

**Note:** The implementation corresponds to

$$\mathbf{V}_i := R_i \sum_{j=1}^N R_j^T D_j \mathbf{U}_j = D_i \mathbf{U}_i + \sum_{j \in \mathcal{O}(i)} R_i R_j^T D_j \mathbf{U}_j$$

where  $\mathcal{O}(i)$  is the set of neighbors of subdomain  $i$ . Therefore, the matrix vector product is computed in three steps:

- concurrent computing of  $D_j \mathbf{U}_j$  for all  $1 \leq j \leq N$ ;
  - neighbor to neighbor MPI-communications ( $R_i R_j^T$ );
  - concurrent sum of neighbor contributions.
- 

## Distributed Matrix and Vector resulting from a variational formulation

The discretization of a variational formulation on the global mesh  $Th$  yields a global matrix  $A$  and a global right hand side **RHS**. Thanks to the sparsity of finite element matrices for partial differential equations and thanks to the overlap between subdomains, the knowledge of the local matrix  $R_i A R_i^T$  on each subdomain  $1 \leq i \leq N$  is sufficient to perform the matrix-vector product  $A \times \mathbf{U}$  for any global vector  $\mathbf{U}$ . Once the problem has been set up by a call to `ffddmsetupOperator(myprefix, myFEprefix, myVarf)`, the matrix-vector product is performed by calling the function `pr#A(Ui)` where `pr` is a user defined prefix that refers to the problem at hand which itself implicitly refers to the triplet (domain decomposition, finite element, variational formulation). See more on problem definition in this [documentation](#) and more on distributed linear algebra in chapter 8 of “An Introduction to Domain Decomposition Methods: algorithms, theory and parallel implementation” SIAM 2015.

## Distributed Linear Solvers

In many cases, we are interested in the solution of the problem in terms of the vector of d.o.f.’s  $\mathbf{X}$  that satisfies:

$$A \mathbf{X} = \mathbf{RHS}.$$

`ffddm` offers two parallel solvers: *direct factorization* and *Schwarz* domain decomposition methods.

### Distributed Direct Solvers

In order to benefit from the sparsity of the matrix arising from a finite element discretization of a partial differential equation, a variant of Gauss elimination, the frontal method, that automatically avoids a large number of operations involving zero terms was developed. A frontal solver builds a *LU* or Cholesky decomposition of a sparse matrix given as the assembly of element matrices by eliminating equations only on a subset of elements at a time. This subset is called the *front* and it is essentially the transition region between the part of the system already finished and the part not touched yet. These methods are basically sequential since the unknowns are processed one after another or one front after another. In order to benefit from multicore processors, a *multifrontal solver* is an improvement of the frontal solver that uses several independent fronts at the same time. The fronts can be worked on by different processors, which enables parallel computing. `ffddm` provides an interface to the parallel sparse direct solver **MUMPS**.

### Schwarz methods

We consider the solve of the equation  $A \mathbf{X} = \mathbf{RHS}$  by a flexible GMRES method preconditioned by domain decomposition methods.

## Restricted Additive Schwarz (RAS)

The RAS preconditioner reads:

$$M_{RAS}^{-1} := \sum_{j=1}^N R_j^T D_j (R_j A R_j^T)^{-1} R_j.$$

Let  $A_i$  denote the local matrix  $(R_i A R_i^T)$ . The application of the operator  $M_{RAS}^{-1}$  to a distributed right hand side  $(\mathbf{RHS}_i)_{i=1}^N$  consists in computing:

$$R_i \sum_{j=1}^N R_j^T D_j A_j^{-1} \mathbf{RHS}_j = D_i A_i^{-1} \mathbf{RHS}_i + \sum_{j \in \mathcal{O}(i)} (R_i R_j^T) D_j A_j^{-1} \mathbf{RHS}_j.$$

This task is performed by first solving concurrently on all subdomains a linear system for  $\mathbf{Y}_j$  for all  $1 \leq j \leq N$ :

$$A_j \mathbf{Y}_j = \mathbf{RHS}_j.$$

Each local vector  $\mathbf{Y}_j$  is weighted by the partition of unity matrix  $D_j$ . Then data transfers between neighboring subdomains implement the  $R_i R_j^T D_j \mathbf{Y}_j$  formula. The contribution from neighboring subdomains are summed locally. This pattern is very similar to that of the [update](#) procedure.

## Optimized Restricted Additive Schwarz (ORAS)

The ORAS preconditioner may be seen as a variant of the RAS preconditioner. It reads:

$$M_{RAS}^{-1} := \sum_{j=1}^N R_j^T D_j B_j^{-1} R_j$$

where  $B_j$  are local matrices of size  $\#\mathcal{N}_j \times \#\mathcal{N}_j$  for  $1 \leq j \leq N$ . This variant is very useful when dealing with wave propagation phenomena such as Helmholtz problems in acoustics or Maxwell system in the frequency domain for electromagnetism. Defining  $B_j$  as the discretization of the physical equation with impedance conditions on the boundary of the subdomain has been proved to be a good choice.

## Two level methods

The RAS and ORAS methods are called a one-level method in the sense that sub-domains only interact with their direct neighbors. For some problems such as Darcy problems or static elasticity problems and when the number of sub-domains is large, such one-level methods may suffer from a slow convergence. The fix is to add to the preconditioner an auxiliary coarse problem that couples all subdomains at each iteration and is inexpensive to calculate.

In mathematical terms, we first choose a full rank rectangular matrix  $Z \in \mathbb{R}^{\#\mathcal{N} \times NC}$  where  $NC \ll \#\mathcal{N}$  denotes the dimension of the coarse space spanned by the columns of  $Z$ . We also pick a coarse matrix  $A_C \in \mathbb{R}^{NC \times NC}$ . A generic one-level method preconditioner  $M_1^{-1}$  is enriched by a solve on the coarse space. The simplest correction formula is additive:

$$M_2^{-1} := Z A_C^{-1} Z^T + M_1^{-1}$$

Other correction formulas are given in [documentation](#).

We consider two ways to build  $Z$  and thus the coarse space and the coarse problem  $A_C$ , see below [Coarse Mesh](#) and [GenEO](#)

## Coarse Mesh

A first possibility is to discretize the problem on a coarse mesh, following the same principle as multi-grid methods. For 3-D problems, a coarsening of the mesh size by a factor 2, reduces by a factor  $2^3 = 8$  the size of the coarse problem which is then easier to solve by a direct method. Then,  $Z$  is the interpolation matrix from the coarse finite element space to the fine one.

## GenEO

For highly heterogeneous or anisotropic problems, two level methods based on coarse meshes might fail and a more sophisticated construction must be used. A provable robust coarse space called GenEO is built by first solving the following local generalized eigenvalue problem in parallel for each subdomain  $1 \leq i \leq N$ , where  $A_i^{\text{Neu}}$  denotes the local matrix resulting from the variational formulation:

$$D_i A_i D_i V_{i,k} = \lambda_{i,k} A_i^{\text{Neu}} V_{i,k}$$

The eigenvectors selected to enter the coarse space correspond to eigenvalues  $\lambda_{i,k} \geq \tau$ , where the threshold parameter  $\tau$  is user-defined. The precise formulas are given in this [documentation](#). From a mathematical point of view, it has been proved that for a symmetric positive definite matrix  $A$ , the spectrum of the preconditioned by the two-level method with a GenEO coarse space lies in the interval  $[\frac{1}{1 + k_1 \tau}, k_0]$ .

**Note** A heuristic that justifies this construction is as follows. We first introduce the Additive Schwarz method (ASM) which can be seen as a symmetrized variant of the RAS preconditioner:

$$M_{\text{ASM}}^{-1} := \sum_{j=1}^N R_j^T A_j^{-1} R_j .$$

It can be proved that the lower bound for the eigenvalue of  $M_{\text{ASM}}^{-1} A$  is close to zero (which is bad for convergence) whereas the upper bound depends only on the number of neighbors of a subdomain (which is good for convergence).

Second, we also introduce the following preconditioner  $M_{NN}^{-1}$ :

$$M_{NN}^{-1} := \sum_{1 \leq j \leq N} D_j (A_j^{\text{Neu}})^{-1} D_j .$$

We have a very good lower bound for the preconditioned operator  $M_{NN}^{-1} A$  that does not depend on the number of subdomains but only on the maximum multiplicity of intersections  $k_1$  (which is good for convergence). But the upper bound for this preconditioner is very large (which is bad for convergence).

Now, if we compare formulas for  $M_{NN}^{-1}$  and  $M_{\text{ASM}}^{-1}$ , we may suspect that vectors  $\mathbf{V}_{ik}$  for which  $D_i (A_i^{\text{Neu}})^{-1} D_i \mathbf{V}_{ik}$  and  $A_i^{-1} \mathbf{V}_{ik}$  have very different values are responsible for the slow convergence and should contribute to the coarse space. This is a way to interpret the above generalized eigenvalue problem which controls the lower bound of the two-level preconditioned system.

## 3.9.2 ffddm documentation

### Minimal example

```

1 macro dimension 3 // EOM           // 2D or 3D
2
3 include "ffddm.idp"
4

```

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```

5 int[int] LL = [2,2, 1,2, 2,2];
6 mesh3 ThGlobal = cube(10, 10, 10, [x, y, z], label = LL);           // global mesh
7
8 macro grad(u) [dx(u), dy(u), dz(u)]// EOM      // three-dimensional gradient
9
10 macro Varf(varfName, meshName, VhName)
11     varf varfName(u,v) = int3d(meshName) (grad(u)' * grad(v)) + int3d(meshName) (v) +
12     on(1, u = 1.0);
13 // EOM
14
15 // Domain decomposition
16 ffddmbuildDmesh( Lap , ThGlobal , mpiCommWorld )
17
18 macro def(i)i// EOM                                // scalar field definition
19 macro init(i)i// EOM                               // scalar field initialization
20 ffddmbuildDfespace( Lap , Lap , real , def , init , P1 )
21
22 ffddmsetupOperator( Lap , Lap , Varf )
23
24 real[int] rhsi(0);
25 ffddmbuildrhs( Lap , Varf , rhsi )
26
27 LapVhi def(ui);
28
29 //Direct solve
30 ui[] = Lapdirectsolve(rhsi);
31
32 Lapwritesummary
33 ffddmplot(Lap,ui,"u");

```

## Overlapping mesh decomposition

```
1 ffddmbuildDmesh(pr,Th,comm)
```

decomposes the mesh **Th** into overlapping submeshes. The mesh will be distributed over the mpi ranks of communicator **comm**. This will create and expose variables whose names will be prefixed by **pr**, see below (# is the concatenation operator). The way the initial mesh **Th** is partitioned depends on the value of *ffddmpartitioner*.

The size of the overlap between subdomains (its width in terms of number of mesh elements) is given by *ffddmoverlap*.

The level of refinement of the resulting submeshes with respect to the input mesh **Th** is given by *ffddmsplit*.

If *ffddmexclude*  $\neq 0$ , the first *ffddmpCS* mpi ranks of **comm** will be excluded from the spatial domain decomposition, in order to dedicate them later to the coarse problem (for two-level preconditioners).

The label of the new border of the submeshes (the interface between the subdomains) is given by *ffddminterfacelabel*.

**defines:**

- int pr#npart number of subdomains for this decomposition; should be equal to mpiSize(**comm**) - *ffddmexclude* \* *ffddmpCS*
- meshN[int] pr#aTh array (size pr#npart) of local meshes of the subdomains. In the standard parallel case, only the local mesh for this mpi rank pr#aTh[mpiRank(pr#commddm)] is defined (unless this mpi rank is excluded from the spatial domain decomposition, i.e. prmesh#excluded = 1, see below). In the sequential case, all local meshes are defined.

- meshN pr#Thi the local mesh of the subdomain for this mpi rank, i.e. pr#aTh[mpiRank(pr#commddm)] in the parallel case - int pr#numberIntersection the number of neighbors for this mpi rank
- int [int] pr#arrayIntersection the list of neighbor ranks in pr#commddm for this mpi rank
- int pr#pCS equal to *ffddmpCS*
- int pr#exclude equal to *ffddmexclude*
- int pr#excluded *true* if *ffddmexclude* is *true* ( $\neq 0$ ) and  $\text{mpiRank}(\text{comm}) < \text{pr}\#\text{pCS}$ . In this case, this mpi rank will be excluded from the spatial domain decomposition and will only work on the coarse problem.
- mpiComm pr#commddm mpi communicator for ranks participating in the spatial domain decomposition (ranks 0 to pr#npart-1 in **comm** if pr#exclude is *false*, ranks pr#pCS to pr#pCS+pr#npart-1 otherwise)
- mpiComm pr#commCS mpi communicator for ranks participating in the assembly and resolution of the coarse problem for two-level preconditioners (ranks 0 to pr#pCS - 1 in **comm**)
- mpiComm pr#commself self mpi communicator (this mpi rank only), used for factorizing local matrices

**Remark for sequential use (see *-seqddm*):**

- meshN [int] pr#aTh array (size pr#npart) of local meshes of the subdomains

**Local finite element spaces**

```
1 ffdmbuildDfespace(pr,prmsh,scalar,def,init,Pk)
```

builds the local finite element spaces and associated distributed operators on top of the mesh decomposition **prmsh**. This will create and expose variables whose names will be prefixed by **pr**, see below. It is assumed that *ffdmbuild-Dmesh* has already been called with prefix **prmsh** in order to build the mesh decomposition.

The local finite element spaces of type **Pk** (where **Pk** is the type of finite element: P1, [P2,P2,P1], ...) are defined on the local meshes of the subdomains based on the mesh decomposition previously created with prefix **prmsh**.

**scalar** determines the type of data for this finite element: *real* or *complex*.

Two macros, **def** and **init**, are needed: **def** specifies how to define a finite element function in the finite element space **Pk**, and **init** specifies how to interpolate a scalar function onto the (possibly multiple) components of **Pk**. Two examples are given below:

For scalar P2 finite elements and complex-valued problems:

```
1 macro def(u) u// EOM
2 macro init(u) u// EOM
3 ffdmbuildDfespace(myFEprefix,mymeshprefix,complex,def,init,P2)
```

For vectorial [P2,P2,P1] finite elements and real-valued problems:

```
1 macro def(u) [u, u#B, u#C]// EOM
2 macro init(u) [u, u, u]// EOM
3 ffdmbuildDfespace(myFEprefix,mymeshprefix,real,def,init,[P2,P2,P1])
```

In practice, this builds the necessary distributed operators associated to the finite element space: the local partition of unity functions  $(D_i)_{i=1,\dots,N}$  (see pr#Dk and pr#Dih below) as well as the function pr#update (see below) which synchronizes local vectors  $(u_i)_{i=1,\dots,N}$  between neighboring subdomains, performing the equivalent of  $u_i = R_i(\sum_{j=1}^N R_j^T u_j)$  or  $u_i = R_i(\sum_{j=1}^N R_j^T D_j u_j)$  in a distributed parallel environment.

pr#scalprod (see below) performs the parallel scalar product for vectors defined on this finite element.

**defines:**

- `pr#prmsh` macro, saves the parent prefix **prmsh** of the mesh decomposition
- `pr#K` macro, saves the type of data **scalar** for this finite element space (*real* or *complex*)
- `func pr#fPk` saves the type of finite element **Pk**, e.g. *P1*, [*P2,P2,P1*], ...
- `fespace pr#Vhi` the local finite element space for this mpi rank, defined on the local mesh `prmsh#Thi`
- `int pr#Ndofglobal` the total number of degrees of freedom *n* for this finite element discretization
- `pr#mdef` macro, saves the macro **def** giving the definition of a finite element function in the finite element space **Pk**
- `pr#minit` macro, saves the macro **init** specifying how to interpolate a scalar function onto the (possibly multiple) components of a finite element function of **Pk**. This is used to create the local partition of unity function in `pr#Vhi`, by interpolating the local *P1* partition of unity function onto the components of `pr#Vhi`. For non Lagrange finite element spaces (e.g. *RT0*, *Edge03d*, ...), see [ffddmbuildDfespaceEdge](#).
- `pr#K[int] [int] pr#Dk` array (size `prmsh#npart`) of local partition of unity vectors in the subdomains, equivalent to  $(D_i)_{i=1,\dots,N}$ . In the standard parallel case, only the local partition of unity vector for this mpi rank `pr#Dk[mpiRank(prmsh#commddm)]` is defined (unless this mpi rank is excluded from the spatial domain decomposition, i. e. `prmsh#excluded = 1`). In the sequential case, all local partition of unity vectors are defined.
- `matrix<pr#K>[int] pr#Dih` array (size `prmsh#npart`) similar to `pr#Dk` but in *matrix* form, allowing for easier *matrix-matrix* multiplications. `pr#Dih[i]` is a diagonal matrix, with the diagonal equal to `pr#Dk[i]`.
- `fespace pr#Vhglob` the global finite element space defined on the global mesh `prmsh#Thglob`. Defined only if `-noGlob` is not used.
- `matrix<pr#K>[int] pr#Rih` array (size `prmsh#npart`) of restriction matrices from the global finite element space to the local finite element spaces on the local submeshes of the subdomains. In the standard parallel case, only the restriction matrix for this mpi rank `pr#Rih[mpiRank(prmsh#commddm)]` is defined (unless this mpi rank is excluded from the spatial domain decomposition, i. e. `prmsh#excluded = 1`). In the sequential case, all restriction matrices `pr#Rih` are defined only if `-noGlob` is not used.
- `func int pr#update(scalar[int] ui, bool scale)` The function `pr#update` synchronizes the local vector *ui* between subdomains by exchanging the values of *ui* shared with neighboring subdomains (in the overlap region) using point-to-point MPI communications. If *scale* is *true*, *ui* is multiplied by the local partition of unity beforehand. This is equivalent to  $u_i = R_i(\sum_{j=1}^N R_j^T u_j)$  when *scale* is *false* and  $u_i = R_i(\sum_{j=1}^N R_j^T D_j u_j)$  when *scale* is *true*.
- `func scalar pr#scalprod(scalar[int] ai, scalar[int] bi)` The function `pr#scalprod` computes the global scalar product of two vectors whose local restriction to the subdomain of this mpi rank are *ai* and *bi*. The result is computed as  $\sum_{j=1}^N (D_j a_j, b_j)$ .

**Define the problem to solve**

```
| ffddmsetupOperator(pr,prfe,Varf)
```

builds the distributed operator associated to the variational problem given by **Varf**, on top of the distributed finite element space **prfe**. This will create and expose variables whose names will be prefixed by **pr**, see below. It is assumed that [ffddmbuildDfespace](#) has already been called with prefix **prfe** in order to define the distributed finite element space.

In practice, this builds the so-called local ‘Dirichlet’ matrices  $A_i = R_i A R_i^T$ , the restrictions of the global operator  $A$  to the subdomains (see `pr#aRd` below). The matrices correspond to the discretization of the bilinear form given by the macro **Varf**, which represents the abstract variational form of the problem. These matrices are then used to implement the action of the global operator  $A$  on a local vector (the parallel matrix-vector product with  $A$ ), see `pr#A` below.

At this point, we already have the necessary data to be able to solve the problem with a parallel direct solver (*MUMPS*), which is the purpose of the function `pr#directsolve` (see below). See [`ffddmbuildrhs`](#) for building the right-hand side.

The macro **Varf** is required to have three parameters: the name of the variational form, the mesh, and the finite element space. The variational form given in this ‘abstract’ format will then be used by `ffddm` to assemble the discrete operators by setting the appropriate mesh and finite element space as parameters. An example is given below:

```

1 macro myVarf(varfName, meshName, VhName)
2   varf varfName(u,v) = int3d(meshName)(grad(u)' * grad(v)) + on(1, u = 1.0);
3 // EOM
4
5 ffdmsetupOperator(myprefix,myFEprefix,myVarf)
```

**Remark** In this simple example, the third parameter *VhName* is not used. However, for more complex cases such as non-linear or time dependent problems where the problem depends on a solution computed at a previous step, it is useful to know for which discrete finite element space the variational form is being used. See for example TODO

**defines:**

- `pr#prfe` macro, saves the parent prefix **prfe** of the finite element space
- `int pr#verbosity` the level of verbosity for this problem, initialized with the value of [`ffdmverbosity`](#)
- `pr#writesummary` macro, prints a summary of timings for this problem, such as the time spent to assemble local matrices or solve the linear system.
- `matrix<prfe#K> pr#Aglobal` the global matrix  $A$  corresponding to the discretization of the variational form given by the macro **Varf** on the global finite element space `prfe#Vhglob`. Defined only in the sequential case.
- `matrix<prfe#K>[int] pr#aRd` array (size `prfe#prmsh#npart`) of so-called local ‘Dirichlet’ matrices in the subdomains; these are the restrictions of the global operator to the subdomains, equivalent to  $A_i = R_i A R_i^T$  with  $A$  the global matrix corresponding to the discretization of the variational form given by the macro **Varf** on the global finite element space. In the standard parallel case, only the local matrix for this mpi rank `pr#aRd[mpiRank(prmesh#commddm)]` is defined (unless this mpi rank is excluded from the spatial domain decomposition, i. e. `prmsh#excluded = 1`). In the sequential case, all local matrices are defined.
- `func prfe#K[int] pr#A(prfe#K[int] &ui)` The function `pr#A` computes the parallel matrix-vector product, i.e. the action of the global operator  $A$  on the local vector  $u_i$ . The computation is equivalent to  $R_i(\sum_{j=1}^N R_j^T D_j A_j u_j)$  and is performed in parallel using local matrices `pr#aRd` and the function `prfe#update`. In the sequential case, the global matrix `pr#Aglobal` is used instead.
- `func prfe#K[int] pr#directsolve(prfe#K[int] &rhsi)` The function `pr#directsolve` allows to solve the linear system  $Ax = b$  in parallel using the parallel direct solver *MUMPS*. The matrix is given to *MUMPS* in distributed form through the local matrices `pr#aRd`. The input `rhsi` is given as a distributed vector (`rhsi` is the restriction of the global right-hand side  $b$  to the subdomain of this mpi rank, see [`ffddmbuildrhs`](#)) and the returned vector is local as well.

```

1 ffdmbuildrhs(pr,Varfrhs,rhs)
```

builds the right-hand side associated to the variational form given by **Varfrhs** for the problem corresponding to prefix **pr**. The resulting right-hand side vector **rhs** corresponds to the discretization of the abstract linear form given by the macro **Varfrhs** (see [`ffdmsetupOperator`](#) for more details on how to define the abstract variational form as a macro).

The input vector **rhs** is resized and contains the resulting local right-hand side  $R_i b$ , the restriction of the global right-hand side  $b$  to the subdomain of this mpi rank. In the sequential case, the global right-hand side vector  $b$  is assembled instead.

An example is given below:

```

1 macro myVarfrhs(varfName, meshName, VhName)
2   varf varfName(u,v) = intN(meshName)(v) + on(1, u = 1.0);
3 // EOM
4
5 real[int] rhsi(0);
6 ffddmBuildrhs(myprefix,myVarfrhs,rhsi)
```

## One level preconditioners

```

1 ffddmSetupPrecond(pr,VarfPrec)
```

builds the one level preconditioner for problem **pr**. This will create and expose variables whose names will be prefixed by **pr**, see below. It is assumed that *ffddmsetupOperator* has already been called with prefix **pr** in order to define the problem to solve.

In practice, this builds and performs the factorization of the local matrices used in the one level preconditioner. The local matrices depend on the choice of *ffddmrecond* and **VarfPrec**, see **pr#aR** below.

**defines:**

- string **pr#prec** equal to *ffddmrecond*. Sets the type of one level preconditioner  $M_1^{-1}$  to be used: “asm” (*Additive Schwarz*), “ras” (*Restricted Additive Schwarz*), “oras” (*Optimized Restricted Additive Schwarz*), “soras” (*Symmetric Optimized Restricted Additive Schwarz*) or “none” (no preconditioner).
- matrix<**pr#prfe#K**[int] **pr#aR** array (size **prfe#prmsh#npart**) of local matrices used for the one level preconditioner. Each mpi rank of the spatial domain decomposition performs the  $LU$  (or  $LDL^T$ ) factorization of the local matrix corresponding to its subdomain using the direct solver *MUMPS*.
  - If **VarfPrec** is not a previously defined macro (just put *null* for example), the matrices **pr#aR** are set to be equal to the so-called local ‘Dirichlet’ matrices **pr#aRd** (see *ffddmsetupOperator*). This is for the classical ASM preconditioner  $M_1^{-1} = M_{\text{ASM}}^{-1} = \sum_{i=1}^N R_i^T A_i^{-1} R_i$  or classical RAS preconditioner  $M_1^{-1} = M_{\text{RAS}}^{-1} = \sum_{i=1}^N R_i^T D_i A_i^{-1} R_i$  (it is assumed that *ffddmrecond* is equal to “asm” or “ras”).
  - If **VarfPrec** is a macro, it is assumed that **VarfPrec** defines an abstract bilinear form (see *ffddmsetupOperator* for more details on how to define the abstract variational form as a macro).
    - \* If *ffddmrecond* is equal to “asm” or “ras”, the matrices **pr#aR** will be assembled as local ‘Dirichlet’ matrices in the same manner as **pr#aRd**, but using the bilinear form defined by **VarfPrec** instead. This defines the ASM preconditioner as  $M_1^{-1} = M_{\text{ASM}}^{-1} = \sum_{i=1}^N R_i^T (A_i^{\text{Prec}})^{-1} R_i$  and the RAS preconditioner as  $M_1^{-1} = M_{\text{RAS}}^{-1} = \sum_{i=1}^N R_i^T D_i (A_i^{\text{Prec}})^{-1} R_i$ , where  $A_i^{\text{Prec}} = R_i A^{\text{Prec}} R_i^T$ .
    - \* If *ffddmrecond* is equal to “oras” or “soras”, the matrices **pr#aR** will correspond to the discretization of the variational form **VarfPrec** in the subdomains  $\Omega_i$ . In particular, various boundary conditions can be imposed at the interface between subdomains (corresponding to mesh boundary of label *ffddminterfacelabel* set by the parent call to *ffddmBuildDmesh*), such as Optimized Robin boundary conditions. We note the ORAS preconditioner as  $M_1^{-1} = M_{\text{ORAS}}^{-1} = \sum_{i=1}^N R_i^T D_i (B_i^{\text{Prec}})^{-1} R_i$  and the SORAS preconditioner as  $M_1^{-1} = M_{\text{SORAS}}^{-1} = \sum_{i=1}^N R_i^T D_i (B_i^{\text{Prec}})^{-1} D_i R_i$ .
- func **pr#prfe#K[int] pr#PREC1(pr#prfe#K[int] &ui)** The function **pr#PREC1** computes the parallel application of the one level preconditioner  $M_1^{-1}$ , i.e. the action of  $M_1^{-1}$  on the local vector  $u_i$ . In the sequential case, it computes the action of  $M_1^{-1}$  on a global vector. The action of the inverse of local matrices **pr#aRd** is computed by forward-backward substitution using their  $LU$  (or  $LDL^T$ ) decomposition.

- `func pr#prfe#K[int] pr#PREC(pr#prfe#K[int] &ui)` The function `pr#PREC` corresponds to the action of the preconditioner  $M^{-1}$  for problem `pr`. It coincides with the one level preconditioner `pr#PREC1` after the call to `ffdmsetupPrecond`. If a second level is subsequently added (see the next section about *Two level preconditioners*), it will then coincide with the two level preconditioner  $M_2^{-1}$  (see `pr#PREC2level`).
- `func pr#prfe#K[int] pr#fGMRES(pr#prfe#K[int]& x0i, pr#prfe#K[int]& bi, real eps, int nbiter, string sprec)` The function `pr#fGMRES` allows to solve the linear system  $Ax = b$  in parallel using the flexible GMRES method preconditioned by  $M^{-1}$ . The action of the global operator  $A$  is given by `pr#A`, the action of the preconditioner  $M^{-1}$  is given by `pr#PREC` and the scalar products are computed by `pr#scalprod`. More details are given in the section *Solving the linear system*.

## Two level preconditioners

The main ingredient of a two level preconditioner is the so-called ‘coarse space’ matrix  $Z$ .

$Z$  is a rectangular matrix of size  $n \times n_c$ , where usually  $n_c \ll n$ .

$Z$  is used to build the ‘coarse space operator’  $E = Z^T AZ$ , a square matrix of size  $n_c \times n_c$ . We can then define the ‘coarse space correction operator’  $Q = ZE^{-1}Z^T = Z(Z^T AZ)^{-1}Z^T$ , which can then be used to enrich the one level preconditioner through a correction formula. The simplest one is the *additive* coarse correction:  $M_2^{-1} = M_1^{-1} + Q$ . See `pr#corr` below for all other available correction formulas.

There are multiple ways to define a relevant coarse space  $Z$  for different classes of problems. `ffdmgeneosetup` defines a coarse space correction operator by building the GenEO coarse space, while `ffdmcoarsemeshsetup` builds the coarse space using a coarse mesh.

After a call to either `ffdmgeneosetup` or `ffdmcoarsemeshsetup`, the following variables and functions are set up:

- `int pr#ncoarsespace` the size of the coarse space  $n_c$ .
- `string pr#corr` initialized with the value of `ffdmcorrection`. Specifies the type of coarse correction formula to use for the two level preconditioner. The possible values are:

"AD"	<i>Additive</i> ,	$M^{-1} = M_2^{-1} = M_1^{-1} + Q$
"BNN"	<i>Balancing Neumann-Neumann</i> ,	$M^{-1} = M_2^{-1} = (I - QA)M_1^{-1}(I - AQ) + Q$
"ADEF1"	<i>Adapted Deflation Variant 1</i> ,	$M^{-1} = M_2^{-1} = M_1^{-1}(I - AQ) + Q$
"ADEF2"	<i>Adapted Deflation Variant 2</i> ,	$M^{-1} = M_2^{-1} = (I - QA)M_1^{-1} + Q$
"RBNN1"	<i>Reduced Balancing Variant 1</i> ,	$M^{-1} = M_2^{-1} = (I - QA)M_1^{-1}(I - AQ)$
"RBNN2"	<i>Reduced Balancing Variant 2</i> ,	$M^{-1} = M_2^{-1} = (I - QA)M_1^{-1}$
"none"	<i>no coarse correction</i> ,	$M^{-1} = M_2^{-1} = M_1^{-1}$

- Note that *AD*, *ADEF1* and *RBNN2* only require one application of  $Q$ , while *BNN*, *ADEF2* and *RBNN1* require two. The default coarse correction is *ADEF1*, which is cheaper and generally as robust as *BNN* or *ADEF2*.
- `func pr#prfe#K[int] pr#Q(pr#prfe#K[int] &ui)` The function `pr#Q` computes the parallel application of the coarse correction operator  $Q$ , i.e. the action of  $Q = ZE^{-1}Z^T$  on the local vector  $u_i$ . In the sequential case, it computes the action of  $Q$  on a global vector. The implementation differs depending on the method used to build the coarse space (with GenEO or using a coarse mesh), but the idea is the same: the action of the transpose of the distributed operator  $Z$  on the distributed vector  $u_i$  is computed in parallel, with the contribution of all subdomains being gathered in a vector of size  $n_c$  in the mpi process of rank 0. The action of the inverse of the coarse space operator  $E$  is then computed by forward-backward substitution using its *LU* (or *LDL<sup>T</sup>*) decomposition previously computed by the first `pr#prfe#prmsh#pCS` ranks of the mpi communicator. The result is then sent back to all subdomains to perform the last application of  $Z$  and obtain the resulting local vector in each subdomain.
- `func pr#prfe#K[int] pr#PREC2level(pr#prfe#K[int] &ui)` The function `pr#PREC2level` computes the parallel application of the two level preconditioner  $M_2^{-1}$ , i.e. the action of  $M_2^{-1}$  on the local vector  $u_i$ . In the sequential case, it computes the action of  $M_2^{-1}$  on a global vector.

The two level preconditioner depends on the choice of the coarse correction formula which is determined by `pr#corr`, see above.

### Building the GenEO coarse space

```
1 ffdmgenosetup(pr,Varf)
```

This builds the GenEO coarse space for problem `pr`. This will create and expose variables whose names will be prefixed by `pr`, see below. It is assumed that `ffdmsetupPrecond` has already been called for prefix `pr` in order to define the one level preconditioner for problem `pr`. The GenEO coarse space is  $Z = (R_i^T D_i V_{i,k})_{\lambda_{i,k} \geq \tau}^{i=1, \dots, N}$ , where  $V_{i,k}$  are eigenvectors corresponding to eigenvalues  $\lambda_{i,k}$  of the following local generalized eigenvalue problem in subdomain  $i$ :

$$D_i A_i D_i V_{i,k} = \lambda_{i,k} A_i^{\text{Neu}} V_{i,k},$$

where  $A_i^{\text{Neu}}$  is the local Neumann matrix of subdomain  $i$  (with Neumann boundary conditions at the subdomain interface).

In practice, this builds and factorizes the local Neumann matrices  $A_i^{\text{Neu}}$  corresponding to the abstract bilinear form given by the macro `Varf` (see `ffdmsetupOperator` for more details on how to define the abstract variational form as a macro). In the GenEO method, the abstract bilinear form `Varf` is assumed to be the same as the one used to define the problem `pr` through the previous call to `ffdmsetupOperator`. The local generalized eigenvalue problem is then solved in each subdomain to find the eigenvectors  $V_{i,k}$  corresponding to the largest eigenvalues  $\lambda_{i,k}$  (see `pr#Z` below). The number of computed eigenvectors  $\nu$  is given by `ffdmnu`. The eigenvectors selected to enter  $Z$  correspond to eigenvalues  $\lambda_{i,k}$  larger than  $\tau$ , where the threshold parameter  $\tau$  is given by `ffdmtau`. If `ffdmtau = 0`, all `ffdmnu` eigenvectors are selected. Finally, the coarse space operator  $E = Z^T AZ$  is assembled and factorized (see `pr#E` below).

**defines:**

- `pr#prfe#K[int] [int] pr#Z` array of local eigenvectors  $Z_{i,k} = D_i V_{i,k}$  obtained by solving the local generalized eigenvalue problem above in the subdomain of this mpi rank using `Arpack`. The number of computed eigenvectors  $\nu$  is given by `ffdmnu`. The eigenvectors selected to enter  $Z$  correspond to eigenvalues  $\lambda_{i,k}$  larger than  $\tau$ , where the threshold parameter  $\tau$  is given by `ffdmtau`. If `ffdmtau = 0`, all `ffdmnu` eigenvectors are selected.
- `matrix<pr#prfe#K> pr#E` the coarse space operator  $E = Z^T AZ$ . The matrix `pr#E` is assembled in parallel and is factorized by the parallel direct solver `MUMPS` using the first `pr#prfe#prmsh#pCS` ranks of the mpi communicator, with mpi rank 0 as the master process. The number of mpi processes dedicated to the coarse problem is set by the underlying mesh decomposition of problem `pr`, which also specifies if these mpi ranks are excluded from the spatial decomposition or not. These parameters are set by `ffdmmpCS` and `ffdmexclude` when calling `ffdmbuildDmesh` (see `ffdmbuildDmesh` for more details).

### Building the coarse space from a coarse mesh

```
1 ffdmcoarsemeshsetup(pr,Thc,VarfEprec,VarfAprec)
```

builds the coarse space for problem `pr` from a coarse problem which corresponds to the discretization of a variational form on a coarser mesh `Thc` of  $\Omega$ . This will create and expose variables whose names will be prefixed by `pr`, see below. It is assumed that `ffdmsetupPrecond` has already been called for prefix `pr` in order to define the one level preconditioner for problem `pr`. The abstract variational form for the coarse problem can differ from the original problem `pr` and is given by macro `VarfEprec` (see `ffdmsetupOperator` for more details on how to define the abstract variational form as a macro). For example, absorption can be added in the preconditioner for wave propagation problems, see examples for Helmholtz and Maxwell equations in the `Examples` section.

The coarse space  $Z$  corresponds to the interpolation operator from the coarse finite element space to the original finite element space of the problem. Thus, the coarse space operator  $E = Z^T A^{\text{Apref}} Z$  corresponds to the matrix of the problem given by **VarfEprec** discretized on the coarse mesh **Thc** and is assembled as such.

Similarly, **VarfAprec** specifies the global operator involved in multiplicative coarse correction formulas. For example,  $M_{2,\text{ADEFI}}^{-1} = M_1^{-1}(I - A^{\text{Apref}}Q) + Q$  (where  $Q = ZE^{-1}Z^T$ ).  $A^{\text{Apref}}$  defaults to  $A$  if **VarfAprec** is not a valid macro (you can put *null* for example).

**defines:**

- `meshN pr#ThCoarse` the coarse mesh **Thc**
- `fespace pr#VhCoarse` the coarse finite element space of type `pr#prfe#fPk` defined on the coarse mesh `pr#ThCoarse`
- `matrix<pr#prfe#K> pr#AglobEprec` the global matrix  $A^{\text{Apref}}$  corresponding to the discretization of the variational form given by the macro **VarfAprec** on the global finite element space `pr#prfe#Vhglob`. Defined only in the sequential case. `pr#AglobEprec` is equal to `pr#Aglobal` if **VarfAprec** is not a valid macro.
- `matrix<pr#prfe#K> pr#aRdEprec` the local ‘Dirichlet’ matrix corresponding to **VarfAprec**; it is the local restriction of the global operator  $A^{\text{Apref}}$  to the subdomain, equivalent to  $A_i^{\text{Apref}} = R_i A^{\text{Apref}} R_i^T$  with  $A^{\text{Apref}}$  the global matrix corresponding to the discretization of the variational form given by the macro **VarfAprec** on the global finite element space. Defined only if this mpi rank is not excluded from the spatial domain decomposition, i. e. `prmsh#excluded = 0`. `pr#aRdEprec` is equal to `pr#aRd[mpiRank (prmsh#comddm)]` if **VarfAprec** is not a valid macro.
- `func pr#prfe#K[int] pr#AEprec(pr#prfe#K[int] &ui)` The function `pr#AEprec` computes the parallel matrix-vector product, i.e. the action of the global operator  $A^{\text{Apref}}$  on the local vector  $u_i$ . The computation is equivalent to  $R_i(\sum_{j=1}^N R_j^T D_j A_j^{\text{Apref}} u_j)$  and is performed in parallel using local matrices `pr#aRdEprec` and the function `pr#prfe#update`. In the sequential case, the global matrix `pr#AglobEprec` is used instead.
- `matrix<pr#prfe#K> pr#ZCM` the interpolation operator  $Z$  from the coarse finite element space `pr#VhCoarse` to the global finite element space `pr#prfe#Vhglob`. Defined only in the sequential case.
- `matrix<pr#prfe#K> pr#ZCMi` the local interpolation operator  $Z_i$  from the coarse finite element space `pr#VhCoarse` to the local finite element space `pr#prfe#Vhi`. Defined only if this mpi rank is not excluded from the spatial domain decomposition, i. e. `prmsh#excluded = 0`. `pr#ZCMi` is used for the parallel application of  $Z$  and  $Z^T$ .
- `matrix<pr#prfe#K> pr#ECM` the coarse space operator  $E = Z^T A^{\text{Apref}} Z$ . The matrix `pr#ECM` is assembled by discretizing the variational form given by **VarfEprec** on the coarse mesh and factorized by the parallel direct solver **MUMPS** using the first `pr#prfe#prmsh#pCS` ranks of the mpi communicator, with mpi rank 0 as the master process. The number of mpi processes dedicated to the coarse problem is set by the underlying mesh decomposition of problem **pr**, which also specifies if these mpi ranks are excluded from the spatial decomposition or not. These parameters are set by `ffddmpCS` and `ffddmexclude` when calling `ffddmbuildDmesh` (see `ffddmbuildDmesh` for more details).

## Solving the linear system

```
func pr#prfe#K[int] pr#fGMRES(pr#prfe#K[int]& x0i, pr#prfe#K[int]& bi, real eps, int_
itmax, string sp)
```

solves the linear system for problem **pr** using the flexible GMRES algorithm with preconditioner  $M^{-1}$  (corresponding to `pr#PREC`). Returns the local vector corresponding to the restriction of the solution to `pr#prfe#Vhi`. **x0i** and **bi** are local distributed vectors corresponding respectively to the initial guess and the right-hand side (see `ffddmbuildrhs`). **eps** is the stopping criterion in terms of the relative decrease in residual norm. If **eps** < 0, the residual norm itself is

used instead. **itmax** sets the maximum number of iterations. **sp** selects between the "left" or "right" preconditioning variants: *left* preconditioned GMRES solves  $M^{-1}Ax = M^{-1}b$ , while *right* preconditioned GMRES solves  $AM^{-1}y = b$  for  $y$ , with  $x = M^{-1}y$ .

### Using HPDDM within ffddm

**ffddm** allows you to use **HPDDM** to solve your problem, effectively replacing the **ffddm** implementation of all parallel linear algebra computations. **ffddm** can then be viewed as a finite element interface for **HPDDM**.

You can use **HPDDM** features unavailable in **ffddm** such as advanced Krylov subspace methods implementing block and recycling techniques.

To switch to **HPDDM**, simply define the macro `pr#withhpddm` before using `ffddmsetupOperator`. You can then pass **HPDDM** options with command-line arguments or directly to the underlying **HPDDM** operator `pr#hpddmOP`:

```

1 macro PBwithhpddm() 1 // EOM
2 ffddmsetupOperator( PB , FE , Varf )
3 set (PBhpddmOP, sparams="-hpddm_krylov_method_gcradr");

```

You can also choose to replace only the Krylov solver, by defining the macro `pr#withhpddmkrylov` before using `ffddmsetupOperator`. Doing so, a call to `pr#fGMRES` will call the **HPDDM** Krylov solver, with **ffddm** providing the operator and preconditioner through `pr#A` and `pr#PREC`.

An example can be found in **Helmholtz-2d-HPDDM-BGMRES.edp**, see the *Examples* section.

### 3.9.3 Parameters

#### Command-line arguments

- `-ffddm_verbosity N`, the level of verbosity of **ffddm**, see `ffddmverbosity` (default 3).
- `-seqddm N` use **ffddm** in sequential mode, with N the number of subdomains.
- `-noGlob` if present, do not define any global quantity (such as saving the global mesh for plotting or building the global restriction matrices). Cannot be used in sequential mode or with plotting.
- `-ffddm_partitioner N` specifies how to partition the initial domain, see `ffddmpartitioner` (default 1, *metis*).
- `-ffddm_overlap N` specifies the width of the overlap region between subdomains, see `ffddmoverlap` (default 1).
- `-ffddm_master_p N`, number of master processes for the coarse problem (for two level preconditioners), see `ffddmpCS` (default 1).
- `-ffddm_master_exclude 0|1` exclude master processes from the domain decomposition, see `ffddmexclude` (default 0).
- `-ffddm_split N`, level of refinement of the local submeshes with respect to the initial global mesh, see `ffddmsplit` (default 1).
- `-ffddm_schwarz_method S`, specifies the type of one level preconditioner  $M_1^{-1}$ : "asm" (*Additive Schwarz*), "ras" (*Restricted Additive Schwarz*), "oras" (*Optimized Restricted Additive Schwarz*), "soras" (*Symmetric Optimized Restricted Additive Schwarz*) or "none" (no preconditioner), see `ffddmrecond` (default "ras").
- `-ffddm_geneo_nu N`, number of local eigenvectors to compute in each subdomain when solving the local generalized eigenvalue problem for the GenEO method, see `ffddmnmu` (default 20).

- `-ffddm_geneo_threshold R`, threshold parameter for selecting local eigenvectors when solving the local generalized eigenvalue problems for the GenEO method, see [ffddmtau](#) (default 0.5). If the command-line parameter `-ffddm_geneo_nu N` is used, then [ffddmtau](#) is initialized to 0.
- `-ffddm_schwarz_coarse_correction S`, specifies the coarse correction formula to use for the two level preconditioner: “AD” (Additive), “BNN” (Balancing Neumann-Neumann), “ADEF1” (Adapted Deflation Variant 1), “ADEF2” (Adapted Deflation Variant 2), “RBNN1” (Reduced Balancing Variant 1), “RBNN2” (Reduced Balancing Variant 2) or “none” (no coarse correction), see [ffddmcorrection](#) (default “ADEF1”).

## Global parameters

- `ffddmverbosity` initialized by command-line argument `-ffddm_verbosity N`, specifies the level of verbosity of **ffddm** (default 3).
- `ffddmpartitioner` initialized by command-line argument `-ffddm_partitioner N`, specifies how to partition the initial domain:
  - `N=0`: user-defined partition through the definition of a macro, see [ffddmbuildDmesh](#)
  - `N=1`: use the automatic graph partitioner *metis* (default)
  - `N=2`: use the automatic graph partitioner *scotch*
- `ffddmoverlap` initialized by command-line argument `-ffddm_overlap N`, specifies the number of layers of mesh elements in the overlap region between subdomains  $N \geq 1$  (default 1). **Remark** The actual width of the overlap region between subdomains is  $2N$ , since each subdomain is extended by  $N$  layers of elements in a symmetric way.
- `ffddminterfacelabel` the label of the new border of the subdomain meshes (the interface between the subdomains) (default 10). Used for imposing problem-dependent boundary conditions at the interface between subdomains for the preconditioner, for example optimized Robin boundary conditions (see ORAS).
- `ffddmpCS` initialized by command-line argument `-ffddm_master_p N`, number of mpi processes used for the assembly and resolution of the coarse problem for two level preconditioners (default 1).
- `ffddmexclude` initialized by command-line argument `-ffddm_master_exclude`, 0 or 1 (default 0). If true, mpi ranks participating in the assembly and resolution of the coarse problem for two level preconditioners will be excluded from the spatial domain decomposition and will only work on the coarse problem.
- `ffddmsplit` initialized by command-line argument `ffddm_split N`, level of refinement of the local submeshes with respect to the initial global mesh (default 1). This is useful for large problems, where we want to avoid working with a very large global mesh. The idea is to start from a coarser global mesh, and generate finer local meshes in parallel during the mesh decomposition step in order to reach the desired level of refinement for the subdomains. For example, calling [ffddmbuildDmesh](#) with `ffddmsplit = 3` will generate local submeshes where each mesh element of the initial mesh is split into  $3^d$  elements.
- `ffddmrecond` initialized by command-line argument `-ffddm_schwarz_method S`, specifies the type of one level preconditioner  $M_1^{-1}$  to build when calling [ffddmsetupPrecond](#): “asm” (Additive Schwarz), “ras” (Restricted Additive Schwarz), “oras” (Optimized Restricted Additive Schwarz), “soras” (Symmetric Optimized Restricted Additive Schwarz) or “none” (no preconditioner). Default is “ras”. See [ffddmsetupPrecond](#) for more details.
- `ffddmnu` initialized by command-line argument `-ffddm_geneo_nu N`, number of local eigenvectors to compute in each subdomain when solving the local generalized eigenvalue problem for the GenEO method (default 20). See [ffddmgeneosetup](#) for more details.
- `ffddmtau` initialized by command-line argument `-ffddm_geneo_threshold R`, threshold parameter for selecting local eigenvectors when solving the local generalized eigenvalue problems for the GenEO method (default 0.5). If the command-line parameter `-ffddm_geneo_nu N` is used, then [ffddmtau](#) is initialized to 0. See [ffddmgeneosetup](#) for more details.

- `ffddmcorrection` initialized by command-line argument `-ffddm_schwarz_coarse_correction S`, specifies the coarse correction formula to use for the two level preconditioner: “AD” (Additive), “BNN” (Balancing Neumann-Neumann), “ADEF1” (Adapted Deflation Variant 1), “ADEF2” (Adapted Deflation Variant 2), “RBNN1” (Reduced Balancing Variant 1), “RBNN2” (Reduced Balancing Variant 2) or “none” (no coarse correction). Default is “ADEF1”. See the section about [Two level preconditioners](#) for more details.

### 3.9.4 Tutorial

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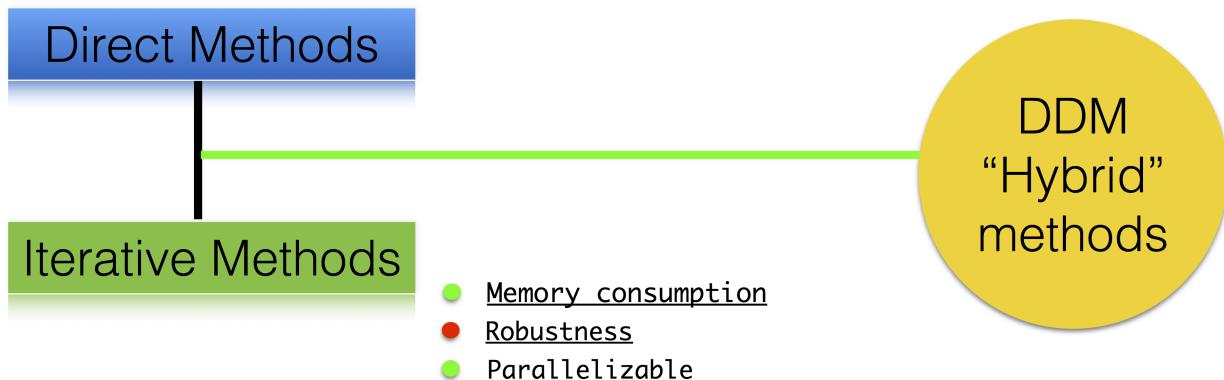
#### What is ffddm ?

- **ffddm** implements a class of parallel solvers in *FreeFEM: overlapping Schwarz domain decomposition methods*
- **The entire ffddm framework is written in the FreeFEM language ffddm** aims at simplifying the use of parallel solvers in *FreeFEM*. You can find the **ffddm** scripts [here](#) ('ffddm\*.idp' files) and examples [here](#)
- **ffddm provides a set of high-level macros and functions to**
  - handle data distribution: distributed meshes and linear algebra
  - build DD preconditioners for your variational problems
  - solve your problem using preconditioned Krylov methods
- **ffddm** implements scalable two level Schwarz methods, with a coarse space correction built either from a coarse mesh or a [GenEO](#) coarse space *Ongoing research*: approximate coarse solves and three level methods
- **ffddm can also act as a wrapper for the HPDDM library.** HPDDM is an efficient C++11 implementation of various domain decomposition methods and Krylov subspace algorithms with advanced block and recycling techniques More details on how to use **HPDDM** within **ffddm** [here](#)

#### Why Domain Decomposition Methods ?

How can we solve a large sparse linear system  $Au = b \in \mathbb{R}^n$  ?

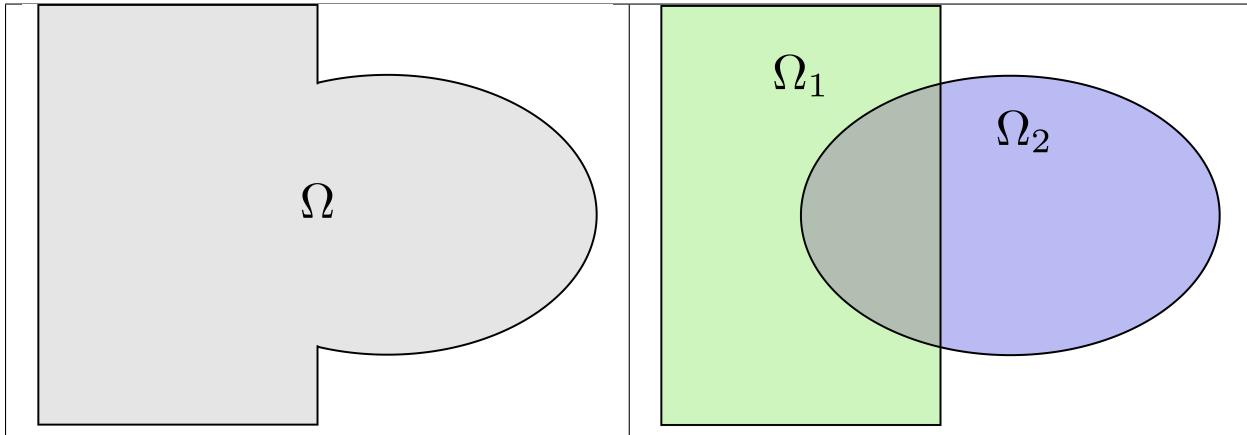
- [Memory consumption](#)
- [Robustness](#)
- [Parallelizable](#)



## Step 1: Decompose the mesh

See [documentation](#)

Build a collection of  $N$  overlapping sub-meshes  $(Th_i)_{i=1}^N$  from the global mesh  $Th$



```
1 ffddmbuildDmesh( prmsh , ThGlobal , comm )
```

- mesh distributed over the MPI processes of communicator **comm**
- initial mesh **ThGlobal** partitioned with *metis* by default
- size of the overlap given by [ffddmoverlap](#) (default 1)

*prmsh#Thi* is the local mesh of the subdomain for each mpi process

```
1 macro dimension 2// EOM           // 2D or 3D
2
3 include "ffddm.idp"
4
5 mesh ThGlobal = square(100,100);    // global mesh
6
7 // Step 1: Decompose the mesh
8 ffddmbuildDmesh( M , ThGlobal , mpiCommWorld )
9
10 medit("Th"+mpirank, MThi);
```

Copy and paste this to a file ‘test.edp’ and run it:

```
1 ff-mpirun -np 2 test.edp -wg
```

## Step 2: Define your finite element

See [documentation](#)

```
1 ffddmbuildDfespace( prfe , prmsh , scalar , def , init , Pk )
```

builds the local finite element spaces and associated distributed operators on top of the mesh decomposition **prmsh**

- **scalar**: type of data for this finite element: *real* or *complex*
- **Pk**: your type of finite element: P1, [P2,P2,P1], ...

- **def, init:** macros specifying how to define and initialize a **Pk** FE function

*pfe#Vhi* is the local FE space defined on *prmesh#Thi* for each mpi process

Example for P2 *complex*:

```

1 macro def(u) u // EOM
2 macro init(u) u // EOM
3 ffddmbuildDfespace( FE, M, complex,
4                     def, init, P2 )

```

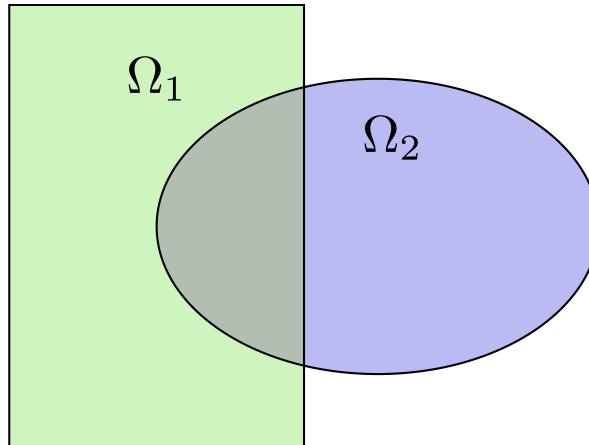
Example for [P2,P2,P1] *real*:

```

1 macro def(u) [u, u#B, u#C] // EOM
2 macro init(u) [u, u, u] // EOM
3 ffddmbuildDfespace( FE, M, real, def,
4                      init, [P2,P2,P1] )

```

## Distributed vectors and restriction operators



Natural decomposition of the set of d.o.f.'s  $\mathcal{N}$  of  $Vh$  into the  $N$  subsets of d.o.f.'s  $(\mathcal{N}_i)_{i=1}^N$  each associated with the local FE space  $Vh_i$

$$\mathcal{N} = \bigcup_{i=1}^N \mathcal{N}_i,$$

but with duplications of the d.o.f.'s in the overlap

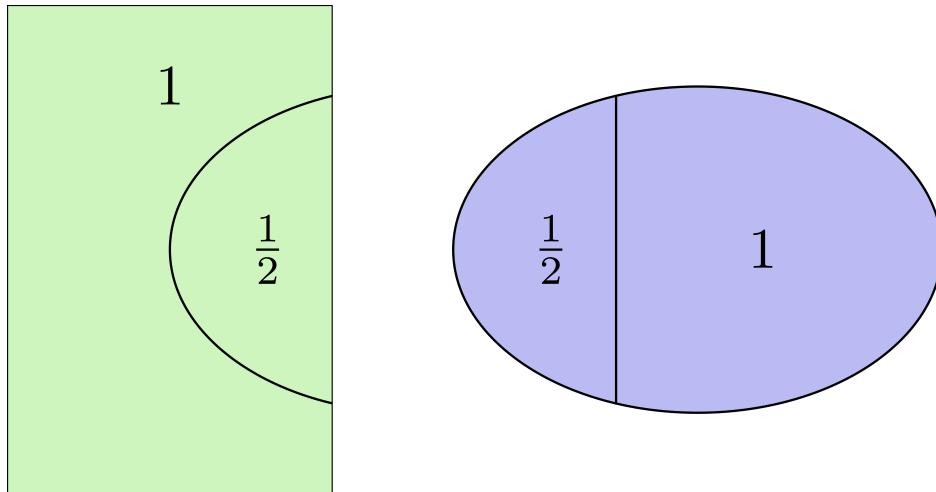
Definition a *distributed vector* is a collection of local vectors  $(\mathbf{V}_i)_{1 \leq i \leq N}$  so that the values on the duplicated d.o.f.'s are the same:

$$\mathbf{V}_i = R_i \mathbf{V}, \quad i = 1, \dots, N$$

where  $\mathbf{V}$  is the corresponding global vector and  $R_i$  is the *restriction operator* from  $\mathcal{N}$  into  $\mathcal{N}_i$

**Remark**  $R_i^T$  is the *extension operator*: extension by 0 from  $\mathcal{N}_i$  into  $\mathcal{N}$

## Partition of unity



Duplicated unknowns coupled via a *partition of unity*:

$$I = \sum_{i=1}^N R_i^T D_i R_i$$

$(D_i)_{1 \leq i \leq N}$  are square diagonal matrices of size  $\#\mathcal{N}_i$

$$\mathbf{V} = \sum_{i=1}^N R_i^T D_i R_i \mathbf{V} = \sum_{i=1}^N R_i^T D_i \mathbf{V}_i$$

## Data exchange between neighbors

```
| func prfe#update(K[int] vi, bool scale)
```

synchronizes local vectors  $\mathbf{V}_i$  between subdomains  $\Rightarrow$  exchange the values of  $\mathbf{V}_i$  shared with neighbors in the overlap region

$$\mathbf{V}_i \leftarrow R_i \left( \sum_{j=1}^N R_j^T D_j \mathbf{V}_j \right) = D_i \mathbf{V}_i + \sum_{j \in \mathcal{O}(i)} R_i R_j^T D_j \mathbf{V}_j$$

where  $\mathcal{O}(i)$  is the set of neighbors of subdomain  $i$ . Exchange operators  $R_i R_j^T$  correspond to neighbor-to-neighbor MPI communications

```
| FEupdate(vi, false);
```

$$\mathbf{V}_i \leftarrow R_i \left( \sum_{j=1}^N R_j^T \mathbf{V}_j \right)$$

```
| FEupdate(vi, true);
```

$$\mathbf{V}_i \leftarrow R_i \left( \sum_{j=1}^N R_j^T D_j \mathbf{V}_j \right)$$

```

1 macro dimension 2// EOM           // 2D or 3D
2
3 include "ffddm.idp"
4
5 mesh ThGlobal = square(100,100);    // global mesh
6
7 // Step 1: Decompose the mesh
8 ffddmbuildDmesh( M , ThGlobal , mpiCommWorld )
9
10 // Step 2: Define your finite element
11 macro def(u) u // EOM
12 macro init(u) u // EOM
13 ffddmbuildDfespace( FE , M , real , def , init , P2 )
14
15 FEVhi vi = x;
16 medit("v"+mpirank, MThi, vi);
17
18 vi[] = FEDk[mpirank];
19 medit("D"+mpirank, MThi, vi);
20
21 vi = 1;
22 FEupdate(vi[],true);
23 ffddmplot(FE,vi,"1")
24
25 FEupdate(vi[],false);
26 ffddmplot(FE,vi,"multiplicity")

```

### Step 3: Define your problem

See *documentation*

```
1 ffddmsetupOperator( pr , prfe , Varf )
```

builds the distributed operator associated to your variational form on top of the distributed FE **prfe**

**Varf** is a macro defining your abstract variational form

```

1 macro Varf(varfName, meshName, VhName)
2   varf varfName(u,v) = int2d(meshName)(grad(u)' * grad(v))
3     + int2d(meshName)(f*v) + on(1, u = 0); // EOM

```

⇒ assemble local ‘Dirichlet’ matrices  $A_i = R_i A R_i^T$

$$A = \sum_{i=1}^N R_i^T D_i A_i R_i$$

**Warning:** only true because  $D_i R_i A = D_i A_i R_i$  due to the fact that  $D_i$  vanishes at the interface !!

*pr#A* applies  $A$  to a distributed vector:  $\mathbf{U}_i \leftarrow R_i \sum_{j=1}^N R_j^T D_j A_j \mathbf{V}_j$

⇒ multiply by  $A_i + prfe\#update$

```

1 macro dimension 2 // EOM           // 2D or 3D
2
3 include "ffddm.idp"
4
5 mesh ThGlobal = square(100,100);    // global mesh
6
7 // Step 1: Decompose the mesh
8 ffddmBuildDmesh( M , ThGlobal , mpiCommWorld )
9
10 // Step 2: Define your finite element
11 macro def(u) u // EOM
12 macro init(u) u // EOM
13 ffddmBuildDfespace( FE , M , real , def , init , P2 )
14
15 // Step 3: Define your problem
16 macro grad(u) [dx(u), dy(u)] // EOM
17 macro Varf(varfName, meshName, VhName)
18   varf varfName(u,v) = int2d(meshName)(grad(u)' * grad(v))
19     + int2d(meshName)(1*v) + on(1, u = 0); // EOM
20 ffddmSetupOperator( PB , FE , Varf )
21
22 FEVhi ui, bi;
23 ffddmBuildRhs( PB , Varf , bi[] )
24
25 ui[] = PBA(bi[]);
26 ffddmPlot(FE, ui, "A*b")

```

## Summary so far: translating your sequential *FreeFEM* script

### Step 1: Decompose the mesh

See *documentation*

```
1 mesh Th = square(100,100);
```

```
1 mesh Th = square(100,100);
2 ffddmBuildDmesh(M, Th, mpiCommWorld)
```

### Step 2: Define your finite element

See *documentation*

```
1 fespace Vh(Th, P1);
```

```
1 macro def(u) u // EOM
2 macro init(u) u // EOM
3 ffddmBuildDfespace(FE, M, real, def, init, P1)
```

### Step 3: Define your problem

See *documentation*

```
1 varf Pb(u, v) = ...
2 matrix A = Pb(Vh, Vh);
```

```

1 macro Varf(varfName, meshName, VhName)
2   varf varfName(u,v) = ... // EOM
3   ffddmsetupOperator(PB, FE, Varf)

```

## Solve the linear system

See documentation

```

1 u[] = A^-1 * b[];
2
3 ui[] = PBdirectsolve(bi[]);

```

## Solve the linear system with the parallel direct solver *MUMPS*

See documentation

```

1 func K[int] pr#directsolve(K[int]& bi)

```

We have  $A$  and  $b$  in distributed form, we can solve the linear system  $Au = b$  using the parallel direct solver *MUMPS*

```

1 // Solve the problem using the direct parallel solver MUMPS
2 ui[] = PBdirectsolve(bi[]);
3 ffddmplot(FE, ui, "u")

```

## Step 4: Define the one level DD preconditioner

See documentation

```

1 ffddmsetupPrecond( pr , VarfPrec )

```

builds the one level preconditioner for problem **pr**.

By default it is the *Restricted Additive Schwarz (RAS)* preconditioner:

$$M_1^{-1} = M_{\text{RAS}}^{-1} = \sum_{i=1}^N R_i^T D_i A_i^{-1} R_i \quad \text{with } A_i = R_i A R_i^T$$

Setup step: compute the *LU* (or  $LDL^T$ ) factorization of local matrices  $A_i$

$pr\#\text{PREC1}$  applies  $M_1^{-1}$  to a distributed vector:  $\mathbf{U}_i \leftarrow R_i \sum_{j=1}^N R_j^T D_j A_j^{-1} \mathbf{V}_i$

$\Rightarrow$  apply  $A_i^{-1}$  (forward/backward substitutions) +  $prfe\#\text{update}$

## Step 5: Solve the linear system with preconditioned GMRES

See documentation

```

1 func K[int] pr#fGMRES(K[int]& x0i, K[int]& bi, real eps, int itmax, string sp)

```

solves the linear system with flexible GMRES with DD preconditioner  $M^{-1}$

- **x0i**: initial guess
- **bi**: right-hand side
- **eps**: relative tolerance
- **itmax**: maximum number of iterations
- **sp**: “left” or “right” preconditioning

*left preconditioning*

solve  $M^{-1}Ax = M^{-1}b$

*right preconditioning*

solve  $AM^{-1}y = b$

$\Rightarrow x = M^{-1}y$

```

1  macro dimension 2 // EOM           // 2D or 3D
2  include "ffddm.idp"
3
4  mesh ThGlobal = square(100,100);    // global mesh
5  // Step 1: Decompose the mesh
6  ffddmBuildDmesh( M , ThGlobal , mpiCommWorld )
7  // Step 2: Define your finite element
8  macro def(u) u // EOM
9  macro init(u) u // EOM
10 ffddmBuildDfespace( FE , M , real , def , init , P2 )
11 // Step 3: Define your problem
12 macro grad(u) [dx(u), dy(u)] // EOM
13 macro Varf(varfName, meshName, VhName)
14     varf varfName(u,v) = int2d(meshName) (grad(u)' * grad(v))
15             + int2d(meshName) (1*v) + on(1, u = 0); // EOM
16 ffddmSetupOperator( PB , FE , Varf )
17
18 FEVhi ui, bi;
19 ffddmBuildrhs( PB , Varf , bi[] )
20
21 // Step 4: Define the one level DD preconditioner
22 ffddmSetupPrecond( PB , Varf )
23
24 // Step 5: Solve the linear system with GMRES
25 FEVhi x0i = 0;
26 ui[] = PBfGMRES(x0i[], bi[], 1.e-6, 200, "right");
27
28 ffddmplot(FE, ui, "u")
29 PBwritesummary

```

## Define a two level DD preconditioner

See *documentation*

**Goal** improve scalability of the one level method

$\Rightarrow$  enrich the one level preconditioner with a *coarse problem* coupling all subdomains

Main ingredient is a rectangular matrix  $Z$  of size  $n \times n_c$ , where  $n_c \ll n$   $Z$  is the *coarse space* matrix

The *coarse space operator*  $E = Z^T A Z$  is a square matrix of size  $n_c \times n_c$

The simplest way to enrich the one level preconditioner is through the *additive coarse correction* formula:

$$M_2^{-1} = M_1^{-1} + \mathbf{Z} \mathbf{E}^{-1} \mathbf{Z}^T$$

How to choose  $\mathbf{Z}$ ?

### Build the GenEO coarse space

See *documentation*

```
1 ffddmgeneosetup( pr , Varf )
```

The *GenEO* method builds a robust coarse space for highly heterogeneous or anisotropic **SPD** problems

⇒ solve a local generalized eigenvalue problem in each subdomain

$$D_i A_i D_i V_{i,k} = \lambda_{i,k} A_i^{\text{Neu}} V_{i,k}$$

with  $A_i^{\text{Neu}}$  the local Neumann matrices built from **Varf** (same **Varf** as *Step 3*)

The GenEO coarse space is  $\mathbf{Z} = (R_i^T D_i V_{i,k})_{\lambda_{i,k} \geq \tau}^{i=1, \dots, N}$ . The eigenvectors  $V_{i,k}$  selected to enter the coarse space correspond to eigenvalues  $\lambda_{i,k} \geq \tau$ , where  $\tau$  is a threshold parameter

**Theorem** the spectrum of the preconditioned operator lies in the interval  $[\frac{1}{1+k_1\tau}, k_0]$  where  $k_0 - 1$  is the # of neighbors and  $k_1$  is the multiplicity of intersections ⇒  $k_0$  and  $k_1$  do not depend on  $N$  nor on the PDE

```
1 macro dimension 2// EOM           // 2D or 3D
2 include "ffddm.idp"
3
4 mesh ThGlobal = square(100,100);    // global mesh
5 // Step 1: Decompose the mesh
6 ffddmbuildDmesh( M , ThGlobal , mpiCommWorld )
7 // Step 2: Define your finite element
8 macro def(u) u // EOM
9 macro init(u) u // EOM
10 ffddmbuildDfespace( FE , M , real , def , init , P2 )
11 // Step 3: Define your problem
12 macro grad(u) [dx(u), dy(u)] // EOM
13 macro Varf(varfName, meshName, VhName)
14     varf varfName(u,v) = int2d(meshName)(grad(u)' * grad(v))
15             + int2d(meshName)(1*v) + on(1, u = 0); // EOM
16 ffddmsetupOperator( PB , FE , Varf )
17
18 FEVhi ui, bi;
19 ffddmbuildrhs( PB , Varf , bi[] )
20
21 // Step 4: Define the one level DD preconditioner
22 ffddmsetupPrecond( PB , Varf )
23
24 // Build the GenEO coarse space
25 ffddmgeneosetup( PB , Varf )
26
27 // Step 5: Solve the linear system with GMRES
28 FEVhi x0i = 0;
29 ui[] = PBfGMRES(x0i[], bi[], 1.e-6, 200, "right");
```

## Build the coarse space from a coarse mesh

See [documentation](#)

```
1 ffddmcoarsemeshsetup( pr , Thc , VarfEprec , VarfAprec )
```

For **non SPD** problems, an alternative is to build the coarse space by discretizing the PDE on a coarser mesh **Thc**

$Z$  will be the *interpolation matrix* from the coarse FE space  $Vh_c$  to the original FE space  $Vh$

$\Rightarrow E = Z^T A Z$  is the matrix of the problem discretized on the coarse mesh

The variational problem to be discretized on **Thc** is given by macro **VarfEprec**

**VarfEprec** can differ from the original **Varf** of the problem

*Example:* added absorption for wave propagation problems

Similarly, **VarfAprec** specifies the global operator involved in multiplicative coarse correction formulas. It defaults to **A** if **VarfAprec** is not defined

```
1 macro dimension 2// EOM           // 2D or 3D
2 include "ffddm.idp"
3
4 mesh ThGlobal = square(100,100);    // global mesh
5 // Step 1: Decompose the mesh
6 ffddmbuildDmesh( M , ThGlobal , mpiCommWorld )
7 // Step 2: Define your finite element
8 macro def(u) u // EOM
9 macro init(u) u // EOM
10 ffddmbuildDfespace( FE , M , real , def , init , P2 )
11 // Step 3: Define your problem
12 macro grad(u) [dx(u), dy(u)] // EOM
13 macro Varf(varfName, meshName, VhName)
14     varf varfName(u,v) = int2d(meshName)(grad(u)' * grad(v))
15         + int2d(meshName)(1*v) + on(1, u = 0); // EOM
16 ffddmsetupOperator( PB , FE , Varf )
17
18 FEVhi ui, bi;
19 ffddmbuildrhs( PB , Varf , bi[] )
20
21 // Step 4: Define the one level DD preconditioner
22 ffddmsetupPrecond( PB , Varf )
23
24 // Build the coarse space from a coarse mesh
25 mesh Thc = square(10,10);
26 ffddmcoarsemeshsetup( PB , Thc , Varf , null )
27
28 // Step 5: Solve the linear system with GMRES
29 FEVhi x0i = 0;
30 ui[] = PBfGMRES(x0i[], bi[], 1.e-6, 200, "right");
```

## Use HPDDM within ffddm

See [documentation](#)

**ffddm** allows you to use **HPDDM** to solve your problem, effectively replacing the **ffddm** implementation of all parallel linear algebra computations

$\Rightarrow$  define your problem with **ffddm**, solve it with **HPDDM**

⇒ **ffddm** acts as a finite element interface for **HPDDM**

- you can use **HPDDM** features unavailable in **ffddm** such as advanced Krylov subspace methods implementing block and recycling techniques
- conversely, some features of **ffddm** such as two level methods built from a coarse mesh are not implemented in **HPDDM**

To switch to **HPDDM**, simply define the macro `pr#withhpddm` before using `ffddmsetupOperator` (*Step 3*). You can then pass **HPDDM** options with command-line arguments or directly to the underlying **HPDDM** operator:

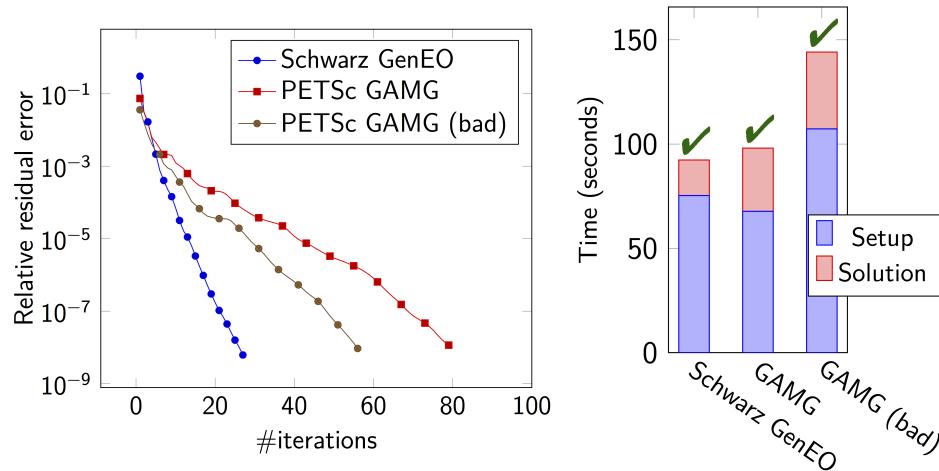
```
1 macro PBwithhpddm() 1 // EOM
2 ffddmsetupOperator( PB , FE , Varf )
3 set (PBhpddmOP, sparams="-hpddm_krylov_method gcrdr");
```

Or, define `pr#withhpdmkrylov` to use **HPDDM** only for the Krylov method

Example [here](#): Helmholtz problem with multiple rhs solved with Block GMRES

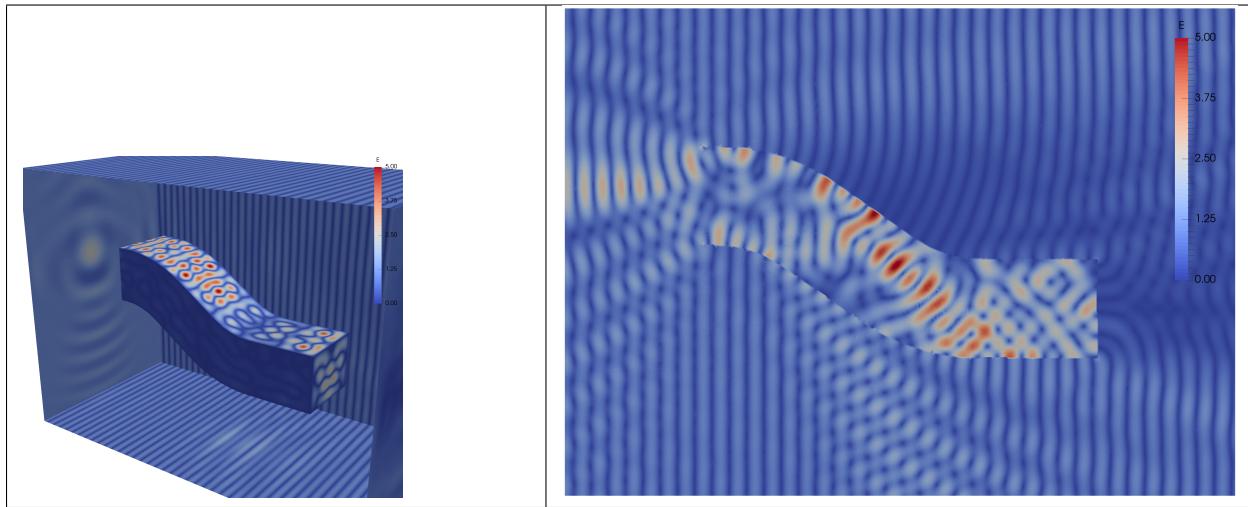
### Some results: Heterogeneous 3D elasticity with GenEO

Heterogeneous 3D linear elasticity equation discretized with P2 FE solved on 4096 MPI processes  $n \approx 262$  million

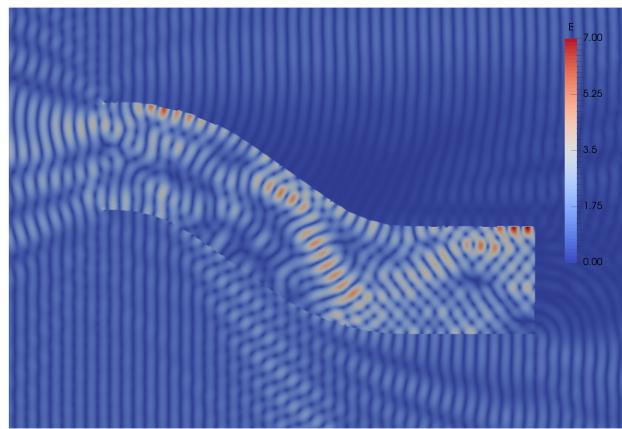


### Some results: 2-level DD for Maxwell equations, scattering from the COBRA cavity

$f = 10$  GHz



$f = 16 \text{ GHz}$



### Some results: 2-level DD for Maxwell equations, scattering from the COBRA cavity

- order 2 Nedelec edge FE
- fine mesh: 10 points per wavelength
- coarse mesh: 3.33 points per wavelength
- two level ORAS preconditioner with added absorption
- $f = 10 \text{ GHz}$ :  $n \approx 107$  million,  $n_c \approx 4$  million

$f = 16 \text{ GHz}$ :  $n \approx 198$  million,  $n_c \approx 7.4$  million

→ coarse problem too large for a direct solver  $\Rightarrow$  inexact coarse solve: GMRES + one level ORAS preconditioner

$f$	$N$	# it	Total # inner it	Total times (seconds)			
				Total	Setup	GMRES	inner
10GHz	1536	32	1527	515.8	383.2	132.6	61.8
10GHz	3072	33	2083	285.0	201.6	83.4	40.6
16GHz	3072	43	3610	549.2	336.8	212.4	118.6
16GHz	6144	46	4744	363.0	210.5	152.5	96.8

speedup of 1.81 from 1536 to 3072 cores at 10GHz

1.51 from 3072 to 6144 cores at 16GHz

You can find the script [here](#)

### 3.9.5 Examples

File name	$M_1^{-1}$	$M_2^{-1}$	inexact CS	comments
diffusion-3d-minimal-direct.edp				direct solver MUMPS
diffusion-3d-minimal-ddm.edp	RAS	GenEO		
diffusion-3d-simple.edp	RAS	GenEO		comparison with direct solver
diffusion-2d-thirddlevelgeneo.edp	RAS	GenEO	RAS + GenEO	
elasticity-3d-simple.edp	RAS	GenEO		
elasticity-3d-thirddlevelgeneo.edp	RAS	GenEO	RAS + GenEO	
Helmholtz-2d-simple.edp	ORAS	Coarse Mesh / DtN		for the DtN coarse space see <a href="#">this paper</a>
Helmholtz-2d-marmousi.edp	ORAS	Coarse Mesh		
Helmholtz-3d-simple.edp	ORAS	Coarse Mesh		
Helmholtz-3d-overthrust.edp	ORAS			
Helmholtz-2d-HPDDM-BGMRES.edp	ORAS			multi-rhs Block GMRES with HPDDM
Navier-2d-marmousi2.edp	ORAS	Coarse Mesh		
Maxwell-3d-simple.edp	ORAS	Coarse Mesh		
Maxwell_Cobracavity.edp	ORAS	Coarse Mesh	ORAS	
natural_convection.edp	ORAS	Coarse Mesh		nonlinear
natural_convection_3D_obstacle.edp	ORAS	Coarse Mesh		nonlinear
Richards-2d.edp	RAS			nonlinear time dependent mesh adaptation
heat-torus-3d-surf.edp	RAS	GenEO		3d surface time dependent



## LANGUAGE REFERENCES

In essence **FreeFEM** is a compiler: its language is typed, polymorphic, with exception and reentrant. Every variable must be declared of a certain type, in a declarative statement; each statement are separated from the next by a semicolon ;.

The language allows the manipulation of basic types integers (`int`), reals (`real`), strings (`string`), arrays (example: `real[int]`), bi-dimensional (2D) finite element meshes (`mesh`), 2D finite element spaces (`fespace`), analytical functions (`func`), arrays of finite element functions (`func[basic_type]`), linear and bilinear operators, sparse matrices, vectors , etc. For example:

```
1 int i, n = 20; //i, n are integer
2 real[int] xx(n), yy(n); //two array of size n
3 for (i = 0; i <= 20; i++){ //which can be used in statements such as
4     xx[i] = cos(i*pi/10);
5     yy[i] = sin(i*pi/10);
6 }
```

The life of a variable is the current block { . . . }, except the `fespace` variable, and the variables local to a block are destroyed at the end of the block as follows.

---

**Tip:** Example

```
1 real r = 0.01;
2 mesh Th = square(10, 10); //unit square mesh
3 fespace Vh(Th, P1); //P1 Lagrange finite element space
4 Vh u = x + exp(y);
5 func f = z*x + r*log(y);
6 plot(u, wait=true);
7 { // new block
8     real r = 2; //not the same r
9     fespace Vh(Th, P1); //error because Vh is a global name
10 } // end of block
11 //here r back to 0.01
```

---

The type declarations are mandatory in **FreeFEM**; in the end this feature is an asset because it is easy to make bugs in a language with many implicit types.

The variable name is just an alphanumeric string, the underscore character \_ is not allowed, because it will be used as an operator in the future.

## 4.1 Types

### 4.1.1 Standard types

#### int

Integer value (equivalent to `long` in C++).

```
1 int i = 0;
```

#### bool

Boolean value.

```
1 bool b = true;
```

---

**Tip:** The result of a comparison is a boolean

```
bool b = (1 < 2);
```

---

#### real

Real value (equivalent to `double` in C++).

```
1 real r = 0.;
```

#### complex

Complex value (equivalent to two `double` or `complex<double>` in C++).

```
1 complex c = 0. + 1i;
```

The imaginary number  $i$  is defined as `1i`

---

**Tip:** Example

```
1 complex a = 1i, b = 2 + 3i;
2 cout << "a + b = " << a + b << endl;
3 cout << "a - b = " << a - b << endl;
4 cout << "a*b = " << a*b << endl;
5 cout << "a/b = " << a/b << endl;
```

The output of this script is:

```
a + b = (2,4)
a - b = (-2,-2)
a*b = (-3,2)
a/b = (0.230769,0.153846)
```

---

**Note:** See [Complex example](#) for a detailed example.

---

## string

String value.

```
1 string s = "this is a string";
```

---

**Note:** string value is enclosed within double quotes.

---

Other types can be concatenate to a string, like:

```
1 int i = 1;
2 real r = 1.;
3 string s = "the int i = " + i + ", the real r = " + r + ", the complex z = " + (1. +
    ↪1i);
```

To append a string in a string at position 4:

```
1 s(4:3) = "+++";
```

To copy a substring in an other string:

```
1 string s2 = s1(5:10);
```

See [String Example](#) for a complete example.

## 4.1.2 Mesh design

### border

Border type.

```
1 border b(t=0., 1.) {x=cos(2.*pi*t); y=sin(2.*pi*t);}
```

Define the 2D geometrical border in parametric coordinates.

---

**Note:** Label

A label can be defined with the border:

```
1 border b(t=0., 1.) {x=cos(2.*pi*t); y=sin(2.*pi*t); label=1;}
```

---



---

**Note:** Inner variable

An inner variable can be defined inside a border:

```
1 border b(t=0., 1.) {real tt=2.*pi*t; x=cos(tt); y=sin(tt);}
```

**Note:** From vector

A border can be defined from two vectors using `P.x` and `P.y`:

```
1 border b(t=0, vectorX.n-1) {P.x=vectorX[t]; P.y=vectorY[t];}
```

## mesh

2D Mesh type (see [Mesh Generation](#)).

```
1 mesh Th;
```

## mesh3

3D mesh type (see [Mesh Generation](#)).

```
1 mesh3 Th;
```

### 4.1.3 Finite element space design

#### fespace

Finite element space type (see [Finite Element](#)).

```
1 fespace Uh(Th, P1);
2 fespace UPh(Th, [P2, P2, P1]);
```

A finite element space is based on a mesh (`Th`) with an element definition, scalar (`P1`) or vector (`[P2, P2, P1]`).

#### Available finite element space:

Generic:

- P0 / P03d
- P0Edge
- P1 / P13d
- P1dc
- P1b / P1b3d
- P1bl / P1bl3d
- P1nc
- P2 / P23d
- P2b
- P2dc

- P2h
- RT0 / RT03d
- RT0Ortho
- Edge03d

Using *Element\_P3*:

- P3

Using *Element\_P3dc*:

- P3dc

Using *Element\_P4*:

- P4

Using *Element\_P4dc*:

- P4dc

Using *Element\_PkEdge*:

- P1Edge
- P2Edge
- P3Edge
- P4Edge
- P5Edge

Using *Morlay*:

- P2Morley

Using *HCT*:

- HCT

Using *BernardiRaugel*:

- P2BR

Using *Element\_Mixte*:

- RT1
- RT1Ortho
- RT2
- RT2Ortho
- BDM1
- BDM1Ortho

Using *Element\_Mixte3d*:

- Edge13d
- Edge23d

Using *Element\_QF*:

- FEQF

A finite element function is defined as follow:

```
1 fespace Uh(Th, P1);
2 Uh u;
3
4 fespace UPh(Th, [P2, P2, P1]);
5 UPh [Ux, Uy, p];
```

#### 4.1.4 Macro design

##### macro

Macro type.

```
1 macro vU() [Ux, Uy] //
2 macro grad(u) [dx(u), dy(u)] //
```

Macro ends with //.

---

**Note:** Macro concatenation

You can use the C concatenation operator ## inside a macro using #.

If  $Ux$  and  $Uy$  are defined as finite element function, you can define:

```
1 macro Grad(U) [grad(U#x), grad(U#y)] // End of macro
```

---

See *Macro example*

##### NewMacro / EndMacro

**Warning:** In developement - Not tested

Set and end a macro

```
1 NewMacro grad(u) [dx(u), dy(u)] EndMacro
```

##### IFMACRO

Check if a macro exists and check its value.

```
1 IFMACRO(AA) //check if macro AA exists
2 ...
3 ENDIFMACRO
4
5 IFMACRO(AA, tt) //check if amcro exists and is equall to tt
6 ...
7 ENDIFMACRO
```

**ENDIFMACRO**

### 4.1.5 Functions design

**func**

Function type.

Function without parameters ( $x$ ,  $y$  and  $z$  are implicitly considered):

```
1 func f = x^2 + y^2;
```

**Note:** Function's type is defined by the expression's type.

Function with parameters:

```
1 func real f (real var) {
2     return x^2 + y^2 + var^2;
3 }
```

**Elementary functions**

Class of basic functions (polynomials, exponential, logarithmic, trigonometric, circular) and the functions obtained from those by the four arithmetic operations

$$f(x) + g(x), f(x) - g(x), f(x)g(x), f(x)/g(x)$$

and by composition  $f(g(x))$ , each applied a finite number of times.

In FreeFEM, all elementary functions can thus be created. The derivative of an elementary function is also an elementary function; however, the indefinite integral of an elementary function cannot always be expressed in terms of elementary functions.

See *Elementary function example* for a complete example.

**Random functions**

FreeFEM includes the Mersenne Twister random number generator. It is a very fast and accurate random number generator of period  $2^{219937} - 1$ .

See `randint32()`, `randint31()`, `randreal1()`, `randreal2()`, `randreal3()`, `randres53()`, `randinit(seed)`.

In addition, the `ffrandom` plugin interface `random`, `srandom` and `srandomdev` functions of the Unix `libc` library. The range is  $0 - 2^{31} - 1$ .

**Note:** If `srandomdev` is not defined, a seed based on the current time is used.

`gsl` plugin equally allows usage of all random functions of the `gsl` external library.

## FE-functions

Finite element functions are also constructed like elementary functions by an arithmetic formula involving elementary functions.

The difference is that they are evaluated at declaration time and **FreeFEM** stores the array of its values at the places associated with the degree of freedom of the finite element type. By opposition, elementary functions are evaluated only when needed. Hence FE-functions are not defined only by their formula but also by the mesh and the finite element which enter in their definitions.

If the value of a FE-function is requested at a point which is not a degree of freedom, an interpolation is used, leading to an interpolation error, while by contrast, an elementary function can be evaluated at any point exactly.

```

1 func f = x2*(1+y)3 + y2;
2 mesh Th = square(20, 20, [-2+4*x, -2+4*y]); // ]-2, 2[^2
3 fespace Vh(Th, P1);
4 Vh fh=f; //fh is the projection of f to Vh (real value)
5 func zf = (x2*(1+y)3 + y2)*exp(x + 1i*y);
6 Vh<complex> zh = zf; //zh is the projection of zf to complex value Vh space

```

The construction of `fh = f` is explained in *Finite Element*.

**Warning:** The `plot` command only works for real or complex FE-functions, not for elementary functions.

## 4.1.6 Problem design

### problem

Problem type.

```

1 problem Laplacian (u, uh) = ...

```

**FreeFEM** needs the variational form in the problem definition.

In order to solve the problem, just call:

```

1 Laplacian;

```

**Note:** Solver

A solver can be specified in the problem definition:

```

1 problem Laplacian(u, uh, solver=CG) = ...

```

The default solver is `sparsesolver` or `LU` if any direct sparse solver is available.

Solvers are listed in the *Global variables* section.

---

**Note:** Stop test

A criterion  $\varepsilon$  can be defined for iterative methods, like CG for example:

```

1 problem Laplacian(u, uh, solver=CG, eps=1.e-6) = ...

```

If  $\varepsilon > 0$ , the stop test is:

$$\|Ax - b\| < \varepsilon$$

Else, the stop test is:

$$\|Ax - b\| < \frac{|\varepsilon|}{\|Ax_0 - b\|}$$


---

#### Note: Reconstruction

The keyword `init` controls the reconstruction of the internal problem matrix.

If `init` is set to `false` or `0`, the matrix is reconstructed at each problem calls (or after a mesh modification), else the previously constructed matrix is used.

```
1 problem Laplacian(u, uh, init=1) = ...
```

---

#### Note: Preconditioning

A preconditioner can be specified in the problem definition:

```
1 problem Laplacian(u, uh, precon=P) = ...
```

---

The preconditioning function must have a prototype like:

```
1 func real[int] P(real[int] &xx);
```

---

#### Note: “Très grande valeur”

The “Très grande valeur” `tgv` (or *Terrible giant value*) used to implement the Dirichlet conditions can be modified in the problem definition:

```
1 problem Laplacian(u, uh, tgv=1e30) = ...
```

---

Refer to [Problem definition](#) for a description of the Dirichlet condition implementation.

---

#### Note: Pivot tolerance

The tolerance of the pivot in UMFPACK, LU, Crout, Cholesky factorization can be modified in the problem definition:

```
1 problem Laplacian(u, uh, solver=LU, tolpivot=1e-20) = ...
```

---

#### Note: UMFPACK

Two specific parameters for the UMFPACK can be modified:

- Tolerance of the pivot sym
- strategy

```
| problem Laplacian(u, uh, solver=LU, tolpivotsym=1e-1, strategy=0) = ...
```

Refer to the [UMFPACK](#) website for more informations.

**Note:** `dimKrylov`

Dimension of the Krylov space

Usage of `problem` is detailed in the [tutorials](#).

### solve

Solve type.

Identical to `problem` but automatically solved.

Usage of `solve` is detailed in the [tutorials](#).

### varf

Variational form type.

```
| varf vLaplacian (u, uh) = ...
```

Directly define a variational form.

This is the other way to define a problem in order to directly manage matrix and right hand side.

Usage of `varf` is detailed in the [tutorial](#).

## 4.1.7 Array

An array stores multiple objects, and there are 2 kinds of arrays:

- the first is similar to vector, i.e. array with integer indices
- the second is array with string indices

In the first case, the size of the array must be known at execution time, and implementation is done with the `KN<>` class and all the vector operator of `KN<>` are implemented.

Arrays can be set like in Matlab or Scilab with the operator `::`, the array generator of `a:c` is equivalent to `a:1:c`, and the array set by `a:b:c` is set to size  $\lfloor (b-a)/c \rfloor + 1$  and the value  $i$  is set by  $a + i(b-a)/c$ .

There are `int`, `real`, `complex` array with, in the third case, two operators (`.im`, `.re`) to generate the real and imaginary real array from the complex array (without copy).

**Note:** Quantiles are points taken at regular intervals from the cumulative distribution function of a random variable. Here the array values are random.

This statistical function `a.quantile(q)` computes  $v$  from an array  $a$  of size  $n$  for a given number  $q \in ]0, 1[$  such that:

$$\#\{i/a[i] < v\} \sim q * n$$

---

it is equivalent to  $v = a[q * n]$  when the array  $a$  is sorted.

For example, to declare, fill and display an array of `real` of size  $n$ :

```

1 int n = 5;
2 real[int] Ai(n);
3 for (int i = 0; i < n; i++)
4     Ai[i] = i;
5 cout << Ai << endl;
```

The output of this script is:

```

5
0   1   2   3   4
```

See the [Array example](#) for a complete example.

## Array index

Array index can be `int` or `string`:

```

1 real[int] Ai = [1, 1, 0, 0];
2 real[string] As = [1, 1, 0, 0];
```

## Array size

The size of an array is obtained using the keyword `n`:

```

1 int ArraySize = Ai.n;
```

## Array sort

To sort an array:

```

1 Ai.sort;
```

## Double array

A double array (matrix) can be defined using two indexes:

```

1 real[int, int] Aii = [[1, 1], [0, 0]];
```

The two sizes are obtained using the keywords `n` and `m`:

```

1 int ArraySize1 = Aii.n;
2 int ArraySize2 = Aii.m;
```

The minimum and maximum values of an array (simple or double) can be obtained using:

```

1 real ArrayMin = Aii.min;
2 real ArrayMax = Aii.max;
```

The minimum and maximum position of an array can be obtained using:

```

1 int mini = Aii.imin;
2 int minj = Aii.jmin;
3
4 int maxi = Aii.imax;
5 int maxj = Aii.jmax;
```

---

**Tip:** An array can be obtained from a finite element function using:

```

1 real[int] aU = U[];
```

where  $U$  is a finite element function.

---

## Array of FE functions

It is also possible to make an array of FE functions, with the same syntax, and we can treat them as vector valued function if we need them.

The syntax for space or vector finite function is

```

1 int n = 100; //size of the array.
2 Vh[int] wh(n); //real scalar case
3 Wh[int] [uh,vh](n); //real vectorial case
4 Vh<complex>[int] cwh(n); //complex scalar case
5 Wh<complex>[int] [cuh, cvh](n); //complex vectorial case
6 [cuh[2], cvh[2]] = [x, y]; //set interpolation of index 2
7
8 // Array of Array
9 real [int][int] V(10);
10 matrix[int] B(10);
11 real [int, int][int] A(10);
```

---

**Tip:** Example

In the following example, Poisson's equation is solved for 3 different given functions  $f = 1, \sin(\pi x) \cos(\pi y), |x - 1||y - 1|$ , whose solutions are stored in an array of FE function.

```

// Mesh
mesh Th = square(20, 20, [2*x, 2*y]);
3
// Fespace
fespace Vh(Th, P1);
Vh u, v, f;
7
// Problem
problem Poisson (u, v)
= int2d(Th) (
    dx(u)*dx(v)
    + dy(u)*dy(v)
)
+ int2d(Th) (
    - f*v
)
```

(continues on next page)

(continued from previous page)

```

17 + on(1, 2, 3, 4, u=0)
18 ;
19
20 Vh[int] uu(3); //an array of FE function
21 // Solve problem 1
22 f = 1;
23 Poisson;
24 uu[0] = u;
25 // Solve problem 2
26 f = sin(pi*x)*cos(pi*y);
27 Poisson;
28 uu[1] = u;
29 // Solve problem 3
30 f = abs(x-1)*abs(y-1);
31 Poisson;
32 uu[2] = u;
33
34 // Plot
35 for (int i = 0; i < 3; i++)
36   plot(uu[i], wait=true);

```

See [FE array example](#).

## Map arrays

```

1 real[string] map; //a dynamic array
2
3 map["1"] = 2.0;
4 map[2] = 3.0; //2 is automatically cast to the string "2"
5
6 cout << "map[\\"1\\"] = " << map["1"] << endl;
7 cout << "map[2] = " << map[2] << endl;

```

It is just a map of the standard template library so no operations on vector are allowed, except the selection of an item.

### 4.1.8 matrix

Defines a sparse matrix.

Matrices can be defined like vectors:

```

1 matrix A = [[1, 2, 3],
2   [4, 5, 6],
3   [7, 8, 9]];

```

or using a variational form type (see [Finite Element](#)):

```

1 matrix Laplacian = vLaplacian(Uh, Uh);

```

or from block of matrices:

```

1 matrix A1, ..., An;
2 matrix A = [[A1, ...], ..., [..., An]];

```

or using sparse matrix set:

```
1 A = [ I, J, C ] ;
```

---

**Note:**  $\mathbf{I}$  and  $\mathbf{J}$  are `int[int]` and  $\mathbf{C}$  is `real[int]`. The matrix is defined as:

$$A = \sum_k C[k] M_{I[k], J[k]}$$

where  $M_{a,b} = (\delta_{ia} \delta_{jb})_{ij}$

$\mathbf{I}, \mathbf{J}$  and  $\mathbf{C}$  can be retrieved using  $[I, J, C] = A$  (arrays are automatically resized).

The size of the matrix is  $n = \mathbf{I}.\max; , m = \mathbf{J}.\max; .$

---

Matrices are designed using templates, so they can be real or complex:

```
1 matrix<real> A = ...
2 matrix<complex> Ai = ...
```

---

**Note:** Solver

See [problem](#).

The default solver is [GMRES](#).

```
1 matrix A = vLaplacian(Uh, Uh, solver=sparsesolver) ;
```

or

```
1 set (A , solver=sparsesolver) ;
```

---

**Note:** Factorize

If true, the factorization is done for LU, Cholesky or Crout.

```
1 matrix A = vLaplacian(Uh, Uh, solver=LU, factorize=1) ;
```

or

```
1 set (A , solver=LU, factorize=1) ;
```

---

**Note:** Stop test

See [problem](#).

---

**Note:** Très grande valeur

See [problem](#).

---

**Note:** Preconditioning

---

See *problem*.

---



---

**Note:** Pivot tolerance

---

See *problem*.

---



---

**Note:** UMFPACK

---

See *problem*.

---



---

**Note:** dimKrylov

---

See *problem*.

---



---

**Note:** datafilename

---

Name of the file containing solver parameters, see *Parallel sparse solvers*

---



---

**Note:** lparams

---

Vector of integer parameters for the solver, see *Parallel sparse solvers*

---



---

**Note:** dparams

---

Vector of real parameters for the solver, see *Parallel sparse solvers*

---



---

**Note:** sparams

---

String parameters for the solver, see *Parallel sparse solvers*

---



---

**Tip:** To modify the `solver`, the stop test,... after the matrix construction, use the `set keyword`.

---

## Matrix size

The size of a matrix is obtain using:

```
1 int NRows = A.n;
2 int NColumns = A.m;
```

## Matrix resize

To resize a matrix, use:

```
1 A.resize(n, m);
```

**Warning:** When resizing, all new terms are set to zero.

## Matrix diagonal

The diagonal of the matrix is obtained using:

```
1 real[int] Aii = A.diag;
```

## Matrix renumbering

```
1 int[int] I(15), J(15);
2 matrix B = A;
3 B = A(I, J);
4 B = A(I^-1, J^-1);
```

## Complex matrix

Use `.im` and `.re` to get the imaginary and real part of a complex matrix, respectively:

```
1 matrix<complex> C = ...
2 matrix R = C.re;
3 matrix I = C.im;
```

## Dot product / Outer product

The dot product of two matrices is realized using:

```
1 real d = A' * B;
```

The outer product of two matrices is realized using:

```
1 matrix C = A * B'
```

See [Matrix operations example](#) for a complete example.

## Matrix inversion

See [Matrix inversion example](#).

## 4.2 Global variables

### 4.2.1 area

Area of the current triangle.

```

1 fespace Vh0(Th, P0);
2 Vh0 A = area;

```

## 4.2.2 ARGV

Array that contains all the command line arguments.

```

1 for (int i = 0; i < ARGV.n; i++)
2   cout << ARGV[i] << endl;

```

See *Command line arguments example* for a complete example.

## 4.2.3 BoundaryEdge

Return 1 if the current edge is on a boundary, 0 otherwise.

```

1 real B = int2d(Th) (BoundaryEdge);

```

## 4.2.4 CG

Conjugate gradient solver.

Usable in *problem* and *solve* definition

```

1 problem Laplacian (U, V, solver=CG) = ...

```

Or in *matrix* construction

```

1 matrix A = vLaplacian(Uh, Uh, solver=CG);

```

Or in *set function*

```

1 set (A, solver=CG);

```

## 4.2.5 Cholesky

Cholesky solver.

## 4.2.6 Crout

Crout solver.

## 4.2.7 edgeOrientation

Sign of  $i - j$  if the current edge is  $[q_i, q_j]$ .

```

1 real S = int1d(Th, 1) (edgeOrientation);

```

## 4.2.8 false

False boolean value.

```
1 bool b = false;
```

## 4.2.9 GMRES

GMRES solver (Generalized minimal residual method).

## 4.2.10 hTriangle

Size of the current triangle.

```
1 fespace Vh(Th, P0);  
2 Vh h = hTriangle;
```

## 4.2.11 include

Include an *external library*.

```
1 include "iovtk"
```

## 4.2.12 InternalEdge

Return 0 if the current edge is on a boundary, 1 otherwise.

```
1 real I = int2d(Th)(InternalEdge);
```

## 4.2.13 label

Label number of a boundary if the current point is on a boundary, 0 otherwise.

```
1 int L = Th(xB, yB).label;
```

## 4.2.14 lenEdge

Length of the current edge.

For an edge  $[q_i, g_j]$ , return  $|q_i - q_j|$ .

```
1 real L = int1d(Th, 1)(lenEdge);
```

## 4.2.15 load

Load a script.

```
1 load "Element_P3"
```

## 4.2.16 LU

LU solver.

## 4.2.17 N

Outward unit normal at the current point if it is on a curve defined by a border. `N.x`, `N.y`, `N.z` are respectively the  $x$ ,  $y$  and  $z$  components of the normal.

```
1 func Nx = N.x;
2 func Ny = N.y;
3 func Nz = N.z;
```

## 4.2.18 nTonEdge

Number of adjacent triangles of the current edge.

```
1 real nTE = int2d(Th) (nTonEdge);
```

## 4.2.19 nuEdge

Index of the current edge in the triangle.

```
1 real nE = int2d(Th) (nuEdge);
```

## 4.2.20 nuTriangle

Index of the current triangle.

```
1 fespace Vh(Th, P0);
2 Vh n = nuTriangle;
```

## 4.2.21 P

Current point.

```
1 real cx = P.x;
2 real cy = P.y;
3 real cz = P.z;
```

## 4.2.22 pi

Pi = 3.14159.

```
1 real Pi = pi;
```

This is a real value.

## 4.2.23 region

Region number of the current point. If the point is outside, then `region == notaregion` where `notaregion` is a **FreeFEM** integer constant.

```
1 int R = Th(xR, yR).region;
```

## 4.2.24 sparsesolver

Sparse matrix solver.

## 4.2.25 true

True boolean value.

```
1 bool b = true;
```

## 4.2.26 verbosity

Verbosity level.

```
1 int Verbosity = verbosity;
2 verbosity = 0;
```

0 = nothing, 1 = little information, 10 = a lot of information, ...

This is an integer value.

## 4.2.27 version

FreeFEM version.

```
1 cout << version << endl;
```

## 4.2.28 volume

Volume of the current tetrahedra.

```
1 fespace Vh0(Th, P0);
2 Vh0 V = volume;
```

## 4.2.29 x

The  $x$  coordinate at the current point.

```
1 real CurrentX = x;
```

This is a real value.

## 4.2.30 y

The  $y$  coordinate at the current point.

```
1 real CurrentY = y;
```

This is a real value.

## 4.2.31 z

The  $z$  coordinate at the current point.

```
1 real CurrentZ = z;
```

This is a real value.

## 4.3 Quadrature formulae

The quadrature formula is like the following:

$$\int_D f(\mathbf{x}) \approx \sum_{\ell=1}^L \omega_\ell f(\boldsymbol{\xi}_\ell)$$

### 4.3.1 int1d

Quadrature formula on an edge.

#### Notations

$|D|$  is the measure of the edge  $D$ .

For a shake of simplicity, we denote:

$$f(\mathbf{x}) = g(t)$$

with  $0 \leq t \leq 1$ ;  $\mathbf{x} = (1 - t)\mathbf{x}_0 + t\mathbf{x}_1$ .

**qf1pE**

```
| int1d(Th, qfe=qf1pE) ( ... )
```

or

```
| int1d(Th, qforder=2) ( ... )
```

This quadrature formula is exact on  $\mathbb{P}_1$ .

$$\int_D f(\mathbf{x}) \approx |D|g\left(\frac{1}{2}\right)$$

**qf2pE**

```
| int1d(Th, qfe=qf2pE) ( ... )
```

or

```
| int1d(Th, qforder=3) ( ... )
```

This quadrature formula is exact on  $\mathbb{P}_3$ .

$$\int_D f(\mathbf{x}) \approx \frac{|D|}{2} \left( g\left(\frac{1 + \sqrt{1/3}}{2}\right) + g\left(\frac{1 - \sqrt{1/3}}{2}\right) \right)$$

**qf3pE**

```
| int1d(Th, qfe=qf3pE) ( ... )
```

or

```
| int1d(Th, qforder=6) ( ... )
```

This quadrature formula is exact on  $\mathbb{P}_5$ .

$$\int_D f(\mathbf{x}) \approx \frac{|D|}{18} \left( 5g\left(\frac{1 + \sqrt{3/5}}{2}\right) + 8g\left(\frac{1}{2}\right) + 5g\left(\frac{1 - \sqrt{3/5}}{2}\right) \right)$$

**qf4pE**

```
| int1d(Th, qfe=qf4pE) ( ... )
```

or

```
| int1d(Th, qforder=8) ( ... )
```

This quadrature formula is exact on  $\mathbb{P}_7$ .

$$\int_D f(\mathbf{x}) \approx \frac{|D|}{72} \left( (18 - \sqrt{30})g\left(\frac{1 - \frac{\sqrt{525+70\sqrt{30}}}{35}}{2}\right) + (18 - \sqrt{30})g\left(\frac{1 + \frac{\sqrt{525+70\sqrt{30}}}{35}}{2}\right) + (18 + \sqrt{30})g\left(\frac{1 - \frac{\sqrt{525-70\sqrt{30}}}{35}}{2}\right) + (18 + \sqrt{30})g\left(\frac{1 + \frac{\sqrt{525-70\sqrt{30}}}{35}}{2}\right) \right)$$

**qf5pE**

```
1 int1d(Th, qfe=qf5pE) ( ... )
```

or

```
1 int1d(Th, qforder=10) ( ... )
```

This quadrature formula is exact on  $\mathbb{P}_9$ .

$$\int_D f(\mathbf{x}) \approx |D| \left( \frac{(332 - 13\sqrt{70})}{1800} g\left(\frac{1 - \frac{\sqrt{245+14\sqrt{70}}}{21}}{2}\right) + \frac{(332 - 13\sqrt{70})}{1800} g\left(\frac{1 + \frac{\sqrt{245+14\sqrt{70}}}{21}}{2}\right) + \frac{64}{225} g\left(\frac{1}{2}\right) + \frac{(332 + 13\sqrt{70})}{1800} \right)$$

**qf1pElump**

```
1 int1d(Th, qfe=qf1pElump) ( ... )
```

This quadrature formula is exact on  $\mathbb{P}_2$ .

$$\int_D f(\mathbf{x}) \approx \frac{|D|}{2} (g(0) + g(1))$$

**4.3.2 int2d**

**Note:** Complete formulas are no longer detailed

**qf1pT**

```
1 int2d(Th, qfe=qf1pT) ( ... )
```

or

```
1 int2d(Th, qforder=2) ( ... )
```

This quadrature formula is exact on  $\mathbb{P}_1$ .

**qf2pT**

```
1 int2d(Th, qfe=qf2pT) ( ... )
```

or

```
1 int2d(Th, qforder=3) ( ... )
```

This quadrature formula is exact on  $\mathbb{P}_2$ .

### qf5pT

```
1 int2d(Th, qfe=qf5pT) ( ... )
```

or

```
1 int2d(Th, qforder=6) ( ... )
```

This quadrature formula is exact on  $\mathbb{P}_5$ .

### qf1pTlump

```
1 int2d(Th, qfe=qf1pTlump) ( ... )
```

This quadrature formula is exact on  $\mathbb{P}_1$ .

### qf2pT4P1

```
1 int2d(Th, qfe=qf2pT4P1) ( ... )
```

This quadrature formula is exact on  $\mathbb{P}_1$ .

### qf7pT

```
1 int2d(Th, qfe=qf7pT) ( ... )
```

or

```
1 int2d(Th, qforder=8) ( ... )
```

This quadrature formula is exact on  $\mathbb{P}_7$ .

### qf9pT

```
1 int2d(Th, qfe=qf9pT) ( ... )
```

or

```
1 int2d(Th, qforder=10) ( ... )
```

This quadrature formula is exact on  $\mathbb{P}_9$ .

## 4.3.3 int3d

### qfV1

```
1 int3d(Th, qfe=qfV1) ( ... )
```

or

```
1 int3d(Th, qforder=2) ( ... )
```

This quadrature formula is exact on  $\mathbb{P}_1$ .

## qfV2

```
1 int3d(Th, qfe=qfV2) ( ... )
```

or

```
1 int3d(Th, qforder=3) ( ... )
```

This quadrature formula is exact on  $\mathbb{P}_2$ .

## qfV5

```
1 int3d(Th, qfe=qfV5) ( ... )
```

or

```
1 int3d(Th, qforder=6) ( ... )
```

This quadrature formula is exact on  $\mathbb{P}_5$ .

## qfV1lump

```
1 int3d(Th, qfe=qfV1lump) ( ... )
```

This quadrature formula is exact on  $\mathbb{P}_1$ .

## 4.4 Operators

### 4.4.1 Addition operator +

```
1 real a = 1. + 2.;
```

Works for int, real, complex, string, mesh, mesh3, array.

### 4.4.2 Increment operator ++

Pre-increment:

```
1 int i = 0;
2 ++i;
```

Post-increment:

```
1 int i = 0;
2 i++;
```

#### 4.4.3 Substraction operator -

```
1 real a = 1. - 2.;
```

Works for int, real, complex, array.

#### 4.4.4 Decrement operator –

Pre-decrement:

```
1 int i = 0;
2 --i;
```

Post-decrement:

```
1 int i = 0;
2 i--;
```

#### 4.4.5 Multiplication operator \*

```
1 real[int] b;
2 matrix A
3 real[int] x = A^-1*b;
```

Works for int, real, complex, array, matrix.

#### 4.4.6 Equal operator =

```
1 real a = 1.;
```

#### 4.4.7 Comparison operator ==

```
1 real a = 1.;
2 real b = 1.;
3
4 cout << (a == b) << endl;
```

#### 4.4.8 Comparison operator !=

```
1 real a = 1.;
2 real b = 2.;
3
4 cout << (a != b) << endl;
```

#### 4.4.9 Comparison operator <, <=

```

1 real a = 1.;
2 real b = 2.;

3
4 cout << (a < b) << endl;
5 cout << (a <= b) << endl;

```

#### 4.4.10 Comparison operator >, >=

```

1 real a = 3.;
2 real b = 2.;

3
4 cout << (a > b) << endl;
5 cout << (a >= b) << endl;

```

#### 4.4.11 Compound operator +=, -=, \*=, /=

```

1 real a = 1;
2 a += 2.;
3 a -= 1.;
4 a *= 3.;
5 a /= 2.;

```

#### 4.4.12 Term by term multiplication .\*

```

1 matrix A = B .* C;

```

#### 4.4.13 Division operator /

```

1 real a = 1. / 2.;

```

Works for int, real, complex.

#### 4.4.14 Term by term division ./

```

1 matrix A = B ./ C;

```

#### 4.4.15 Remainder from the division %

```

1 int a = 1 % 2;

```

Works for int, real.

#### 4.4.16 Power operator ^

```
1 real a = 2.^2;
```

Works for int, real, complex, matrix.

#### 4.4.17 Inverse of a matrix ^-1

```
1 real[int] Res = A^-1 * b;
```

**Warning:** This operator can not be used to directly create a matrix, see [Matrix inversion](#).

#### 4.4.18 Transpose operator '

```
1 real[int] a = b';
```

Works for array and matrix.

**Note:** For matrix<complex>, the ::freefem““ operator return the Hermitian transpose.

#### 4.4.19 Tensor scalar product :

$$A : B = \sum_{i,j} A_{ij} B_{ij}$$

#### 4.4.20 C++ arithmetical if expression ? :

a ? b : c is equal to b if the a is true, c otherwise.

**Tip:** Example with int

```
1 int a = 12; int b = 5;
2
3 cout << a << " + " << b << " = " << a + b << endl;
4 cout << a << " - " << b << " = " << a - b << endl;
5 cout << a << " * " << b << " = " << a * b << endl;
6 cout << a << " / " << b << " = " << a / b << endl;
7 cout << a << " % " << b << " = " << a % b << endl;
8 cout << a << " ^ " << b << " = " << a ^ b << endl;
9 cout << "(" << a << " < " << b << " ? " << a << " : " << b << ")" = " << (a < b ? a : b) << endl;
```

The output of this script is:

```

12 + 5 = 17
12 - 5 = 7
12 * 5 = 60
12 / 5 = 2
12 % 5 = 2
12 ^ 5 = 248832
( 12 < 5 ? 12 : 5) = 5

```

**Tip:** Example with `real`

```

1 real a = qsrt(2.); real b = pi;
2
3 cout << a << " + " << b << " = " << a + b << endl;
4 cout << a << " - " << b << " = " << a - b << endl;
5 cout << a << " * " << b << " = " << a * b << endl;
6 cout << a << " / " << b << " = " << a / b << endl;
7 cout << a << " % " << b << " = " << a % b << endl;
8 cout << a << " ^ " << b << " = " << a ^ b << endl;
9 cout << "( " << a << " < " << b << " ? " << a << " : " << b << ")" = " << (a < b ? a : b) << endl;

```

The output of this script is:

```

1.41421 + 3.14159 = 4.55581
1.41421 - 3.14159 = -1.72738
1.41421 * 3.14159 = 4.44288
1.41421 / 3.14159 = 0.450158
1.41421 % 3.14159 = 1
1.41421 ^ 3.14159 = 2.97069

```

## 4.5 Loops

See *Loop example*.

### 4.5.1 for

For loop.

```

1 for (int i = 0; i < N; ++i){
2     ...
3 }

```

### 4.5.2 if

If condition.

```

1 if (condition){
2     ...

```

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```

3 }
4 else{
5     ...
6 }
```

### 4.5.3 else

See [if](#).

### 4.5.4 while

While loop.

```

1 while (condition){
2     ...
3 }
```

### 4.5.5 continue

Continue a loop.

```

1 for (int i = 0; i < N; ++i) {
2     ...
3     if (condition) continue;
4     ...
5 }
```

### 4.5.6 break

Break a loop.

```

1 while (condition1){
2     ...
3     if (condition) break;
4     ...
5 }
```

### 4.5.7 try

Try a part of code.

```

1 try{
2     ...
3 }
4 catch(...){
5     ...
6 }
```

See [Basic error handling example](#) and [Error handling example](#).

## 4.5.8 catch

Catch an error, see [try](#)

## 4.5.9 Implicit loop

Array with one index:

```
1 for [i, ai : a]
```

If `real[int] a(10)`, then `i=0:9` and `ai` is a reference to `a[i]`.

Array with two indices or matrix:

```
1 for [i, j, aij : a]
```

If `real[int] a(10, 11)`, then `i=0:9, j=1:10` and `aij` is a reference to `a(i, j)`.

See [Implicit loop example](#).

## 4.6 I/O

See [I/O example](#)

See [File stream example](#).

### 4.6.1 cout

Standard C++ output device (default: console).

```
1 cout << "Some text" << endl;
```

### 4.6.2 cin

Standard C++ input device (default: keyboard).

```
1 cin >> var;
```

### 4.6.3 endl

End of line.

```
1 cout << "Some text" << endl;
```

### 4.6.4 ifstream

Open a file in read mode.

```
| ifstream file("file.txt");
```

---

**Note:** A file is closed at the end of a block.

---

#### 4.6.5 ofstream

Open a file in write mode.

```
| ofstream file("file.txt");
```

---

**Note:** A file is closed at the end of a block.

---

#### 4.6.6 append

Append data to an existing file.

```
| ofstream file("file.txt", append);
```

#### 4.6.7 binary

Write a file in binary.

```
| ofstream file("file.btxt", binary);
```

#### 4.6.8 seekg

Set the file position.

```
| file.seekg(Pos);
```

#### 4.6.9 tellg

Get the file position.

```
| int Pos = file.tellg();
```

#### 4.6.10 flush

Flush the buffer of the file.

```
| file.flush
```

### 4.6.11 getline

Get the current line.

```
1 string s;
2 getline(file, s);
```

### 4.6.12 Output format

In the descriptions below, `f` is an output stream, for example `cout` or a `ofstream`.

All this methods, excepted the first, return a stream, so they can be chained:

```
1 cout.scientific.showpos << 3 << endl;
```

#### precision

Set the number of digits printed to the right of the decimal point. This applies to all subsequent floating point numbers written to that output stream. However, this won't make floating-point "integers" print with a decimal point. It's necessary to use `fixed` for that effect.

```
1 int np = f.precision(n)
```

#### scientific

Formats floating-point numbers in scientific notation

```
1 f.scientific
```

#### fixed

Used fixed point notation for floating-point numbers. Opposite of scientific.

```
1 f.fixed
```

#### showbase

Converts insertions to an external form that can be read according to the C++ lexical conventions for integral constants. By default, `showbase` is not set.

```
1 f.showbase
```

#### noshowbase

Unset `showbase` flags.

```
1 f.noshowbase
```

## showpos

Inserts a plus sign (+) into a decimal conversion of a positive integral value.

```
f.showpos
```

## noshowpos

Unset showpos flags.

```
f.noshowpos
```

## default

Reset all the previous flags to the default expect precision.

```
f.default
```

## setw

Behaves as if member width were called with n as argument on the stream on which it is inserted as a manipulator (it can be inserted on output streams).

```
f.setw(n)
```

## 4.7 Functions

### 4.7.1 abs

Return the absolute value.

```
real a = abs(b);
```

Parameters:

- b (int, real, complex, fespace function, real[int] or real[int, int])

Output:

- a (int, real, real[int] or real[int, int])

### 4.7.2 acos

arccos function.

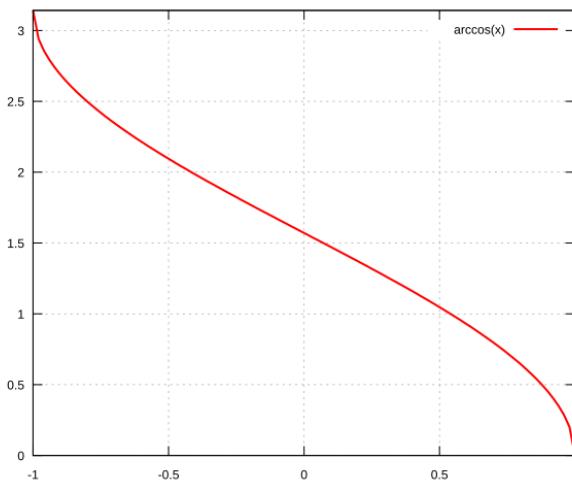
```
real theta = acos(x);
```

Parameter:

- x (real, real[int] or real[int, int])

Output:

- `theta(real, real[int] or real[int, int])`



**Fig. 4.1:** arccos function

### 4.7.3 acosh

```
1 real theta = acosh(x);
```

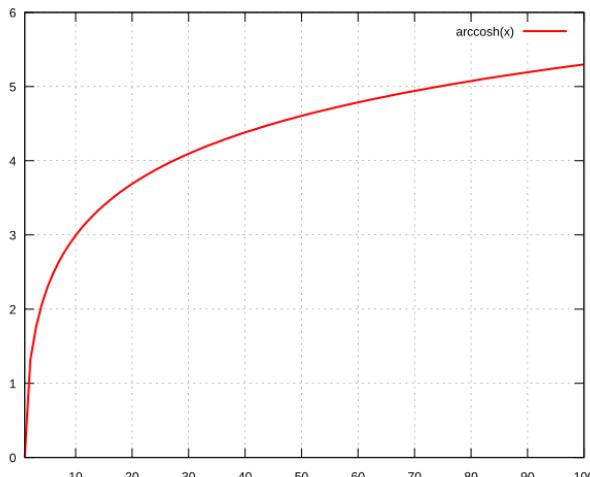
$$\operatorname{arccosh}(x) = \ln \left( x + \sqrt{x^2 - 1} \right)$$

Parameter:

- $x$  (real)

Output:

- `theta(real)`



**Fig. 4.2:** arccosh function

## 4.7.4 adaptmesh

Mesh adaptation function.

```
mesh Thnew = adaptmesh(Th, [fx, fy], hmin=HMin, hmax=HMax, err=Err, errg=ErrG,
    ↵ nbvx=NbVx, nbsmooth=NbSmooth, nbjacoby=NbJacoby, ratio=Ratio, omega=Omega, iso=Iso,
    ↵ abserror=AbsError, cutoff=CutOff, verbosity=Verbosity, inquire=Inquire,
    ↵ splitpbedge=SplitPbEdge, maxsubdiv=MaxSubdiv, rescaling=Rescaling,
    ↵ keepbackvertices=KeepBackVertices, IsMetric=isMetric, power=Power,
    ↵ thetamax=ThetaMax, splitin2=SplitIn2, metric=Metric,
    ↵ nomeshgeneration=NoMeshGeneration, periodic=Periodic);
```

Parameters:

- Th (mesh) Mesh to refine
- [fx, fy] (func or fespace function), scalar or vectorial Function to follow for the mesh adaptation
- hmin= (real) Minimum edge size
- hmax= (real) Maximum edge size
- err= (real) Error level (P1 interpolation)
- errg= (real) Relative geometrical error
- nbvx= (int) Maximum number of vertices
- nbsmooth= (int) Number of smoothing iterations
- nbjacoby= (int) Number of iterations for the smoothing procedure
- ratio= (real) Ratio of the triangles
- omega= (real) Relaxation parameter for the smoothing procedure
- iso= (bool) Isotropic adaptation (if true)
- abserror= (bool) Error (if true) - Relative error (if false)
- cutoff= (real) Lower limit of the relative error evaluation
- verbosity= (real) Verbosity level
- inquire= (bool) If true, inquire graphically
- splitpbedge= (bool) If true, split all internal edges in half
- maxsubdiv= (int) Bound the maximum subdivisions
- rescaling= (bool) Rescale the function in [0, 1]
- keepbackvertices= (bool) If true, try to keep vertices of the original mesh
- IsMetric= (bool) If true, the metric is defined explicitly
- power= (int) Exponent of the Hessian
- thetamax= (int) Minimum corner angle (in degree)
- splitin2= (bool) Split all triangles into 4 sub-triangles if true
- metric= ([real[int], real[int], real[int]]) Array of 3 real arrays defining the metric
- nomeshgeneration= (bool) If true, the mesh is not generated
- periodic= (real[int, int]) Build an adapted periodic mesh

Output:

- Thnew (mesh or mesh3)

### 4.7.5 adj

Adjacent triangle of the triangle  $k$  by the edge  $e$

```
1 int T = Th[k].adj(e);
```

Parameter:

- e (int) Edge number

Output:

- T (int) Triangle number

### 4.7.6 AffineCG

Affine conjugate gradient solver

Used to solve a problem like  $Ax = b$

```
1 int Conv = AffineCG(A, x, precon=Precon, nbiter=NbIter, eps=Eps, veps=VEps,
  ↵stop=Stop);
```

Parameters:

- A (matrix) Matrix of the problem  $Ax = b$
- x (real[int]) Solution vector
- precon= (real[int]) Preconditionning function
- nbiter= (int) Maximum number of iterations
- eps= (real)

Convergence criterion

If  $\varepsilon > 0$ : test  $\|A(x)\|_p \leq \varepsilon \|A(x_0)\|_p$

If  $\varepsilon < 0$ : test  $\|A(x)\|_p^2 \leq |\varepsilon|$

- veps= (real) Same as eps, but return -eps
- stop= (func) Convergence criterion as a function

Prototype is func bool StopFunc (int Iter, real[int] U, real[int] g)

u: current solution, g: current gradient (not preconditionned)

Output:

- Conv (int) 0: converged - !0: not converged

### 4.7.7 AffineGMRES

Affine GMRES solver

Parameters and output are the same as [AffineCG](#)

## 4.7.8 arg

Return the argument of a complex number.

```
1 real a = arg(c);
```

Parameters:

- c (complex)

Output:

- r (real)

## 4.7.9 asin

arcsin function.

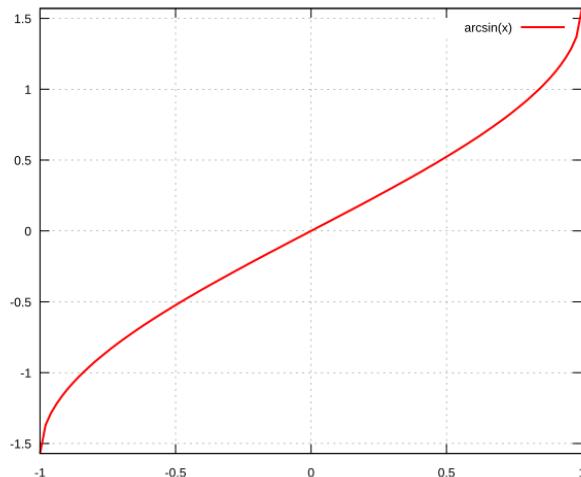
```
1 real theta = asin(x);
```

Parameter:

- x (real, real[int] or real[int, int])

Output:

- theta (real, real[int] or real[int, int])



**Fig. 4.3:** arcsin function

## 4.7.10 asinh

```
1 real theta = asinh(x);
```

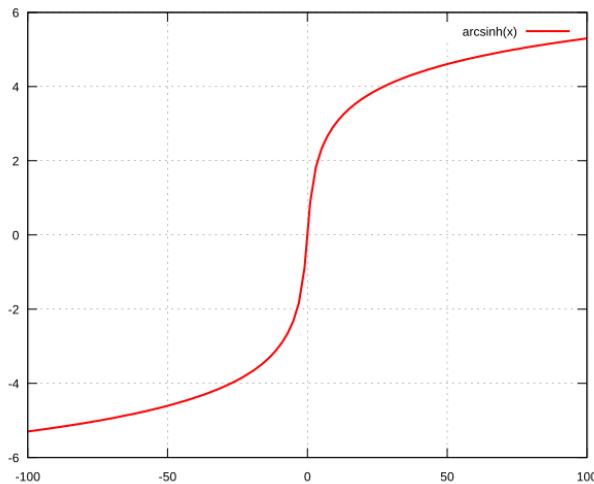
$$\operatorname{arcsinh}(x) = \ln \left( x + \sqrt{x^2 + 1} \right)$$

Parameter:

- x (real)

Output:

- `theta (real)`



**Fig. 4.4:** arcsinh function

### 4.7.11 assert

Verify if a condition is true (same as C), if not the program stops.

```
1 assert (x==0)
```

Parameter:

- Boolean condition

Output:

- None

### 4.7.12 atan

arctan function.

```
1 real theta = atan(x);
```

Parameter:

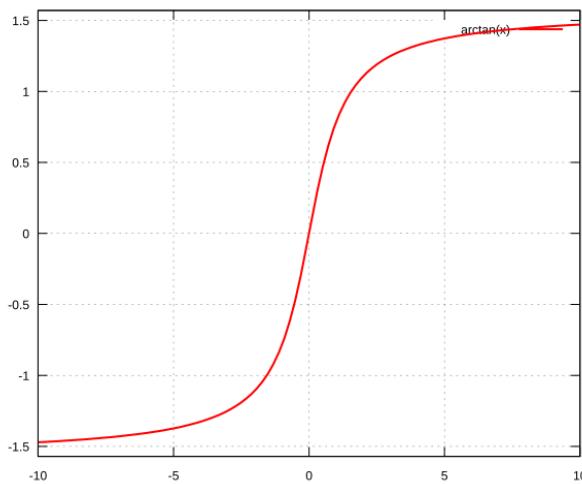
- `x (real)`

Output:

- `theta (real)`

### 4.7.13 atan2

$\arctan\left(\frac{y}{x}\right)$  function, returning the correct sign for  $\theta$ .

**Fig. 4.5:** arctan function

```
1 real theta = atan2(y, x)
```

Parameter:

- **x** (real)

Output:

- theta (real)

#### 4.7.14 atanh

```
1 real theta = atanh(x);
```

Parameter:

- **x** (real)

Output:

- theta (real)

#### 4.7.15 atoi

Convert a string to an interger.

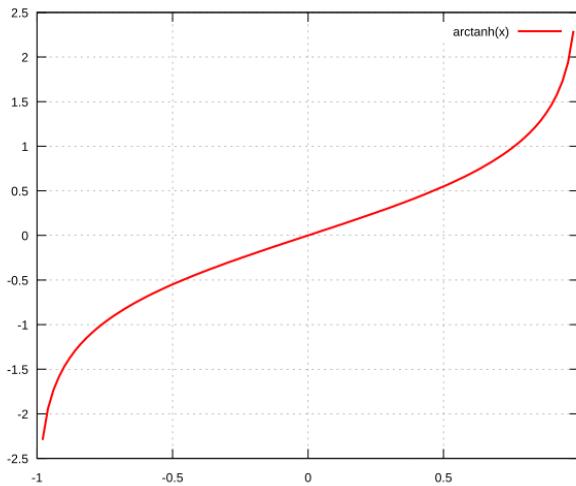
```
1 int a = atoi(s);
```

Parameter:

- **s** (string)

Output:

- **a** (int)

**Fig. 4.6:** arctanh function

### 4.7.16 atof

Convert a string to a real.

```
1 real a = atof(s);
```

Parameter:

- s (string)

Output:

- a (real)

### 4.7.17 BFGS

---

**Todo:** todo

---

### 4.7.18 buildmesh

Build a 2D mesh using border elements.

```
1 mesh Th = buildmesh(b1(nn) + b2(nn) + b3(nn) + b4(nn), [points=Points], [nbvx=Nbvx],  
↪ [fixedborder=FixedBorder]);
```

Parameters:

- b1, b2, b3, b4 (border)

Geometry border, b1(nn) means b1 border discretized by nn vertices

- points (real[int, int]) [Optional]

Specify a set of points

The size of Points array is (nbp, 2), containing a set of nbp points with x and y coordinates

- nbvx= (int) *[Optional]*  
Maximum number of vertices Default: 9000
- fixedborder= (bool) *[Optional]*  
If true, mesh generator cannot change the boundary mesh  
Default: false

Output:

- Th (mesh) Resulting mesh

### 4.7.19 ceil

Round fractions up of  $x$ .

```
1 int c = ceil(x);
```

Parameter:

- x (real)

Output:

- c (int)

### 4.7.20 change

Change a property of a mesh.

```
1 int[int] L = [0, 1];
2 Thnew = change(Th, label=L);
```

Parameters:

- Th (mesh) Original mesh
- label=L (int[int]) Pair of old and new label
- region=R (int[int]) Pair of old and new region
- flabel=l(func int) Function of int given the new label
- fregion=r(func int) Function of int given the new region

Output:

- Thnew (mesh) Mesh with changed parameters

### 4.7.21 checkmovemesh

Check a *movemesh* without mesh generation.

```
1 real minT = checkmovemesh(Th, [Dx, Dy]);
```

Parameters:

Same as *movemesh*

Output:

- `minT` (real) Minimum triangle area

## 4.7.22 chi

Characteristic function of a mesh.

```
1 int IsInMesh = chi(Th)(x, y);
```

Parameters:

- `Th` (mesh or mesh3)
- `x` (real) Position *x*
- `y` (real) Position *y*

Output:

- `IsInMesh` (int) 1 if  $(x, y) \in Th$  0 if  $(x, y) \notin Th$

## 4.7.23 clock

Get the clock in second.

```
1 real t = clock();
```

Parameter:

- None

Output:

- `t` (real) Current CPU time

## 4.7.24 complexEigenValue

Same as *EigenValue* for complex problems.

## 4.7.25 conj

Caculate the conjuguate of a complex number.

```
1 complex C1 = 1 + 1i;
2 complex C2 = conj(C1);
```

Parameter:

- `C1` (complex) Complex number

Output:

- `C2` (complex) Conjuguate of `C1`

## 4.7.26 convect

Characteristics Galerkin method.

```
1 real cgm = convect([Ux, Uy], dt, c);
2 real cgm = convect([Ux, Uy, Uz], dt, c);
```

Compute  $c \circ \mathbf{X}$  with  $\mathbf{X}(\mathbf{x}) = \mathbf{x}_\tau$  and  $\mathbf{x}_\tau$  is the solution of:

$$\begin{aligned}\dot{\mathbf{x}}_\tau &= \mathbf{u}(\mathbf{x}_\tau) \\ \mathbf{x}_\tau &= \mathbf{x}\end{aligned}$$

Parameters:

- ux (fespace function) Velocity:  $x$  component
- uy (fespace function) Velocity:  $y$  component
- uz (fespace function) **3D only**  
Velocity:  $z$  component
- dt (real) Time step
- c (fespace function) Function to convect

Output:

- cgm (real) Result

## 4.7.27 copysign

C++ copysign function.

```
1 real s = copysign(a, b);
```

## 4.7.28 cos

cos function.

```
1 real x = cos(theta);
```

Parameters:

- theta (real or complex)

Output:

- x (real or complex)

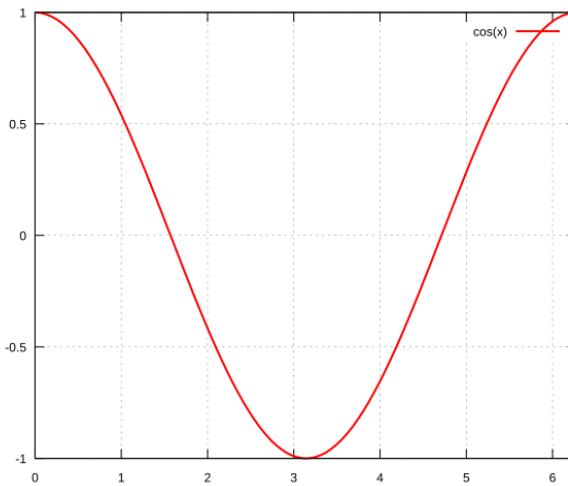
## 4.7.29 cosh

cosh function.

```
1 real x = cosh(theta);
```

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

Parameters:

**Fig. 4.7:** cos function

- `theta(real)`

Output:

- `x(real)`

### 4.7.30 diffnp

Arithmetic useful function.

```
1 diffnp(a, b) = (a<0) & (0<b) ? (b-a) : 0;
```

### 4.7.31 diffpos

Arithmetic useful function.

```
1 diffpos(a, b) = max(b-a, 0);
```

### 4.7.32 dist

Arithmetic useful function.

```
1 dist(a, b) = sqrt(a^2 + b^2);
2 dist(a, b, c) = sqrt(a^2 + b^2 + c^2);
```

### 4.7.33 dumptable

Show all types, operators and functions in **FreeFEM**.

```
1 dumptable(out);
```

Parameters:

- `out` (`ostream`) `cout` of `ofstream` file.

Output:

- None

### 4.7.34 `dx`

*x* derivative.

```
1 Uh up = dx(u);
```

$$\frac{\partial u}{\partial x}$$

Parameters:

- `u` (`fespace` function)

Output:

- `up` (`fespace` function)

### 4.7.35 `dxx`

*x* double derivative.

```
1 Uh upp = dxx(u);
```

$$\frac{\partial^2 u}{\partial x^2}$$

Parameters:

- `u` (`fespace` function)

Output:

- `upp` (`fespace` function)

### 4.7.36 `dxy`

*xy* derivative.

```
1 Uh upp = dxy(u);
```

$$\frac{\partial^2 u}{\partial x \partial y}$$

Parameters:

- `u` (`fespace` function)

Output:

- `upp` (`fespace` function)

### 4.7.37 dxz

*xz* derivative.

```
1 Uh upp = dxz(u);
```

$$\frac{\partial^2 u}{\partial x \partial z}$$

Parameters:

- *u* (fespace function)

Output:

- *upp* (fespace function)

### 4.7.38 dy

*y* derivative.

```
1 Uh up = dy(u);
```

$$\frac{\partial u}{\partial y}$$

Parameters:

- *u* (fespace function)

Output:

- *upp* (fespace function)

### 4.7.39 dyx

*yx* derivative.

```
1 Uh upp = dyx(u);
```

$$\frac{\partial^2 u}{\partial y \partial x}$$

Parameters:

- *u* (fespace function)

Output:

- *upp* (fespace function)

### 4.7.40 dyy

*y* double derivative.

```
1 Uh upp = dyy(u);
```

$$\frac{\partial^2 u}{\partial x^2}$$

Parameters:

- `u` (fespace function)

Output:

- `upp` (fespace function)

#### 4.7.41 `dyz`

*yz* derivative.

```
1 Uh upp = dyz(u);
```

$$\frac{\partial^2 u}{\partial y \partial z}$$

Parameters:

- `u` (fespace function)

Output:

- `upp` (fespace function)

#### 4.7.42 `dz`

*z* derivative.

```
1 Uh up = dz(u);
```

$$\frac{\partial u}{\partial z}$$

Parameters:

- `u` (fespace function)

Output:

- `upp` (fespace function)

#### 4.7.43 `dzx`

*zx* derivative.

```
1 Uh upp = dzx(u);
```

$$\frac{\partial^2 u}{\partial z \partial x}$$

Parameters:

- `u` (fespace function)

Output:

- `upp` (fespace function)

### 4.7.44 dzy

*zy* derivative.

```
1 Uh upp = dzy(u);
```

$$\frac{\partial^2 u}{\partial z \partial y}$$

Parameters:

- *u* (fespace function)

Output:

- *upp* (fespace function)

### 4.7.45 dzz

*z* double derivative.

```
1 Uh upp = dzz(u);
```

$$\frac{\partial^2 u}{\partial z^2}$$

Parameters:

- *u* (fespace function)

Output:

- *upp* (fespace function)

### 4.7.46 EigenValue

Compute the generalized eigenvalue of  $Au = \lambda Bu$ . The shifted-inverse method is used by default with `sigma=σ`, the shift of the method. The function `EigenValue` can be used for either matrices or functions returning a matrix vector product. The use of the matrix version is shown below.

```
1 int k = EigenValue(A,B,nev= , sigma= );
```

Parameters:

- *A*, *B*: matrices of same size
- *nev=n*: number of desired eigenvalues given by an integer *n*
- *sym=*: the problem is symmetric or not
- *tol=*: the relative accuracy to which eigenvalues are to be determined
- *value=*: an array to store the real part of the eigenvalues
- *ivalue=*: an array to store the imaginary part of the eigenvalues
- *vector=*: a Finite Element function array to store the eigenvectors
- *sigma=*: the shift value
- Other parameters are available for more advanced use: see the ARPACK documentation.

Output: The output is the number of converged eigenvalues, which can be different than the number of requested eigenvalues given by `nev=`. Note that the eigenvalues and the eigenvectors are stored for further purposes using the parameters `value=` and `vector=`.

#### 4.7.47 emptymesh

Build an empty mesh.

Useful to handle Lagrange multipliers in mixed and Mortar methods.

```
1 mesh eTh = emptymesh(Th, ssd);
```

Parameters:

- `Th (mesh)` Mesh to empty
- `ssd(int [int])` Pseudo subregion label

Output:

- `eTh (mesh)` Empty mesh

#### 4.7.48 erf

The error function:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt$$

```
1 real err = erf(x);
```

Parameters:

- `x (real)`

Output:

- `err (real)`

#### 4.7.49 erfc

Complementary of the *error function*:

$$\text{erfc}(x) = 1 - \text{erf}(x)$$

```
1 real errc = erfc(x);
```

Parameters:

- `x (real)`

Output:

- `err (real)`

### 4.7.50 exec

Execute an external command.

```
1 int v = exec(command);
```

Parameters:

- command (string) Command to execute

Output:

- v (int) Value returned by the command

### 4.7.51 exit

Exit function, equivalent to `return`.

```
1 exit(N);
```

Parameters:

- N (int) Return value

Output:

- None

### 4.7.52 exp

Exponential function.

```
1 real a = exp(b);
```

Parameters:

- b (real or complex)

Output:

- a (real or complex)

### 4.7.53 fdim

Positive difference (`cmath` function).

```
1 real fd = fdim(a, b);
```

Parameters:

- a (real)
- b (real)

Output:

- fd (real) If  $x > y$ , return  $x - y$ . If  $x \leq y$ , return 0

## 4.7.54 floor

Floor function.

```
1 real a = floor(b);
```

Return the largest integer value not greater than b.

Parameters:

- b (real)

Output:

- a (real)

## 4.7.55 fmax

Maximum (cmath function).

```
1 real Max = fmax(a, b);
```

Parameters:

- a (real)
- b (real)

Output:

- Max (real)

## 4.7.56 fmin

Minimum (cmath function).

```
1 real Min = fmin(a, b);
```

Parameters:

- a (real)
- b (real)

Output:

- Min (real)

## 4.7.57 fmod

Remainder of  $a/b$  (cmath function).

```
1 real Mod = fmod(a, b);
```

Parameters:

- a (real)
- b (real)

Output:

- Mod (real)

## 4.7.58 imag

Imaginary part of a complex number.

```
1 complex c = 1. + 1i;
2 real Im = imag(c);
```

## 4.7.59 int1d

1D integral.

```
1 int1d(Th, [Label], [qfe=Qfe], [qforder=Qforder]) (
2     ...
3 )
```

Used in *problem*, *solve* or *varf* definition to impose a boundary condition only (**FreeFEM** does not support 1D simulation), or outside to calculate a quantity.

Parameters:

- Th (mesh) Mesh where the integral is calculated
- Label (int) [*Optional*]  
Label of the 1D border Default: all borders of the mesh
- qfe= (*quadrature formula*) [*Optional*]  
Quadrature formula, see *quadrature formulae*
- qforder= (*quadrature formula*) [*Optional*]  
Quadrature order, see *quadrature formulae*

Output:

- Depending on the situation: In a problem, solve or varf definition: Non relevant.  
Outside: real (example: real l = int1d(Th, 1)(1.);).

**Warning:** In a problem, solve or varf definition, the content of int1d must be a linear or bilinear form.

## 4.7.60 int2d

2D integral.

```
1 int2d(Th, [Region], [qfe=Qfe], [qforder=Qforder]) (
2     ...
3 )
```

Or

```

1 int2d(Th, [Label], [qfe=Qfe], [qforder=Qforder]) (
2     ...
3 )

```

Used in *problem*, *solve* or *varf* definition to: - Calculate integral in 2D simulation - Impose a boundary condition in 3D simulation Or outside to calculate a quantity.

Parameters:

- Th (mesh or mesh3) Mesh where the integral is calculated
- Region (int) [*Optional*] Label of the 2D region (2D simulation) Default: all regions of the mesh
- Label (int) [*Optional*] Label of the 2D border (3D simulation) Default: all borders of the mesh
- qfe= (*quadrature formula*) [*Optional*]  
Quadrature formula, see *quadrature formulae*
- qforder= (*quadrature formula*) [*Optional*]  
Quadrature order, see *quadrature formulae*

Output:

- Depending on the situation: In a *problem*, *solve* or *varf* definition: Non relevant. Outside: *real* (example: *real s = int2d(Th, 1) (1.);*).

**Warning:** In a *problem*, *solve* or *varf* definition, the content of the *int2d* must be a linear or bilinear form.

#### 4.7.61 int3d

3D integral.

```

1 int3d(Th, [Region], [qfe=Qfe], [qforder=Qforder]) (
2     ...
3 )

```

Used in *problem*, *solve* or *varf* definition to calculate integral in 3D simulation, or outside to calculate a quantity.

Parameters:

- Th (mesh3) Mesh where the integral is calculated
- Region (int) [*Optional*]  
Label of the 3D region  
Default: all regions of the mesh
- qfe= (*quadrature formula*) [*Optional*]  
Quadrature formula, see *quadrature formulae*
- qforder= (*quadrature formula*) [*Optional*]  
Quadrature order, see *quadrature formulae*

Output:

- Depending on the situation: In a *problem*, *solve* or *varf* definition: Non relevant. Outside: *real* (example: *real v = int3d(Th, 1) (1.);*).

**Warning:** In a problem, solve or varf definition, the content of the `int 3d` must be a linear or bilinear form.

## 4.7.62 intalledges

Integral on all edges.

```
1 intalledges (Th, [Region]) (
2     ...
3 )
```

Parameters:

- Th (mesh) Mesh where the integral is calculated
- Region (int) [*Optional*]

Label of the region

Default: all regions of the mesh

Output:

- Non relevant

## 4.7.63 intallfaces

Integral on all faces.

Same as `intalledges` for mesh3.

## 4.7.64 interpolate

Interpolation operator from a finite element space to another.

```
1 matrix I = interpolate (Wh, Vh, [inside=Inside], [t=T], [op=Op], [U2Vc=U2VC]);
```

Parameters:

- Wh (fespace) Target finite element space
- Vh (fespace) Original finite element space
- inside= (bool) If true, create a zero extension outside the Vh domain
- t= (bool) If true, return the transposed matrix
- op= (int) 0: interpolate the function (default value) 1: interpolate  $\partial_x$  2: interpolate  $\partial_y$  3: interpolate  $\partial_z$
- U2Vc= (int [int]) Array of the same size of Wh describing which component of Vh is interpolated in Wh

Output:

- I (matrix) Interpolation matrix operator

## 4.7.65 invdiff

Arithmetic useful function.

```
1 invdiff(a, b) = (abs(a-b) < 10^(-30)) ? (a-b) : 1/(a-b)
2 invdiff(a, b, e) = (abs(a-b) < e) ? (a-b) : 1/(a-b)
```

## 4.7.66 invdiffnp

Arithmetic useful function.

```
1 invdiffnp(a, b) = (a<0) & (0<b) ? 1/(b-a) : 0
```

## 4.7.67 invdiffpos

Arithmetic useful function.

```
1 invdiffpos(a, b) = (a<b) ? 1./(b-a) : 0
```

## 4.7.68 isInf

The C++ `isInf` function.

```
1 bool b = isInf(a);
```

## 4.7.69 isNaN

The C++ `isNaN` function.

```
1 bool b = isNaN(a);
```

## 4.7.70 isNormal

The C++ `isNormal` function.

## 4.7.71 j0

Bessel function of first kind, order 0.

```
1 real b = j0(x);
```

Parameters:

- `x` (real)

Output:

- `b` (real)

## 4.7.72 j1

Bessel function of first kind, order 1.

```
1 real b = j1(x);
```

Parameters:

- x (real)

Output:

- b (real)

## 4.7.73 jn

Bessel function of first kind, order n.

```
1 real b = jn(n, x);
```

$$J_n(x) = \sum_{p=0}^{\infty} \frac{(1)^p}{p!(n+p)!} \left(\frac{x}{2}\right)^{2p+n}$$

Parameters:

- n (int)
- x (real)

Output:

- b (real)

## 4.7.74 jump

Jump function across an edge.

```
1 intalledges (
2     ... jump(c) ...
3 )
```

Parameters:

- c (fespace function) Discontinuous function

Output:

- Non relevant

## 4.7.75 LinearCG

Linear CG solver

Parameters and output are the same as *AffineCG*

## 4.7.76 LinearGMRES

Linear GMRES solver

Parameters and output are the same as *AffineCG*

## 4.7.77 Igamma

Natural logarithm of the absolute value of the  $\Gamma$  function of  $x$ .

```
1 real lg = lgamma(x);
```

Parameters:

- $x$  (real)

Output:

- $lg$  (real)

## 4.7.78 log

Natural logarithm.

```
1 real l = log(x);
```

Parameters:

- $x$  (real or complex)

Output:

- $l$  (real or complex)

---

**Note:** Complex value

For complex value, the `log` function is defined as:

$$\log(z) = \log(|z|) + i \arg(z)$$

---

## 4.7.79 log10

Common logarithm.

```
1 real l = log10(x);
```

Parameters:

- $x$  (real)

Output:

- $l$  (real)

### 4.7.80 lrint

Integer value nearest to  $x$ .

```
1 int l = lrint(a);
```

Parameters:

- a (real)

Output:

- l (int)

### 4.7.81 lround

Round a value, and return an integer value.

```
1 int l = lround(a);
```

Parameters:

- a (real)

Output:

- l (int)

### 4.7.82 ltime

Return the current time since *the Epoch*.

```
1 int t = ltime();
```

Parameter:

- None

Output:

- t (int)

### 4.7.83 max

Maximum value of two, three or four values.

```
1 real m = max(a, b);
2 real m = max(a, b, c);
3 real m = max(a, b, c, d);
```

Parameters:

- a (int or real)
- b (int or real)
- c (int or real) [Optional]
- d (int or real) [Optional]

Output:

- b (int or real)

## 4.7.84 min

Minimum value of two, three or four values.

```
1 real m = min(a, b);
2 real m = min(a, b, c);
3 real m = min(a, b, c, d);
```

Parameters:

- a (int or real)
- b (int or real)
- c (int or real) [Optional]
- d (int or real) [Optional]

Output:

- b (int or real)

## 4.7.85 movemesh

Move a mesh.

```
1 mesh MovedTh = movemesh(Th, [Dx, Dy]);
2 mesh3 MovedTh = movemesh(Th, [Dx, Dy, Dz], [region=Region], [label=Label],  
→ [facemerge=FaceMerge], [ptmerge=PtMerge], [orientation=Orientation]);
```

Parameters:

- Th (mesh or mesh3) Mesh to move
- Dx (fespace function) Displacement along  $x$
- Dy (fespace function) Displacement along  $y$
- Dz (fespace function) **3D only**  
Displacement along  $z$
- region= (int) [Optional] **3D only**  
Set label to tetrahedra
- label= (int [int]) [Optional] **3D only**  
Set label of faces (see [change](#) for more information)
- facemerge= (int) [Optional] **3D only**  
If equal to 1, some faces can be merged during the mesh moving Default: 1
- ptmerge= (real) [Optional] **3D only**  
Criteria to define when two points merge

- `orientation=(int)` [Optional] **3D only**  
If equal to 1, allow orientation reverse if tetrahedra is not positive Default: 1

Output:

- `MovedTh (mesh or mesh3)` Moved mesh

## 4.7.86 NaN

C++ nan function.

```
1 real n = NaN([String]);
```

Parameters:

- `String (string)` Default: ""

## 4.7.87 NLCG

Non-linear conjugate gradient.

Parameters and output are the same as [AffineCG](#)

## 4.7.88 on

Dirichlet condition function.

```
1 problem (u, v)
2     ...
3     + on (Label, u=uD)
4     ...
```

**Warning:** Used only in problem, solve and varf

Parameters:

- `Label (int or border in 2D)`  
Boundary reference where to impose the Dirichlet condition
- `uD (fespace function, func or real or int)`  
Dirichlet condition (`u` is an unknown of the problem)

Output:

- Non relevant

## 4.7.89 plot

Plot meshes and results.

```
1 plot ([Th], [u], [[Ux, Uy, Uz]], [wait=Wait], [ps=PS], [coef=Coef], [fill=Fill], [cmm=Cmm], [value=Value], [aspectratio=AspectRatio], [bb=Bb], [nbiso=NbIso], [nbarrow=NbArrow], [viso=VIso], [varrow=VArrow], [bw=Bw], [grey=Grey], (onmesh on next page) [boundary=Boundary], [dim=Dim], [prev=Prev], [WindowIndex=WI]);
```

(continued from previous page)

---

**Note:** Only one of Th, u or [Ux, Uy] / [Ux, Uy, Uz] is needed for the plot command.

---

Parameters:

- Th (mesh or mesh3) Mesh to display
- u (fespace function) Scalar fespace function to display
- [Ux, Uy] / [Ux, Uy, Uz] (fespace function array) Vectorial fespace function to display
- [Ux, Uy] ([real[int], real[int]]) Couple a real array to display a curve
- wait= (bool) If true, wait before continue
- ps= (string) Name of the file to save the plot (.ps or .eps format)
- coef= (real) Arrow size
- fill= (bool) If true, fill color between isovalue (usable with scalar fespace function only)
- cmm= (string) Text comment in the graphic window
- value= (bool) If true, show the value scale
- aspectratio= (bool) If true, preserve the aspect ratio
- bb= ([real[int], real[int]]) Specify a bounding box using two corner points
- nbiso= (int) Number of isovales
- nbarrow= (int) Number of colors of arrows values
- viso= (real[int]) Specify an array of isovales
- varrow= (real[int]) Specify an array of arrows values color
- bw= (bool) If true, the plot is in black and white
- grey= (bool) If true, the plot is in grey scale
- hsv= (real[int]) Array of  $3 \times n$  values defining HSV color model  $[h_1, s_1, v_1, \dots, h_n, s_n, v_n]$
- boundary= (bool) If true, display the boundary of the domain
- dim= (int) Set the dimension of the plot: 2 or 3
- prev= (bool) Use the graphic state of the previous state
- WindowIndex= (int) Specify window index for multiple windows graphics

Output:

- None

See the [plot](#) section for in-graphic commands.

## 4.7.90 polar

Polar coordinates.

```
| complex p = polar(a, b);
```

Parameters:

- a (real)
- b (real)

Output:

- p (complex)

### 4.7.91 pow

Power function.

```
1 real p = pow(a, b);
```

$$p = a^b$$

Parameters:

- a (real)
- b (real)

Output:

- p (real)

### 4.7.92 projection

Arithmetic useful function.

```
1 real p = projection(a, b, x);
```

Projection is equivalent to:

```
1 projection(a, b, x) = min(max(a, x), b) * (a < b) + min(max(b, x), a) * (1 - (a < b));
```

Parameters:

- a (real)
- b (real)
- x (real)

Output:

- p (real)

### 4.7.93 randinit

Initialize the state vector by using a seed.

```
1 randinit(seed);
```

Parameters:

- seed (int)

Output:

- None

#### 4.7.94 randint31

Generate `unsigned int` (31 bits) random number.

```
1 int r = randint31();
```

Parameters:

- None

Output:

- `r (int)`

#### 4.7.95 randint32

Generate `unsigned int` (32 bits) random number.

```
1 int r = randint32();
```

Parameters:

- None

Output:

- `r (int)`

#### 4.7.96 randreal1

Generate uniform `real` in  $[0, 1]$  (32 bits).

```
1 real r = randreal1();
```

Parameters:

- None

Output:

- `r (real)`

#### 4.7.97 randreal2

Generate uniform `real` in  $[0, 1]$  (32 bits).

```
1 real r = randreal2();
```

Parameters:

- None

Output:

- r (real)

### 4.7.98 randreal3

Generate uniform real in (0, 1) (32 bits).

```
1 real r = randreal3();
```

Parameters:

- None

Output:

- r (real)

### 4.7.99 randres53

Generate uniform real in [0, 1) (53 bits).

```
1 real r = randres53();
```

Parameters:

- None

Output:

- r (real)

### 4.7.100 readmesh

Read a 2D mesh file at different formats (see *Mesh Generation*).

```
1 mesh Th = readmesh(MeshFileName);
```

Parameters:

- MeshFileName (string)

Output:

- Th (mesh)

### 4.7.101 readmesh3

Read a 3D mesh file at different formats (see *Mesh Generation*).

```
1 mesh3 Th = readmesh3(MeshFileName);
```

Parameters:

- MeshFileName (string)

Output:

- Th (mesh3)

### 4.7.102 real

Return the real part of a complex number.

```
1 real r = real(c);
```

Parameters:

- c (complex)

Output:

- r (real)

### 4.7.103 rint

Integer value nearest to  $x$  (real value).

```
1 real r = rint(a);
```

Parameters:

- a (real)

Output:

- r (real)

### 4.7.104 round

Round a value (real value).

```
1 real r = round(a);
```

Parameters:

- a (real)

Output:

- r (real)

### 4.7.105 savemesh

Save a 2D or 3D mesh in different formats (see *Mesh Generation 2D* and *Mesh Generation 3D*).

```
1 savemesh(Th, MeshFileName);
```

Parameters:

- Th (mesh or mesh3)
- MeshFileName (string)

Output:

- None

### 4.7.106 set

Set a property to a matrix. See [matrix](#).

### 4.7.107 sign

Sign of a value.

```
1 int s = sign(a);
```

Parameters:

- a (real or int)

Output:

- s (int)

### 4.7.108 signbit

C++ signbit function

```
1 int s = signbit(a);
```

### 4.7.109 sin

sin function.

```
1 real x = sin(theta);
```

Parameter:

- theta (real or complex)

Output:

- x (real or complex)

### 4.7.110 sinh

sinh function.

```
1 real x = sinh(theta);
```

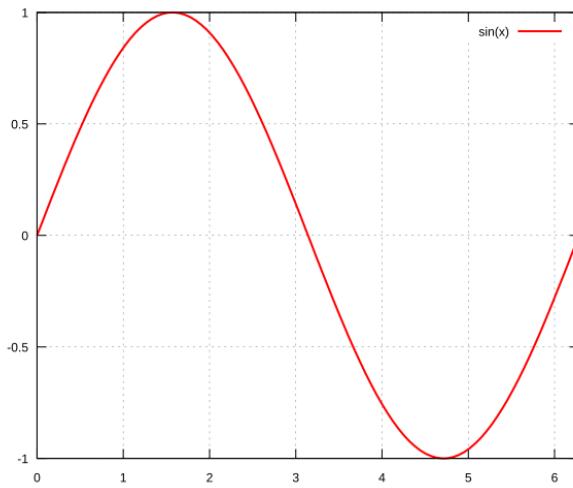
$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

Parameter:

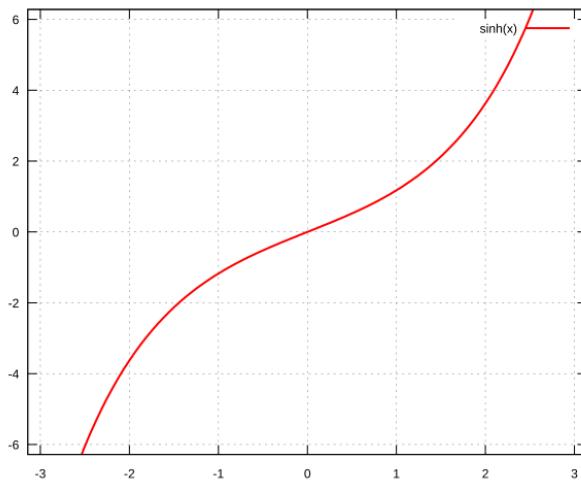
- theta (real)

Output:

- x (real)



**Fig. 4.8:**  $\sin$  function



**Fig. 4.9:**  $\sinh$  function

### 4.7.111 sort

Sort two array in parallel

```
1 sort(A, B);
```

Parameters:

- A (real[int])
- B (int[int])

Output:

- None

A is sorted in ascending order, B is sorted as A.

### 4.7.112 splitmesh

Split mesh triangles according to a function.

```
1 Th = splitmesh(Th0, f);
```

Parameters:

- Th0 (mesh)
- f (func or fespace function)

Output:

- Th (mesh)

### 4.7.113 sqrt

Square root

```
1 real s = sqrt(a);
```

Parameter:

- a (real)

Output:

- s (real)

### 4.7.114 square

1. Square of a number.

```
1 real S = square(a);
```

Parameter:

- a (real)

Output:

- S (real)
2. Build a structured square mesh.

```
mesh Th = square(nnX, nnY, [L*x, H*y], [flags=Flags], [label=Labels],  
→ [region=Region]);
```

Parameters:

- nnX (int) Discretization along  $x$
- nnY (int) Discretization along  $y$
- L (real) [Optional] Length along  $x$
- H (real) [Optional] Height along  $y$
- flags= (int) [Optional]
- label= (int [int]) [Optional]
- region= (int) [Optional]

Structured mesh type, see [Mesh Generation chapter](#) for more information

Output:

- Th (mesh)

#### 4.7.115 storagetotal

```
int total = storagetotal();
```

#### 4.7.116 storageused

```
int used = storageused();
```

#### 4.7.117 strtod

C++ *strtod* function

```
1 string text = "10.5";  
2 real number = strtod(text);
```

Parameter:

- text (string)

Output:

- number (real)

#### 4.7.118 strtol

C++ *strtol* function

```

1 string text = "10";
2 int number = strtol(text);
3
4 int base = 16;
5 int number = strtol(text, base);

```

Parameter:

- text (string)
- base (int) Base [*Optional*]

Output:

- number (int)

### 4.7.119 swap

Swap values.

```

1 swap(a, b);

```

Parameters:

- a (real)
- b (real)

Output:

- None

### 4.7.120 system

Execute a system command.

```

1 int Res = system(Command);

```

Parameter:

- Command (string) System command

Output:

- Res (int) Value returned by the system command

---

**Note:** On Windows, the full path of the command is needed. For example, to execute `ls.exe`:

```

1 int Res = exec("C:\\cygwin\\bin\\ls.exe");

```

---

### 4.7.121 tan

tan function.

```

1 real x = tan(theta);

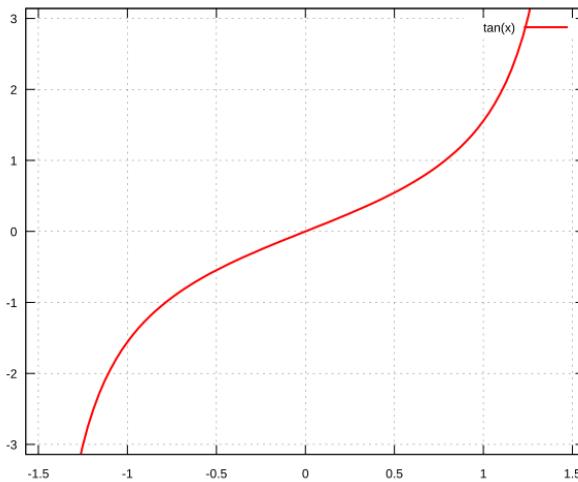
```

Parameter:

- theta (real)

Output:

- x (real)



**Fig. 4.10:** tan function

#### 4.7.122 tanh

tanh function.

```
1 real x = tanh(theta);
```

Parameter:

- theta (real)

Output:

- x (real)

#### 4.7.123 tgamma

Calculate the  $\Gamma$  function of  $x$ .

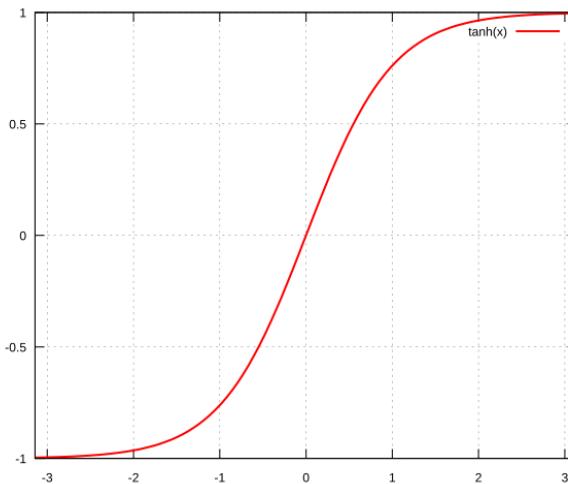
```
1 real tg = tgamma(x);
```

Parameter:

- x (real)

Output:

- tg (real)

**Fig. 4.11:** tanh function

#### 4.7.124 time

Return the current time (C++ function).

```
1 real t = time();
```

Parameter:

- None

Output:

- t (real)

#### 4.7.125 trace

Matrix trace

```
1 real tr = trace([[1, 2], [3, 4]]);
```

Parameters:

- Matrix

Output:

- Trace of the matrix (real)

#### 4.7.126 trunc

Split triangle of a mesh.

```
1 mesh Th = trunc(Th0, R, [split=Split], [label=Label]);
```

Parameters:

- Th0 (mesh)

- R (bool or int) Split triangles where R is true or different from 0

- split=(int) [*Optional*]

Level of splitting Default: 1

- label=(int) [*Optional*]

Label number of new boundary item Default: 1

Output:

- Th (mesh)

## 4.7.127 y0

Bessel function of second kind, order 0.

```
real B = y0(x);
```

Parameters:

- x (real)

Output:

- b (real)

## 4.7.128 y1

Bessel function of second kind, order 1.

```
real B = y1(x);
```

Parameters:

- x (real)

Output:

- b (real)

## 4.7.129 yn

Bessel function of second kind, order n.

```
real B = yn(n, x);
```

$$Y_n(x) = \lim_{\lambda \rightarrow n} \frac{J_\lambda(x) \cos(\lambda\pi) - J_{-\lambda}(x)}{\sin(\lambda\pi)}$$

Parameters:

- n (int)
- x (real)

Output:

- b (real)

## 4.8 External libraries

### 4.8.1 aniso

**boundaniso**

---

**Todo:** todo

---

### 4.8.2 BEC

**BECtrap**

---

**Todo:** todo

---

**GPvortex**

---

**Todo:** todo

---

**dxGPVortex**

---

**Todo:** todo

---

**dyGPVortex**

---

**Todo:** todo

---

### 4.8.3 Binary I/O

**LoadVec**

---

**Todo:** todo

---

**LoadFlag**

---

**Todo:** todo

---

**SaveVec**

---

**Todo:** todo

---

**flag**

---

**Todo:** todo

---

#### 4.8.4 buildlayer

**buildlayers**

---

**Todo:** todo

---

#### 4.8.5 ClosePoints

**radiusSearch**

---

**Todo:** todo

---

**Voisinage**

---

**Todo:** todo

---

**neighborhood**

---

**Todo:** todo

---

## ClosePoints2

---

**Todo:** todo

---

## ClosePoint

---

**Todo:** todo

---

## ClosePoints1

---

**Todo:** todo

---

## 4.8.6 Curvature

### extractborder

Extract a border of a mesh.

```
| int Res = extractborder(Th, Label, Points);
```

Parameters:

- Th (mesh or mesh3)
- Label (int) Label of the border to extract
- Points (real[int, int]) Extracted points Must be allocated as real[int, int] Points(3, 1);

Output:

- Res (real) Length of the extracted border

### curvature

---

**Todo:** todo

---

### raxicurvature

---

**Todo:** todo

---

**curves**

---

**Todo:** todo

---

**setecurveabcisse**

---

**Todo:** todo

---

**equiparameter**

---

**Todo:** todo

---

**Tresca**

---

**Todo:** todo

---

**VonMises**

---

**Todo:** todo

---

**4.8.7 dfft**

Refer to the [FFTW documentation](#) for more informations.

**plandfft**

---

**Todo:** todo

---

**execute**

---

**Todo:** todo

---

**delete**

---

**Todo:** todo

---

**dfft**

---

**Todo:** todo

---

**map**

---

**Todo:** todo

---

## 4.8.8 distance

Need

```
1 load "distance"
```

**distance**

```
1 distance(Th, d, dist, [distmax=DistMax]);
```

Parameters:

- Th (mesh)
- d
- dist (real[int])

Output:

- 

---

**Todo:** todo

---

**checkdist**

---

**Todo:** todo

---

#### 4.8.9 DxWriter

**Dxaddmesh**

---

**Todo:** todo

---

**Dxaddtimeseries**

---

**Todo:** todo

---

**Dxaddsol2ts**

---

**Todo:** todo

---

#### 4.8.10 Element\_P1bl

**expert**

---

**Todo:** todo

---

#### 4.8.11 exactpartition

**exactpartition**

---

**Todo:** todo

---

#### 4.8.12 ff-AiryBiry

**airy**

---

**Todo:** todo

---

**biry**

---

**Todo:** todo

---

### 4.8.13 ff-cmaes

**cmaes**

---

**Todo:** todo

---

### 4.8.14 ff\_gsl\_awk

Refer to the [GSL documentation](#) for more informations

**gslcdfugaussianP**

Link to:

```
| gsl_cdf_ugaussian_P(a)
```

**gslcdfugaussianQ**

Link to:

```
| gsl_cdf_ugaussian_Q(a)
```

**gslcdfugaussianPinv**

Link to:

```
| gsl_cdf_ugaussian_Pinv(a)
```

**gslcdfugaussianQinv**

Link to:

```
| gsl_cdf_ugaussian_Qinv(a)
```

**gslcdfgaussianP**

Link to:

```
| gsl_cdf_gaussian_P(a, b)
```

**gslcdfgaussianQ**

Link to:

```
| gsl_cdf_gaussian_Q(a, b)
```

### **gslcdfgaussianPinv**

Link to:

```
| gsl_cdf_gaussian_Pinv(a, b)
```

### **gslcdfgaussianQinv**

Link to:

```
| gsl_cdf_gaussian_Qinv(a, b)
```

### **gslcdfgammaP**

Link to:

```
| gsl_cdf_gamma_P(a, b, c)
```

### **gslcdfgammaQ**

Link to:

```
| gsl_cdf_gamma_Q(a, b, c)
```

### **gslcdfgammaPinv**

Link to:

```
| gsl_cdf_gamma_Pinv(a, b, c)
```

### **gslcdfgammaQinv**

Link to:

```
| gsl_cdf_gamma_Pinv(a, b, c)
```

### **gslcdfcauchyP**

Link to:

```
| gsl_cdf_cauchy_P(a, b)
```

### **gslcdfcauchyQ**

Link to:

```
| gsl_cdf_cauchy_Q(a, b)
```

**gslcdfcauchyPinv**

Link to:

```
| gsl_cdf_cauchy_Pinv(a, b)
```

**gslcdfcauchyQinv**

Link to:

```
| gsl_cdf_cauchy_Qinv(a, b)
```

**gslcdflaplaceP**

Link to:

```
| gsl_cdf_lapalce_P(a, b)
```

**gslcdflaplaceQ**

Link to:

```
| gsl_cdf_lapalce_Q(a, b)
```

**gslcdflaplacePinv**

Link to:

```
| gsl_cdf_lapalce_Pinv(a, b)
```

**gslcdflaplaceQinv**

Link to:

```
| gsl_cdf_lapalce_Qinv(a, b)
```

**gslcdfrayleighP**

Link to:

```
| gsl_cdf_rayleigh_P(a, b)
```

**gslcdfrayleighQ**

Link to:

```
| gsl_cdf_rayleigh_Q(a, b)
```

### **gslcdfrayleighPinv**

Link to:

```
| gsl_cdf_rayleigh_Pinv(a, b)
```

### **gslcdfrayleighQinv**

Link to:

```
| gsl_cdf_rayleigh_Qinv(a, b)
```

### **gslcdfchisqP**

Link to:

```
| gsl_cdf_chisq_P(a, b)
```

### **gslcdfchisqQ**

Link to:

```
| gsl_cdf_chisq_Q(a, b)
```

### **gslcdfchisqPinv**

Link to:

```
| gsl_cdf_chisq_Pinv(a, b)
```

### **gslcdfchisqQinv**

Link to:

```
| gsl_cdf_chisq_Qinv(a, b)
```

### **gslcdfexponentialP**

Link to:

```
| gsl_cdf_exponential_P(a, b)
```

### **gslcdfexponentialQ**

Link to:

```
| gsl_cdf_exponential_Q(a, b)
```

**gslcdfexponentialPinv**

Link to:

```
| gsl_cdf_exponential_Pinv(a, b)
```

**gslcdfexponentialQinv**

Link to:

```
| gsl_cdf_exponential_Qinv(a, b)
```

**gslcdfexppowP**

Link to:

```
| gsl_cdf_exppow_P(a, b, c)
```

**gslcdfexppowQ**

Link to:

```
| gsl_cdf_exppow_Q(a, b, c)
```

**gslcdftdistP**

Link to:

```
| gsl_cdf_t_dist_P(a, b)
```

**gslcdftdistQ**

Link to:

```
| gsl_cdf_t_dist_Q(a, b)
```

**gslcdftdistPinv**

Link to:

```
| gsl_cdf_t_dist_Pinv(a, b)
```

**gslcdftdistQinv**

Link to:

```
| gsl_cdf_t_dist_Qinv(a, b)
```

### **gslcdffdistP**

Link to:

```
| gsl_cdf_fdist_P(a, b, c)
```

### **gslcdffdistQ**

Link to:

```
| gsl_cdf_fdist_Q(a, b, c)
```

### **gslcdffdistPinv**

Link to:

```
| gsl_cdf_fdist_Pinv(a, b, c)
```

### **gslcdffdistQinv**

Link to:

```
| gsl_cdf_fdist_Qinv(a, b, c)
```

### **gslcdfbetaP**

Link to:

```
| gsl_cdf_beta_P(a, b, c)
```

### **gslcdfbetaQ**

Link to:

```
| gsl_cdf_beta_Q(a, b, c)
```

### **gslcdfbetaPinv**

Link to:

```
| gsl_cdf_beta_Pinv(a, b, c)
```

### **gslcdfbetaQinv**

Link to:

```
| gsl_cdf_beta_Qinv(a, b, c)
```

**gslcdfflatP**

Link to:

```
| gsl_cdf_flat_P(a, b, c)
```

**gslcdfflatQ**

Link to:

```
| gsl_cdf_flat_Q(a, b, c)
```

**gslcdfflatPinv**

Link to:

```
| gsl_cdf_flat_Pinv(a, b, c)
```

**gslcdfflatQinv**

Link to:

```
| gsl_cdf_flat_Qinv(a, b, c)
```

**gslcdflognormalP**

Link to:

```
| gsl_cdf_lognormal_P(a, b, c)
```

**gslcdflognormalQ**

Link to:

```
| gsl_cdf_lognormal_Q(a, b, c)
```

**gslcdflognormalPinv**

Link to:

```
| gsl_cdf_lognormal_Pinv(a, b, c)
```

**gslcdflognormalQinv**

Link to:

```
| gsl_cdf_lognormal_Qinv(a, b, c)
```

### **gslcdfgumbel1P**

Link to:

```
| gsl_cdf_gumbel1_P(a, b, c)
```

### **gslcdfgumbel1Q**

Link to:

```
| gsl_cdf_gumbel1_Q(a, b, c)
```

### **gslcdfgumbel1Pinv**

Link to:

```
| gsl_cdf_gumbel1_Pinv(a, b, c)
```

### **gslcdfgumbel1Qinv**

Link to:

```
| gsl_cdf_gumbel1_Qinv(a, b, c)
```

### **gslcdfgumbel2P**

Link to:

```
| gsl_cdf_gumbel2_P(a, b, c)
```

### **gslcdfgumbel2Q**

Link to:

```
| gsl_cdf_gumbel2_Q(a, b, c)
```

### **gslcdfgumbel2Pinv**

Link to:

```
| gsl_cdf_gumbel2_Pinv(a, b, c)
```

### **gslcdfgumbel2Qinv**

Link to:

```
| gsl_cdf_gumbel2_Qinv(a, b, c)
```

**gslcdfweibullP**

Link to:

```
| gsl_cdf_weibull_P(a, b, c)
```

**gslcdfweibullQ**

Link to:

```
| gsl_cdf_weibull_Q(a, b, c)
```

**gslcdfweibullPinv**

Link to:

```
| gsl_cdf_weibull_Pinv(a, b, c)
```

**gslcdfweibullQinv**

Link to:

```
| gsl_cdf_weibull_Qinv(a, b, c)
```

**gslcdfparetoP**

Link to:

```
| gsl_cdf_pareto_P(a, b, c)
```

**gslcdfparetoQ**

Link to:

```
| gsl_cdf_pareto_Q(a, b, c)
```

**gslcdfparetoPinv**

Link to:

```
| gsl_cdf_pareto_Pinv(a, b, c)
```

**gslcdfparetoQinv**

Link to:

```
| gsl_cdf_pareto_Qinv(a, b, c)
```

### **gslcdflogisticP**

Link to:

```
| gsl_cdf_logistic_P(a, b)
```

### **gslcdflogisticQ**

Link to:

```
| gsl_cdf_logistic_Q(a, b)
```

### **gslcdflogisticPinv**

Link to:

```
| gsl_cdf_logistic_Pinv(a, b)
```

### **gslcdflogisticQinv**

Link to:

```
| gsl_cdf_logistic_Qinv(a, b)
```

### **gslcdfbinomialP**

Link to:

```
| gsl_cdf_binomial_P(a, b, c)
```

### **gslcdfbinomialQ**

Link to:

```
| gsl_cdf_binomial_Q(a, b, c)
```

### **gslcdfpoissonP**

Link to:

```
| gsl_cdf_poisson_P(a, b)
```

### **gslcdfpoissonQ**

Link to:

```
| gsl_cdf_poisson_Q(a, b)
```

**gslcdfgeometricP**

Link to:

```
| gsl_cdf_geometric_P(a, b)
```

**gslcdfgeometricQ**

Link to:

```
| gsl_cdf_geometric_Q(a, b)
```

**gslcdfnegativebinomialP**

Link to:

```
| gsl_cdf_negative_binomial_P(a, b, c)
```

**gslcdfnegativebinomialQ**

Link to:

```
| gsl_cdf_negative_binomial_Q(a, b, c)
```

**gslcdfpascalP**

Link to:

```
| gsl_cdf_pascal_P(a, b, c)
```

**gslcdfpascalQ**

Link to:

```
| gsl_cdf_pascal_Q(a, b, c)
```

**gsiranbernoullipdf**

Link to:

```
| gsl_ran_bernoulli_pdf(a, b)
```

**gsiranbeta**

Link to:

```
| gsl_ran_beta(a, b, c)
```

### **gslranbetapdf**

Link to:

```
| gsl_ran_beta_pdf(a, b, c)
```

### **gslranbinomialpdf**

Link to:

```
| gsl_ran_binomial_pdf(a, b, c)
```

### **gslranexponential**

Link to:

```
| gsl_ran_exponential(a, b)
```

### **gslranexponentialpdf**

Link to:

```
| gsl_ran_exponential_pdf(a, b)
```

### **gslranexppow**

Link to:

```
| gsl_ran_exppow(a, b, c)
```

### **gslranexppowpdf**

Link to:

```
| gsl_ran_exppow_pdf(a, b, c)
```

### **gslrancauchy**

Link to:

```
| gsl_ran_cauchy(a, b)
```

### **gslrancauchypdf**

Link to:

```
| gsl_ran_cauchy_pdf(a, b)
```

**gslranchisq**

Link to:

```
| gsl_ran_chisq(a, b)
```

**gslranchisqpdf**

Link to:

```
| gsl_ran_chisq_pdf(a, b)
```

**gslranerlang**

Link to:

```
| gsl_ran_erlang(a, b, c)
```

**gslranerlangpdf**

Link to:

```
| gsl_ran_erlang_pdf(a, b, c)
```

**gslranfdist**

Link to:

```
| gsl_ran_fdist(a, b, c)
```

**gslranfdistpdf**

Link to:

```
| gsl_ran_fdist_pdf(a, b, c)
```

**gslranflat**

Link to:

```
| gsl_ran_flat(a, b, c)
```

**gslranflatpdf**

Link to:

```
| gsl_ran_flat_pdf(a, b, c)
```

## gslrangamma

Link to:

```
| gsl_ran_gamma(a, b, c)
```

## gslrangammaint

Link to:

```
| gsl_ran_gamma_int(a, b, c)
```

## gslrangammapdf

Link to:

```
| gsl_ran_gamma_pdf(a, b, c)
```

## gslrangammamt

Link to:

```
| gsl_ran_gamma_mt(a, b, c)
```

## gslrangammaknuth

Link to:

```
| gsl_ran_gamma_knuth(a, b, c)
```

## gslrangaussian

Link to:

```
| gsl_ran_gaussian(a, b)
```

## gslrangaussianratiomethod

Link to:

```
| gsl_ran_gaussian_ratio_method(a, b)
```

## gslrangaussianziggurat

Link to:

```
| gsl_ran_gaussian_ziggurat(a, b)
```

**gslrangaussianpdf**

Link to:

```
| gsl_ran_gaussian_pdf(a, b)
```

**gsiranugaussian**

Link to:

```
| gsl_ran_ugaussian(a)
```

**gsiranugaussianratiomethod**

Link to:

```
| gsl_ran_ugaussian_ratio_method(a)
```

**gslranugaussianpdf**

Link to:

```
| gsl_ran_ugaussian_pdf(a)
```

**gslrangaussiantail**

Link to:

```
| gsl_ran_gaussian_tail(a, b, c)
```

**gslrangaussiantailpdf**

Link to:

```
| gsl_ran_gaussian_tail_pdf(a, b, c)
```

**gsiranugaussiantail**

Link to:

```
| gsl_ran_ugaussian_tail(a, b)
```

**gsiranugaussiantailpdf**

Link to:

```
| gsl_ran_ugaussian_tail_pdf(a, b)
```

### **gslranlandau**

Link to:

```
| gsl_ran_landau(a)
```

### **gslranlandaupdf**

Link to:

```
| gsl_ran_landau_pdf(a)
```

### **gslrangeometricpdf**

Link to:

```
| gsl_ran_geometric_pdf(a, b)
```

### **gslrangumbel1**

Link to:

```
| gsl_ran_gumbel1(a, b, c)
```

### **gslrangumbel1pdf**

Link to:

```
| gsl_ran_gumbel1_pdf(a, b, c)
```

### **gslrangumbel2**

Link to:

```
| gsl_ran_gumbel2(a, b, c)
```

### **gslrangumbel2pdf**

Link to:

```
| gsl_ran_gumbel2_pdf(a, b, c)
```

### **gslranlogistic**

Link to:

```
| gsl_ran_logistic(a, b)
```

**gslranlogisticpdf**

Link to:

```
| gsl_ran_logistic_pdf(a, b)
```

**gslranlognormal**

Link to:

```
| gsl_ran_lognormal(a, b, c)
```

**gslranlognormalpdf**

Link to:

```
| gsl_ran_lognormal_pdf(a, b, c)
```

**gslranlogarithmicpdf**

Link to:

```
| gsl_ran_logarithmic_pdf(a, b)
```

**gslrannegativebinomialpdf**

Link to:

```
| gsl_ran_negative_binomial_pdf(a, b, c)
```

**gslranpascalpdf**

Link to:

```
| gsl_ran_pascal_pdf(a, b, c)
```

**gslranpareto**

Link to:

```
| gsl_ran_pareto(a, b, c)
```

**gslranparetopdf**

Link to:

```
| gsl_ran_pareto_pdf(a, b, c)
```

### **gslranpoissonpdf**

Link to:

```
| gsl_ran_poisson_pdf(a, b)
```

### **gslranrayleigh**

Link to:

```
| gsl_ran_rayleigh(a, b)
```

### **gslranrayleighpdf**

Link to:

```
| gsl_ran_rayleigh_pdf(a, b)
```

### **gslranrayleightail**

Link to:

```
| gsl_ran_rayleigh_tail(a, b, c)
```

### **gslranrayleightailpdf**

Link to:

```
| gsl_ran_rayleigh_tail_pdf(a, b, c)
```

### **gslrantdist**

Link to:

```
| gsl_ran_tdsit(a, b)
```

### **gslrantdistpdf**

Link to:

```
| gsl_ran_tdsit_pdf(a, b)
```

### **gslranlaplace**

Link to:

```
| gsl_ran_laplace(a, b)
```

**gslranlaplacepdf**

Link to:

```
| gsl_ran_laplace_pdf(a, b)
```

**gslranlevy**

Link to:

```
| gsl_ran_levy(a, b, c)
```

**gslranweibull**

Link to:

```
| gsl_ran_weibull(a, b, c)
```

**gslranweibullpdf**

Link to:

```
| gsl_ran_weibull_pdf(a, b, c)
```

**gslsfairyAi**

Link to:

```
| gsl_sf_airy_Ai(a, b)
```

**gslsfairyBi**

Link to:

```
| gsl_sf_airy_Bi(a, b)
```

**gslsfairyAiscaled**

Link to:

```
| gsl_sf_airy_Ai_scaled(a, b)
```

**gslsfairyBiscaled**

Link to:

```
| gsl_sf_airy_Bi_scaled(a, b)
```

### **gsl\_sfairyAideriv**

Link to:

```
| gsl_sf_airy_Ai_deriv(a, b)
```

### **gsl\_sfairyBideriv**

Link to:

```
| gsl_sf_airy_Bi_deriv(a, b)
```

### **gsl\_sfairyAiderivscaled**

Link to:

```
| gsl_sf_airy_Ai_deriv_scaled(a, b)
```

### **gsl\_sfairyBiderivscaled**

Link to:

```
| gsl_sf_airy_Bi_deriv_scaled(a, b)
```

### **gsl\_sfairyzeroAi**

Link to:

```
| gsl_sf_airy_Ai(a, b)
```

### **gsl\_sfairyzeroBi**

Link to:

```
| gsl_sf_airy_aero_Bi(a)
```

### **gsl\_sfairyzeroAideriv**

Link to:

```
| gsl_sf_airy_aero_Ai_deriv(a)
```

### **gsl\_sfairyzeroBideriv**

Link to:

```
| gsl_sf_airy_aero_Bi_deriv(a)
```

**gsl\_sf\_besselJ0**

Link to:

```
| gsl_sf_bessel_J0(a)
```

**gsl\_sf\_besselJ1**

Link to:

```
| gsl_sf_bessel_J1(a)
```

**gsl\_sf\_besselJn**

Link to:

```
| gsl_sf_bessel_Jn(a, b)
```

**gsl\_sf\_besselY0**

Link to:

```
| gsl_sf_bessel_Y0(a)
```

**gsl\_sf\_besselY1**

Link to:

```
| gsl_sf_bessel_Y1(a)
```

**gsl\_sf\_besselYn**

Link to:

```
| gsl_sf_bessel_Yn(a, b)
```

**gsl\_sf\_besselI0**

Link to:

```
| gsl_sf_bessel_I0(a)
```

**gsl\_sf\_besselI1**

Link to:

```
| gsl_sf_bessel_I1(a)
```

### **gsl\_sf\_besselIn**

Link to:

```
| gsl_sf_bessel_In(a, b)
```

### **gsl\_sf\_besselI0scaled**

Link to:

```
| gsl_sf_bessel_I0_scaled(a)
```

### **gsl\_sf\_besselI1scaled**

Link to:

```
| gsl_sf_bessel_I1_scaled(a)
```

### **gsl\_sf\_besselInscaled**

Link to:

```
| gsl_sf_bessel_In_scaled(a, b)
```

### **gsl\_sf\_besselK0**

Link to:

```
| gsl_sf_bessel_K0(a)
```

### **gsl\_sf\_besselK1**

Link to:

```
| gsl_sf_bessel_K1(a)
```

### **gsl\_sf\_besselKn**

Link to:

```
| gsl_sf_bessel_Kn(a, b)
```

### **gsl\_sf\_besselK0scaled**

Link to:

```
| gsl_sf_bessel_K0_scaled(a)
```

**gsl\_sf\_besselK1scaled**

Link to:

```
| gsl_sf_bessel_K1_scaled(a)
```

**gsl\_sf\_besselKnscaled**

Link to:

```
| gsl_sf_bessel_Kn_scaled(a, b)
```

**gsl\_sf\_besselj0**

Link to:

```
| gsl_sf_bessel_j0(a)
```

**gsl\_sf\_besselj1**

Link to:

```
| gsl_sf_bessel_j1(a)
```

**gsl\_sf\_besselj2**

Link to:

```
| gsl_sf_bessel_j2(a)
```

**gsl\_sf\_besseljl**

Link to:

```
| gsl_sf_bessel_jl(a, b)
```

**gsl\_sf\_bessely0**

Link to:

```
| gsl_sf_bessel_y0(a)
```

**gsl\_sf\_bessely1**

Link to:

```
| gsl_sf_bessel_y1(a)
```

## **gsl\_sf\_bessely2**

Link to:

```
| gsl_sf_bessel_y0(a)
```

## **gsl\_sf\_bessely1**

Link to:

```
| gsl_sf_bessel_j1(a, b)
```

## **gsl\_sf\_bessel\_i0scaled**

Link to:

```
| gsl_sf_bessel_i0_scaled(a)
```

## **gsl\_sf\_bessel\_i1scaled**

Link to:

```
| gsl_sf_bessel_i1_scaled(a)
```

## **gsl\_sf\_bessel\_i2scaled**

Link to:

```
| gsl_sf_bessel_i2_scaled(a)
```

## **gsl\_sf\_bessel\_il\_scaled**

Link to:

```
| gsl_sf_bessel_il_scaled(a, b)
```

## **gsl\_sf\_bessel\_k0scaled**

Link to:

```
| gsl_sf_bessel_k0_scaled(a)
```

## **gsl\_sf\_bessel\_k1scaled**

Link to:

```
| gsl_sf_bessel_k1_scaled(a)
```

**gsl\_sf\_besselk2scaled**

Link to:

```
| gsl_sf_bessel_k2_scaled(a)
```

**gsl\_sf\_besselkliscaled**

Link to:

```
| gsl_sf_bessel_kl_scaled(a, b)
```

**gsl\_sf\_besselJnu**

Link to:

```
| gsl_sf_bessel_Jnu(a, b)
```

**gsl\_sf\_besselYnu**

Link to:

```
| gsl_sf_bessel_Ynu(a, b)
```

**gsl\_sf\_besselInuscaled**

Link to:

```
| gsl_sf_bessel_Inu_scaled(a, b)
```

**gsl\_sf\_besselInu**

Link to:

```
| gsl_sf_bessel_Inu(a, b)
```

**gsl\_sf\_besselKnuscaled**

Link to:

```
| gsl_sf_bessel_Knu_scaled(a, b)
```

**gsl\_sf\_besselKnu**

Link to:

```
| gsl_sf_bessel_Knu(a, b)
```

### **gsl\_sf\_besselInKnu**

Link to:

```
| gsl_sf_bessel_lnKnu(a, b)
```

### **gsl\_sf\_besselzeroJ0**

Link to:

```
| gsl_sf_bessel_zero_J0(a)
```

### **gsl\_sf\_besselzeroJ1**

Link to:

```
| gsl_sf_bessel_zero_J1(a)
```

### **gsl\_sf\_besselzeroJnu**

Link to:

```
| gsl_sf_bessel_zero_Jnu(a, b)
```

### **gsl\_sf\_clausen**

Link to:

```
| gsl_sf_clausen(a)
```

### **gsl\_sf\_hydrogenicR1**

Link to:

```
| gsl_sf_hydrogenicR_1(a, b)
```

### **gsl\_sf\_dawson**

Link to:

```
| gsl_sf_dawson(a)
```

### **gsl\_sf\_debye1**

Link to:

```
| gsl_sf_debye_1(a)
```

**gsl\_sf\_debye2**

Link to:

```
| gsl_sf_debye_2(a)
```

**gsl\_sf\_debye3**

Link to:

```
| gsl_sf_debye_3(a)
```

**gsl\_sf\_debye4**

Link to:

```
| gsl_sf_debye_4(a)
```

**gsl\_sf\_debye5**

Link to:

```
| gsl_sf_debye_5(a)
```

**gsl\_sf\_debye6**

Link to:

```
| gsl_sf_debye_6(a)
```

**gsl\_sf\_dilog**

Link to:

```
| gsl_sf_dilog(a)
```

**gsl\_sf\_multiply**

Link to:

```
| gsl_sf_multiply(a, b)
```

**gsl\_sf\_ellint\_Kcomp**

Link to:

```
| gsl_sf_ellint_Kcomp(a, b)
```

### **gslsfallintEcomp**

Link to:

```
| gsl_sf_ellint_Ecomp(a, b)
```

### **gslsfallintPcomp**

Link to:

```
| gsl_sf_ellint_Pcomp(a, b, c)
```

### **gslsfallintDcomp**

Link to:

```
| gsl_sf_ellint_Dcomp(a, b)
```

### **gslsfallintF**

Link to:

```
| gsl_sf_ellint_F(a, b, c)
```

### **gslsfallintE**

Link to:

```
| gsl_sf_ellint_E(a, b, c)
```

### **gslsfallintRC**

Link to:

```
| gsl_sf_ellint_RC(a, b, c)
```

### **gslsferfc**

Link to:

```
| gsl_sf_erfc(a)
```

### **gslsflogercf**

Link to:

```
| gsl_sf_log_erfc(a)
```

**gslsferf**

Link to:

```
| gsl_sf_erf(a)
```

**gslsferfZ**

Link to:

```
| gsl_sf_erf_Z(a)
```

**gslsferfQ**

Link to:

```
| gsl_sf_erf_Q(a)
```

**gslsfhazard**

Link to:

```
| gsl_sf_hazard(a)
```

**gslsfexp**

Link to:

```
| gsl_sf_exp(a)
```

**gslsfexpmult**

Link to:

```
| gsl_sf_exp_mult(a, b)
```

**gslsfexpm1**

Link to:

```
| gsl_sf_expm1(a)
```

**gslsfexprel**

Link to:

```
| gsl_sf_exprel(a)
```

## gsl\_sfexprel2

Link to:

```
| gsl_sf_exprel_2(a)
```

## gsl\_sfexpreln

Link to:

```
| gsl_sf_exprel_n(a, b)
```

## gsl\_sfexpintE1

Link to:

```
| gsl_sf_expint_E1(a)
```

## gsl\_sfexpintE2

Link to:

```
| gsl_sf_expint_E2(a)
```

## gsl\_sfexpintEn

Link to:

```
| gsl_sf_expint_En(a, b)
```

## gsl\_sfexpintE1scaled

Link to:

```
| gsl_sf_expint_E1_scaled(a)
```

## gsl\_sfexpintE2scaled

Link to:

```
| gsl_sf_expint_E2_scaled(a)
```

## gsl\_sfexpintEnscaled

Link to:

```
| gsl_sf_expint_En_scaled(a, b)
```

**gsl\_sfexpintEi**

Link to:

```
| gsl_sf_expint_Ei(a)
```

**gsl\_sfexpintEiscaled**

Link to:

```
| gsl_sf_expint_Ei_scaled(a)
```

**gsl\_sfShi**

Link to:

```
| gsl_sf_Shi(a)
```

**gsl\_sfChi**

Link to:

```
| gsl_sf_Chi(a)
```

**gsl\_sfexpint3**

Link to:

```
| gsl_sf_expint_3(a)
```

**gsl\_sfSi**

Link to:

```
| gsl_sf_Si(a)
```

**gsl\_sfCi**

Link to:

```
| gsl_sf_Ci(a)
```

**gsl\_sfatanint**

Link to:

```
| gsl_sf_atanint(a)
```

### **gslsffermidiracm1**

Link to:

```
| gsl_sf_fermi_dirac_m1(a)
```

### **gslsffermidirac0**

Link to:

```
| gsl_sf_fermi_dirac_0(a)
```

### **gslsffermidirac1**

Link to:

```
| gsl_sf_fermi_dirac_1(a)
```

### **gslsffermidirac2**

Link to:

```
| gsl_sf_fermi_dirac_2(a)
```

### **gslsffermidiracint**

Link to:

```
| gsl_sf_fermi_dirac_int(a, b)
```

### **gslsffermidiracmhalf**

Link to:

```
| gsl_sf_fermi_dirac_mhalf(a)
```

### **gslsffermidirachalf**

Link to:

```
| gsl_sf_fermi_dirac_half(a)
```

### **gslsffermidirac3half**

Link to:

```
| gsl_sf_fermi_dirac_3half(a)
```

**gsl\_sf\_fermi\_dirac\_inc\_0**

Link to:

```
| gsl_sf_fermi_dirac_inc_0(a, b)
```

**gsl\_sf\_lngamma**

Link to:

```
| gsl_sf_lngamma(a)
```

**gsl\_sf\_gamma**

Link to:

```
| gsl_sf_gamma(a)
```

**gsl\_sf\_gammastar**

Link to:

```
| gsl_sf_gammastar(a)
```

**gsl\_sf\_gammainv**

Link to:

```
| gsl_sf_gammainv(a)
```

**gsl\_sf\_taylorcoeff**

Link to:

```
| gsl_sf_taylorcoeff(a, b)
```

**gsl\_sf\_fact**

Link to:

```
| gsl_sf_fact(a)
```

**gsl\_sf\_doublefact**

Link to:

```
| gsl_sf_doublefact(a)
```

### **gsl\_sf\_lnfact**

Link to:

```
| gsl_sf_lnfact(a)
```

### **gsl\_sf\_ldoublefact**

Link to:

```
| gsl_sf_ldoublefact(a)
```

### **gsl\_sf\_lnchoose**

Link to:

```
| gsl_sf_lnchoose(a, b)
```

### **gsl\_sf\_choose**

Link to:

```
| gsl_sf_choose(a, b)
```

### **gsl\_sf\_lnpoch**

Link to:

```
| gsl_sf_lnpoch(a, b)
```

### **gsl\_sf\_poch**

Link to:

```
| gsl_sf_poch(a, b)
```

### **gsl\_sf\_pochrel**

Link to:

```
| gsl_sf_pochrel(a, b)
```

### **gsl\_sf\_gammaincQ**

Link to:

```
| gsl_sf_gamma_inc_Q(a, b)
```

**gsl\_sf\_gammaincP**

Link to:

```
| gsl_sf_gamma_inc_P(a, b)
```

**gsl\_sf\_gammainc**

Link to:

```
| gsl_sf_gamma_inc(a, b)
```

**gsl\_sf\_lnbeta**

Link to:

```
| gsl_sf_lnbeta(a, b)
```

**gsl\_sf\_beta**

Link to:

```
| gsl_sf_beta(a, b)
```

**gsl\_sf\_betainc**

Link to:

```
| gsl_sf_betainc(a, b, c)
```

**gsl\_sf\_gegenpoly1**

Link to:

```
| gsl_sf_gegenpoly_1(a, b)
```

**gsl\_sf\_gegenpoly2**

Link to:

```
| gsl_sf_gegenpoly_2(a, b)
```

**gsl\_sf\_gegenpoly3**

Link to:

```
| gsl_sf_gegenpoly_3(a, b)
```

### gsl\_sf\_egenpoly

Link to:

```
| gsl_sf_egenpoly_n(a, b, c)
```

### gsl\_sf\_hyperg0F1

Link to:

```
| gsl_sf_hyperg_0F1(a, b)
```

### gsl\_sf\_hyperg1F1int

Link to:

```
| gsl_sf_hyperg_1F1_inc(a, b, c)
```

### gsl\_sf\_hyperg1F1

Link to:

```
| gsl_sf_hyperg_1F1(a, b, c)
```

### gsl\_sf\_hypergUint

Link to:

```
| gsl_sf_hyperg_U_inc(a, b, c)
```

### gsl\_sf\_hypergU

Link to:

```
| gsl_sf_hyperg_U(a, b, c)
```

### gsl\_sf\_hyperg2F0

Link to:

```
| gsl_sf_hyperg_U_2F0(a, b, c)
```

### gsl\_sf\_laguerre1

Link to:

```
| gsl_sf_laguerre_1(a, b)
```

**gsl\_sf\_laguerre2**

Link to:

```
| gsl_sf_laguerre_2(a, b)
```

**gsl\_sf\_laguerre3**

Link to:

```
| gsl_sf_laguerre_3(a, b)
```

**gsl\_sf\_laguerre\_n**

Link to:

```
| gsl_sf_laguerre_n(a, b, c)
```

**gsl\_sf\_lambertW0**

Link to:

```
| gsl_sf_lambert_W0(a)
```

**gsl\_sf\_lambertWm1**

Link to:

```
| gsl_sf_lambert_Wm1(a)
```

**gsl\_sf\_legendreP1**

Link to:

```
| gsl_sf_legendre_P1(a, b)
```

**gsl\_sf\_legendreP1**

Link to:

```
| gsl_sf_legendre_P1(a)
```

**gsl\_sf\_legendreP2**

Link to:

```
| gsl_sf_legendre_P2(a)
```

### **gsl\_sf\_legendreP3**

Link to:

```
| gsl_sf_legendre_P3(a)
```

### **gsl\_sf\_legendreQ0**

Link to:

```
| gsl_sf_legendre_Q0(a)
```

### **gsl\_sf\_legendreQ1**

Link to:

```
| gsl_sf_legendre_Q1(a)
```

### **gsl\_sf\_legendreQl**

Link to:

```
| gsl_sf_legendre_Ql(a, b)
```

### **gsl\_sf\_legendrePlm**

Link to:

```
| gsl_sf_legendre_Plm(a, b, c)
```

### **gsl\_sf\_legendresphPlm**

Link to:

```
| gsl_sf_legendre_sphPlm(a, b, c)
```

### **gsl\_sf\_legendrearray\_size**

Link to:

```
| gsl_sf_legendre_array_size(a, b)
```

### **gsl\_sf\_conicalPhalf**

Link to:

```
| gsl_sf_conicalP_half(a, b)
```

**gsl\_sf\_conicalPmhalf**

Link to:

```
| gsl_sf_conicalP_mhalf(a, b)
```

**gsl\_sf\_conicalP0**

Link to:

```
| gsl_sf_conicalP_0(a, b)
```

**gsl\_sf\_conicalP1**

Link to:

```
| gsl_sf_conicalP_1(a, b)
```

**gsl\_sf\_conicalPsphreg**

Link to:

```
| gsl_sf_conicalP_sph_reg(a, b, c)
```

**gsl\_sf\_conicalPcylreg**

Link to:

```
| gsl_sf_conicalP_cyl_reg(a, b, c)
```

**gsl\_sf\_legendreH3d0**

Link to:

```
| gsl_sf_legendre_H3d_0(a, b)
```

**gsl\_sf\_legendreH3d1**

Link to:

```
| gsl_sf_legendre_H3d_1(a, b)
```

**gsl\_sf\_legendreH3d**

Link to:

```
| gsl_sf_legendre_H3d(a, b, c)
```

### **gsl\_sf\_log**

Link to:

```
| gsl_sf_log(a)
```

### **gsl\_sf\_logabs**

Link to:

```
| gsl_sf_log_abs(a)
```

### **gsl\_sf\_log1plusx**

Link to:

```
| gsl_sf_log_1plusx(a)
```

### **gsl\_sf\_log1plusx\_mx**

Link to:

```
| gsl_sf_log_1plusx_mx(a)
```

### **gsl\_sf\_powint**

Link to:

```
| gsl_sf_pow_int(a, b)
```

### **gsl\_sf\_psiint**

Link to:

```
| gsl_sf_psi_int(a)
```

### **gsl\_sf\_psi**

Link to:

```
| gsl_sf_psi(a)
```

### **gsl\_sf\_psi1piy**

Link to:

```
| gsl_sf_psi_1piy(a)
```

**gslsfpsi1int**

Link to:

```
| gsl_sf_psi_1_int(a)
```

**gslsfpsi1**

Link to:

```
| gsl_sf_psi_1(a)
```

**gslsfpsin**

Link to:

```
| gsl_sf_psi_n(a, b)
```

**gslsfsynchrotron1**

Link to:

```
| gsl_sf_synchrotron_1(a)
```

**gslsfsynchrotron2**

Link to:

```
| gsl_sf_synchrotron_2(a)
```

**gslsftransport2**

Link to:

```
| gsl_sf_transport_2(a)
```

**gslsftransport3**

Link to:

```
| gsl_sf_transport_3(a)
```

**gslsftransport4**

Link to:

```
| gsl_sf_transport_4(a)
```

## gsl\_sftransport5

Link to:

```
| gsl_sf_transport_5(a)
```

## gslsfsin

Link to:

```
| gsl_sf_sin(a)
```

## gslfcos

Link to:

```
| gsl_sf_cos(a)
```

## gslfhypot

Link to:

```
| gsl_sf_hypot(a, b)
```

## gslfsinc

Link to:

```
| gsl_sf_sinc(a)
```

## gslflnsinh

Link to:

```
| gsl_sf_lnsinh(a)
```

## gslflncosh

Link to:

```
| gsl_sf_lncosh(a)
```

## gslfanglerestrictsymm

Link to:

```
| gsl_sf_andle_restrict_symm(a)
```

**gsl\_sf\_angle\_restrict\_pos**

Link to:

```
| gsl_sf_angle_restrict_pos(a)
```

**gsl\_sf\_zetaint**

Link to:

```
| gsl_sf_zeta_int(a)
```

**gsl\_sf\_zeta**

Link to:

```
| gsl_sf_zeta(a)
```

**gsl\_sf\_zetam1**

Link to:

```
| gsl_sf_zetam1(a)
```

**gsl\_sf\_zetam1int**

Link to:

```
| gsl_sf_zetam1_int(a)
```

**gsl\_sf\_hzeta**

Link to:

```
| gsl_sf_hzeta(a, b)
```

**gsl\_sf\_etaint**

Link to:

```
| gsl_sf_eta_int(a)
```

**gsl\_sf\_eta**

Link to:

```
| gsl_sf_eta(a)
```

### 4.8.15 ff-lpopt

Refer to the [Ipopt documentation](#) for more informations.

#### IPOPT

---

**Todo:** todo

---

### 4.8.16 fflapack

Refer to the [LAPACK documentation](#) for more informations.

#### inv

---

**Todo:** todo

---

#### dgeev

---

**Todo:** todo

---

#### zgeev

---

**Todo:** todo

---

#### geev

---

**Todo:** todo

---

#### dggev

---

**Todo:** todo

---

#### dggev

---

**Todo:** todo

---

**zggev**

---

**Todo:** todo

---

**dsygvd**

---

**Todo:** todo

---

**dgesdd**

---

**Todo:** todo

---

**zhgev**

---

**Todo:** todo

---

**dsyev**

---

**Todo:** todo

---

**zheev**

---

**Todo:** todo

---

#### 4.8.17 ff-mmap-semaphore

**Wait**

---

**Todo:** todo

---

**trywait**

---

**Todo:** todo

---

**Post**

---

**Todo:** todo

---

**msync**

---

**Todo:** todo

---

**Read**

---

**Todo:** todo

---

**Write**

---

**Todo:** todo

---

### 4.8.18 ffnewuoa

**newuoa**

---

**Todo:** todo

---

### 4.8.19 ff-NLopt

Refer to the [NLOPT documentation](#) for more informations.

**nloptDIRECT**

---

**Todo:** todo

---

**nloptDIRECTL**

---

**Todo:** todo

---

### nloptDIRECTRand

---

**Todo:** todo

---

### nloptDIRECTScal

---

**Todo:** todo

---

### nloptDIRECTNoScal

---

**Todo:** todo

---

### nloptDIRECTLNoScal

---

**Todo:** todo

---

### nloptDIRECTRandNoScal

---

**Todo:** todo

---

### nloptOrigDIRECT

---

**Todo:** todo

---

### nloptOrigDIRECTL

---

**Todo:** todo

---

### nloptStoGO

---

**Todo:** todo

---

### nloptStoGORand

---

**Todo:** todo

---

### nloptLBFGS

---

**Todo:** todo

---

### nloptPRAXIS

---

**Todo:** todo

---

### nloptVar1

---

**Todo:** todo

---

### nloptVar2

---

**Todo:** todo

---

### nloptTNewton

---

**Todo:** todo

---

### nloptTNewtonRestart

---

**Todo:** todo

---

### nloptTNewtonPrecond

---

**Todo:** todo

---

**nloptNewtonPrecondRestart**

---

**Todo:** todo

---

**nloptCRS2**

---

**Todo:** todo

---

**nloptMMA**

---

**Todo:** todo

---

**nloptCOBYLA**

---

**Todo:** todo

---

**nloptNEWUOA**

---

**Todo:** todo

---

**nloptNEWUOABound**

---

**Todo:** todo

---

**nloptNelderMead**

---

**Todo:** todo

---

**nloptSbpIx**

---

**Todo:** todo

---

## nloptBOBYQA

---

**Todo:** todo

---

## nloptSRES

---

**Todo:** todo

---

## nloptSLSQP

---

**Todo:** todo

---

## nloptMLSL

---

**Todo:** todo

---

## nloptMLSLLDS

---

**Todo:** todo

---

## nloptAUGLAG

---

**Todo:** todo

---

## nloptAUGLAGEQ

---

**Todo:** todo

---

## 4.8.20 ffrandom

### srandomdev

---

**Todo:** todo

---

**srandom**

---

**Todo:** todo

---

**random**

---

**Todo:** todo

---

## 4.8.21 FreeFemQA

**MeshGenQA**

---

**Todo:** todo

---

## 4.8.22 freeyams

**freeyams**

---

**Todo:** todo

---

## 4.8.23 gmsh

Need

```
1 load "gsmh"
```

The gmsh software is available [here](#)

**gmshload**

Load a 2D mesh build with Gmsh.

```
1 mesh Th = gmshload(MeshFile, [reftri=RefTri], [renum=Renum]);
```

Parameters:

- MeshFile (string) Mesh file name
- reftri=(.. todo:: todo)
- renum=(.. todo:: todo)

Output:

- Th (mesh)

## gmshload3

Load a 3D mesh build with Gmsh.

```
mesh3 Th = gmshload(MeshFile, [reftet=RefTet], [renum=Renum]);
```

Parameters:

- MeshFile (string) Mesh file name
- reftet=(.. todo:: todo)
- renum=(.. todo:: todo)

Output:

- Th (mesh3)

## savegmsh

---

**Todo:** todo

---

## 4.8.24 gsl

### gslpolysolvequadratic

---

**Todo:** todo

---

### gslpolysolvecubic

---

**Todo:** todo

---

### gslpolycomplexsolve

---

**Todo:** todo

---

### gslrnguniform

---

**Todo:** todo

---

**gslrnguniformpos**

---

**Todo:** todo

---

**gslname**

---

**Todo:** todo

---

**gslrngget**

---

**Todo:** todo

---

**gslrngmin**

---

**Todo:** todo

---

**gslrngmax**

---

**Todo:** todo

---

**gslrngset**

---

**Todo:** todo

---

**gslrngtype**

---

**Todo:** todo

---

**4.8.25 ilut****applyIlutPrecond**

---

**Todo:** todo

---

**makellutPrecond**

---

**Todo:** todo

---

**4.8.26 iohdf5****savehdf5sol**

---

**Todo:** todo

---

**4.8.27 iovtk****savevtk**

Save mesh or solution in vtk/vtu format.

```
1 savevtk(FileName, Th, [Ux, Uy, Uz], p, [dataname=DataManager],  
2   ↪ [withsurfacemesh=WithSurfaceMesh], [order=Order], [floatmesh=FloatMesh],  
3   ↪ [floatsol=FloatSol], [bin=Bin], [swap=Swap]);
```

Parameters:

- `FileName` (string) File name: \*.vtk or \*.vtu
- `Th` (mesh or mesh3)
- `[Ux, Uy, Uz]`, `p` (fespace function of vector of fespace functions) Solutions to save, as much as wanted
- `dataname=`(string) Name of solutions, separated by a space
- `withsurfacemesh=`(bool) .. todo:: todo
- `order=`(int[int]) Order of solutions.

Available: 0 or 1

- `floatmesh=`(bool) .. todo:: todo
- `floatsol=`(bool) .. todo:: todo
- `bin=`(bool) If true, save file in binary format
- `swap`(bool) .. todo:: todo

Output:

- None

**vtkload**

---

**Todo:** todo

---

## vtkload3

---

**Todo:** todo

---

## 4.8.28 isoline

Need

```
| load "isoline"
```

### isoline

```
| int N = isoline(Th, u, xy, iso=Iso, close=Close, smoothing=Smoothing, ratio=Ratio,_
| ↵eps=Eps, beginend=BeginEnd, file=File);
```

---

**Todo:** todo

---

### Curve

---

**Todo:** todo

---

### Area

---

**Todo:** todo

---

### findallocalmin

---

**Todo:** todo

---

## 4.8.29 lapack

### inv

---

**Todo:** todo

---

**dgeev**

---

**Todo:** todo

---

**zgeev**

---

**Todo:** todo

---

**geev**

---

**Todo:** todo

---

**dggev**

---

**Todo:** todo

---

**zggev**

---

**Todo:** todo

---

**dsygvd**

---

**Todo:** todo

---

**dgesdd**

---

**Todo:** todo

---

**zhegv**

---

**Todo:** todo

---

**dsyev**

---

**Todo:** todo

---

**zheev**

---

**Todo:** todo

---

**dgelsy**

---

**Todo:** todo

---

#### 4.8.30 lgbmo

**bmo**

---

**Todo:** todo

---

#### 4.8.31 mat\_dervieux

**MatUpWind1**

---

**Todo:** todo

---

#### 4.8.32 mat\_psi

**MatUpWind0**

---

**Todo:** todo

---

#### 4.8.33 medit

**medit**

---

**Todo:** todo

---

**savesol**

---

**Todo:** todo

---

**readsol**

---

**Todo:** todo

---

#### 4.8.34 metis

**metisnodal**

---

**Todo:** todo

---

**metisdual**

---

**Todo:** todo

---

#### 4.8.35 MetricKuate

**MetricKuate**

---

**Todo:** todo

---

#### 4.8.36 MetricPk

**MetricPk**

---

**Todo:** todo

---

#### 4.8.37 mmg3d

**mmg3d**

---

**Todo:** todo

---

#### 4.8.38 mmg3d-v4.0

**mmg3d**

---

**Todo:** todo

---

#### 4.8.39 msh3

**change**

---

**Todo:** todo

---

#### movemesh23

---

**Todo:** todo

---

#### movemesh2D3Dsurf

---

**Todo:** todo

---

#### movemesh3

---

**Todo:** todo

---

#### movemesh

---

**Todo:** todo

---

#### movemesh3D

---

**Todo:** todo

---

**displacement**

---

**Todo:** todo

---

**checkbemesh**

---

**Todo:** todo

---

**buildlayers**

---

**Todo:** todo

---

**bcube**

---

**Todo:** todo

---

**cube**

Construct a cubic mesh.

```
mesh3 Th = cube(nnX, nnY, nnZ, [X(x), Y(y), Z(z)], [label=Label], [flags=Flags],  
→ [region=Region]);
```

Parameters:

- nnX (int) Number of discretization point along  $x$
- nnY (int) Number of discretization point along  $y$
- nnZ (int) Number of discretization point along  $z$
- X (x) (func) [Optional] Affine function of  $x$  to define the length Default:  $x$
- Y (y) (func) [Optional] Affine function of  $y$  to define the width Default:  $y$
- Z (z) (func) [Optional] Affine function of  $z$  to define the height Default:  $z$
- label=(int [int]) [Optional]

List of surface labels Default: [1, 2, 3, 4, 5, 6]

- flags=(int) [Optional]

Refer to [square](#)

- region=(int) [Optional]

Region number of the cube volume Default: 0

Output:

- Th (mesh3) Cube mesh

**trunc**

---

**Todo:** todo

---

**gluemesh**

---

**Todo:** todo

---

**extract**

---

**Todo:** todo

---

**showborder**

---

**Todo:** todo

---

**getborder**

---

**Todo:** todo

---

**AddLayers**

---

**Todo:** todo

---

#### 4.8.40 mshmet

**mshmet**

---

**Todo:** todo

---

#### 4.8.41 MUMPS

**defaulttoMUMPSseq**

---

**Todo:** todo

---

#### 4.8.42 MUMPS\_seq

**defaulttoMUMPSseq**

---

**Todo:** todo

---

#### 4.8.43 netgen

**netg**

---

**Todo:** todo

---

**netgstl**

---

**Todo:** todo

---

**netgload**

---

**Todo:** todo

---

#### 4.8.44 NewSolver

**defaulttoUMFPACK**

---

**Todo:** todo

---

#### 4.8.45 PARDISO

**defaulttoPARDISO**

---

**Todo:** todo

---

**ompsetnumthreads**

---

**Todo:** todo

---

**ompgetnumthreads**

---

**Todo:** todo

---

**ompgetmaxthreads**

---

**Todo:** todo

---

#### 4.8.46 pcm2rnm

**readpcm**

---

**Todo:** todo

---

#### 4.8.47 pipe

**flush**

---

**Todo:** todo

---

**sleep**

---

**Todo:** todo

---

**usleep**

---

**Todo:** todo

---

#### 4.8.48 qf11to25

**QF1d**

---

**Todo:** todo

---

**QF2d**

---

**Todo:** todo

---

**QF3d**

---

**Todo:** todo

---

**tripleQF**

#### 4.8.49 scotch

**scotch**

---

**Todo:** todo

---

#### 4.8.50 shell

**readdir**

---

**Todo:** todo

---

**unlink**

---

**Todo:** todo

---

**rmdir**

---

**Todo:** todo

---

**cddir**

---

**Todo:** todo

---

**chdir**

---

**Todo:** todo

---

**basename**

---

**Todo:** todo

---

**dirname**

---

**Todo:** todo

---

**mkdir**

---

**Todo:** todo

---

**chmod**

---

**Todo:** todo

---

**cpfile**

---

**Todo:** todo

---

**stat**

---

**Todo:** todo

---

**isdir**

---

**Todo:** todo

---

**getenv**

---

**Todo:** todo

---

**setenv**

---

**Todo:** todo

---

**unsetenv**

---

**Todo:** todo

---

#### 4.8.51 **splitedges**

**SplitedgeMesh**

---

**Todo:** todo

---

#### 4.8.52 **splitmesh12**

**splitmesh12**

---

**Todo:** todo

---

#### 4.8.53 splitmesh3

**splitmesh3**

---

**Todo:** todo

---

#### 4.8.54 splitmesh4

**splimesh4**

---

**Todo:** todo

---

#### 4.8.55 splitmesh6

**splitmesh6**

---

**Todo:** todo

---

#### 4.8.56 SuperLu

**defaulttoSuperLu**

---

**Todo:** todo

---

#### 4.8.57 symmetrizeCSR

**symmetrizeCSR**

---

**Todo:** todo

---

#### 4.8.58 tetgen

Refer to the [Tetgen](#) documentation for more informations.

## tetgconvexhull

---

**Todo:** todo

---

## tetgtransfo

---

**Todo:** todo

---

## tetg

Build a 3D mesh from a surface.

```
mesh3 Th = tetg(Th0, [reftet=RefTet], [label=Label], [switch=Switch],  
  ↪[nbofholes=NbOfHoles], [holelist=HoleList], [nbofregions=NbOfRegions],  
  ↪[regionlist=RegionList], [nboffacetcl=NbOfFaceTcl], [facetcl=FaceTcl])
```

---

**Todo:** todo

---

## tetgreconstruction

---

**Todo:** todo

---

## 4.8.59 UMFPACK64

### defaulttoUMFPACK64

---

**Todo:** todo

---

## 4.8.60 VTK\_writer\_3d

### Vtkaddmesh

---

**Todo:** todo

---

Vtkaddscalar

---

**Todo:** todo

---

#### 4.8.61 VTK\_writer

Vtkaddmesh

---

**Todo:** todo

---

Vtkaddscalar



## MATHEMATICAL MODELS

*Summary:*

This chapter goes deeper into a number of problems that **FreeFEM** can solve. It is a complement to the *Tutorial part* which was only an introduction.

**Users are invited to contribute to make this models database grow.**

### 5.1 Static problems

#### 5.1.1 Soap Film

Our starting point here will be the mathematical model to find the shape of **soap film** which is glued to the ring on the  $xy$ -plane:

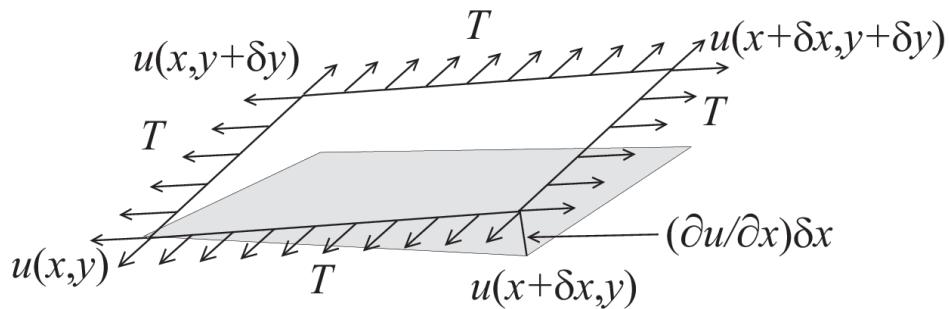
$$C = \{(x, y); x = \cos t, y = \sin t, 0 \leq t \leq 2\pi\}$$

We assume the shape of the film is described by the graph  $(x, y, u(x, y))$  of the vertical displacement  $u(x, y)$  ( $x^2 + y^2 < 1$ ) under a vertical pressure  $p$  in terms of force per unit area and an initial tension  $\mu$  in terms of force per unit length.

Consider the “small plane” ABCD, A:  $(x, y, u(x, y))$ , B:  $(x, y, u(x + \delta x, y))$ , C:  $(x, y, u(x + \delta x, y + \delta y))$  and D:  $(x, y, u(x, y + \delta y))$ .

Denote by  $\vec{n}(x, y) = (n_x(x, y), n_y(x, y), n_z(x, y))$  the normal vector of the surface  $z = u(x, y)$ . We see that the vertical force due to the tension  $\mu$  acting along the edge AD is  $-\mu n_x(x, y)\delta y$  and the vertical force acting along the edge AD is:

$$\mu n_x(x + \delta x, y)\delta y \simeq \mu \left( n_x(x, y) + \frac{\partial n_x}{\partial x} \delta x \right) (x, y)\delta y$$



Similarly, for the edges AB and DC we have:

$$-\mu n_y(x, y)\delta x, \quad \mu(n_y(x, y) + \partial n_y/\partial y)(x, y)\delta x$$

The force in the vertical direction on the surface ABCD due to the tension  $\mu$  is given by:

$$\mu(\partial n_x/\partial x)\delta x\delta y + T(\partial n_y/\partial y)\delta y\delta x$$

Assuming small displacements, we have:

$$\begin{aligned} \nu_x &= (\partial u/\partial x)/\sqrt{1+(\partial u/\partial x)^2+(\partial u/\partial y)^2} \simeq \partial u/\partial x, \\ \nu_y &= (\partial u/\partial y)/\sqrt{1+(\partial u/\partial x)^2+(\partial u/\partial y)^2} \simeq \partial u/\partial y \end{aligned}$$

Letting  $\delta x \rightarrow dx$ ,  $\delta y \rightarrow dy$ , we have the equilibrium of the vertical displacement of soap film on ABCD by  $p$ :

$$\mu dxdy\partial^2 u/\partial x^2 + \mu dxdy\partial^2 u/\partial y^2 + pdxdy = 0$$

Using the Laplace operator  $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ , we can find the virtual displacement write the following:

$$-\Delta u = f \quad \text{in } \Omega$$

where  $f = p/\mu$ ,  $\Omega = \{(x, y); x^2 + y^2 < 1\}$ .

*Poisson's equation* appears also in **electrostatics** taking the form of  $f = \rho/\epsilon$  where  $\rho$  is the charge density,  $\epsilon$  the dielectric constant and  $u$  is named as electrostatic potential.

The soap film is glued to the ring  $\partial\Omega = C$ , then we have the boundary condition:

$$u = 0 \quad \text{on } \partial\Omega$$

If the force is gravity, for simplify, we assume that  $f = -1$ .

```

1 // Parameters
2 int nn = 50;
3 func f = -1;
4 func ue = (x^2+y^2-1)/4; //ue: exact solution
5
6 // Mesh
7 border a(t=0, 2*pi) {x=cos(t); y=sin(t); label=1;};
8 mesh disk = buildmesh(a(nn));
9 plot(disk);
10
11 // Fespace
12 fespace fempl(disk, P1);
13 fempl u, v;
14
15 // Problem
16 problem laplace (u, v)
17   = int2d(disk) ( //bilinear form
18     dx(u)*dx(v)
19     + dy(u)*dy(v)
20   )
21   - int2d(disk) ( //linear form
22     f*v
23   )
24   + on(1, u=0) //boundary condition
25 ;

```

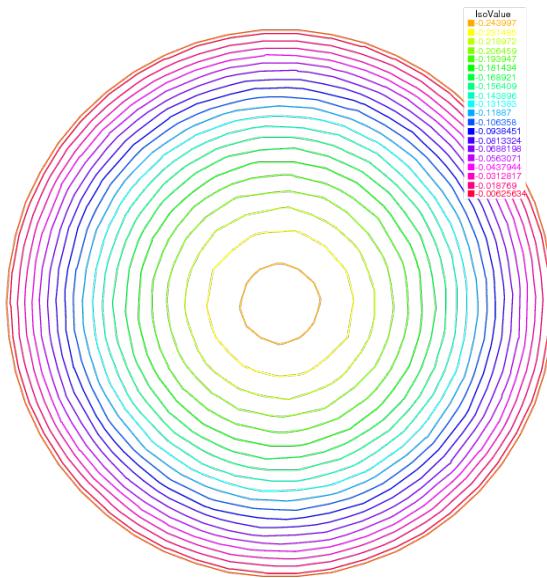
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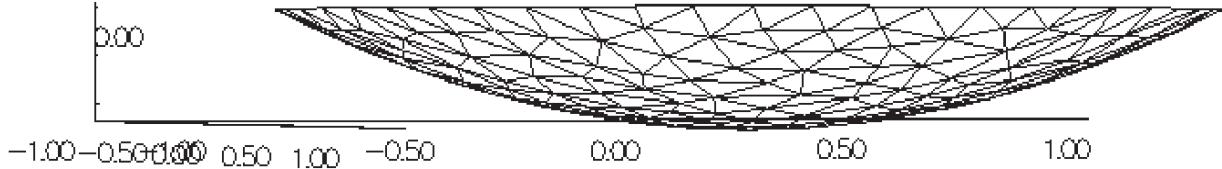
(continued from previous page)

```

27 // Solve
28 laplace;
29
30 // Plot
31 plot (u, value=true, wait=true);
32
33 // Error
34 femp1 err = u - ue;
35 plot(err, value=true, wait=true);
36
37 cout << "error L2 = " << sqrt( int2d(disk)(err^2) )<< endl;
38 cout << "error H10 = " << sqrt( int2d(disk)((dx(u)-x/2)^2) + int2d(disk)((dy(u)-y/2)^
39 << 2) )<< endl;
40
41 /// Re-run with a mesh adaptation ///
42
43 // Mesh adaptation
44 disk = adaptmesh(disk, u, err=0.01);
45 plot(disk, wait=true);
46
47 // Solve
48 laplace;
49 plot (u, value=true, wait=true);
50
51 // Error
52 err = u - ue; //become FE-function on adapted mesh
53 plot(err, value=true, wait=true);
54
55 cout << "error L2 = " << sqrt( int2d(disk)(err^2) )<< endl;
56 cout << "error H10 = " << sqrt( int2d(disk)((dx(u)-x/2)^2) + int2d(disk)((dy(u)-y/2)^
57 << 2) )<< endl;

```

**Fig. 5.1:** Isovalue of  $u$

**Fig. 5.2:** A side view of  $u$ 

In the 37th line, the  $L^2$ -error estimation between the exact solution  $u_e$ ,

$$\|u_h - u_e\|_{0,\Omega} = \left( \int_{\Omega} |u_h - u_e|^2 \, dx dy \right)^{1/2}$$

and in the following line, the  $H^1$ -error seminorm estimation:

$$|u_h - u_e|_{1,\Omega} = \left( \int_{\Omega} |\nabla u_h - \nabla u_e|^2 \, dx dy \right)^{1/2}$$

are done on the initial mesh. The results are  $\|u_h - u_e\|_{0,\Omega} = 0.000384045$ ,  $|u_h - u_e|_{1,\Omega} = 0.0375506$ .

After the adaptation, we have  $\|u_h - u_e\|_{0,\Omega} = 0.000109043$ ,  $|u_h - u_e|_{1,\Omega} = 0.0188411$ . So the numerical solution is improved by adaptation of mesh.

### 5.1.2 Electrostatics

We assume that there is no current and a time independent charge distribution. Then the electric field  $\mathbf{E}$  satisfies:

$$\begin{aligned} \operatorname{div} \mathbf{E} &= \rho/\epsilon \\ \operatorname{curl} \mathbf{E} &= 0 \end{aligned} \tag{5.1}$$

where  $\rho$  is the charge density and  $\epsilon$  is called the permittivity of free space.

From the equation (5.1) We can introduce the electrostatic potential such that  $\mathbf{E} = -\nabla\phi$ . Then we have Poisson's equation  $-\Delta\phi = f$ ,  $f = -\rho/\epsilon$ .

We now obtain the equipotential line which is the level curve of  $\phi$ , when there are no charges except conductors  $\{C_i\}_{1,\dots,K}$ . Let us assume  $K$  conductors  $C_1, \dots, C_K$  within an enclosure  $C_0$ .

Each one is held at an electrostatic potential  $\varphi_i$ . We assume that the enclosure  $C_0$  is held at potential 0. In order to know  $\varphi(x)$  at any point  $x$  of the domain  $\Omega$ , we must solve:

$$-\Delta\varphi = 0 \quad \text{in } \Omega$$

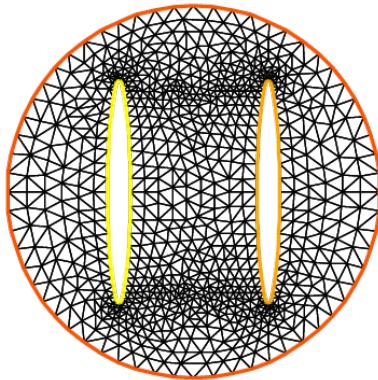
where  $\Omega$  is the interior of  $C_0$  minus the conductors  $C_i$ , and  $\Gamma$  is the boundary of  $\Omega$ , that is  $\sum_{i=0}^N C_i$ .

Here  $g$  is any function of  $x$  equal to  $\varphi_i$  on  $C_i$  and to 0 on  $C_0$ . The boundary equation is a reduced form for:

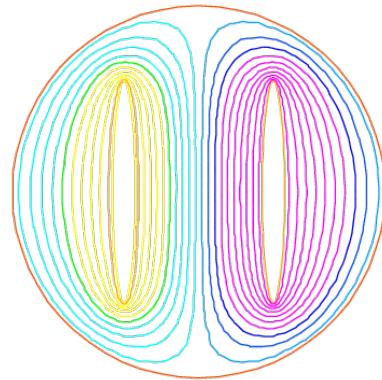
$$\varphi = \varphi_i \text{ on } C_i, \quad i = 1 \dots N, \quad \varphi = 0 \text{ on } C_0.$$

First we give the geometrical informations;  $C_0 = \{(x, y); x^2 + y^2 = 5^2\}$ ,  $C_1 = \{(x, y) : \frac{1}{0.3^2}(x-2)^2 + \frac{1}{3^2}y^2 = 1\}$ ,  $C_2 = \{(x, y) : \frac{1}{0.3^2}(x+2)^2 + \frac{1}{3^2}y^2 = 1\}$ .

Let  $\Omega$  be the disk enclosed by  $C_0$  with the elliptical holes enclosed by  $C_1$  and  $C_2$ . Note that  $C_0$  is described counterclockwise, whereas the elliptical holes are described clockwise, because the boundary must be oriented so that the computational domain is to its left.



(a) Disk with two elliptical holes

(b) Equipotential lines where  $C_1$  is located in right hand side

```

1 // Mesh
2 border C0(t=0, 2*pi){x=5*cos(t); y=5*sin(t);}
3 border C1(t=0, 2*pi){x=2+0.3*cos(t); y=3*sin(t);}
4 border C2(t=0, 2*pi){x=-2+0.3*cos(t); y=3*sin(t);}

5
6 mesh Th = buildmesh(C0(60) + C1(-50) + C2(-50));
7 plot(Th);

8
9 // Fespace
10 fespace Vh(Th, P1);
11 Vh uh, vh;

12
13 // Problem
14 problem Electro(uh, vh)
15   = int2d(Th)( //bilinear
16     dx(uh)*dx(vh)
17     + dy(uh)*dy(vh)
18   )
19   + on(C0, uh=0) //boundary condition on C_0
20   + on(C1, uh=1) //+1 volt on C_1
21   + on(C2, uh=-1) // -1 volt on C_2
22 ;
23
24 // Solve
25 Electro;
26 plot(uh);

```

### 5.1.3 Aerodynamics

Let us consider a wing profile  $S$  in a uniform flow. Infinity will be represented by a large circle  $\Gamma_\infty$ . As previously, we must solve:

$$\Delta\varphi = 0 \quad \text{in } \Omega, \quad \varphi|_S = c, \quad \varphi|_{\Gamma_\infty} = u_\infty 1_x - u_\infty 2_x \quad (5.2)$$

where  $\Omega$  is the area occupied by the fluid,  $u_\infty$  is the air speed at infinity,  $c$  is a constant to be determined so that  $\partial_n \varphi$  is continuous at the trailing edge  $P$  of  $S$  (so-called Kutta-Joukowski condition). Lift is proportional to  $c$ .

To find  $c$  we use a superposition method. As all equations in (5.2) are linear, the solution  $\varphi_c$  is a linear function of  $c$

$$\varphi_c = \varphi_0 + c\varphi_1$$

where  $\varphi_0$  is a solution of (5.2) with  $c = 0$  and  $\varphi_1$  is a solution with  $c = 1$  and zero speed at infinity.

With these two fields computed, we shall determine  $c$  by requiring the continuity of  $\partial\varphi/\partial n$  at the trailing edge. An equation for the upper surface of a NACA0012 (this is a classical wing profile in aerodynamics; the rear of the wing is called the trailing edge) is:

$$y = 0.17735\sqrt{x} - 0.075597x - 0.212836x^2 + 0.17363x^3 - 0.06254x^4$$

Taking an incidence angle  $\alpha$  such that  $\tan \alpha = 0.1$ , we must solve:

$$-\Delta\varphi = 0 \quad \text{in } \Omega, \quad \varphi|_{\Gamma_1} = y - 0.1x, \quad \varphi|_{\Gamma_2} = c$$

where  $\Gamma_2$  is the wing profile and  $\Gamma_1$  is an approximation of infinity. One finds  $c$  by solving:

$$\begin{aligned} -\Delta\varphi_0 &= 0 \quad \text{in } \Omega, \quad \varphi_0|_{\Gamma_1} = y - 0.1x, \quad \varphi_0|_{\Gamma_2} = 0, \\ -\Delta\varphi_1 &= 0 \quad \text{in } \Omega, \quad \varphi_1|_{\Gamma_1} = 0, \quad \varphi_1|_{\Gamma_2} = 1 \end{aligned}$$

The solution  $\varphi = \varphi_0 + c\varphi_1$  allows us to find  $c$  by writing that  $\partial_n\varphi$  has no jump at the trailing edge  $P = (1, 0)$ .

We have  $\partial_n\varphi - (\varphi(P^+) - \varphi(P))/\delta$  where  $P^+$  is the point just above  $P$  in the direction normal to the profile at a distance  $\delta$ . Thus the jump of  $\partial_n\varphi$  is  $(\varphi_0|_{P^+} + c(\varphi_1|_{P^+} - 1)) + (\varphi_0|_{P^-} + c(\varphi_1|_{P^-} - 1))$  divided by  $\delta$  because the normal changes sign between the lower and upper surfaces. Thus

$$c = -\frac{\varphi_0|_{P^+} + \varphi_0|_{P^-}}{(\varphi_1|_{P^+} + \varphi_1|_{P^-} - 2)},$$

which can be programmed as:

$$c = -\frac{\varphi_0(0.99, 0.01) + \varphi_0(0.99, -0.01)}{(\varphi_1(0.99, 0.01) + \varphi_1(0.99, -0.01) - 2)}.$$

```

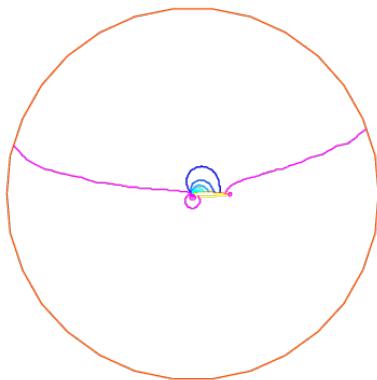
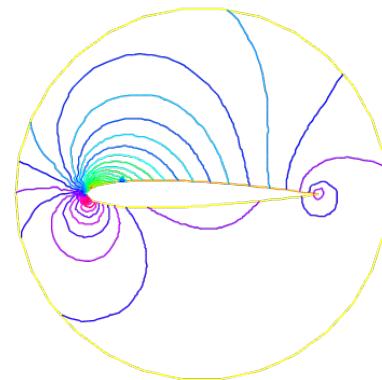
1 // Mesh
2 border a(t=0, 2*pi) {x=5*cos(t); y=5*sin(t);}
3 border upper(t=0, 1) {
4     x=t;
5     y=0.17735*sqrt(t)-0.075597*t - 0.212836*(t^2) + 0.17363*(t^3) - 0.06254*(t^4);
6 }
7 border lower(t=1, 0) {
8     x=t;
9     y=-(0.17735*sqrt(t) - 0.075597*t - 0.212836*(t^2) + 0.17363*(t^3) - 0.06254*(t^4));
10 }
11 border c(t=0, 2*pi) {x=0.8*cos(t)+0.5; y=0.8*sin(t);}

12 mesh Zoom = buildmesh(c(30) + upper(35) + lower(35));
13 mesh Th = buildmesh(a(30) + upper(35) + lower(35));

14 // Fespace
15 fespace Vh(Th, P2);
16 Vh psi0, psil, vh;
17
18 fespace ZVh(Zoom, P2);
19
20
21

```

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(a) Isovalue of  $cp = -(\partial_x \psi)^2 - (\partial_y \psi)^2$ (b) Zooming of  $cp$ 

(continued from previous page)

```

22 // Problem
23 solve Joukowski0(psi0, vh)
24   = int2d(Th) (
25     dx(psi0)*dx(vh)
26     + dy(psi0)*dy(vh)
27   )
28   + on(a, psi0=y-0.1*x)
29   + on(upper, lower, psi0=0)
30 ;
31
32 plot(psi0);
33
34 solve Joukowskil(psil,vh)
35   = int2d(Th) (
36     dx(psil)*dx(vh)
37     + dy(psil)*dy(vh)
38   )
39   + on(a, psil=0)
40   + on(upper, lower, psil=1);
41
42 plot(psil);
43
44 //continuity of pressure at trailing edge
45 real beta = psi0(0.99,0.01) + psi0(0.99,-0.01);
46 beta = -beta / (psil(0.99,0.01) + psil(0.99,-0.01)-2);
47
48 Vh psi = beta*psil + psi0;
49 plot(psi);
50
51 ZVh Zpsi = psi;
52 plot(Zpsi, bw=true);
53
54 ZVh cp = -dx(psi)^2 - dy(psi)^2;
55 plot(cp);
56
57 ZVh Zcp = cp;
58 plot(Zcp, nbiso=40);

```

### 5.1.4 Error estimation

There are famous estimation between the numerical result  $u_h$  and the exact solution  $u$  of the *Poisson's problem*:

If triangulations  $\{\mathcal{T}_h\}_{h \downarrow 0}$  is regular (see [Regular Triangulation](#)), then we have the estimates:

$$\begin{aligned} |\nabla u - \nabla u_h|_{0,\Omega} &\leq C_1 h \\ \|u - u_h\|_{0,\Omega} &\leq C_2 h^2 \end{aligned} \quad (5.3)$$

with constants  $C_1, C_2$  independent of  $h$ , if  $u$  is in  $H^2(\Omega)$ . It is known that  $u \in H^2(\Omega)$  if  $\Omega$  is convex.

In this section we check (5.3). We will pick up numerically error if we use the numerical derivative, so we will use the following for (5.3).

$$\begin{aligned} \int_{\Omega} |\nabla u - \nabla u_h|^2 \, dx \, dy &= \int_{\Omega} \nabla u \cdot \nabla(u - 2u_h) \, dx \, dy + \int_{\Omega} \nabla u_h \cdot \nabla u_h \, dx \, dy \\ &= \int_{\Omega} f(u - 2u_h) \, dx \, dy + \int_{\Omega} f u_h \, dx \, dy \end{aligned}$$

The constants  $C_1, C_2$  are depend on  $\mathcal{T}_h$  and  $f$ , so we will find them by **FreeFEM**.

In general, we cannot get the solution  $u$  as a elementary functions even if speical functions are added. Instead of the exact solution, here we use the approximate solution  $u_0$  in  $V_h(\mathcal{T}_h, P_2)$ ,  $h \sim 0$ .

```

1 // Parameters
2 func f = x*y;
3
4 // Mesh
5 mesh Th0 = square(100, 100);
6
7 // Fespace
8 fespace V0h(Th0, P2);
9 V0h u0, v0;
10
11 // Problem
12 solve Poisson0 (u0, v0)
13   = int2d(Th0) (
14     dx(u0) * dx(v0)
15     + dy(u0) * dy(v0)
16   )
17   - int2d(Th0) (
18     f * v0
19   )
20   + on(1, 2, 3, 4, u0=0)
21   ;
22 plot(u0);
23
24 // Error loop
25 real[int] errL2(10), errH1(10);
26 for (int i = 1; i <= 10; i++) {
27   // Mesh
28   mesh Th = square(5+i*3, 5+i*3);
29
30   // Fespace
31   fespace Vh(Th, P1);
32   Vh u, v;
33   fespace Ph(Th, P0);
34   Ph h = hTriangle; //get the size of all triangles
35
36   // Problem
37   solve Poisson (u, v)

```

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```

38     = int2d(Th) (
39         dx(u)*dx(v)
40         + dy(u)*dy(v)
41     )
42     - int2d(Th) (
43         f*v
44     )
45     + on(1, 2, 3, 4, u=0)
46 ;
47
48 // Error
49 V0h uu = u; //interpolate solution on first mesh
50 errL2[i-1] = sqrt( int2d(Th0) ((uu - u0)^2) )/h[].max^2;
51 errH1[i-1] = sqrt( int2d(Th0) (f*(u0 - 2*uu + uu)) )/h[].max;
52 }
53
54 // Display
55 cout << "C1 = " << errL2.max << "("<<errL2.min<<")" << endl;
56 cout << "C2 = " << errH1.max << "("<<errH1.min<<")" << endl;

```

We can guess that  $C_1 = 0.0179253(0.0173266)$  and  $C_2 = 0.0729566(0.0707543)$ , where the numbers inside the parentheses are minimum in calculation.

### 5.1.5 Periodic Boundary Conditions

We now solve the Poisson equation:

$$-\Delta u = \sin(x + \pi/4.) * \cos(y + \pi/4.)$$

on the square  $]0, 2\pi[^2$  under bi-periodic boundary condition  $u(0, y) = u(2\pi, y)$  for all  $y$  and  $u(x, 0) = u(x, 2\pi)$  for all  $x$ .

These boundary conditions are achieved from the definition of the periodic finite element space.

```

1 // Parameters
2 func f = sin(x+pi/4.)*cos(y+pi/4.); //right hand side
3
4 // Mesh
5 mesh Th = square(10, 10, [2*x*pi, 2*y*pi]);
6
7 // Fespace
8 //defined the fespace with periodic condition
9 //label: 2 and 4 are left and right side with y the curve abscissa
10 //      1 and 2 are bottom and upper side with x the curve abscissa
11 fespace Vh(Th, P2, periodic=[[2, y], [4, y], [1, x], [3, x]]);
12 Vh uh, vh;
13
14 // Problem
15 problem laplace (uh, vh)
16     = int2d(Th) (
17         dx(uh)*dx(vh)
18         + dy(uh)*dy(vh)
19     )
20     + int2d(Th) (
21         - f*vh

```

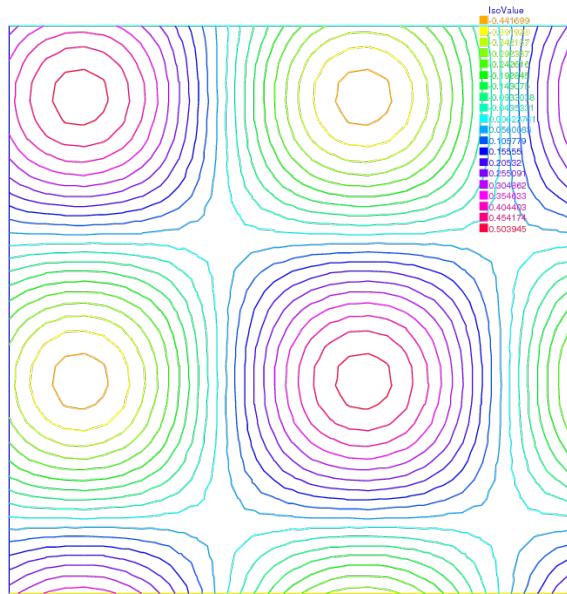
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```

22     )
23 ;
24
25 // Solve
26 laplace;
27
28 // Plot
29 plot(uh, value=true);

```

**Fig. 5.5:** The isovalue of solution  $u$  with periodic boundary condition

The periodic condition does not necessarily require parallel boundaries. The following example give such example.

**Tip:** Periodic boundary conditions - non-parallel boundaries

```

1 // Parameters
2 int n = 10;
3 real r = 0.25;
4 real r2 = 1.732;
5 func f = (y+x-1)*(y-x-1)*(y-x+1)*(y-x-1);
6
7 // Mesh
8 border a(t=0, 1){x=-t+1; y=t; label=1;};
9 border b(t=0, 1){x=-t; y=1-t; label=2;};
10 border c(t=0, 1){x=t-1; y=-t; label=3;};
11 border d(t=0, 1){x=t; y=-1+t; label=4;};
12 border e(t=0, 2*pi){x=r*cos(t); y=-r*sin(t); label=0;};
13 mesh Th = buildmesh(a(n) + b(n) + c(n) + d(n) + e(n));
14 plot(Th, wait=true);
15
16 // Fespace
17 //warning for periodic condition:
18 //side a and c

```

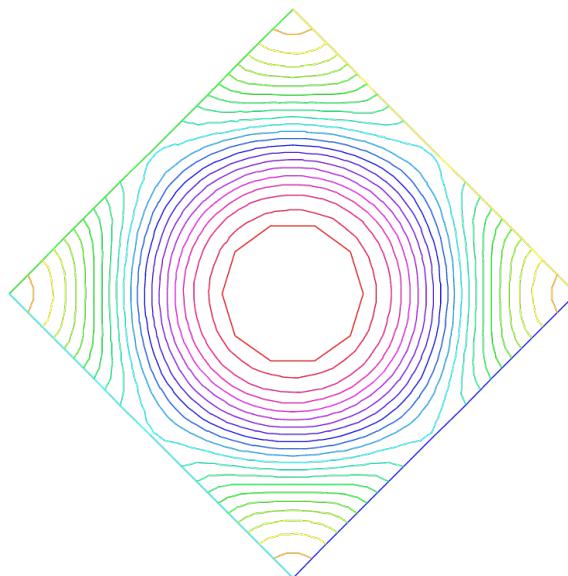
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```

19 //on side a (label 1) $ x \in [0,1] $ or $ x-y\in [-1,1] $
20 //on side c (label 3) $ x \in [-1,0]$ or $ x-y\in[-1,1] $
21 //so the common abscissa can be respectively $x$ and $x+1$
22 //or you can try curviline abscissa $x-y$ and $x+y$
23 //1 first way
24 //fespace Vh(Th, P2, periodic=[[2, 1+x], [4, x], [1, x], [3, 1+x]]);
25 //2 second way
26 fespace Vh(Th, P2, periodic=[[2, x+y], [4, x+y], [1, x-y], [3, x-y]]);
27 Vh uh, vh;
28
29 // Problem
30 real intf = int2d(Th)(f);
31 real mTh = int2d(Th)(1);
32 real k = intf / mTh;
33 problem laplace (uh, vh)
34   = int2d(Th) (
35     dx(uh)*dx(vh)
36     + dy(uh)*dy(vh)
37   )
38   + int2d(Th) (
39     (k-f)*vh
40   )
41 ;
42
43 // Solve
44 laplace;
45
46 // Plot
47 plot(uh, wait=true);

```



**Fig. 5.6:** The isovalue of solution  $u$  for  $\Delta u = ((y+x)^2 + 1)((y-x)^2 + 1) - k$ , in  $\Omega$  and  $\partial_n u = 0$  on hole, and with two periodic boundary condition on external border

An other example with no equal border, just to see if the code works.

---

**Tip:** Periodic boundary conditions - non-equal border

```

1 // Macro
2 //irregular boundary condition to build border AB
3 macro LINEBORDER(A, B, lab)
4     border A#B(t=0,1) { real t1=1.-t;
5     x=A*x*t1+B*x*t;
6     y=A*y*t1+B*y*t;
7     label=lab; } //EOM
8 // compute |||AB||| A=(ax,ay) et B =(bx,by)
9 macro dist(ax, ay, bx, by)
10    sqrt(square((ax)-(bx)) + square((ay)-(by))) //EOM
11 macro Grad(u) [dx(u), dy(u)] //EOM
12
13 // Parameters
14 int n = 10;
15 real Ax = 0.9, Ay = 1;
16 real Bx = 2, By = 1;
17 real Cx = 2.5, Cy = 2.5;
18 real Dx = 1, Dy = 2;
19 real gx = (Ax+Bx+Cx+Dx)/4.;
20 real gy = (Ay+By+Cy+Dy)/4.;

21
22 // Mesh
23 LINEBORDER(A,B,1)
24 LINEBORDER(B,C,2)
25 LINEBORDER(C,D,3)
26 LINEBORDER(D,A,4)
27 mesh Th=buildmesh(AB(n)+BC(n)+CD(n)+DA(n), fixedborder=1);

28
29 // Fespace
30 real 11 = dist(Ax,Ay,Bx,By);
31 real 12 = dist(Bx,By,Cx,Cy);
32 real 13 = dist(Cx,Cy,Dx,Dy);
33 real 14 = dist(Dx,Dy,Ax,Ay);
34 func s1 = dist(Ax,Ay,x,y)/11; //abscisse on AB = |||AX|||/|||AB|||
35 func s2 = dist(Bx,By,x,y)/12; //abscisse on BC = |||BX|||/|||BC|||
36 func s3 = dist(Cx,Cy,x,y)/13; //abscisse on CD = |||CX|||/|||CD|||
37 func s4 = dist(Dx,Dy,x,y)/14; //abscisse on DA = |||DX|||/|||DA|||
38 verbosity = 6; //to see the abscisse value of the periodic condition
39 fespace Vh(Th, P1, periodic=[[1, s1], [3, s3], [2, s2], [4, s4]]);
40 verbosity = 1; //reset verbosity
41 Vh u, v;
42
43 real cc = 0;
44 cc = int2d(Th) ((x-gx)*(y-gy)-cc)/Th.area;
45 cout << "compatibility = " << int2d(Th) ((x-gx)*(y-gy)-cc) <<endl;
46
47 // Problem
48 solve Poisson (u, v)
49     = int2d(Th) (
50         Grad(u)'*Grad(v)
51         + 1e-10*u*v
52     )
53     -int2d(Th) (

```

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```

54     10*v*( (x-gx)*(y-gy)-cc)
55   )
56 ;
57
58 // Plot
59 plot(u, value=true);

```

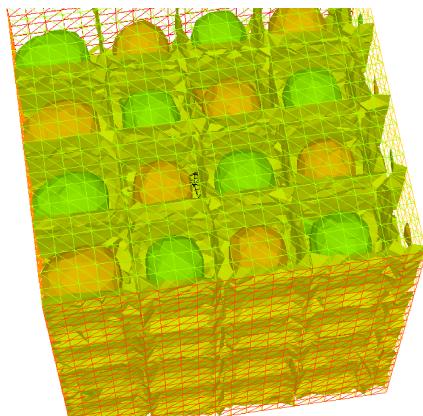
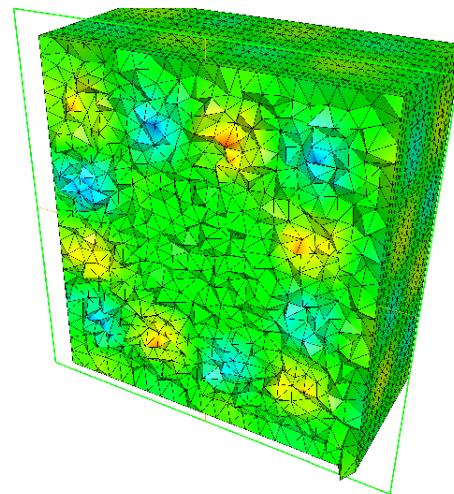
**Tip:** Periodic boundary conditions - Poisson cube-balloon

```

1 load "msh3" load "tetgen" load "medit"
2
3 // Parameters
4 real hs = 0.1; //mesh size on sphere
5 int[int] N = [20, 20, 20];
6 real [int,int] B = [[-1, 1], [-1, 1], [-1, 1]];
7 int [int,int] L = [[1, 2], [3, 4], [5, 6]];
8
9 real x0 = 0.3, y0 = 0.4, z0 = 0.6;
10 func f = sin(x*2*pi+x0)*sin(y*2*pi+y0)*sin(z*2*pi+z0);
11
12 // Mesh
13 bool buildTh = 0;
14 mesh3 Th;
15 try { //a way to build one time the mesh or read it if the file exist
16   Th = readmesh3("Th-hex-sph.mesh");
17 }
18 catch (...) {
19   buildTh = 1;
20 }
21
22 if (buildTh) {
23   include "MeshSurface.idp"
24
25   // Surface Mesh
26   mesh3 ThH = SurfaceHex(N, B, L, 1);
27   mesh3 ThS = Sphere(0.5, hs, 7, 1);
28
29   mesh3 ThHS = ThH + ThS;
30
31   real voltet = (hs^3)/6.;
32   real[int] domain = [0, 0, 0, 1, voltet, 0, 0, 0.7, 2, voltet];
33   Th = tetg(ThHS, switch="pqaAYYQ", nbofregions=2, regionlist=domain);
34
35   savemesh(Th, "Th-hex-sph.mesh");
36 }
37
38 // Fespace
39 fespace Ph(Th, P0);
40 Ph reg = region;
41 cout << " centre = " << reg(0,0,0) << endl;
42 cout << " exterieur = " << reg(0,0,0.7) << endl;
43
44 verbosity = 50;
45 fespace Vh(Th, P1, periodic=[[3, x, z], [4, x, z], [1, y, z], [2, y, z], [5, x, y], [6, x, y]]);

```

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(a) View of the surface isovalue of periodic solution  $uh$ (b) View a the cut of the solution  $uh$  with ffmedit

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```

46 verbosity = 1;
47 Vh uh,vh;
48
49 // Macro
50 macro Grad(u) [dx(u),dy(u),dz(u)] // EOM
51
52 // Problem
53 problem Poisson (uh, vh)
54     = int3d(Th, 1)(Grad(uh)'*Grad(vh)*100
55     )
56     + int3d(Th, 2)(Grad(uh)'*Grad(vh)*2
57     )
58     + int3d(Th) (
59         vh*f
60     )
61     ;
62
63
64
65 // Solve
66 Poisson;
67
68 // Plot
69 plot(uh, wait=true, nbiso=6);
70 medit("uh", Th, uh);

```

## 5.1.6 Poisson Problems with mixed boundary condition

Here we consider the Poisson equation with mixed boundary conditions:

For given functions  $f$  and  $g$ , find  $u$  such that:

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= g && \text{on } \Gamma_D \\ \partial u / \partial n &= 0 && \text{on } \Gamma_N \end{aligned}$$

where  $\Gamma_D$  is a part of the boundary  $\Gamma$  and  $\Gamma_N = \Gamma \setminus \overline{\Gamma_D}$ .

The solution  $u$  has the singularity at the points  $\{\gamma_1, \gamma_2\} = \overline{\Gamma_D} \cap \overline{\Gamma_N}$ .

When  $\Omega = \{(x, y); -1 < x < 1, 0 < y < 1\}$ ,  $\Gamma_N = \{(x, y); -1 \leq x < 0, y = 0\}$ ,  $\Gamma_D = \partial\Omega \setminus \Gamma_N$ , the singularity will appear at  $\gamma_1 = (0, 0)$ ,  $\gamma_2(-1, 0)$ , and  $u$  has the expression:

$$u = K_i u_S + u_R, u_R \in H^2(\text{near } \gamma_i), i = 1, 2$$

with a constants  $K_i$ .

Here  $u_S = r_j^{1/2} \sin(\theta_j/2)$  by the local polar coordinate  $(r_j, \theta_j)$  at  $\gamma_j$  such that  $(r_1, \theta_1) = (r, \theta)$ .

Instead of polar coordinate system  $(r, \theta)$ , we use that  $r = \sqrt{x^2 + y^2}$  and  $\theta = \text{atan2}(y, x)$  in FreeFEM.

Assume that  $f = -2 \times 30(x^2 + y^2)$  and  $g = u_e = 10(x^2 + y^2)^{1/4} \sin([\tan^{-1}(y/x)]/2) + 30(x^2 y^2)$ , where  $u_e$  is the exact solution.

```

1 // Parameters
2 func f = -2*30*(x^2+y^2); //given function
3 //the singular term of the solution is K*us (K: constant)
4 func us = sin(atan2(y,x)/2)*sqrt(sqrt(x^2+y^2));
5 real K = 10.;
6 func ue = K*us + 30*(x^2*y^2);
7
8 // Mesh
9 border N(t=0, 1){x=-1+t; y=0; label=1;};
10 border D1(t=0, 1){x=t; y=0; label=2;};
11 border D2(t=0, 1){x=1; y=t; label=2;};
12 border D3(t=0, 2){x=1-t; y=1; label=2;};
13 border D4(t=0, 1){x=-1; y=1-t; label=2;};
14
15 mesh T0h = buildmesh(N(10) + D1(10) + D2(10) + D3(20) + D4(10));
16 plot(T0h, wait=true);
17
18 // Fespace
19 fespace V0h(T0h, P1);
20 V0h u0, v0;
21
22 //Problem
23 solve Poisson0 (u0, v0)
24   = int2d(T0h) (
25     dx(u0)*dx(v0)
26     + dy(u0)*dy(v0)
27   )
28   - int2d(T0h) (
29     f*v0
30   )
31   + on(2, u0=ue)
32   ;
33
34 // Mesh adaptation by the singular term
35 mesh Th = adaptmesh(T0h, us);
36 for (int i = 0; i < 5; i++)
37   mesh Th = adaptmesh(Th, us);
38
39 // Fespace
40 fespace Vh(Th, P1);
41 Vh u, v;
42
```

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```

43 // Problem
44 solve Poisson (u, v)
45   = int2d(Th) (
46     dx(u)*dx(v)
47     + dy(u)*dy(v)
48   )
49   - int2d(Th) (
50     f*v
51   )
52   + on(2, u=ue)
53 ;
54
55 // Plot
56 plot(Th);
57 plot(u, wait=true);
58
59 // Error in H1 norm
60 Vh ue = ue;
61 real H1e = sqrt( int2d(Th) (dx(ue)^2 + dy(ue)^2 + ue^2) );
62 Vh err0 = u0 - ue;
63 Vh err = u - ue;
64 Vh H1err0 = int2d(Th) (dx(err0)^2 + dy(err0)^2 + err0^2);
65 Vh H1err = int2d(Th) (dx(err)^2 + dy(err)^2 + err^2);
66 cout << "Relative error in first mesh = "<< int2d(Th) (H1err0)/H1e << endl;
67 cout << "Relative error in adaptive mesh = "<< int2d(Th) (H1err)/H1e << endl;

```

From line 35 to 37, mesh adaptations are done using the base of singular term.

In line 61,  $H1e = |u_e|_{1,\Omega}$  is calculated.

In lines 64 and 65, the relative errors are calculated, that is:

$$\begin{aligned}\|u_h^0 - u_e\|_{1,\Omega} / H1e &= 0.120421 \\ \|u_h^a - u_e\|_{1,\Omega} / H1e &= 0.0150581\end{aligned}$$

where  $u_h^0$  is the numerical solution in T0h and  $u_h^a$  is u in this program.

### 5.1.7 Poisson with mixed finite element

Here we consider the Poisson equation with mixed boundary value problems:

For given functions  $f$ ,  $g_d$ ,  $g_n$ , find  $p$  such that

$$\begin{aligned}-\Delta p &= 1 & \text{in } \Omega \\ p &= g_d & \text{on } \Gamma_D \\ \partial p / \partial n &= g_n & \text{on } \Gamma_N\end{aligned}$$

where  $\Gamma_D$  is a part of the boundary  $\Gamma$  and  $\Gamma_N = \Gamma \setminus \overline{\Gamma_D}$ .

The mixed formulation is: find  $p$  and  $\mathbf{u}$  such that:

$$\begin{aligned}\nabla p + \mathbf{u} &= \mathbf{0} & \text{in } \Omega \\ \nabla \cdot \mathbf{u} &= f & \text{in } \Omega \\ p &= g_d & \text{on } \Gamma_D \\ \partial \mathbf{u} \cdot \mathbf{n} &= \mathbf{g}_n \cdot \mathbf{n} & \text{on } \Gamma_N\end{aligned}$$

where  $\mathbf{g}_n$  is a vector such that  $\mathbf{g}_n \cdot \mathbf{n} = g_n$ .

The variational formulation is:

$$\begin{aligned} \forall \mathbf{v} \in \mathbb{V}_0 : \int_{\Omega} p \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} &= \int_{\Gamma_d} g_d \mathbf{v} \cdot \mathbf{n} \\ \forall q \in \mathbb{P} : \int_{\Omega} q \nabla \cdot u &= \int_{\Omega} q f \\ &\quad \partial u \cdot \mathbf{n} = \mathbf{g}_n \cdot \mathbf{n} \quad \text{on } \Gamma_N \end{aligned}$$

where the functional space are:

$$\mathbb{P} = L^2(\Omega), \quad \mathbb{V} = H(\text{div}) = \{\mathbf{v} \in L^2(\Omega)^2, \nabla \cdot \mathbf{v} \in L^2(\Omega)\}$$

and:

$$\mathbb{V}_0 = \{\mathbf{v} \in \mathbb{V}; \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_N\}$$

To write the **FreeFEM** example, we have just to choose the finites elements spaces.

Here  $\mathbb{V}$  space is discretize with Raviart-Thomas finite element **RT0** and  $\mathbb{P}$  is discretize by constant finite element **P0**.

### Example 9.10 LaplaceRT.edp

```

1 // Parameters
2 func gd = 1.;
3 func g1n = 1.;
4 func g2n = 1.;
5
6 // Mesh
7 mesh Th = square(10, 10);
8
9 // Fespace
10 fespace Vh(Th, RT0);
11 Vh [u1, u2];
12 Vh [v1, v2];
13
14 fespace Ph(Th, P0);
15 Ph p, q;
16
17 // Problem
18 problem laplaceMixte ([u1, u2, p], [v1, v2, q], solver=GMRES, eps=1.0e-10, tgv=1e30,
19   ↳dimKrylov=150)
20   = int2d(Th) (
21     p*q*1e-15 //this term is here to be sure
22     // that all sub matrix are inversible (LU requirement)
23     + u1*v1
24     + u2*v2
25     + p*(dx(v1)+dy(v2))
26     + (dx(u1)+dy(u2))*q
27   )
28   + int2d(Th) (
29     q
30   )
31   - int1d(Th, 1, 2, 3) (
32     gd*(v1*N.x + v2*N.y)
33   )
34   + on(4, u1=g1n, u2=g2n)
35   ;
36
37 // Solve
38 laplaceMixte;

```

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```

39 // Plot
40 plot([u1, u2], coef=0.1, wait=true, value=true);
41 plot(p, fill=1, wait=true, value=true);

```

### 5.1.8 Metric Adaptation and residual error indicator

We do metric mesh adaption and compute the classical residual error indicator  $\eta_T$  on the element  $T$  for the Poisson problem.

First, we solve the same problem as in a previous example.

```

1 // Parameters
2 real[int] viso(21);
3 for (int i = 0; i < viso.n; i++)
4   viso[i] = 10.^(+ (i-16.) / 2.);
5 real error = 0.01;
6 func f = (x-y);
7
8 // Mesh
9 border ba(t=0, 1.0) {x=t; y=0; label=1;}
10 border bb(t=0, 0.5) {x=1; y=t; label=2;}
11 border bc(t=0, 0.5) {x=1-t; y=0.5; label=3;}
12 border bd(t=0.5, 1) {x=0.5; y=t; label=4;}
13 border be(t=0.5, 1) {x=1-t; y=1; label=5;}
14 border bf(t=0.0, 1) {x=0; y=1-t; label=6;}
15 mesh Th = buildmesh(ba(6) + bb(4) + bc(4) + bd(4) + be(4) + bf(6));
16
17 // Fespace
18 fespace Vh(Th, P2);
19 Vh u, v;
20
21 fespace Nh(Th, P0);
22 Nh rho;
23
24 // Problem
25 problem Probem1 (u, v, solver=CG, eps=1.0e-6)
26   = int2d(Th, qforder=5) (
27     u*v*1.0e-10
28     + dx(u)*dx(v)
29     + dy(u)*dy(v)
30   )
31   + int2d(Th, qforder=5) (
32     - f*v
33   )
34 ;

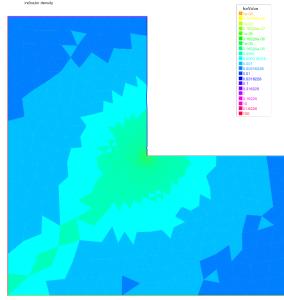
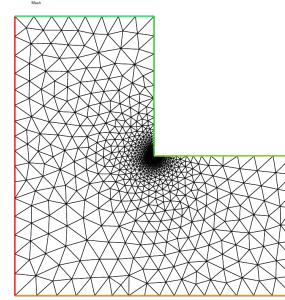
```

Now, the local error indicator  $\eta_T$  is:

$$\eta_T = \left( h_T^2 \|f + \Delta u_h\|_{L^2(T)}^2 + \sum_{e \in \mathcal{E}_T} h_e \left\| \left[ \frac{\partial u_h}{\partial n_k} \right] \right\|_{L^2(e)}^2 \right)^{\frac{1}{2}}$$

where  $h_T$  is the longest edge of  $T$ ,  $\mathcal{E}_T$  is the set of  $T$  edge not on  $\Gamma = \partial\Omega$ ,  $n_T$  is the outside unit normal to  $K$ ,  $h_e$  is the length of edge  $e$ ,  $[g]$  is the jump of the function  $g$  across edge (left value minus right value).

Of course, we can use a variational form to compute  $\eta_T^2$ , with test function constant function in each triangle.

(a) Density of the error indicator with isotropic  $P_2$  metric(b) Density of the error indicator with isotropic  $P_2$  metric

```

1 // Error
2 varf indicator2 (uu, chiK)
3   = intalledges(Th) (
4     chiK*lenEdge*square(jump(N.x*dx(u) + N.y*dy(u)))
5   )
6   + int2d(Th) (
7     chiK*square(hTriangle*(f + dxx(u) + dyy(u)))
8   )
9 ;
10
11 // Mesh adaptation loop
12 for (int i = 0; i < 4; i++) {
13   // Solve
14   Probem1;
15   cout << u[].min << " " << u[].max << endl;
16   plot(u, wait=true);
17
18   // Error
19   rho[] = indicator2(0, Nh);
20   rho = sqrt(rho);
21   cout << "rho = min " << rho[].min << " max=" << rho[].max << endl;
22   plot(rho, fill=true, wait=true, cmm="indicator density", value=true, viso=viso,
23       nbiso=viso.n);
24
25   // Mesh adaptation
26   plot(Th, wait=true, cmm="Mesh (before adaptation)");
27   Th = adaptmesh(Th, [dx(u), dy(u)], err=error, anisomax=1);
28   plot(Th, wait=true, cmm="Mesh (after adaptation)");
29   u = u;
30   rho = rho;
31   error = error/2;
}

```

If the method is correct, we expect to look the graphics by an almost constant function  $\eta$  on your computer as in Fig. 5.8a and Fig. 5.8b.

### 5.1.9 Adaptation using residual error indicator

In the previous example we compute the error indicator, now we use it, to adapt the mesh. The new mesh size is given by the following formulae:

$$h_{n+1}(x) = \frac{h_n(x)}{f_n(\eta_K(x))}$$

where  $\eta_n(x)$  is the level of error at point  $x$  given by the local error indicator,  $h_n$  is the previous “mesh size” field, and  $f_n$  is a user function define by  $f_n = \min(3, \max(1/3, \eta_n/\eta_n^*))$  where  $\eta_n^* = \text{mean}(\eta_n)c$ , and  $c$  is an user coefficient generally close to one.

First a macro `MeshSizecomputation` is defined to get a  $P_1$  mesh size as the average of edge length.

```

1 // macro to get the current mesh size parameter
2 // in:
3 // Th the mesh
4 // Vh P1 fespace on Th
5 // out :
6 // h: the Vh finite element finite set to the current mesh size
7 macro MeshSizecomputation (Th, Vh, h)
8 {
9     real[int] count(Th.nv);
10    /*mesh size (lenEdge = integral(e) 1 ds)*/
11    varf vmeshsizen (u, v) = intalldges(Th, qfnbpE=1) (v);
12    /*number of edges per vertex*/
13    varf vedgecount (u, v) = intalldges(Th, qfnbpE=1) (v/lenEdge);
14    /*mesh size*/
15    count = vedgecount(0, Vh);
16    h[] = 0.;
17    h[] = vmeshsizen(0, Vh);
18    cout << "count min = " << count.min << " max = " << count.max << endl;
19    h[] = h[]./count;
20    cout << "-- bound meshsize = " << h[].min << " " << h[].max << endl;
21 } //
```

A second macro to re-mesh according to the new mesh size.

```

1 // macro to remesh according the de residual indicator
2 // in:
3 // Th the mesh
4 // Ph P0 fespace on Th
5 // Vh P1 fespace on Th
6 // vindicator the varf to evaluate the indicator
7 // coef on etameam
8 macro ReMeshIndicator (Th, Ph, Vh, vindicator, coef)
9 {
10     Vh h=0;
11     /*evaluate the mesh size*/
12     MeshSizecomputation(Th, Vh, h);
13     Ph etak;
14     etak[] = vindicator(0, Ph);
15     etak[] = sqrt(etak[]);
16     real etastar= coef*(etak[].sum/etak[].n);
17     cout << "etastar = " << etastar << " sum = " << etak[].sum << " " << endl;
18
19     /*etaK is discontinous*/
20     /*we use P1 L2 projection with mass lumping*/
21     Vh fn, sigma;
22     varf veta(unused, v) = int2d(Th) (etak*v);
23     varf vun(unused, v) = int2d(Th) (1*v);
24     fn[] = veta(0, Vh);
25     sigma[] = vun(0, Vh);
26     fn[] = fn[]./ sigma[];
27     fn = max(min(fn/etastar, 3.), 0.3333);
```

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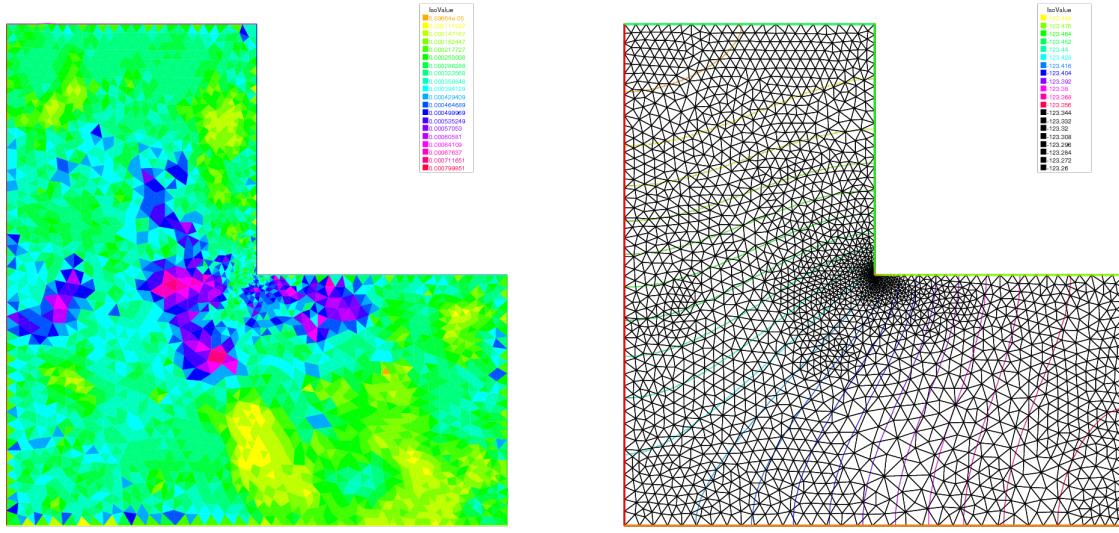
```

29  /*new mesh size*/
30  h = h / fn;
31  /*build the mesh*/
32  Th = adaptmesh(Th, IsMetric=1, h, splitpbedge=1, nbvx=10000);
33 } //
```

```

1  // Parameters
2  real hinit = 0.2; //initial mesh size
3  func f=(x-y);
4
5  // Mesh
6  border ba(t=0, 1.0){x=t; y=0; label=1;}
7  border bb(t=0, 0.5){x=1; y=t; label=2;}
8  border bc(t=0, 0.5){x=1-t; y=0.5; label=3;}
9  border bd(t=0.5, 1){x=0.5; y=t; label=4;}
10 border be(t=0.5, 1){x=1-t; y=1; label=5;}
11 border bf(t=0.0, 1){x=0; y=1-t; label=6;}
12 mesh Th = buildmesh(ba(6) + bb(4) + bc(4) + bd(4) + be(4) + bf(6));
13
14 // Fespace
15 fespace Vh(Th, P1); //for the mesh size and solution
16 Vh h = hinit; //the FE function for the mesh size
17 Vh u, v;
18
19 fespace Ph(Th, P0); //for the error indicator
20
21 //Build a mesh with the given mesh size hinit
22 Th = adaptmesh(Th, h, IsMetric=1, splitpbedge=1, nbvx=10000);
23 plot(Th, wait=1);
24
25 // Problem
26 problem Poisson (u, v)
27   = int2d(Th, qforder=5) (
28     u*v*1.0e-10
29     + dx(u)*dx(v)
30     + dy(u)*dy(v)
31   )
32   - int2d(Th, qforder=5) (
33     f*v
34   )
35 ;
36
37 varf indicator2 (unused, chiK)
38   = intalledges(Th) (
39     chiK*lenEdge*square(jump(N.x*dx(u) + N.y*dy(u)))
40   )
41   + int2d(Th) (
42     chiK*square(hTriangle*(f + dxx(u) + dyy(u)))
43   )
44 ;
45
46 // Mesh adaptation loop
47 for (int i = 0; i < 10; i++) {
48   u = u;
49
50   // Solve
```

(continues on next page)

(a) The error indicator with isotropic  $P_1$ 

(b) The mesh and isovalue of the solution

(continued from previous page)

```

51 Poisson;
52 plot(Th, u, wait=true);
53
54 real cc = 0.8;
55 if (i > 5) cc=1;
56 ReMeshIndicator(Th, Ph, Vh, indicator2, cc);
57 plot(Th, wait=true);
58 }
```

## 5.2 Elasticity

Consider an elastic plate with undeformed shape  $\Omega \times ]-h, h[$  in  $\mathbb{R}^3$ ,  $\Omega \subset \mathbb{R}^2$ .

By the deformation of the plate, we assume that a point  $P(x_1, x_2, x_3)$  moves to  $\mathcal{P}(\xi_1, \xi_2, \xi_3)$ . The vector  $\mathbf{u} = (u_1, u_2, u_3) = (\xi_1 - x_1, \xi_2 - x_2, \xi_3 - x_3)$  is called the *displacement vector*.

By the deformation, the line segment  $\overline{\mathbf{x}, \mathbf{x} + \tau \Delta \mathbf{x}}$  moves approximately to  $\overline{\mathbf{x} + \mathbf{u}(\mathbf{x}), \mathbf{x} + \tau \Delta \mathbf{x} + \mathbf{u}(\mathbf{x} + \tau \Delta \mathbf{x})}$  for small  $\tau$ , where  $\mathbf{x} = (x_1, x_2, x_3)$ ,  $\Delta \mathbf{x} = (\Delta x_1, \Delta x_2, \Delta x_3)$ .

We now calculate the ratio between two segments:

$$\eta(\tau) = \tau^{-1} |\Delta \mathbf{x}|^{-1} (|\mathbf{u}(\mathbf{x} + \tau \Delta \mathbf{x}) - \mathbf{u}(\mathbf{x}) + \tau \Delta \mathbf{x}| - \tau |\Delta \mathbf{x}|)$$

then we have (see e.g. [NECAS2017], p.32)

$$\lim_{\tau \rightarrow 0} \eta(\tau) = (1 + 2e_{ij}\nu_i\nu_j)^{1/2} - 1, \quad 2e_{ij} = \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} + \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

where  $\nu_i = \Delta x_i |\Delta \mathbf{x}|^{-1}$ . If the deformation is *small*, then we may consider that:

$$(\partial u_k / \partial x_i)(\partial u_k / \partial x_i) \approx 0$$

and the following is called *small strain tensor*:

$$\varepsilon_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

The tensor  $e_{ij}$  is called *finite strain tensor*.

Consider the small plane  $\Delta\Pi(\mathbf{x})$  centered at  $\mathbf{x}$  with the unit normal direction  $\mathbf{n} = (n_1, n_2, n_3)$ , then the surface on  $\Delta\Pi(\mathbf{x})$  at  $\mathbf{x}$  is:

$$(\sigma_{1j}(\mathbf{x})n_j, \sigma_{2j}(\mathbf{x})n_j, \sigma_{3j}(\mathbf{x})n_j)$$

where  $\sigma_{ij}(\mathbf{x})$  is called *stress tensor* at  $\mathbf{x}$ . Hooke's law is the assumption of a linear relation between  $\sigma_{ij}$  and  $\varepsilon_{ij}$  such as:

$$\sigma_{ij}(\mathbf{x}) = c_{ijkl}(\mathbf{x})\varepsilon_{ij}(\mathbf{x})$$

with the symmetry  $c_{ijkl} = c_{jikl}, c_{ijkl} = c_{ijlk}, c_{ijkl} = c_{klji}$ .

If Hooke's tensor  $c_{ijkl}(\mathbf{x})$  do not depend on the choice of coordinate system, the material is called *isotropic* at  $\mathbf{x}$ .

If  $c_{ijkl}$  is constant, the material is called *homogeneous*. In homogeneous isotropic case, there is *Lamé constants*  $\lambda, \mu$  (see e.g. [NECAS2017], p.43) satisfying

$$\sigma_{ij} = \lambda\delta_{ij}\operatorname{div}\mathbf{u} + 2\mu\varepsilon_{ij}$$

where  $\delta_{ij}$  is Kronecker's delta.

We assume that the elastic plate is fixed on  $\Gamma_D \times ]-h, h[$ ,  $\Gamma_D \subset \partial\Omega$ . If the body force  $\mathbf{f} = (f_1, f_2, f_3)$  is given in  $\Omega \times ]-h, h[$  and surface force  $\mathbf{g}$  is given in  $\Gamma_N \times ]-h, h[$ ,  $\Gamma_N = \partial\Omega \setminus \overline{\Gamma_D}$ , then the equation of equilibrium is given as follows:

$$\begin{aligned} -\partial_j\sigma_{ij} &= f_i \text{ in } \Omega \times ]-h, h[, \quad i = 1, 2, 3 \\ \sigma_{ij}n_j &= g_i \text{ on } \Gamma_N \times ]-h, h[, \quad u_i = 0 \text{ on } \Gamma_D \times ]-h, h[, \quad i = 1, 2, 3 \end{aligned} \tag{5.4}$$

We now explain the plain elasticity.

- **Plain strain:**

On the end of plate, the contact condition  $u_3 = 0, g_3 = 0$  is satisfied.

In this case, we can suppose that  $f_3 = g_3 = u_3 = 0$  and  $\mathbf{u}(x_1, x_2, x_3) = \bar{\mathbf{u}}(x_1, x_2)$  for all  $-h < x_3 < h$ .

- **Plain stress:**

The cylinder is assumed to be very thin and subjected to no load on the ends  $x_3 = \pm h$ , that is,

$$\sigma_{3i} = 0, \quad x_3 = \pm h, \quad i = 1, 2, 3$$

The assumption leads that  $\sigma_{3i} = 0$  in  $\Omega \times ]-h, h[$  and  $\mathbf{u}(x_1, x_2, x_3) = \bar{\mathbf{u}}(x_1, x_2)$  for all  $-h < x_3 < h$ .

- **Generalized plain stress:**

The cylinder is subjected to no load at  $x_3 = \pm h$ . Introducing the mean values with respect to thickness,

$$\bar{u}_i(x_1, x_2) = \frac{1}{2h} \int_{-h}^h u(x_1, x_2, x_3) dx_3$$

and we derive  $\bar{u}_3 \equiv 0$ . Similarly we define the mean values  $\bar{f}, \bar{g}$  of the body force and surface force as well as the mean values  $\bar{\varepsilon}_{ij}$  and  $\bar{\sigma}_{ij}$  of the components of stress and strain, respectively.

In what follows we omit the overlines of  $\bar{u}, \bar{f}, \bar{g}, \bar{\varepsilon}_{ij}$  and  $\bar{\sigma}_{ij}$ . Then we obtain similar equation of equilibrium given in (5.4) replacing  $\Omega \times ]-h, h[$  with  $\Omega$  and changing  $i = 1, 2$ . In the case of plane stress,  $\sigma_{ij} = \lambda^* \delta_{ij} \operatorname{div} \mathbf{u} + 2\mu \varepsilon_{ij}$ ,  $\lambda^* = (2\lambda\mu)/(\lambda + \mu)$ .

The equations of elasticity are naturally written in variational form for the displacement vector  $\mathbf{u}(\mathbf{x}) \in V$  as:

$$\int_{\Omega} [2\mu \epsilon_{ij}(\mathbf{u}) \epsilon_{ij}(\mathbf{v}) + \lambda \epsilon_{ii}(\mathbf{u}) \epsilon_{jj}(\mathbf{v})] = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma} \mathbf{g} \cdot \mathbf{v}, \forall \mathbf{v} \in V$$

where  $V$  is the linear closed subspace of  $H^1(\Omega)^2$ .

### Tip: Beam

Consider an elastic plate with the undeformed rectangle shape  $]0, 10[\times]0, 2[$ . The body force is the gravity force  $\mathbf{f}$  and the boundary force  $\mathbf{g}$  is zero on lower and upper side. On the two vertical sides of the beam are fixed.

```

1 // Parameters
2 real E = 21.5;
3 real sigma = 0.29;
4 real gravity = -0.05;
5
6 // Mesh
7 border a(t=2, 0){x=0; y=t; label=1;}
8 border b(t=0, 10){x=t; y=0; label=2;}
9 border c(t=0, 2){ x=10; y=t; label=1;}
10 border d(t=0, 10){ x=10-t; y=2; label=3;}
11 mesh th = buildmesh(b(20) + c(5) + d(20) + a(5));
12
13 // Fespace
14 fespace Vh(th, [P1, P1]);
15 Vh [uu, vv];
16 Vh [w, s];
17
18 // Macro
19 real sqrt2 = sqrt(2.);
20 macro epsilon(u1, u2) [dx(u1), dy(u2), (dy(u1)+dx(u2))/sqrt2] //
21 macro div(u,v) (dx(u) + dy(v)) //
22
23 // Problem
24 real mu = E/(2*(1+sigma));
25 real lambda = E*sigma/((1+sigma)*(1-2*sigma));
26 solve Elasticity ([uu, vv], [w, s])
27   = int2d(th)
28     lambda*div(w,s)*div(uu,vv)
29     + 2.*mu*( epsilon(w,s)'*epsilon(uu,vv) )
30   )
31   + int2d(th)
32     - gravity*s
33   )
34   + on(1, uu=0, vv=0)
35 ;
36
37 // Plot
38 plot([uu, vv], wait=true);
39 plot([uu, vv], wait=true, bb=[[-0.5, 2.5], [2.5, -0.5]]);
40
41 // Movemesh
42 mesh th1 = movemesh(th, [x+uu, y+vv]);
43 plot(th1, wait=true);

```

### Tip: Beam 3D

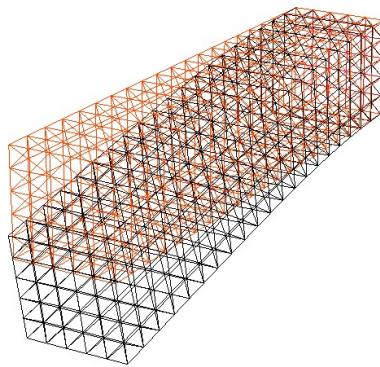
Consider elastic box with the undeformed parallelepiped shape  $]0, 5[\times]0, 1[\times]0, 1[$ . The body force is the gravity force  $\mathbf{f}$  and the boundary force  $\mathbf{g}$  is zero on all face except one the one vertical left face where the beam is fixed.

```

1  include "cube.idp"
2
3 // Parameters
4 int[int] Nxyz = [20, 5, 5];
5 real [int, int] Bxyz = [[0., 5.], [0., 1.], [0., 1.]];
6 int [int, int] Lxyz = [[1, 2], [2, 2], [2, 2]];
7
8 real E = 21.5e4;
9 real sigma = 0.29;
10 real gravity = -0.05;
11
12 // Mesh
13 mesh3 Th = Cube(Nxyz, Bxyz, Lxyz);
14
15 // Fespace
16 fespace Vh(Th, [P1, P1, P1]);
17 Vh [u1, u2, u3], [v1, v2, v3];
18
19 // Macro
20 real sqrt2 = sqrt(2.);
21 macro epsilon(u1, u2, u3) [
22   dx(u1), dy(u2), dz(u3),
23   (dz(u2) + dy(u3))/sqrt2,
24   (dz(u1) + dx(u3))/sqrt2,
25   (dy(u1) + dx(u2))/sqrt2] //
26 macro div(u1, u2, u3) (dx(u1) + dy(u2) + dz(u3)) //
27
28 // Problem
29 real mu = E/(2*(1+sigma));
30 real lambda = E*sigma/((1+sigma)*(1-2*sigma));
31
32 solve Lame ([u1, u2, u3], [v1, v2, v3])
33   = int3d(Th)
34     lambda*div(u1, u2, u3)*div(v1, v2, v3)
35     + 2.*mu*(epsilon(u1, u2, u3)'*epsilon(v1, v2, v3))
36   )
37   - int3d(Th) (
38     gravity*v3
39   )
40   + on(1, u1=0, u2=0, u3=0)
41 ;
42
43 // Display
44 real dmax = u1[].max;
45 cout << "max displacement = " << dmax << endl;
46
47 // Movemesh
48 real coef = 0.1/dmax;
49 int[int] ref2 = [1, 0, 2, 0];
50 mesh3 Thm = movemesh3(Th, transfo=[x+u1*coef, y+u2*coef, z+u3*coef], label=ref2);
51 Thm = change(Thm, label=ref2);
52
53 // Plot
54 plot(Th, Thm, wait=true, cmm="coef amplification = "+coef);

```

coef amplification = 9997.95



**Fig. 5.10:** 3d Beam deformed and undeformed box

### 5.2.1 Fracture Mechanics

Consider the plate with the crack whose undeformed shape is a curve  $\Sigma$  with the two edges  $\gamma_1, \gamma_2$ .

We assume the stress tensor  $\sigma_{ij}$  is the state of plate stress regarding  $(x, y) \in \Omega_\Sigma = \Omega \setminus \Sigma$ . Here  $\Omega$  stands for the undeformed shape of elastic plate without crack.

If the part  $\Gamma_N$  of the boundary  $\partial\Omega$  is fixed and a load  $\mathcal{L} = (\mathbf{f}, \mathbf{g}) \in L^2(\Omega)^2 \times L^2(\Gamma_N)^2$  is given, then the displacement  $\mathbf{u}$  is the minimizer of the potential energy functional:

$$\mathcal{E}(\mathbf{v}; \mathcal{L}, \Omega_\Sigma) = \int_{\Omega_\Sigma} \{w(x, \mathbf{v}) - \mathbf{f} \cdot \mathbf{v}\} - \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{v}$$

over the functional space  $V(\Omega_\Sigma)$ ,

$$V(\Omega_\Sigma) = \{\mathbf{v} \in H^1(\Omega_\Sigma)^2; \mathbf{v} = 0 \quad \text{on } \Gamma_D = \partial\Omega \setminus \overline{\Gamma_N}\},$$

where  $w(x, \mathbf{v}) = \sigma_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{v})/2$ ,

$$\sigma_{ij}(\mathbf{v}) = C_{ijkl}(x)\varepsilon_{kl}(\mathbf{v}), \quad \varepsilon_{ij}(\mathbf{v}) = (\partial v_i / \partial x_j + \partial v_j / \partial x_i)/2, \quad (C_{ijkl} : \text{Hooke's tensor}).$$

If the elasticity is homogeneous isotropic, then the displacement  $\mathbf{u}(x)$  is decomposed in an open neighborhood  $U_k$  of  $\gamma_k$  as in (see e.g. [OHTSUKA2000])

$$\mathbf{u}(x) = \sum_{l=1}^2 K_l(\gamma_k) r_k^{1/2} S_{kl}^C(\theta_k) + \mathbf{u}_{k,R}(x) \quad \text{for } x \in \Omega_\Sigma \cap U_k, k = 1, 2 \quad (5.5)$$

with  $\mathbf{u}_{k,R} \in H^2(\Omega_\Sigma \cap U_k)^2$ , where  $U_k$ ,  $k = 1, 2$  are open neighborhoods of  $\gamma_k$  such that  $\partial L_1 \cap U_1 = \gamma_1$ ,  $\partial L_m \cap U_2 = \gamma_2$ , and

$$\begin{aligned} S_{k1}^C(\theta_k) &= \frac{1}{4\mu} \frac{1}{(2\pi)^{1/2}} \begin{bmatrix} [2\kappa - 1] \cos(\theta_k/2) - \cos(3\theta_k/2) \\ -[2\kappa + 1] \sin(\theta_k/2) + \sin(3\theta_k/2) \end{bmatrix}, \\ S_{k2}^C(\theta_k) &= \frac{1}{4\mu} \frac{1}{(2\pi)^{1/2}} \begin{bmatrix} -[2\kappa - 1] \sin(\theta_k/2) + 3 \sin(3\theta_k/2) \\ -[2\kappa + 1] \cos(\theta_k/2) + \cos(3\theta_k/2) \end{bmatrix}. \end{aligned}$$

where  $\mu$  is the shear modulus of elasticity,  $\kappa = 3 - 4\nu$  ( $\nu$  is the Poisson's ratio) for plane strain and  $\kappa = \frac{3-\nu}{1+\nu}$  for plane stress.

The coefficients  $K_1(\gamma_i)$  and  $K_2(\gamma_i)$ , which are important parameters in fracture mechanics, are called stress intensity factors of the opening mode (mode I) and the sliding mode (mode II), respectively.

For simplicity, we consider the following simple crack

$$\Omega = \{(x, y) : -1 < x < 1, -1 < y < 1\}, \quad \Sigma = \{(x, y) : -1 \leq x \leq 0, y = 0\}$$

with only one crack tip  $\gamma = (0, 0)$ . Unfortunately, **FreeFEM** cannot treat crack, so we use the modification of the domain with U-shape channel (see *U-shape example*, Fig. 3.20) with  $d = 0.0001$ . The undeformed crack  $\Sigma$  is approximated by

$$\Sigma_d = \{(x, y) : -1 \leq x \leq -10 * d, -d \leq y \leq d\} \cup \{(x, y) : -10 * d \leq x \leq 0, -d + 0.1 * x \leq y \leq d - 0.1 * x\}$$

and  $\Gamma_D = R$  in *U-shape example*, Fig. 3.20.

In this example, we use three technique:

- Fast Finite Element Interpolator from the mesh Th to Zoom for the scale-up of near  $\gamma$ .
- After obtaining the displacement vector  $\mathbf{u} = (u, v)$ , we shall watch the deformation of the crack near  $\gamma$  as follows,

```
1 mesh Plate = movemesh(Zoom, [x+u, y+v]);
2 plot(Plate);
```

- Adaptivity is an important technique here, because a large singularity occurs at  $\gamma$  as shown in (5.5).

The first example creates mode I deformation by the opposed surface force on B and T in the vertical direction of  $\Sigma$ , and the displacement is fixed on R.

In a laboratory, fracture engineers use photoelasticity to make stress field visible, which shows the principal stress difference

$$\sigma_1 - \sigma_2 = \sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2}$$

where  $\sigma_1$  and  $\sigma_2$  are the principal stresses.

In opening mode, the photoelasticity make symmetric pattern concentrated at  $\gamma$ .

**Tip:** Crack Opening,  $K_2(\gamma) = 0$

```
1 //Parameters
2 real d = 0.0001; int n = 5; real cb = 1, ca = 1, tip = 0.0;
3
4 real E = 21.5;
5 real sigma = 0.29;
6
7 // Mesh
8 border L1(t=0, ca-d){x=-cb; y=-d-t;};
9 border L2(t=0, ca-d){x=-cb; y=ca-t;};
10 border B(t=0, 2){x=cb*(t-1); y=-ca;};
11 border C1(t=0, 1){x=-ca*(1-t)+(tip-10*d)*t; y=d;};
12 border C21(t=0, 1){x=(tip-10*d)*(1-t)+tip*t; y=d*(1-t);};
13 border C22(t=0, 1){x=(tip-10*d)*t+tip*(1-t); y=-d*t;};
14 border C3(t=0, 1){x=(tip-10*d)*(1-t)-ca*t; y=-d;};
15 border C4(t=0, 2*d){x=-ca; y=-d+t;};
16 border R(t=0, 2){x=cb; y=cb*(t-1);};
17 border T(t=0, 2){x=cb*(1-t); y=ca;};
18 mesh Th = buildmesh(L1(n/2) + L2(n/2) + B(n)
19   + C1(n) + C21(3) + C22(3) + C3(n) + R(n) + T(n));
20 plot(Th, wait=true);
```

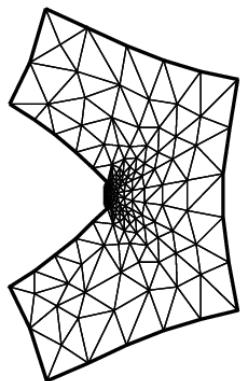
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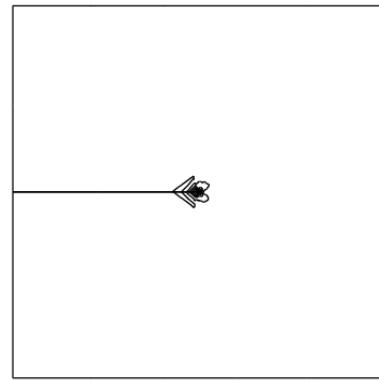
```

22 cb=0.1; ca=0.1;
23 mesh Zoom = buildmesh(L1(n/2) + L2(n/2) + B(n) + C1(n
24 + C21(3) + C22(3) + C3(n) + R(n) + T(n));
25 plot(Zoom, wait=true);
26
27 // Fespace
28 fespace Vh(Th, [P2, P2]);
29 Vh [u, v];
30 Vh [w, s];
31
32 fespace zVh(Zoom, P2);
33 zVn Sx, Sy, Sxy, N;
34
35 // Problem
36 real mu = E/(2*(1+sigma));
37 real lambda = E*sigma/((1+sigma)*(1-2*sigma));
38 solve Problem ([u, v], [w, s])
39 = int2d(Th)(
40     2*mu*(dx(u)*dx(w) + ((dx(v)+dy(u))*(dx(s)+dy(w)))/4)
41     + lambda*(dx(u) + dy(v))*(dx(w) + dy(s))/2
42 )
43 -int1d(Th, T)(
44     0.1*(1-x)*s
45 )
46 +int1d(Th, B)(
47     0.1*(1-x)*s
48 )
49 +on(R, u=0, v=0)
50 ;
51
52 // Loop
53 for (int i = 1; i <= 5; i++) {
54     mesh Plate = movemesh(Zoom, [x+u, y+v]); //deformation near gamma
55     Sx = lambda*(dx(u) + dy(v)) + 2*mu*dx(u);
56     Sy = lambda*(dx(u) + dy(v)) + 2*mu*dy(v);
57     Sxy = mu*(dy(u) + dx(v));
58     N = 0.1*sqrt((Sx-Sy)^2 + 4*Sxy^2); //principal stress difference
59     if (i == 1){
60         plot(Plate, bw=1);
61         plot(N, bw=1);
62     }
63     else if (i == 5){
64         plot(Plate, bw=1);
65         plot(N, bw=1);
66         break;
67     }
68
69 // Adaptmesh
70 Th = adaptmesh(Th, [u, v]);
71
72 // Solve
73 Problem;
74 }
```

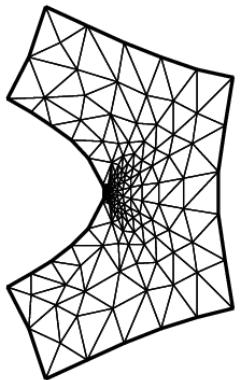
It is difficult to create mode II deformation by the opposed shear force on B and T that is observed in a laboratory. So



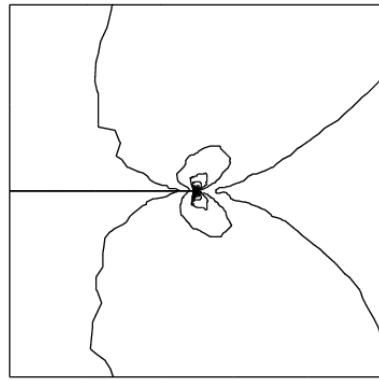
(a) Crack open displacement (COD) on the first mesh



(b) Principal stress difference on the first mesh



(c) COD on the last adaptive mesh



(d) Principal stress difference on the last adaptive mesh

we use the body shear force along  $\Sigma$ , that is, the  $x$ -component  $f_1$  of the body force  $\mathbf{f}$  is given by

$$f_1(x, y) = H(y - 0.001) * H(0.1 - y) - H(-y - 0.001) * H(y + 0.1)$$

where  $H(t) = 1$  if  $t > 0$ ;  $= 0$  if  $t < 0$ .

---

**Tip:** Crack Sliding,  $K_2(\gamma) = 0$

```

1 // Parameters
2 real d = 0.0001; int n = 5; real cb = 1, ca = 1, tip = 0.0;
3
4 real E = 21.5;
5 real sigma = 0.29;
6
7 // Mesh
8 border L1(t=0, ca-d){x=-cb; y=-d-t; }
9 border L2(t=0, ca-d){x=-cb; y=ca-t; }
10 border B(t=0, 2){x=cb*(t-1); y=-ca; }
11 border C1(t=0, 1){x=-ca*(1-t)+(tip-10*d)*t; y=d; }
12 border C21(t=0, 1){x=(tip-10*d)*(1-t)+tip*t; y=d*(1-t); }
13 border C22(t=0, 1){x=(tip-10*d)*t+tip*(1-t); y=-d*t; }
14 border C3(t=0, 1){x=(tip-10*d)*(1-t)-ca*t; y=-d; }
15 border C4(t=0, 2*d){x=-ca; y=-d+t; }
16 border R(t=0, 2){x=cb; y=cb*(t-1); }
17 border T(t=0, 2){x=cb*(1-t); y=ca; }
18 mesh Th = buildmesh(L1(n/2) + L2(n/2) + B(n)
19     + C1(n) + C21(3) + C22(3) + C3(n) + R(n) + T(n));
20 plot(Th, wait=true);
21
22 cb=0.1; ca=0.1;
23 mesh Zoom = buildmesh(L1(n/2) + L2(n/2) + B(n) + C1(n)
24     + C21(3) + C22(3) + C3(n) + R(n) + T(n));
25 plot(Zoom, wait=true);
26
27 // Fespace
28 fespace Vh(Th, [P2, P2]);
29 Vh [u, v];
30 Vh [w, s];
31
32 fespace zVh(Zoom, P2);
33 zVh Sx, Sy, Sxy, N;
34
35 fespace Vh1(Th, P1);
36 Vh1 fx = ((y>0.001)*(y<0.1)) - ((y<-0.001)*(y>-0.1));
37
38 // Problem
39 real mu = E/(2*(1+sigma));
40 real lambda = E*sigma/((1+sigma)*(1-2*sigma));
41 solve Problem ([u, v], [w, s])
42     = int2d(Th) (
43         2*mu*(dx(u)*dx(w) + ((dx(v) + dy(u))*(dx(s) + dy(w)))/4)
44         + lambda*(dx(u) + dy(v))*(dx(w) + dy(s))/2
45     )
46     -int2d(Th) (
47         fx*w
48     )
49     +on (R, u=0, v=0)

```

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```

50 ;
51
52 // Loop
53 for (int i = 1; i <= 3; i++) {
54     mesh Plate = movemesh(Zoom, [x+u, y+v]); //deformation near gamma
55     Sx = lambda*(dx(u) + dy(v)) + 2*mu*dx(u);
56     Sy = lambda*(dx(u) + dy(v)) + 2*mu*dy(v);
57     Sxy = mu*(dy(u) + dx(v));
58     N = 0.1*sqrt((Sx-Sy)^2 + 4*Sxy^2); //principal stress difference
59     if (i == 1) {
60         plot(Plate, bw=1);
61         plot(N, bw=1);
62     }
63     else if (i == 3) {
64         plot(Plate, bw=1);
65         plot(N, bw=1);
66         break;
67     }
68
69 // Adaptmesh
70 Th=adaptmesh(Th, [u, v]);
71
72 // Solve
73 Problem;
74 }
```

## 5.3 Non-linear static problems

Here we propose to solve the following non-linear academic problem of minimization of a functional:

$$J(u) = \int_{\Omega} \frac{1}{2} f(|\nabla u|^2) - u * b$$

where  $u$  is function of  $H_0^1(\Omega)$  and  $f$  defined by:

$$f(x) = a * x + x - \ln(1+x), \quad f'(x) = a + \frac{x}{1+x}, \quad f''(x) = \frac{1}{(1+x)^2}$$

### 5.3.1 Newton-Raphson algorithm

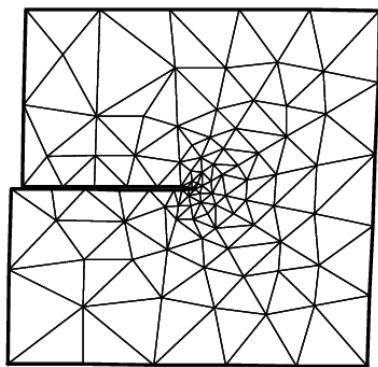
Now, we solve the Euler problem  $\nabla J(u) = 0$  with Newton-Raphson algorithm, that is:

$$u^{n+1} = u^n - (\nabla^2 J(u^n))^{-1} * \nabla J(u^n)$$

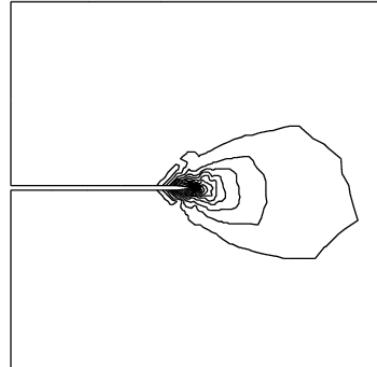
```

1 // Parameters
2 real a = 0.001;
3 func b = 1.;
4
5 // Mesh
6 mesh Th = square(10, 10);
7 Th = adaptmesh(Th, 0.05, IsMetric=1, splitpbedge=1);
```

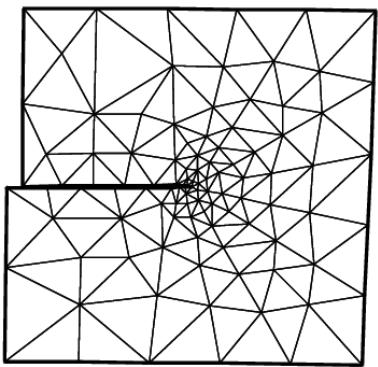
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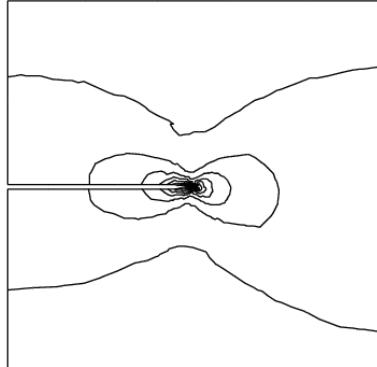
(a) COD on the first mesh



(b) Principal stress difference in the first mesh



(c) COD on the last adaptive mesh



(d) Principal stress difference on the last adaptive mesh

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```

8 plot(Th, wait=true);
9
10 // Fespace
11 fespace Vh(Th, P1);
12 Vh u=0;
13 Vh v, w;
14
15 fespace Ph(Th, P1dc);
16 Ph alpha; //to store |nabla u|^2
17 Ph dalpha; //to store 2f''(|nabla u|^2)
18
19 // Function
20 func real f (real u){
21     return u*a + u - log(1.+u);
22 }
23 func real df (real u){
24     return a +u/(1.+u);
25 }
26 func real ddf (real u){
27     return 1. / ((1.+u)*(1.+u));
28 }
29
30 // Problem
31 //the variational form of evaluate dJ = nabla J
32 //dJ = f'()*(dx(u)*dx(vh) + dy(u)*dy(vh))
33 varf vdJ (uh, vh)
34 = int2d(Th) (
35     alpha*(dx(u)*dx(vh) + dy(u)*dy(vh))
36     - b*vh
37 )
38 + on(1, 2, 3, 4, uh=0)
39 ;
40
41 //the variational form of evaluate ddJ = nabla^2 J
42 //hJ(uh,vh) = f'()*(dx(uh)*dx(vh) + dy(uh)*dy(vh))
43 // + 2*f''()(dx(u)*dx(uh) + dy(u)*dy(uh)) * (dx(u)*dx(vh) + dy(u)*dy(vh))
44 varf vhJ (uh, vh)
45 = int2d(Th) (
46     alpha*(dx(uh)*dx(vh) + dy(uh)*dy(vh))
47     + dalPHA*(dx(u)*dx(vh) + dy(u)*dy(vh))*(dx(u)*dx(uh) + dy(u)*dy(uh))
48 )
49 + on(1, 2, 3, 4, uh=0)
50 ;
51
52 // Newton algorithm
53 for (int i = 0; i < 100; i++) {
54     // Compute f' and f''
55     alpha = df(dx(u)*dx(u) + dy(u)*dy(u));
56     dalPHA = 2*ddf(dx(u)*dx(u) + dy(u)*dy(u));
57
58     // nabla J
59     v[] = vdJ(0, Vh);
60
61     // Residual
62     real res = v[]'*v[];
63     cout << i << " residu^2 = " << res << endl;
64     if( res < 1e-12) break;

```

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```

65 // HJ
66 matrix H = vhJ(Vh, Vh, factorize=1, solver=LU);
67
68 // Newton
69 w[] = H^-1*v[];
70 u[] -= w[];
71 }
72
73 // Plot
74 plot (u, wait=true, cmm="Solution with Newton-Raphson");
75

```

## 5.4 Eigen value problems

This section depends on your installation of **FreeFEM**; you need to have compiled ARPACK. This tool is available in **FreeFEM** if the word eigenvalue appears in line `Load:`, like:

```

1 -- FreeFem++ v*.** (date *** *** ** *:**:** CET ****)
2 file : ***.edp
3 Load: lg_fem lg_mesh eigenvalue

```

This tool is based on `arpack++`, the object-oriented version of ARPACK eigenvalue package [LEHOUQC1998].

The function `EigenValue` computes the generalized eigenvalue of  $Au = \lambda Bu$ . The Shift-invert method is used by default, with `sigma = \sigma` the shift of the method.

The matrix  $OP$  is defined with  $A - \sigma B$ .

The return value is the number of converged eigenvalues (can be greater than the number of requested eigenvalues `nev`)

```

1 int k = EigenValue(OP, B, nev=Nev, sigma=Sigma);

```

where the matrix  $OP = A - \sigma B$  with a solver and boundary condition, and the matrix  $B$ .

There is also a functional interface:

```

1 int k = EigenValue(n, FOP1, FB, nev=Nev, sigma=Sigma);

```

where  $n$  is the size of the problem, and the operators are now defined through functions, defining respectively the matrix product of  $OP^{-1}$  and  $B$ , as in

```

1 int n = OP1.n;
2 func real[int] FOP1(real[int] & u){ real[int] Au = OP^-1*u; return Au; }
3 func real[int] FB(real[int] & u){ real[int] Au = B*u; return Au; }

```

If you want finer control over the method employed in ARPACK, you can specify which mode ARPACK will work with (`mode`), see ARPACK documentation [LEHOUQC1998]). The operators necessary for the chosen mode can be passed through the optional parameters `A=`, `A1=`, `B=`, `B1=`, (see below).

- `mode=1`: Regular mode for solving  $Au = \lambda u$

```

1 int k = EigenValue(n, A=FOP, mode=1, nev=Nev);

```

where the function `FOP` defines the matrix product of `A`

- mode=2: Regular inverse mode for solving  $Au = \lambda Bu$

```
int k = EigenValue(n, A=FOP, B=FB, B1=FB1, mode=2, nev=Nev);
```

where the functions FOP, FB and FB1 define respectively the matrix product of  $A$ ,  $B$  and  $B^{-1}$

- mode=3: Shift-invert mode for solving  $Au = \lambda Bu$

```
int k = EigenValue(n, A1=FOP1, B=FB, mode=3, sigma=Sigma, nev=Nev);
```

where the functions FOP1 and FB define respectively the matrix product of  $OP^{-1} = (A - \sigma B)^{-1}$  and  $B$

You can also specify which subset of eigenvalues you want to compute (which=). The default value is which="LM", for eigenvalues with largest magnitude. "SM" is for smallest magnitude, "LA" for largest algebraic value, "SA" for smallest algebraic value, and "BE" for both ends of the spectrum.

Remark: For complex problems, you need to use the keyword `complexEigenValue` instead of `EigenValue` when passing operators through functions.

#### Note: Boundary condition and Eigenvalue Problems

The locking (Dirichlet) boundary condition is made with exact penalization so we put `1e30=tgv` on the diagonal term of the locked degree of freedom (see [Finite element chapter](#)). So take Dirichlet boundary condition just on  $A$  and not on  $B$  because we solve  $w = OP^{-1} * B * v$ .

If you put locking (Dirichlet) boundary condition on  $B$  matrix (with key work `on`) you get small spurious modes ( $10^{-30}$ ), due to boundary condition, but if you forget the locking boundary condition on  $B$  matrix (no keyword `on`) you get huge spurious ( $10^{30}$ ) modes associated to these boundary conditions. We compute only small mode, so we get the good one in this case.

- sym= The problem is symmetric (all the eigen values are real)
- nev= The number desired eigenvalues (nev) close to the shift.
- value= The array to store the real part of the eigenvalues
- ivalue= The array to store the imaginary part of the eigenvalues
- vector= The FE function array to store the eigenvectors
- rawvector= An array of type `real[int, int]` to store eigenvectors by column.

For real non symmetric problems, complex eigenvectors are given as two consecutive vectors, so if eigenvalue  $k$  and  $k + 1$  are complex conjugate eigenvalues, the  $k$ th vector will contain the real part and the  $k + 1$ th vector the imaginary part of the corresponding complex conjugate eigenvectors.

- tol= The relative accuracy to which eigenvalues are to be determined;
- sigma= The shift value;
- maxit= The maximum number of iterations allowed;
- ncv= The number of Arnoldi vectors generated at each iteration of ARPACK;
- mode= The computational mode used by ARPACK (see above);
- which= The requested subset of eigenvalues (see above).

#### Tip: Laplace eigenvalue

In the first example, we compute the eigenvalues and the eigenvectors of the Dirichlet problem on square  $\Omega = ]0, \pi[^2$ .

The problem is to find:  $\lambda$ , and  $\nabla u_\lambda$  in  $\mathbb{R} \times H_0^1(\Omega)$

$$\int_{\Omega} \nabla u_\lambda \nabla v = \lambda \int_{\Omega} uv \quad \forall v \in H_0^1(\Omega)$$

The exact eigenvalues are  $\lambda_{n,m} = (n^2 + m^2)$ ,  $(n, m) \in \mathbb{N}_*^2$  with the associated eigenvectors are  $u_{m,n} = \sin(nx) * \sin(my)$ .

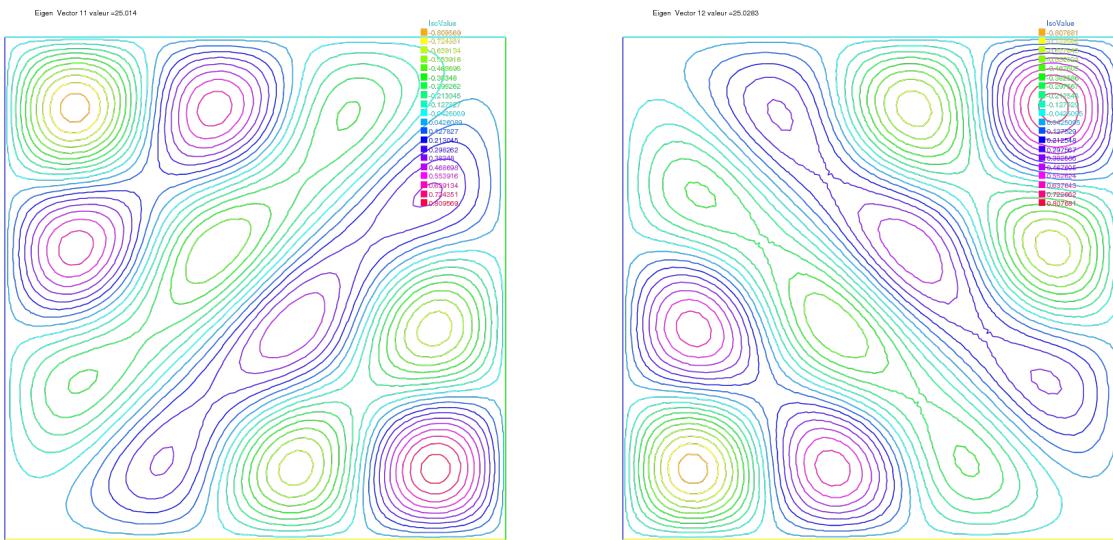
We use the generalized inverse shift mode of the *arpack++* library, to find 20 eigenvalues and eigenvectors close to the shift value  $\sigma = 20$ .

```

1 // Parameters
2 verbosity=0;
3 real sigma = 20; //value of the shift
4 int nev = 20; //number of computed eigen value close to sigma
5
6 // Mesh
7 mesh Th = square(20, 20, [pi*x, pi*y]);
8
9 // Fespace
10 fespace Vh(Th, P2);
11 Vh u1, u2;
12
13 // Problem
14 // OP = A - sigma B ; // the shifted matrix
15 varf op (u1, u2)
16   = int2d(Th) (
17     dx(u1)*dx(u2)
18     + dy(u1)*dy(u2)
19     - sigma* u1*u2
20   )
21   + on(1, 2, 3, 4, u1=0)
22 ;
23
24 varf b ([u1], [u2]) = int2d(Th) (u1*u2); //no boundary condition
25
26 matrix OP = op(Vh, Vh, solver=Crout, factorize=1); //crout solver because the matrix
27 //in not positive
28 matrix B = b(Vh, Vh, solver=CG, eps=1e-20);
29
30 // important remark:
31 // the boundary condition is make with exact penalization:
32 // we put 1e30=tgv on the diagonal term of the lock degree of freedom.
33 // So take Dirichlet boundary condition just on $a$ variational form
34 // and not on $b$ variational form.
35 // because we solve $ w=OP^{-1}*B*v $
36
37 // Solve
38 real[int] ev(nev); //to store the nev eigenvalue
39 Vh[int] eV(nev); //to store the nev eigenvector
40
41 int k = EigenValue(OP, B, sym=true, sigma=sigma, value=ev, vector=eV,
42   tol=1e-10, maxit=0, ncv=0);
43
44 // Display & Plot
45 for (int i = 0; i < k; i++) {
46   u1 = eV[i];
47   real gg = int2d(Th) (dx(u1)*dx(u1) + dy(u1)*dy(u1));
48   real mm = int2d(Th) (u1*u1) ;

```

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(a) Isovalue of 11th eigenvector  $u_{4,3} - u_{3,4}$ (b) Isovalue of 12th eigenvector  $u_{4,3} + u_{3,4}$ 

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```

48   cout << "lambda[" << i << "] = " << ev[i] << ", err= " << int2d(Th) (dx(u1)*dx(u1) -
49   << dy(u1)*dy(u1) - (ev[i])*u1*u1) << endl;
50   plot(ev[i], cmm="Eigen Vector "+i+" value =" + ev[i], wait=true, value=true);
}

```

The output of this example is:

```

1 lambda[0] = 5.0002, err= -1.46519e-11
2 lambda[1] = 8.00074, err= -4.05158e-11
3 lambda[2] = 10.0011, err= 2.84925e-12
4 lambda[3] = 10.0011, err= -7.25456e-12
5 lambda[4] = 13.002, err= -1.74257e-10
6 lambda[5] = 13.0039, err= 1.22554e-11
7 lambda[6] = 17.0046, err= -1.06274e-11
8 lambda[7] = 17.0048, err= 1.03883e-10
9 lambda[8] = 18.0083, err= -4.05497e-11
10 lambda[9] = 20.0096, err= -2.21678e-13
11 lambda[10] = 20.0096, err= -4.16212e-14
12 lambda[11] = 25.014, err= -7.42931e-10
13 lambda[12] = 25.0283, err= 6.77444e-10
14 lambda[13] = 26.0159, err= 3.19864e-11
15 lambda[14] = 26.0159, err= -4.9652e-12
16 lambda[15] = 29.0258, err= -9.99573e-11
17 lambda[16] = 29.0273, err= 1.38242e-10
18 lambda[17] = 32.0449, err= 1.2522e-10
19 lambda[18] = 34.049, err= 3.40213e-11
20 lambda[19] = 34.0492, err= 2.41751e-10

```

## 5.5 Evolution problems

FreeFEM also solves evolution problems such as the heat equation:

$$\begin{aligned}\frac{\partial u}{\partial t} - \mu \Delta u &= f && \text{in } \Omega \times ]0, T[ \\ u(\mathbf{x}, 0) &= u_0(\mathbf{x}) && \text{in } \Omega \\ (\partial u / \partial n)(\mathbf{x}, t) &= 0 && \text{on } \partial\Omega \times ]0, T[\end{aligned}\tag{5.6}$$

with a positive viscosity coefficient  $\mu$  and homogeneous Neumann boundary conditions.

We solve (5.6) by FEM in space and finite differences in time.

We use the definition of the partial derivative of the solution in the time derivative:

$$\frac{\partial u}{\partial t}(x, y, t) = \lim_{\tau \rightarrow 0} \frac{u(x, y, t) - u(x, y, t - \tau)}{\tau}$$

which indicates that  $u^m(x, y) = u(x, y, m\tau)$  will satisfy approximatively:

$$\frac{\partial u}{\partial t}(x, y, m\tau) \simeq \frac{u^m(x, y) - u^{m-1}(x, y)}{\tau}$$

The time discretization of heat equation (5.6) is as follows,  $\forall m = 0, \dots, [T/\tau]$ :

$$\begin{aligned}\frac{u^{m+1} - u^m}{\tau} - \mu \Delta u^{m+1} &= f^{m+1} && \text{in } \Omega \\ u^0(\mathbf{x}) &= u_0(\mathbf{x}) && \text{in } \Omega \\ \partial u^{m+1} / \partial n(\mathbf{x}) &= 0 && \text{on } \partial\Omega\end{aligned}$$

which is so-called *backward Euler method* for (5.6).

To obtain the variational formulation, multiply with the test function  $v$  both sides of the equation:

$$\int_{\Omega} \{u^{m+1}v - \tau \Delta u^{m+1}v\} = \int_{\Omega} \{u^m + \tau f^{m+1}\}v$$

By the divergence theorem, we have:

$$\int_{\Omega} \{u^{m+1}v + \tau \nabla u^{m+1} \cdot \nabla v\} - \int_{\partial\Omega} \tau (\partial u^{m+1} / \partial n)v = \int_{\Omega} \{u^m v + \tau f^{m+1}v\}$$

By the boundary condition  $\partial u^{m+1} / \partial n = 0$ , it follows that:

$$\int_{\Omega} \{u^{m+1}v + \tau \nabla u^{m+1} \cdot \nabla v\} - \int_{\Omega} \{u^m v + \tau f^{m+1}v\} = 0\tag{5.7}$$

Using the identity just above, we can calculate the finite element approximation  $u_h^m$  of  $u^m$  in a step-by-step manner with respect to  $t$ .

### Tip: Example

We now solve the following example with the exact solution  $u(x, y, t) = tx^4$ ,  $\Omega = ]0, 1[^2$ .

$$\begin{aligned}\frac{\partial u}{\partial t} - \mu \Delta u &= x^4 - \mu 12tx^2 && \text{in } \Omega \times ]0, 3[ \\ u(x, y, 0) &= 0 && \text{on } \Omega \\ u|_{\partial\Omega} &= t * x^4\end{aligned}$$

```

1 // Parameters
2 real dt = 0.1;
3 real mu = 0.01;
4
5 // Mesh
6 mesh Th = square(16, 16);
7
8 // Fespace
9 fespace Vh(Th, P1);
10 Vh u, v, uu, f, g;
11
12 // Problem
13 problem dHeat (u, v)
14   = int2d(Th) (
15     u*v
16     + dt*mu*(dx(u)*dx(v) + dy(u)*dy(v))
17   )
18   + int2d(Th) (
19     - uu*v
20     - dt*f*v
21   )
22   + on(1, 2, 3, 4, u=g)
23 ;
24
25 // Time loop
26 real t = 0;
27 uu = 0;
28 for (int m = 0; m <= 3/dt; m++) {
29   // Update
30   t = t+dt;
31   f = x^4 - mu*t*12*x^2;
32   g = t*x^4;
33   uu = u;
34
35   // Solve
36   dHeat;
37
38   // Plot
39   plot(u, wait=true);
40   cout << "t=" << t << " - L^2-Error=" << sqrt(int2d(Th)((u-t*x^4)^2)) << endl;
41 }

```

In the last statement, the  $L^2$ -error  $\left(\int_{\Omega} |u - tx^4|^2\right)^{1/2}$  is calculated at  $t = m\tau, \tau = 0.1$ . At  $t = 0.1$ , the error is 0.000213269. The errors increase with  $m$  and 0.00628589 at  $t = 3$ .

The iteration of the backward Euler (5.7) is made by `for loop`.

---

**Note:** The stiffness matrix in the loop is used over and over again. FreeFEM support reuses of stiffness matrix.

---

### 5.5.1 Mathematical Theory on Time Difference Approximations.

In this section, we show the advantage of implicit schemes. Let  $V, H$  be separable Hilbert space and  $V$  is dense in  $H$ . Let  $a$  be a continuous bilinear form over  $V \times V$  with coercivity and symmetry.

Then  $\sqrt{a(v, v)}$  become equivalent to the norm  $\|v\|$  of  $V$ .

**Problem Ev(f,Omega):** For a given  $f \in L^2(0, T; V')$ ,  $u^0 \in H$

$$\begin{aligned}\frac{d}{dt}(u(t), v) + a(u(t), v) &= (f(t), v) \quad \forall v \in V, \quad a.e. t \in [0, T] \\ u(0) &= u^0\end{aligned}$$

where  $V'$  is the dual space of  $V$ .

Then, there is an unique solution  $u \in L^\infty(0, T; H) \cap L^2(0, T; V)$ .

Let us denote the time step by  $\tau > 0$ ,  $N_T = [T/\tau]$ . For the discretization, we put  $u^n = u(n\tau)$  and consider the time difference for each  $\theta \in [0, 1]$

$$\begin{aligned}\frac{1}{\tau} (u_h^{n+1} - u_h^n, \phi_i) + a(u_h^{n+\theta}, \phi_i) &= \langle f^{n+\theta}, \phi_i \rangle \\ i &= 1, \dots, m, \quad n = 0, \dots, N_T \\ u_h^{n+\theta} &= \theta u_h^{n+1} + (1-\theta)u_h^n, \\ f^{n+\theta} &= \theta f^{n+1} + (1-\theta)f^n\end{aligned}$$

Formula (5.8) is the *forward Euler scheme* if  $\theta = 0$ , *Crank-Nicolson scheme* if  $\theta = 1/2$ , the *backward Euler scheme* if  $\theta = 1$ .

Unknown vectors  $u^n = (u_1^n, \dots, u_m^n)^T$  in

$$u_h^n(x) = u_1^n \phi_1(x) + \dots + u_m^n \phi_m(x), \quad u_1^n, \dots, u_m^n \in \mathbb{R}$$

are obtained from solving the matrix

$$\begin{aligned}(M + \theta \tau A)u^{n+1} &= \{M - (1-\theta)\tau A\}u^n + \tau \{\theta f^{n+1} + (1-\theta)f^n\} \\ M &= (m_{ij}), \quad m_{ij} = (\phi_j, \phi_i), \quad A = (a_{ij}), \quad a_{ij} = a(\phi_j, \phi_i)\end{aligned}$$

Refer [TABATA1994], pp.70–75 for solvability of (5.8). The stability of (5.8) is in [TABATA1994], Theorem 2.13:

Let  $\{\mathcal{T}_h\}_{h \downarrow 0}$  be regular triangulations (see *Regular Triangulation*). Then there is a number  $c_0 > 0$  independent of  $h$  such that,

$$|u_h^n|^2 \leq \begin{cases} \frac{1}{\delta} \left\{ |u_h^0|^2 + \tau \sum_{k=0}^{n-1} \|f^{k+\theta}\|_{V'_h}^2 \right\} & \theta \in [0, 1/2] \\ |u_h^0|^2 + \tau \sum_{k=0}^{n-1} \|f^{k+\theta}\|_{V'_h}^2 & \theta \in [1/2, 1] \end{cases}$$

if the following are satisfied:

1. When  $\theta \in [0, 1/2)$ , then we can take a time step  $\tau$  in such a way that

$$\tau < \frac{2(1-\delta)}{(1-2\theta)c_0^2} h^2$$

for arbitrary  $\delta \in (0, 1)$ .

2. When  $1/2 \leq \theta \leq 1$ , we can take  $\tau$  arbitrary.

---

**Tip:** Example

```

1 // Parameters
2 real tau = 0.1; real
3 theta = 0.;
4
5 // Mesh
```

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(continued from previous page)

```

6  mesh Th = square(12, 12);
7
8 // Fespace
9 fespace Vh(Th, P1);
10 Vh u, v, oldU;
11 Vh f1, f0;
12
13 fespace Ph(Th, P0);
14 Ph h = hTriangle; // mesh sizes for each triangle
15
16 // Function
17 func real f (real t){
18     return x^2*(x-1)^2 + t*(-2 + 12*x - 11*x^2 - 2*x^3 + x^4);
19 }
20
21 // File
22 ofstream out("err02.csv"); //file to store calculations
23 out << "mesh size = " << h[].max << ", time step = " << tau << endl;
24 for (int n = 0; n < 5/tau; n++)
25     out << n*tau << ",";
26 out << endl;
27
28 // Problem
29 problem aTau (u, v)
30     = int2d(Th) (
31         u*v
32         + theta*tau*(dx(u)*dx(v) + dy(u)*dy(v) + u*v)
33     )
34     - int2d(Th) (
35         oldU*v
36         - (1-theta)*tau*(dx(oldU)*dx(v) + dy(oldU)*dy(v) + oldU*v)
37     )
38     - int2d(Th) (
39         tau*(theta*f1 + (1-theta)*f0)*v
40     )
41 ;
42
43 // Theta loop
44 while (theta <= 1.0) {
45     real t = 0;
46     real T = 3;
47     oldU = 0;
48     out << theta << ",";
49     for (int n = 0; n < T/tau; n++) {
50         // Update
51         t = t + tau;
52         f0 = f(n*tau);
53         f1 = f((n+1)*tau);
54
55         // Solve
56         aTau;
57         oldU = u;
58
59         // Plot
60         plot(u);
61
62         // Error

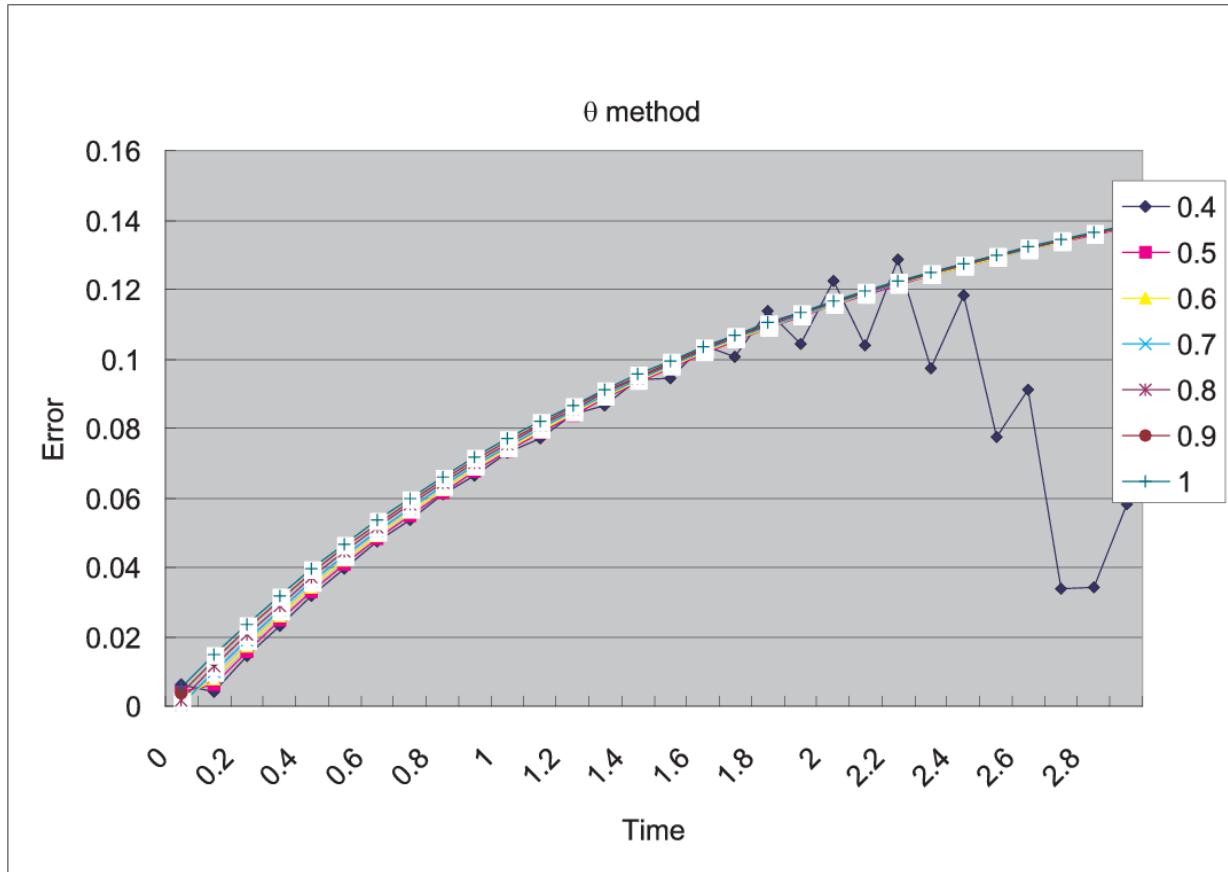
```

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```

63     Vh uex = t*x^2*(1-x)^2; //exact solution = tx^2(1-x)^2
64     Vh err = u - uex; // err = FE-sol - exact
65     out << abs(err[].max) / abs(uex[].max) << ", ";
66   }
67   out << endl;
68   theta = theta + 0.1;
69 }
```



**Fig. 5.14:**  $\max_{x \in \Omega} |u_h^n(\theta) - u_{ex}(n\tau)| / \max_{x \in \Omega} |u_{ex}(n\tau)|$  at  $n = 0, 1, \dots, 29$

We can see in Fig. 5.14 that  $u_h^n(\theta)$  become unstable at  $\theta = 0.4$ , and figures are omitted in the case  $\theta < 0.4$ .

## 5.5.2 Convection

The hyperbolic equation

$$\partial_t u + \alpha \cdot \nabla u = f; \text{ for a vector-valued function } \alpha \quad (5.8)$$

appears frequently in scientific problems, for example in the Navier-Stokes equations, in the Convection-Diffusion equation, etc.

In the case of 1-dimensional space, we can easily find the general solution  $(x, t) \mapsto u(x, t) = u^0(x - \alpha t)$  of the

following equation, if  $\alpha$  is constant,

$$\begin{aligned}\partial_t u + \alpha \partial_x u &= 0 \\ u(x, 0) &= u^0(x),\end{aligned}\tag{5.9}$$

because  $\partial_t u + \alpha \partial_x u = -\alpha \dot{u}^0 + a \dot{u}^0 = 0$ , where  $\dot{u}^0 = du^0(x)/dx$ .

Even if  $\alpha$  is not constant, the construction works on similar principles. One begins with the ordinary differential equation (with the convention that  $\alpha$  is prolonged by zero apart from  $(0, L) \times (0, T)$ ):

$$\dot{X}(\tau) = +\alpha(X(\tau), \tau), \quad \tau \in (0, t) \quad X(t) = x$$

In this equation  $\tau$  is the variable and  $x, t$  are parameters, and we denote the solution by  $X_{x,t}(\tau)$ . Then it is noticed that  $(x, t) \rightarrow v(X(\tau), \tau)$  in  $\tau = t$  satisfies the equation

$$\partial_t v + \alpha \partial_x v = \partial_t X \dot{v} + a \partial_x X \dot{v} = 0$$

and by the definition  $\partial_t X = \dot{X} = +\alpha$  and  $\partial_x X = \partial_x x$  in  $\tau = t$ , because if  $\tau = t$  we have  $X(\tau) = x$ .

The general solution of (5.9) is thus the value of the boundary condition in  $X_{x,t}(0)$ , that is to say  $u(x, t) = u^0(X_{x,t}(0))$  where  $X_{x,t}(0)$  is on the  $x$  axis,  $u(x, t) = u^0(X_{x,t}(0))$  if  $X_{x,t}(0)$  is on the axis of  $t$ .

In higher dimension  $\Omega \subset R^d$ ,  $d = 2, 3$ , the equation for the convection is written

$$\partial_t u + \alpha \cdot \nabla u = 0 \text{ in } \Omega \times (0, T)$$

where  $\mathbf{a}(x, t) \in \mathbb{R}^d$ .

**FreeFEM** implements the Characteristic-Galerkin method for convection operators. Recall that the equation (5.8) can be discretized as

$$\frac{Du}{Dt} = f \text{ i.e. } \frac{du}{dt}(X(t), t) = f(X(t), t) \text{ where } \frac{dX}{dt}(t) = \alpha(X(t), t)$$

where  $D$  is the total derivative operator. So a good scheme is one step of backward convection by the method of Characteristics-Galerkin

$$\frac{1}{\tau} (u^{m+1}(x) - u^m(X^m(x))) = f^m(x) \tag{5.10}$$

where  $X^m(x)$  is an approximation of the solution at  $t = m\tau$  of the ordinary differential equation

$$\frac{d\mathbf{X}}{dt}(t) = \alpha^m(\mathbf{X}(t)), \quad \mathbf{X}((m+1)\tau) = x.$$

where  $\alpha^m(x) = (\alpha_1(x, m\tau), \alpha_2(x, m\tau))$ . Because, by Taylor's expansion, we have

$$\begin{aligned}u^m(\mathbf{X}(m\tau)) &= u^m(\mathbf{X}((m+1)\tau)) - \tau \sum_{i=1}^d \frac{\partial u^m}{\partial x_i}(\mathbf{X}((m+1)\tau)) \frac{\partial X_i}{\partial t}((m+1)\tau) + o(\tau) \\ &= u^m(x) - \tau \alpha^m(x) \cdot \nabla u^m(x) + o(\tau)\end{aligned}$$

where  $X_i(t)$  are the  $i$ -th component of  $\mathbf{X}(t)$ ,  $u^m(x) = u(x, m\tau)$  and we used the chain rule and  $x = \mathbf{X}((m+1)\tau)$ . From (5.11), it follows that

$$u^m(X^m(x)) = u^m(x) - \tau \alpha^m(x) \cdot \nabla u^m(x) + o(\tau)$$

Also we apply Taylor's expansion for  $t \rightarrow u^m(x - \alpha^m(x)t)$ ,  $0 \leq t \leq \tau$ , then

$$u^m(x - \alpha\tau) = u^m(x) - \tau \alpha^m(x) \cdot \nabla u^m(x) + o(\tau).$$

Putting

$$\text{convect}(\alpha, -\tau, u^m) \approx u^m(x - \alpha^m \tau)$$

we can get the approximation

$$u^m(X^m(x)) \approx \text{convect}([a_1^m, a_2^m], -\tau, u^m) \text{ by } X^m \approx x \mapsto x - \tau[a_1^m(x), a_2^m(x)]$$

A classical convection problem is that of the “rotating bell” (quoted from [LUCQUIN1998], p.16).

Let  $\Omega$  be the unit disk centered at 0, with its center rotating with speed  $\alpha_1 = y$ ,  $\alpha_2 = -x$ . We consider the problem (5.8) with  $f = 0$  and the initial condition  $u(x, 0) = u^0(x)$ , that is, from (5.10)

$$u^{m+1}(x) = u^m(X^m(x)) \approx \text{convect}(\alpha, -\tau, u^m)$$

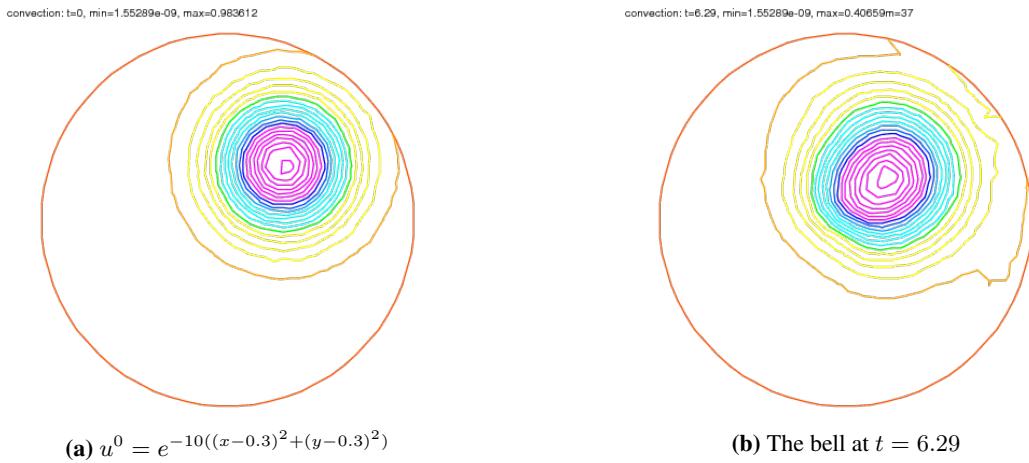
The exact solution is  $u(x, t) = u(\mathbf{X}(t))$  where  $\mathbf{X}$  equals  $x$  rotated around the origin by an angle  $\theta = -t$  (rotate in clockwise). So, if  $u^0$  in a 3D perspective looks like a bell, then  $u$  will have exactly the same shape, but rotated by the same amount. The program consists in solving the equation until  $T = 2\pi$ , that is for a full revolution and to compare the final solution with the initial one; they should be equal.

### Tip: Convect

```

1 // Parameters
2 real dt = 0.17;
3
4 // Mesh
5 border C(t=0, 2*pi) {x=cos(t); y=sin(t);}
6 mesh Th = buildmesh(C(70));
7
8 // Fespace
9 fespace Vh(Th, P1);
10 Vh u0;
11 Vh a1 = -y, a2 = x; //rotation velocity
12 Vh u;
13
14 // Initialization
15 u = exp(-10*((x-0.3)^2 + (y-0.3)^2));
16
17 // Time loop
18 real t = 0.;
19 for (int m = 0; m < 2*pi/dt; m++) {
20     // Update
21     t += dt;
22     u0 = u;
23
24     // Convect
25     u = convect([a1, a2], -dt, u0); //u^{m+1}=u^m(X^m(x))
26
27     // Plot
28     plot(u, cmm=" t=" + t + ", min=" + u[].min + ", max=" + u[].max);
29 }
```

**Note:** The scheme `convect` is unconditionally stable, then the bell become lower and lower (the maximum of  $u^{37}$  is 0.406 as shown in Fig. 5.15a.



### 5.5.3 2D Black-Scholes equation for an European Put option

In mathematical finance, an option on two assets is modeled by a Black-Scholes equations in two space variables, (see for example [WILMOTT1995] or [ACHDOU2005]).

$$\begin{aligned} \partial_t u + \frac{(\sigma_1 x)^2}{2} \frac{\partial^2 u}{\partial x^2} + \frac{(\sigma_2 y)^2}{2} \frac{\partial^2 u}{\partial y^2} \\ + \rho x y \frac{\partial^2 u}{\partial x \partial y} + r S_1 \frac{\partial u}{\partial x} + r S_2 \frac{\partial u}{\partial y} - r P = 0 \end{aligned}$$

which is to be integrated in  $(0, T) \times \mathbb{R}^+ \times \mathbb{R}^+$  subject to, in the case of a put

$$u(x, y, T) = (K - \max(x, y))^+$$

Boundary conditions for this problem may not be so easy to device. As in the one dimensional case the PDE contains boundary conditions on the axis  $x_1 = 0$  and on the axis  $x_2 = 0$ , namely two one dimensional Black-Scholes equations driven respectively by the data  $u(0, +\infty, T)$  and  $u(+\infty, 0, T)$ . These will be automatically accounted for because they are embedded in the PDE. So if we do nothing in the variational form (i.e. if we take a Neumann boundary condition at these two axis in the strong form) there will be no disturbance to these. At infinity in one of the variable, as in 1D, it makes sense to impose  $u = 0$ . We take

$$\sigma_1 = 0.3, \quad \sigma_2 = 0.3, \quad \rho = 0.3, \quad r = 0.05, \quad K = 40, \quad T = 0.5$$

An implicit Euler scheme is used and a mesh adaptation is done every 10 time steps. To have an unconditionally stable scheme, the first order terms are treated by the Characteristic Galerkin method, which, roughly, approximates

$$\frac{\partial u}{\partial t} + a_1 \frac{\partial u}{\partial x} + a_2 \frac{\partial u}{\partial y} \approx \frac{1}{\tau} (u^{n+1}(x) - u^n(x - \alpha\tau))$$

**Tip:** Black-Scholes

```

1 // Parameters
2 int m = 30; int L = 80; int LL = 80; int j = 100; real sigx = 0.3; real sigy = 0.3;
  ↵real rho = 0.3; real r = 0.05; real K = 40; real dt = 0.01;
3
4 // Mesh
5 mesh th = square(m, m, [L*x, LL*y]);
6
7 // Fespace
8 fespace Vh(th, P1);

```

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```

9 Vh u = max(K-max(x,y),0.);
10 Vh xveloc, yveloc, v, uold;
11
12 // Time loop
13 for (int n = 0; n*dt <= 1.0; n++) {
14     // Mesh adaptation
15     if (j > 20) {
16         th = adaptmesh(th, u, verbosity=1, abserror=1, nbjacobi=2,
17                         err=0.001, nbvx=5000, omega=1.8, ratio=1.8, nbsmooth=3,
18                         splitpedge=1, maxsubdiv=5, rescaling=1);
19         j = 0;
20         xveloc = -x*r + x*sigx^2 + x*rho*sigx*sigy/2;
21         yveloc = -y*r + y*sigy^2 + y*rho*sigx*sigy/2;
22         u = u;
23     }
24
25     // Update
26     uold = u;
27
28     // Solve
29     solve eq1(u, v, init=j, solver=LU)
30     = int2d(th) (
31         u*v*(r+1/dt)
32         + dx(u)*dx(v)*(x*sigx)^2/2
33         + dy(u)*dy(v)*(y*sigy)^2/2
34         + (dy(u)*dx(v) + dx(u)*dy(v))*rho*sigx*sigy*x*y/2
35     )
36     - int2d(th) (
37         v*convect([xveloc, yveloc], dt, uold)/dt
38     )
39     + on(2, 3, u=0)
40     ;
41
42     // Update
43     j = j+1;
44 }
45
46 // Plot
47 plot(u, wait=true, value=true);

```

Results are shown on Fig. 5.16a and Fig. 5.16b.

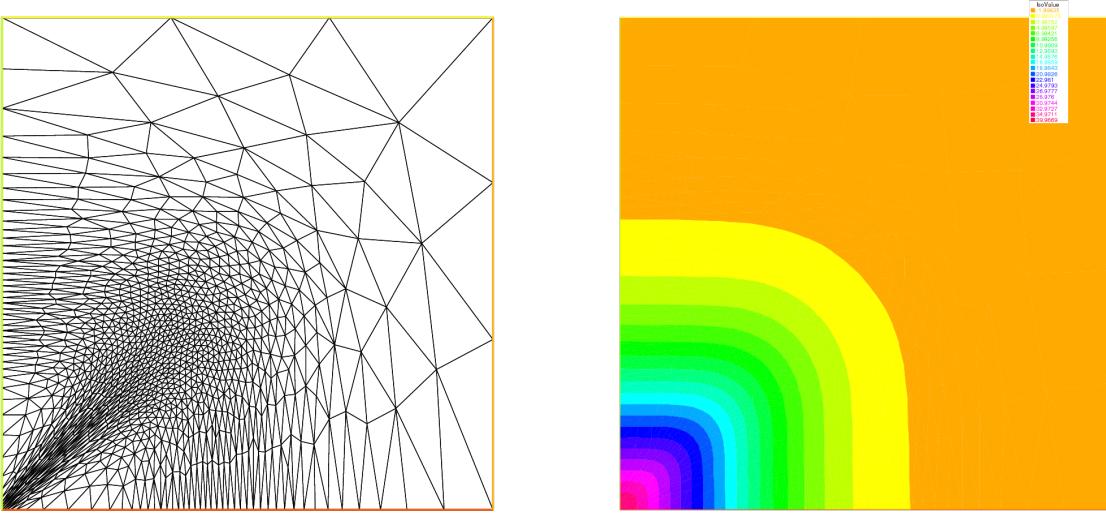
## 5.6 Navier-Stokes equations

The Stokes equations are: for a given  $\mathbf{f} \in L^2(\Omega)^2$ :

$$\left. \begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned} \right\} \quad \text{in } \Omega \quad (5.11)$$

where  $\mathbf{u} = (u_1, u_2)$  is the velocity vector and  $p$  the pressure. For simplicity, let us choose Dirichlet boundary conditions on the velocity,  $\mathbf{u} = \mathbf{u}_\Gamma$  on  $\Gamma$ .

In [TEMAM1977], Theorem 2.2, there is a weak form of (5.11):



(a) The adapted triangulation

(b) The level line of the European basquet put option

Find  $\mathbf{v} = (v_1, v_2) \in \mathbf{V}(\Omega)$ :

$$\mathbf{V}(\Omega) = \{\mathbf{w} \in H_0^1(\Omega)^2 \mid \operatorname{div} \mathbf{w} = 0\}$$

which satisfy:

$$\sum_{i=1}^2 \int_{\Omega} \nabla u_i \cdot \nabla v_i = \int_{\Omega} \mathbf{f} \cdot \mathbf{w} \quad \text{for all } v \in V$$

Here it is used the existence  $p \in H^1(\Omega)$  such that  $\mathbf{u} = \nabla p$ , if:

$$\int_{\Omega} \mathbf{u} \cdot \mathbf{v} = 0 \quad \text{for all } \mathbf{v} \in V$$

Another weak form is derived as follows: We put:

$$\mathbf{V} = H_0^1(\Omega)^2; \quad W = \left\{ q \in L^2(\Omega) \mid \int_{\Omega} q = 0 \right\}$$

By multiplying the first equation in (5.11) with  $v \in V$  and the second with  $q \in W$ , subsequent integration over  $\Omega$ , and an application of Green's formula, we have:

$$\begin{aligned} \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} - \int_{\Omega} \operatorname{div} \mathbf{v} p &= \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \\ \int_{\Omega} \operatorname{div} \mathbf{u} q &= 0 \end{aligned}$$

This yields the weak form of (5.11):

Find  $(\mathbf{u}, p) \in \mathbf{V} \times W$  such that:

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) \\ b(\mathbf{u}, q) &= 0 \end{aligned}$$

for all  $(\mathbf{v}, q) \in V \times W$ , where:

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} = \sum_{i=1}^2 \int_{\Omega} \nabla u_i \cdot \nabla v_i \\ b(\mathbf{u}, q) &= - \int_{\Omega} \operatorname{div} \mathbf{u} q \end{aligned}$$

Now, we consider finite element spaces  $\mathbf{V}_h \subset \mathbf{V}$  and  $W_h \subset W$ , and we assume the following basis functions:

$$\begin{aligned}\mathbf{V}_h &= V_h \times V_h, \quad V_h = \{v_h \mid v_h = v_1\phi_1 + \cdots + v_{M_V}\phi_{M_V}\}, \\ W_h &= \{q_h \mid q_h = q_1\varphi_1 + \cdots + q_{M_W}\varphi_{M_W}\}\end{aligned}$$

The discrete weak form is: Find  $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$  such that:

$$\begin{aligned}a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p) &= (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h \\ b(\mathbf{u}_h, q_h) &= 0, \quad \forall q_h \in W_h\end{aligned}\tag{5.12}$$

---

**Note:** Assume that:

1. There is a constant  $\alpha_h > 0$  such that:

$$a(\mathbf{v}_h, \mathbf{v}_h) \geq \alpha \|\mathbf{v}_h\|_{1,\Omega}^2 \quad \text{for all } \mathbf{v}_h \in Z_h$$

where:

$$Z_h = \{\mathbf{v}_h \in \mathbf{V}_h \mid b(\mathbf{w}_h, q_h) = 0 \quad \text{for all } q_h \in W_h\}$$

2. There is a constant  $\beta_h > 0$  such that:

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{1,\Omega}} \geq \beta_h \|q_h\|_{0,\Omega} \quad \text{for all } q_h \in W_h$$

Then we have an unique solution  $(\mathbf{u}_h, p_h)$  of (5.12) satisfying:

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \leq C \left( \inf_{\mathbf{v}_h \in \mathbf{V}_h} \|u - v_h\|_{1,\Omega} + \inf_{q_h \in W_h} \|p - q_h\|_{0,\Omega} \right)$$

with a constant  $C > 0$  (see e.g. [ROBERTS1993], Theorem 10.4).

---

Let us denote that:

$$\begin{aligned}A &= (A_{ij}), A_{ij} = \int_{\Omega} \nabla \phi_j \cdot \nabla \phi_i \quad i, j = 1, \dots, M_V \\ \mathbf{B} &= (Bx_{ij}, By_{ij}), Bx_{ij} = - \int_{\Omega} \partial \phi_j / \partial x \varphi_i \quad By_{ij} = - \int_{\Omega} \partial \phi_j / \partial y \varphi_i \\ &\quad i = 1, \dots, M_W; j = 1, \dots, M_V\end{aligned}$$

then (5.12) is written by:

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}^* \\ \mathbf{B} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{U}_h \\ \{p_h\} \end{pmatrix} = \begin{pmatrix} \mathbf{F}_h \\ 0 \end{pmatrix}$$

where:

$$\mathbf{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad \mathbf{B}^* = \begin{Bmatrix} Bx^T \\ By^T \end{Bmatrix} \quad \mathbf{U}_h = \begin{Bmatrix} \{u_{1,h}\} \\ \{u_{2,h}\} \end{Bmatrix} \quad \mathbf{F}_h = \begin{Bmatrix} \{\int_{\Omega} f_1 \phi_i\} \\ \{\int_{\Omega} f_2 \phi_i\} \end{Bmatrix}$$

**Penalty method:** This method consists of replacing (5.12) by a more regular problem:

Find  $(\mathbf{v}_h^\epsilon, p_h^\epsilon) \in \mathbf{V}_h \times \tilde{W}_h$  satisfying:

$$\begin{aligned}a(\mathbf{u}_h^\epsilon, \mathbf{v}_h) + b(\mathbf{v}_h, p_h^\epsilon) &= (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h \\ b(\mathbf{u}_h^\epsilon, q_h) - \epsilon(p_h^\epsilon, q_h) &= 0, \quad \forall q_h \in \tilde{W}_h\end{aligned}\tag{5.13}$$

where  $\tilde{W}_h \subset L^2(\Omega)$ . Formally, we have:

$$\operatorname{div} \mathbf{u}_h^\epsilon = \epsilon p_h^\epsilon$$

and the corresponding algebraic problem:

$$\begin{pmatrix} \mathbf{A} & B^* \\ B & -\epsilon I \end{pmatrix} \begin{pmatrix} \mathbf{U}_h^\epsilon \\ \{p_h^\epsilon\} \end{pmatrix} = \begin{pmatrix} \mathbf{F}_h \\ 0 \end{pmatrix}$$

**Note:** We can eliminate  $p_h^\epsilon = (1/\epsilon)B\mathbf{U}_h^\epsilon$  to obtain:

$$(A + (1/\epsilon)B^*B)\mathbf{U}_h^\epsilon = \mathbf{F}_h^\epsilon \quad (5.14)$$

Since the matrix  $A + (1/\epsilon)B^*B$  is symmetric, positive-definite, and sparse, (5.14) can be solved by known technique. There is a constant  $C > 0$  independent of  $\epsilon$  such that:

$$\|\mathbf{u}_h - \mathbf{u}_h^\epsilon\|_{1,\Omega} + \|p_h - p_h^\epsilon\|_{0,\Omega} \leq C\epsilon$$

(see e.g. [ROBERTS1993], 17.2)

### Tip: Cavity

The driven cavity flow problem is solved first at zero Reynolds number (Stokes flow) and then at Reynolds 100. The velocity pressure formulation is used first and then the calculation is repeated with the stream function vorticity formulation.

We solve the driven cavity problem by the penalty method (5.13) where  $\mathbf{u}_\Gamma \cdot \mathbf{n} = 0$  and  $\mathbf{u}_\Gamma \cdot \mathbf{s} = 1$  on the top boundary and zero elsewhere ( $\mathbf{n}$  is the unit normal to  $\Gamma$ , and  $\mathbf{s}$  the unit tangent to  $\Gamma$ ).

The mesh is constructed by:

```
1 mesh Th = square(8, 8);
```

We use a classical Taylor-Hood element technique to solve the problem:

The velocity is approximated with the  $P_2$  FE ( $X_h$  space), and the pressure is approximated with the  $P_1$  FE ( $M_h$  space), where:

$$X_h = \{\mathbf{v} \in H^1([0, 1]^2) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_2\}$$

and:

$$M_h = \{v \in H^1([0, 1]^2) \mid \forall K \in \mathcal{T}_h \quad v|_K \in P_1\}$$

The FE spaces and functions are constructed by:

```
1 fespace Xh(Th, P2); //definition of the velocity component space
2 fespace Mh(Th, P1); //definition of the pressure space
3 Xh u2, v2;
4 Xh u1, v1;
5 Mh p, q;
```

The Stokes operator is implemented as a system-solve for the velocity  $(u1, u2)$  and the pressure  $p$ . The test function for the velocity is  $(v1, v2)$  and  $q$  for the pressure, so the variational form (5.12) in freefem language is:

```

1 solve Stokes (u1, u2, p, v1, v2, q, solver=Croult)
2   = int2d(Th) (
3     (
4       dx(u1)*dx(v1)
5       + dy(u1)*dy(v1)
6       + dx(u2)*dx(v2)
7       + dy(u2)*dy(v2)
8     )
9     - p*q*(0.000001)
10    - p*dx(v1) - p*dy(v2)
11    - dx(u1)*q - dy(u2)*q
12  )
13  + on(3, u1=1, u2=0)
14  + on(1, 2, 4, u1=0, u2=0)
15 ;

```

Each unknown has its own boundary conditions.

If the streamlines are required, they can be computed by finding  $\psi$  such that  $\text{rot}\psi = u$  or better:

$$-\Delta\psi = \nabla \times u$$

```

1 Xh psi, phi;
2
3 solve streamlines (psi, phi)
4   = int2d(Th) (
5     dx(psi)*dx(phi)
6     + dy(psi)*dy(phi)
7   )
8   + int2d(Th) (
9     - phi*(dy(u1) - dx(u2))
10   )
11  + on(1, 2, 3, 4, psi=0)
12 ;

```

Now the Navier-Stokes equations are solved:

$$\frac{\partial u}{\partial t} + u \cdot \nabla u - \nu \Delta u + \nabla p = 0, \nabla \cdot u = 0$$

with the same boundary conditions and with initial conditions  $u = 0$ .

This is implemented by using the convection operator `convect` for the term  $\frac{\partial u}{\partial t} + u \cdot \nabla u$ , giving a discretization in time

$$\begin{aligned} \frac{1}{\tau}(u^{n+1} - u^n \circ X^n) - \nu \Delta u^{n+1} + \nabla p^{n+1} &= 0, \\ \nabla \cdot u^{n+1} &= 0 \end{aligned}$$

The term  $u^n \circ X^n(x) \approx u^n(x - u^n(x)\tau)$  will be computed by the operator `convect`, so we obtain:

```

1 int i=0;
2 real alpha=1/dt;
3 problem NS (u1, u2, p, v1, v2, q, solver=Croult, init=i)
4   = int2d(Th) (
5     alpha*(u1*v1 + u2*v2)
6     + nu * (
7       dx(u1)*dx(v1) + dy(u1)*dy(v1)
8       + dx(u2)*dx(v2) + dy(u2)*dy(v2)

```

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```

9      )
10     - p*q*(0.000001)
11     - p*dx(v1) - p*dy(v2)
12     - dx(u1)*q - dy(u2)*q
13   )
14   + int2d(Th) (
15     - alpha*convect([up1,up2],-dt,up1)*v1
16     - alpha*convect([up1,up2],-dt,up2)*v2
17   )
18   + on(3, u1=1, u2=0)
19   + on(1, 2, 4,u1=0, u2=0)
20 ;
21
22 // Time loop
23 for (i = 0; i <= 10; i++) {
24   // Update
25   up1 = u1;
26   up2 = u2;
27
28   // Solve
29   NS;
30
31   // Plot
32   if (!(i % 10))
33     plot(coef=0.2, cmm="[u1,u2] and p", p, [u1, u2]);
34 }
```

Notice that the stiffness matrices are reused (keyword `init=i`)

The complete script is available in *cavity example*.

### 5.6.1 Uzawa Algorithm and Conjugate Gradients

We solve Stokes problem without penalty. The classical iterative method of Uzawa is described by the algorithm (see e.g. [ROBERTS1993], 17.3, [GLOWINSKI1979], 13 or [GLOWINSKI1985], 13):

- **Initialize:** Let  $p_h^0$  be an arbitrary chosen element of  $L^2(\Omega)$ .
- **Calculate :math:`\mathbf{u}\_h^n`:** Once  $p_h^n$  is known,  $\mathbf{v}_h^n$  is the solution of:

$$\mathbf{u}_h^n = A^{-1}(\mathbf{f}_h - \mathbf{B}^* p_h^n)$$

- **Advance :math:`\mathbf{p}\_h^n`:** Let  $p_h^{n+1}$  be defined by;

$$p_h^{n+1} = p_h^n + \rho_n \mathbf{B} \mathbf{u}_h^n$$

There is a constant  $\alpha > 0$  such that  $\alpha \leq \rho_n \leq 2$  for each  $n$ , then  $\mathbf{u}_h^n$  converges to the solution  $\mathbf{u}_h$ , and then  $B\mathbf{v}_h^n \rightarrow 0$  as  $n \rightarrow \infty$  from the Advance  $p_h$ . This method in general converges quite slowly.

First we define mesh, and the Taylor-Hood approximation. So  $X_h$  is the velocity space, and  $M_h$  is the pressure space.

**Tip:** Stokes Uzawa

```

1 // Mesh
2 mesh Th = square(10, 10);
3
4 // Fespace
5 fespace Xh(Th, P2);
6 Xh u1, u2;
7 Xh bc1, bc2;
8 Xh b;
9
10 fespace Mh(Th, P1);
11 Mh p;
12 Mh ppp; //ppp is a working pressure
13
14 // Problem
15 varf bx (u1, q) = int2d(Th)(-(dx(u1)*q));
16 varf by (u1, q) = int2d(Th)(-(dy(u1)*q));
17 varf a (u1, u2)
= int2d(Th)(
    dx(u1)*dx(u2)
    + dy(u1)*dy(u2)
)
18     + on(3, u1=1)
19     + on(1, 2, 4, u1=0);
20 //remark: put the on(3,u1=1) before on(1,2,4,u1=0)
21 //because we want zero on intersection
22
23 matrix A = a(Xh, Xh, solver=CG);
24 matrix Bx = bx(Xh, Mh); //B=(Bx, By)
25 matrix By = by(Xh, Mh);
26
27 bc1[] = a(0,Xh); //boundary condition contribution on u1
28 bc2 = 0; //no boundary condition contribution on u2
29
30
31 //p_h^n -> B A^-1 - B^* p_h^n = -div u_h
32 //is realized as the function divup
33 func real[int] divup (real[int] & pp){
34     //compute u1(pp)
35     b[] = Bx'*pp;
36     b[] *= -1;
37     b[] += bc1[];
38     u1[] = A^-1*b[];
39     //compute u2(pp)
40     b[] = By'*pp;
41     b[] *= -1;
42     b[] += bc2[];
43     u2[] = A^-1*b[];
44     //u^n = (A^-1 Bx^T p^n, By^T p^n)^T
45     ppp[] = Bx*u1[]; //ppp = Bx u_1
46     ppp[] += By*u2[]; //+ By u_2
47
48     return ppp[];
49 }
50
51 // Initialization
52 p=0; //p_h^0 = 0
53 LinearCG(divup, p[], eps=1.e-6, nbiter=50); //p_h^{n+1} = p_h^n + B u_h^n
54 // if n> 50 or |p_h^{n+1} - p_h^n| <= 10^-6, then the loop end
55
56
57

```

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```

58 divup(p[]); //compute the final solution
59
60 plot([u1, u2], p, wait=1, value=true, coef=0.1);

```

## 5.6.2 NSUzawaCahouetChabart.edp

In this example we solve the Navier-Stokes equation past a cylinder with the Uzawa algorithm preconditioned by the Cahouet-Chabart method (see [GLOWINSKI2003] for all the details).

The idea of the preconditioner is that in a periodic domain, all differential operators commute and the Uzawa algorithm comes to solving the linear operator  $\nabla \cdot ((\alpha Id + \nu \Delta)^{-1} \nabla)$ , where  $Id$  is the identity operator. So the preconditioner suggested is  $\alpha \Delta^{-1} + \nu Id$ .

To implement this, we do:

### Tip: NS Uzawa Cahouet Chabart

```

1 // Parameters
2 verbosity = 0;
3 real D = 0.1;
4 real H = 0.41;
5 real cx0 = 0.2;
6 real cy0 = 0.2; //center of cylinder
7 real xa = 0.15;
8 real ya = 0.2;
9 real xe = 0.25;
10 real ye = 0.2;
11 int nn = 15;
12
13 //TODO
14 real Um = 1.5; //max velocity (Rey 100)
15 real nu = 1e-3;
16
17 func U1 = 4.*Um*y*(H-y)/(H*H); //Boundary condition
18 func U2 = 0.;
19 real T=2;
20 real dt = D/nn/Um; //CFL = 1
21 real epspq = 1e-10;
22 real eps = 1e-6;
23
24 // Variables
25 func Ub = Um*2./3.;
26 real alpha = 1/dt;
27 real Rey = Ub*D/nu;
28 real t = 0.;
29
30 // Mesh
31 border fr1(t=0, 2.2){x=t; y=0; label=1;};
32 border fr2(t=0, H){x=2.2; y=t; label=2;};
33 border fr3(t=2.2, 0){x=t; y=H; label=1;};
34 border fr4(t=H, 0){x=0; y=t; label=1;};
35 border fr5(t=2*pi, 0){x=cx0+D*sin(t)/2; y=cy0+D*cos(t)/2; label=3;};
36 mesh Th = buildmesh(fr1(5*nn) + fr2(nn) + fr3(5*nn) + fr4(nn) + fr5(-nn*3));
37

```

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```

38 // Fespace
39 fespace Mh(Th, [P1]);
40 Mh p;
41
42 fespace Xh(Th, [P2]);
43 Xh u1, u2;
44
45 fespace Wh(Th, [P1dc]);
46 Wh w; //vorticity
47
48 // Macro
49 macro grad(u) [dx(u), dy(u)] //
50 macro div(u1, u2) (dx(u1) + dy(u2)) //
51
52 // Problem
53 varf von1 ([u1, u2, p], [v1, v2, q])
54     = on(3, u1=0, u2=0)
55     + on(1, u1=U1, u2=U2)
56 ;
57
58 //remark : the value 100 in next varf is manualy fitted, because free outlet.
59 varf vA (p, q) =
60     int2d(Th) (
61         grad(p)' * grad(q)
62     )
63     + int1d(Th, 2) (
64         100*p*q
65     )
66 ;
67
68 varf vM (p, q)
69     = int2d(Th, qft=qf2pT) (
70         p*q
71     )
72     + on(2, p=0)
73 ;
74
75 varf vu ([u1], [v1])
76     = int2d(Th) (
77         alpha*(u1*v1)
78         + nu*(grad(u1)' * grad(v1))
79     )
80     + on(1, 3, u1=0)
81 ;
82
83 varf vul ([p], [v1]) = int2d(Th) (p*dx(v1));
84 varf vu2 ([p], [v1]) = int2d(Th) (p*dy(v1));
85
86 varf vonu1 ([u1], [v1]) = on(1, u1=U1) + on(3, u1=0);
87 varf vonu2 ([u1], [v1]) = on(1, u1=U2) + on(3, u1=0);
88
89 matrix pAM = vM(Mh, Mh, solver=UMFPACK);
90 matrix pAA = vA(Mh, Mh, solver=UMFPACK);
91 matrix AU = vu(Xh, Xh, solver=UMFPACK);
92 matrix B1 = vul(Mh, Xh);
93 matrix B2 = vu2(Mh, Xh);
94

```

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```

95 real[int] brhs1 = vonu1(0, Xh);
96 real[int] brhs2 = vonu2(0, Xh);
97
98 varf vrhs1(uu, vv) = int2d(Th)(convect([u1, u2], -dt, u1)*vv*alpha) + vonu1;
99 varf vrhs2(v2, v1) = int2d(Th)(convect([u1, u2], -dt, u2)*v1*alpha) + vonu2;
100
101 // Uzawa function
102 func real[int] JUzawa (real[int] & pp){
103     real[int] b1 = brhs1; b1 += B1*pp;
104     real[int] b2 = brhs2; b2 += B2*pp;
105     u1[] = AU^-1 * b1;
106     u2[] = AU^-1 * b2;
107     pp = B1'*u1[];
108     pp += B2'*u2[];
109     pp = -pp;
110     return pp;
111 }
112
113 // Preconditioner function
114 func real[int] Precon (real[int] & p){
115     real[int] pa = pAA^-1*p;
116     real[int] pm = pAM^-1*p;
117     real[int] pp = alpha*pa + nu*pm;
118     return pp;
119 }
120
121 // Initialization
122 p = 0;
123
124 // Time loop
125 int ndt = T/dt;
126 for(int i = 0; i < ndt; ++i){
127     // Update
128     brhs1 = vrhs1(0, Xh);
129     brhs2 = vrhs2(0, Xh);
130
131     // Solve
132     int res = LinearCG(JUzawa, p[], precon=Precon, nbiter=100, verbosity=10,
133     ↳veps=eps);
134     assert(res==1);
135     eps = -abs(eps);
136
137     // Vorticity
138     w = -dy(u1) + dx(u2);
139     plot(w, fill=true, wait=0, nbiso=40);
140
141     // Update
142     dt = min(dt, T-t);
143     t += dt;
144     if(dt < 1e-10*T) break;
145 }
146
147 // Plot
148 plot(w, fill=true, nbiso=40);
149
150 // Display
cout << "u1 max = " << u1[].linfty

```

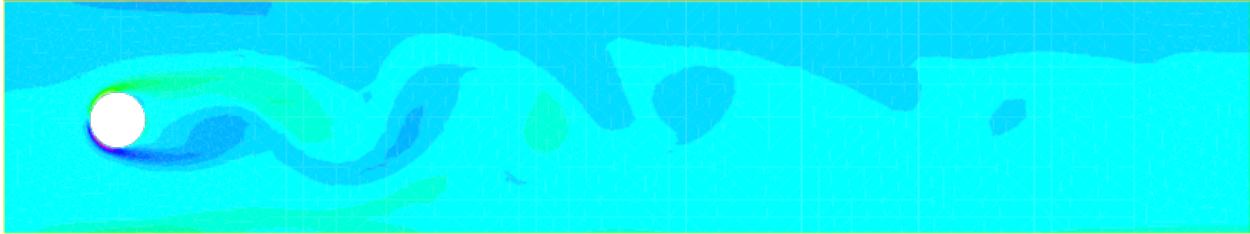
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```
151    << ", u2_max = " << u2[].linfty
152    << ", p_max = " << p[].max << endl;
```

**Warning:** Stop test of the conjugate gradient

Because we start from the previous solution and the end the previous solution is close to the final solution, don't take a relative stop test to the first residual, take an absolute stop test (negative here).



**Fig. 5.17:** The vorticity at Reynolds number 100 a time 2s with the Cahouet-Chabart method.

## 5.7 Variational Inequality

We present, a classical example of variational inequality.

Let us denote  $\mathcal{C} = \{u \in H_0^1(\Omega), u \leq g\}$

The problem is:

$$u = \arg \min_{u \in \mathcal{C}} J(u) = \frac{1}{2} \int_{\Omega} \nabla u \cdot \nabla u - \int_{\Omega} f u$$

where  $f$  and  $g$  are given function.

The solution is a projection on the convex  $\mathcal{C}$  of  $f^*$  for the scalar product  $((v, w)) = \int_{\Omega} \nabla v \cdot \nabla w$  of  $H_0^1(\Omega)$  where  $f^*$  is solution of:

$$(f^*, v) = \int_{\Omega} f v, \forall v \in H_0^1(\Omega)$$

The projection on a convex satisfy clearly  $\forall v \in \mathcal{C}, ((u - v, u - \tilde{f})) \leq 0$ , and after expanding, we get the classical inequality:

$$\forall v \in \mathcal{C}, \int_{\Omega} \nabla(u - v) \cdot \nabla u \leq \int_{\Omega} (u - v) f$$

We can also rewrite the problem as a saddle point problem:

Find  $\lambda, u$  such that:

$$\max_{\lambda \in L^2(\Omega), \lambda \geq 0} \min_{u \in H_0^1(\Omega)} \mathcal{L}(u, \lambda) = \frac{1}{2} \int_{\Omega} \nabla u \cdot \nabla u - \int_{\Omega} f u + \int_{\Omega} \lambda(u - g)^+$$

where  $((u - g)^+ = \max(0, u - g))$ .

This saddle point problem is equivalent to find  $u, \lambda$  such that:

$$\begin{cases} \int_{\Omega} \nabla u \cdot \nabla v + \lambda v^+ d\omega = \int_{\Omega} f v, & \forall v \in H_0^1(\Omega) \\ \int_{\Omega} \mu(u - g)^+ = 0, & \forall \mu \in L^2(\Omega), \mu \geq 0, \lambda \geq 0, \end{cases}$$

An algorithm to solve the previous problem is:

1.  $k=0$ , and choose  $\lambda_0$  belong  $H^{-1}(\Omega)$
2. Loop on  $k = 0, \dots$ 
  - set  $I_k = \{x \in \Omega / \lambda_k + c * (u_{k+1} - g) \leq 0\}$
  - $V_{g,k+1} = \{v \in H_0^1(\Omega) / v = g \text{ on } I_k\}$ ,
  - $V_{0,k+1} = \{v \in H_0^1(\Omega) / v = 0 \text{ on } I_k\}$ ,
  - Find  $u_{k+1} \in V_{g,k+1}$  and  $\lambda_{k+1} \in H^{-1}(\Omega)$  such that

$$\begin{cases} \int_{\Omega} \nabla u_{k+1} \cdot \nabla v_{k+1} d\omega = \int_{\Omega} f v_{k+1}, & \forall v_{k+1} \in V_{0,k+1} \\ \langle \lambda_{k+1}, v \rangle = \int_{\Omega} \nabla u_{k+1} \cdot \nabla v - f v d\omega \end{cases}$$

where  $\langle \cdot, \cdot \rangle$  is the duality bracket between  $H_0^1(\Omega)$  and  $H^{-1}(\Omega)$ , and  $c$  is a penalty constant (large enough).

You can find all the mathematics about this algorithm in [ITO2003] [HINTERMULLER2002].

Now how to do that in **FreeFEM**? The full example is:

---

**Tip:** Variational inequality

```

1  load "medit"
2
3 // Parameters
4  real eps = 1e-5;
5  real c = 1000; //penalty parameter of the algoritm
6  real tgv = 1e30; //a huge value for exact penalization
7  func f = 1; //right hand side function
8  func fd = 0; //Dirichlet boundary condition function
9
10 // Mesh
11 mesh Th = square(20, 20);
12
13 // Fespace
14 fespace Vh(Th, P1);
15 int n = Vh.ndof; //number of degree of freedom
16 Vh uh, uhp; //u^{n+1} and u^n
17 Vh Ik; //to define the set where the containt is reached.
18 Vh g = 0.05; //discret function g
19 Vh lambda = 0;
20
21 // Problem
22 varf a (uh, vh)
23   = int2d(Th) (
24     dx(uh)*dx(vh)
25     + dy(uh)*dy(vh)
26   )

```

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```

27   - int2d(Th) (
28     f*vh
29   )
30   + on(1, 2, 3, 4, uh=fd)
31 ;
32
33 //the mass Matrix construction
34 varf vM (uh, vh) = int2d(Th) (uh*vh);
35
36 //two versions of the matrix of the problem
37 matrix A = a(Vh, Vh, tgv=tgv, solver=CG); //one changing
38 matrix AA = a(Vh, Vh, solver=CG); //one for computing residual
39
40 matrix M = vM(Vh, Vh); //to do a fast computing of L^2 norm : sqrt(u'* (w=M*u))
41
42 real[int] Aiin(n);
43 real[int] Aii = A.diag; //get the diagonal of the matrix
44 real[int] rhs = a(0, Vh, tgv=tgv);
45
46 // Initialization
47 Ik = 0;
48 uhp = -tgv;
49
50 // Loop
51 for(int iter = 0; iter < 100; ++iter){
52   // Update
53   real[int] b = rhs; //get a copy of the Right hand side
54   real[int] Ak(n); //the complementary of Ik ( !Ik = (Ik-1) )
55   Ak = 1.; Ak -= Ik[];
56   //adding new locking condition on b and on the diagonal if (Ik ==1 )
57   b = Ik[] .* g[]; b *= tgv; b -= Ak .* rhs;
58   Aiin = Ik[] * tgv; Aiin += Ak .* Aii; //set Aii= tgv i in Ik
59   A.diag = Aiin; //set the matrix diagonal
60   set(A, solver=CG); //important to change preconditioning for solving
61
62   // Solve
63   uh[] = A^-1* b; //solve the problem with more locking condition
64
65   // Residual
66   lambda[] = AA * uh[]; //compute the residual (fast with matrix)
67   lambda[] += rhs; //remark rhs = -\int f v
68
69   Ik = (lambda + c*( g- uh)) < 0.; //the new locking value
70
71   // Plot
72   plot(Ik, wait=true, cmm=" lock set ", value=true, fill=true);
73   plot(uh, wait=true, cmm="uh");
74
75   // Error
76   //trick to compute L^2 norm of the variation (fast method)
77   real[int] diff(n), Mdiff(n);
78   diff = uh[] - uhp[];
79   Mdiff = M*diff;
80   real err = sqrt(Mdiff'*diff);
81   cout << "|| u_{k=1} - u_{k} ||_2 = " << err << endl;
82
83   // Stop test

```

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```

84     if(err < eps) break;
85
86     // Update
87     uhp[] = uh[];
88 }
89
90 // Plot
91 medit ("uh", Th, uh);

```

**Note:** As you can see on this example, some vector, or matrix operator are not implemented so a way is to skip the expression and we use operator  $+=$ ,  $-=$  to merge the result.

## 5.8 Domain decomposition

We present three classic examples of domain decomposition technique: first, Schwarz algorithm with overlapping, second Schwarz algorithm without overlapping (also call Shur complement), and last we show to use the conjugate gradient to solve the boundary problem of the Shur complement.

### 5.8.1 Schwarz overlapping

To solve:

$$-\Delta u = f, \text{ in } \Omega = \Omega_1 \cup \Omega_2 \quad u|_{\Gamma} = 0$$

the Schwarz algorithm runs like this:

$$\begin{aligned} -\Delta u_1^{n+1} &= f \text{ in } \Omega_1 \quad u_1^{n+1}|_{\Gamma_1} = u_2^n \\ -\Delta u_2^{n+1} &= f \text{ in } \Omega_2 \quad u_2^{n+1}|_{\Gamma_2} = u_1^n \end{aligned}$$

where  $\Gamma_i$  is the boundary of  $\Omega_i$  and on the condition that  $\Omega_1 \cap \Omega_2 \neq \emptyset$  and that  $u_i$  are zero at iteration 1.

Here we take  $\Omega_1$  to be a quadrangle,  $\Omega_2$  a disk and we apply the algorithm starting from zero.

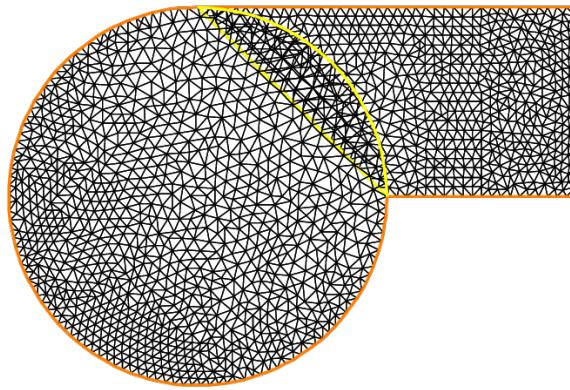
**Tip:** Schwarz overlapping

```

1 // Parameters
2 int inside = 2; //inside boundary
3 int outside = 1; //outside boundary
4 int n = 4;
5
6 // Mesh
7 border a(t=1, 2){x=t; y=0; label=outside;};
8 border b(t=0, 1){x=2; y=t; label=outside;};
9 border c(t=2, 0){x=t; y=1; label=outside;};
10 border d(t=1, 0){x=1-t; y=t; label=inside;};
11 border e(t=0, pi/2){x=cos(t); y=sin(t); label=inside;};
12 border e1(t=pi/2, 2*pi){x=cos(t); y=sin(t); label=outside;};
13 mesh th = buildmesh(a(5*n) + b(5*n) + c(10*n) + d(5*n));
14 mesh TH = buildmesh(e(5*n) + e1(25*n));

```

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**Fig. 5.18:** The 2 overlapping mesh TH and th

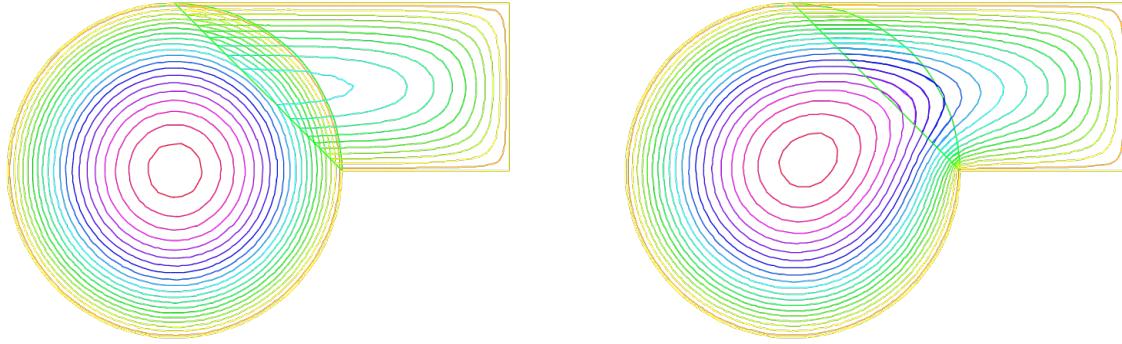
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```

15 plot(th, TH, wait=true); //to see the 2 meshes
16
17 // Fespace
18 fespace vh(th, P1);
19 vh u=0, v;
20
21 fespace VH(TH, P1);
22 VH U, V;
23
24 // Problem
25 int i = 0;
26 problem PB (U, V, init=i, solver=Cholesky)
27   = int2d(TH) (
28     dx(U)*dx(V)
29     + dy(U)*dy(V)
30   )
31   + int2d(TH) (
32     - V
33   )
34   + on(inside, U=u)
35   + on(outside, U=0)
36   ;
37
38 problem pb (u, v, init=i, solver=Cholesky)
39   = int2d(th) (
40     dx(u)*dx(v)
41     + dy(u)*dy(v)
42   )
43   + int2d(th) (
44     - v
45   )

```

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(a) Isovalues of the solution at iteration 0

(b) Isovalues of the solution at iteration 0

**Fig. 5.19:** Schwarz overlapping

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```

46   + on (inside, u=U)
47   + on (outside, u=0)
48 ;
49
50 // Calculation loop
51 for (i = 0 ; i < 10; i++) {
52   // Solve
53   PB;
54   pb;
55
56   // Plot
57   plot(U, u, wait=true);
58 }
```

## 5.8.2 Schwarz non overlapping Scheme

To solve:

$$-\Delta u = f \text{ in } \Omega = \Omega_1 \cup \Omega_2 \quad u|_{\Gamma} = 0$$

the Schwarz algorithm for domain decomposition without overlapping runs like this

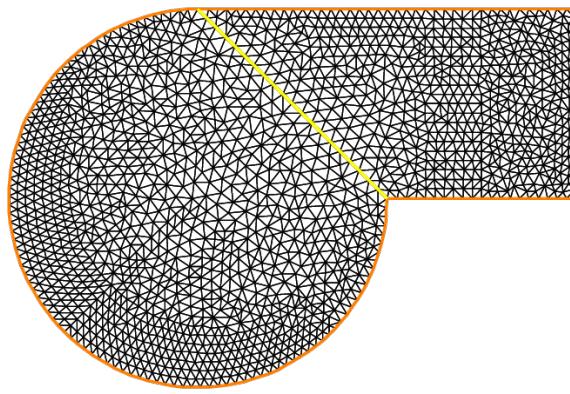
Let introduce  $\Gamma_i$  is common the boundary of  $\Omega_1$  and  $\Omega_2$  and  $\Gamma_e^i = \partial\Omega_i \setminus \Gamma_i$ .

The problem find  $\lambda$  such that  $(u_1|_{\Gamma_i} = u_2|_{\Gamma_i})$  where  $u_i$  is solution of the following Laplace problem:

$$-\Delta u_i = f \text{ in } \Omega_i \quad u_i|_{\Gamma_i} = \lambda \quad u_i|_{\Gamma_e^i} = 0$$

To solve this problem we just make a loop with upgrading  $\lambda$  with

$$\lambda = \lambda \pm \frac{(u_1 - u_2)}{2}$$



**Fig. 5.20:** The two none overlapping mesh  $\text{TH}$  and  $\text{th}$

where the sign + or - of  $\pm$  is choose to have convergence.

**Tip:** Schwarz non-overlapping

```

1 // Parameters
2 int inside = 2; int outside = 1; int n = 4;
3
4 // Mesh
5 border a(t=1, 2){x=t; y=0; label=outside;};
6 border b(t=0, 1){x=2; y=t; label=outside;};
7 border c(t=2, 0){x=t; y=1; label=outside;};
8 border d(t=1, 0){x=1-t; y=t; label=inside;};
9 border e(t=0, 1){x=1-t; y=t; label=inside;};
10 border e1(t=pi/2, 2*pi){x=cos(t); y=sin(t); label=outside;};
11 mesh th = buildmesh(a(5*n) + b(5*n) + c(10*n) + d(5*n));
12 mesh TH = buildmesh(e(5*n) + e1(25*n));
13 plot(th, TH, wait=true);
14
15 // Fespace
16 fespace vh(th, P1);
17 vh u=0, v;
18 vh lambda=0;
19
20 fespace VH(TH, P1);
21 VH U, V;
22
23 // Problem
24 int i = 0;
25 problem PB (U, V, init=i, solver=Cholesky)
26   = int2d(TH) (
27     dx(U)*dx(V)
28     + dy(U)*dy(V)

```

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```

29     )
30     + int2d(TH) (
31         - V
32     )
33     + int1d(TH, inside) (
34         lambda*V
35     )
36     + on(outside, U= 0 )
37 ;
38
39 problem pb (u, v, init=i, solver=Cholesky)
40 = int2d(th) (
41     dx(u)*dx(v)
42     + dy(u)*dy(v)
43 )
44 + int2d(th) (
45     - v
46 )
47 + int1d(th, inside) (
48     - lambda*v
49 )
50 + on(outside, u=0)
51 ;
52
53 for (i = 0; i < 10; i++) {
54     // Solve
55     PB;
56     pb;
57     lambda = lambda - (u-U)/2;
58
59     // Plot
60     plot(U,u,wait=true);
61 }
62
63 // Plot
64 plot(U, u);

```

### 5.8.3 Schwarz conjuguate gradient

To solve  $-\Delta u = f$  in  $\Omega = \Omega_1 \cup \Omega_2$   $u|_{\Gamma} = 0$  the Schwarz algorithm for domain decomposition without overlapping runs like this

Let introduce  $\Gamma_i$  is common the boundary of  $\Omega_1$  and  $\Omega_2$  and  $\Gamma_e^i = \partial\Omega_i \setminus \Gamma_i$ .

The problem find  $\lambda$  such that  $(u_1|_{\Gamma_i} = u_2|_{\Gamma_i})$  where  $u_i$  is solution of the following Laplace problem:

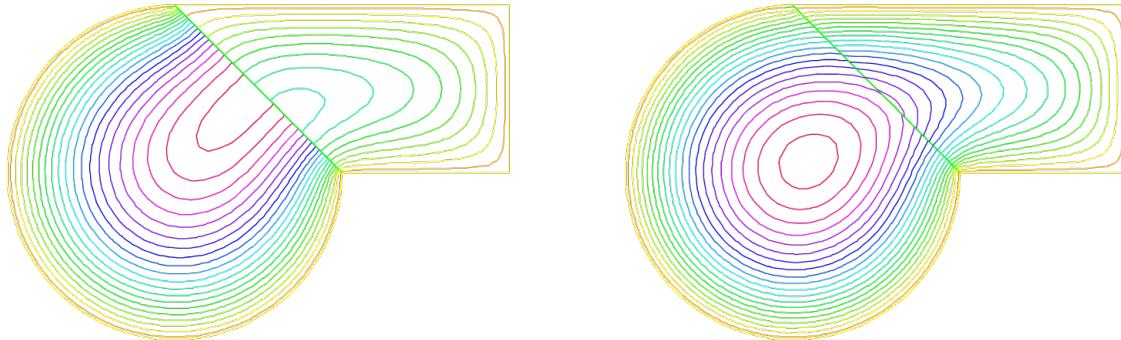
$$-\Delta u_i = f \text{ in } \Omega_i \quad u_i|_{\Gamma_i} = \lambda \quad u_i|_{\Gamma_e^i} = 0$$

The version of this example uses the Shur complement. The problem on the border is solved by a conjugate gradient method.

---

**Tip:** Schwarz conjuguate gradient

First, we construct the two domains:



(a) Isovalues of the solution at iteration 0 without overlapping

(b) Isovalues of the solution at iteration 9 without overlapping

```

1 // Parameters
2 int inside = 2; int outside = 1; int n = 4;
3
4 // Mesh
5 border Gamma1(t=1, 2){x=t; y=0; label=outside;};
6 border Gamma2(t=0, 1){x=2; y=t; label=outside;};
7 border Gamma3(t=2, 0){x=t; y=1; label=outside;};
8 border GammaInside(t=1, 0){x=1-t; y=t; label=inside;};
9 border GammaArc(t=pi/2, 2*pi){x=cos(t); y=sin(t); label=outside;};
10 mesh Th1 = buildmesh(Gamma1(5*n) + Gamma2(5*n) + GammaInside(5*n) + Gamma3(5*n));
11 mesh Th2 = buildmesh(GammaInside(-5*n) + GammaArc(25*n));
12 plot(Th1, Th2);

```

Now, define the finite element spaces:

```

1 // Fespace
2 fespace Vh1(Th1, P1);
3 Vh1 u1, v1;
4 Vh1 lambda;
5 Vh1 p=0;
6
7 fespace Vh2(Th2, P1);
8 Vh2 u2, v2;

```

**Note:** It is impossible to define a function just on a part of boundary, so the  $\lambda$  function must be defined on the all domain  $\Omega_1$  such as:

```
1 Vh1 lambda;
```

The two Poisson's problems:

```

1 problem Pb1 (u1, v1, init=i, solver=Cholesky)
2   = int2d(Th1) (
3     dx(u1)*dx(v1)
4     + dy(u1)*dy(v1)
5   )
6   + int2d(Th1) (
7     - v1
8   )
9   + int1d(Th1, inside) (
10    lambda*v1
11  )
12  + on(outside, u1=0)
13 ;
14
15 problem Pb2 (u2, v2, init=i, solver=Cholesky)
16   = int2d(Th2) (
17     dx(u2)*dx(v2)
18     + dy(u2)*dy(v2)
19   )
20   + int2d(Th2) (
21     - v2
22   )
23   + int1d(Th2, inside) (
24     - lambda*v2
25   )
26   + on(outside, u2=0)
27 ;

```

And, we define a border matrix, because the  $\lambda$  function is none zero inside the domain  $\Omega_1$ :

```

1 varf b(u2, v2, solver=CG) = int1d(Th1, inside) (u2*v2);
2 matrix B = b(Vh1, Vh1, solver=CG);

```

The boundary problem function,

$$\lambda \longrightarrow \int_{\Gamma_i} (u_1 - u_2)v_1$$

```

1 // Boundary problem function
2 func real[int] BoundaryProblem (real[int] &l){
3   lambda[] = l; //make FE function form 1
4   Pb1;
5   Pb2;
6   i++; //no refactorization i != 0
7   v1 = -(u1-u2);
8   lambda[] = B*v1[];
9   return lambda[];
10 }

```

---

**Note:** The difference between the two notations  $v_1$  and  $v_1[]$  is:  $v_1$  is the finite element function and  $v_1[]$  is the vector in the canonical basis of the finite element function  $v_1$ .

---

```

1 // Solve
2 real cpu=clock();

```

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```

3 LinearCG(BoundaryProblem, p[], eps=1.e-6, nbiter=100);
4 //compute the final solution, because CG works with increment
5 BoundaryProblem(p[]); //solve again to have right u1, u2
6
7 // Display & Plot
8 cout << " -- CPU time schwarz-gc:" << clock()-cpu << endl;
9 plot(u1, u2);

```

## 5.9 Fluid-structure coupled problem

This problem involves the Lamé system of elasticity and the Stokes system for viscous fluids with velocity  $\mathbf{u}$  and pressure  $p$ :

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= 0 & \text{in } \Omega \\ \nabla \cdot \mathbf{u} &= 0 & \text{in } \Omega \\ \mathbf{u} &= \mathbf{u}_\Gamma & \text{on } \Gamma = \partial\Omega \end{aligned}$$

where  $u_\Gamma$  is the velocity of the boundaries. The force that the fluid applies to the boundaries is the normal stress

$$\mathbf{h} = (\nabla \mathbf{u} + \nabla \mathbf{u}^T) \mathbf{n} - p \mathbf{n}$$

Elastic solids subject to forces deform: a point in the solid at  $(x,y)$  goes to  $(X,Y)$  after. When the displacement vector  $\mathbf{v} = (v_1, v_2) = (X - x, Y - y)$  is small, Hooke's law relates the stress tensor  $\sigma$  inside the solid to the deformation tensor  $\epsilon$ :

$$\sigma_{ij} = \lambda \delta_{ij} \nabla \cdot \mathbf{v} + 2\mu \epsilon_{ij}, \quad \epsilon_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

where  $\delta$  is the Kronecker symbol and where  $\lambda, \mu$  are two constants describing the material mechanical properties in terms of the modulus of elasticity, and Young's modulus:

The equations of elasticity are naturally written in variational form for the displacement vector  $v(x) \in V$  as:

$$\int_{\Omega} [2\mu \epsilon_{ij}(\mathbf{v}) \epsilon_{ij}(\mathbf{w}) + \lambda \epsilon_{ii}(v) \epsilon_{jj}(\mathbf{w})] = \int_{\Omega} \mathbf{g} \cdot \mathbf{w} + \int_{\Gamma} \mathbf{h} \cdot \mathbf{w}, \quad \forall \mathbf{w} \in V$$

The data are the gravity force  $\mathbf{g}$  and the boundary stress  $\mathbf{h}$ .

**Tip:** Fluide-structure In our example, the Lamé system and the Stokes system are coupled by a common boundary on which the fluid stress creates a displacement of the boundary and hence changes the shape of the domain where the Stokes problem is integrated. The geometry is that of a vertical driven cavity with an elastic lid. The lid is a beam with weight so it will be deformed by its own weight and by the normal stress due to the fluid reaction. The cavity is the  $10 \times 10$  square and the lid is a rectangle of height  $l = 2$ .

A beam sits on a box full of fluid rotating because the left vertical side has velocity one. The beam is bent by its own weight, but the pressure of the fluid modifies the bending.

The bending displacement of the beam is given by  $(uu, vv)$  whose solution is given as follows.

```

1 // Parameters
2 int bottombeam = 2; //label of bottombeam
3 real E = 21.5;
4 real sigma = 0.29;

```

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```

5 real gravity = -0.05;
6 real coef = 0.2;
7
8 // Mesh (solid)
9 border a(t=2, 0){x=0; y=t; label=1;}
10 border b(t=0, 10){x=t; y=0; label=bottombeam;}
11 border c(t=0, 2){x=10; y=t; label=1;}
12 border d(t=0, 10){x=10-t; y=2; label=3;}
13 mesh th = buildmesh(b(20) + c(5) + d(20) + a(5));
14
15 // Fespace (solid)
16 fespace Vh(th, P1);
17 Vh uu, w, vv, s;
18
19 // Macro
20 real sqrt2 = sqrt(2.);
21 macro epsilon(u1, u2) [dx(u1), dy(u2), (dy(u1)+dx(u2))/sqrt2] //
22 macro div(u1, u2) (dx(u1) + dy(u2)) //
23
24 // Problem (solid)
25 real mu = E/(2*(1+sigma));
26 real lambda = E*sigma/((1+sigma)*(1-2*sigma));
27 solve Elasticity([uu, vv], [w, s])
28 = int2d(th) (
29     lambda*div(w,s)*div(uu,vv)
30     + 2.*mu*(epsilon(w,s)'*epsilon(uu,vv))
31 )
32 + int2d(th) (
33     - gravity*s
34 )
35 + on(1, uu=0, vv=0)
36 ;
37
38 plot([uu, vv], wait=true);
39 mesh th1 = movemesh(th, [x+uu, y+vv]);
40 plot(th1, wait=true);

```

Then Stokes equation for fluids at low speed are solved in the box below the beam, but the beam has deformed the box (see border h):

```

1 // Mesh (fluid)
2 border e(t=0, 10){x=t; y=-10; label= 1;}
3 border f(t=0, 10){x=10; y=-10+t ; label= 1;}
4 border g(t=0, 10){x=0; y=-t; label= 2;}
5 border h(t=0, 10){x=t; y=vv(t,0)*( t>=0.001 )*( t <= 9.999 ); label=3;}
6 mesh sh = buildmesh(h(-20) + f(10) + e(10) + g(10));
7 plot(sh, wait=true);

```

We use the Uzawa conjugate gradient to solve the Stokes problem like in *Navier-Stokes equations*.

```

1 // Fespace (fluid)
2 fespace Xh(sh, P2);
3 Xh ul, u2;
4 Xh bc1;
5 Xh brhs;
6 Xh bcx=0, bcy=1;

```

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```

7   fespace Mh(sh, P1);
8   Mh p, ppp;
9
10 // Problem (fluid)
11 varf bx (u1, q) = int2d(sh) (-(dx(u1)*q));
12 varf by (u1, q) = int2d(sh) (-(dy(u1)*q));
13 varf Lap (u1, u2)
14   = int2d(sh) (
15     dx(u1)*dx(u2)
16     + dy(u1)*dy(u2)
17   )
18   + on(2, u1=1)
19   + on(1, 3, u1=0)
20 ;
21
22 bc1[] = Lap(0, Xh);
23
24 matrix A = Lap(Xh, Xh, solver=CG);
25 matrix Bx = bx(Xh, Mh);
26 matrix By = by(Xh, Mh);
27
28
29 func real[int] divup (real[int] & pp) {
30   int verb = verbosity;
31   verbosity = 0;
32   brhs[] = Bx'*pp;
33   brhs[] += bc1[] .*bcx[];
34   u1[] = A^-1*brhs[];
35   brhs[] = By'*pp;
36   brhs[] += bc1[] .*bcy[];
37   u2[] = A^-1*brhs[];
38   ppp[] = Bx*u1[];
39   ppp[] += By*u2[];
40   verbosity = verb;
41   return ppp[];
42 }
43

```

do a loop on the two problems

```

1 // Coupling loop
2 for(int step = 0; step < 10; ++step) {
3   // Solve (fluid)
4   LinearCG(divup, p[], eps=1.e-3, nbiter=50);
5   divup(p[]);

```

Now the beam will feel the stress constraint from the fluid:

```

1 // Forces
2 Vh sigma11, sigma22, sigma12;
3 Vh uu1=uu, vv1=vv;
4
5 sigma11([x+uu, y+vv]) = (2*dx(u1) - p);
6 sigma22([x+uu, y+vv]) = (2*dy(u2) - p);
7 sigma12([x+uu, y+vv]) = (dx(u1) + dy(u2));

```

which comes as a boundary condition to the PDE of the beam:

```

1 // Solve (solid)
2 solve Elasticity2 ([uu, vv], [w, s], init=step)
3 = int2d(th) (
4     lambda*div(w,s)*div(uu,vv)
5     + 2.*mu*(epsilon(w,s)'*epsilon(uu,vv))
6 )
7 + int2d(th) (
8     - gravity*s
9 )
10 + int1d(th, bottombeam) (
11     - coef*(sigma11*N.x*w + sigma22*N.y*s + sigma12*(N.y*w+N.x*s))
12 )
13 + on(1, uu=0, vv=0)
14 ;
15
16 // Plot
17 plot([uu, vv], wait=1);
18
19 // Error
20 real err = sqrt(int2d(th)((uu-uu1)^2 + (vv-vv1)^2));
21 cout << "Erreure L2 = " << err << endl;

```

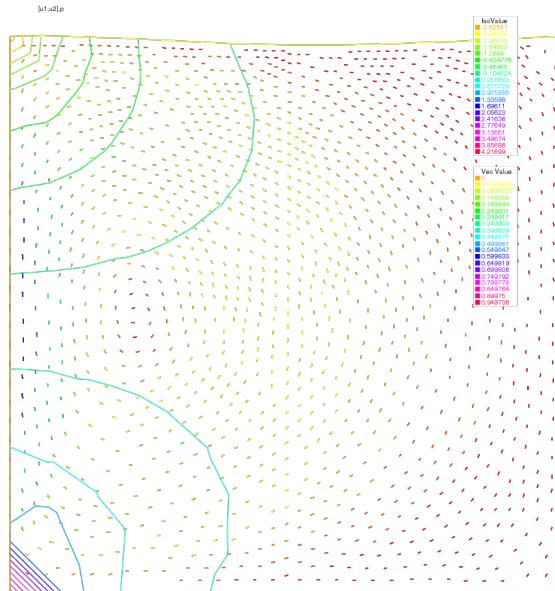
Notice that the matrix generated by `Elasticity2` is reused (see `init=i`). Finally we deform the beam:

```

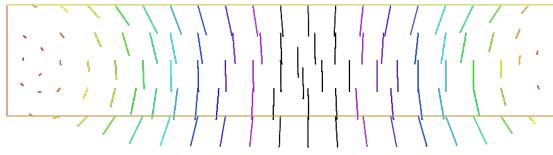
1 // Movemesh
2 th1 = movemesh(th, [x+0.2*uu, y+0.2*vv]);
3 plot(th1, wait=true);

```

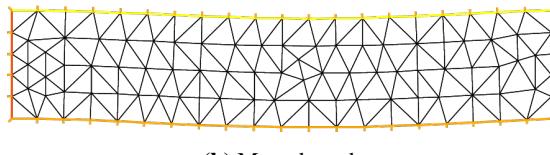
Fluid velocity and pressure, displacement vector of the structure and displaced geometry in the fluid-structure interaction of a soft side and a driven cavity are shown Fig. 5.22, Fig. 5.23a and Fig. 5.23b



**Fig. 5.22:** Velocity and pressure



(a) Displacement



(b) Moved mesh

## 5.10 Transmission problem

Consider an elastic plate whose displacement change vertically, which is made up of three plates of different materials, welded on each other.

Let  $\Omega_i$ ,  $i = 1, 2, 3$  be the domain occupied by  $i$ -th material with tension  $\mu_i$  (see [Soap film](#)).

The computational domain  $\Omega$  is the interior of  $\overline{\Omega_1} \cup \overline{\Omega_2} \cup \overline{\Omega_3}$ . The vertical displacement  $u(x, y)$  is obtained from:

$$\begin{aligned} -\mu_i \Delta u &= f && \text{in } \Omega_i \\ \mu_i \partial_n u|_{\Gamma_i} &= -\mu_j \partial_n u|_{\Gamma_j} && \text{on } \overline{\Omega_i} \cap \overline{\Omega_j} \text{ if } 1 \leq i < j \leq 3 \end{aligned} \quad (5.15)$$

where  $\partial_n u|_{\Gamma_i}$  denotes the value of the normal derivative  $\partial_n u$  on the boundary  $\Gamma_i$  of the domain  $\Omega_i$ .

By introducing the characteristic function  $\chi_i$  of  $\Omega_i$ , that is:

$$\chi_i(x) = 1 \text{ if } x \in \Omega_i; \chi_i(x) = 0 \text{ if } x \notin \Omega_i$$

we can easily rewrite (5.15) to the weak form. Here we assume that  $u = 0$  on  $\Gamma = \partial\Omega$ .

Transmission problem: For a given function  $f$ , find  $u$  such that:

$$a(u, v) = \ell(f, v) \text{ for all } v \in H_0^1(\Omega)$$

$$\begin{aligned} a(u, v) &= \int_{\Omega} \mu \nabla u \cdot \nabla v \\ \ell(f, v) &= \int_{\Omega} f v \end{aligned}$$

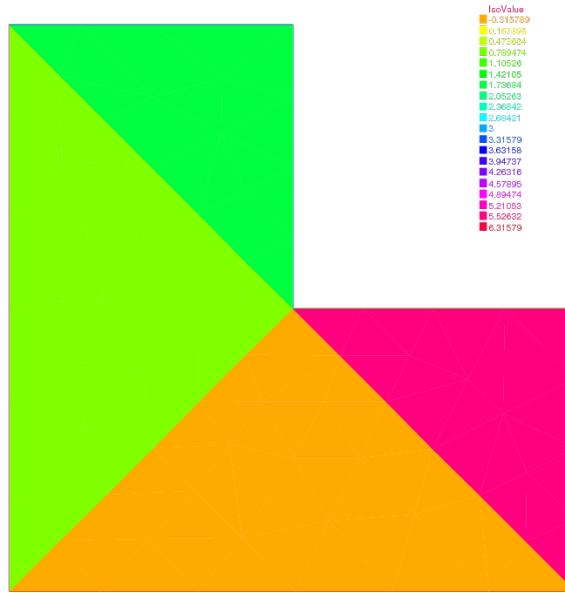
where  $\mu = \mu_1 \chi_1 + \mu_2 \chi_2 + \mu_3 \chi_3$ . Here we notice that  $\mu$  become the discontinuous function.

This example explains the definition and manipulation of *region*, i.e. sub-domains of the whole domain. Consider this L-shaped domain with 3 diagonals as internal boundaries, defining 4 sub-domains:

```

1 // Mesh
2 border a(t=0, 1){x=t; y=0;};
3 border b(t=0, 0.5){x=1; y=t;};
4 border c(t=0, 0.5){x=1-t; y=0.5;};
5 border d(t=0.5, 1){x=0.5; y=t;};
6 border e(t=0.5, 1){x=1-t; y=1;};
7 border f(t=0, 1){x=0; y=1-t;};
8 border i1(t=0, 0.5){x=t; y=1-t;};
9 border i2(t=0, 0.5){x=t; y=t;};
10 border i3(t=0, 0.5){x=1-t; y=t;};
11 mesh th = buildmesh(a(6) + b(4) + c(4) + d(4) + e(4)
12   + f(6) + i1(6) + i2(6) + i3(6));
13
14 // Fespace
15 fespace Ph(th, P0); //constant discontinuous functions / element
16 Ph reg=region; //defined the P0 function associated to region number
17
18 // Plot
19 plot(reg, fill=true, wait=true, value=true);

```

**Fig. 5.24:** The function reg

region is a keyword of **FreeFEM** which is in fact a variable depending of the current position (is not a function today, use Ph reg=region; to set a function). This variable value returned is the number of the sub-domain of the current position. This number is defined by buildmesh which scans while building the mesh all its connected component.

So to get the number of a region containing a particular point one does:

```

1 // Characteristic function
2 int nupper = reg(0.4, 0.9); //get the region number of point (0.4,0.9)
3 int nlower = reg(0.9, 0.1); //get the region number of point (0.4,0.1)
4 cout << "nlower = " << nlower << ", nupper = " << nupper<< endl;
5 Ph nu = 1 + 5*(region==nlower) + 10*(region==nupper);
6
7 // Plot
8 plot(nu, fill=true,wait=true);

```

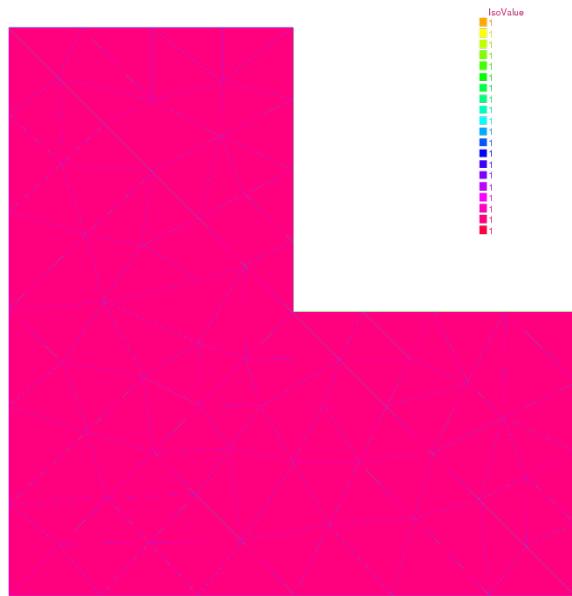
This is particularly useful to define discontinuous functions such as might occur when one part of the domain is copper and the other one is iron, for example.

We this in mind we proceed to solve a Laplace equation with discontinuous coefficients ( $\nu$  is 1, 6 and 11 below).

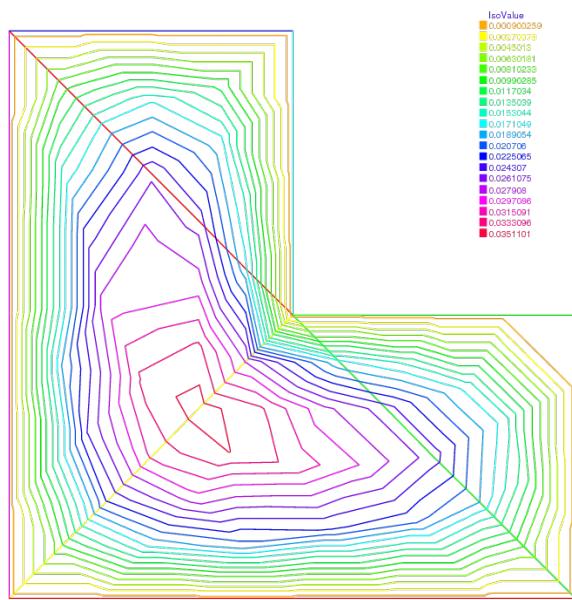
```

1 // Problem
2 solve lap (u, v)
3   = int2d(th) (
4     nu* (dx(u)*dx(v) + dy(u)*dy(v))
5   )
6   + int2d(th) (
7     - 1*v
8   )
9   + on(a, b, c, d, e, f, u=0)
10 ;
11
12 // Plot
13 plot(u);

```



**Fig. 5.25:** The function  $\nu$



**Fig. 5.26:** The isovalue of the solution  $u$

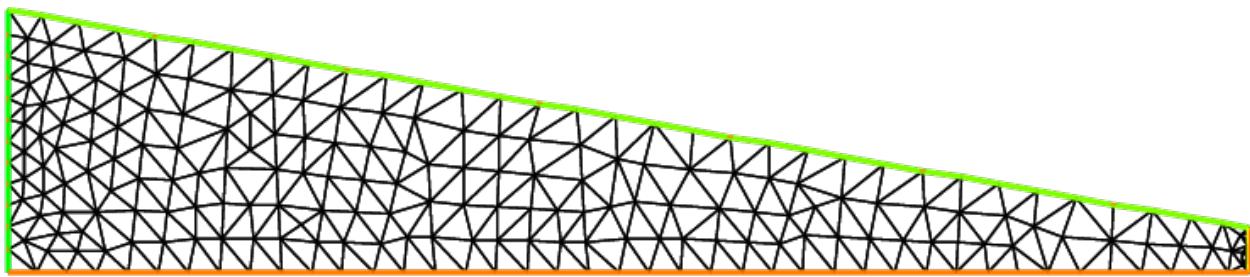
## 5.11 Free boundary problems

The domain  $\Omega$  is defined with:

```

1 // Parameters
2 real L = 10; //length
3 real hr = 2.1; //left height
4 real hl = 0.35; //right height
5 int n = 4;
6
7 // Mesh
8 border a(t=0, L){x=t; y=0;}; //bottom: Gamma_a
9 border b(t=0, hr){x=L; y=t;}; //right: Gamma_b
10 border f(t=L, 0){x=t; y=t*(hr-hl)/(L+hl);}; //free surface: Gamma_f
11 border d(t=hl, 0){x=0; y=t;}; // left: Gamma_d
12 mesh Th = buildmesh(a(10*n) + b(6*n) + f(8*n) + d(3*n));
13 plot(Th);

```



**Fig. 5.27:** The mesh of the domain  $\Omega$

The free boundary problem is:

Find  $u$  and  $\Omega$  such that:

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = y & \text{on } \Gamma_b \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_d \cup \Gamma_a \\ \frac{\partial u}{\partial n} = \frac{q}{K} n_x & \text{on } \Gamma_f \\ u = y & \text{on } \Gamma_f \end{cases}$$

We use a fixed point method;

$$\Omega^0 = \Omega$$

In two step, fist we solve the classical following problem:

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega^n \\ u = y & \text{on } \Gamma_b^n \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_d^n \cup \Gamma_a^n \\ u = y & \text{on } \Gamma_f^n \end{cases}$$

The variational formulation is:

Find  $u$  on  $V = H^1(\Omega^n)$ , such than  $u = y$  on  $\Gamma_b^n$  and  $\Gamma_f^n$

$$\int_{\Omega^n} \nabla u \nabla u' = 0, \forall u' \in V \text{ with } u' = 0 \text{ on } \Gamma_b^n \cup \Gamma_f^n$$

And secondly to construct a domain deformation  $\mathcal{F}(x, y) = [x, y - v(x, y)]$  where  $v$  is solution of the following problem:

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega^n \\ v = 0 & \text{on } \Gamma_a^n \\ \frac{\partial v}{\partial n} = 0 & \text{on } \Gamma_b^n \cup \Gamma_d^n \\ \frac{\partial v}{\partial n} = \frac{\partial u}{\partial n} - \frac{q}{K} n_x & \text{on } \Gamma_f^n \end{cases}$$

The variational formulation is:

Find  $v$  on  $V$ , such than  $v = 0$  on  $\Gamma_a^n$ :

$$\int_{\Omega^n} \nabla v \nabla v' = \int_{\Gamma_f^n} \left( \frac{\partial u}{\partial n} - \frac{q}{K} n_x \right) v', \quad \forall v' \in V \text{ with } v' = 0 \text{ on } \Gamma_a^n$$

Finally the new domain  $\Omega^{n+1} = \mathcal{F}(\Omega^n)$

---

### Tip: Free boundary

The FreeFEM implementation is:

```

1 // Parameters
2 real L = 10; //length
3 real hr = 2.1; //left height
4 real hl = 0.35; //right height
5 int n = 4;
6
7 real q = 0.02; //incoming flow
8 real K = 0.5; //permeability
9
10 // Mesh
11 border a(t=0, L){x=t; y=0}; //bottom: Gamma_a
12 border b(t=0, hr){x=L; y=t;}; //right: Gamma_b
13 border f(t=L, 0){x=t; y=t*(hr-hl)/L+hl;}; //free surface: Gamma_f
14 border d(t=hl, 0){x=0; y=t;}; // left: Gamma_d
15 mesh Th = buildmesh(a(10*n) + b(6*n) + f(8*n) + d(3*n));
16 plot(Th);
17
18 // Fespace
19 fespace Vh(Th, P1);
20 Vh u, v;
21 Vh uu, vv;
22
23 // Problem
24 problem Pu (u, uu, solver=CG)
25   = int2d(Th) (
26     dx(u)*dx(uu)
27     + dy(u)*dy(uu)
28   )
29   + on(b, f, u=y)
30 ;
31
32 problem Pv (v, vv, solver=CG)
33   = int2d(Th) (
34     dx(v)*dx(vv)
35     + dy(v)*dy(vv)
36   )
37   + on(a, v=0)

```

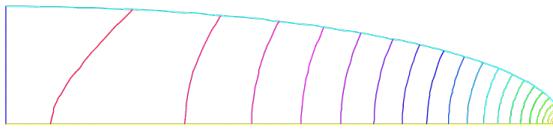
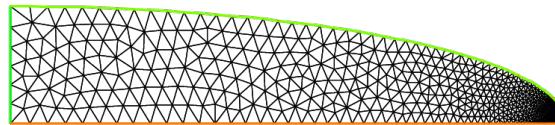
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```

38     + int1d(Th, f) (
39         vv*(q/K)*N.y - (dx(u)*N.x + dy(u)*N.y)
40     )
41 ;
42
43 // Loop
44 int j = 0;
45 real errv = 1.;
46 real erradap = 0.001;
47 while (errv > 1e-6){
48     // Update
49     j++;
50
51     // Solve
52     Pu;
53     Pv;
54
55     // Plot
56     plot(Th, u, v);
57
58     // Error
59     errv = int1d(Th, f) (v*v);
60
61     // Movemesh
62     real coef = 1.;
63     real mintcc = checkmovemesh(Th, [x, y])/5.;
64     real mint = checkmovemesh(Th, [x, y-v*coef]);
65
66     if (mint < mintcc || j%10==0){ //mesh too bad => remeshing
67         Th = adaptmesh(Th, u, err=erradap);
68         mintcc = checkmovemesh(Th, [x, y])/5.;
69     }
70
71     while (1){
72         real mint = checkmovemesh(Th, [x, y-v*coef]);
73
74         if (mint > mintcc) break;
75
76         cout << "min |T| = " << mint << endl;
77         coef /= 1.5;
78     }
79
80     Th=movemesh(Th, [x, y-coef*v]);
81
82     // Display
83     cout << endl << j << " - errv = " << errv << endl;
84 }
85
86 // Plot
87 plot(Th);
88 plot(u, wait=true);

```

(a) The final solution on the new domain  $\Omega^{72}$ (b) The adapted mesh of the domain  $\Omega^{72}$ 

## 5.12 Non-linear elasticity

The nonlinear elasticity problem is: find the displacement  $(u_1, u_2)$  minimizing  $J$ :

$$\min J(u_1, u_2) = \int_{\Omega} f(F2) - \int_{\Gamma_p} P_a u_2$$

where  $F2(u_1, u_2) = A(E[u_1, u_2], E[u_1, u_2])$  and  $A(X, Y)$  is bilinear symmetric positive form with respect two matrix  $X, Y$ .

where  $f$  is a given  $C^2$  function, and  $E[u_1, u_2] = (E_{ij})_{i=1,2, j=1,2}$  is the Green-Saint Venant deformation tensor defined with:

$$E_{ij} = 0.5((\partial_i u_j + \partial_j u_i) + \sum_k \partial_i u_k \times \partial_j u_k)$$

Denote  $\mathbf{u} = (u_1, u_2)$ ,  $\mathbf{v} = (v_1, v_2)$ ,  $\mathbf{w} = (w_1, w_2)$ . So, the differential of  $J$  is:

$$DJ(\mathbf{u})(\mathbf{v}) = \int DF2(\mathbf{u})(\mathbf{v}) f'(F2(\mathbf{u})) - \int_{\Gamma_p} P_a v_2$$

where  $DF2(\mathbf{u})(\mathbf{v}) = 2 A(DE[\mathbf{u}](\mathbf{v}), E[\mathbf{u}])$  and  $DE$  is the first differential of  $E$ .

The second order differential is:

$$\begin{aligned} D^2J(\mathbf{u})((\mathbf{v}), (\mathbf{w})) &= \int DF2(\mathbf{u})(\mathbf{v}) DF2(\mathbf{u})(\mathbf{w}) f''(F2(\mathbf{u})) \\ &+ \int D^2F2(\mathbf{u})(\mathbf{v}, \mathbf{w}) f'(F2(\mathbf{u})) \end{aligned}$$

where:

$$D^2F2(\mathbf{u})(\mathbf{v}, \mathbf{w}) = 2 A(D^2E[\mathbf{u}](\mathbf{v}, \mathbf{w}), E[\mathbf{u}]) + 2 A(DE[\mathbf{u}](\mathbf{v}), DE[\mathbf{u}](\mathbf{w})).$$

and  $D^2E$  is the second differential of  $E$ .

So all notations can be define with *macro*:

```

1  macro EL(u, v) [dx(u), (dx(v)+dy(u)), dy(v)] //is [epsilon_11, 2epsilon_12, epsilon_
2  ↪22]
3
4  macro ENL(u, v) [
5    (dx(u)*dx(u) + dx(v)*dx(v))*0.5,
6    (dx(u)*dy(u) + dx(v)*dy(v)),
7    (dy(u)*dy(u) + dy(v)*dy(v))*0.5
8  ] //
9
10 macro dENL(u, v, uu, vv) [
11   (dx(u)*dx(uu) + dx(v)*dx(vv)),
12   (dx(u)*dy(uu) + dx(v)*dy(vv) + dx(uu)*dy(u) + dx(vv)*dy(v)),
13   (dy(u)*dy(uu) + dy(v)*dy(vv))
14

```

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```

13 ] //
14
15 macro E(u, v) (EL(u,v) + ENL(u,v)) //is [E_11, 2E_12, E_22]
16 macro dE(u, v, uu, vv) (EL(uu, vv) + dENL(u, v, uu, vv)) //
17 macro ddE(u, v, uu, vv, uuu, vvv) dENL(uuu, vvv, uu, vv) //
18 macro F2(u, v) (E(u, v)' *A*E(u, v)) //
19 macro df2(u, v, uu, vv) (E(u, v)' *A*dE(u, v, uu, vv)*2.) //
20 macro ddF2(u, v, uu, vv, uuu, vvv) (
21     (dE(u, v, uu, vv)' *A*dE(u, v, uuu, vvv))*2.
22     + (E(u, v)' *A*ddE(u, v, uu, vv, uuu, vvv))*2.
23 ) //

```

The Newton Method is:

choose  $n = 0$ , and  $u_0, v_0$  the initial displacement

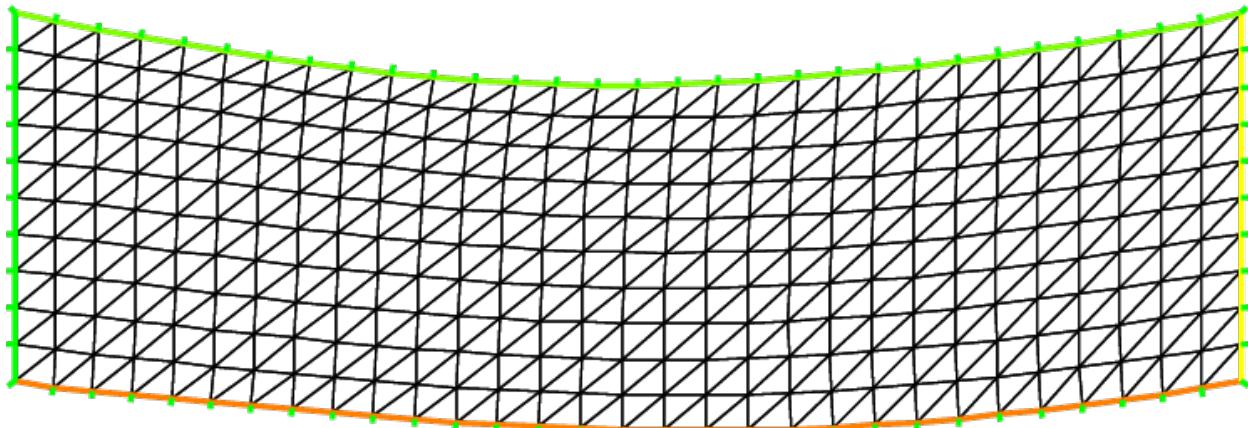
- loop:
  - find  $(du, dv)$  : solution of

$$D^2 J(u_n, v_n)((w, s), (du, dv)) = D J(u_n, v_n)(w, s), \quad \forall w, s$$

- $un = un - du, \quad vn = vn - dv$
- until  $(du, dv)$  small is enough

The way to implement this algorithm in **FreeFEM** is use a macro tool to implement  $A$  and  $F2, f, f', f''$ .

A macro is like in **ccp** preprocessor of C++, but this begin by **macro** and the end of the macro definition is before the comment **//**. In this case the macro is very useful because the type of parameter can be change. And it is easy to make automatic differentiation.



**Fig. 5.29:** The deformed domain

```

1 // Macro
2 macro EL(u, v) [dx(u), (dx(v)+dy(u)), dy(v)] //is [epsilon_11, 2epsilon_12, epsilon_
3   ↪22]
4
5 macro ENL(u, v) [
6   (dx(u)*dx(u) + dx(v)*dx(v))*0.5,
7   (dx(u)*dy(u) + dx(v)*dy(v)),

```

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```

7   (dy(u)*dy(u) + dy(v)*dy(v))*0.5
8   ] //
9
10 macro dENL(u, v, uu, vv) [
11   (dx(u)*dx(uu) + dx(v)*dx(vv)),
12   (dx(u)*dy(uu) + dx(v)*dy(vv) + dx(uu)*dy(u) + dx(vv)*dy(v)),
13   (dy(u)*dy(uu) + dy(v)*dy(vv))
14 ] //
15
16 macro E(u, v) (EL(u,v) + ENL(u,v)) //is [E_11, 2E_12, E_22]
17 macro dE(u, v, uu, vv) (EL(uu, vv) + dENL(u, v, uu, vv)) //
18 macro ddE(u, v, uu, vv, uuu, vvv) dENL(uuu, vvv, uu, vv) //
19 macro F2(u, v) (E(u, v)'*A*E(u, v)) //
20 macro dF2(u, v, uu, vv) (E(u, v)'*A*dE(u, v, uu, vv)*2.) //
21 macro ddF2(u, v, uu, vv, uuu, vvv) (
22   (dE(u, v, uu, vv)'*A*dE(u, v, uuu, vvv))*2.
23   + (E(u, v)'*A*ddE(u, v, uu, vv, uuu, vvv))*2.
24 ) //
25
26 macro f(u) ((u)*(u)*0.25) //
27 macro df(u) ((u)*0.5) //
28 macro ddf(u) (0.5) //
29
30 // Parameters
31 real mu = 0.012e5; //kg/cm^2
32 real lambda = 0.4e5; //kg/cm^2
33 real Pa = 1e2;
34
35 // sigma = 2 mu E + lambda tr(E) Id
36 // A(u,v) = sigma(u):E(v)
37 //
38 // ( a b )
39 // ( b c )
40 //
41 // tr*Id : (a,b,c) -> (a+c,0,a+c)
42 // so the associed matrix is:
43 // ( 1 0 1 )
44 // ( 0 0 0 )
45 // ( 1 0 1 )
46
47 real a11 = 2*mu + lambda;
48 real a22 = mu; //because [0, 2*t12, 0]' A [0, 2*s12, 0] = 2*mu*(t12*s12 + t21*s21) =_
49   ↪ 4*mu*t12*s12
50 real a33 = 2*mu + lambda;
51 real a12 = 0;
52 real a13 = lambda;
53 real a23 = 0;
54 // symmetric part
55 real a21 = a12;
56 real a31 = a13;
57 real a32 = a23;
58
59 //the matrix A
60 func A = [[a11, a12, a13], [a21, a22, a23], [a31, a32, a33]];
61
62 // Mesh
63 int n = 30;

```

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```

63 int m = 10;
64 mesh Th = square(n, m, [x, .3*y]); //label: 1 bottom, 2 right, 3 up, 4 left;
65 int bottom = 1, right = 2, upper = 3, left = 4;
66 plot(Th);
67
68 // Fespace
69 fespace Wh(Th, P1dc);
70 Wh e2, fe2, dfe2, ddfe2;
71
72 fespace Vh(Th, [P1, P1]);
73 Vh [uu, vv] = [0, 0], [w, s], [un, vn] = [0, 0];
74
75 fespace Sh(Th, P1);
76 Sh u1, v1;
77
78 // Problem
79 varf vmass ([uu, vv], [w, s], solver=CG) = int2d(Th) (uu*w + vv*s);
80 matrix M = vmass(Vh, Vh);
81 problem NonLin([uu, vv], [w, s], solver=LU)
82   = int2d(Th, qforder=1)( //( $D^2 J(un)$ )
83     dF2(un, vn, uu, vv)*dF2(un, vn, w, s)*ddfe2
84     + ddF2(un, vn, uu, vv, w, s)*ddfe2
85   )
86   - int1d(Th, upper)(
87     Pa*s
88   )
89   - int2d(Th, qforder=1)( //( $D J(un)$ )
90     dF2(un, vn, w, s)*dfe2
91   )
92   + on(right, left, uu=0, vv=0)
93 ;
94
95 // Newton's method
96 for (int i = 0; i < 10; i++) {
97   cout << "Loop " << i << endl;
98
99   // Update
100  e2 = F2(un, vn);
101  dfe2 = df(e2) ;
102  ddfe2 = ddf(e2) ;
103  cout << "e2 max = " << e2[].max << ", min = " << e2[].min << endl;
104  cout << "dfe2 max = " << dfe2[].max << ", min = " << dfe2[].min << endl;
105  cout << "ddfe2 max = " << ddfe2[].max << ", min = " << ddfe2[].min << endl;
106
107  // Solve
108  NonLin;
109  w[] = M*uu[];
110
111  // Residual
112  real res = sqrt(w[]' * uu[]); //L^2 norm of [uu, vv]
113  cout << " L^2 residual = " << res << endl;
114
115  // Update
116  v1 = vv;
117  u1 = uu;
118  cout << "u1 min = " << u1[].min << ", u1 max = " << u1[].max << endl;
119  cout << "v1 min = " << v1[].min << ", v2 max = " << v1[].max << endl;

```

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```

120
121 // Plot
122 plot([uu, vv], wait=true, cmm="uu, vv");
123
124 // Update
125 un[] -= uu[];
126 plot([un, vn], wait=true, cmm="displacement");
127
128 if (res < 1e-5) break;
129 }
130
131 // Plot
132 plot([un, vn], wait=true);
133
134 // Movemesh
135 mesh th1 = movemesh(Th, [x+un, y+vn]);
136
137 // Plot
138 plot(th1, wait=true);

```

## 5.13 Compressible Neo-Hookean materials

Author: *Alex Sadovsky*

### 5.13.1 Notation

In what follows, the symbols  $\mathbf{u}$ ,  $\mathbf{F}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$ ,  $\underline{\sigma}$  denote, respectively, the displacement field, the deformation gradient, the left Cauchy-Green strain tensor  $\mathbf{B} = \mathbf{FF}^T$ , the right Cauchy-Green strain tensor  $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ , and the Cauchy stress tensor.

We also introduce the symbols  $I_1 := \text{tr } \mathbf{C}$  and  $J := \det \mathbf{F}$ . Use will be made of the identity:

$$\frac{\partial J}{\partial \mathbf{C}} = J \mathbf{C}^{-1}$$

The symbol  $\mathbf{I}$  denotes the identity tensor. The symbol  $\Omega_0$  denotes the reference configuration of the body to be deformed. The unit volume in the reference (resp., deformed) configuration is denoted  $dV$  (resp.,  $dV_0$ ); these two are related by:

$$dV = J dV_0,$$

which allows an integral over  $\Omega$  involving the Cauchy stress  $\mathbf{T}$  to be rewritten as an integral of the Kirchhoff stress  $\kappa = JT$  over  $\Omega_0$ .

### 5.13.2 Recommended References

For an exposition of nonlinear elasticity and of the underlying linear and tensor algebra, see [OGDEN1984]. For an advanced mathematical analysis of the Finite Element Method, see [RAVIART1998].

### 5.13.3 A Neo-Hookean Compressible Material

*Constitutive Theory and Tangent Stress Measures*

The strain energy density function is given by:

$$W = \frac{\mu}{2}(I_1 - \text{tr } \mathbf{I} - 2 \ln J)$$

(see [HORGAN2004], formula (12)).

The corresponding 2nd Piola-Kirchoff stress tensor is given by:

$$\mathbf{S}_n := \frac{\partial W}{\partial \mathbf{E}}(\mathbf{F}_n) = \mu(\mathbf{I} - \mathbf{C}^{-1})$$

The Kirchhoff stress, then, is:

$$\kappa = \mathbf{F} \mathbf{S} \mathbf{F}^T = \mu(\mathbf{B} - \mathbf{I})$$

The tangent Kirchhoff stress tensor at  $\mathbf{F}_n$  acting on  $\delta \mathbf{F}_{n+1}$  is, consequently:

$$\frac{\partial \kappa}{\partial \mathbf{F}}(\mathbf{F}_n) \delta \mathbf{F}_{n+1} = \mu [\mathbf{F}_n (\delta \mathbf{F}_{n+1})^T + \delta \mathbf{F}_{n+1} (\mathbf{F}_n)^T]$$

*The Weak Form of the BVP in the Absence of Body (External) Forces*

The  $\Omega_0$  we are considering is an elliptical annulus, whose boundary consists of two concentric ellipses (each allowed to be a circle as a special case), with the major axes parallel. Let  $P$  denote the dead stress load (traction) on a portion  $\partial \Omega_0^t$  (= the inner ellipse) of the boundary  $\partial \Omega_0$ . On the rest of the boundary, we prescribe zero displacement.

The weak formulation of the boundary value problem is:

$$0 = \left. \begin{aligned} & \int_{\Omega_0} \kappa[\mathbf{F}] : \{(\nabla \otimes \mathbf{w})(\mathbf{F})^{-1}\} \\ & - \int_{\partial \Omega_0^t} P \cdot \hat{N}_0 \end{aligned} \right\}$$

For brevity, in the rest of this section we assume  $P = 0$ . The provided FreeFEM code, however, does not rely on this assumption and allows for a general value and direction of  $P$ .

Given a Newton approximation  $\mathbf{u}_n$  of the displacement field  $\mathbf{u}$  satisfying the BVP, we seek the correction  $\delta \mathbf{u}_{n+1}$  to obtain a better approximation:

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \delta \mathbf{u}_{n+1}$$

by solving the weak formulation:

$$0 = \left. \begin{aligned} & \int_{\Omega_0} \kappa[\mathbf{F}_n + \delta \mathbf{F}_{n+1}] : \{(\nabla \otimes \mathbf{w})(\mathbf{F}_n + \delta \mathbf{F}_{n+1})^{-1}\} - \int_{\partial \Omega_0} P \cdot \hat{N}_0 \\ & = \int_{\Omega_0} \{\kappa[\mathbf{F}_n] + \frac{\partial \kappa}{\partial \mathbf{F}}[\mathbf{F}_n] \delta \mathbf{F}_{n+1}\} : \{(\nabla \otimes \mathbf{w})(\mathbf{F}_n + \delta \mathbf{F}_{n+1})^{-1}\} \\ & = \int_{\Omega_0} \{\kappa[\mathbf{F}_n] + \frac{\partial \kappa}{\partial \mathbf{F}}[\mathbf{F}_n] \delta \mathbf{F}_{n+1}\} : \{(\nabla \otimes \mathbf{w})(\mathbf{F}_n^{-1} + \mathbf{F}_n^{-2} \delta \mathbf{F}_{n+1})\} \\ & = \int_{\Omega_0} \kappa[\mathbf{F}_n] : \{(\nabla \otimes \mathbf{w}) \mathbf{F}_n^{-1}\} \\ & - \int_{\Omega_0} \kappa[\mathbf{F}_n] : \{(\nabla \otimes \mathbf{w}) (\mathbf{F}_n^{-2} \delta \mathbf{F}_{n+1})\} \\ & + \int_{\Omega_0} \{\frac{\partial \kappa}{\partial \mathbf{F}}[\mathbf{F}_n] \delta \mathbf{F}_{n+1}\} : \{(\nabla \otimes \mathbf{w}) \mathbf{F}_n^{-1}\} \end{aligned} \right\} \quad \text{for all test functions } \mathbf{w},$$

where we have taken:

$$\delta \mathbf{F}_{n+1} = \nabla \otimes \delta \mathbf{u}_{n+1}$$

---

**Note:** Contrary to standard notational use, the symbol  $\delta$  here bears no variational context. By  $\delta$  we mean simply an increment in the sense of Newton's Method. The role of a variational virtual displacement here is played by  $\mathbf{w}$ .

---

### 5.13.4 An Approach to Implementation in FreeFEM

Introducing the code-like notation, where a string in <>’s is to be read as one symbol, the individual components of the tensor:

$$\langle TanK \rangle := \frac{\partial \kappa}{\partial \mathbf{F}} [\mathbf{F}_n] \delta \mathbf{F}_{n+1}$$

will be implemented as the macros  $\langle TanK11 \rangle, \langle TanK12 \rangle, \dots$

The individual components of the tensor quantities:

$$\mathbf{D}_1 := \mathbf{F}_n (\delta \mathbf{F}_{n+1})^T + \delta \mathbf{F}_{n+1} (\mathbf{F}_n)^T,$$

$$\mathbf{D}_2 := \mathbf{F}_n^{-T} \delta \mathbf{F}_{n+1},$$

$$\mathbf{D}_3 := (\nabla \otimes \mathbf{w}) \mathbf{F}_n^{-2} \delta \mathbf{F}_{n+1},$$

and

$$\mathbf{D}_4 := (\nabla \otimes \mathbf{w}) \mathbf{F}_n^{-1},$$

will be implemented as the macros:

$$\left. \begin{aligned} &\langle d1Aux11 \rangle, \langle d1Aux12 \rangle, \dots, \langle d1Aux22 \rangle, \\ &\langle d2Aux11 \rangle, \langle d2Aux12 \rangle, \dots, \langle d2Aux22 \rangle \\ &\langle d3Aux11 \rangle, \langle d3Aux12 \rangle, \dots, \langle d3Aux22 \rangle \\ &\langle d4Aux11 \rangle, \langle d4Aux12 \rangle, \dots, \langle d4Aux22 \rangle \end{aligned} \right\},$$

respectively.

In the above notation, the tangent Kirchhoff stress term becomes

$$\frac{\partial \kappa}{\partial \mathbf{F}} (\mathbf{F}_n) \delta \mathbf{F}_{n+1} = \mu \mathbf{D}_1$$

while the weak BVP formulation acquires the form:

$$0 = \left. \begin{aligned} &\int_{\Omega_0} \kappa[\mathbf{F}_n] : \mathbf{D}_4 \\ &- \int_{\Omega_0} \kappa[\mathbf{F}_n] : \mathbf{D}_3 \\ &+ \int_{\Omega_0} \left\{ \frac{\partial \kappa}{\partial \mathbf{F}} [\mathbf{F}_n] \delta \mathbf{F}_{n+1} \right\} : \mathbf{D}_4 \end{aligned} \right\} \quad \text{for all test functions w}$$

```

1 // Macro
2 //Macros for the gradient of a vector field (u1, u2)
3 macro grad11(u1, u2) (dx(u1)) //
4 macro grad21(u1, u2) (dy(u1)) //
5 macro grad12(u1, u2) (dx(u2)) //
6 macro grad22(u1, u2) (dy(u2)) //

7
8 //Macros for the deformation gradient
9 macro F11(u1, u2) (1.0 + grad11(u1, u2)) //
10 macro F12(u1, u2) (0.0 + grad12(u1, u2)) //
11 macro F21(u1, u2) (0.0 + grad21(u1, u2)) //
12 macro F22(u1, u2) (1.0 + grad22(u1, u2)) //

13
14 //Macros for the incremental deformation gradient
15 macro dF11(varu1, varu2) (grad11(varu1, varu2)) //
16 macro dF12(varu1, varu2) (grad12(varu1, varu2)) //

```

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```

17 macro dF21(varu1, varu2) (grad21(varu1, varu2)) //
18 macro dF22(varu1, varu2) (grad22(varu1, varu2)) //
19
20 //Macro for the determinant of the deformation gradient
21 macro J(u1, u2) (
22     F11(u1, u2)*F22(u1, u2)
23     - F12(u1, u2)*F21(u1, u2)
24 ) //
25
26 //Macros for the inverse of the deformation gradient
27 macro Finv11 (u1, u2) (
28     F22(u1, u2) / J(u1, u2)
29 ) //
30 macro Finv22 (u1, u2) (
31     F11(u1, u2) / J(u1, u2)
32 ) //
33 macro Finv12 (u1, u2) (
34     - F12(u1, u2) / J(u1, u2)
35 ) //
36 macro Finv21 (u1, u2) (
37     - F21(u1, u2) / J(u1, u2)
38 ) //
39
40 //Macros for the square of the inverse of the deformation gradient
41 macro FFinv11 (u1, u2) (
42     Finv11(u1, u2)^2
43     + Finv12(u1, u2)*Finv21(u1, u2)
44 ) //
45
46 macro FFinv12 (u1, u2) (
47     Finv12(u1, u2)*(Finv11(u1, u2)
48     + Finv22(u1, u2)))
49 ) //
50
51 macro FFinv21 (u1, u2) (
52     Finv21(u1, u2)*(Finv11(u1, u2)
53     + Finv22(u1, u2)))
54 ) //
55
56 macro FFinv22 (u1, u2) (
57     Finv12(u1, u2)*Finv21(u1, u2)
58     + Finv22(u1, u2)^2
59 ) //
60
61 //Macros for the inverse of the transpose of the deformation gradient
62 macro FinvT11(u1, u2) (Finv11(u1, u2)) //
63 macro FinvT12(u1, u2) (Finv21(u1, u2)) //
64 macro FinvT21(u1, u2) (Finv12(u1, u2)) //
65 macro FinvT22(u1, u2) (Finv22(u1, u2)) //
66
67 //The left Cauchy-Green strain tensor
68 macro B11(u1, u2) (
69     F11(u1, u2)^2 + F12(u1, u2)^2
70 ) //
71
72 macro B12(u1, u2) (
73     F11(u1, u2)*F21(u1, u2)

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74     + F12(u1, u2)*F22(u1, u2)
75 )///
76
77 macro B21(u1, u2) (
78     F11(u1, u2)*F21(u1, u2)
79     + F12(u1, u2)*F22(u1, u2)
80 )///
81
82 macro B22(u1, u2) (
83     F21(u1, u2)^2 + F22(u1, u2)^2
84 )///
85
86 //The macros for the auxiliary tensors (D0, D1, D2, ...): Begin
87 //The tensor quantity D0 = F{n} (delta F{n+1})^T
88 macro d0Aux11 (u1, u2, varu1, varu2) (
89     dF11(varu1, varu2) * F11(u1, u2)
90     + dF12(varu1, varu2) * F12(u1, u2)
91 )///
92
93 macro d0Aux12 (u1, u2, varu1, varu2) (
94     dF21(varu1, varu2) * F11(u1, u2)
95     + dF22(varu1, varu2) * F12(u1, u2)
96 )///
97
98 macro d0Aux21 (u1, u2, varu1, varu2) (
99     dF11(varu1, varu2) * F21(u1, u2)
100    + dF12(varu1, varu2) * F22(u1, u2)
101 )///
102
103 macro d0Aux22 (u1, u2, varu1, varu2) (
104     dF21(varu1, varu2) * F21(u1, u2)
105     + dF22(varu1, varu2) * F22(u1, u2)
106 )///
107
108 //The tensor quantity D1 = D0 + D0^T
109 macro d1Aux11 (u1, u2, varu1, varu2) (
110     2.0 * d0Aux11 (u1, u2, varu1, varu2)
111 )///
112
113 macro d1Aux12 (u1, u2, varu1, varu2) (
114     d0Aux12 (u1, u2, varu1, varu2)
115     + d0Aux21 (u1, u2, varu1, varu2)
116 )///
117
118 macro d1Aux21 (u1, u2, varu1, varu2) (
119     d1Aux12 (u1, u2, varu1, varu2)
120 )///
121
122 macro d1Aux22 (u1, u2, varu1, varu2) (
123     2.0 * d0Aux22 (u1, u2, varu1, varu2)
124 )///
125
126 //The tensor quantity D2 = F^{-T}_{n} dF_{n+1}
127 macro d2Aux11 (u1, u2, varu1, varu2) (
128     dF11(varu1, varu2) * FinvT11(u1, u2)
129     + dF21(varu1, varu2) * FinvT12(u1, u2)
130 )///

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131
132 macro d2Aux12 (u1, u2, varu1, varu2) (
133     dF12(varu1, varu2) * FinvT11(u1, u2)
134     + dF22(varu1, varu2) * FinvT12(u1, u2)
135 ) //
136
137 macro d2Aux21 (u1, u2, varu1, varu2) (
138     dF11(varu1, varu2) * FinvT21(u1, u2)
139     + dF21(varu1, varu2) * FinvT22(u1, u2)
140 ) //
141
142 macro d2Aux22 (u1, u2, varu1, varu2) (
143     dF12(varu1, varu2) * FinvT21(u1, u2)
144     + dF22(varu1, varu2) * FinvT22(u1, u2)
145 ) //
146
147 ///The tensor quantity D3 = F^{-2}_{n} dF_{n+1}
148 macro d3Aux11 (u1, u2, varu1, varu2, w1, w2) (
149     dF11(varu1, varu2) * FFinv11(u1, u2) * grad11(w1, w2)
150     + dF21(varu1, varu2) * FFinv12(u1, u2) * grad11(w1, w2)
151     + dF11(varu1, varu2) * FFinv21(u1, u2) * grad12(w1, w2)
152     + dF21(varu1, varu2) * FFinv22(u1, u2) * grad12(w1, w2)
153 ) //
154
155 macro d3Aux12 (u1, u2, varu1, varu2, w1, w2) (
156     dF12(varu1, varu2) * FFinv11(u1, u2) * grad11(w1, w2)
157     + dF22(varu1, varu2) * FFinv12(u1, u2) * grad11(w1, w2)
158     + dF12(varu1, varu2) * FFinv21(u1, u2) * grad12(w1, w2)
159     + dF22(varu1, varu2) * FFinv22(u1, u2) * grad12(w1, w2)
160 ) //
161
162 macro d3Aux21 (u1, u2, varu1, varu2, w1, w2) (
163     dF11(varu1, varu2) * FFinv11(u1, u2) * grad21(w1, w2)
164     + dF21(varu1, varu2) * FFinv12(u1, u2) * grad21(w1, w2)
165     + dF11(varu1, varu2) * FFinv21(u1, u2) * grad22(w1, w2)
166     + dF21(varu1, varu2) * FFinv22(u1, u2) * grad22(w1, w2)
167 ) //
168
169 macro d3Aux22 (u1, u2, varu1, varu2, w1, w2) (
170     dF12(varu1, varu2) * FFinv11(u1, u2) * grad21(w1, w2)
171     + dF22(varu1, varu2) * FFinv12(u1, u2) * grad21(w1, w2)
172     + dF12(varu1, varu2) * FFinv21(u1, u2) * grad22(w1, w2)
173     + dF22(varu1, varu2) * FFinv22(u1, u2) * grad22(w1, w2)
174 ) //
175
176 ///The tensor quantity D4 = (grad w) * Finv
177 macro d4Aux11 (w1, w2, u1, u2) (
178     Finv11(u1, u2)*grad11(w1, w2)
179     + Finv21(u1, u2)*grad12(w1, w2)
180 ) //
181
182 macro d4Aux12 (w1, w2, u1, u2) (
183     Finv12(u1, u2)*grad11(w1, w2)
184     + Finv22(u1, u2)*grad12(w1, w2)
185 ) //
186
187 macro d4Aux21 (w1, w2, u1, u2) (

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188     Finv11(u1, u2)*grad21(w1, w2)
189     + Finv21(u1, u2)*grad22(w1, w2)
190 )///
191
192 macro d4Aux22 (w1, w2, u1, u2) (
193     Finv12(u1, u2)*grad21(w1, w2)
194     + Finv22(u1, u2)*grad22(w1, w2)
195 )///
196 //The macros for the auxiliary tensors (D0, D1, D2, ...): End
197
198 //The macros for the various stress measures: BEGIN
199 //The Kirchhoff stress tensor
200 macro StressK11(u1, u2) (
201     mu * (B11(u1, u2) - 1.0)
202 )///
203
204 //The Kirchhoff stress tensor
205 macro StressK12(u1, u2) (
206     mu * B12(u1, u2)
207 )///
208
209 //The Kirchhoff stress tensor
210 macro StressK21(u1, u2) (
211     mu * B21(u1, u2)
212 )///
213
214 //The Kirchhoff stress tensor
215 macro StressK22(u1, u2) (
216     mu * (B22(u1, u2) - 1.0)
217 )///
218
219 //The tangent Kirchhoff stress tensor
220 macro TanK11(u1, u2, varu1, varu2) (
221     mu * d1Aux11(u1, u2, varu1, varu2)
222 )///
223
224 macro TanK12(u1, u2, varu1, varu2) (
225     mu * d1Aux12(u1, u2, varu1, varu2)
226 )///
227
228 macro TanK21(u1, u2, varu1, varu2) (
229     mu * d1Aux21(u1, u2, varu1, varu2)
230 )///
231
232 macro TanK22(u1, u2, varu1, varu2) (
233     mu * d1Aux22(u1, u2, varu1, varu2)
234 )///
235 //The macros for the stress tensor components: END
236
237 // Parameters
238 real mu = 5.e2; //Elastic coefficients (kg/cm^2)
239 real D = 1.e3; ///(1 / compressibility)
240 real Pa = -3.e2; //Stress loads
241
242 real InnerRadius = 1.e0; //The wound radius
243 real OuterRadius = 4.e0; //The outer (truncated) radius
244 real tol = 1.e-4; //Tolerance (L^2)

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245 real InnerEllipseExtension = 1.e0; //Extension of the inner ellipse ((major axis) -_
246 // (minor axis))
247
248 int m = 40, n = 20;
249
250 // Mesh
251 border InnerEdge(t=0, 2.*pi) {x=(1.0 + InnerEllipseExtension)*InnerRadius*cos(t);_
252 // y=InnerRadius*sin(t); label=1;};
253 border OuterEdge(t=0, 2.*pi) {x=(1.0 + 0.0*InnerEllipseExtension)*OuterRadius*cos(t);_
254 // y=OuterRadius*sin(t); label=2;};
255 mesh Th = buildmesh(InnerEdge(-m) + OuterEdge(n));
256 int bottom = 1, right = 2, upper = 3, left = 4;
257 plot(Th);
258
259 // Fespace
260 fespace Wh(Th, P1dc);
261 fespace Vh(Th, [P1, P1]);
262 Vh [w1, w2], [u1n, u2n], [varu1, varu2];
263 Vh [ehat1x, ehat1y], [ehat2x, ehat2y];
264 Vh [auxVec1, auxVec2]; //The individual elements of the total 1st Piola-Kirchoff_
265 // stress
266 Vh [ef1, ef2];
267
268 // fespace
269 fespace Sh(Th, P1);
270 Sh p, ppp;
271 Sh StrK11, StrK12, StrK21, StrK22;
272 Sh u1, u2;
273
274 // Problem
275 varf vfMass1D(p, q) = int2d(Th)(p*q);
276 matrix Mass1D = vfMass1D(Sh, Sh, solver=CG);
277
278 p[] = 1;
279 ppp[] = Mass1D * p[];
280
281 real DomainMass = ppp[].sum;
282 cout << "DomainMass = " << DomainMass << endl;
283
284 varf vmass ([u1, u2], [v1, v2], solver=CG)
285 = int2d(Th)( (u1*v1 + u2*v2) / DomainMass );
286
287 matrix Mass = vmass(Vh, Vh);
288
289 matrix Id = vmass(Vh, Vh);
290
291 //Define the standard Euclidean basis functions
292 [ehat1x, ehat1y] = [1.0, 0.0];
293 [ehat2x, ehat2y] = [0.0, 1.0];
294
295 real ContParam, dContParam;
296
297 problem neoHookeanInc ([varu1, varu2], [w1, w2], solver=LU)
298 = int2d(Th, qforder=1)(
299 - (
300     StressK11 (u1n, u2n) * d3Aux11(u1n, u2n, varu1, varu2, w1, w2)
301     + StressK12 (u1n, u2n) * d3Aux12(u1n, u2n, varu1, varu2, w1, w2)
302     + StressK21 (u1n, u2n) * d3Aux21(u1n, u2n, varu1, varu2, w1, w2)

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298     + StressK22 (u1n, u2n) * d3Aux22(u1n, u2n, varu1, varu2, w1, w2)
299   )
300   + TanK11 (u1n, u2n, varu1, varu2) * d4Aux11(w1, w2, u1n, u2n)
301   + TanK12 (u1n, u2n, varu1, varu2) * d4Aux12(w1, w2, u1n, u2n)
302   + TanK21 (u1n, u2n, varu1, varu2) * d4Aux21(w1, w2, u1n, u2n)
303   + TanK22 (u1n, u2n, varu1, varu2) * d4Aux22(w1, w2, u1n, u2n)
304   )
305   + int2d(Th, qforder=1) (
306     StressK11 (u1n, u2n) * d4Aux11(w1, w2, u1n, u2n)
307     + StressK12 (u1n, u2n) * d4Aux12(w1, w2, u1n, u2n)
308     + StressK21 (u1n, u2n) * d4Aux21(w1, w2, u1n, u2n)
309     + StressK22 (u1n, u2n) * d4Aux22(w1, w2, u1n, u2n)
310   )
311   //Choose one of the following two boundary conditions involving Pa:
312   // Load vectors normal to the boundary:
313   - int1d(Th, 1) (
314     Pa * (w1*N.x + w2*N.y)
315   )
316   //Load vectors tangential to the boundary:
317   //int1d(Th, 1) (
318   //  Pa * (w1*N.y - w2*N.x)
319   //)
320   + on(2, varu1=0, varu2=0)
321   ;
322
323 //Auxiliary variables
324 matrix auxMat;
325
326 // Newton's method
327 ContParam = 0.;
328 dContParam = 0.01;
329
330 //Initialization:
331 [varu1, varu2] = [0., 0.];
332 [u1n, u2n] = [0., 0.];
333 real res = 2. * tol;
334 real eforceres;
335 int loopcount = 0;
336 int loopmax = 45;
337
338 // Iterations
339 while (loopcount <= loopmax && res >= tol) {
340   loopcount++;
341   cout << "Loop " << loopcount << endl;
342
343   // Solve
344   neoHookeanInc;
345
346   // Update
347   u1 = varu1;
348   u2 = varu2;
349
350   // Residual
351   w1[] = Mass*varu1[];
352   res = sqrt(w1[]' * varu1[]); //L^2 norm of [varu1, varu2]
353   cout << " L^2 residual = " << res << endl;
354

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355 // Newton
356 u1n[] += varu1[];
357
358 // Plot
359 plot([u1n,u2n], cmm="displacement");
360 }
361
362 // Plot
363 plot(Th, wait=true);
364
365 // Movemesh
366 mesh Th1 = movemesh(Th, [x+u1n, y+u2n]);
367
368 // Plot
369 plot(Th1, wait=true);
370 plot([u1n,u2n]);

```

## 5.14 Whispering gallery modes

Author: I. S. Grudinin

In whispering gallery mode (WGM) resonators, which are typically spheres or disks, electromagnetic field is trapped by total internal reflections from the boundary. Modes of such resonators are distinguished by compact volume and record high quality factors ( $Q$ ) in a broad range of frequencies.

Modern applications of such resonators include microwave and optical cavities for atomic clocks, cavity optomechanics, nonlinear and quantum optics. Analytical solutions for WG modes are only available for a limited number of idealized geometries, such as sphere or ellipsoid. Since resonator dimensions are typically much larger than optical wavelength, direct application of numerical 3D finite difference time domain (FDTD) or finite element methods (FEM) is not practical. It's possible to solve the vectorial wave equation by reducing it to a two dimensional case by taking axial symmetry into account.

Such reduction leads to a system of 3 equations to be solved in a 2D “ $\rho - z$ ” section of a resonator. Please refer to [[OXBORROW2007](#)] for a detailed derivation and to [[GRUDININ2012](#)] for an example of using **FreeFEM** to compute WGMs.

### 5.14.1 Wave equation for the WGMs

Since electric field is discontinuous on the surface of a dielectric and magnetic field is typically not, we derive our equations for the magnetic field. The electric field can be easily derived at a later stage from  $\vec{E} = \frac{i}{\omega\epsilon_0}\hat{\epsilon}^{-1}\nabla \times \vec{H}$ . Following a standard procedure starting with Maxwell equations we derive a wave equation in a single-axis anisotropic medium such as an optical crystal:

$$\nabla \times (\hat{\epsilon}^{-1} \nabla \times \vec{H}) - k_0^2 \vec{H} - \alpha \nabla (\nabla \cdot \vec{H}) = 0 \quad (5.16)$$

Here  $k_0 = \omega/c$  is the wavenumber,  $\alpha$  is the penalty term added to fight spurious FEM solutions. For anisotropic single-axis medium with  $\partial\hat{\epsilon}/\partial\phi = 0$  in cylindrical system of coordinates we have:

$$\hat{\epsilon} = \begin{pmatrix} \epsilon_\rho & 0 & 0 \\ 0 & \epsilon_\rho & 0 \\ 0 & 0 & \epsilon_z \end{pmatrix}.$$

We now assume axial symmetry of our electromagnetic fields and insert an imaginary unity in front of the  $H_\phi$  to allow all field components to be real numbers and also to account for the phase shift of this component  $\vec{H}(\rho, \phi, z) = \{H_\rho(\rho, z), iH_\phi(\rho, z), H_z(\rho, z)\} \times e^{im\phi}$ .

We write the wave equation (5.16) explicitly in cylindrical coordinates, thus obtaining a set of three differential equations for the domain  $\Omega$  given by the resonator's cross section and some space outside:

$$\begin{aligned} A_1\{H_\rho^t, H_\phi^t, H_z^t\} &= 0 \\ A_2\{H_\rho^t, H_\phi^t, H_z^t\} &= 0 \\ A_3\{H_\rho^t, H_\phi^t, H_z^t\} &= 0 \end{aligned}$$

The numerical solutions of these equations and boundary conditions can be found with **FreeFEM** if we write the system in the weak, or integral form.

### 5.14.2 Weak formulation

In general, to obtain the integral or “weak” statements equivalent to system (5.17) and boundary conditions we form a scalar dot product between an arbitrary magnetic field test function  $\mathbf{H}^t = \{H_\rho^t, H_\phi^t, H_z^t\}$  and the components of our vectorial equation  $A_1, A_2, A_3$ , and integrate over the resonator's cross section domain  $\Omega$  (and its boundary for the boundary conditions):

$$\int_{\Omega} (H_\rho^t A_1 + H_\phi^t A_2 + H_z^t A_3) d\Omega$$

We can reduce the order of partial derivatives in this integral by using the Green's formula for integration by parts. For example:

$$\int_{\Omega} H_z^t \frac{\partial^2 H_z}{\partial \rho^2} d\Omega = - \int_{\Omega} \frac{\partial H_z^t}{\partial \rho} \frac{\partial H_z}{\partial \rho} d\Omega + \oint H_z^t \frac{\partial H_z}{\partial \rho} n_\rho d\Gamma$$

Thus converting equations (5.17) we obtain a large expression for the weak form.

### 5.14.3 A dielectric sphere example with FreeFEM

We now compute the fundamental mode frequency for a fused silica sphere. The sphere is 36 micrometer in diameter, the refractive index is 1.46, the boundary condition is the magnetic wall (which can actually be omitted as it holds automatically).

```

1 // Parameters
2 real radius = 36; //approximate radius of the cavity
3 real yb = -10, yt = -yb; //window yb=bottom and yt=top coordinates
4 real xl = radius-5, xr = radius+3; //window xl=left and xr=right coordinates
5 real angle = asin((yt)/radius); //angle of the sphere segment to model in radians
6 int Nm = 60; //number of mesh vertices per border
7 real ne = 1.46; //n_e-extraordinary refractive index (root of permittivity parallel_
8   ↪to z-axis, epara)
9 real no = 1.46; //n_o-ordinary refractive index (root of permittivity orthogonal to z-
10  ↪axis, eortho)
11 real nm = 1; //refractive index of surrounding medium (air)
12
13 int nev = 4; // number of eigen values to find
14 int M = 213; //azimuthal mode order ~ 2Pi*n*R/lambda
real alpha = 1; //penalty term

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15
16 // Mesh
17 border W1l(t=0, 1){x=xl+(radius*cos(angle)-xl)*(1-t); y=yt; label=1;}
18 border W1r(t=0, 1){x=xr-(xr-radius*cos(angle))*(t); y=yt; label=1;}
19 border W2(t=0, 1){x=xr; y=yb+(yt-yb)*t; label=1;}
20 border W3l(t=0, 1){x=xl+(radius*cos(angle)-xl)*(t); y=yb; label=1;}
21 border W3r(t=0, 1){x=xr-(xr-radius*cos(angle))*(1-t); y=yb; label=1;}
22 border W4(t=0, 1){x=xl; y=yt-(yt-yb)*t; label=1;}
23 border S(t=0, 1){x=radius*cos((t-0.5)*2*angle); y=radius*sin((t-0.5)*2*angle); ↵
   ↵label=2;}
24 mesh Th = buildmesh(W1r(Nm/4) + W1l(Nm/4) + W4(Nm) + W3l(Nm/4) + W3r(Nm/4) + W2(Nm) + ↵
   ↵S(Nm));
25 plot(Th, WindowIndex=0);

26
27 // Fespace
28 fespace Ph(Th, P0);
29 Ph reg = region;
30
31 int ncav = reg(xl+1, 0); // cavity
32 int nair = reg(xr-1, 0); // air
33 Ph eorto = no^2*(region==ncav) + nm^2*(region==nair); // subdomains for epsilon values ↵
   ↵inside and outside the resonators
34 Ph epara = ne^2*(region==ncav) + nm^2*(region==nair); // subdomains for epsilon values ↵
   ↵inside and outside the resonators
35
36 // supplementary variables to store eigenvectors, defined on mesh Th with P2 elements - ↵
   ↵Largange quadratic.
37 fespace Supp(Th, P2);
38 Supp eHsqr;
39
40 // 3d vector FE space
41 fespace Vh(Th, [P2, P2, P2]);
42 Vh [Hr, Hphi, Hz], [vHr, vHphi, vHz]; // magnetic field components on Vh space and ↵
   ↵test functions vH
43
44 // Macro
45 // boundary condition macros
46 macro EWall(Hr, Hphi, Hz) (
47     dy(Hr) - dx(Hz) + Hr*N.x + Hz*N.y
48     - epara*(Hz*M - dy(Hphi)*x)*N.y
49     + eorto*(Hphi - Hr*M+dx(Hphi)*x)*N.x) //
50 macro MWall(Hr, Hphi, Hz) (
51     Hphi + Hz*N.x - Hr*N.y
52     + epara*(Hz*M - dy(Hphi)*x)*N.x
53     + eorto*(Hphi - Hr*M+dx(Hphi)*x)*N.y) //
54
55 // Problem
56 real sigma = (M/(ne*radius))^2+2; // value of the shift ( $k^2$ ), where the modes will be ↵
   ↵found
57 varf b ([Hr, Hphi, Hz], [vHr, vHphi, vHz])
58 = int2d(Th) (
59     x*(Hr*vHr+Hphi*vHphi+Hz*vHz)
60 )
61 ;
62 // OP = A - sigma B ; // the shifted matrix
63 varf op ([Hr, Hphi, Hz], [vHr, vHphi, vHz])=
64     int2d(Th) (

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106
    (
        (eorto*(vHphi*Hphi - M*(vHphi*Hr + Hphi*vHr) + M^2*vHr*Hr) + epara*M^
    ↪2*vHz*Hz) /x //A/r
        + eorto*(dx(vHphi)*(Hphi - M*Hr) + dx(Hphi)*(vHphi - M*vHr)) -_
    ↪epara*M*(vHz*dy(Hphi) + Hz*dy(vHphi)) //B
        + x*(eorto*dx(vHphi)*dx(Hphi) + epara*((dx(vHz) - dy(vHr))*(dx(Hz) -_
    ↪dy(Hr)) + dy(vHphi)*dy(Hphi))) //C
    ) / (eorto*epara)
    + alpha*(
        (vHr*Hr - M*(vHphi*Hr + Hphi*vHr) + M^2*vHphi*Hphi) /x //D/r
        + (dx(vHr) + dy(vHz))*(Hr - M*Hphi) + (vHr - M*vHphi)*(dx(Hr) + dy(Hz)) //
    ↪E
        + x*(dx(vHr) + dy(vHz))*(dx(Hr) + dy(Hz)) //F
    )
    - sigma*x*(vHr*Hr + vHphi*Hphi + vHz*Hz)
)
//electric wall boundary condition on the boundary of computation domain
+int1d(Th, 1)(
    EWall(Hr, Hphi, Hz)*EWall(vHr, vHphi, vHz)
)
;
//setting sparse matrices and assigning the solver UMFPACK to solve eigenvalue problem
matrix B = b(Vh, Vh, solver=UMFPACK);
matrix OP = op(Vh, Vh, solver=UMFPACK);

// Solve
real[int] ev(nev); //to store the nev eigenvalue
Vh[int] [eHr, eHphi, eHz](nev); //to store the nev eigenvector
//calling ARPACK on sparse matrices with the assigned solver UMFPACK:
int k = EigenValue(OP, B, sym=true, sigma=sigma, value=ev, vector=eHr, tol=1e-10,
    ↪maxit=0, ncv=0);

k = min(k, nev); //sometimes the number of converged eigen values
//can be greater than nev

//file to output mode values
ofstream f("modes.txt");
//setting number of digits in the file output
int nold = f.precision(11);

// Plot & Save
for (int i = 0; i < k; i++){
    real lambda = 2*pi/sqrt(ev[i]);
    eHsqr = (sqrt(eHr[i]^2 + eHphi[i]^2 + eHz[i]^2)); //intensity from magnetic field
    ↪components
    plot(eHsqr, WindowIndex=i, value=1, nbiso=20, LabelColors=1, aspectratio=1, cmm=
    ↪"Mode "+i+", lambda="+lambda+", F="+((299792.458/lambda));
    f << "Mode "<< i << ", ka=" << sqrt(ev[i])*radius << endl;
}
}

```

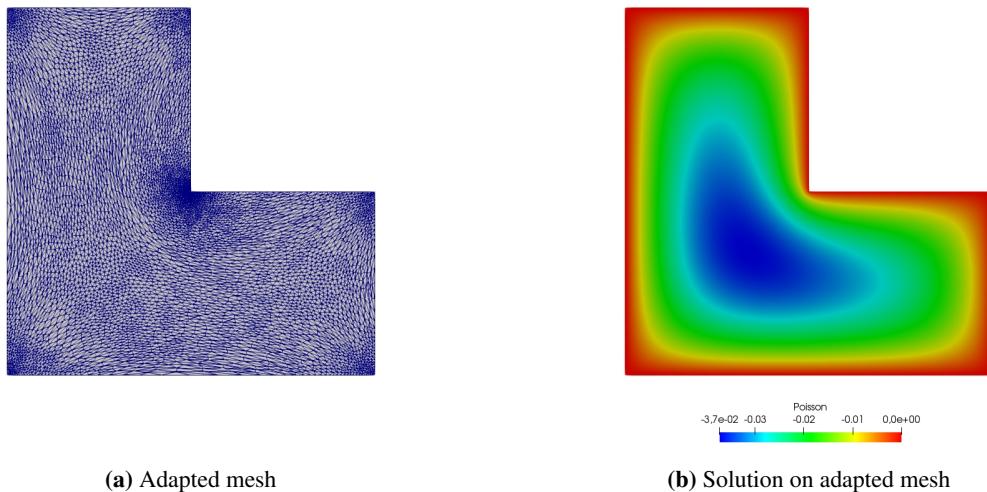
## EXAMPLES

### 6.1 Misc

#### 6.1.1 Poisson's Equation

```
1 // Parameters
2 int nn = 20;
3 real L = 1.;
4 real H = 1.;
5 real l = 0.5;
6 real h = 0.5;
7
8 func f = 1.;
9 func g = 0.;
10
11 int NAdapt = 10;
12
13 // Mesh
14 border b1(t=0, L){x=t; y=0;};
15 border b2(t=0, h){x=L; y=t;};
16 border b3(t=L, l){x=t; y=h;};
17 border b4(t=h, H){x=l; y=t;};
18 border b5(t=l, 0){x=t; y=H;};
19 border b6(t=H, 0){x=0; y=t;};
20
21 mesh Th = buildmesh(b1(nn*L) + b2(nn*h) + b3(nn*(L-l)) + b4(nn*(H-h)) + b5(nn*l) +_
22 ↪ b6(nn*H));
23
24 // Fespace
25 fespace Vh(Th, P1); // Change P1 to P2 to test P2 finite element
26 Vh u, v;
27
28 // Macro
29 macro grad(u) [dx(u), dy(u)] //
30
31 // Problem
32 problem Poisson (u, v, solver=CG, eps=-1.e-6)
33   = int2d(Th)(
34     grad(u)' * grad(v)
35   )
36   + int2d(Th)(
37     f * v
38   )
```

(continues on next page)

**Fig. 6.1:** Poisson

(continued from previous page)

```

38     + on(b1, b2, b3, b4, b5, b6, u=g)
39
40
41 // Mesh adaptation iterations
42 real error = 0.1;
43 real coef = 0.1^(1./5.);
44 for (int i = 0; i < NAdapt; i++) {
45     // Solve
46     Poisson;
47
48     // Plot
49     plot(Th, u);
50
51     // Adaptmesh
52     Th = adaptmesh(Th, u, inquire=1, err=error);
53     error = error * coef;
54 }
```

## 6.1.2 Poisson's equation 3D

```

1 load "tetgen"
2
3 // Parameters
4 real hh = 0.1;
5 func ue = 2.*x*x + 3.*y*y + 4.*z*z + 5.*x*y + 6.*x*z + 1.;
6 func f= -18.;
7
8 // Mesh
9 mesh Th = square(10, 20, [x*pi-pi/2, 2*y*pi]); // ]-pi/2, pi/2[X]0,2pi[
10 func f1 = cos(x)*cos(y);
11 func f2 = cos(x)*sin(y);
```

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```

12 func f3 = sin(x);
13 func f1x = sin(x)*cos(y);
14 func f1y = -cos(x)*sin(y);
15 func f2x = -sin(x)*sin(y);
16 func f2y = cos(x)*cos(y);
17 func f3x = cos(x);
18 func f3y = 0;
19 func m11 = f1x^2 + f2x^2 + f3x^2;
20 func m21 = f1x*f1y + f2x*f2y + f3x*f3y;
21 func m22 = f1y^2 + f2y^2 + f3y^2;
22 func perio = [[4, y], [2, y], [1, x], [3, x]];
23 real vv = 1/square(hh);
24 Th = adaptmesh(Th, m11*vv, m21*vv, m22*vv, IsMetric=1, periodic=perio);
25 Th = adaptmesh(Th, m11*vv, m21*vv, m22*vv, IsMetric=1, periodic=perio);
26 plot(Th);
27
28 real[int] domain = [0., 0., 0., 1, 0.01];
29 mesh3 Th3 = tetgtransfo(Th, transfo=[f1, f2, f3], nbofregions=1, regionlist=domain);
30 plot(Th3);
31
32 border cc(t=0, 2*pi) {x=cos(t); y=sin(t); label=1;};
33 mesh Th2 = buildmesh(cc(50));
34
35 // Fespace
36 fespace Vh(Th3, P23d);
37 Vh u, v;
38 Vh uhe = ue;
39 cout << "uhe min: " << uhe[].min << " - max: " << uhe[].max << endl;
40 cout << uhe(0.,0.,0.) << endl;
41
42 fespace Vh2(Th2, P2);
43 Vh2 u2, u2e;
44
45 // Macro
46 macro Grad3(u) [dx(u), dy(u), dz(u)] //
47
48 // Problem
49 problem Lap3d (u, v, solver=CG)
50   = int3d(Th3) (
51     Grad3(v)' * Grad3(u)
52   )
53   - int3d(Th3) (
54     f * v
55   )
56   + on(0, 1, u=ue)
57 ;
58
59 // Solve
60 Lap3d;
61 cout << "u min: " << u[].min << " - max: " << u[].max << endl;
62
63 // Error
64 real err = int3d(Th3)(square(u-ue));
65 cout << int3d(Th3)(1.) << " = " << Th3.measure << endl;
66 Vh d = ue - u;
67 cout << " err = " << err << " - diff l^intfy = " << d[].linfty << endl;
68

```

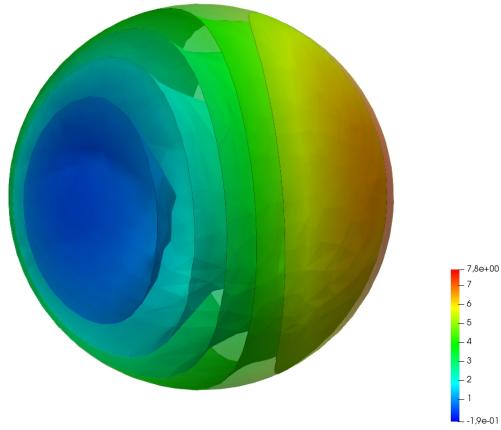
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```

69 // Plot
70 u2 = u;
71 u2e = ue;
72 plot(u2, wait=true);
73 plot(u2, u2e,wait=true);

```

**Fig. 6.2:** Iso-surfaces of the solution

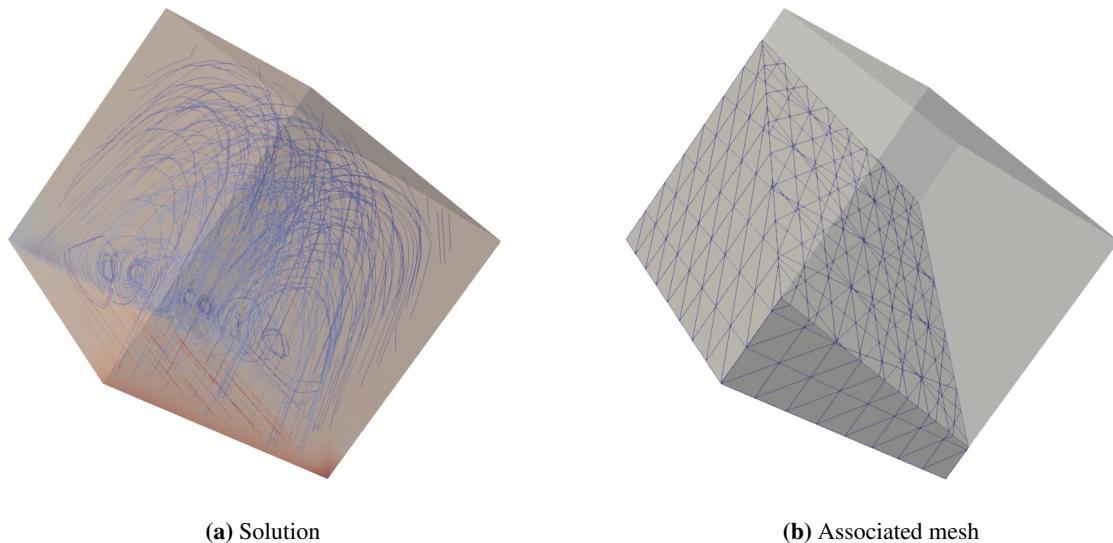
### 6.1.3 Stokes Equation on a cube

```

1 load "msh3"
2 load "medit" // Dynamically loaded tools for 3D
3
4 // Parameters
5 int nn = 8;
6
7 // Mesh
8 mesh Th0 = square(nn, nn);
9 int[int] rup = [0, 2];
10 int[int] rdown = [0, 1];
11 int[int] rmid = [1, 1, 2, 1, 3, 1, 4, 1];
12 real zmin = 0, zmax = 1;
13 mesh3 Th = buildlayers(Th0, nn, zbound=[zmin, zmax],
14     reffacemid=rmid, reffaceup=rup, reffacelow=rdown);
15
16 medit("c8x8x8", Th); // 3D mesh visualization with medit
17
18 // Fespaces
19 fespace Vh2(Th0, P2);
20 Vh2 ux, uz, p2;
21
22 fespace VVh(Th, [P2, P2, P2, P1]);
23 VVh [u1, u2, u3, p];
24 VVh [v1, v2, v3, q];
25
26 // Macro
27 macro Grad(u) [dx(u), dy(u), dz(u)] //

```

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**Fig. 6.3:** Stokes

(continued from previous page)

```

28 macro div(u1,u2,u3) (dx(u1) + dy(u2) + dz(u3)) //
29
30 // Problem (directly solved)
31 solve vStokes ([u1, u2, u3, p], [v1, v2, v3, q])
32   = int3d(Th, qforder=3)(
33     Grad(u1)' * Grad(v1)
34     + Grad(u2)' * Grad(v2)
35     + Grad(u3)' * Grad(v3)
36     - div(u1, u2, u3) * q
37     - div(v1, v2, v3) * p
38     + 1e-10 * q * p
39   )
40   + on(2, u1=1., u2=0, u3=0)
41   + on(1, u1=0, u2=0, u3=0)
42   ;
43
44 // Plot
45 plot(p, wait=1, nbiso=5); // 3D visualization of pressure isolines
46
47 // See 10 plan of the velocity in 2D
48 for(int i = 1; i < 10; i++){
49   // Cut plane
50   real yy = i/10.;
51   // 3D to 2D interpolation
52   ux = u1(x,yy,y);
53   uz = u3(x,yy,y);
54   p2 = p(x,yy,y);
55   // Plot
56   plot([ux, uz], p2, cmm="cut y = "+yy, wait= 1);
57 }
```

## 6.1.4 Cavity

```

1 //Parameters
2 int m = 300;
3 real L = 1;
4 real rho = 500.;
5 real mu = 0.1;
6
7 real uin = 1;
8 func fx = 0;
9 func fy = 0;
10 int[int] noslip = [1, 2, 4];
11 int[int] inflow = [3];
12
13 real dt = 0.1;
14 real T = 50;
15
16 real eps = 1e-3;
17
18 //Macros
19 macro div(u) (dx(u#x) + dy(u#y))//
20 macro grad(u) [dx(u), dy(u)]//
21 macro Grad(u) [grad(u#x), grad(u#y)]//
22
23 //Time
24 real cpu;
25 real tabcpu;
26
27 //mesh
28 border C1(t = 0, L){ x = t; y = 0; label = 1; }
29 border C2(t = 0, L){ x = L; y = t; label = 2; }
30 border C3(t = 0, L){ x = L-t; y = L; label = 3; }
31 border C4(t = 0, L){ x = 0; y = L-t; label = 4; }
32 mesh th = buildmesh( C1(m) + C2(m) + C3(m) + C4(m) );
33
34 fespace UPh(th, [P2,P2,P1]);
UPh [ux, uy, p];
UPh [uhx, uhy, ph];
UPh [upx, upy, pp];
35
36 //Solve
37 varf navierstokes([ux, uy, p], [uhx, uhy, ph])
= int2d(th) (
38     rho/dt* [ux, uy]' * [uhx, uhy]
39     + mu* (Grad(u):Grad(uh))
40     - p* div(uh)
41     - ph* div(u)
42     - 1e-10 *p*ph
43 )
44
45 + int2d(th) (
46     [fx, fy]' * [uhx, uhy]
47     + rho/dt* [convect([upx, upy], -dt, upx), convect([upx, upy], -dt, upy)]' * [uhx,
48     uhy]
49 )
50
51 + on(noslip, ux=0, uy=0)

```

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```

55 + on(inflow, ux=uin, uy=0)
56 ;
57
58 //Initialization
59 [ux, uy, p]=[0, 0, 0];
60
61 matrix<real> NS = navierstokes(UPh, UPh, solver=sparse);
62 real[int] NSrhs = navierstokes(0, UPh);
63
64 //Time loop
65 for(int j = 0; j < T/dt; j++){
66   [upx, upy, pp]=[ux, uy, p];
67
68   NSrhs = navierstokes(0, UPh);
69   ux[] = NS^-1 * NSrhs;
70
71   plot( [ux,uy], p, wait=0, cmm=j);
72 }
73
74 //CPU
75 cout << " CPU = " << clock()-cpu << endl ;
76 tabcpu = clock()-cpu;

```

## 6.2 Mesh Generation

### 6.2.1 Square mesh

```

1 mesh Th0 = square(10 ,10);
2
3 mesh Th1 = square(4, 5);
4
5 real x0 = 1.2;
6 real x1 = 1.8;
7 real y0 = 0;
8 real y1 = 1;
9 int n = 5;
10 real m = 20;
11 mesh Th2 = square(n, m, [x0+(x1-x0)*x, y0+(y1-y0)*y]);
12
13 for (int i = 0; i < 5; ++i){
14   int[int] labs = [11, 12, 13, 14];
15   mesh Th3 = square(3, 3, flags=i, label=labs, region=10);
16   plot(Th3, wait=1, cmm="square flags = "+i );
17 }

```

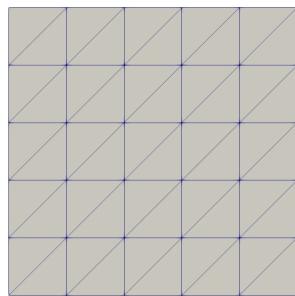
### 6.2.2 Mesh adaptation

```

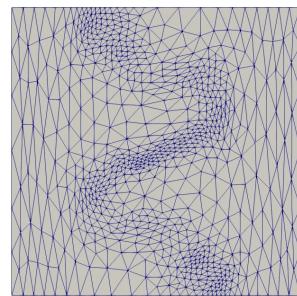
1 // Parameters
2 real eps = 0.0001;
3 real h = 1;
4 real hmin = 0.05;

```

(continues on next page)



(a) Initial mesh



(b) Adapted mesh

**Fig. 6.4:** Mesh adaptation

(continued from previous page)

```

5 func f = 10.0*x^3 + y^3 + h*atan2(eps, sin(5.0*y)-2.0*x);
6
7 // Mesh
8 mesh Th = square(5, 5, [-1+2*x, -1+2*y]);
9
10 // Fespace
11 fespace Vh(Th,P1);
12 Vh fh = f;
13 plot(fh);
14
15 // Adaptmesh
16 for (int i = 0; i < 2; i++){
17     Th = adaptmesh(Th, fh);
18     fh = f; //old mesh is deleted
19     plot(Th, fh, wait=true);
20 }
```

### 6.2.3 Mesh adaptation for the Poisson's problem

```

1 // Parameters
2 real error = 0.1;
3
4 // Mesh
5 border ba(t=0, 1){x=t; y=0; label=1;}
6 border bb(t=0, 0.5){x=1; y=t; label=1;}
7 border bc(t=0, 0.5){x=1-t; y=0.5; label=1;}
8 border bd(t=0.5, 1){x=0.5; y=t; label=1;}
9 border be(t=0.5, 1){x=1-t; y=1; label=1;}
10 border bf(t=0, 1){x=0; y=1-t; label=1;}
11 mesh Th = buildmesh(ba(6) + bb(4) + bc(4) + bd(4) + be(4) + bf(6));
12
13 // Fespace
14 fespace Vh(Th, P1);
15 Vh u, v;
16
```

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```

17 // Function
18 func f = 1;
19
20 // Problem
21 problem Poisson(u, v, solver=CG, eps=1.e-6)
22   = int2d(Th) (
23     dx(u)*dx(v)
24     + dy(u)*dy(v)
25   )
26   - int2d(Th) (
27     f*v
28   )
29   + on(1, u=0);
30
31 // Adaptmesh loop
32 for (int i = 0; i < 4; i++) {
33   Poisson;
34   Th = adaptmesh(Th, u, err=error);
35   error = error/2;
36 }
37
38 // Plot
39 plot(u);

```

## 6.2.4 Uniform mesh adaptation

```

1 mesh Th = square(2, 2); // The initial mesh
2 plot(Th, wait=true);
3
4 Th = adaptmesh(Th, 1./30., IsMetric=1, nbvx=10000);
5 plot(Th, wait=true);
6
7 Th = adaptmesh(Th, 1./30., IsMetric=1, nbvx=10000); // More than one time due to the
8 Th = adaptmesh(Th, 1./30., IsMetric=1, nbvx=10000); // adaptation bound `maxsubdiv=
9 plot(Th, wait=true);

```

## 6.2.5 Borders

```

1 {
2   int upper = 1;
3   int others = 2;
4   int inner = 3;
5
6   border C01(t=0, 1){x=0; y=-1+t; label=upper;}
7   border C02(t=0, 1){x=1.5-1.5*t; y=-1; label=upper;}
8   border C03(t=0, 1){x=1.5; y=-t; label=upper;}
9   border C04(t=0, 1){x=1+0.5*t; y=0; label=others;}
10  border C05(t=0, 1){x=0.5+0.5*t; y=0; label=others;}
11  border C06(t=0, 1){x=0.5*t; y=0; label=others;}
12  border C11(t=0, 1){x=0.5; y=-0.5*t; label=inner;}
13  border C12(t=0, 1){x=0.5+0.5*t; y=-0.5; label=inner;}
14  border C13(t=0, 1){x=1; y=-0.5+0.5*t; label=inner;}

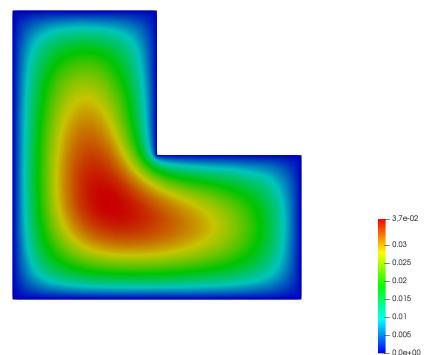
```

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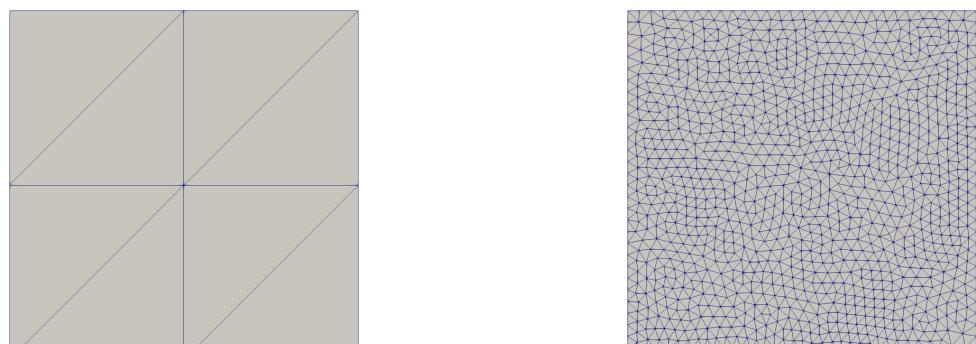
(a) Initial mesh

(b) Adapted mesh



(c) Solution on adapted mesh

**Fig. 6.5:** Mesh adaptation (Poisson)



(a) Initial mesh

(b) Adapted mesh

**Fig. 6.6:** Uniform mesh adaptation

(continued from previous page)

```

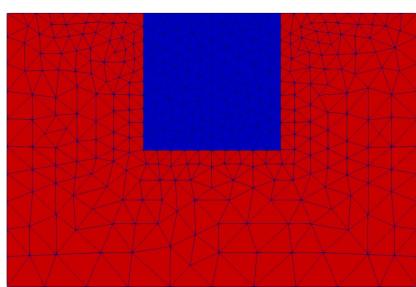
15
16   int n = 10;
17   plot(C01(-n) + C02(-n) + C03(-n) + C04(-n) + C05(-n)
18       + C06(-n) + C11(n) + C12(n) + C13(n), wait=true);
19
20   mesh Th = buildmesh(C01(-n) + C02(-n) + C03(-n) + C04(-n) + C05(-n)
21       + C06(-n) + C11(n) + C12(n) + C13(n));
22
23   plot(Th, wait=true);
24
25   cout << "Part 1 has region number " << Th(0.75, -0.25).region << endl;
26   cout << "Part 2 has region number " << Th(0.25, -0.25).region << endl;
27 }
28
29 {
30   border a(t=0, 2*pi){x=cos(t); y=sin(t); label=1;}
31   border b(t=0, 2*pi){x=0.3+0.3*cos(t); y=0.3*sin(t); label=2;}
32   plot(a(50) + b(30)); //to see a plot of the border mesh
33   mesh Thwithouthole = buildmesh(a(50) + b(30));
34   mesh Thwithhole = buildmesh(a(50) + b(-30));
35   plot(Thwithouthole);
36   plot(Thwithhole);
37 }
38
39 {
40   real r=1;
41   border a(t=0, 2*pi){x=r*cos(t); y=r*sin(t); label=1;}
42   r=0.3;
43   border b(t=0, 2*pi){x=r*cos(t); y=r*sin(t); label=1;}
44   // mesh Thwithhole = buildmesh(a(50) + b(-30)); // do not do this because the two
45   // circles have the same radius = $0.3$
46 }
```

## 6.2.6 Change

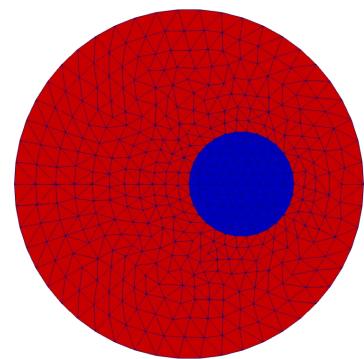
```

1  verbosity=3;
2
3 // Mesh
4 mesh Th1 = square(10, 10);
5 mesh Th2 = square(20, 10, [x+1, y]);
6
7 int[int] r1=[2, 0];
8 plot(Th1, wait=true);
9
10 Th1 = change(Th1, label=r1); // Change edges' label from 2 to 0
11 plot(Th1, wait=true);
12
13 int[int] r2=[4, 0];
14 Th2 = change(Th2, label=r2); // Change edges' label from 4 to 0
15 plot(Th2, wait=true);
16
17 mesh Th = Th1 + Th2; // 'gluing together' Th1 and Th2 meshes
18 cout << "nb lab = " << int1d(Th1,1,3,4)(1./lenEdge)+int1d(Th2,1,2,3)(1./lenEdge)
19     << " == " << int1d(Th,1,2,3,4)(1./lenEdge) << " == " << ((10+20)+10)*2 << endl;
20 plot(Th, wait=true);
```

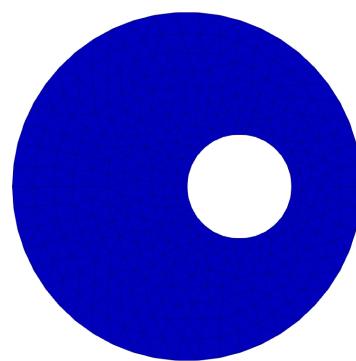
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(a) Mesh with two regions



(b) Mesh without a hole



(c) Mesh with a hole

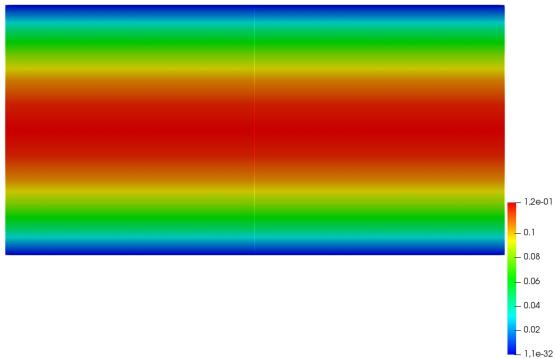
**Fig. 6.7:** Borders

(continued from previous page)

```

21
22 fespace Vh(Th, P1);
23 Vh u, v;
24
25 macro Grad(u) [dx(u),dy(u)] // Definition of a macro
26
27 solve P(u, v)
28   = int2d(Th) (
29     Grad(u)'*Grad(v)
30   )
31   -int2d(Th) (
32     v
33   )
34   + on(1, 3, u=0)
35 ;
36
37 plot(u, wait=1);

```

**Fig. 6.8:** Result

### 6.2.7 Cube

```

1 load "msh3"
2
3 int[int] l6 = [37, 42, 45, 40, 25, 57];
4 int r11 = 11;
5 mesh3 Th = cube(4, 5, 6, [x*2-1, y*2-1, z*2-1], label=l6, flags=3, region=r11);
6
7 cout << "Volume = " << Th.measure << ", border area = " << Th.bordermeasure << endl;
8
9 int err = 0;
10 for(int i = 0; i < 100; ++i){
11   real s = int2d(Th,i)(1.);
12   real sx = int2d(Th,i)(x);
13   real sy = int2d(Th,i)(y);
14   real sz = int2d(Th,i)(z);
15

```

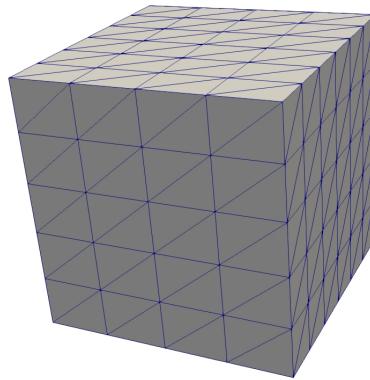
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```

16 if(s){
17     int ix = (sx/s+1.5);
18     int iy = (sy/s+1.5);
19     int iz = (sz/s+1.5);
20     int ii = (ix + 4*(iy+1) + 16*(iz+1) );
21     //value of ix,iy,iz => face min 0, face max 2, no face 1
22     cout << "Label = " << i << ", s = " << s << " " << ix << iy << iz << " : " <<
23     ii << endl;
24     if( i != ii ) err++;
25 }
26 real volr11 = int3d(Th,r11)(1.);
27 cout << "Volume region = " << 11 << ":" << volr11 << endl;
28 if((volr11 - Th.measure )>1e-8) err++;
29 plot(Th, fill=false);
30 cout << "Nb err = " << err << endl;
31 assert(err==0);

```

**Fig. 6.9:** Cube

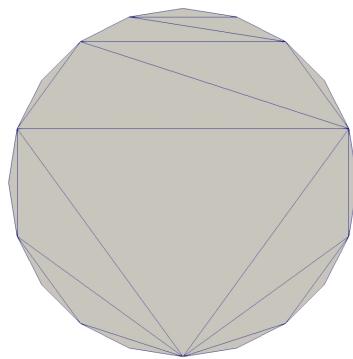
### 6.2.8 Empty mesh

```

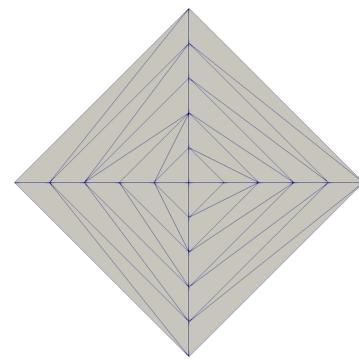
1 {
2     border a(t=0, 2*pi){x=cos(t); y=sin(t); label=1;}
3     mesh Th = buildmesh(a(20));
4     Th = emptymesh(Th);
5     plot(Th);
6 }
7 {
8     mesh Th = square(10, 10);
9     int[int] ssd(Th.nt);
10    // Builds the pseudo region numbering
11    for(int i = 0; i < ssd.n; i++){
12        int iq = i/2; // Because we have 2 triangles per quad
13        int ix = iq%10;
14        int iy = iq/10;
15        ssd[i] = 1 + (ix>=5) + (iy>=5)*2;

```

(continues on next page)



(a) Empty square



(b) Empty diamond

**Fig. 6.10:** Empty mesh

(continued from previous page)

```

16 }
17 // Builds an empty mesh with all edges that satisfy e=T1 cap T2 and ssd[T1] !=_
18 ↪ssd[T2]
19 Th = emptymesh(Th, ssd);
20 // Plot
21 plot(Th);
}

```

## 6.2.9 3 points

```

1 // Square for Three-Point Bend Specimens fixed on Fix1, Fix2
2 // It will be loaded on Load
3 real a = 1, b = 5, c = 0.1;
4 int n = 5, m = b*n;
5 border Left(t=0, 2*a) {x=-b; y=a-t; }
6 border Bot1(t=0, b/2-c) {x=-b+t; y=-a; }
7 border Fix1(t=0, 2*c) {x=-b/2-c+t; y=-a; }
8 border Bot2(t=0, b-2*c) {x=-b/2+c+t; y=-a; }
9 border Fix2(t=0, 2*c) {x=b/2-c+t; y=-a; }
10 border Bot3(t=0, b/2-c) {x=b/2+c+t; y=-a; }
11 border Right(t=0, 2*a) {x=b; y=-a+t; }
12 border Top1(t=0, b-c) {x=b-t; y=a; }
13 border Load(t=0, 2*c) {x=c-t; y=a; }
14 border Top2(t=0, b-c) {x=-c-t; y=a; }
15
16 mesh Th = buildmesh(Left(n) + Bot1(m/4) + Fix1(5) + Bot2(m/2)
17 + Fix2(5) + Bot3(m/4) + Right(n) + Top1(m/2) + Load(10) + Top2(m/2));
18 plot(Th, bw=true);

```

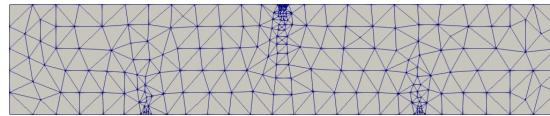
## 6.2.10 Bezier

```

1 // A cubic Bezier curve connecting two points with two control points
2 func real bzi(real p0, real p1, real q1, real q2, real t){

```

(continues on next page)

**Fig. 6.11:** 3 Points

(continued from previous page)

```

3   return p0*(1-t)^3 + q1*3*(1-t)^2*t + q2*3*(1-t)*t^2 + p1*t^3;
4 }
5
6 real[int] p00 = [0, 1], p01 = [0, -1], q00 = [-2, 0.1], q01 = [-2, -0.5];
7 real[int] p11 = [1,-0.9], q10 = [0.1, -0.95], q11=[0.5, -1];
8 real[int] p21 = [2, 0.7], q20 = [3, -0.4], q21 = [4, 0.5];
9 real[int] q30 = [0.5, 1.1], q31 = [1.5, 1.2];
10 border G1(t=0, 1){
11   x=bzi(p00[0], p01[0], q00[0], q01[0], t);
12   y=bzi(p00[1], p01[1], q00[1], q01[1], t);
13 }
14 border G2(t=0, 1){
15   x=bzi(p01[0], p11[0], q10[0], q11[0], t);
16   y=bzi(p01[1], p11[1], q10[1], q11[1], t);
17 }
18 border G3(t=0, 1){
19   x=bzi(p11[0], p21[0], q20[0], q21[0], t);
20   y=bzi(p11[1], p21[1], q20[1], q21[1], t);
21 }
22 border G4(t=0, 1){
23   x=bzi(p21[0], p00[0], q30[0], q31[0], t);
24   y=bzi(p21[1], p00[1], q30[1], q31[1], t);
25 }
26 int m = 5;
27 mesh Th = buildmesh(G1(2*m) + G2(m) + G3(3*m) + G4(m));
28 plot(Th, bw=true);

```

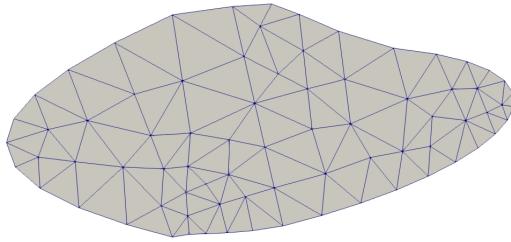
### 6.2.11 Build layer mesh

```

1 load "msh3"
2 load "tetgen"
3 load "medit"
4
5 // Parameters
6 int C1 = 99;

```

(continues on next page)

**Fig. 6.12:** Bezier

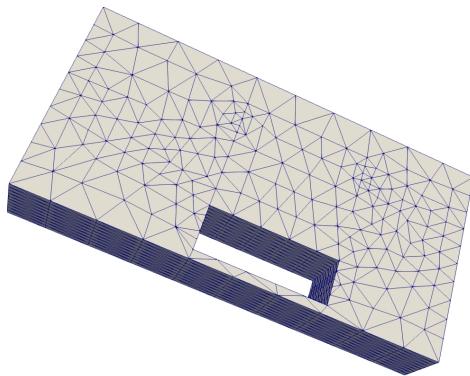
(continued from previous page)

```

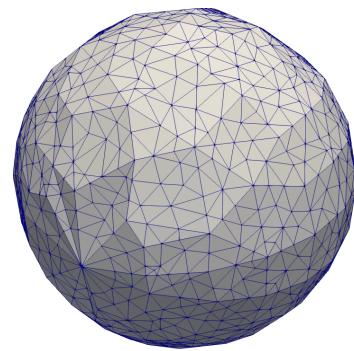
7 int C2 = 98;
8
9 // 2D mesh
10 border C01(t=0, pi){x=t; y=0; label=1;}
11 border C02(t=0, 2*pi){ x=pi; y=t; label=1;}
12 border C03(t=0, pi){ x=pi-t; y=2*pi; label=1;}
13 border C04(t=0, 2*pi){ x=0; y=2*pi-t; label=1;}
14
15 border C11(t=0, 0.7){x=0.5+t; y=2.5; label=C1;}
16 border C12(t=0, 2){x=1.2; y=2.5+t; label=C1;}
17 border C13(t=0, 0.7){x=1.2-t; y=4.5; label=C1;}
18 border C14(t=0, 2){x=0.5; y=4.5-t; label=C1;}
19
20 border C21(t=0, 0.7){x=2.3+t; y=2.5; label=C2;}
21 border C22(t=0, 2){x=3; y=2.5+t; label=C2;}
22 border C23(t=0, 0.7){x=3-t; y=4.5; label=C2;}
23 border C24(t=0, 2){x=2.3; y=4.5-t; label=C2;}
24
25 mesh Th = buildmesh(C01(10) + C02(10) + C03(10) + C04(10)
26   + C11(5) + C12(5) + C13(5) + C14(5)
27   + C21(-5) + C22(-5) + C23(-5) + C24(-5));
28
29 mesh Ths = buildmesh(C01(10) + C02(10) + C03(10) + C04(10)
30   + C11(5) + C12(5) + C13(5) + C14(5));
31
32 // Construction of a box with one hole and two regions
33 func zmin = 0.;
34 func zmax = 1.;
35 int MaxLayer = 10;
36
37 func XX = x*cos(y);
38 func YY = x*sin(y);
39 func ZZ = z;
40
41 int[int] r1 = [0, 41], r2 = [98, 98, 99, 99, 1, 56];
42 int[int] r3 = [4, 12]; // Change upper surface mesh's triangles labels
43 // generated by the 2D mesh's triangles Th

```

(continues on next page)



(a) Box with a hole



(b) Sphere

**Fig. 6.13:** Build layer mesh

(continued from previous page)

```

44 // from label 4 to label 12
45 int[int] r4 = [4, 45]; // Change lower surface mesh's triangles labels
46 // generated by the 2D mesh's triangles Th
47 // from label 4 to label 45
48
49 mesh3 Th3 = buildlayers(Th, MaxLayer, zbound=[zmin, zmax], region=r1,
50   labelmid=r2, labelup=r3, labeledown=r4);
51 medit("box 2 regions 1 hole", Th3);
52
53 // Construction of a sphere with TetGen
54 func XX1 = cos(y)*sin(x);
55 func YY1 = sin(y)*sin(x);
56 func ZZ1 = cos(x);
57
58 real[int] domain = [0., 0., 0., 0, 0.001];
59 string test = "paACQ";
60 cout << "test = " << test << endl;
61 mesh3 Th3sph = tetgtransfo(Ths, transfo=[XX1, YY1, ZZ1],
62   switch=test, nbofregions=1, regionlist=domain);
63 medit("sphere 2 regions", Th3sph);

```

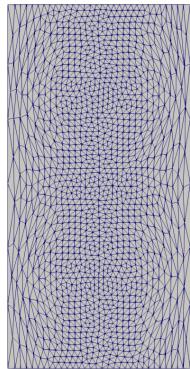
## 6.2.12 Sphere

```

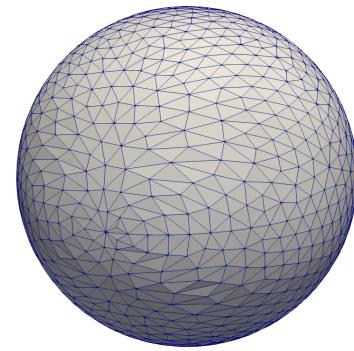
1 // Parameter
2 real hh = 0.1;
3
4 // Mesh 2D
5 mesh Th = square(10, 20, [x*pi-pi/2, 2*y*pi]); // ]-pi/2, pi/2[X]0, 2pi[
6 // A parametrization of a sphere
7 func f1 = cos(x)*cos(y);
8 func f2 = cos(x)*sin(y);
9 func f3 = sin(x);
10 // Partial derivative of the parametrization DF
11 func f1x = sin(x)*cos(y);
12 func f1y = -cos(x)*sin(y);

```

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(a) Initial mesh



(b) Sphere

**Fig. 6.14:** Sphere

(continued from previous page)

```

13 func f2x = -sin(x)*sin(y);
14 func f2y = cos(x)*cos(y);
15 func f3x = cos(x);
16 func f3y = 0;
17 //M = DF^t DF
18 func m11 = f1x^2 + f2x^2 + f3x^2;
19 func m21 = f1x*f1y + f2x*f2y + f3x*f3y;
20 func m22 = f1y^2 + f2y^2 + f3y^2;
21
22 // Periodic condition
23 func perio = [[4, y, [2, y, [1, x, [3, x]]];
24
25 // Mesh adaptation
26 real vv = 1/square(hh);
27 Th = adaptmesh(Th, m11*vv, m21*vv, m22*vv, IsMetric=1, inquire=1, periodic=perio);
28 Th = adaptmesh(Th, m11*vv, m21*vv, m22*vv, IsMetric=1, periodic=perio);
29 Th = adaptmesh(Th, m11*vv, m21*vv, m22*vv, IsMetric=1, periodic=perio);
30 Th = adaptmesh(Th, m11*vv, m21*vv, m22*vv, IsMetric=1, periodic=perio);
31
32 // Sphere
33 mesh3 Th3 = movemesh23(Th, transfo=[f1, f2, f3]);
34 plot(Th3);

```

## 6.3 Finite Element

### 6.3.1 Periodic 3D

```

1 load "msh3"
2 load "medit"
3
4 // Parameters
5 searchMethod=1; // More safe search algo
6 real a = 1, d = 0.5, h = 0.5;
7 int nnb = 7, nni = 10;

```

(continues on next page)

(continued from previous page)

```

8 int nz = 3;
9 func zmin = 0;
10 func zmax = h;
11
12 // Mesh 2D
13 border b1(t=0.5, -0.5) {x=a*t; y=-a/2; label=1;}
14 border b2(t=0.5, -0.5) {x=a/2; y=a*t; label=2;}
15 border b3(t=0.5, -0.5) {x=a*t; y=a/2; label=3;}
16 border b4(t=0.5, -0.5) {x=-a/2; y=a*t; label=4;}
17 border i1(t=0, 2.*pi) {x=d/2*cos(t); y=-d/2*sin(t); label=7;}
18 mesh Th = buildmesh(b1(-nnb) + b3(nnb) + b2(-nnb) + b4(nnb) + i1(nni));
19
20 { // Cleaning the memory correctly
21     int[int] old2new(0:Th.nv-1);
22     fespace Vh2(Th, P1);
23     Vh2 sorder = x + y;
24     sort(sorder[], old2new);
25     int[int] new2old = old2new^-1; // Inverse permutation
26     Th = change(Th, renumv=new2old);
27     sorder[] = 0:Th.nv-1;
28 }
29 {
30     fespace Vh2(Th, P1);
31     Vh2 nu;
32     nu[] = 0:Th.nv-1;
33     plot(nu, cmm="nu=", wait=true);
34 }
35
36 // Mesh 3D
37 int[int] rup = [0, 5], rlow = [0, 6], rmid = [1, 1, 2, 2, 3, 3, 4, 4, 7, 7], rtet =_
38     [0, 41];
39 mesh3 Th3 = buildlayers(Th, nz, zbound=[zmin, zmax],
40     reftet=rtet, reffacemid=rmid, reffaceup=rup, reffacelow=rlow);
41 for(int i = 1; i <= 6; ++i)
42     cout << " int " << i << " : " << int2d(Th3,i)(1.) << " " << int2d(Th3,i)(1./area)
43     << endl;
44
45 plot(Th3, wait=true);
46 medit("Th3", Th3);
47
48 fespace Vh(Th3, P2, periodic=[[1, x, z], [3, x, z], [2, y, z], [4, y, z], [5, x, y],_
49     [6, x, y]]);

```

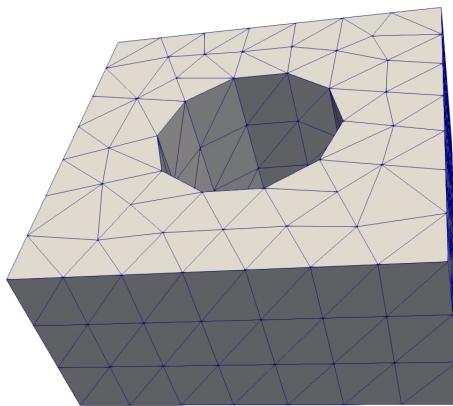
### 6.3.2 Lagrange multipliers

```

1 // Parameters
2 func f = 1 + x - y;
3
4 // Mesh
5 mesh Th = square(10, 10);
6
7 // Fespace
8 fespace Vh(Th, P1);
9 int n = Vh.ndof;
10 int n1 = n+1;

```

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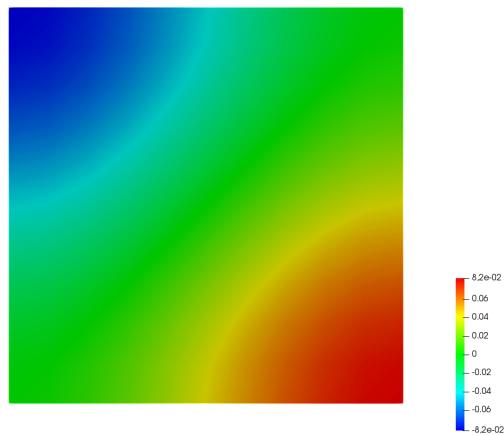
**Fig. 6.15:** Periodic mesh

(continued from previous page)

```

11 Vh uh, vh;
12
13 // Problem
14 varf va (uh, vh)
15   = int2d(Th) (
16     dx(uh)*dx(vh)
17     + dy(uh)*dy(vh)
18   )
19 ;
20
21 varf vL (uh, vh) = int2d(Th) (f*vh);
22 varf vb (uh, vh) = int2d(Th) (1.*vh);
23
24 matrix A = va(Vh, Vh);
25 real[int] b = vL(0, Vh);
26 real[int] B = vb(0, Vh);
27
28 // Block matrix
29 matrix AA = [ [ A, B ], [ B', 0 ] ];
30 set(AA, solver=sparseSolver);
31
32 real[int] bb(n+1), xx(n+1), b1(1), l(1);
33 b1 = 0;
34 // Builds the right hand side block
35 bb = [b, b1];
36
37 // Solve
38 xx = AA^-1 * bb;
39
40 // Set values
41 [uh[],l] = xx;
42
43 // Display
44 cout << " l = " << l(0) << " , b(u, 1) =" << B'*uh[] << endl;
45
46 // Plot
47 plot(uh);

```

**Fig. 6.16:** Result

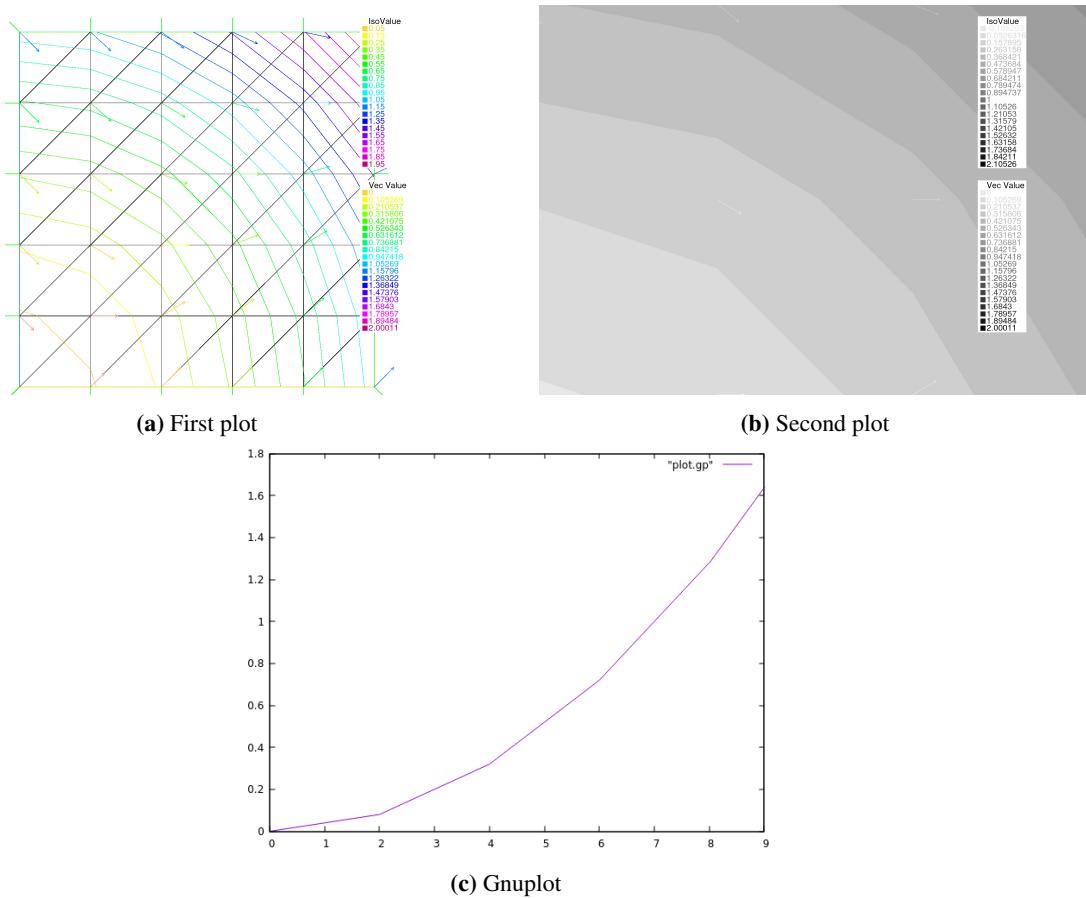
## 6.4 Visualization

### 6.4.1 Plot

```

1 mesh Th = square(5,5);
2 fespace Vh(Th, P1);
3
4 // Plot scalar and vectorial FE function
5 Vh uh=x*x+y*y, vh=-y^2+x^2;
6 plot(Th, uh, [uh, vh], value=true, wait=true);
7
8 // Zoom on box defined by the two corner points [0.1,0.2] and [0.5,0.6]
9 plot(uh, [uh, vh], bb=[[0.1, 0.2], [0.5, 0.6]],
10      wait=true, grey=true, fill=true, value=true);
11
12 // Compute a cut
13 int n = 10;
14 real[int] xx(10), yy(10);
15 for (int i = 0; i < n; i++) {
16     x = i/real(n);
17     y = i/real(n);
18     xx[i] = i;
19     yy[i] = uh; // Value of uh at point (i/10., i/10.)
20 }
21 plot([xx, yy], wait=true);
22
23 { // File for gnuplot
24     ofstream gnu("plot.gp");
25     for (int i = 0; i < n; i++)
26         gnu << xx[i] << " " << yy[i] << endl;
27 }
28
29 // Calls the gnuplot command, waits 5 seconds and generates a postscript plot (UNIX_
30 // ONLY)
exec("echo 'plot \"plot.gp\" w 1 \n pause 5 \n set term postscript \n set output \
31      \"gnuplot.eps\" \n replot \n quit' | gnuplot");

```

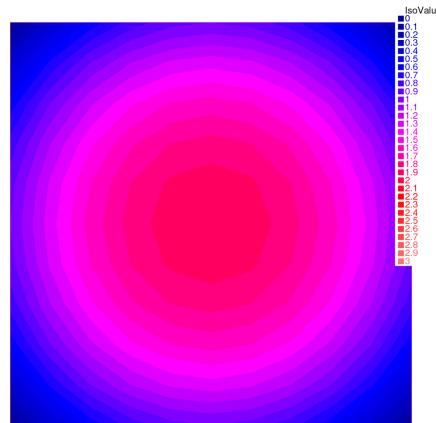
**Fig. 6.17:** Plot

## 6.4.2 HSV

```

1 // From: http://en.wikipedia.org/wiki/HSV_color_space
2 // The HSV (Hue, Saturation, Value) model defines a color space
3 // in terms of three constituent components:
4 // HSV color space as a color wheel
5 // Hue, the color type (such as red, blue, or yellow):
6 // Ranges from 0-360 (but normalized to 0-100% in some applications like here)
7 // Saturation, the "vibrancy" of the color: Ranges from 0-100%
8 // The lower the saturation of a color, the more "grayness" is present
9 // and the more faded the color will appear.
10 // Value, the brightness of the color: Ranges from 0-100%
11
12 mesh Th = square(10, 10, [2*x-1, 2*y-1]);
13
14 fespace Vh(Th, P1);
15 Vh uh=2-x*x-y*y;
16
17 real[int] colorhsv=[ // Color hsv model
18     4./6., 1 , 0.5, // Dark blue
19     4./6., 1 , 1, // Blue
20     5./6., 1 , 1, // Magenta
21     1, 1. , 1, // Red
22     1, 0.5 , 1 // Light red
23 ];
24 real[int] viso(31);
25
26 for (int i = 0; i < viso.n; i++)
27     viso[i] = i*0.1;
28
29 plot(uh, viso=viso(0:viso.n-1), value=true, fill=true, wait=true, hsv=colorhsv);

```



**Fig. 6.18:** Result

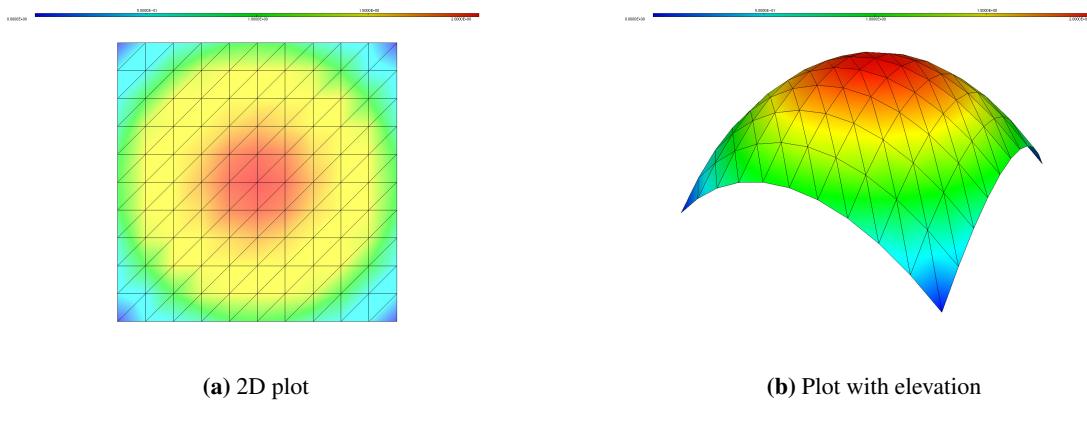
## 6.4.3 Medit

```

1 load "medit"
2
3 mesh Th = square(10, 10, [2*x-1, 2*y-1]);

```

(continues on next page)

**Fig. 6.19:** Medit

(continued from previous page)

```

4
5 fespace Vh(Th, P1);
6 Vh u=2-x*x-y*y;
7
8 medit("u", Th, u);
9
10 // Old way
11 savemesh(Th, "u", [x, y, u*.5]); // Saves u.points and u.faces file
12 // build a u.bb file for medit
13 {
14     ofstream file("u.bb");
15     file << "2 1 1 " << u[].n << " 2 \n";
16     for (int j = 0; j < u[].n; j++)
17         file << u[] [j] << endl;
18 }
19 // Calls medit command
20 exec("ffmedit u");
21 // Cleans files on unix-like OS
22 exec("rm u.bb u.faces u.points");

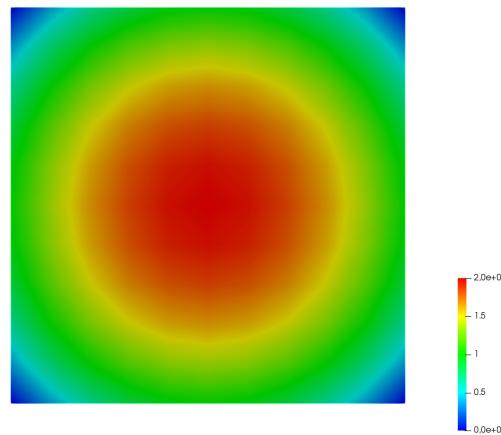
```

#### 6.4.4 Paraview

```

1 load "iovtk"
2
3 mesh Th = square(10, 10, [2*x-1, 2*y-1]);
4
5 fespace Vh(Th, P1);
6 Vh u=2-x*x-y*y;
7
8 int[int] Order = [1];
9 string DataName = "u";
10 savevtk("u.vtu", Th, u, dataname=DataName, order=Order);

```

**Fig. 6.20:** Result

## 6.5 Algorithms & Optimizations

### 6.5.1 Algorithms

```

1 // Parameters
2 int nerr = 0;
3 int debugJ = 0;
4 int debugdJ = 0;
5 real umax = 0;
6
7 // Algorithms tests
8 {
9     func bool stop (int iter, real[int] u, real[int] g){
10         cout << " stop = " << iter << " " << u.linfy << " " << g.linfy << endl;
11         return g.linfy < 1e-5 || iter > 15;
12     }
13     // minimization of J(u) = 1./2 * sum (i+1) u_i^2 - b_i
14     real[int] b(10), u(10);
15
16     //J
17     func real J (real[int] & u) {
18         real s = 0;
19         for (int i = 0; i < u.n; i++)
20             s += (i+1)*u[i]*u[i]*0.5 - b[i]*u[i];
21         if (debugJ)
22             cout << "J = " << s << ", u = " << u[0] << " " << u[1] << endl;
23         return s;
24     }
25
26     //the gradiant of J (this is a affine version (the RHS is in)
27     func real[int] DJ (real[int] &u) {
28         for (int i = 0; i < u.n; i++)
29             u[i] = (i+1)*u[i];
30         if (debugdJ)
31             cout << "dJ: u = " << u[0] << " " << u[1] << " " << u[2] << endl;
32         u -= b;
33         if (debugdJ)

```

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```

34     cout << "dJ-b: u = " << u[0] << " " << u[1] << " " << u[2] << endl;
35     return u; //return of global variable ok
36 }
37
38 //the gradiant of the bilinear part of J (the RHS is remove)
39 func real[int] DJ0 (real[int] &u){
40     for (int i = 0 ; i < u.n; i++)
41         u[i] = (i+1)*u[i];
42     if(debugdJ)
43         cout << "dJ0: u =" << u[0] << " " << u[1] << " " << u[2] << endl;
44     return u; //return of global variable ok
45 }
46
47 //erro calculation
48 func real error (real[int] & u, real[int] & b){
49     real s = 0;
50     for (int i = 0; i < u.n; i++)
51         s += abs((i+1)*u[i] - b[i]);
52     return s;
53 }
54
55 func real[int] matId (real[int] &u){ return u; }
56
57 int verb=5; //verbosity
58 b = 1.; //set right hand side
59 u = 0.; //set initial gest
60
61 LinearCG(DJ, u, eps=1.e-6, nbiter=20, precon=matId, verbosity=verb);
62 cout << "LinearGC (Affine) : J(u) = " << J(u) << ", err = " << error(u, b) <<
63 endl;
64 nerr += !(error(u,b) < 1e-5);
65 if(nerr) cout << "sol: u = " << u[0] << " " << u[1] << " " << u[2] << endl;
66
67 b = 1;
68 u = 0;
69 LinearCG(DJ, u, eps=1.e-15, nbiter=20, precon=matId, verbosity=verb, stop=stop);
70 cout << "LinearGC (Affine with stop) : J(u) = " << J(u) << ", err = " << error(u,
71 b) << endl;
72 nerr += !(error(u,b) < 1e-5);
73 if(nerr) cout << "sol: u = " << u[0] << " " << u[1] << " " << u[2] << endl;
74
75 b = 1;
76 u = 0;
77 LinearCG(DJ0, u, b, eps=1.e-6, nbiter=20, precon=matId, verbosity=verb);
78 cout << "LinearGC (Linear) : J(u) = " << J(u) << ", err = " << error(u, b) <<
79 endl;
80 nerr += !(error(u,b) < 1e-5);
81 if(nerr) cout << "sol: u = " << u[0] << " " << u[1] << " " << u[2] << endl;
82
83 b = 1;
84 u = 0;
85 AffineGMRES(DJ, u, eps=1.e-6, nbiter=20, precon=matId, verbosity=verb);
86 cout << "AffineGMRES (Affine) : J(u) = " << J(u) << ", err = " << error(u, b) <<
87 endl;
88 nerr += !(error(u,b) < 1e-5);
89 if(nerr) cout << "sol: u = " << u[0] << " " << u[1] << " " << u[2] << endl;

```

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```

87
88     b=1;
89     u=0;
90     LinearGMRES(DJ0, u, b, eps=1.e-6, nbiter=20, precon=matId, verbosity=verb);
91     cout << "LinearGMRES (Linear) : J(u) = " << J(u) << ", err = " << error(u, b) << endl;
92     nerr += !(error(u,b) < 1e-5);
93     if(nerr) cout << "sol: u = " << u[0] << " " << u[1] << " " << u[2] << endl;
94
95
96     b=1;
97     u=0;
98     NLCG(DJ, u, eps=1.e-6, nbiter=20, precon=matId, verbosity=verb);
99     cout << "NLCG: J(u) = " << J(u) << ", err = " << error(u, b) << endl;
100    nerr += !(error(u,b) < 1e-5);
101    if(nerr) cout << "sol: u = " << u[0] << " " << u[1] << " " << u[2] << endl;
102
103
104 //warning: BFGS use a full matrix of size nxn (where n=u.n)
105 b=1;
106 u=2;
107 BFGS(J, DJ, u, eps=1.e-6, nbiter=20, nbiterline=20);
108 cout << "BFGS: J(u) = " << J(u) << ", err = " << error(u, b) << endl;
109 assert(error(u,b) < 1e-5);
110 if(nerr) cout << "sol: u = " << u[0] << " " << u[1] << " " << u[2] << endl;
111
112 assert(nerr==0);
113 }
114
115 { // A real non linear test
116     // Parameters
117     real a = 0.001;
118     real eps = 1e-6;
119     //f(u) = a*u + u-ln(1+u), f'(u) = a+ u/(1+u), f''(u) = 1/(1+u)^2
120     func real f(real u) { return u*a+u-log(1+u); }
121     func real df(real u) { return a+u/(1+u); }
122     func real ddf(real u) { return 1/((1+u)*(1+u)); }
123
124     // Mesh
125     mesh Th = square(20, 20);
126
127     // Fespace
128     fespace Vh(Th, P1);
129     Vh b = 1;
130     Vh u = 0;
131
132     fespace Ph(Th, P0);
133     Ph alpha; //store df(/nabla u)^2
134
135     // The functionnal J
136     //J(u) = 1/2 int_Omega f(/nabla u)^2 - int_Omega u b
137     func real J (real[int] & u) {
138         Vh w;
139         w[] = u;
140         real r = int2d(Th) (0.5*f(dx(w)*dx(w) + dy(w)*dy(w)) - b*w);
141         cout << "J(u) = " << r << " " << u.min << " " << u.max << endl;
142         return r;

```

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```

143 }
144
145 // The gradient of J
146 func real[int] dJ (real[int] & u) {
147     Vh w;
148     w[] = u;
149     alpha = df(dx(w)*dx(w) + dy(w)*dy(w));
150     varf au (uh, vh)
151         = int2d(Th) (
152             alpha*(dx(w)*dx(vh) + dy(w)*dy(vh))
153             - b*vh
154         )
155         + on(1, 2, 3, 4, uh=0)
156     ;
157
158     u = au(0, Vh);
159     return u; //warning: no return of local array
160 }
161
162 // Problem
163 alpha = df(dx(u)*dx(u) + dy(u)*dy(u));
164 varf alap (uh, vh)
165     = int2d(Th) (
166         alpha*(dx(uh)*dx(vh) + dy(uh)*dy(vh))
167     )
168     + on(1, 2, 3, 4, uh=0)
169 ;
170
171 varf amass(uh, vh)
172     = int2d(Th) (
173         uh*vh
174     )
175     + on(1, 2, 3, 4, uh=0)
176 ;
177
178 matrix Amass = amass(Vh, Vh, solver=CG);
179 matrix Alap= alap(Vh, Vh, solver=Cholesky, factorize=1);
180
181 // Preconditionner
182 func real[int] C(real[int] & u) {
183     real[int] w = u;
184     u = Alap^-1*w;
185     return u; //warning: no return of local array variable
186 }
187
188 // Solve
189 int conv=0;
190 for(int i = 0; i < 20; i++) {
191     conv = NLCG(dJ, u[], nbiter=10, precon=C, veps=eps, verbosity=5);
192     if (conv) break;
193
194     alpha = df(dx(u)*dx(u) + dy(u)*dy(u));
195     Alap = alap(Vh, Vh, solver=Cholesky, factorize=1);
196     cout << "Restart with new preconditionner " << conv << ", eps =" << eps <<
197     endl;
198 }
```

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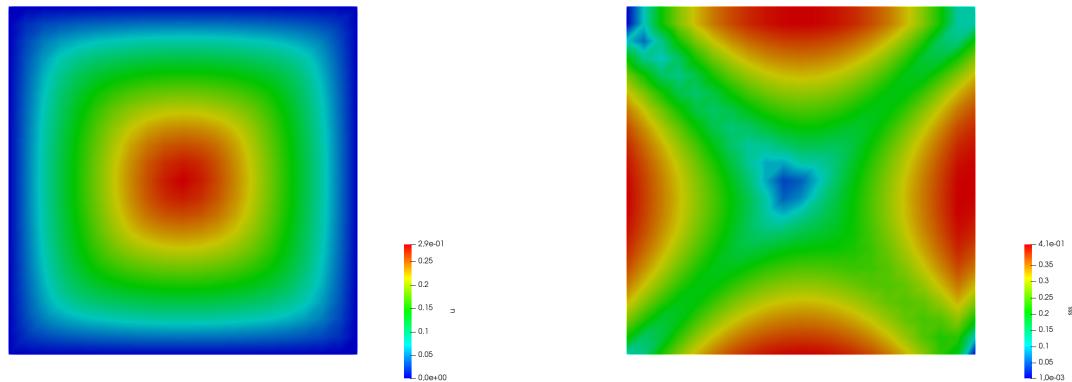
(a) Result  $u$ (b)  $df(dx(u) * dx(u) + dy(u) * dy(u))$ 

Fig. 6.21: Algorithms

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```

199 // Plot
200 plot (u, wait=true, cmm="solution with NLCG");
201 umax = u[].max;
202
203 Vh sss= df(dx(u)*dx(u) + dy(u)*dy(u));
204 plot (sss, fill=true, value=true);
205 }
206
207 assert (nerr==0);

```

## 6.5.2 CMAES variational inequality

```

1 load "ff-cmaes"
2
3 // Parameters
4 int NN = 7;
5 func f1 = 1.;
6 func f2 = -1.;
7 func g1 = 0.;
8 func g2 = 0.1;
9 int iter = 0;
10 int nadapt = 1;
11 real starttol = 1e-10;
12 real bctol = 6.e-12;
13 real pena = 1000.;

14
15 // Mesh
16 mesh Th = square(NN, NN);
17
18 // Fespace
19 fespace Vh(Th, P1);
20 Vh ou1, ou2;
21
22 // Mesh adaptation loops
23 for (int al = 0; al < nadapt; ++al) {

```

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```

24 // Problem
25 varf BVF (v, w)
26   = int2d(Th) (
27     0.5*dx(v)*dx(w)
28     + 0.5*dy(v)*dy(w)
29   )
30 ;
31 varf LVF1 (v, w) = int2d(Th) (f1*w);
32 varf LVF2 (v, w) = int2d(Th) (f2*w);
33
34 matrix A = BVF(Vh, Vh);
35 real[int] b1 = LVF1(0, Vh);
36 real[int] b2 = LVF2(0, Vh);
37
38 varf Vbord (v, w) = on(1, 2, 3, 4, v=1);
39
40 Vh In, Bord;
41 Bord[] = Vbord(0, Vh, tgv=1);
42 In[] = Bord[] ? 0:1;
43 Vh gh1 = Bord*g1;
44 Vh gh2 = Bord*g2;
45
46 // Function which creates a vector of the search space type from
47 // two finite element functions
48 func int FEFToSSP (real[int] &fef1, real[int] &fef2, real[int] &ssp){
49   int kX = 0;
50   for (int i = 0; i < Vh.ndof; ++i){
51     if (In[i]){
52       ssp[kX] = fef1[i];
53       ssp[kX+In.sum] = fef2[i];
54       ++kX;
55     }
56   }
57   return 1;
58 }
59
60 // Splits a vector from the search space and fills
61 // two finite element functions with it
62 func int SSPToFEF (real[int] &fef1, real[int] &fef2, real[int] &ssp){
63   int kX = 0;
64   for (int i = 0; i < Vh.ndof; ++i){
65     if (In[i]){
66       fef1[i] = ssp[kX];
67       fef2[i] = ssp[kX+In.sum];
68       ++kX;
69     }
70     else{
71       fef1[i] = gh1[i];
72       fef2[i] = gh2[i];
73     }
74   }
75   return 1;
76 }
77
78 func real IneqC (real[int] &X){
79   real[int] constraints(In.sum);
80   for (int i = 0; i < In.sum; ++i){

```

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```

81     constraints[i] = X[i] - X[i+In[].sum];
82     constraints[i] = constraints[i] <= 0 ? 0. : constraints[i];
83 }
84 return constraints.12;
85 }

86
87 func real J (real[int] &X) {
88     Vh u1, u2;
89     SSPToFEF(u1[], u2[], X);
90     iter++;
91     real[int] Au1 = A*u1[], Au2 = A*u2[];
92     Au1 -= b1;
93     Au2 -= b2;
94     real val = u1[]'*Au1 + u2[]'*Au2;
95     val += pena * IneqC(X);
96     if (iter%200 == 199)
97         plot(u1, u2, nbiso=30, fill=1, dim=3, cmm="adapt level "+al+" - iteration
98           "+iter+" - J = "+val, value=1);
99     return val ;
100 }
101
102 // Solve
103 real[int] start(2*In[].sum);
104
105 if (al == 0){
106     start(0:In[].sum-1) = 0.;
107     start(In[].sum:2*In[].sum-1) = 0.1;
108 }
109 else
110     FEFToSSP(ou1[], ou2[], start);
111
112 real mini = cmaes(J, start, stopMaxFunEval=10000*(al+1), stopTolX=1.e-3/
113   (10*(al+1)), initialStdDev=(0.025/(pow(100.,al))));
114 Vh best1, best2;
115 SSPToFEF(best1[], best2[], start);
116
117 // Mesh adaptation
118 Th = adaptmesh(Th, best1, best2);
119 ou1 = best1;
ou2 = best2;
}

```

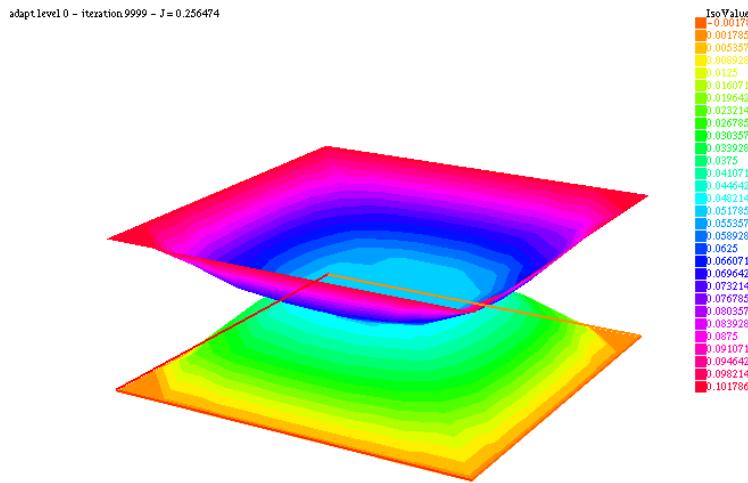
### 6.5.3 IPOPT minimal surface & volume

```

1 load "msh3";
2 load "medit";
3 load "ff-Ipopt";
4
5 // Parameters
6 int nadapt = 3;
7 real alpha = 0.9;
8 int np = 30;
9 real regtest;
10 int shapeswitch = 1;
11 real sigma = 2*pi/40.;


```

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**Fig. 6.22:** Results

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```

12 real threshold = 0.1;
13 real e = 0.1;
14 real r0 = 0.25;
15 real rr = 2-r0;
16 real E = 1./(e*e);
17 real RR = 1./(rr*rr);
18
19 // Mesh
20 mesh Th = square(2*np, np, [2*pi*x, pi*y]);
21
22 // Fespace
23 fespace Vh(Th, P1, periodic=[[2, y], [4, y]]);
24 //Initial shape definition
25 //outside of the mesh adaptation loop to initialize with the previous optimial shape
//found on further iterations
26 Vh startshape = 5;
27 Vh uz = 1., lz = 1.;
28
29 // Mesh adaptation loop
30 real[int] lm = [1];
31 for(int kkk = 0; kkk < nadapt; ++kkk){
32     int iter=0;
33     func sin2 = square(sin(y));
34
35     // A function which transform Th in 3d mesh (r=rho)
36     //a point (theta,phi) of Th becomes ( r(theta,phi)*cos(theta)*sin(phi) , r(theta,
//phi)*sin(theta)*sin(phi) , r(theta,phi)*cos(phi) )
37     //then displays the resulting mesh with medit
38     func int Plot3D (real[int] &rho, string cmm, bool ffplot){
39         Vh rhoo;
40         rhoo[] = rho;
41         //mesh sTh = square(np, np/2, [2*pi*x, pi*y]);
42         //fespace svh(sTh, P1);

```

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```

43 //Vh rho;
44 try{
45     mesh3 Sphere = movemesh23(Th, transfo=[rho(x,y)*cos(x)*sin(y), rho(x,
46     -y)*sin(x)*sin(y), rho(x,y)*cos(y)]);
47     if(ffplot)
48         plot(Sphere);
49     else
50         medit(cmm, Sphere);
51 }
52 catch(...){
53     cout << "PLOT ERROR" << endl;
54 }
55 return 1;
56 }

// Surface computation
//Maybe is it possible to use movemesh23 to have the surface function less
57 //complicated
//However, it would not simplify the gradient and the hessian
58 func real Area (real[int] &X) {
59     Vh rho;
60     rho[] = X;
61     Vh rho2 = square(rho);
62     Vh rho4 = square(rho2);
63     real res = int2d(Th)(sqrt(rho4*sin2 + rho2*square(dx(rho)) +
64     rho2*sin2*square(dy(rho))))+
65     ++iter;
66     if(1)
67         plot(rho, value=true, fill=true, cmm="rho(theta,phi) on [0,2pi]x[0,pi] -"
68     S=+res, dim=3);
69     else
70         Plot3D(rho[], "shape_evolution", 1);
71     return res;
72 }

73 func real[int] GradArea (real[int] &X) {
74     Vh rho, rho2;
75     rho[] = X;
76     rho2[] = square(X);
77     Vh sqrtPsi, alpha;
78     {
79         Vh dxrho2 = dx(rho)*dx(rho), dyrho2 = dy(rho)*dy(rho);
80         sqrtPsi = sqrt(rho2*rho2*sin2 + rho2*dxrho2 + rho2*dyrho2*sin2);
81         alpha = 2.*rho2*rho*sin2 + rho*dxrho2 + rho*dyrho2*sin2;
82     }
83     varf dArea (u, v)
84     = int2d(Th)(
85         1./sqrtPsi * (alpha*v + rho2*dx(rho)*dx(v) + rho2*dy(rho)*sin2*dy(v))
86     )
87     ;
88

89     real[int] grad = dArea(0, Vh);
90     return grad;
91 }

92 matrix hessianA;
93 func matrix HessianArea (real[int] &X) {

```

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```

96     Vh rho, rho2;
97     rho[] = X;
98     rho2 = square(rho);
99     Vh sqrtPsi, sqrtPsi3, C00, C01, C02, C11, C12, C22, A;
100    {
101        Vh C0, C1, C2;
102        Vh dxrho2 = dx(rho)*dx(rho), dyrho2 = dy(rho)*dy(rho);
103        sqrtPsi = sqrt( rho2*rho2*sin2 + rho2*dxrho2 + rho2*dyrho2*sin2 );
104        sqrtPsi3 = (rho2*rho2*sin2 + rho2*dxrho2 + rho2*dyrho2*sin2)*sqrtPsi;
105        C0 = 2*rho2*rho*sin2 + rho*dxrho2 + rho*dyrho2*sin2;
106        C1 = rho2*dx(rho);
107        C2 = rho2*sin2*dy(rho);
108        C00 = square(C0);
109        C01 = C0*C1;
110        C02 = C0*C2;
111        C11 = square(C1);
112        C12 = C1*C2;
113        C22 = square(C2);
114        A = 6.*rho2*sin2 + dxrho2 + dyrho2*sin2;
115    }
116    varf d2Area (w, v)
117    =int2d(Th) (
118        1./sqrtPsi * (
119            A*w*v
120            + 2*rho*dx(rho)*dx(w)*v
121            + 2*rho*dx(rho)*w*dx(v)
122            + 2*rho*dy(rho)*sin2*dy(w)*v
123            + 2*rho*dy(rho)*sin2*w*dy(v)
124            + rho2*dx(w)*dx(v)
125            + rho2*sin2*dy(w)*dy(v)
126        )
127        + 1./sqrtPsi3 * (
128            C00*w*v
129            + C01*dx(w)*v
130            + C01*w*dx(v)
131            + C02*dy(w)*v
132            + C02*w*dy(v)
133            + C11*dx(w)*dx(v)
134            + C12*dx(w)*dy(v)
135            + C12*dy(w)*dx(v)
136            + C22*dy(w)*dy(v)
137        )
138    )
139    ;
140    hessianA = d2Area(Vh, Vh);
141    return hessianA;
142}
143
144 // Volume computation
145 func real Volume (real[int] &X) {
146     Vh rho;
147     rho[] = X;
148     Vh rho3 = rho*rho*rho;
149     real res = 1./3.*int2d(Th) (rho3*sin(y));
150     return res;
151 }
152

```

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```

153 func real[int] GradVolume (real[int] &X) {
154     Vh rho;
155     rho[] = X;
156     varf dVolume(u, v) = int2d(Th) (rho*rho*sin(y)*v);
157     real[int] grad = dVolume(0, Vh);
158     return grad;
159 }
160 matrix hessianV;
161 func matrix HessianVolume(real[int] &X) {
162     Vh rho;
163     rho[] = X;
164     varf d2Volume(w, v) = int2d(Th) (2*rho*sin(y)*v*w);
165     hessianV = d2Volume(Vh, Vh);
166     return hessianV;
167 }
168
169 //if we want to use the volume as a constraint function
170 //we must wrap it in some freefem functions returning the appropriate type
171 //The lagrangian hessian also have to be wrapped since the Volume is not linear
172 ↪with
173     //respect to rho, it will contribute to the hessian.
174 func real[int] ipVolume (real[int] &X){ real[int] vol = [Volume(X)]; return vol; }
175 matrix mdV;
176 func matrix ipGradVolume (real[int] &X) { real[int,int] dvol(1,Vh.ndof); dvol(0,
177 ↪:) = GradVolume(X); mdV = dvol; return mdV; }
178 matrix HLagrangian;
179 func matrix ipHessianLag (real[int] &X, real objfact, real[int] &lambda) {
180     HLagrangian = objfact*HessianArea(X) + lambda[0]*HessianVolume(X);
181     return HLagrangian;
182 }
183
184 //building struct for GradVolume
185 int[int] gvi(Vh.ndof), gvj=0:Vh.ndof-1;
186 gvi = 0;
187
188 Vh rc = startshape; //the starting value
189 Vh ub = 1.e19; //bounds definition
190 Vh lb = 0;
191
192 func real Gaussian (real X, real Y, real theta, real phi){
193     real deltax2 = square((X-theta)*sin(Y)), deltay2 = square(Y-phi);
194     return exp(-0.5 * (deltax2 + deltay2) / (sigma*sigma));
195 }
196
197 func disc1 = sqrt(1./(RR+RR)*cos(y)*cos(y))*(1+0.1*cos(7*x));
198 func disc2 = sqrt(1./(RR+RR)*cos(x)*cos(x)*sin2);
199
200 if(1){
201     lb = r0;
202     for (int q = 0; q < 5; ++q){
203         func f = rr*Gaussian(x, y, 2*q*pi/5., pi/3.);
204         func g = rr*Gaussian(x, y, 2*q*pi/5.+pi/5., 2.*pi/3.);
205         lb = max(max(lb, f), g);
206     }
207     lb = max(lb, rr*Gaussian(x, y, 2*pi, pi/3));
208 }
209 lb = max(lb, max(disc1, disc2));

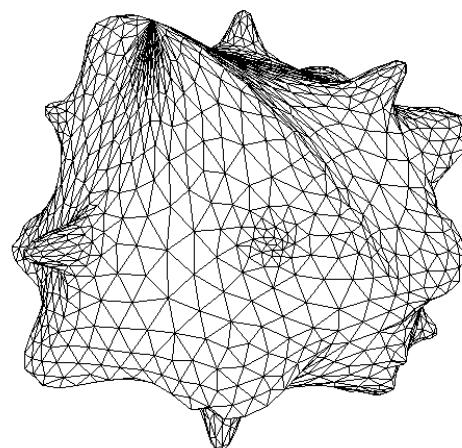
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```

208 real Vobj = Volume(lb[]);
209 real Vnvc = 4./3.*pi*pow(lb[].linfty,3);
210
211 if(1)
212     Plot3D(lb[], "object_inside", 1);
213 real[int] clb = 0., cub = [(1-alpha)*Vobj + alpha*Vnvc];
214
215 // Call IPOPT
216 int res = IPOPT(Area, GradArea, ipHessianLag, ipVolume, ipGradVolume,
217                 rc[], ub=ub[], lb=lb[], clb=clb, cub=cub, checkindex=1, maxiter=kkk
218 ↪<nadapt-1 ? 40:150,
219                 warmstart=kkk, lm=lm, uz=uz[], lz=lz[], tol=0.00001, structjacc=[gvi,
220 ↪gvj]);
221     cout << "IPOPT: res =" << res << endl ;
222
223 // Plot
224 Plot3D(rc[], "Shape_at_"+kkk, 1);
225 Plot3D(GradArea(rc[]), "ShapeGradient", 1);
226
227 // Mesh adaptation
228 if (kkk < nadapt-1){
229     Th = adaptmesh(Th, rc*cos(x)*sin(y), rc*sin(x)*sin(y), rc*cos(y),
230                   nbvx=50000, periodic=[[2, y], [4, y]]);
231     plot(Th, wait=true);
232     startshape = rc;
233     uz = uz;
234     lz = lz;
235 }
236
237 regtest = rc[]'*rc[];
238 }
```

**Fig. 6.23:** Mesh

## 6.5.4 CMAES MPI variational inequality

Command:

```
1 ff-mpirun -np 4 CMAESMPIVariationalInequality.edp -glut ff	glut
```

```

1 load "mpi-cmaes"
2
3 // Parameters
4 int NN = 10;
5 func f1 = 1.;
6 func f2 = -1.;
7 func g1 = 0.;
8 func g2 = 0.1;
9 int iter = 0;
10 int nadapt = 1;
11 real starttol = 1e-10;
12 real bctol = 6.e-12;
13 real pena = 1000;
14
15 // Mesh
16 mesh Th = square(NN, NN);
17
18 // Fespace
19 fespace Vh(Th, P1);
20 Vh ou1, ou2;
21
22 // Mesh adaptation loop
23 for (int al = 0; al < nadapt; ++al) {
24     // Problem
25     varf BVF (v, w)
26         = int2d(Th) (
27             0.5*dx(v)*dx(w)
28             + 0.5*dy(v)*dy(w)
29         )
30         ;
31     varf LVF1 (v, w) = int2d(Th) (f1*w);
32     varf LVF2 (v, w) = int2d(Th) (f2*w);
33     matrix A = BVF(Vh, Vh);
34     real[int] b1 = LVF1(0, Vh);
35     real[int] b2 = LVF2(0, Vh);
36
37     varf Vbord (v, w) = on(1, 2, 3, 4, v=1);
38
39     Vh In, Bord;
40     Bord[] = Vbord(0, Vh, tgv=1);
41     In[] = Bord[] ? 0:1;
42     Vh gh1 = Bord*g1, gh2 = Bord*g2;
43
44 //Function which create a vector of the search space type from
45 //two finite element functions
46 func int FEFToSSP (real[int] &fef1, real[int] &fef2, real[int] &ssp) {
47     int kX = 0;
48     for (int i = 0; i < Vh.ndof; ++i) {
49         if (In[i]) {
50             ssp[kX] = fef1[i];
51             ssp[kX+In.sum] = fef2[i];
52         }
53     }
54 }
```

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```

52         ++kX;
53     }
54 }
55 return 1;
56 }

57
58 //Function splitting a vector from the search space and fills
59 //two finite element functions with it
60 func int SSPToFEF (real[int] &fef1, real[int] &fef2, real[int] &ssp) {
61     int kX = 0;
62     for (int i = 0; i < Vh.ndof; ++i){
63         if (In[] [i]){
64             fef1[i] = ssp[kX];
65             fef2[i] = ssp[kX+In[] .sum];
66             ++kX;
67         }
68         else{
69             fef1[i] = gh1[] [i];
70             fef2[i] = gh2[] [i];
71         }
72     }
73     return 1;
74 }

75
76 func real IneqC (real[int] &X) {
77     real[int] constraints(In[] .sum);
78     for (int i = 0; i < In[] .sum; ++i){
79         constraints[i] = X[i] - X[i+In[] .sum];
80         constraints[i] = constraints[i] <= 0 ? 0. : constraints[i];
81     }
82     return constraints.12;
83 }

84
85 func real J (real[int] &X) {
86     Vh u1, u2;
87     SSPToFEF(u1[], u2[], X);
88     iter++;
89     real[int] Au1 = A*u1[], Au2 = A*u2[];
90     Au1 -= b1;
91     Au2 -= b2;
92     real val = u1[]'*Au1 + u2[]'*Au2;
93     val += pena * IneqC(X);
94     plot(u1, u2, nbiso=30, fill=1, dim=3, cmm="adapt level "+al+" - iteration
95     ↪"+iter+" - J = "+val, value=1);
96     return val ;
97 }

98 // Solve
99 real[int] start(2*In[] .sum);

100
101 if (al==0){
102     start(0:In[] .sum-1) = 0.;
103     start(In[] .sum:2*In[] .sum-1) = 0.1;
104 }
105 else
106     FEFToSSP(ou1[], ou2[], start);
107

```

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```

108   real mini = cmaesMPI(J, start, stopMaxFunEval=10000*(al+1), stopTolX=1.e-4/
109   ↪(10*(al+1)), initialStdDev=(0.025/(pow(100.,al))));  

110   Vh best1, best2;  

111   SSPToFEF(best1[], best2[], start);  

112  

113   // Mesh adaptation  

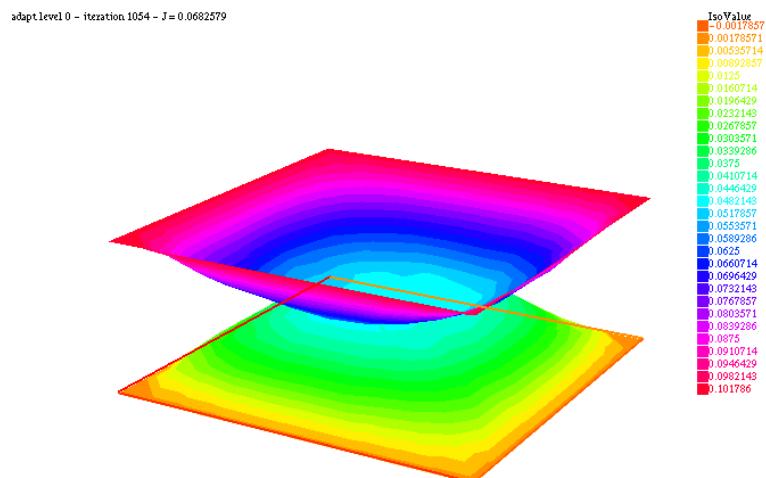
114   Th = adaptmesh(Th, best1, best2);  

115   ou1 = best1;  

116   ou2 = best2;  

}

```

**Fig. 6.24:** Result

## 6.6 Parallelization

### 6.6.1 MPI-GMRES 2D

To launch this script, use for example:

```
1 ff-mpirun -np 12 MPIGMRES2D.edp -d 1 -k 1 -gmres 2 -n 50
```

```

1 //usage :  

2 //ff-mpirun [mpi parameter] MPIGMRES2d.edp [-glut ff	glut] [-n N] [-k K] [-d D] [-ns] ↪  

3 //[-gmres [0/1]]  

4 //arguments:  

5 //glut ff	glut : to see graphically the process  

6 // -n N: set the mesh cube split NxNxN  

7 // -d D: set debug flag D must be one for mpiplot  

8 // -k K: to refined by K all element  

9 // -ns: remove script dump  

10 // -gmres  

11 // 0: use iterative schwarz algo.

```

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```

11 //1: Algo GMRES on residu of schwarz algo
12 //2: DDM GMRES
13 //3: DDM GMRES with coarse grid preconditionner (Good one)
14
15 load "MPICG"
16 load "medit"
17 load "metis"
18 include "getARGV.idp"
19 include "MPIplot.idp"
20 include "MPIGMRESmacro.idp"
21
22 searchMethod = 0; //more safe search algo (warning can be very expensive in case of ↴
23             ↴ lot of outside point)
24 assert(version >= 3.11); //need at least v3.11
25 real[int] ttt(10);
26 int ittt=0;
27 macro sett {ttt[ittt++] = mpiWtime();} //
28
29 // Arguments
30 verbosity = getARGV("-vv", 0);
31 int vdebug = getARGV("-d", 1);
32 int ksplit = getARGV("-k", 3);
33 int nloc = getARGV("-n", 10);
34 string sff = getARGV("-p", "");
35 int gmres = getARGV("-gmres", 2);
36 bool dplot = getARGV("-dp", 0);
37 int nC = getARGV("-N", max(nloc/10, 4));
38
39 if (mpirank==0 && verbosity) {
40     cout << "ARGV: ";
41     for (int i = 0; i < ARGV.n; ++i)
42         cout << ARGV[i] << " ";
43     cout << endl;
44 }
45 if(mpirank==0 && verbosity)
46     cout << " vdebug: " << vdebug << ", kspilt "<< ksplit << ", nloc "<< nloc << ", sff "<<
47             ↴ sff << "." << endl;
48
49 // Parameters
50 int withplot = 0;
51 bool withmetis = 1;
52 bool RAS = 1;
53 string sPk = "P2-2gd";
54 func Pk = P2;
55 int sizeoverlaps = 1; //size of overlap
56 int[int] l111 = [1, 1, 1, 1]; //mesh labels
57
58 // MPI function
59 func bool plotMPIall(mesh &Th, real[int] &u, string cm) {
60     if(vdebug)
61         PLOTMPIALL(mesh, Pk, Th, u, {cmm=cm, nbiso=20, fill=1, dim=3, value=1});
62     return 1;
63 }
64
65 // MPI
66 mpiComm comm(MPICommWorld, 0, 0); //trick : make a no split mpiWorld

```

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```

66
67 int npart = mpiSize(comm); //total number of partition
68 int ipart = mpiRank(comm); //current partition number
69
70 int njpart = 0; //Number of part with intersection (a jpart) with ipart without ipart
71 int[int] jpart(npart); //list of jpart
72 if(ipart==0)
73     cout << " Final N = " << ksplit*nloc << ", nloc = " << nloc << ", split = " <<
74     ↵ksplit << endl;
75 settt
76
77 // Mesh
78 mesh Thg = square(nloc, nloc, label=1111);
79 mesh ThC = square(nC, nC, label=1111); // Coarse mesh
80
81 mesh Thi, Thin; //with overlap, without olverlap
82
83 // Fespace
84 fespace Phg(Thg, P0);
85 Phg part;
86
87 fespace Vhg(Thg, P1);
88 Vhg unssd; //boolean function: 1 in the subdomain, 0 elsewhere
89
90 fespace VhC(ThC, P1); // of the coarse problem
91
92 // Partitioning
93 {
94     int[int] nupart(Thg.nt);
95     nupart = 0;
96     if (npart > 1 && ipart == 0)
97         metisDual(nupart, Thg, npart);
98
99     broadcast(processor(0, comm), nupart);
100    for(int i = 0; i < nupart.n; ++i)
101        part[i][i] = nupart[i];
102 }
103
104 if (withplot > 1)
105     plot(part, fill=1, cmm="dual", wait=1);
106
107 // Overlapping partition
108 Phg suppi = abs(part-ipart) < 0.1;
109
110 Thin = trunc(Thg, suppi>0, label=10); // non-overlapping mesh, interfaces have label
111     ↵10
112 int nnn = sizeoverlaps*2;// to be sure
113 AddLayers(Thg, suppi[], nnn, unssd[]); //see above! suppi and unssd are modified
114 unssd[] *= nnn; //to put value nnn a 0
115 real nnn0 = nnn - sizeoverlaps + 0.001;
116 Thi = trunc(Thg, unssd>nnn0, label=10); //overlapping mesh, interfaces have label 10
117
118 settt
119
120 // Fespace
121 fespace Vhi(Thi,P1);
122 int npij = npart;

```

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```

176         if(dplot)
177             plot(pij[njpart], wait=1, cmm=" j = "+ i + " " + njpart);
178             sumphi[] += pij[njpart][];
179             if(dplot)
180                 plot(sumphi, wait=1, cmm=" sum j = "+ i + " " + njpart);
181                 jpart[njpart++] = i;
182             }
183         }
184     }
185
186     if(dplot)
187         plot(sumphi, wait=1, dim=3, cmm="sum ", fill=1);
188     pii[] = pii[] ./ sumphi[];
189     for (int j = 0; j < njpart; ++j)
190         pij[j][] = pij[j][] ./ sumphi[];
191     jpart.resize(njpart);
192     for (int j = 0; j < njpart; ++j)
193         assert(pij[j][]{max} <= 1);
194     {
195         cout << ipart << " number of jpart " << njpart << " : ";
196         for (int j = 0; j < njpart; ++j)
197             cout << jpart[j] << " ";
198         cout << endl;
199     }
200     sumphi[] = pii[];
201     for (int j = 0; j < njpart; ++j)
202         sumphi[] += pij[j]++;
203     if(vdebug > 2)
204         cout << "sum min " << sumphi[].min << " " << sumphi[].max << endl;
205     assert(sumphi[].min > 1.-1e-6 && sumphi[].max < 1.+1e-6);
206     }
207 } //This is remove here
// end of the construction of the local partition of the unity ...
// on Thi
208 if (ipart == 0)
209     cout << "End build partition" << endl;
210
211 // Computation of number of intersection
212 //here pii and the pij is the local partition of the unit on
213 //Thi (mesh with overlap)
214 if ( dplot){
215     plot(Thi, wait=1);
216     for(int j = 0; j < njpart; ++j)
217         plot(pij[j], cmm=" j="+j, wait=1);
218 }
219
220 //Partition of the unity on Thi
221 //computation of message
222 //all j > we have to receive
223 //data on intersection of the support of pij[0] and pij[j]
224 settt
225
226 if(vdebug)
227     plotMPIall(Thi, pii[], "pi_i");
228
229 mesh[int] aThij(njpart);
230 matrix Pii;
231
232

```

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```

233 matrix[int] sMj(njpart); //M of send to j
234 matrix[int] rMj(njpart); //M to recv from j
235 fespace Whi(Thi, Pk);
236 mesh Thij = Thi;
237 fespace Whij(Thij, Pk);//

238
239 //construction of the mesh intersect i,j part
240 for(int jp = 0; jp < njpart; ++jp)
241     aThij[jp] = trunc(Thi, pij[jp] > 1e-6, label=10); //mesh of the supp of pij
242
243 for(int jp = 0; jp < njpart; ++jp)
244     aThij[jp] = trunc(aThij[jp], 1, split=ksplit);
245
246 Thi = trunc(Thi, 1, split=ksplit);
247
248 settt
249
250 if (ipart == 0)
251     cout << "End build mesh intersection" << endl;
252
253 // Construction of transfert matrix
254 {
255     Whi wpii = pii;
256     Pii = wpii[];
257     for(int jp = 0; jp < njpart; ++jp){
258         int j = jpart[jp];
259         Thij = aThij[jp];
260         matrix I = interpolate(Whij, Whi); //Whji <- Whi
261         sMj[jp] = I*Pii; //Whi -> s Whij
262         rMj[jp] = interpolate(Whij, Whi, t=1); //Whji -> Whi
263         if(vdebug > 10){
264             {Whi uuu=1; Whij vvv=-1; vvv[]+=I*uuu[]; cout << jp << " %% " << vvv[].
265             ~linfty << endl; assert(vvv[].linfty < 1e-6);}
266             {Whi uuu=1; Whij vvv=-1; vvv[]+=rMj[jp]*uuu[]; cout << jp << " ### " <<_
267             ~vvv[].linfty << endl; assert(vvv[].linfty < 1e-6);}
268         }
269     }
270 }
271 if (ipart == 0)
272     cout << "End build transfert matrix" << endl;
273
274 // Allocate array of send and recv data
275 InitU(njpart, Whij, Thij, aThij, Usend) //initU(n, Vh, Th, aTh, U)
276 InitU(njpart, Whij, Thij, aThij, Vrecv)
277 if (ipart == 0)
278     cout << "End init data for send/recv" << endl;
279
280 Whi ui, vi;
281
282 func bool Update(real[int] &ui, real[int] &vi){
283     for(int j = 0; j < njpart; ++j)
284         Usend[j][] = sMj[j]*ui;
285     SendRecvUV(comm, jpart, Usend, Vrecv)
286     vi = Pii*ui;
287     for(int j = 0; j < njpart; ++j)
288         vi += rMj[j]*Vrecv[j][];
289     return true;

```

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```

288 }
289
290 // Definition of the Problem
291 func G = x*0.1;
292 func F = 1.;
293 macro grad(u) [dx(u),dy(u)] //
294 varf vBC (U, V) = on(1, U=G);
295 varf vPb (U, V) = int2d(Thi)(grad(U)'*grad(V)) + int2d(Thi)(F*V) + on(10, U=0) + on(1,
296 ↵ U=G);
297 varf vPbC (U, V) = int2d(ThC)(grad(U)'*grad(V)) + on(1, U=0);
298 varf vPbon (U, V) = on(10, U=1) + on(1, U=1);
299 varf vPbon10only (U, V) = on(10, U=1) + on(1, U=0);
300 //remark the order is important we want 0 part on 10 and 1
301
302 matrix Ai = vPb(Whi, Whi, solver=sparse);
303 matrix AC, Rci, Pci;
304
305 if (mpiRank(comm) == 0)
306     AC = vPbC(VhC, VhC, solver=sparse);
307
308 Pci = interpolate(Whi, VhC);
309 Rci = Pci'*Pii;
310
311 real[int] onG10 = vPbon10only(0, Whi);
312 real[int] onG = vPbon(0, Whi);
313 real[int] Bi=vPb(0, Whi);
314
315 int kiter = -1;
316
317 func bool CoarseSolve(real[int] &V, real[int] &U, mpiComm &comm) {
318     //solving the coarse problem
319     real[int] Uc(Rci.n), Bc(Uc.n);
320     Uc = Rci*U;
321     mpiReduce(Uc, Bc, processor(0, comm), mpiSUM);
322     if (mpiRank(comm) == 0)
323         Uc = AC^-1*Bc;
324     broadcast(processor(0, comm), Uc);
325     V = Pci*Uc;
326 }
327
328 func real[int] DJ (real[int] &U) {
329     ++kiter;
330     real[int] V(U.n);
331     V = Ai*U;
332     V = onG10 ? 0.: V; //remove internal boundary
333     return V;
334 }
335
336 func real[int] PDJ (real[int] &U) {
337     real[int] V(U.n);
338
339     real[int] b = onG10 ? 0. : U;
340     V = Ai^-1*b;
341     Update(V, U);
342     return U;
343 }
```

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```

344 func real[int] PDJC (real[int] &U) {
345     real[int] V(U.n);
346     CoarseSolve(V, U, comm);
347     V = -V; // -C2 * Uo
348     U += Ai*V; // U = (I - A C2) Uo
349     real[int] b = onG10 ? 0. : U;
350     U = Ai^-1*b; // (C1( I - A C2) Uo
351     V = U - V;
352     Update(V, U);
353     return U;
354 }
355
356 func real[int] DJ0(real[int] &U) {
357     ++kiter;
358     real[int] V(U.n);
359     real[int] b = onG .* U;
360     b = onG ? b : Bi ;
361     V = Ai^-1*b;
362     Update(V, U);
363     V -= U;
364     return V;
365 }
366
367 Whi u = 0, v;
368 { //verification
369     Whi u = 1, v;
370     Update(u[], v[]);
371     u[] -= v[];
372     assert(u[].linfty < 1e-6);
373 }
374
375 settt
376 u[] = vBC(0, Whi, tgv=1); //set u with tgv BC value
377
378 real epss = 1e-6;
379 int rgmres = 0;
380 if (gmres == 1) {
381     rgmres = MPIAffineGMRES(DJ0, u[], veps=epss, nbiter=300, comm=comm, dimKrylov=100,
382     ↪ verbosity=ipart ? 0: 50);
383     real[int] b = onG .* u[];
384     b = onG ? b : Bi;
385     v[] = Ai^-1*b;
386     Update(v[], u[]);
387 }
388 else if (gmres == 2)
389     rgmres = MPILinearGMRES(DJ, precon=PDJ, u[], Bi, veps=epss, nbiter=300, comm=comm,
390     ↪ dimKrylov=100, verbosity=ipart ? 0: 50);
391 else if (gmres == 3)
392     rgmres = MPILinearGMRES(DJ, precon=PDJC, u[], Bi, veps=epss, nbiter=300, ↪
393     comm=comm, dimKrylov=100, verbosity=ipart ? 0: 50);
394 else // algo Schwarz for demo
395     for(int iter = 0; iter < 10; ++iter){
396         real[int] b = onG .* u[];
397         b = onG ? b : Bi ;
398         v[] = Ai^-1*b;
399
400         Update(v[], u[]);

```

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```

398     if(vdebug)
399         plotMPIall(Thi, u[], "u-"+iter);
400     v[] -= u[];
401
402     real err = v[].linfty;
403     real umax = u[].max;
404     real[int] aa = [err, umax], bb(2);
405     mpiAllReduce(aa, bb, comm, mpiMAX);
406     real errg = bb[0];
407     real umaxg = bb[1];
408
409     if (ipart == 0)
410         cout << ipart << " err = " << errg << " u. max " << umaxg << endl;
411     if (errg < 1e-5) break;
412 }
413
414 if (vdebug)
415     plotMPIall(Thi, u[], "u-final");
416
417 settt
418
419 real errg = 1, umaxg;
420 {
421     real umax = u[].max, umaxg;
422     real[int] aa = [umax], bb(1);
423     mpiAllReduce(aa, bb, comm, mpiMAX);
424     errg = bb[0];
425     if (ipart == 0)
426         cout << "umax global = " << bb[0] << " Wtime = " << (ttt[ittt-1]-ttt[ittt-2])
427         << " s " << " " << kiter << endl;
428 }
429
430 if (sff != "") {
431     ofstream ff(sff+".txt", append);
432     cout << "++++ ";
433     cout << mpirank << "/" << mpisize << " k=" << ksplit << " n= " << nloc << " " <<
434     sizeoverlaps << " it= " << kiter;
435     for (int i = 1; i < ittt; ++i)
436         cout << " " << ttt[i]-ttt[i-1] << " ";
437     cout << epss << " " << Ai.nbcoef << " " << Ai.n << endl;
438
439 /*
440 1 mpirank
441 2 mpisize
442 3 ksplit
443 4 nloc
444 5 sizeoverlaps
445 6 kiter
446 7 mesh & part build
447 8 build the partion
448 9 build mesh, transfere , and the fine mesh ..
449 10 build the matrix, the trans matrix, factorizatioon
450 11 GMRES
451 */
452
453 ff << mpirank << " " << mpisize << " " << sPk << " ";
454 ff << ksplit << " " << nloc << " " << sizeoverlaps << " " << kiter;

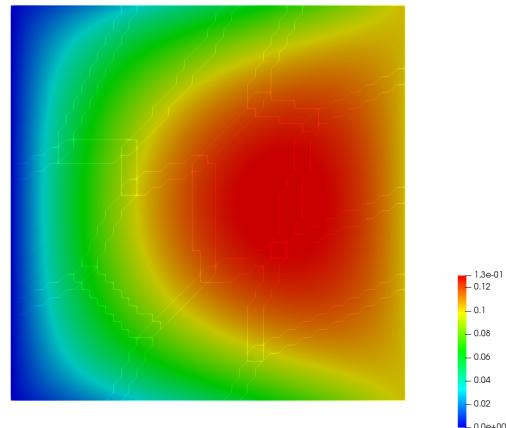
```

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```

453     for (int i = 1; i < ittt; ++i)
454         ff << " " << ttt[i]-ttt[i-1] << " ";
455     ff << epss << " " << Ai.nbcoef << " " << Ai.n << " " << gmres << endl;
456 }
```

**Fig. 6.25:** Results

## 6.6.2 MPI-GMRES 3D

---

**Todo:** todo

---

## 6.6.3 Direct solvers

```

1  load "MUMPS_FreeFem"
2  //default solver: real-> MUMPS, complex -> MUMPS
3  load "real_SuperLU_DIST_FreeFem"
4  default solver: real-> SuperLU_DIST, complex -> MUMPS
5  load "real_pastix_FreeFem"
6  //default solver: real-> pastix, complex -> MUMPS
7
8  // Solving with pastix
9  {
10    matrix A =
11      [[1, 2, 2, 1, 1],
12       [2, 12, 0, 10, 10],
13       [2, 0, 1, 0, 2],
14       [1, 10, 0, 22, 0.],
15       [1, 10, 2, 0., 22]];
16
17    real[int] xx = [1, 32, 45, 7, 2], x(5), b(5), di(5);
18    b = A*xx;
19    cout << "b = " << b << endl;
20    cout << "xx = " << xx << endl;
```

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```

22 set(A, solver=sparse solver, datafilename="ffpastix_iparm_dparm.txt");
23 cout << "solve" << endl;
24 x = A^-1*b;
25 cout << "b = " << b << endl;
26 cout << "x = " << endl;
27 cout << x << endl;
28 di = xx - x;
29 if (mpirank == 0) {
30     cout << "x-xx = " << endl;
31     cout << "Linf = " << di.linfty << ", L2 = " << di.L2 << endl;
32 }
33 }

34

35 // Solving with SuperLU_DIST
36 realdefaulttoSuperLUDist();
37 //default solver: real-> SuperLU_DIST, complex -> MUMPS
38 {
39     matrix A =
40         [[1, 2, 2, 1, 1],
41          [2, 12, 0, 10, 10],
42          [2, 0, 1, 0, 2],
43          [1, 10, 0, 22, 0.],
44          [1, 10, 2, 0., 22]];
45
46     real[int] xx = [1, 32, 45, 7, 2], x(5), b(5), di(5);
47     b = A*xx;
48     cout << "b = " << b << endl;
49     cout << "xx = " << xx << endl;
50
51     set(A, solver=sparse solver, datafilename="ffsuperlu_dist_fileparam.txt");
52     cout << "solve" << endl;
53     x = A^-1*b;
54     cout << "b = " << b << endl;
55     cout << "x = " << endl;
56     cout << x << endl;
57     di = xx - x;
58     if (mpirank == 0) {
59         cout << "x-xx = " << endl;
60         cout << "Linf = " << di.linfty << ", L2 = " << di.L2 << endl;
61     }
62 }

63

64 // Solving with MUMPS
65 defaulttoMUMPS();
66 //default solver: real-> MUMPS, complex -> MUMPS
67 {
68     matrix A =
69         [[1, 2, 2, 1, 1],
70          [2, 12, 0, 10, 10],
71          [2, 0, 1, 0, 2],
72          [1, 10, 0, 22, 0.],
73          [1, 10, 2, 0., 22]];
74
75     real[int] xx = [1, 32, 45, 7, 2], x(5), b(5), di(5);
76     b = A*xx;
77     cout << "b = " << b << endl;
78     cout << "xx = " << xx << endl;

```

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```

79     set(A, solver=sparseSolver, datafilename="ffmumps_fileparam.txt");
80     cout << "solving solution" << endl;
81     x = A^-1*b;
82     cout << "b = " << b << endl;
83     cout << "x = " << endl;
84     cout << x << endl;
85     di = xx - x;
86     if (mpirank == 0){
87         cout << "x-xx = " << endl;
88         cout << "Linf = " << di.linfty << ", L2 " << di.L2 << endl;
89     }
90 }
91 }
```

## 6.6.4 Solver MUMPS

```

1 load "MUMPS_FreeFem"
2
3 // Parameters
4 int[int] ICNTL(40); //declaration of ICNTL parameter for MUMPS
5
6 //get value of ICNTL from file
7 if (mpirank == 0){
8     ifstream ff("ffmumps_fileparam.txt");
9     string line;
10    getline(ff, line);
11    getline(ff, line);
12    for (int iii = 0; iii < 40; iii++){
13        ff >> ICNTL[iii];
14        getline(ff, line);
15    }
16 }
17
18 broadcast(processor(0), ICNTL);
19
20 // Given data of MUMPS solver in array lparams(SYM, PAR, ICNTL)
21 // There is no symmetric storage for a matrix associated with a sparse solver.
22 // Therefore, the matrix will be considered unsymmetric for parallel sparse solver,
23 // even if symmetric.
24 {
25     // Problem
26     int SYM = 0;
27     int PAR = 1;
28     matrix A =
29     [
30         [40, 0, 45, 0, 0],
31         [0, 12, 0, 0, 0],
32         [0, 0, 40, 0, 0],
33         [12, 0, 0, 22, 0],
34         [0, 0, 20, 0, 22]
35     ];
36
37     // Construction of integer parameter for MUMPS
38     int[int] MumpsLParams(42);
39     MumpsLParams[0] = SYM;
```

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```

39     MumpsLParams[1] = PAR;
40     for (int ii = 0; ii < 40; ii++)
41         MumpsLParams[ii+2] = ICNTL[ii]; //ICNTL begin with index 0 here
42
43     real[int] xx = [1, 32, 45, 7, 2], x(5), b(5), di(5);
44     b = A*xx;
45     if (mpirank == 0)
46         cout << "xx = " << xx << endl;
47
48     set(A, solver=sparsesolver, lparams=MumpsLParams); //we take the default value
49     → for CNTL MUMPS parameter
50
51     // Solve
52     if (mpirank == 0)
53         cout << "Solve" << endl;
54     x = A-1*b;
55     if (mpirank == 0)
56         cout << "b = " << b << endl;
57     if (mpirank == 0)
58         cout << "x = " << endl; cout << x << endl;
59     di = xx-x;
60     if (mpirank == 0) {
61         cout << "x-xx = " << endl;
62         cout << "Linf = " << di.linfty << ", L2 = " << di.L2 << endl;
63     }
64
65 // Read parameter of MUMPS solver in file ffmumps_fileparam.txt
66 {
67     // Problem
68     matrix A =
69     [
70         [40, 0, 45, 0, 0],
71         [0, 12, 0, 0, 0],
72         [0, 0, 40, 0, 0],
73         [12, 0, 0, 22, 0],
74         [0, 0, 20, 0, 22]
75     ];
76
77     real[int] xx = [1, 32, 45, 7000, 2], x(5), b(5), di(5);
78     b = A*xx;
79     if (mpirank == 0) {
80         cout << "b = " << b << endl;
81         cout << "xx = " << xx << endl;
82     }
83
84     set(A, solver=sparsesolver, datafilename="ffmumps_fileparam.txt");
85
86     // Solve
87     if (mpirank == 0)
88         cout << "Solve" << endl;
89     x = A-1*b;
90
91     if (mpirank == 0) {
92         cout << "b = " << b << endl;
93         cout << "x = " << x << endl;
94     }

```

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```

95   di = xx-x;
96   if (mpirank == 0) {
97     cout << "x-xx = " << endl;
98     cout << "Linf = " << di.linfty << ", L2 = " << di.L2 << endl;
99   }
100 }
```

## 6.6.5 Solver superLU\_DIST

---

**Todo:** write code (SuperLU\_DIST seems to have a bug)

---

## 6.6.6 Solver PaStiX

---

**Todo:** write code (PaStiX seems to have a bug)

---

## 6.7 Developers

### 6.7.1 FFT

```

1  load "dfft"
2
3 // Parameters
4 int nx = 32;
5 real ny = 16;
6 real N = nx*ny;
7 func f1 = cos(2*x*2*pi)*cos(3*y*2*pi);
8
9 // Mesh
10 //warning: the fourier space is not exactly the unit square due to periodic condition
11 mesh Th = square(nx-1, ny-1, [(nx-1)*x/nx, (ny-1)*y/ny]);
12 //warning: the numbering of the vertices (x,y) is
13 //given by i = x/nx + nx*y/ny
14
15 // Fespace
16 fespace Vh(Th,P1);
17 Vh<complex> u = f1, v;
18 Vh w = f1;
19 Vh ur, ui;
20
21 // FFT
22 //in dfft the matrix n, m is in row-major order and array n, m is
23 //store j + m*i (the transpose of the square numbering)
24 v[] = dfft(u[], ny, -1);
25 u[] = dfft(v[], ny, +1);
26 cout << "| |u| |_\infty " << u[].linfty << endl;
```

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```

28 u[] *= 1./N;
29 cout << "| |u| |_\infty " << u[].linfty << endl;
30
31 ur = real(u);
32
33 // Plot
34 plot(w, wait=1, value=1, cmm="w");
35 plot(ur, wait=1, value=1, cmm="u");
36 V = w - u;
37 cout << "diff = " << v[].max << " " << v[].min << endl;
38 assert( norm(v[].max) < 1e-10 && norm(v[].min) < 1e-10);
39
40 // Other example
41 //FFT Lapacian
42 //-\Delta u = f with biperiodic condition
43 func f = cos(3*2*pi*x)*cos(2*2*pi*y);
44 func ue = (1./(square(2*pi)*13.))*cos(3*2*pi*x)*cos(2*2*pi*y); //the exact solution
45 Vh<complex> ff = f;
46 Vh<complex> fhat;
47 Vh<complex> wij;
48
49 // FFT
50 fhat[] = dfft(ff[], ny, -1);
51
52 //warning in fact we take mode between -nx/2, nx/2 and -ny/2, ny/2
53 //thanks to the operator ?:
54 wij = square(2.*pi)*(square((x<0.5?x*nx:(x-1)*nx)) + square((y<0.5?y*ny:(y-1)*ny)));
55 wij[0] = 1e-5; //to remove div / 0
56 fhat[] = fhat[] ./ wij[];
57 u[] = dfft(fhat[], ny, 1);
58 u[] /= complex(N);
59 ur = real(u); //the solution
60 w = real(ue); //the exact solution
61
62 // Plot
63 plot(w, ur, value=1, cmm="ue", wait=1);
64
65 // Error
66 w[] -= ur[];
67 real err = abs(w[].max) + abs(w[].min);
68 cout << "err = " << err << endl;
69 assert(err < 1e-6);
70
71 fftwplan p1 = planfft(u[], v[], ny, -1);
72 fftwplan p2 = planfft(u[], v[], ny, 1);
73 real ccc = square(2.*pi);
74 cout << "ny = " << ny << endl;
75 map(wij[], ny, ccc*(x*x+y*y));
76 wij[0] = 1e-5;
77 plot(wij, cmm="wij");

```

## 6.7.2 Complex

```

1 real a = 2.45, b = 5.33;
2 complex z1 = a + b*1i, z2 = a + sqrt(2.)*1i;

```

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```

3
4 func string pc(complex z){
5     string r = "(" + real(z);
6     if (imag(z) >= 0) r = r + "+";
7     return r + imag(z) + "i)";
8 }
9
10 func string toPolar(complex z){
11     return "";//abs(z) + "*(" + cos(arg(z)) + ") + i * sin(" + arg(z) + ")";
12 }
13
14 cout << "Standard output of the complex " << pc(z1) << " is the pair: " << z1 << endl;
15 cout << pc(z1) << " + " << pc(z2) << " = " << pc(z1+z2) << endl;
16 cout << pc(z1) << " - " << pc(z2) << " = " << pc(z1-z2) << endl;
17 cout << pc(z1) << " * " << pc(z2) << " = " << pc(z1*z2) << endl;
18 cout << pc(z1) << " / " << pc(z2) << " = " << pc(z1/z2) << endl;
19 cout << "Real part of " << pc(z1) << " = " << real(z1) << endl;
20 cout << "Imaginary part of " << pc(z1) << " = " << imag(z1) << endl;
21 cout << "abs(" << pc(z1) << ") = " << abs(z1) << endl;
22 cout << "Polar coordinates of " << pc(z2) << " = " << toPolar(z2) << endl;
23 cout << "de Moivre formula: " << pc(z2) << "^3 = " << toPolar(z2^3) << endl;
24 cout << " and polar(" << abs(z2) << ", " << arg(z2) << ") = " << pc(polar(abs(z2),_
25     arg(z2))) << endl;
26 cout << "Conjugate of " << pc(z2) << " = " << pc(conj(z2)) << endl;
27 cout << pc(z1) << " ^ " << pc(z2) << " = " << pc(z1^z2) << endl;

```

Output of this script is:

```

1 Standard output of the complex (2.45+5.33i) is the pair: (2.45,5.33)
2 (2.45+5.33i) + (2.45+1.41421i) = (4.9+6.74421i)
3 (2.45+5.33i) - (2.45+1.41421i) = (0+3.91579i)
4 (2.45+5.33i) * (2.45+1.41421i) = (-1.53526+16.5233i)
5 (2.45+5.33i) / (2.45+1.41421i) = (1.692+1.19883i)
6 Real part of (2.45+5.33i) = 2.45
7 Imaginary part of (2.45+5.33i) = 5.33
8 abs((2.45+5.33i)) = 5.86612
9 Polar coordinates of (2.45+1.41421i) =
10 de Moivre formula: (2.45+1.41421i)^3 =
11 and polar(2.82887, 0.523509) = (2.45+1.41421i)
12 Conjugate of (2.45+1.41421i) = (2.45-1.41421i)
13 (2.45+5.33i) ^ (2.45+1.41421i) = (8.37072-12.7078i)

```

### 6.7.3 String

```

1 // Concatenation
2 string tt = "totol" + 1 + " -- 77";
3
4 // Append
5 string t1 = "0123456789";
6 t1(4:3) = "abcdefghijklm-";
7
8 // Sub string
9 string t55 = t1(4:14);
10
11 cout << "tt = " << tt << endl;

```

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```

12
13 cout << "t1 = " << t1 << endl;
14 cout << "t1.find(abc) = " << t1.find("abc") << endl;
15 cout << "t1.rfind(abc) = " << t1.rfind("abc") << endl;
16 cout << "t1.find(abc, 10) = " << t1.find("abc",10) << endl;
17 cout << "t1.ffind(abc, 10) = " << t1.ffind("abc",10) << endl;
18 cout << "t1.length = " << t1.length << endl;
19
20 cout << "t55 = " << t55 << endl;

```

The output of this script is:

```

1 tt = toto11 -- 77
2 t1 = 0123abcdefgijk-456789
3 t1.find(abc) = 4
4 t1.rfind(abc) = 4
5 t1.find(abc, 10) = -1
6 t1.ffind(abc, 10) = 4
7 t1.length = 22
8 t55 = abcdefgijk

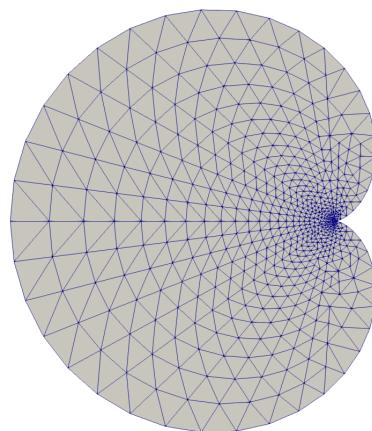
```

## 6.7.4 Elementary function

```

1 real b = 1.;
2 real a = b;
3 func real phix(real t){
4     return (a+b)*cos(t) - b*cos(t*(a+b)/b);
5 }
6 func real phiy(real t){
7     return (a+b)*sin(t) - b*sin(t*(a+b)/b);
8 }
9
10 border C(t=0, 2*pi){x=phix(t); y=phiy(t);}
11 mesh Th = buildmesh(C(50));
12 plot(Th);

```



**Fig. 6.26:** Mesh

## 6.7.5 Array

```

1 real[int] tab(10), tab1(10); //2 array of 10 real
2 //real[int] tab2; //bug: array with no size
3
4 tab = 1.03; //set all the array to 1.03
5 tab[1] = 2.15;
6
7 cout << "tab: " << tab << endl;
8 cout << "min: " << tab.min << endl;
9 cout << "max: " << tab.max << endl;
10 cout << "sum: " << tab.sum << endl;
11
12 tab.resize(12); //change the size of array tab to 12 with preserving first value
13 tab(10:11) = 3.14; //set values 10 & 11
14 cout << "resized tab: " << tab << endl;
15
16 tab.sort ; //sort the array tab
17 cout << "sorted tab:" << tab << endl;
18
19 real[string] tt; //array with string index
20 tt["+"] = 1.5;
21 cout << "tt[\\"a\\"] = " << tt["a"] << endl;
22 cout << "tt[\\"+\\"] = " << tt["+"] << endl;
23
24 real[int] a(5), b(5), c(5), d(5);
25 a = 1;
26 b = 2;
27 c = 3;
28 a[2] = 0;
29 d = ( a ? b : c ); //for i = 0, n-1 : d[i] = a[i] ? b[i] : c[i]
30 cout << " d = ( a ? b : c ) is " << d << endl;
31 d = ( a ? 1 : c ); //for i = 0, n-1: d[i] = a[i] ? 1 : c[i]
32 d = ( a ? b : 0 ); //for i = 0, n-1: d[i] = a[i] ? b[i] : 0
33 d = ( a ? 1 : 0 ); //for i = 0, n-1: d[i] = a[i] ? 0 : 1
34
35 int[int] ii(0:d.n-1); //set array ii to 0, 1, ..., d.n-1
36 d = -1:-5; //set d to -1, -2, ..., -5
37
38 sort(d, ii); //sort array d and ii in parallel
39 cout << "d: " << d << endl;
40 cout << "ii: " << ii << endl;
41
42 {
43     int[int] A1(2:10); //2, 3, 4, 5, 6, 7, 8, 9, 10
44     int[int] A2(2:3:10); //2, 5, 8
45     cout << "A1(2:10): " << A1 << endl;
46     cout << "A2(2:3:10): " << A2 << endl;
47     A1 = 1:2:5;
48     cout << "1:2:5 => " << A1 << endl;
49 }
50 {
51     real[int] A1(2:10); //2, 3, 4, 5, 6, 7, 8, 9, 10
52     real[int] A2(2:3:10); //2, 5, 8
53     cout << "A1(2:10): " << A1 << endl;
54     cout << "A2(2:3:10): " << A2 << endl;

```

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```

56   A1 = 1.:0.5:3.999;
57   cout << "1.:0.5:3.999 => " << A1 << endl;
58 }
59 {
60   complex[int] A1(2.+0i:10.+0i); //2, 3, 4, 5, 6, 7, 8, 9, 10
61   complex[int] A2(2.:3.:10.); //2, 5, 8
62   cout << " A1(2.+0i:10.+0i): " << A1 << endl;
63   cout << " A2(2.:3.:10.): " << A2 << endl;
64   cout << " A1.re real part array: " << A1.re << endl ;
65   // he real part array of the complex array
66   cout << " A1.im imag part array: " << A1.im << endl ;
67   //the imaginary part array of the complex array
68 }

69
70 // Integer array operators
71 {
72   int N = 5;
73   real[int] a(N), b(N), c(N);
74   a = 1;
75   a(0:4:2) = 2;
76   a(3:4) = 4;
77   cout << "a: " << a << endl;
78   b = a + a;
79   cout << "b = a + a: " << b << endl;
80   b += a;
81   cout << "b += a: " << b << endl;
82   b += 2*a;
83   cout << "b += 2*a: " << b << endl;
84   b /= 2;
85   cout << " b /= 2: " << b << endl;
86   b *= a; // same as b = b .* a
87   cout << "b *= a: " << b << endl;
88   b ./= a; //same as b = b ./ a
89   cout << "b ./= a: " << b << endl;
90   c = a + b;
91   cout << "c = a + b: " << c << endl;
92   c = 2*a + 4*b;
93   cout << "c = 2*a + 4b: " << c << endl;
94   c = a + 4*b;
95   cout << "c = a + 4b: " << c << endl;
96   c = -a + 4*b;
97   cout << "c = -a + 4b: " << c << endl;
98   c = -a - 4*b;
99   cout << "c = -a - 4b: " << c << endl;
100  c = -a - b;
101  cout << "c = -a -b: " << c << endl;

102
103  c = a .* b;
104  cout << "c = a .* b: " << c << endl;
105  c = a ./ b;
106  cout << "c = a ./ b: " << c << endl;
107  c = 2 * b;
108  cout << "c = 2 * b: " << c << endl;
109  c = b * 2;
110  cout << "c = b * 2: " << c << endl;

111
112 //this operator do not exist

```

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```

113 //c = b/2;
114 //cout << "c = b / 2: " << c << endl;
115
116 //Array methods
117 cout << "||a||_1 = " << a.11 << endl;
118 cout << "||a||_2 = " << a.12 << endl;
119 cout << "||a||_infty = " << a.linfty << endl;
120 cout << "sum a_i = " << a.sum << endl;
121 cout << "max a_i = " << a.max << " a[ " << a.imax << " ] = " << a[a.imax] << endl;
122 cout << "min a_i = " << a.min << " a[ " << a.imin << " ] = " << a[a.imin] << endl;
123
124 cout << "a' * a = " << (a'*a) << endl;
125 cout << "a quantile 0.2 = " << a.quantile(0.2) << endl;
126
127 //Array mapping
128 int[int] I = [2, 3, 4, -1, 3];
129 b = c = -3;
130 b = a(I); //for (i = 0; i < b.n; i++) if (I[i] >= 0) b[i] = a[I[i]];
131 c(I) = a; //for (i = 0; i < I.n; i++) if (I[i] >= 0) C(I[i]) = a[i];
132 cout << "b = a(I) : " << b << endl;
133 cout << "c(I) = a " << c << endl;
134 c(I) += a; //for (i = 0; i < I.n; i++) if (I[i] >= 0) C(I[i]) += a[i];
135 cout << "b = a(I) : " << b << endl;
136 cout << "c(I) = a " << c << endl;
137
138 }
139
140 {
141 // Array versus matrix
142 int N = 3, M = 4;
143
144 real[int, int] A(N, M);
145 real[int] b(N), c(M);
146 b = [1, 2, 3];
147 c = [4, 5, 6, 7];
148
149 complex[int, int] C(N, M);
150 complex[int] cb = [1, 2, 3], cc = [10i, 20i, 30i, 40i];
151
152 b = [1, 2, 3];
153
154 int [int] I = [2, 0, 1];
155 int [int] J = [2, 0, 1, 3];
156
157 A = 1; //set all the matrix
158 A(2, :) = 4; //the full line 2
159 A(:, 1) = 5; //the full column 1
160 A(0:N-1, 2) = 2; //set the column 2
161 A(1, 0:2) = 3; //set the line 1 from 0 to 2
162
163 cout << "A = " << A << endl;
164
165 //outer product
166 C = cb * cc';
167 C += 3 * cb * cc';
168 C -= 5i * cb * cc';
169 cout << "C = " << C << endl;

```

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```

170
171 //this transforms an array into a sparse matrix
172 matrix B;
173 B = A;
174 B = A(I, J); //B(i, j) = A(I(i), J(j))
175 B = A(I^-1, J^-1); //B(I(i), J(j)) = A(i, j)
176
177 //outer product
178 A = 2. * b * c';
179 cout << "A = " << A << endl;
180 B = b*c'; //outer product B(i, j) = b(i)*c(j)
181 B = b*c'; //outer product B(i, j) = b(i)*c(j)
182 B = (2*b*c')(I, J); //outer product B(i, j) = b(I(i))*c(J(j))
183 B = (3.*b*c')(I^-1,J^-1); //outer product B(I(i), J(j)) = b(i)*c(j)
184 cout << "B = (3.*b*c')(I^-1,J^-1) = " << B << endl;
185
186 //row and column of the maximal coefficient of A
187 int i, j, ii, jj;
188 ijmax(A, ii, jj);
189
190 i = A.imax;
191 j = A.jmax;
192
193 cout << "Max " << i << " " << j << ", = " << A.max << endl;
194
195 //row and column of the minimal coefficient of A
196 ijmin(A, i, j);
197
198 ii = A.imin;
199 jj = A.jmin;
200
201 cout << "Min " << ii << " " << jj << ", = " << A.min << endl;
202 }
```

The output os this script is:

```

1 tab: 10
2     1.03    2.15    1.03    1.03    1.03
3     1.03    1.03    1.03    1.03    1.03
4
5 min: 1.03
6 max: 2.15
7 sum: 11.42
8 resized tab: 12
9     1.03    2.15    1.03    1.03    1.03
10    1.03    1.03    1.03    1.03    1.03
11    3.14    3.14
12 sorted tab:12
13    1.03    1.03    1.03    1.03    1.03
14    1.03    1.03    1.03    1.03    2.15
15    3.14    3.14
16 tt["a"] = 0
17 tt["+"] = 1.5
18 d = ( a ? b : c ) is 5
19     2    2    3    2    2
20
21 d: 5
```

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```

22      -5   -4   -3   -2   -1
23
24 ii: 5
25      4    3    2    1    0
26
27 A1(2:10): 9
28      2    3    4    5    6
29      7    8    9   10
30 A2(2:3:10): 9
31      2    3    4    5    6
32      7    8    9   10
33 1:2:5 => 3
34      1    3    5
35 A1(2:10): 9
36      2    3    4    5    6
37      7    8    9   10
38 A2(2:3:10): 9
39      2    3    4    5    6
40      7    8    9   10
41 1.:0.5:3.999 => 6
42      1 1.5    2 2.5    3
43      3.5
44 A1(2.+0i:10.+0i): 9
45      (2,0)  (3,0)  (4,0)  (5,0)  (6,0)
46      (7,0)  (8,0)  (9,0)  (10,0)
47 A2(2.:3.:10.):= 3
48      (2,0)  (5,0)  (8,0)
49 A1.re real part array: 9
50      2    3    4    5    6
51      7    8    9   10
52 A1.im imag part array: 9
53      0    0    0    0    0
54      0    0    0    0
55 a: 5
56      2    1    2    4    4
57
58 b = a + a: 5
59      4    2    4    8    8
60
61 b += a: 5
62      6    3    6   12   12
63
64 b += 2*a: 5
65      10   5   10   20   20
66
67 b /= 2: 5
68      5 2.5    5   10   10
69
70 b .*= a: 5
71      10 2.5    10   40   40
72
73 b ./= a: 5
74      5 2.5    5   10   10
75
76 c = a + b: 5
77      7 3.5    7   14   14
78

```

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```

79 c = 2*a + 4b: 5
80     24 12 24 48 48
81
82 c = a + 4b: 5
83     22 11 22 44 44
84
85 c = -a + 4b: 5
86     18 9 18 36 36
87
88 c = -a - 4b: 5
89     -22 -11 -22 -44 -44
90
91 c = -a -b: 5
92     -7 -3.5      -7 -14 -14
93
94 c = a .* b: 5
95     10 2.5 10 40 40
96
97 c = a ./ b: 5
98     0.4 0.4 0.4 0.4 0.4
99
100 c = 2 * b: 5
101    10 5 10 20 20
102
103 c = b * 2: 5
104    10 5 10 20 20
105
106 ||a||_1 = 13
107 ||a||_2 = 6.40312
108 ||a||_infty = 4
109 sum a_i = 13
110 max a_i = 4 a[ 3 ] = 4
111 min a_i = 1 a[ 1 ] = 1
112 a' * a = 41
113 a quantile 0.2 = 2
114 b = a(I) : 5
115     2 4 4 -3 4
116
117 c(I) = a 5
118     -3 -3 2 4 2
119
120 b = a(I) : 5
121     2 4 4 -3 4
122
123 c(I) = a 5
124     -3 -3 4 9 4
125
126 A = 3 4
127     1 5 2 1
128     3 3 3 1
129     4 5 2 4
130
131 C = 3 4
132     (-50,-40) (-100,-80) (-150,-120) (-200,-160)
133     (-100,-80) (-200,-160) (-300,-240) (-400,-320)
134     (-150,-120) (-300,-240) (-450,-360) (-600,-480)
135

```

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```

136 A = 3 4
137     8 10 12 14
138     16 20 24 28
139     24 30 36 42
140
141 B = (3.*b*c') (I^-1, J^-1) = # Sparse Matrix (Morse)
142 # first line: n m (is symmetric) nbcoef
143 # after for each nonzero coefficient: i j a_ij where (i,j) \in {1,...,n}x{1,...,m}
144 3 4 0 12
145     1      1 10
146     1      2 12
147     1      3 8
148     1      4 14
149     2      1 15
150     2      2 18
151     2      3 12
152     2      4 21
153     3      1 5
154     3      2 6
155     3      3 4
156     3      4 7

```

## 6.7.6 Block matrix

```

1 // Parameters
2 real f1 = 1.;
3 real f2 = 1.5;
4
5 // Mesh
6 mesh Th1 = square(10, 10);
7 mesh Th2 = square(10, 10, [1+x, -1+y]);
8 plot(Th1, Th2);
9
10 // Fespace
11 fespace Uh1(Th1, P1);
12 Uh1 u1;
13
14 fespace Uh2(Th2, P2);
15 Uh2 u2;
16
17 // Macro
18 macro grad(u) [dx(u), dy(u)] //
19
20 // Problem
21 varf vPoisson1 (u, v)
22     = int2d(Th1)(
23         grad(u)' * grad(v)
24     )
25     - int2d(Th1)(
26         f1 * v
27     )
28     + on(1, 2, 3, 4, u=0)
29 ;
30
31 varf vPoisson2 (u, v)

```

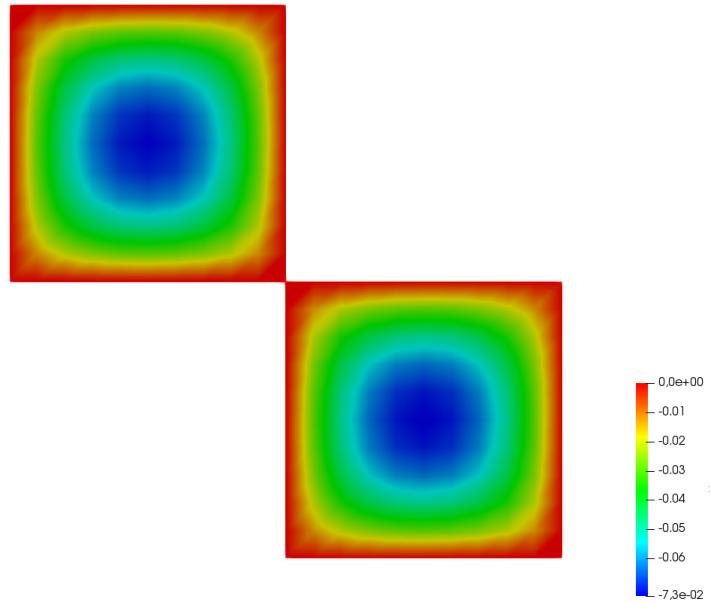
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```

32     = int2d(Th2) (
33         grad(u)' * grad(v)
34     )
35     - int2d(Th2) (
36         f1 * v
37     )
38     + on(1, 2, 3, 4, u=0)
39 ;
40 matrix<real> Poisson1 = vPoisson1(Uh1, Uh1);
41 real[int] Poisson1b = vPoisson1(0, Uh1);
42
43 matrix<real> Poisson2 = vPoisson2(Uh2, Uh2);
44 real[int] Poisson2b = vPoisson2(0, Uh2);
45
46 //block matrix
47 matrix<real> G = [[Poisson1, 0], [0, Poisson2]];
48 set(G, solver=sparse);
49
50 //block right hand side
51 real[int] Gb = [Poisson1b, Poisson2b];
52
53 // Solve
54 real[int] sol = G^-1 * Gb;
55
56 // Dispatch
57 [u1[], u2[]] = sol;
58
59 // Plot
60 plot(u1, u2);

```

**Fig. 6.27:** Result

## 6.7.7 Matrix operations

```

1 // Mesh
2 mesh Th = square(2, 1);
3
4 // Fespace
5 fespace Vh(Th, P1);
6 Vh f, g;
7 f = x*y;
8 g = sin(pi*x);
9
10 Vh<complex> ff, gg; //a complex valued finite element function
11 ff= x*(y+1i);
12 gg = exp(pi*x*1i);
13
14 // Problem
15 varf mat (u, v)
16   = int2d(Th) (
17     1*dx(u)*dx(v)
18     + 2*dx(u)*dy(v)
19     + 3*dy(u)*dx(v)
20     + 4*dy(u)*dy(v)
21   )
22   + on(1, 2, 3, 4, u=1)
23 ;
24
25 varf mati (u, v)
26   = int2d(Th) (
27     1*dx(u)*dx(v)
28     + 2i*dx(u)*dy(v)
29     + 3*dy(u)*dx(v)
30     + 4*dy(u)*dy(v)
31   )
32   + on(1, 2, 3, 4, u=1)
33 ;
34
35 matrix A = mat(Vh, Vh);
36 matrix<complex> AA = mati(Vh, Vh); //a complex sparse matrix
37
38 // Operations
39 Vh m0; m0[] = A*f[];
40 Vh m01; m01[] = A'*f[];
41 Vh m1; m1[] = f[].*g[];
42 Vh m2; m2[] = f[]./g[];
43
44 // Display
45 cout << "f = " << f[] << endl;
46 cout << "g = " << g[] << endl;
47 cout << "A = " << A << endl;
48 cout << "m0 = " << m0[] << endl;
49 cout << "m01 = " << m01[] << endl;
50 cout << "m1 = " << m1[] << endl;
51 cout << "m2 = " << m2[] << endl;
52 cout << "dot Product = "<< f[]'*g[] << endl;
53 cout << "hermitien Product = "<< ff[]'*gg[] << endl;
54 cout << "outer Product = "<< (A=f[]*g[]) << endl;
55 cout << "hermitien outer Product = "<< (AA=ff[]*gg[]) << endl;
```

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```

56
57 // Diagonal
58 real[int] diagofA(A.n);
59 diagofA = A.diag; //get the diagonal of the matrix
60 A.diag = diagofA ; //set the diagonal of the matrix
61
62 // Sparse matrix set
63 int[int] I(1), J(1);
64 real[int] C(1);
65
66 [I, J, C] = A; //get the sparse term of the matrix A (the array are resized)
67 cout << "I = " << I << endl;
68 cout << "J = " << J << endl;
69 cout << "C = " << C << endl;
70
71 A = [I, J, C]; //set a new matrix
72 matrix D = [diagofA]; //set a diagonal matrix D from the array diagofA
73 cout << "D = " << D << endl;

```

The output of this script is:

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```

37      -1.25    -2.25    0.5    0  5e+29
38      1e+30
39 m01 = 6
40      -1.25    -2.25    0  0.25    5e+29
41      1e+30
42 m1 = 6
43      0    0    0    0  0.5
44      1.224646799e-16
45 m2 = 6
46      -nan    0    0  -nan    0.5
47      8.165619677e+15
48 dot Product = 0.5
49 hermitien Product = (1.11022e-16,2.5)
50 outer Product = # Sparse Matrix (Morse)
51 # first line: n m (is symmetric) nbcoef
52 # after for each nonzero coefficient: i j a_ij where (i,j) \in {1,...,n}x{1,...,m}
53 6 6 0 8
54      5    2  0.5
55      5    3 6.1232339957367660359e-17
56      5    5  0.5
57      5    6 6.1232339957367660359e-17
58      6    2  1
59      6    3 1.2246467991473532072e-16
60      6    5  1
61      6    6 1.2246467991473532072e-16
62
63 hermitien outer Product = # Sparse Matrix (Morse)
64 # first line: n m (is symmetric) nbcoef
65 # after for each nonzero coefficient: i j a_ij where (i,j) \in {1,...,n}x{1,...,m}
66 6 6 0 24
67      2    1 (0,0.5)
68      2    2 (0.5,3.0616169978683830179e-17)
69      2    3 (6.1232339957367660359e-17,-0.5)
70      2    4 (0,0.5)
71      2    5 (0.5,3.0616169978683830179e-17)
72      2    6 (6.1232339957367660359e-17,-0.5)
73      3    1 (0,1)
74      3    2 (1,6.1232339957367660359e-17)
75      3    3 (1.2246467991473532072e-16,-1)
76      3    4 (0,1)
77      3    5 (1,6.1232339957367660359e-17)
78      3    6 (1.2246467991473532072e-16,-1)
79      5    1 (0.5,0.5)
80      5    2 (0.5,-0.4999999999999994449)
81      5    3 (-0.4999999999999994449,-0.50000000000000011102)
82      5    4 (0.5,0.5)
83      5    5 (0.5,-0.4999999999999994449)
84      5    6 (-0.4999999999999994449,-0.50000000000000011102)
85      6    1 (1,1)
86      6    2 (1,-0.9999999999999988898)
87      6    3 (-0.9999999999999988898,-1.000000000000000222)
88      6    4 (1,1)
89      6    5 (1,-0.9999999999999988898)
90      6    6 (-0.9999999999999988898,-1.000000000000000222)
91
92 I = 8
93      4    4    4    4    5

```

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```

94      5   5   5
95 J = 8
96      1   2   4   5   1
97      2   4   5
98 C = 8
99      0.5 6.123233996e-17 0.5 6.123233996e-17    1
100     1.224646799e-16  1 1.224646799e-16
101 -- Raw Matrix   nxm =6x6 nb none zero coef. 8
102 -- Raw Matrix   nxm =6x6 nb none zero coef. 6
103 D = # Sparse Matrix (Morse)
104 # first line: n m (is symmetric) nbcoef
105 # after for each nonzero coefficient: i j a_ij where (i,j) \in {1,...,n}x{1,...,m}
106 6 6 1 6
107      1       1 0
108      2       2 0
109      3       3 0
110      4       4 0
111      5       5 0.5
112      6       6 1.2246467991473532072e-16

```

**Warning:** Due to Fortran indices starting at one, the output of a diagonal matrix `D` is indexed from 1. but in FreeFEM, the indices start from 0.

## 6.7.8 Matrix inversion

```

1 load "lapack"
2 load "fflapack"
3
4 // Matrix
5 int n = 5;
6 real[int, int] A(n, n), A1(n, n), B(n, n);
7 for (int i = 0; i < n; ++i)
8     for (int j = 0; j < n; ++j)
9         A(i, j) = (i == j) ? n+1 : 1;
10 cout << A << endl;
11
12 // Inversion (lapack)
13 A1 = A^-1; //def in "lapack"
14 cout << A1 << endl;
15
16 B = 0;
17 for (int i = 0; i < n; ++i)
18     for (int j = 0; j < n; ++j)
19         for (int k = 0; k < n; ++k)
20             B(i, j) += A(i, k)*A1(k, j);
21 cout << B << endl;
22
23 // Inversion (fflapack)
24 inv(A1); //def in "fflapack"
25 cout << A1 << endl;

```

The output of this script is:

```

1 5 5
2      6   1   1   1   1
3      1   6   1   1   1
4      1   1   6   1   1
5      1   1   1   6   1
6      1   1   1   1   6
7
8 5 5
9      0.18 -0.02 -0.02 -0.02 -0.02
10     -0.02 0.18 -0.02 -0.02 -0.02
11     -0.02 -0.02 0.18 -0.02 -0.02
12     -0.02 -0.02 -0.02 0.18 -0.02
13     -0.02 -0.02 -0.02 -0.02 0.18
14
15 5 5
16      1 1.040834086e-17 1.040834086e-17 1.734723476e-17 2.775557562e-17
17      3.469446952e-18 1 -1.734723476e-17 1.734723476e-17 2.775557562e-17
18      2.428612866e-17 -3.122502257e-17 1 1.734723476e-17 2.775557562e-17
19      2.081668171e-17 -6.938893904e-17 -3.469446952e-17 1 0
20      2.775557562e-17 -4.163336342e-17 -2.775557562e-17 0 1
21
22 5 5
23      6   1   1   1   1
24      1   6   1   1   1
25      1   1   6   1   1
26      1   1   1   6   1
27      1   1   1   1   6

```

**Tip:** To compile `lapack.cpp` and `fflapack.cpp`, you must have the `lapack` library on your system and compile the plugin with the command:

```
1 ff-c++ lapack.cpp -llapack      ff-c++ fflapack.cpp -llapack
```

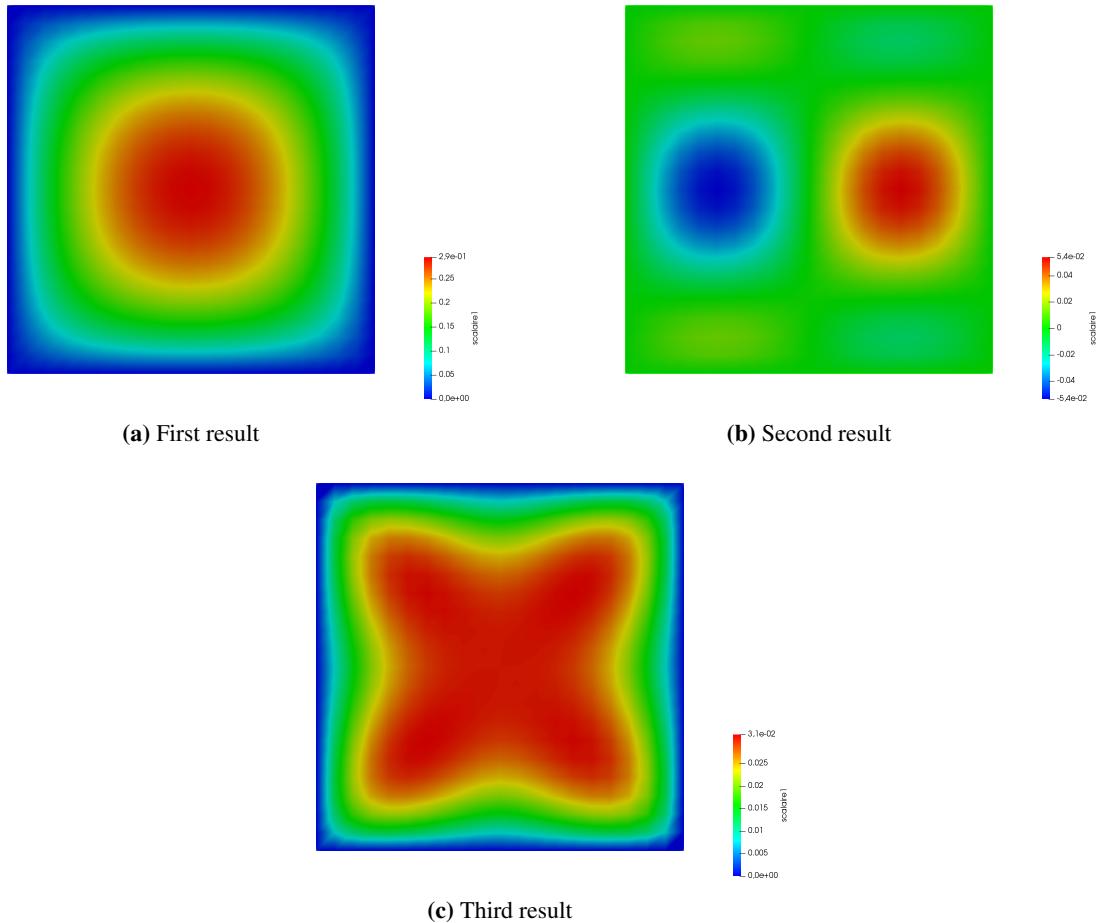
## 6.7.9 FE array

```

1 // Mesh
2 mesh Th = square(20, 20, [2*x, 2*y]);
3
4 // Fespace
5 fespace Vh(Th, P1);
6 Vh u, v, f;
7
8 // Problem
9 problem Poisson (u, v)
10    = int2d(Th) (
11        dx(u)*dx(v)
12        + dy(u)*dy(v)
13    )
14    + int2d(Th) (
15        - f*v
16    )
17    + on(1, 2, 3, 4, u=0)
18 ;

```

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**Fig. 6.28:** Finite element array

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```

19 Vh[int] uu(3); //an array of FE function
20 // Solve problem 1
21 f = 1;
22 Poisson;
23 uu[0] = u;
24 // Solve problem 2
25 f = sin(pi*x)*cos(pi*y);
26 Poisson;
27 uu[1] = u;
28 // Solve problem 3
29 f = abs(x-1)*abs(y-1);
30 Poisson;
31 uu[2] = u;
32
33 // Plot
34 for (int i = 0; i < 3; i++)
35     plot(uu[i], wait=true);
36

```

## 6.7.10 Loop

```

1  for (int i = 0; i < 10; i=i+1)
2      cout << i << endl;
3
4  real eps = 1.;
5  while (eps > 1e-5){
6      eps = eps/2;
7      if (i++ < 100)
8          break;
9      cout << eps << endl;
10 }
11
12 for (int j = 0; j < 20; j++) {
13     if (j < 10) continue;
14     cout << "j = " << j << endl;
15 }
```

## 6.7.11 Implicit loop

```

1  real [int, int] a(10, 10);
2  real [int] b(10);
3
4  for [i, bi : b]{
5      bi = i+1;
6      cout << i << " " << bi << endl;
7  }
8  cout << "b = " << b << endl;
9
10 for [i, j, aij : a]{
11     aij = 1./(2+i+j);
12     if (abs(aij) < 0.2) aij = 0;
13 }
14 cout << "a = " << a << endl;
15
16 matrix A = a;
17 string[string] ss; //a map
18 ss["1"] = 1;
19 ss["2"] = 2;
20 ss["3"] = 5;
21 for [i, bi : ss]
22     bi = i + 6 + "-dddd";
23 cout << "ss = " << ss << endl;
24
25 int[string] si;
26 si[1] = 2;
27 si[50] = 1;
28 for [i, vi : si]{
29     cout << " i " << setw(3) << i << " " << setw(10) << vi << endl;
30     vi = atoi(i)*2;
31 }
32 cout << "si = " << si << endl;
33
34 for [i, j, aij : A]{
35     cout << i << " " << j << " " << aij << endl;
```

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```

36     aij = -aij;
37 }
38 cout << A << endl;
```

The output of this script is:

```

1 0 1
2 1 2
3 2 3
4 3 4
5 4 5
6 5 6
7 6 7
8 7 8
9 8 9
10 9 10
11 b = 10
12   1   2   3   4   5
13   6   7   8   9   10
14
15 a = 10 10
16   0.5 0.3333333333 0.25 0.2   0   0   0   0   0   0
17   0.3333333333 0.25 0.2   0   0   0   0   0   0   0
18   0.25 0.2   0   0   0   0   0   0   0   0   0
19   0.2   0   0   0   0   0   0   0   0   0   0
20   0   0   0   0   0   0   0   0   0   0   0
21   0   0   0   0   0   0   0   0   0   0   0
22   0   0   0   0   0   0   0   0   0   0   0
23   0   0   0   0   0   0   0   0   0   0   0
24   0   0   0   0   0   0   0   0   0   0   0
25   0   0   0   0   0   0   0   0   0   0   0
26
27 ss = 1 1
28 2 2
29 3 5
30
31 i   1           2
32 i   50          1
33 si = 1 2
34 50 100
35
36 0 0 0.5
37 0 1 0.333333
38 0 2 0.25
39 0 3 0.2
40 1 0 0.333333
41 1 1 0.25
42 1 2 0.2
43 2 0 0.25
44 2 1 0.2
45 3 0 0.2
46 # Sparse Matrix (Morse)
47 # first line: n m (is symmetric) nbcoef
48 # after for each nonzero coefficient: i j a_ij where (i,j) \in {1,...,n}x{1,...,m}
49 10 10 0 10
50   1           1 -0.5
51   1           2 -0.3333333333333331483
```

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```

52      1      3 -0.25
53      1      4 -0.2000000000000000111
54      2      1 -0.3333333333333331483
55      2      2 -0.25
56      2      3 -0.2000000000000000111
57      3      1 -0.25
58      3      2 -0.2000000000000000111
59      4      1 -0.2000000000000000111

```

## 6.7.12 I/O

```

1 int i;
2 cout << "std-out" << endl;
3 cout << " enter i = ?";
4 cin >> i;
5
6 {
7     ofstream f("toto.txt");
8     f << i << "hello world'\n";
9 } //close the file f because the variable f is delete
10
11 {
12     ifstream f("toto.txt");
13     f >> i;
14 }
15
16 {
17     ofstream f("toto.txt", append);
18     //to append to the existing file "toto.txt"
19     f << i << "hello world'\n";
20 } //close the file f because the variable f is delete
21
22 cout << i << endl;

```

## 6.7.13 File stream

```

1 int where;
2 real[int] f = [0, 1, 2, 3, 4, 5];
3 real[int] g(6);
4
5 {
6     ofstream file("f.txt", binary);
7     file.precision(16);
8     file << f << endl;
9     where = file.tellp();
10    file << 0.1 ;
11
12    cout << "Where in file " << where << endl;
13    file << "# comment bla bla ... 0.3 \n";
14    file << 0.2 << endl;
15    file.flush; //to flush the buffer of file
16 }

```

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```

17 //Function to skip comment starting with # in a file
18 func ifstream skipcomment(ifstream &ff) {
19     while(1) {
20         int where = ff.tellg(); //store file position
21         string comment;
22         ff >> comment;
23         if (!ff.good()) break;
24         if (comment(0:0)=="#") {
25             getline(ff, comment);
26             cout << " -- #" << comment << endl;
27         }
28         else{
29             ff.seekg(where); //restore file position
30             break;
31         }
32     }
33     return ff;
34 }
35 {
36     real xx;
37     ifstream file("f.txt", binary);
38     cout << "Where " << file.seekg << endl;
39     file.seekg(where);
40     file >> xx;
41     cout << " xx = " << xx << " good ? " << file.good() << endl;
42     assert(xx == 0.1);
43     skipcomment(file) >> xx;
44     assert(xx == 0.2);
45     file.seekg(0); //rewind
46     cout << "Where " << file.tellg() << " " << file.good() << endl;
47     file >> g;
48 }
49 }
```

## 6.7.14 Command line arguments

When using the command:

```
1 FreeFem++ script.edp arg1 arg2
```

The arguments can be used in the script with:

```
1 for (int i = 0; i < ARGV.n; i++)
2     cout << ARGV[i] << endl;
```

When using the command:

```
1 FreeFem++ script.edp -n 10 -a 1. -d 42.
```

The arguments can be used in the script with:

```
1 include "getARGV.idp"
2
3 int n = getARGV("-n", 1);
```

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```

4 real a = getARGV("-a", 1.);
5 real d = getARGV("-d", 1.);
```

## 6.7.15 Macro

```

1 // Macro without parameters
2 macro xxx() {
3     real i = 0;
4     int j = 0;
5     cout << i << " " << j << endl;
6 } //
7
8 XXX
9
10 // Macro with parameters
11 macro toto(i) i //
12
13 toto({real i = 0; int j = 0; cout << i << " " << j << endl; })
14
15 // Macro as parameter of a macro
16 real[int,int] CC(7, 7), EE(6, 3), EEpss(4, 4);
17
18 macro VIL6(v, i) [v(1,i), v(2,i), v(4,i), v(5,i), v(6,i)] //
19 macro VIL3(v, i) [v(1,i), v(2,i)] //
20 macro VV6(v, vv) [
21     v(vv,1), v(vv,2),
22     v(vv,4), v(vv,5),
23     v(vv,6)] //
24 macro VV3(v, vv) [v(vv,1), v(vv,2)] //
25
26 func C5x5 = VV6(VIL6, CC);
27 func E5x2 = VV6(VIL3, EE);
28 func Eps = VV3(VIL3, EEpss);
29
30 // Macro concatenation
31 mesh Th = square(2, 2);
32 fespace Vh(Th, P1);
33 Vh Ux=x, Uy=y;
34
35 macro div(V) (dx(V#x) + dy(V#y)) //
36
37 cout << int2d(Th)(div(U)) << endl;
38
39 // Verify the quoting
40 macro foo(i, j, k) i j k //
41 foo(, , )
42 foo({int[], {int} a(10), {}})
43
44 //NewMacro - EndMacro
45 NewMacro grad(u) [dx(u), dy(u)] EndMacro
46 cout << int2d(Th)(grad(Ux)' * grad(Uy)) << endl;
47
48 // IFMACRO - ENDIFMACRO
49 macro AA CAS1 //
```

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```

51 IFMACRO(AA,CAS1 )
52   cout << "AA = " << Stringification(AA) << endl;
53 macro CASE file1.edp//  

54 ENDIFMACRO
55 IFMACRO(AA, CAS2)
56 macro CASE file2.edp//  

57 ENDIFMACRO
58
59 cout << "CASE = " << Stringification(CASE) << endl;
60
61 IFMACRO(CASE)
62   include Stringification(CASE)
63 ENDIFMACRO
64
65 // FILE - LINE
66 cout << "In " << FILE << ", line " << LINE << endl;

```

The output script generated with macros is:

```

1 : // Macro without parameters
2 : macro xxx {
3 :   real i = 0;
4 :   int j = 0;
5 :   cout << i << " " << j << endl;
6 : }//
7 :
8 :
9 :
10:
11:
12: {
13:   real i = 0;
14:   int j = 0;
15:   cout << i << " " << j << endl;
16: }
17:
18: // Macro with parameters
19: macro toto(i )    i //
20:
21:           real i = 0; int j = 0; cout << i << " " << j << endl;
22:
23: // Macro as parameter of a macro
24: real[int,int] CC(7, 7), EE(6, 3), EEps(4, 4);
25:
26: macro VIL6(v,i ) [v(1,i), v(2,i), v(4,i), v(5,i), v(6,i)] //
27: macro VIL3(v,i ) [v(1,i), v(2,i)] //
28: macro VV6(v,vv ) [
29:   v(vv,1), v(vv,2),
30:   v(vv,4), v(vv,5),
31:   v(vv,6)] //
32: macro VV3(v,vv ) [v(vv,1), v(vv,2)] //
33:
34: func C5x5 =
35:
36:
37: [
38:   [ CC(1,1), CC(2,1), CC(4,1), CC(5,1), CC(6,1) ] , [ CC(1,
    ↵2), CC(2,2), CC(4,2), CC(5,2), CC(6,2) ] , (continues on next page)

```

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```

39 2 : [ CC(1,4), CC(2,4), CC(4,4), CC(5,4), CC(6,4) ] , [ CC(1,
40   ↪5), CC(2,5), CC(4,5), CC(5,5), CC(6,5) ] ,
41 3 : [ CC(1,6), CC(2,6), CC(4,6), CC(5,6), CC(6,6) ] ;
42 27 : func E5x2 =
43 1 :
44 2 :
45 3 : [
46 1 :   [ EE(1,1), EE(2,1) ] , [ EE(1,2), EE(2,2) ] ,
47 2 :   [ EE(1,4), EE(2,4) ] , [ EE(1,5), EE(2,5) ] ,
48 3 :   [ EE(1,6), EE(2,6) ] ] ;
49 28 : func Eps = [ [ EEps(1,1), EEps(2,1) ] , [ EEps(1,2), EEps(2,2) ] ] ;
50   ↪;
51 29 :
52 30 : // Macro concatenation
53 31 : mesh Th = square(2, 2);
54 32 : fespace Vh(Th, P1);
55 33 : Vh Ux=x, Uy=y;
56 34 :
57 35 : macro div(V) (dx(V#x) + dy(V#y)) //
58 36 :
59 37 : cout << int2d(Th)( dx(Ux) + dy(Uy) ) << endl;
60 38 :
61 39 : // Verify the quoting
62 40 : macro foo(i,j,k) i j k //
63 41 :
64 42 : int[ int] a(10 );
65 43 :
66 44 : //NewMacro - EndMacro
67 45 : macro grad(u) [dx(u), dy(u)]
68 46 : cout << int2d(Th)( [dx(Ux), dy(Ux)] ' * [dx(Uy), dy(Uy)] ) << endl;
69 47 :
70 48 : // IFMACRO - ENDIFMACRO
71 49 : macro AACAS1 //
72 50 :
73 51 :
74 52 : cout << "AA = " << Stringification( CAS1 ) << endl;
75 53 : macro CASEfile1.edp//  

76 54 : cout << "CASE = " << Stringification(file1.edp) << endl;
77 55 :
78 56 :
79 1 : include Stringification(file1.edp)cout << "This is the file 1" << endl;
80 2 :
81 2 :
82 57 :
83 58 : // FILE - LINE
84 59 : cout << "In " << FILE << ", line " << LINE << endl;

```

The output os this script is:

```

1 AA = CAS1
2 CASE = file1.edp
3 This is the file 1
4 In Macro.edp, line 59

```

## 6.7.16 Basic error handling

```

1 real a;
2 try{
3     a = 1./0.;
4 }
5 catch (...) //all exceptions can be caught
6 {
7     cout << "Catch an ExecError" << endl;
8     a = 0.;
9 }
```

The output of this script is:

```

1 1/0 : d d d
2     current line = 3
3 Exec error : Div by 0
4     -- number :1
5 Catch an ExecError
```

## 6.7.17 Error handling

```

// Parameters
1 int nn = 5;
2 func f = 1; //right hand side function
3 func g = 0; //boundary condition function
4
5
// Mesh
6 mesh Th = square(nn, nn);
7
8
// Fespace
9 fespace Vh(Th, P1);
10 Vh uh, vh;
11
12
// Problem
13 real cpu = clock();
14 problem laplace (uh, vh, solver=Cholesky, tolpivot=1e-6)
15     = int2d(Th) (
16         dx(uh)*dx(vh)
17         + dy(uh)*dy(vh)
18     )
19     + int2d(Th) (
20         - f*vh
21     )
22     ;
23
24
try{
25     cout << "Try Cholesky" << endl;
26
27
28     // Solve
29     laplace;
30
31     // Plot
32     plot(uh);
```

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```

34 // Display
35   cout << "laplacian Cholesky " << nn << ", x_" << nn << " : " << -cpu+clock() << " "
36   ↳s, max = " << uh[].max << endl;
37 }
38 catch(...) { //catch all error
39   cout << " Catch cholesky PB " << endl;
}

```

The output of this script is:

```

1 Try Cholesky
2 ERREUR choleskypivot (35)= -6.43929e-15 < 1e-06
3   current line = 29
4 Exec error : FATAL ERREUR dans ../../femlib/MatriceCreuse_tpl.hpp
5 cholesky line:
6   -- number :688
7 catch an erreur in solve => set sol = 0 !!!!
8 Catch cholesky PB

```



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