Second order linear Equations

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1 Make the problem clear

A second order ordinary differential equation has the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right) \tag{1}$$

It is said to be **linear** if the function f has the form

$$f\left(t, y, \frac{dy}{dt}\right) = g(t) - p(t)\frac{dy}{dt} - q(t)y \tag{2}$$

here g, p, q are specified funcitons of t, the independent variable. We urally rewirite the Eq.(1) as

$$y'' + p(t)y' + q(t)y = g(t) \tag{3}$$

Also we often see the equation

$$P(t)y'' + Q(t)y' + R(t)y = G(t) \tag{4}$$

If $P(t) \neq 0$ we can divided Eq.(4) by $P(t)$ to obtain Eq.(3).

homogeneous and nonhomogeneous

A second order linear equation is said to be **homogeneous** if the term $g(t)$ in Eq.(3) or the term $G(t)$ in Eq.(4) is zero for all t. Otherwise the equation is called **nonhomogeneous**. The term $g(t)$ or $G(t)$ is sometimes called the nonhomogeneous term.

2 Homogeneous Equatioins with Constant Coefficients

The equation we discussed in this section has the form

$$ay'' + by' + cy = 0 \tag{5}$$

where a, b, c are give constants.

2.1 characteristic equation

We start by seeking exponential solutions of the frm $y = e^{rt}$, r is a parameter to be determined. Then it follows that $y' = re^{rt}$ and $y'' = r^2e^{rt}$. Substitute them for y, y' and y'' in Eq.(5), we obtain

$$(ar^2 + br + c)e^{rt} = 0$$

or, since $e^{rt} \neq 0$,

$$ar^2 + br + c = 0 \tag{6}$$

Eq.(6) is called the characteristic equation for the differential equation (5). Then we begin to discuss the root of Eq.(6).

2.2 $b^2 - 4ac > 0$ two different real roots

Suppose the two different roots of the characteristic equation is r_1 and r_2. Then $y_1(t) = e^{r_1t}$ and $y_2(t) = 3^{r_2t}$ are two solutions of Eq.(5). Moreover, the general solution can be written as

$$y = c_1y_1(t) + c_2y_2(t) = c_1e^{r_1t} + c_2e^{r_2t} \tag{7}$$

Since the general solutioni is in such form, the solution has a relatively simple geometrical behavior:

> As t increases, the magnitude of the solution either tends to zero (both exponents are negative) or else grows rapidly (at least one exponent is positive).

If we have the initial conditions that

$$y(t_0) = y_0,\ y'(t_0) = y'_0$$

By substituting $t = t_0$ and $y = y_0$ in Eq.(7) we obtain

$$c_1e^{r_1t_0} + c_2e^{r_2t_0} = y_0 \tag{8}$$

and

$$c_1r_1e^{r_1t_0} + c_2r_2e^{r_2t_0} = y'_0 \tag{9}$$

So solving the two equations, we can find that:

$$c_1 = \frac{y'_0 - y_0r_2}{r_1 - r_2}e^{-r_1t_0},\ c_2 = \frac{y_0r_1 - y'_0}{r_1 - r_2}e^{-r_2t_0} \tag{10}$$

Recall that $r_1 - r_2 \neq 0$, so Eq.(10) always make sense.

2.3 $b^2 - 4ac < 0$ complex and conjugate roots

Suppose now that $b^2 - 4ac$ is negative. Then the roots of Eq.(6) are conjugate complex numbers. We denote them by

$$r_1 = \lambda + i\mu,\ r_2 = \lambda - i\mu \tag{11}$$

Where λ and μ are real. Then the corresponding expressions for y are

$$y_1(t) = exp\left[(\lambda + i\mu)t\right],\ y_2(t) = exp\left[(\lambda - i\mu)t\right] \tag{12}$$

Euler's Formula

From Taylor's series, we have

$$\begin{aligned} e^{it} &= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!} \\ &= cost + isint \end{aligned}$$

So

$$e^{(\lambda + i\mu)t} = e^{\lambda t} cos\mu t + i e^{\lambda t} sin\mu t \tag{13}$$

Chosing the real part of of either $y_1(t)$ and $y_2(t)$, we obtain the general solution of Eq.(5)

$$y = c_1 e^{\lambda t} cos\mu t + c_2 e^{\lambda t} sin\mu t \tag{14}$$

2.4 $b^2 - 4ac = 0$ repeated root