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Hypothesis tests in the proportional inverse Gaussian distribution

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[摘要]: Lijoi 等 [1] 第一次提出 NIG 分布（正则化逆高斯分布）并将其作为贝叶斯分层模型中的先验分布。刘鹏懿博士和田国梁教授 [2] 在发现其可作为一种新的分析比例数据的工具后，进一步研究了这个分布的性质和统计推断的方法并将其命名为 PIG 分布（比例逆高斯分布）。本文首先简要介绍连续比例数据和一些常用于分析这种数据的方法，并且回顾了 PIG 分布和它的统计推断方法。为了进一步更好地利用这个模型，本文提出了涉及此模型的四个假设检验问题，包括检验参数是否相等，检验单样本和多样本的均值，检验 PIG 回归系数和检验 PIG 回归模型充分性，以及适用于这些假设检验的新的推断方法。之后我使用了仿真实验测试了这几种假设检验的有效性和算法效率，我也使用了蒙特卡洛方法估计检验的势，附录中提供了实现仿真实验的代码。

[关键词]: 假设检验; MM 算法; PIG 分布; PIG 回归

[ABSTRACT]: The normalized inverse Gaussian (NIG) distribution was first proposed by Lijoi et al. [1] as a prior distribution to be used in Bayesian hierarchical models. Liu & Tian [2] provided some distributional properties and statistical inference methods for this distribution and renamed it as proportional inverse Gaussian (PIG) distribution, aiming to develop it as a new tool for modeling continuous proportional data in unit interval (0,1). This thesis first simply introduces the continuous proportional data and some commonly used models for analyzing such kind data. Then I review the PIG distribution and its inference methods. To exploit this model further, four hypothesis tests for PIG distribution: test for parameter difference, test on mean for one sample and two samples, test for a single PIG regression coefficient and test for regression overall adequacy, are proposed, as well as some new inference methods. Simulations studies are conducted to assess the performance and efficiency of these hypothesis tests. I also used the Monte Carlo methods to estimate the power of tests. R codes are provided in the Appendix.

[Keywords]: Hypothesis Tests; MM algorithm; PIG distribution; PIG regression

Contents

1 Introduction

1.1 Background

The normalized inverse Gaussian (NIG) distribution was first proposed by Lijoi et al. [1] as a prior distribution to be used in Bayesian hierarchical models. They found that for the NIG prior the weights of observations depend on the number of ties in the sample heavily, which makes it a great replacement of Dirichlet prior. Then, Liu & Tian [2] provided some distributional properties and statistical inference methods for this distribution and renamed it as proportional inverse Gaussian (PIG) distribution, aiming to develop it as a new tool for modeling continuous proportional data in unit interval (0,1). In this thesis, I first simply review the PIG distribution and some of its inference methods, then I discuss the hypothesis tests of the PIG model, like testing hypothesis on parameter difference, sample mean and regression model. Some simulations are also conducted.

1.2 Continuous proportional data and related models

Data in the form of proportions, rates, fractions and percentages which are observed in the unit interval (0,1) are called continuous proportional data. For example, concentration of a drug, proportion of area damaged in a forest fire. While the binomial distribution is generally applicable to data that represents proportions from aggregated binary responses, it might not apply to underlying continuous data. The binomial distribution is typically used when the proportions represent a count of events resulting from a known number of trials. Response measurements that are proportions relative to a standard (such as proportion of control) or percentages of a total amount (such as proportion of chemical absorbed) might not be binomial since they do not represent a set of independent trials.

A commonly used solution is to apply a transformation to the data which maps values from the zero to one interval to the entire real number line. After transformation the usual suite of statistical models, assuming a normally distributed response, can be applied. For many years the arcsine transformation was advised, but in recent decades, techniques that avoid this transformation step have been developed. These methods replace the normal distribution with a probability distribution which is explicitly constrained to the zero to one interval. In cases where proportional data are derived from continuous (non-count) measurements, the beta, simplex and Dirichlet distributions are the ideal candidates. The beta distribution is the most widely used one and the beta regression models have been researched exhaustively. The simplex distribution, belonging to the non-exponential family, can also model continuous proportional data. And the simulation studies of Zhang & Qiu [3] showed that simplex regression model is more robust when beta regression model' s distributional assumptions are violated. Dirichlet regression models can be used to analyze a set of variables lying in a bounded interval that sum up to a constant exhibiting skewness and heteroscedasticity.[4] Liu & Tian [2] have developed the PIG distribution as a new tool for modeling continuous proportional data. Some efficient statistical inference methods for the PIG distribution with and without covariates have been provided.

1.3 Proportional inverse Gaussian distribution

1.3.1 Inverse Gaussian distribution

The inverse Gaussian distribution, also known as the Wald distribution, is a two-parameter family of continuous probability distribution with support $(0, \infty)$. The probability density function (pdf) of an inverse Gaussian random variable $X \sim \text{IG}(\mu, \lambda)$ with location parameter $\mu\ (>0)$ and shape parameter $\lambda\ (>0)$ is given by

$$f_{\text{IG}}(x; \mu, \lambda) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left[-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right]. \tag{1.1}$$

As the shape parameter λ tends to infinity, the inverse Gaussian distribution becomes more like a normal distribution. The inverse Gaussian distribution describes the distribution of the time that a Brownian motion with positive drift takes to reach a fixed positive level.[5] When $\lambda = \mu^2$, we can see that the single-parameter inverse Gaussian distribution $\text{IG}(\mu, \mu^2)$ has equi-dispersion. And its pdf can be written as

$$f_{\text{IG}}\left(x; \mu, \mu^2\right) = \frac{\mu}{\sqrt{2\pi x^3}} \exp\left[-\frac{(x-\mu)^2}{2x}\right]. \tag{1.2}$$

The cumulative distribution function (cdf) of the single-parameter inverse Gaussian distribution is related to the standard normal distribution by

$$\begin{aligned} \Pr(X \leqslant x) &= \Phi\left(z_1\right) + e^{\mu}\Phi\left(z_2\right), \quad &\text{for} \quad 0 < x \leqslant \mu, \\ \Pr(X > x) &= \Phi\left(-z_1\right) - e^{\mu}\Phi\left(z_2\right), \quad &\text{for} \quad x > \mu, \end{aligned} \tag{1.3}$$

where $z_1 = \frac{\mu}{x^{1/2}} - x^{1/2}$, $z_2 = \frac{\mu}{x^{1/2}} + x^{1/2}$, and Φ is the cdf of standard normal distribution $N(0,1)$. Note that z_1 and z_2 are related to each other by $z_2^2 = z_1^2 + 4\mu$.

1.3.2 Proportional inverse Gaussian distribution

Based on two independent single-parameter inverse Gaussian random variables, we can construct a new distribution for modeling continuous proportional data described in section 1.2. Suppose that $Y_1 \sim \mathrm{IG}\left(\theta_1, \theta_1^2\right)$, $Y_2 \sim \mathrm{IG}\left(\theta_2, \theta_2^2\right)$, and they are independent (denoted by $Y_1 \perp Y_2$). Let

$$X = \frac{Y_1}{Y_1 + Y_2}, \tag{1.4}$$

we say that X follows the PIG distribution with parameters $\theta_1 > 0$ and $\theta_2 > 0$, denoted by $X \sim \mathrm{PIG}(\theta_1, \theta_2)$ or $X \sim \mathrm{PIG}(\boldsymbol{\theta})$ with $\boldsymbol{\theta} = (\theta_1, \theta_2)^\top$. Its pdf is given by

$$\begin{aligned} f_{\mathrm{PIG}}(x|\boldsymbol{\theta}) &= c(\boldsymbol{\theta})[x(1-x)]^{-\frac{3}{2}} \int_0^\infty s^{-2} \exp\left[-\frac{s + b(x;\boldsymbol{\theta})/s}{2}\right] \mathrm{d}s \\ &= \frac{c(\boldsymbol{\theta})[x(1-x)]^{-\frac{3}{2}} \cdot 2K_1(\sqrt{b(x;\boldsymbol{\theta})})}{\sqrt{b(x;\boldsymbol{\theta})}} \end{aligned} \tag{1.5}$$

for $0 < x < 1$, where

$$c(\boldsymbol{\theta}) = \frac{\theta_1\theta_2}{2\pi} e^{\theta_1+\theta_2}, \quad b(x;\boldsymbol{\theta}) = \frac{\theta_1^2}{x} + \frac{\theta_2^2}{1-x}, \tag{1.6}$$

and $K_1(\cdot)$ is the modified Bessel function of the second kind. From the stochastic representation (SR) (1.4), we can obtain the following property:

$$\text{If } X \sim \mathrm{PIG}(\theta_1, \theta_2), \text{ then } 1 - X \sim \mathrm{PIG}(\theta_2, \theta_1). \tag{1.7}$$

The mean and variance of the PIG distribution with parameters $\boldsymbol{\theta} = (\theta_1, \theta_2)^\top$ are:

$$\mathrm{E}(X) = \frac{\theta_1}{\theta_1 + \theta_2}, \quad \mathrm{Var}(X) = \theta_1\theta_2 \mathrm{e}^{\theta_1+\theta_2} \Gamma\left(-2, \theta_1 + \theta_2\right), \tag{1.8}$$

where

$$\Gamma\left(-2, \theta_1 + \theta_2\right) = \int_{\theta_1+\theta_2}^\infty t^{-2-1} \mathrm{e}^{-t} \mathrm{d}t$$

is the incomplete gamma function.

Assuming that $X_i \overset{\text{iid}}{\sim} \mathrm{PIG}(\boldsymbol{\theta}), i = 1, \ldots, n$, and $Y_{\text{obs}} = \{x_i\}_{i=1}^n$ denote the observed data, where x_i is the realization of X_i. The log-likelihood function of $\boldsymbol{\theta}$ based on the observed data is

$$\begin{aligned} \ell_1\left(\boldsymbol{\theta}|Y_{\text{obs}}\right) =& c_1 + n\log\left(\theta_1\right) + n\log\left(\theta_2\right) + n\left(\theta_1 + \theta_2\right) \\ &+ \sum_{i=1}^n \log\left[\int_0^\infty h_1\left(s|x_i, \boldsymbol{\theta}\right) \mathrm{d}s\right], \end{aligned} \tag{1.9}$$

where $c_1 = -n\log(2\pi) - \frac{3}{2}\sum_{i=1}^n \log[x_i(1-x_i)]$ is a constant free from $\boldsymbol{\theta}$ and

$$\begin{aligned} h_1(s|x_i, \boldsymbol{\theta}) &= s^{-2} \exp\left[-\frac{s + b\left(x_i;\boldsymbol{\theta}\right)/s}{2}\right] \\ &\overset{(1.6)}{=} s^{-2} \exp\left\{-\frac{1}{2}\left[s + \left(\frac{\theta_1^2}{x_i} + \frac{\theta_2^2}{1-x_i}\right)\frac{1}{s}\right]\right\}. \end{aligned} \tag{1.10}$$

The MLEs of parameters $\boldsymbol{\theta} = (\theta_1, \theta_2)^\top$ can be obtained by following MM algorithm which is proposed by Liu & Tian [2]:

$$\theta_k^{(t+1)} = \frac{n + \sqrt{n^2 + 4na_k^{(t)}}}{2a_k^{(t)}}, \quad k = 1, 2, \tag{1.11}$$

where

$$
a_1^{(t)} = \sum_{i=1}^{n} \frac{B_1(x_i, \boldsymbol{\theta}^{(t)})}{x_i}, \quad a_2^{(t)} = \sum_{i=1}^{n} \frac{B_1(x_i, \boldsymbol{\theta}^{(t)})}{1 - x_i},
$$

$$
B_1(x_i, \boldsymbol{\theta}^{(t)}) = \int_0^{\infty} \frac{g_1(s|x_i, \boldsymbol{\theta}^{(t)})}{s} \mathrm{d}s, \quad g_1\left(s|x_i, \boldsymbol{\theta}\right) = \frac{h_1\left(s|x_i, \boldsymbol{\theta}\right)}{\int_0^{\infty} h_1\left(t|x_i, \boldsymbol{\theta}\right) \mathrm{d}t}, \quad s > 0,
$$

1.4 PIG regression

Liu & Tian [2] has proposed a PIG regression model to investgate the relationship between the mean parameter of the PIG distribution and a set of predictors. Reparameterizing the $\mathrm{PIG}(\theta_1, \theta_2)$ by

$$
\eta = \frac{\theta_1}{\theta_1 + \theta_2} \quad \text{and} \quad \phi = \theta_1 + \theta_2. \tag{1.12}
$$

Then $\theta_1 = \eta\phi$ and $\theta_2 = (1-\eta)\phi$, where $0 < \eta < 1$ is the mean of the PIG distribution and $\phi > 0$. The considered PIG regresion model is:

$$
\begin{aligned}
X_i &\overset{\text{iid}}{\sim} \mathrm{PIG}\left(\theta_{i1}, \theta_{i2}\right) = \mathrm{PIG}\left(\eta_i\phi, (1-\eta_i)\,\phi\right), \quad i = 1, \ldots, n, \\
\log\left(\frac{\eta_i}{1-\eta_i}\right) &= \boldsymbol{z}_i^{\top}\boldsymbol{\alpha}, \quad \text{or} \quad \eta_i = \frac{\exp\left(\boldsymbol{z}_i^{\top}\boldsymbol{\alpha}\right)}{1 + \exp\left(\boldsymbol{z}_i^{\top}\boldsymbol{\alpha}\right)},
\end{aligned} \tag{1.13}
$$

where $\boldsymbol{z}_i = (1, z_{i1}, \ldots, z_{ip})^{\top}$ is the vector of covariates, and $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_p)^{\top}$ is the vector of regression coefficients, with $p + 1 < n$. Let $Y_{\text{obs}} = \{x_i\}_{i=1}^n$ denote the observed data and $\boldsymbol{\beta} = \left(\boldsymbol{\alpha}^{\top}, \phi\right)^{\top}$. The log-likelihood function of $\boldsymbol{\beta}$ is

$$
\begin{aligned}
\ell_2\left(\boldsymbol{\beta}|Y_{\text{obs}}\right) =& c_2 + 2n\log(\phi) + n\phi + \sum_{i=1}^{n} \boldsymbol{z}_i^{\top}\boldsymbol{\alpha} - 2\sum_{i=1}^{n} \log\left(1 + e^{\boldsymbol{z}_i^{\top}\boldsymbol{\alpha}}\right) \\
&+ \sum_{i=1}^{n} \log\left[\int_0^{\infty} h_2\left(s|x_i, \boldsymbol{z}_i, \boldsymbol{\beta}\right) \mathrm{d}s\right],
\end{aligned} \tag{1.14}
$$

where $c_2 = -n\log(2\pi) - \frac{3}{2}\sum_{i=1}^{n}\log[x_i(1-x_i)]$ is a constant not depending on $\boldsymbol{\beta}$ and

$$
h_2\left(s|x_i, \boldsymbol{z}_i, \boldsymbol{\beta}\right) = s^{-2}\exp\left\{-\frac{1}{2}\left[s + \frac{\eta_i^2 - 2x_i\eta_i + x_i}{x_i\left(1-x_i\right)} \cdot \frac{\phi^2}{s}\right]\right\}. \tag{1.15}
$$

MLEs of $\boldsymbol{\beta}$ can be obtained by an MM algorithm aided by the gradient ascent algorithm. The surrogate function is

$$
\begin{aligned}
Q_1(\boldsymbol{\beta}|\boldsymbol{\beta}^{(t)}) =& c_3 + 2n\log(\phi) + n\phi + \sum_{i=1}^{n} \boldsymbol{z}_i^{\top}\boldsymbol{\alpha} - 2\sum_{i=1}^{n} \frac{\exp(\boldsymbol{z}_i^{\top}\boldsymbol{\alpha})}{1 + \exp(\boldsymbol{z}_i^{\top}\boldsymbol{\alpha}^{(t)})} \\
&- \frac{\phi^2}{2}\sum_{i=1}^{n} \frac{B_2(x_i, \boldsymbol{z}_i, \boldsymbol{\beta}^{(t)})}{x_i\left(1-x_i\right)}\left(\eta_i^2 - 2x_i\eta_i + x_i\right),
\end{aligned} \tag{1.16}
$$

which minorizes $\ell_2\left(\boldsymbol{\beta}|Y_{\text{obs}}\right)$ at $\boldsymbol{\beta}=\boldsymbol{\beta}^{(t)}$, where c_3 is a constant and

$$\begin{aligned}
B_2(x_i,\boldsymbol{z}_i,\boldsymbol{\beta}^{(t)})&\hat{=}\int_0^\infty \frac{g_2(s|x_i,\boldsymbol{z}_i,\boldsymbol{\beta}^{(t)})}{s}\mathrm{d}s,\\
g_2\left(s|x_i,\boldsymbol{z}_i,\boldsymbol{\beta}\right)&\hat{=}\frac{h_2\left(s|x_i,\boldsymbol{z}_i,\boldsymbol{\beta}\right)}{\int_0^\infty h_2\left(t|x_i,\boldsymbol{z}_i,\boldsymbol{\beta}\right)\mathrm{d}t},\quad s>0.
\end{aligned}\tag{1.17}$$

The algorithm is:

$$\boldsymbol{\beta}^{(t+1)}=\boldsymbol{\beta}^{(t)}+s^{(t)}\nabla Q_1(\boldsymbol{\beta}^{(t)}|\boldsymbol{\beta}^{(t)}),\tag{1.18}$$

where $s^{(t)}$ is the step size defined by

$$s^{(t)}=\frac{|[\boldsymbol{\beta}^{(t)}-\boldsymbol{\beta}^{(t-1)}]^\top[\nabla Q_1(\boldsymbol{\beta}^{(t)}\boldsymbol{\beta}^{(t)})-\nabla Q_1(\boldsymbol{\beta}^{(t-1)}|\boldsymbol{\beta}^{(t-1)})]|}{\|\nabla Q_1(\boldsymbol{\beta}^{(t)}|\boldsymbol{\beta}^{(t)})-\nabla Q_1(\boldsymbol{\beta}^{(t-1)}|\boldsymbol{\beta}^{(t-1)})||^2},\tag{1.19}$$

and

$$\begin{aligned}
\nabla Q_1(\boldsymbol{\beta}|\boldsymbol{\beta}^{(t)})&=\begin{pmatrix}\dfrac{\partial Q_1(\boldsymbol{\beta}|\boldsymbol{\beta}^{(t)})}{\partial\boldsymbol{\alpha}}\\ \dfrac{\partial Q_1(\boldsymbol{\beta}|\boldsymbol{\beta}^{(t)})}{\partial\phi}\end{pmatrix},\\
\frac{\partial Q_1(\boldsymbol{\beta}|\boldsymbol{\beta}^{(t)})}{\partial\boldsymbol{\alpha}}&=\sum_{i=1}^n\boldsymbol{z}_i-2\sum_{i=1}^n\frac{\exp(\boldsymbol{z}_i^\top\boldsymbol{\alpha})}{1+\exp(\boldsymbol{z}_i^\top\boldsymbol{\alpha}^{(t)})}\boldsymbol{z}_i\\
&\quad-\phi^2\sum_{i=1}^n\frac{B_2(x_i,\boldsymbol{z}_i,\boldsymbol{\beta}^{(t)})\eta_i\left(1-\eta_i\right)\left(\eta_i-x_i\right)}{x_i\left(1-x_i\right)}\boldsymbol{z}_i,\\
\frac{\partial Q_1(\boldsymbol{\beta}|\boldsymbol{\beta}^{(t)})}{\partial\phi}&=\frac{2n}{\phi}+n-\phi\sum_{i=1}^n\frac{B_2(x_i,\boldsymbol{z}_i,\boldsymbol{\beta}^{(t)})}{x_i\left(1-x_i\right)}\left(\eta_i^2-2x_i\eta_i+x_i\right).
\end{aligned}$$

2 Hypothesis tests in the PIG model

2.1 Testing hypothesis on parameter difference

Since PIG(θ_1,θ_2) density with equal θ_1 and θ_2 is symmetrical about $x=0.5$, it's meaningful for us to check whether the two unknown parameters are equal by hypothesis test. Assuming that $X_i\overset{\text{iid}}{\sim}$ PIG$(\theta_1,\theta_2), i=1,\ldots,n$, and $Y_{\text{obs}}=\{x_i\}_{i=1}^n$ denote the observed data, where x_i is the realization of X_i. The null hypothesis and alternative hypothesis are

$$H_0:\theta_1=\theta_2\quad\text{against}\quad H_1:\theta_1\neq\theta_2.$$

Applying the likelihood ratio test, the likelihood ratio statistic is

$$T_1=-2\left\{\ell_1(\hat{\theta}_{1,H_0},\hat{\theta}_{1,H_0}|Y_{\text{obs}})-\ell_1(\hat{\boldsymbol{\theta}}|Y_{\text{obs}})\right\},\tag{2.1}$$

where $\hat{\theta}_{1,H_0}$ is the constrained MLE of θ_1 under H_0, and $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \hat{\theta}_2)^\top$ are the unconstrained MLEs of $\boldsymbol{\theta}$, which could be obtained by the MM algorithm (1.11). The constrained MLE $\hat{\theta}_{1,H_0}$ can be obtained by following procedure which is simply modified from (1.11):

$$\theta_1^{(t+1)} = \frac{n + \sqrt{n^2 + 2na_3^{(t)}}}{a_3^{(t)}}, \tag{2.2}$$

where

$$a_3^{(t)} = \sum_{i=1}^{n} \frac{B_3(x_i, \theta^{(t)})}{x_i(1-x_i)}, \quad B_3(x_i, \theta^{(t)}) = \int_0^\infty \frac{g_3(s|x_i, \theta^{(t)})}{s} \mathrm{d}s,$$

$$g_3(s|x_i, \theta^{(t)}) = \frac{h_3(s|x_i, \theta^{(t)}))}{\int_0^\infty h_3(t|x_i, \theta^{(t)}) \mathrm{d}t},$$

$$h_3(s|x_i, \theta^{(t)}) = s^{-2} \exp\left[-\frac{1}{2}\left(s + \frac{\theta^{2(t)}}{sx_i(1-x_i)}\right)\right].$$

The derivation of (2.2) is given in Appendix A.1. As the sample size $n \to \infty$, the test statistic, under H_0, asymptotically follows chi-square distribution with one degree of freedom, i.e. $T_1 \dot{\sim} \chi^2(1)$. Then the corresponding p-value is given by

$$p = \Pr(T_1 > t|H_0).$$

2.2 Testing hypothesis for mean

Consider the reparameterized PIG(η, ϕ) distribution introduced in (1.12). The parameter η is exactly mean of the distribution. We would like to test whether the population mean equals to a specific value for one sample case and whether two independent samples have the same means.

2.2.1 One sample case

Let $X_1, \dots, X_n \overset{\text{iid}}{\sim} \text{PIG}(\eta, \phi)$ and $Y_{\text{obs}} = \{x_i\}_{i=1}^n$ denote the observed data. The null hypothesis and alternative hypothesis are

$$H_0 : \eta = \eta_0 \quad \text{against} \quad H_1 : \eta \neq \eta_0,$$

where η_0 is a pre-specified value. Using the likelihood ratio test, the test statistic is

$$T_2 = -2\left\{\ell_4(\eta_0, \hat{\phi}_{H_0}|Y_{\text{obs}}) - \ell_4(\hat{\eta}, \hat{\phi}|Y_{\text{obs}})\right\}, \tag{2.3}$$

where $\ell_4(\eta, \phi|Y_{\text{obs}})$ is defined by (2.6), $\hat{\phi}_{H_0}$ is the constrained MLE of ϕ under H_0, $(\hat{\eta}, \hat{\phi})^\top$ are the unconstrained MLEs of $(\eta, \phi)^\top$, which could be obtained from the MLEs of $\boldsymbol{\theta} = (\theta_1, \theta_2)^\top$ according to the invariance property of MLE, i.e.,

$$\hat{\eta} = \frac{\hat{\theta}_1}{\hat{\theta}_1 + \hat{\theta}_2} \quad \text{and} \quad \hat{\phi} = \hat{\theta}_1 + \hat{\theta}_2. \tag{2.4}$$

To calculate the constrained MLE $\hat{\phi}_{H_0}$, which is equal to $\hat{\theta}_{1,H_0} + \hat{\theta}_{2,H_0}$, we need to find the MLEs of $\boldsymbol{\theta}$ with constraint $\theta_1/(\theta_1+\theta_2) = \eta_0$. We have following MM algorithm

$$\theta_{1,H_0}^{(t+1)} = \frac{\tau + \sqrt{\tau^2 + 8na_4^{(t)}}}{2a_4^{(t)}} \quad \text{and} \quad \theta_{2,H_0}^{(t+1)} = \frac{1-\eta_0}{\eta_0}\theta_{1,H_0}^{(t+1)}, \tag{2.5}$$

where

$$\begin{aligned}
\tau &= \frac{n}{\eta_0}\log\left(\frac{1}{\eta_0}-1\right), \quad a_4^{(t)} = \sum_{i=1}^{n}\left(\frac{1}{x_i} + \frac{(1/\eta_0 - 1)^2}{1-x_i}\right) B_4(x_i, \theta_1^{(t)}), \\
B_4(x_i,\theta) &= \int_0^\infty \frac{g_4(s|x_i,\theta)}{s}\mathrm{d}s, \quad g_4(s|x_i,\theta) = \frac{h_4(s|x_i,\theta)}{\int_0^\infty h_4(t|x_i,\theta)\mathrm{d}t}, \\
h_4(s|x_i,\theta) &= s^{-2}\exp\left\{-\frac{1}{2}\left[s + \frac{1}{s}\left(\frac{\theta^2}{x_i} + \frac{\theta^2(1/\eta_0-1)^2}{1-x_i}\right)\right]\right\}.
\end{aligned}$$

The derivation of (2.5) is given in Appendix A.2. The log-likelihood function of η and ϕ based on the observed data is

$$\begin{aligned}
\ell_4(\eta,\phi|Y_{\text{obs}}) =& c_6 + n\phi + n\log[\phi^2\eta(1-\eta)] \\
&+ \sum_{i=1}^{n}\log\left[\int_0^\infty h_5(s|x_i,\eta,\phi)\mathrm{d}s\right],
\end{aligned} \tag{2.6}$$

where $c_6 = -n\log(2\pi) - \frac{3}{2}\sum_{i=1}^{n}\log[x_i(1-x_i)]$ is a constant free from η and ϕ, and

$$h_5(s|x_i,\eta,\phi) = s^{-2}\exp\left\{-\frac{1}{2}\left[s + \frac{\eta^2 - 2x_i\eta + x_i}{x_i(1-x_i)}\cdot\frac{\phi^2}{s}\right]\right\}.$$

The test statistic asymptotically follows $\chi^2(1)$ and the p-value is given by

$$p = \Pr(T_2 > t|H_0),$$

where $T_2 \dot{\sim} \chi^2(1)$, and t is the realization of the test statistic.

2.2.2 Two independent samples case

Let $X_{1i} \overset{\text{iid}}{\sim} \text{PIG}(\eta_1,\phi_1),\ i=1,\ldots,n_1,\ X_{2j} \overset{\text{iid}}{\sim} \text{PIG}(\eta_2,\phi_2),\ j=1,\ldots,n_2$, and the two samples are independent, where we assume that $\phi_1 = \phi_2 = \phi$. Let $Y_{\text{obs}} = \{x_{11},\ldots,x_{1n_1}, x_{21},\ldots,x_{2n_2}\}$ denote the observed data, where x_{ij} is the realization of X_{ij}. The null hypothesis and alternative hypothesis are

$$H_0: \eta_1 = \eta_2 \quad \text{against} \quad H_1: \eta_1 \neq \eta_2.$$

Using the likelihood ratio test, the test statistic is

$$T_3 = -2\left\{\ell_6(\hat{\eta}_{1,H_0}, \hat{\eta}_{1,H_0}, \hat{\phi}_{H_0}|Y_{\text{obs}}) - \ell_6(\hat{\eta}_1,\hat{\eta}_2,\hat{\phi}|Y_{\text{obs}})\right\}, \tag{2.7}$$

where $\ell_6(\eta_1, \eta_2, \phi|Y_{\text{obs}})$ is given by (2.8), $(\hat{\eta}_{1,H_0}, \hat{\phi}_{H_0})^\top$ are the constrained MLEs of $(\eta_1, \phi)^\top$ under H_0, while $(\hat{\eta}_1, \hat{\eta}_2, \hat{\phi})^\top$ are the unconstrained MLEs of $(\eta_1, \eta_2, \phi)^\top$. The log-likelihood function of η_1, η_2 and ϕ based on the observed data is

$$\begin{aligned}\ell_6(\eta_1, \eta_2, \phi|Y_{\text{obs}}) =& c_9 + (n_1 + n_2)[\phi + 2\log(\phi)] + \sum_{k=1}^{2} n_k \log[\eta_k(1-\eta_k)] \\ &+ \sum_{k=1}^{2}\sum_{i=1}^{n_k} \log\left[\int_0^\infty h_5(s|x_{ki}, \eta_k, \phi)\,\mathrm{d}s\right],\end{aligned} \tag{2.8}$$

where

$$c_9 = -(n_1+n_2)\log(2\pi) - \frac{3}{2}\sum_{k=1}^{2}\sum_{i=1}^{n_k}\log[x_{ki}(1-x_{ki})]$$

is a constant free from η_1, η_2 and ϕ.

Since the two PIG models have same parameters under H_0, we can merge the two samples into one sample with sample size n_1+n_2. So the constrained MLEs $(\hat{\eta}_{1,H_0}, \hat{\phi}_{H_0})^\top$ can be obtained by applying (2.4) from the MLEs of $\boldsymbol{\theta} = (\theta_1, \theta_2)^\top$ based on the entire observed data Y_{obs}. To obtain the unconstrained MLEs of $(\eta_1, \eta_2, \phi)^\top$, we could use the following MM algorithm

$$\begin{aligned}\eta_k^{(t+1)} =& \frac{\gamma_k^{(t)}}{3\cdot 2^{1/3}\alpha_k^{(t)}} - \frac{2^{1/3}(-\alpha_k^{2(t)} + \alpha_k^{(t)}\beta_k^{(t)} - 6n\alpha_k^{(t)} - \beta_k^{2(t)})}{3\alpha_k^{(t)}\gamma_k^{(t)}} \\ &+ \frac{\alpha_k^{(t)} + \beta_k^{(t)}}{3\alpha_k^{(t)}}, \quad \text{for } k = 1, 2, \\ \phi^{(t+1)} =& \frac{(n_1+n_2) + \sqrt{(n_1+n_2)^2 + 16(n_1+n_2)a_5^{(t)}}}{4a_5^{(t)}},\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}\alpha_k^{(t)} &= \sum_{i=1}^{n_k}\frac{\phi^{2(t)}B_5(x_{ki}, \eta_k^{(t)}, \phi^{(t)})}{x_{ki}(1-x_{ki})}, \quad \beta_k^{(t)} = \sum_{i=1}^{n_k}\frac{\phi^{2(t)}B_5(x_{ki}, \eta_k^{(t)}, \phi^{(t)})}{1-x_{ki}}, \\ \gamma_{k1}^{(t)} &= 2\alpha_k^{3(t)} - 3\alpha_k^{2(t)}\beta_k^{(t)} - 9n\alpha_k^{2(t)}, \quad \gamma_{k2}^{(t)} = -3\alpha_k^{(t)}\beta_k^{2(t)} + 18n\alpha_k^{(t)}\beta_k^{(t)} + 2\beta_k^{3(t)}, \\ \gamma_k^{(t)} &= \left\{\gamma_{k1}^{(t)} + \sqrt{4\left[-\alpha_k^{2(t)} + \alpha_k^{(t)}\beta_k^{(t)} - 6n\alpha_k^{(t)} - \beta_k^{2(t)}\right]^3 + (\gamma_{k1}^{(t)} + \gamma_{k2}^{(t)})^2} + \gamma_{k2}^{(t)}\right\}^{1/3}, \\ a_5^{(t)} &= \sum_{k=1}^{2}\sum_{i=1}^{n_k}\nu_{ki}^{(t)}\phi^2 B_5(x_{ki}, \eta_k^{(t)}, \phi^{(t)}), \ B_5(x_{ki}, \eta_k^{(t)}, \phi^{(t)}) = \int_0^\infty \frac{g_5(s|x_{ki}, \eta_k^{(t)}, \phi^{(t)})}{s}\,\mathrm{d}s, \\ \nu_{ki}^{(t)} &= \frac{\eta_k^{2(t)} - 2x_{ki}\eta_k^{(t)} + x_{ki}}{2x_{ki}(1-x_{ki})}, \quad g_5(s|x, \eta, \phi) = \frac{h_5(s|x, \eta, \phi)}{\int_0^\infty h_5(t|x, \eta, \phi)\,\mathrm{d}t}.\end{aligned}$$

The derivation of (2.9) is given in Appendix A.3. The test statistic, under the null hypothesis, asymptotically follows $\chi^2(1)$, i.e., $T_3 \dot{\sim} \chi^2(1)$. And the corresponding p-value is given by

$$p = \Pr(T_3 > t|H_0).$$

2.2.3 Testing for the equality of ϕ in two independent samples

Since we assumed the $\phi_1 = \phi_2 = \phi$ in section 2.2.2, it's necessary to check if it really holds. The null hypothesis and alternative hypothesis are

$$H_0 : \phi_1 = \phi_2 \quad \text{against} \quad H_1 : \phi_1 \neq \phi_2.$$

Applying the likelihood ratio test, the likelihood ratio statistic is

$$T_4 = -2\left\{\ell_4(\hat{\eta}_{1,H_0}, \hat{\eta}_{2,H_0}, \hat{\phi}_{H_0}|Y_{\text{obs}}) \quad - \ell_3(\hat{\eta}_1, \hat{\phi}_1|Y_{\text{obs}}) - \ell_3(\hat{\eta}_2, \hat{\phi}_2|Y_{\text{obs}})\right\}, \tag{2.10}$$

where $(\hat{\eta}_{1,H_0}, \hat{\eta}_{2,H_0}, \hat{\phi}_{H_0})^\top$ are the constrained MLEs of $(\eta_1, \eta_2, \phi)^\top$ under H_0, which can be obtained by the MM algorithm (2.9), and the unconstrained MLEs $(\hat{\eta}_1, \hat{\eta}_2, \hat{\phi}_1, \hat{\phi}_2)^\top$ could be easily obtained respectively from two samples. The test statistic asymptotically follows $\chi^2(1)$ and the p-value is given by

$$p = \Pr(T_4 > t|H_0),$$

where $T_4 \dot{\sim} \chi^2(1)$, and t is the realization of the test statistic.

2.3 Testing a single PIG regression coefficient

Suppose we want to test

$$H_0 : \alpha_i = 0 \quad \text{against} \quad H_1 : \alpha_i \neq 0,$$

where α_i is a regression coefficient of PIG regression model defined in (1.13), $i = 1, \ldots, p$. To use the likelihood ratio test, we need to fit two models: the full model and the reduced model under H_0. The likelihood ratio test statistic is

$$T_5 = -2\left\{\ell_2(\hat{\boldsymbol{\beta}}_{H_0}|Y_{\text{obs}}) - \ell_2(\hat{\boldsymbol{\beta}}|Y_{\text{obs}})\right\}, \tag{2.11}$$

where $\hat{\boldsymbol{\beta}}_{H_0}$ is the constrained MLEs of $\boldsymbol{\beta}$ under H_0, and $\hat{\boldsymbol{\beta}}$ is the unconstrained MLEs. The log-likelihood function is given by (1.14). As the sample size $n \to \infty$, the test statistic, under the null hypothesis, asymptotically follows $\chi^2(1)$, i.e. $T_5 \dot{\sim} \chi^2(1)$. Then the corresponding p-value is given by

$$p = \Pr\left(T_5 > t|H_0\right).$$

Applying this procedure on regression coefficients of each covariates, we could identify influential covariates by comparing the obtained p-values.

2.4 Test for significance of regression-overall adequacy

To determine the goodness of fit, we need a global test of model adequacy. The null hypothesis and alternative hypothesis are

$$H_0 : \alpha_1 = \ldots = \alpha_p = 0 \quad \text{against} \quad H_1 : \alpha_j \neq 0 \text{ for at least one } j.$$

Rejection of this null hypothesis implies that at least one of the regressors $z_{i1}, z_{i2}, \ldots, z_{ip}$ contributes significantly to the model.

It's natural to use likelihood ratio test here, with test statistic

$$T_6 = -2\left\{\ell_2(\hat{\boldsymbol{\beta}}_0|Y_{\text{obs}}) - \ell_2(\hat{\boldsymbol{\beta}}|Y_{\text{obs}})\right\}, \tag{2.12}$$

where $\hat{\boldsymbol{\beta}}_0$ is a vector with all coefficients being zero besides intercept's. The log-likelihood function is given by (1.14). The test statistic asymptotically follows $\chi^2(p)$, where p is the number of explanatory variables excluding intercept. The p-value of the test is

$$p = \Pr(T_6 > t|H_0),$$

where $T_6 \dot{\sim} \chi^2(p)$, and t is the realization of test statistic.

We can define a pseudo R^2 as an analogue of the coefficient of determination R^2 in liner regression, which measures how well the explanatory variables can explain the model, by

$$R_L^2 = \frac{\ell_2(\hat{\boldsymbol{\beta}}|Y_{\text{obs}}) - \ell_2(\hat{\boldsymbol{\beta}}_0|Y_{\text{obs}})}{\ell_(\hat{\boldsymbol{\beta}}_0|Y_{\text{obs}})}. \tag{2.13}$$

It represents the proportional increment of log-likelihood, form the model with only intercept to the full model. A pseudo R^2 which is close to one indicates the covariates could explain the model well. The pseudo R^2 allows us to draw comparisons between distinct PIG regression models.

3 Simulations

3.1 Generating data from PIG distribution

An efficient algorithm for generating random variates from inverse gaussian distribution IG(μ, λ) has been proposed by Michel et al.[6] First, generate a random variate from chi-square distribution with one degree of freedom, i.e. $\chi^2(1)$, by square a standard normal random variate.

$$z \sim N(0, 1), \quad y = z^2 \sim \chi^2(1),$$

Let

$$x = \mu + \frac{\mu^2 y}{2\lambda} - \frac{\mu}{2\lambda}\sqrt{4\mu\lambda y + \mu^2 y^2},$$

Generate another random variate from uniform distribution $\mathrm{U}(0,1)$, i.e.,

$$u \sim \mathrm{U}(0,1).$$

If $u \le \frac{\mu}{\mu+x}$ then return x else return $\frac{\mu^2}{x}$. Then, according to the SR (1.4) which defines the PIG distribution, we can easily generate random variates from PIG(θ_1, θ_2) distribution by following procedure: First generate two random variates,

$$Y_1 \sim \mathrm{IG}(\theta_1, \theta_1^2), \text{ and } Y_2 \sim \mathrm{IG}(\theta_2, \theta_2^2),$$

then return

$$X = \frac{Y_1}{Y_1 + Y_2}.$$

3.2 Calculating MLEs by MM algorithms

Liu & Tian [2] s' simulation studies showed that the differences between MLE of $\boldsymbol{\theta}$ and its true values become smaller in tendency as the sample size n increases. And the mean square error (MSEs) of the estimators $\hat{\boldsymbol{\theta}}$

$$\mathrm{MSE}(\hat{\boldsymbol{\theta}}) = \mathrm{E}\|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_2^2$$

also become smaller gradually as the sample size increases. With the help of R language, we can conduct the MM algorithm easily, with some basic built-in commands and few extra codes. It guarantees our further simulation study on hypothesis tests in the PIG model.

3.3 Testing on parameter difference

To evaluate the performance of the algorithm for testing hypothesis on parameter difference, we conduct some simulations. Consider the sample size $n = 30, 50, 100, 500$. The true values of parameters are set as $(\theta_1, \theta_2) = (5,5), (1,0.5)$. Using the algorithm in section 3.1, independently generate $\{x_i^k\}_{i=1}^n$ for $k = 1, 2 \ldots 10$ with parameters $(5,5)$ and $(1, 0.5)$. Then we conduct the tests on generated data and average the calculated test statistics and p-values. The results are reported in table 3.1.

From table 3.1, we have observed that the average p-values for symmetrical case are all greater than 0.1, indicating that we cannot reject the null hypothesis with significance level $\alpha = 0.05$. The average p-values for non-symmetrical case are all statistically significant, indicating strong evidences against the null hypothesis. There are no type I or type II errors occured in simulations. And the average p-values for non-symmetrical case become smaller in tendency as the sample size n increases.

Table 3.1: The average test statistic and average p-value of each simulation of testing on symmetry

n	θ_1	θ_2	A-LRT	Ap-value	θ_1	θ_2	A-LRT	Ap-value
30	5	5	1.00365	0.402789	1	0.5	11.11526	0.042894
50			1.269869	0.337374			22.12728	5.95E-05
100			1.377354	0.344165			38.24866	9.23E-09
500			0.627986	0.576512			221.0034	3.48E-40

3.3.1 Estimating power of the test

Power of a test is the probability of being able to detect a specified difference from a null hypothesis. Here I use Monte Carlo methods to estimate power of the test on parmater difference for PIG(θ_1, θ_2) distribution, with test size $\alpha = 0.05$. I repeat the test 1000 times on batches of size n. Procedure of our simulation is

procedure test.PIG.symmetry.power(θ_1, θ_2, n)
    power ← 0
    for i in 1:1000 **do**
        Y ← **rPIG**(n,θ_1, θ_2)
        p.value ← **test.PIG.symmetry**(Y)
        if p.value < 0.05 **then**
            power ← power + 1
        end if
    end for
    return power/1000
end procedure

where **rPIG** and **test.PIG.symmetry** are procedures for generating random variates from PIG distribution and conducting test on parameter difference.

I conduct the procedure with PIG$(\boldsymbol{\theta})$ distributions of three pairs of parameters and different batch sizes. The estimated powers are reported in table 3.2. From the result we observe that the power increases as the sample size grows. And greater the difference between two parameters, the larger the power.

Table 3.2: The estimated powers of tests with different parameters and sample size

n	$\boldsymbol{\theta}$	power	$\boldsymbol{\theta}$	power	$\boldsymbol{\theta}$	power
30	$(1, 0.5)^\top$	0.947	$(0.6, 0.5)^\top$	0.162	$(6, 5)^\top$	0.437
50		0.996		0.234		0.662
100		1		0.382		0.915

Table 3.3: The average test statistics and p-values of simulation of testing on mean

n	η_0	A-LRT	Ap-value	η_0	A-LRT	Ap-value
30		67.24819	2.39E-16		50.2693	0.001979
50	0.3	140.5696	2.00E-32	0.45	102.0867	1.86E-22
100		236.3922	2.93E-53		202.0562	7.90E-43
30		2.618474	0.355645		4.391624	0.216629
50	0.48	2.56266	0.314923	0.5	5.455161	0.115371
100		5.740138	0.045279		5.333357	0.091281

3.4 Testing for mean

Consider the sample size $n = 30, 50, 100$. Set the parameters of PIG(η, ϕ) distribution as $(0.5, 10)^\top$. Independently generate $\{x_i^{(k)}\}_{i=1}^n$ for $k = 1, \dots, 10$. Then we conduct the tests on generated data with several η_0s. The averaged test statistics and p-values are reported in table 3.3.

From table 3.3, we have observed that, fixing the sample sizes, the p-value becomes larger as the specified value η_0 getting closer to true mean $\eta = 0.5$. As the sample size growing, the p-value gets more significant. When sample size is small, there exists no type I error but some type II errors.

4 Conclusion

This thesis first reviewed the continuous proportional data and some related models, as well as the PIG distribution and its inference methods. Then, five hypothesis tests for PIG model with or without covariates are introduced. Some new inference methods are also proposed. Simulations are conducted to assess the performance and efficiency of these tests. Powers of tests were estimated by the Monte Carlo methods. When discussing the PIG regression model's overall adequacy, a pseudo R^2 is suggested to act as measure of explanatory power of the model. In fact, we could also construct hypothesis tests on multivariate version of the PIG distributions and the mixture distribution of PIG and Bernoulli distribution, which could capture zero and one observations.

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A Some technical derivations

A.1 MM algorithm for symmetrical PIG distribution

Let $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} \text{PIG}(\theta_1, \theta_2)$ and $Y_{\text{obs}} = \{x_1, \dots, x_n\}$, $\theta = \theta_1 = \theta_2$. The pdf of $X \sim \text{PIG}(\theta, \theta)$ can be rewritten from (1.5) as

$$f_{\text{PIG}}(x|\theta) = \frac{\theta^2 e^{2\theta}}{2\pi}[x(1-x)]^{-\frac{3}{2}} \int_0^\infty s^{-2} \exp\left[-\frac{1}{2}\left(s + \frac{\theta^2}{sx(1-x)}\right)\right] \text{d}s$$

for $0 < x < 1$. The log-likelihood function is given by

$$\ell_3(\theta|Y_{\text{obs}}) = c_4 + 2n[\theta + \log(\theta)] + \sum_{i=1}^n \log\left[\int_0^\infty h_3(s|x_i, \theta)\text{d}s\right], \tag{A.1}$$

where c_4 is a constant free from θ and

$$h_3(s|x_i, \theta) = s^{-2} \exp\left[-\frac{1}{2}\left(s + \frac{\theta^2}{sx_i(1-x_i)}\right)\right].$$

Based on $h_3(s|x_i, \theta)$, define a new density function with support $(0, \infty)$

$$g_3(s|x_i, \theta) \hat{=} \frac{h_3(s|x_i, \theta)}{\int_0^\infty h_3(t|x_i, \theta)\text{d}t}, \quad s > 0,$$

Therefore, we have

$$\begin{aligned}
\log\left[\int_0^\infty h_3(s|x_i, \theta)\text{d}s\right] &= \log\left[\int_0^\infty \frac{h_3(s|x_i, \theta)}{g_3(s|x_i, \theta^{(t)})} g_3(s|x_i, \theta^{(t)})\text{d}s\right] \\
&\geq \int_0^\infty \log\left[\frac{h_3(s|x_i, \theta)}{g_3(s|x_i, \theta^{(t)})}\right] g_3(s|x_i, \theta^{(t)})\text{d}s \\
&= c_{i1}^{(t)} + \int_0^\infty \log\left[h_3(s|x_i, \theta)\right] g_3(s|x_i, \theta^{(t)})\text{d}s \\
&= c_{i2}^{(t)} - \frac{1}{2}\int_0^\infty \left[s + \frac{\theta^2}{sx_i(1-x_i)}\right] g_3(s|x_i, \theta^{(t)})\text{d}s \\
&= c_{i3}^{(t)} - \frac{\theta^2}{2x_i(1-x_i)} \int_0^\infty \frac{g_3(s|x_i, \theta^{(t)})}{s}\text{d}s \\
&= c_{i3}^{(t)} - \frac{\theta^2}{2x_i(1-x_i)} B_3(x_i, \theta^{(t)}),
\end{aligned} \tag{A.2}$$

where $\left\{c_{ik}^{(t)}\right\}_{k=1}^3$ are constants not depending on θ, and

$$B_3(x_i, \theta^{(t)}) = \int_0^\infty \frac{g_3(s|x_i, \theta^{(t)})}{s}\text{d}s. \tag{A.3}$$

Then we can construct the surrogate function from (A.1) and (A.2) as

$$Q_2(\theta|\theta^{(t)}) = c_5 + 2n[\theta + \log(\theta)] - \frac{1}{2}\sum_{i=1}^{n} \frac{\theta^2}{x_i(1-x_i)} B_3(x_i, \theta^{(t)}), \tag{A.4}$$

where c_5 is a constant. Obviously, $Q_2(\theta|\theta^{(t)})$ satisfies:

$$\begin{aligned} Q_2(\theta|\theta^{(t)}) &\leqslant \ell_3(\theta|Y_{\text{obs}}), \quad \forall \theta, \theta^{(t)} \in \mathbb{R}_+, \\ Q_2(\theta^{(t)}|\theta^{(t)}) &= \ell_5(\theta^{(t)}|Y_{\text{obs}}). \end{aligned}$$

That is, $Q_2(\theta|\theta^{(t)})$ minorizes $\ell_3(\theta|Y_{\text{obs}})$ at $\theta = \theta^{(t)}$. By the MM principle, given $\theta^{(t)}$ the $(t+1)$-th approximation of $\hat{\theta}$ is updated by:

$$\theta^{(t+1)} = \arg\max_{\boldsymbol{\theta} \in \mathbb{R}_+} Q_2\left(\theta|\theta^{(t)}\right).$$

Let

$$\frac{\mathrm{d}Q_2(\theta|\theta^{(t)})}{\mathrm{d}\theta} = 2n + \frac{2n}{\theta} - \theta \sum_{i=1}^{n} \frac{B_3(x_i, \theta^{(t)})}{x_i(1-x_i)} = 0,$$

we obtain the following MM iteration:

$$\theta_1^{(t+1)} = \frac{n + \sqrt{n^2 + 2na_3^{(t)}}}{a_3^{(t)}}, \tag{A.5}$$

where

$$a_3^{(t)} = \sum_{i=1}^{n} \frac{B_3(x_i, \theta^{(t)})}{x_i(1-x_i)},$$

and $B_3(x_i, \theta^{(t)})$ is given by (A.3).

A.2 MM algorithm for the PIG model with linear constraint

Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{PIG}(\theta_1, \theta_2)$, where $\theta_1/(\theta_1 + \theta_2) = \eta_0$, i.e., $\theta_2 = \theta_1(\eta_0^{-1} - 1)$, and $Y_{\text{obs}} = \{x_i\}_{i=1}^n$. The pdf of $X \sim \text{PIG}(\theta_1, \theta_1(\eta_0^{-1} - 1))$ can be rewritten from (1.5) as

$$f_{\text{PIG}}(x|\theta_1) = c_1(\theta_1)[x(1-x)]^{-\frac{3}{2}} \int_0^\infty s^{-2} \exp\left[-\frac{s + b_1(x;\theta_1)/s}{2}\right] \mathrm{d}s, \tag{A.6}$$

$0 < s < 1$, where

$$c_1(\theta_1) = \frac{\theta_1^2 e^{\theta_1/\eta_0}}{2\pi}\left(\frac{1}{\eta_0} - 1\right), \quad b_1(x;\theta_1) = \frac{\theta_1^2}{x} + \frac{\theta_1^2(1/\eta_0 - 1)^2}{1-x}.$$

The log-likelihood function is

$$\ell_5(\theta_1|Y_{\text{obs}}) = c_7 + 2n\log(\theta_1) + \frac{n\theta_1}{\eta_0}\log\left(\frac{1}{\eta_0} - 1\right) + \sum_{i=1}^{n} \log\left[\int_0^\infty h_4(s|x_i, \theta_1)\mathrm{d}s\right], \tag{A.7}$$

where c_7 is a constant free from θ_1 and

$$h_4(s|x_i,\theta_1) = s^{-2}\exp\left\{-\frac{1}{2}\left[s+\frac{1}{s}\left(\frac{\theta_1^2}{x_i}+\frac{\theta_1^2(1/\eta_0-1)^2}{1-x_i}\right)\right]\right\}.$$

Based on $h_4(s|x_i,\theta_1)$, we can define a density function $g_4(s|x_i,\theta_1)$ with support $(0,\infty)$,

$$g_4(s|x_i,\theta_1)\hat{=}\frac{h_4(s|x_i,\theta_1)}{\int_0^\infty h_4(t|x_i,\theta_1)\mathrm{d}t}, \quad s>0.$$

Therefore we have

$$\begin{aligned}
\log\left[\int_0^\infty h_4(s|x_i,\theta_1)\mathrm{d}s\right] &= \log\left[\int_0^\infty \frac{h_4(s|x_i,\theta_1)}{g_4(s|x_i,\theta_1^{(t)})}g_4(s|x_i,\theta_1^{(t)})\mathrm{d}s\right] \\
&\geq \int_0^\infty \log\left[\frac{h_4(s|x_i,\theta_1)}{g_4(s|x_i,\theta_1^{(t)})}\right] g_4(s|x_i,\theta_1^{(t)})\mathrm{d}s \\
&= c_{i4}^{(t)} + \int_0^\infty \log\left[h_4(s|x_i,\theta_1)\right] g_4(s|x_i,\theta_1^{(t)})\mathrm{d}s \\
&= c_{i5}^{(t)} - \frac{1}{2}\int_0^\infty \left[s+\frac{1}{s}\left(\frac{\theta_1^2}{x_i}+\frac{\theta_1^2(1/\eta_0-1)^2}{1-x_i}\right)\right] g_4(s|x_i,\theta_1^{(t)})\mathrm{d}s \\
&= c_{i6}^{(t)} - \frac{1}{2}\left(\frac{\theta_1^2}{x_i}+\frac{\theta_1^2(1/\eta_0-1)^2}{1-x_i}\right)\int_0^\infty \frac{g_4(s|x_i,\theta_1^{(t)})}{s}\mathrm{d}s \\
&= c_{i6}^{(t)} - \frac{1}{2}\left(\frac{\theta_1^2}{x_i}+\frac{\theta_1^2(1/\eta_0-1)^2}{1-x_i}\right) B_4(x_i,\theta_1^{(t)}),
\end{aligned} \tag{A.8}$$

where $\left\{c_{ik}^{(t)}\right\}_{k=4}^{6}$ are constants not depending on θ_1, and

$$B_4(x_i,\theta_1^{(t)}) = \int_0^\infty \frac{g_4(s|x_i,\theta_1^{(t)})}{s}\mathrm{d}s. \tag{A.9}$$

Then we can construct the surrogate function from (A.7) and (A.8)

$$\begin{aligned}
Q_3(\theta_1|\theta_1^{(t)}) =& c_8 + 2n\log(\theta_1) + \frac{n\theta_1}{\eta_0}\log\left(\frac{1}{\eta_0}-1\right) \\
&- \frac{\theta_1^2}{2}\sum_{i=1}^n\left(\frac{1}{x_i}+\frac{(1/\eta_0-1)^2}{1-x_i}\right) B_4(x_i,\theta_1^{(t)}),
\end{aligned} \tag{A.10}$$

where c_8 is a constant. It minorizes $\ell_5(\theta_1|Y_{\text{obs}})$ at $\theta_1=\theta_1^{(t)}$. So given $\theta_1^{(t)}$, the $(t+1)$-th approximation of $\hat{\theta}_1$ is updated by:

$$\theta_1^{(t+1)} = \arg\max_{\theta_1\in\mathbb{R}_+} Q_3(\theta_1|\theta_1^{(t)}).$$

Let

$$\frac{\mathrm{d}Q_3(\theta_1|\theta_1^{(t)})}{\mathrm{d}\theta_1} = \frac{2n}{\theta_1} + \frac{n\log(1/\eta_0-1)}{\eta_0} - \theta_1\sum_{i=1}^n\left(\frac{1}{x_i}+\frac{(1/\eta_0-1)^2}{1-x_i}\right) B_4(x_i,\theta_1^{(t)}) = 0,$$

we obtain the following MM iteration:

$$\theta_1^{(t+1)} = \frac{\tau + \sqrt{\tau^2 + 8na_4^{(t)}}}{2a_4^{(t)}}, \quad \theta_2^{(t+1)} = \frac{1-\eta_0}{\eta_0}\theta_1^{(t+1)}, \tag{A.11}$$

where

$$\tau = \frac{n\log(1/\eta_0 - 1)}{\eta_0}, \quad a_4^{(t)} = \sum_{i=1}^{n}\left(\frac{1}{x_i} + \frac{(1/\eta_0 - 1)^2}{1-x_i}\right) B_4(x_i, \theta_1^{(t)})$$

and $B_4(x_i, \theta_1^{(t)})$ is defined in (A.9).

A.3 MM algorithm for the PIG model of two independent samples

Let $X_{1i} \overset{\text{iid}}{\sim} \text{PIG}(\eta_1, \phi_1),\ i = 1, \ldots, n_1$, $X_{2j} \overset{\text{iid}}{\sim} \text{PIG}(\eta_2, \phi_2),\ j = 1, \ldots, n_2$, and two samples are independent, and $\phi_1 = \phi_2 = \phi$. Let $Y_{\text{obs}} = \{x_{11}, \ldots, x_{1n_1}, x_{21}, \ldots, x_{2n_2}\}$ denote the observed data. The log-likelihood function of $(\eta_1, \eta_2, \phi)^\top$, based on the observed data, is

$$\begin{aligned}
\ell_6(\eta_1, \eta_2, \phi|Y_{\text{obs}}) =& c_9 + (n_1+n_2)[\phi + 2\log(\phi)] + \sum_{k=1}^{2} n_k \log[\eta_k(1-\eta_k)] \\
&+ \sum_{k=1}^{2}\sum_{i=1}^{n_k} \log\left[\int_0^\infty h_5(s|x_{ki}, \eta_k, \phi)\,\mathrm{d}s\right],
\end{aligned} \tag{A.12}$$

where c_9 is a constant and

$$h_5(s|x, \eta, \phi) = s^{-2}\exp\left\{-\frac{1}{2}\left[s + \frac{\eta^2 - 2x\eta + x}{x(1-x)} \cdot \frac{\phi^2}{s}\right]\right\}.$$

Define a density function based on $h_5(s|x, \eta, \phi)$ on $(0, \infty)$,

$$g_5(s|x, \eta, \phi) \hat{=} \frac{h_5(s|x, \eta, \phi)}{\int_0^\infty h_5(t|x, \eta, \phi)\,\mathrm{d}t}, \quad s > 0.$$

Therefore,

$$\begin{aligned}
\log\left[\int_0^\infty h_5(s|x_{ki}, \eta_k, \phi)\,\mathrm{d}s\right] &= \log\left[\int_0^\infty \frac{h_5(s|x_{ki}, \eta_k, \phi)}{g_5(s|x_{ki}, \eta_k^{(t)}, \phi^{(t)})} g_5(s|x_{ki}, \eta_k^{(t)}, \phi^{(t)})\,\mathrm{d}s\right] \\
&\geqslant \int_0^\infty \log\left[\frac{h_5(s|x_{ki}, \eta_k, \phi)}{g_5(s|x_{ki}, \eta_k^{(t)}, \phi^{(t)})}\right] g_5(s|x_{ki}, \eta_k^{(t)}, \phi^{(t)})\,\mathrm{d}s \\
&= c_{ik7}^{(t)} + \int_0^\infty \log\left[h_5(s|x_{ki}, \eta_k, \phi)\right] g_5(s|x_{ki}, \eta_k^{(t)}, \phi^{(t)})\,\mathrm{d}s \\
&= c_{ik8}^{(t)} - \frac{1}{2}\int_0^\infty \left[s + \frac{\eta_k^2 - 2x_{ki}\eta_k + x_{ki}}{x_{ki}(1-x_{ki})} \cdot \frac{\phi^2}{s}\right] g_5(s|x_{ki}, \eta_k^{(t)}, \phi^{(t)})\,\mathrm{d}s \\
&= c_{ik9}^{(t)} - \frac{\eta_k^2 - 2x_{ki}\eta_k + x_{ki}}{2x_{ki}(1-x_{ki})}\phi^2 \int_0^\infty \frac{g_5(s|x_{ki}, \eta_k^{(t)}, \phi^{(t)})}{s}\,\mathrm{d}s \\
&= c_{ik9}^{(t)} - \nu_{ki}\phi^2 B_5(x_{ki}, \eta_k^{(t)}, \phi^{(t)}),
\end{aligned} \tag{A.13}$$

for $k = 1, 2$ and $i = 1, \cdots, n_k$, where $\{c_{ikl}^{(t)}\}_{l=1}^{3}$ are constants and

$$\nu_{ki} = \frac{\eta_k^2 - 2x_{ki}\eta_k + x_{ki}}{2x_{ki}(1 - x_{ki})}, \quad B_5(x_{ki}, \eta_k^{(t)}, \phi^{(t)}) = \int_0^\infty \frac{g_5(s|x_{ki}, \eta_k^{(t)}, \phi^{(t)})}{s} \, \mathrm{d}s.$$

Then we can construct the surrogate function from (A.12) and (A.13)

$$\begin{aligned} Q_4(\eta_1, \eta_2, \phi|\eta_1^{(t)}, \eta_2^{(t)}, \phi^{(t)}) =& c_{10} + (n_1 + n_2)[\phi + 2\log(\phi)] + \sum_{k=1}^{2} n_k \log[\eta_k(1 - \eta_k)] \\ & - \sum_{k=1}^{2}\sum_{i=1}^{n_k} \nu_{ki}\phi^2 B_5(x_{ki}, \eta_k^{(t)}, \phi^{(t)}), \end{aligned}$$

where c_{10} is a constant free from $(\eta_1, \eta_2, \phi)^\top$. It minorizes $\ell_6(\eta_1, \eta_2, \phi|Y_{\text{obs}})$ at $(\eta_1, \eta_2, \phi)^\top = (\eta_1^{(t)}, \eta_2^{(t)}, \phi^{(t)})^\top$. So, given $(\eta_1^{(t)}, \eta_2^{(t)}, \phi^{(t)})^\top$, the $(t+1)$-th approximations of $(\hat{\eta}_1, \hat{\eta}_2, \hat{\phi})^\top$ are updated by:

$$(\eta_1^{(t+1)}, \eta_2^{(t+1)}, \phi^{(t+1)})^\top = \arg\max Q_4(\eta_1, \eta_2, \phi|\eta_1^{(t)}, \eta_2^{(t)}, \phi^{(t)}),$$

subject to $\eta_1 \in (0, 1), \eta_2 \in (0, 1)$ and $\phi \in \mathbb{R}_+$. Let

$$\begin{aligned} \left.\frac{\partial Q_4}{\partial \eta_k}\right|_{\phi=\phi^{(t)}} =& \frac{n_k}{\eta_k} - \frac{n_k}{1 - \eta_k} - \eta_k \sum_{i=1}^{n_k} \frac{\phi^{2(t)}}{x_{ki}(1 - x_{ki})} B_5(x_{ki}, \eta_k^{(t)}, \phi^{(t)}) \\ & + \sum_{i=1}^{n_k} \frac{\phi^{2(t)}}{1 - x_{ki}} B_5(x_{ki}, \eta_k^{(t)}, \phi^{(t)}) \\ =& 0, \quad \text{for } k = 1, 2, \\ \left.\frac{\partial Q_4}{\partial \phi}\right|_{\eta_1=\eta_1^{(t)}, \eta_2=\eta_2^{(t)}} =& (n_1 + n_2)\left(1 + \frac{2}{\phi}\right) - 2\phi \sum_{k=1}^{2}\sum_{i=1}^{n_k} \nu_{ki}^{(t)} B_5(x_{ki}, \eta_k^{(t+1)}, \phi^{(t)}) \\ =& 0, \end{aligned}$$

Solving these equations we obtain the following MM iteration:

$$\begin{aligned} \eta_k^{(t+1)} =& \frac{\gamma_k^{(t)}}{3 \cdot 2^{1/3}\alpha_k^{(t)}} - \frac{2^{1/3}(-\alpha_k^{2(t)} + \alpha_k^{(t)}\beta_k^{(t)} - 6n\alpha_k^{(t)} - \beta_k^{2(t)})}{3\alpha_k^{(t)}\gamma_k^{(t)}} \\ & + \frac{\alpha_k^{(t)} + \beta_k^{(t)}}{3\alpha_k^{(t)}}, \quad \text{for } k = 1, 2 \end{aligned}$$

$$\phi^{(t+1)} = \frac{(n_1 + n_2) + \sqrt{(n_1 + n_2)^2 + 16(n_1 + n_2)a_5^{(t)}}}{4a_5^{(t)}}, \tag{A.14}$$

where

$$\alpha_k^{(t)} = \sum_{i=1}^{n_k} \frac{\phi^{2(t)} B_5(x_{ki}, \eta_k^{(t)}, \phi^{(t)})}{x_{ki}(1-x_{ki})}, \quad \beta_k^{(t)} = \sum_{i=1}^{n_k} \frac{\phi^{2(t)} B_5(x_{ki}, \eta_k^{(t)}, \phi^{(t)})}{1-x_{ki}},$$

$$\gamma_{k1}^{(t)} = 2\alpha_k^{3(t)} - 3\alpha_k^{2(t)}\beta_k^{(t)} - 9n\alpha_k^{2(t)}, \quad \gamma_{k2}^{(t)} = -3\alpha_k^{(t)}\beta_k^{2(t)} + 18n\alpha_k^{(t)}\beta_k^{(t)} + 2\beta_k^{3(t)},$$

$$\gamma_k^{(t)} = \left\{ \gamma_{k1}^{(t)} + \sqrt{4\left[-\alpha_k^{2(t)} + \alpha_k^{(t)}\beta_k^{(t)} - 6n\alpha_k^{(t)} - \beta_k^{2(t)}\right]^3 + (\gamma_{k1}^{(t)} + \gamma_{k2}^{(t)})^2} + \gamma_{k2}^{(t)} \right\}^{1/3},$$

$$a_5^{(t)} = \sum_{k=1}^{2} \sum_{i=1}^{n_k} \nu_{ki}^{(t)} \phi^2 B_5(x_{ki}, \eta_k^{(t)}, \phi^{(t)}), \quad \nu_{ki}^{(t)} = \frac{\eta_k^{2(t)} - 2x_{ki}\eta_k^{(t)} + x_{ki}}{2x_{ki}(1-x_{ki})}.$$

B Programs

```
library(cubature)

# Generate Data from IG distribution
rIG = function(n, mu, lambda){
    v = rnorm(n, 0, 1)
    y = v^2
    x = mu + mu^2*y/(2*lambda) - mu/(2*lambda) *
        sqrt(4*mu*lambda*y+mu^2*y^2)
    z = runif(n, 0, 1)
    res = ifelse(z <= mu/(mu+x), x, mu^2/x)
    return(res)
}

# Generate Data from PIG distribution
rPIG = function(n, theta1, theta2){
    Y1 = rIG(n, theta1, theta1^2)
    Y2 = rIG(n, theta2, theta2^2)
    X = Y1/(Y1+Y2)
    return(X)
}

# Generate Data from reparmeterized PIG distribution
rPIG.rp = function(n, eta, phi){
  t1 = eta*phi
  t2 = (1-eta)*phi
  return(rPIG(n, t1, t2))
}

# Probability density function of PIG
dPIG = function(x, theta1, theta2){
    if(sum(x <= 0)+sum(x >= 1) > 0){
        stop("variable should be in (0,1)")
        return(FALSE)
    }
    t1 = theta1
    t2 = theta2
    c = t1*t2/(2*pi) * exp(t1+t2)
    b = t1^2/x+t2^2/(1-x)
    f = (c* (x*(1-x))^(-1.5) * 2*besselK(sqrt(b), 1, FALSE)) / sqrt(b)
    return(f)
}

# Log-likelihood function of PIG distribution
logLik.PIG = function(X, theta_1, theta_2){
    n = length(X)
    theta = c(theta_1, theta_2)
    int.h = sapply(1:n, function(i) integrate(
    function(s) s^(-2) * exp(-0.5*(s+(theta[1]^2/X[i] +
    theta[2]^2/(1-X[i]))/s)), lower=0, upper=Inf)$value)
    logLik = n*sum(log(theta)+theta) + sum(log(int.h))
```

```
    return(logLik)
}

# MM algorithm for estimating MLEs of PIG model
MM.MLE.PIG = function(X, debug=TRUE){
    max.iters = 300
    precision = 0.0005

    theta = matrix(rep(0,2*max.iters), ncol=2)

    # initial value
    n = length(X)
    t10 = runif(1,1,5)
    t20 = mean(X)/t10 - t10 #runif(1,1,5)

    theta[1,1] = t10
    theta[1,2] = ifelse(t20 > 0, t20, runif(1,1,5))

    iters = 1
    stop.indc = 1
    if(debug){
        cat("iter:", iters, "t1:", theta[iters,1], "t2:", theta[iters,2], "\n")
    }
    while(stop.indc > precision && iters < max.iters){
        int.h = sapply(1:n, function(i) integrate(
        function(s) s^(-2) * exp(-0.5*(s+(theta[iters,1]^2/X[i] + theta[iters,2]^2/
        (1-X[i]))/s)),lower=0, upper=Inf)$value)
        Bi = sapply(1:n, function(i) integrate(
        function(s) 1/(s*int.h[i]) * s^(-2) * exp(-0.5*(s+(theta[iters,1]^2/
        X[i] +  theta[iters,2]^2/(1-X[i]))/s)),lower=0, upper=Inf)$value)
        a = c(sum(Bi/X), sum(Bi/(1-X)))
        theta[iters+1,1] = (n+sqrt(n^2+4*n*a[1])) / (2*a[1])
        theta[iters+1,2] = (n+sqrt(n^2+4*n*a[2])) / (2*a[2])
        iters = iters+1
        stop.indc = sqrt((theta[iters,1]-theta[iters-1,1])^2+
            (theta[iters,2]-theta[iters-1,2])^2)
    }
    if(debug){
        cat(c(iters, "iterations;", "stop.indc:", stop.indc, "\n"))
        cat(c("theta1:", theta[iters-1,1], "theta2:", theta[iters-1,2], "\n"))
    }
    return(theta[iters-1,1:2])
}

# MM algorithm for estimating MLE of symmetrical PIG model
MM.MLE.PIG.S = function(X, debug=TRUE){
    max.iters = 300
    precision = 0.0005

    theta = rep(0,max.iters)

    # initial value
    n = length(X)

    theta[1] = runif(1,1,5)

    iters = 1
    stop.indc = 1
```

```r
    if(debug){
        cat("iter:", iters, "theta:", theta[iters], "\n")
    }
    while(stop.indc > precision && iters < max.iters){
        int.h = sapply(1:n, function(i) integrate(
        function(s) s^(-2) * exp(-0.5*(s+(theta[iters]^2/(s*X[i]*(1-X[i]))))),
        lower=0, upper=Inf)$value)
        Bi = sapply(1:n, function(i) integrate(
        function(s) 1/(s*int.h[i]) * s^(-2) * exp(-0.5*(s+(theta[iters]^2/
        (s*X[i]*(1-X[i])) ))), lower=0, upper=Inf)$value)
        a = sum(Bi/(X*(1-X)))
        theta[iters+1] = (n+sqrt(n^2+2*n*a)) / a
        iters = iters+1
        stop.indc = sqrt((theta[iters]-theta[iters-1])^2)
    }
    if(debug){
        cat(c(iters, "iterations;", "stop.indc:", stop.indc, "\n"))
        cat(c("theta:", theta[iters-1],"\n"))
    }
    return(theta[iters-1])
}

# MM algorithm for estimating MLEs of PIG model with linear constraints
MM.MLE.PIG.lincons = function(X, eta0, debug=FALSE){
  max.iters = 300
  precision = 0.0005

  theta = rep(0,max.iters)

  # initial value
  n = length(X)
  theta[1] = runif(1, 1, 10)
  tau = n/eta0*log(1/eta0-1)
  iters = 1
  stop.indc = 1
  if(debug){
    cat("iter:", iters, "phi:", phi[iters], "\n")
  }
  while(stop.indc > precision && iters < max.iters){
    int.h4 = sapply(1:n, function(i) cubintegrate(
      function(s) s^(-2)*exp(-0.5*(s+(1/s)*(theta[iters]^2/X[i] +theta[iters]^2*
      (1/eta0-1)^2/(1-X[i])))), lower=0, upper=Inf)$integral)
    B4 = sapply(1:n, function(i) cubintegrate(
      function(s) s^(-3)*exp(-0.5*(s+(1/s)*(theta[iters]^2/X[i] +theta[iters]^2*
      (1/eta0-1)^2/(1-X[i])))) / int.h4[i], lower=0, upper=Inf)$integral)
    a4 = sum(B4)
    theta[iters+1] = (tau+sqrt(tau^2+8*n*a4))/(2*a4)
    iters = iters+1
    stop.indc = abs(theta[iters]-theta[iters-1])
    stop.indc = ifelse(is.nan(stop.indc), 0.1, stop.indc)
    # cat(c(iters, "iterations;", "stop.indc:", stop.indc, "\n"))
  }
  if(debug){
    cat(c(iters, "iterations;", "stop.indc:", stop.indc, "\n"))
    cat(c("theta1:", theta[iters-1], "theta2", (1-eta0)/eta0*theta[iters-1], "\n"))
  }
  return(c(theta[iters-1], (1-eta0)/eta0 * theta[iters-1]))
}
```

```r
# Testing hypothesis on parameter difference
test.symmetry.PIG = function(X, debug=TRUE){
    theta.uncon = MM.MLE.PIG(X, debug=FALSE)
    theta.con = MM.MLE.PIG.S(X, debug=FALSE)
    LRT = -2*(logLik.PIG(X,theta.con, theta.con) -
    logLik.PIG(X,theta.uncon[1], theta.uncon[2]))
    p.value = pchisq(LRT, 1, lower.tail = FALSE)
    if(debug){
        cat("Test statistic:", LRT, "P-value", p.value,"\n")
    }
    return(list(LRT=LRT, p.value=p.value))
}

# Estimating the power of the test on parameter difference
test.symmetry.PIG.power = function(t1, t2, rept=500, n = 100, alpha=0.05){
  power = 0
  for(i in 1:rept){
    x = rPIG(n, t1, t2)
    pv = test.symmetry.PIG(x, debug=FALSE)$p.value
    power = power + ifelse(pv < alpha, 1, 0)
  }
  return(power/rept)
}

# Testing hypothesis on mean
test.mean.PIG = function(X, eta0, debug=FALSE){
  n = length(X)
  theta.uncon = MM.MLE.PIG(X, debug=FALSE)
  eta.uncon = theta.uncon[1]/sum(theta.uncon)
  phi.uncon = sum(theta.uncon)
  theta.con = MM.MLE.PIG.lincons(X, eta0)
  eta.con = eta0
  phi.con = sum(theta.con)

  LRT = -2*(logLik.PIG.repar(X, eta.con, phi.con) -
            logLik.PIG.repar(X, eta.uncon, phi.uncon))
  p.value = pchisq(LRT, 1, lower.tail = FALSE)
  if(debug){
    cat("LRT:", LRT, "p-value", p.value, "\n")
  }
  return(list(LRT = LRT, p.value=p.value))
}
```