Draft: 6 Oct 2006

A Cosmology Calculator for the World Wide Web

E. L. Wright[1],

wright@astro.ucla.edu

ABSTRACT

A cosmology calculator that computes times and distances as a function of redshift for user-defined cosmological parameters is available at http://www.astro.ucla.edu/∼wright/CosmoCalc.html. This note gives the formulae used by the cosmology calculator and discusses some of its implementation. A version of the calculator that allows one to specify the equation of state parameter w and w' is at ACC.html, and a version for converting the light travel times usually given in the popular press into redshifts is at DlttCalc.html.

Subject headings: cosmology: miscellaneous – methods: miscellaneous

1. INTRODUCTION

There are many definitions of distances in cosmology, and it is often frustrating to calculate the kind of distance needed for a given situation. In particular, press releases (Clavin, 2006) very often use the *light travel time distance*, $D_{ltt} = c(t_\circ - t_{em})$, even though this distance has very many undesirable characteristics. I wrote the cosmology calculator in order to be able to calculate D_{ltt} along with the observable angular and luminosity distances D_A and D_L. It also computes the proper radial distance which is the only distance that is compatible with the common definition of distance for moving objects: the spatial separation at a common time. The common time used is now, or $t_\circ$. This it is called D_{now} below. D_{now}, also known as the comoving radial distance, is not measurable from a single position, but it can in principle be measured by a family of comoving observers. It is the one distance that satisfies the Hubble Law exactly: $dD/dt = H(t)D$ without any corrections. Other distances are defined using quantities taken from more than one time and generally only satisfy the Hubble Law to first order.

[1]UCLA Astronomy, PO Box 951547, Los Angeles CA 90095-1547, USA

2. EQUATIONS

The following results have been given many times before. Hogg (1999) gives equations equivalent to the ones in this paper. But for completeness in documenting the cosmology calculator the equations are spelled out below.

The metric is given by

$$ds^2 = c^2dt^2 - a(t)^2 R_\circ^2(d\psi^2 + S^2(\psi)[d\theta^2 + \sin^2\theta d\phi^2]) \tag{1}$$

where $S(x)$ is $\sinh(x)$ if $\Omega_{tot} < 1$ and $\sin(x)$ for $\Omega_{tot} > 1$. The radius of curvature is given by $R_\circ = (c/H_\circ)/\sqrt{|1-\Omega_{tot}|}$ since I use the normalization that $a = 1$ at the current time so the redshift is given by $1 + z = 1/a$. The past light cone is given by $cdt = a(t)R_\circ d\psi$ so the comoving radial distance is

$$D_{now} = R_\circ\psi = \int \frac{cdt}{a} = \int_{1/(1+z)}^{1} \frac{cda}{a\dot{a}} \tag{2}$$

and of course the light travel time distance is given by

$$D_{ltt} = \int cdt = \int_{1/(1+z)}^{1} \frac{cda}{\dot{a}} = c(t_\circ - t(z)) \tag{3}$$

The exact dynamics of the Universe can be captured using the the energy equation from the Newtonian approximation to cosmology, which gives

$$\frac{\dot{a}^2}{2} - \frac{GM(<a)}{a} = \text{const.} \tag{4}$$

GR modifies the acceleration equation by including the pressure as a source of gravity, but this just cancels the variation of the enclosed mass $M(<a)$ caused by the pressure, so the energy equation is exact. So we can write $\dot{a}$ as $H_\circ\sqrt{X}$ with

$$X(a) = \Omega_m/a + \Omega_r/a^2 + \Omega_v a^2 + (1 - \Omega_{tot}) \approx 1 + 2q_\circ z + \ldots \tag{5}$$

where $q_\circ$ is the deceleration parameter. Let us define a quantity Z that is the comoving distance divided by the Hubble radius $c/H_\circ$. Then

$$Z = \int_{1/(1+z)}^{1} \frac{da}{a\sqrt{X}} = \int_0^z \frac{dz}{(1+z)\sqrt{X}} \approx z - z^2(1+q_\circ)/2 + \ldots \tag{6}$$

To first order this agrees with the redshift, so $Z = z + \ldots$ This definition gives $D_{now} = (cZ/H_\circ)$ and $D_{ltt} = (c/H_\circ)\int_{1/(1+z)}^{1} da/\sqrt{X}$. Then the angular size distance, which is defined as $D_A = R/\theta$ where R is the transverse proper length of an object that subtends an angle θ on the sky, is given by:

$$\begin{aligned}
D_A &= \frac{c}{H_\circ}\frac{S(\sqrt{|1-\Omega_{tot}|}Z)}{(1+z)\sqrt{|1-\Omega_{tot}|}} \\
&= \frac{D_{now}}{(1+z)}\left(1+\frac{1}{6}(1-\Omega_{tot})Z^2+\frac{1}{120}(1-\Omega_{tot})^2Z^4+\ldots\right) \qquad (7)
\end{aligned}$$

We can define a function $J(x)$ given by

$$J(x) = \begin{cases} \sin(\sqrt{-x})/\sqrt{-x}, & x<0; \\ \sinh(\sqrt{x})/\sqrt{x}, & x>0; \\ 1+x/6+x^2/120+\ldots+x^n/(2n+1)!+\ldots, & x\approx 0. \end{cases} \qquad (8)$$

Then

$$\begin{aligned}
D_{ltt} &= \frac{c}{H_\circ}\int_{1/(1+z)}^{1}\frac{da}{\sqrt{X}} \\
D_{now} &= \frac{c}{H_\circ}\int_{1/(1+z)}^{1}\frac{da}{a\sqrt{X}} = \frac{cZ}{H_\circ} \\
D_A &= D_{now}\frac{J([1-\Omega_{tot}]Z^2)}{1+z} \\
D_L &= (1+z)^2 D_A \qquad (9)
\end{aligned}$$

Note that the luminosity distance D_L is defined to make the inverse square law work perfectly for bolometric fluxes, so that $F_{bol} = L/(4\pi D_L^2)$ for an object of luminosity L.

The enclosed volume calculation requires the integral of either $\sin^2\psi d\psi$ or $\sinh^2\psi d\psi$. The closed case becomes

$$V(<z) = 4\pi R_\circ^3\int_0^\psi \frac{1-\cos 2\psi'}{2}d\psi' = R_\circ^3\left(\frac{\psi}{2}-\frac{\sin 2\psi}{4}\right) \qquad (10)$$

Note that $\psi = \sqrt{|1-\Omega_{tot}|}Z(z)$. The open case gives

$$V(<z) = 4\pi R_\circ^3\left(\frac{\sinh 2\psi}{4}-\frac{\psi}{2}\right) \qquad (11)$$

The ratio of $V(<z)$ to the Euclidean formula $(4\pi/3)(R_\circ\psi)^3$ is given by

$$\begin{aligned}
\frac{V(<z)}{(4\pi/3)(R_\circ\psi)^3} &= 1+\sum_{n=1}\frac{6\times 4^n}{(2n+3)!}[(1-\Omega_{tot})Z(z)^2]^n \\
&= 1+\frac{1}{5}[(1-\Omega_{tot})Z(z)^2]+\frac{2}{105}[(1-\Omega_{tot})Z(z)^2]^2+\ldots \qquad (12)
\end{aligned}$$

in both the open and closed cases.

3. IMPLEMENTATION

The cosmology calculator is implemented as a Web page (CosmoCalc.html) that has a large number of Javascript definitions in the header, followed by immediately executed Javascript that writes a frameset to the current document. The frameset calls for the input form (CCform.html) and the output page (CCout.html). If Javascript is not enabled, or if there is an error in the Javascript, then the body of CosmoCalc.html is displayed. This body is just an error message saying that Javascript must be enabled.

I have received several requests for the code used in the cosmology calculator. But since the code is in Javascript, it is in the HTML files in ASCII form. It is easy to save the page source to get the code, and it is easy to modify the code using any text editor. If your modifications introduce an error, then you will see the error message saying that Javascript must be enabled. This only means that you must find the error in your modified CosmoCalc.html. I have had to do this dozens of times, so don't be discouraged.

Even if you do not intend to modify the code, downloading the three HTML files will let you run the calculator locally when not connected to the Internet.

The numerical evaluation of the integrals for Z and t is done using the mid-point rule with 1000 panels. While this is not a very efficient use of CPU time, it was very easy to code. And with Javascript, the CPU time is consumed on the user's computer instead of the server. The functions being integrated go smoothly to zero as a goes to zero as long as $\Omega_r > 0$.

Another hidden aspect of the cosmology calculator is that it automatically sets the radiation density $\Omega_r h^2$ to the value appropriate for $T_\circ = 2.72528\,\mathrm{K}$ and three massless neutrino species, $\Omega_r h^2 = 4.165 \times 10^{-5}$. Here $h = H_\circ/100$ km/sec/Mpc, and this factor includes a small ($< 1\%$) boost in the neutrino density due to a slight transfer to e^+e^- annihilation energy into neutrinos (Hannestad & Madsen, 1995). So if you want to verify some simple cases, like the empty Universe, you should use a large value of the Hubble constant which reduces the relative importance of this radiation term. For example, the open button with $H_\circ = 97.78$ and $\Omega_M = 0$ gives an age of the Universe of 9.934 Gyr which is 0.7% from the expected 10 Gyr. But using $H_\circ = 977.8$ gives an age of 999 Myr which is only 0.1% from the expected 1 Gyr.

For very early times, the age of the Universe is adjusted to allow for the existence of e^+, e^-, μ^+, μ^-, etc. using the g_* and g_{*S} for the standard model of particle physics given by Kolb & Turner (1990).

4. Light travel time inversion

A slightly modified version of the cosmology calculator is at
http://www.astro.ucla.edu/ ∼wright/DlttCalc.html.
The input form asks for the light travel time in Gyr instead of redshift. The redshift is found by evaluating the integral for light travel time in steps of -0.001 in a starting from $a = 1$ until the input value is exceeded, and then interpolating to get a and thus z. Once z is known the calculations proceed as in CosmoCalc.html.

5. MORE OPTIONS

Since the cosmology calculator was first written, there have been two developments that change the kinds of models that people want to run. I have created a new version of the cosmology calculator with more options and placed it in
http://www.astro.ucla.edu/∼wright/ACC.html
One development is the discovery of neutrino oscillations, indicating that the assumption of massless neutrinos is not correct. The neutrino temperature is $(4/11)^{1/3}T_\circ = 1.95\,\mathrm{K}$, and a typical momentum for a thermal neutrino is $\approx 3kT/c$. This corresponds to the rest energy of a neutrino with $mc^2 \approx 0.0005\,\mathrm{eV}$. Since the neutrinos thermalized while still relativistic, their distribution is that of a relativistic Fermi-Dirac particle, so the neutrino density is given by

$$\begin{aligned} \rho_\nu &= 4\pi g_s h^{-3} \sum_{e,\mu,\tau} \left(\int \sqrt{m_\nu^2 + p^2/c^2}\, \frac{p^2 dp}{\exp(pc/kT_\nu) + 1} \right) \\ &= 4\pi g_s \left(\frac{kT_\nu}{hc} \right)^3 \sum_{e,\mu,\tau} \left(\int \sqrt{m_\nu^2 + (xkT_\nu)^2/c^4}\, \frac{x^2 dx}{\exp(x) + 1} \right) \end{aligned} \tag{13}$$

The number of spin states for a neutrino is one, but to allow for anti-neutrinos one should use $g_s = 2$.

The integral over neutrino momentum has to be evaluated for each step of the integration over a so it needs to be done efficiently, even when the work is done on the user's computer. In the low temperature limit when $kT_\nu << m_\nu c^2$ the integral over x evaluates to $m_\nu(3/4)\zeta(3)\Gamma(3)$, while in the high temperature limit $kT_\nu >> m_\nu c^2$ the integral evaluates to $(kT_\nu/c^2)(7/8)\zeta(4)\Gamma(4)$. Both limits can be evaluated correctly by approximating the integral using

$$\int \sqrt{m_\nu^2 + x^2(kT_\nu)^2/c^4}\, \frac{x^2 dx}{\exp(x) + 1} \approx m_\nu(3/4)\zeta(3)\Gamma(3)\sqrt{1 + (x_1 kT_\nu/m_\nu c^2)^2} \tag{14}$$

with the single integration knot at $x_1 = [(7/8)\zeta(4)\Gamma(4)]/[(3/4)\zeta(3)\Gamma(3)] = 3.151$. This approximation has a maximum error of $< 3\%$. But a better approximation is

$$\int \sqrt{m_\nu^2 + x^2(kT_\nu)^2/c^4}\ \frac{x^2dx}{\exp(x)+1} = m_\nu(3/4)\zeta(3)\Gamma(3)f(x_1kT_\nu/m_\nu c^2) \quad (15)$$

with

$$f(y) \approx (1+y^\beta)^{1/\beta} \quad (16)$$

and $\beta = 1.842$ which has a maximum error $< 0.3\%$. The mass at which neutrinos are semi-relativistic at the current epoch is

$$m_{rel}c^2 = x_1kT_\nu = 0.000531(T_\circ/2.72528\text{ K})\text{ eV} \quad (17)$$

The final result is that the effective neutrino density can be written as

$$\Omega_\nu(z)h^2 = \left(\frac{T_\circ}{2.72528\text{ K}}\right)^3 \frac{\sum_{e,\mu,\tau} m_\nu f(m_{rel}(1+z)/m_\nu)}{93.14\text{ eV}/c^2} \quad (18)$$

The normalization of 93.14 eV is from Mangano *et al.* (2005) and is 1.05% higher than the nominal due to residual coupling of annihilation energy to the neutrinos. In the relativistic limit the density is 1.53% higher. Increasing T_ν by 0.48% over the nominal $(4/11)^{1/3}T_\circ$ when computing m_{rel} allows for the relativistic limit, and this boost is included in Eq(17).

This $\Omega_\nu(z)$ is just a function of z which gives Ω_ν at $z = 0$ and should not be confused with the actual Ω_ν at z's other than zero. In ACC.html the neutrino density at $z = 0$ is subtracted from the input Ω_M giving separate $\Omega_{CM} = \Omega_M - \Omega_\nu(z = 0)$ for the CDM and baryons, and Ω_ν for the neutrinos. Of course the neutrinos are not included in the radiation so $\Omega_r h^2 = 2.778 \times 10^{-5}(T_\circ/2.72528)^4$.

For a hierarchical neutrino mass pattern, with masses $\approx 0.001,\ 0.009\ \&\ 0.049$ eV, the change in distances introduced by neutrino masses is negligible. At redshifts up to 5 in the WMAP concordance flat ΛCDM cosmology with $\Omega_M = 0.27$ and $H_\circ = 71$, the changes are less than 0.01%. This mass pattern is the default when ACC.html is loaded. But even the more massive nearly degenerate neutrino mass patterns such as $\approx 0.33,\ 0.33\ \&\ 0.33$ eV have a minimal effect on the distances and times.

The second change in the paradigm is the introduction of the equation of state $w(z) = P/\rho c^2$ as a parameter in the model. I have implemented $w(z)$ as $w = w_\circ + 2w'(1-a)$ following Linder (2003) who added the factor of "2" normalization to the simple form used by Chevallier & Polarski (2001). This functional form behaves well at high redshift and it allows an analytic solution for the dark energy density as a function of z. This solution is

$$\rho_{DE} = \rho_{DE,\circ}\, a^{-(3+3w_\circ+6w')}\ \exp(-6w'[1-a]) \quad (19)$$

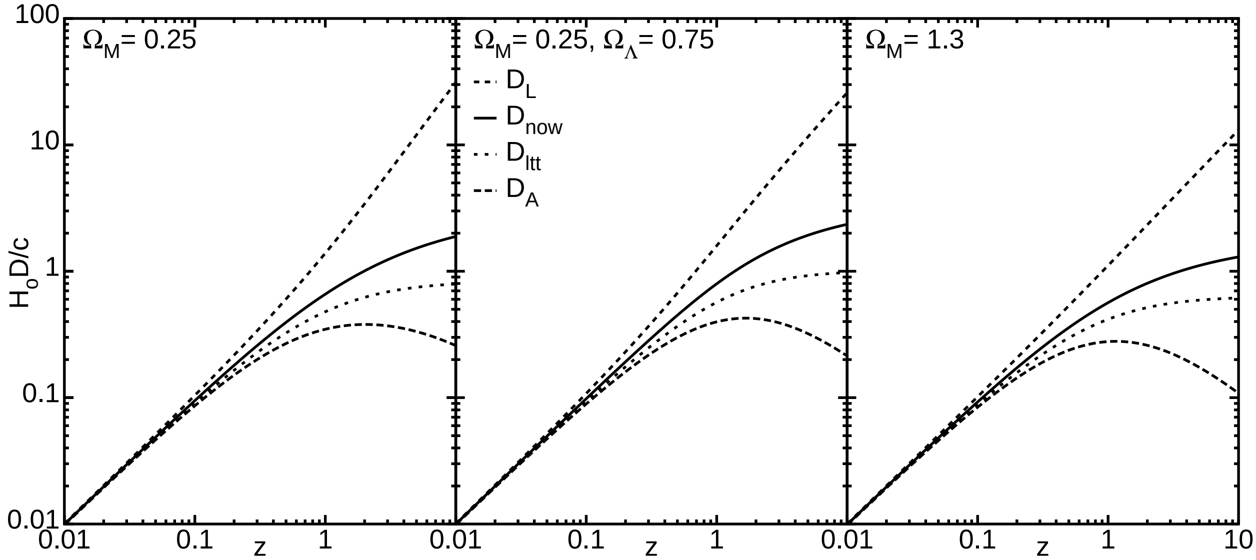

Fig. 1.— The 4 distance measures D_A, D_{ltt}, D_{now} and D_L from bottom to top plotted against a fifth distance measure: the redshift z, for 3 different cosmological models: an open CDM model, a flat ΛCDM model, and a closed CDM model. The center and right panels are consistent with the CMB angular power spectrum, while the left and center panels are consistent with large scale structure. Only the center panel is consistent with supernova data.

The defaults are $w = -1$ and $w' = 0$ and in that case ACC.html reduces to the CosmoCalc.html case.

Unlike the neutrino masses, changes in w have substantial effects on distances and ages. Changing w to -0.7 instead of the default -1 changes the age of the Universe by 6% and the luminosity distance at $z = 1$ by 7% when $\Omega_M = 0.27$, $\Omega_{DE} = 0.73$, and $H_\circ = 71$ are left unchanged. But for $w = -0.7$ the model consistent with both supernovae (Riess *et al.* 2004 gold+silver) and the CMB (Page *et al.* 2003 peak positions) changes to an open model with $\Omega_M = 0.19$, $\Omega_{DE} = 0.74$ and $H_\circ = 84$, and this gives an age change of 17% and a luminosity distance change of 21% at $z = 1$. However, if Ω_M and Ω_{DE} are allowed to vary as free parameters, the observable supernova signal in D_L is reduced to only 0.3% (Wright 2005).

Finally ACC.html allows one to input $T_\circ$, allowing for easier tests of simple cases.

With these changes, the X function is

$$X(a) = (\Omega_{CM} + \Omega_\nu(z))/a + \Omega_r/a^2 + \Omega_{DE}\; a^{-(1+3w_\circ+6w')}\; \exp(-6w'[1-a]) + (1-\Omega_{tot}) \quad (20)$$

The rest of the calculation of distances is unchanged.

6. CONCLUSION

These cosmology calculators are suitable for interactive use, providing fairly quick answers for single cases. Users who wish to use the code for large scale calculations should translate it into a compiled language and change the quadrature formula. In particular, if fitting to datasets with redshifts and distance, the data should be sorted by redshift and the distance integrals evaluated for all objects in a single pass through the sorted data. Plotting figures is an obvious case where redshifts are computed in order: Figure 1 shows the distance measures discussed in this paper as a function of z for 3 different models. The formulae presented here were translated into Postscript for this figure, resulting in a 4 kB file.

REFERENCES

Chevallier, M. & Polarski, D. 2001, Int. J. Mod. Phys., D10, 213.

Clavin, W. 2006, JPL press release, http://www.nasa.gov/centers/jpl/news/spitzerf-20060321.html

Hannestad, S. & Madsen, J. 1995, PRD, 52, 1764-1769.

Hogg, D. 1999, astro-ph/9905116v4

Kolb, E. & Turner, M. 1990, "The Early Universe", (Addison Wesley)

Linder, E. 2003, PRL, 90, 091301

Mangano, G., Miele, G., Pastor, S., Pinto, T., Pisanti, O. & Serpico, P. 2005, Nucl.Phys. B, 729, 221-234.

Page, L. *et al.*, 2003, ApJS, 148, 233.

Riess, A. *et al.*., 2004, ApJ, 607, 665-687.

Wright, E. 2005, New Astronomy Reviews, 49, 407.