

1. Let $\mathbf{x} = (x_1, x_2, x_3)$ be a realisation of $\mathbf{X} = (X_1, X_2, X_3)$ with distribution given on slide 30 of Parameter Estimation II.
 - (a) Simplify $\Pr(\mathbf{X} = \mathbf{x})$. Hence obtain the refined loglikelihood function $\ell(\theta)$.
 - (b) Show that the ML estimate of θ is $\frac{x_2 + 2x_3}{2n}$.
 - (c) Find two estimators of θ using MOM. Which method seems better, MOM or ML?
 - (d) Calculate the information on θ contained in \mathbf{X} .

2. Let X_1, \dots, X_n be IID $\text{Normal}(\mu, \nu)$ random variables.

- (a) Show that

$$\mathcal{I}(\mu, \nu) = \begin{bmatrix} \frac{1}{\nu} & 0 \\ 0 & \frac{1}{2\nu^2} \end{bmatrix}$$

Hence obtain the approximate variance of the ML estimators $(\hat{\mu}, \hat{\nu})$, for large n .

- (b) What is the exact variance of $(\hat{\mu}, \hat{\nu})$?

3. Let $(X_1, X_2, X_3) \sim \text{Multinomial}(n, (p_1, p_2, p_3))$. Let $\hat{\mathbf{p}}$ be the ML estimator of $\mathbf{p} = (p_1, p_2, p_3)$.

- (a) Derive $\text{var}(\hat{\mathbf{p}})$ from the variance of (X_1, X_2, X_3) .

- (b) Derive \mathcal{I} , the information on (p_1, p_2) contained in the $\text{Multinomial}(1, (p_1, p_2, p_3))$ distribution.

- (c) $\frac{\mathcal{I}^{-1}}{n}$ is a certain submatrix of $\text{var}(\hat{\mathbf{p}})$. Make a guess to obtain an expression for \mathcal{I}^{-1} , and verify it by multiplying with \mathcal{I} .

4. Use the Fisher information to estimate the SE of the ML estimate of θ in Tutorial 8 Question 4.

5. Let X_1, \dots, X_n be IID $N(0, \sigma^2)$ RV's.

- (a) Derive the MOM estimator of σ .

- (b) Derive the ML estimator of σ .

- (c) Derive $\mathcal{I}(\sigma)$.

- (d) For large n , approximately, what is the variance of the ML estimator of σ ?

- (e) What is the distribution of $\sqrt{n}\bar{X}/\sigma^2$? Hence or otherwise, show that $n\bar{X}^2/S^2 \sim F_{1, n-1}$, where

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$