ST2132 Tutorial 3 Solution 2210

1. (a) Intuitively, X and Y are symmetric in the bivariate norma distributionl, in the sense that (i) and (ii) in Tutorial 2 Question 5 remain true if X and Y are swapped. The swapped version of (i) gives the marginal distribution of Y.

   [A complete argument uses the symmetry in the joint density function: not needed.]

   (b) $\text{cov}(X,Y) = \rho\sigma_X\sigma_Y$. Since $\sigma_X\sigma_Y > 0$, $\text{cov}(X,Y) = 0$ implies $\rho = 0$. From (ii), the conditional distribution of Y given $X = x$ is $\text{N}(\mu_Y, \sigma_Y^2)$, for any real number x. Therefore, X and Y are independent.

2. (a) From (B), the standardised $\bar{X}$:

$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} = \sqrt{n}\frac{\bar{X}-\mu}{\sigma}$$

   has the N(0,1) distribution. Its square has the χ_1^2 distribution, by definition.

   (b) (C) says that $(n-1)S^2/\sigma^2 \sim \chi_{n-1}^2$ and by (A), it is independent of $(\bar{X}-\mu)/(\sigma/\sqrt{n})$. Therefore, by the definition of the t distribution,

$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \div \frac{S}{\sigma} = \frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$$

   (c) In view of (C), the idea is to calculate E and var of $R = (n-1)S^2/\sigma^2$, then move the constants. Since

$$\text{E}(R) = n-1, \qquad \text{var}(R) = 2(n-1)$$

   $\text{E}(S^2) = \sigma^2$, $\text{var}(S^2) = 2\sigma^4/(n-1)$.

3. (a) From

$$\sum_{i=1}^{n}(X_i-\bar{X})^2 = (n-1)S^2$$

   Tutorial 1 Question 6 gives

$$(n-1)S^2 = \sum_{i=1}^{n}(X_i-\mu)^2 - n(\bar{X}-\mu)^2$$

   Since $\text{E}\{(X_i-\mu)^2\} = \sigma^2$ and $\text{E}\{(\bar{X}-\mu)^2\} = \sigma^2/n$, by taking expectation we get

$$(n-1)\text{E}(S^2) = n\sigma^2 - n(\sigma^2/n) = (n-1)\sigma^2$$

   Hence $\text{E}(S^2) = \sigma^2$.

If $\mathrm{E}(S) = \sigma$, then $\mathrm{var}(S) = \mathrm{E}(S^2) - \{\mathrm{E}(S)\}^2 = 0$, but this is wrong, since S is not a constant. Hence $\mathrm{E}(S) \neq \sigma$; in fact $\mathrm{E}(S) < \sigma$.

(b) Since $\hat{\sigma}^2 = (n-1)S^2/n$,

$$\mathrm{E}(\hat{\sigma}^2) = \frac{n-1}{n}\sigma^2$$

Since $\hat{\sigma} \leq S$, $\mathrm{E}(\hat{\sigma}) \leq \mathrm{E}(S) < \sigma$.

[You can also show that assuming $\mathrm{E}(\hat{\sigma}) = \sigma$ leads to a contradiction like in (a).]

4. The numbers of a's and b's are respectively Np and $N(1-p)$. By definition,

$$\mu = \frac{1}{N}\{Np \cdot a + N(1-p) \cdot b\} = pa + (1-p)b$$

$$\sigma^2 = \frac{1}{N}\left\{Np \cdot (a-\mu)^2 + N(1-p) \cdot (b-\mu)^2\right\}$$

Now $a - \mu = (1-p)(a-b)$, $b - \mu = p(b-a)$. Substitute these into the above to get the result.

5. (a) By exchangeability, $(X_i, X_j, \ldots, X_N)$ (X_i and X_1 are swapped, and X_j and X_2 are swapped,) has the same distribution as $(X_1, X_2, \ldots, X_N)$. Hence, (X_i, X_j) has the same distribution as (X_1, X_2), so $\mathrm{cov}(X_i, X_j) = \mathrm{cov}(X_1, X_2)$. This is true for any $i \neq j$.

(b) Notice that T is the sum of all population values, so is a constant. Let c be the covariance between X_i and X_j, $i \neq j$. Hence $0 = \mathrm{var}(T) = N\sigma^2 + N(N-1)c$, giving the answer.

(c) By Tutorial 1 Question 1(b),

$$\begin{aligned}
\mathrm{var}\left(\sum_{i=1}^{n} X_i\right) &= \sum_{1 \leq i,j \leq n} \mathrm{cov}(X_i, X_j) \\
&= \sum_{i=1}^{n} \mathrm{var}(X_i) + \sum_{i \neq j} \mathrm{cov}(X_i, X_j) \\
&= n\sigma^2 + n(n-1)\left(-\frac{\sigma^2}{N-1}\right) \\
&= n\sigma^2 \frac{N-n}{N-1}
\end{aligned}$$

Dividing by n^2 gives the result.