ST2132 Tutorial 3 2210

1. Refer to Tutorial 2 Question 5.
   (a) State the marginal distribution of Y.
   (b) Using (ii), show that if $\mathrm{cov}(X, Y) = 0$, then X and Y are independent.

2. Let $X_1, \ldots, X_n$ be IID $\mathrm{N}(\mu, \sigma^2)$ random variables. Assume (A), (B), (C) on slide 20 of Probability Review II hold.
   (a) Show that
   $$n\left(\frac{\bar{X} - \mu}{\sigma}\right)^2 \sim \chi_1^2$$
   (b) Show that
   $$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$$
   (c) Calculate $\mathrm{E}(S^2)$ and $\mathrm{var}(S^2)$.

3. Let $X_1, \ldots, X_n$ be IID RV's with expectation μ and variance $\sigma^2 > 0$. Let
   $$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2, \qquad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$
   (a) Show that $\mathrm{E}(S^2) = \sigma^2$. Is $\mathrm{E}(S) = \sigma$?
   (b) Find $\mathrm{E}(\hat{\sigma}^2)$. Is $\mathrm{E}(\hat{\sigma}) = \sigma$?

4. In a population of size N, a fraction p of the individuals have value a, and the others have value b. Show that the mean and variance are
   $$\mu = pa + (1-p)b, \qquad \sigma^2 = p(1-p)(a-b)^2$$

5. Let $X_1, \ldots, X_N$ be random draws without replacement from a population of size N with mean μ and variance σ^2. You may assume that the variables are exchangeable.
   (a) Explain why $\mathrm{cov}(X_i, X_j)$ has the same value for any $i \neq j$.
   (b) By calculating the variance of $T = \sum_{i=1}^{N} X_i$, or otherwise, show that
   $$\mathrm{cov}(X_i, X_j) = -\frac{\sigma^2}{N-1}, \qquad i \neq j$$
   (c) Let $\bar{X}$ be the mean of $X_1, \ldots, X_n$. Show that
   $$\mathrm{var}(\bar{X}) = \frac{N-n}{N-1}\frac{\sigma^2}{n}$$