1 Probability Review

Multinomial Distribution

$$\Pr(X_1 = x_1, \ldots, X_r = x_r) = \binom{n}{x_1, \ldots, x_r} \prod_{i=1}^{r} p_i^{x_i}$$

Mean Square Error (MSE)

$$\mathsf{E}\{(Y-c)^2\} = \mathsf{var}(Y) + \{\mathsf{E}(Y) - c\}^2$$

$$\mathsf{E}\{(Y-c)^2|x\} = \mathsf{var}[Y|x] + \{\mathsf{E}[Y|x] - c\}^2$$

which are special cases of $\mathsf{E}(Y^2) = \mathsf{var}(Y) + [\mathsf{E}(Y)]^2$. MSE is minimized if and only if $c = \mathsf{E}(Y)$ or $\mathsf{E}[Y|x]$.
Usually the formula for $\mathsf{E}[Y|x] = f(x)$ is determined from observations/data and x can be a vector of realisations from covariates.

$$\mathsf{MSE}_{\mathsf{empirical}} = \frac{1}{n}\sum_{i=1}^{n}\{\mathsf{E}[Y|x_i] - y_i\}^2$$

In the real world, we have different realisations x_i of the random variable X, hence the mean MSE is

$$\frac{1}{n}\sum_{i=1}^{n}\mathsf{var}[Y|x_i] \approx \mathsf{E}(\mathsf{var}[Y|X]) \leq \mathsf{var}(Y)$$

Analysis of Variance (ANOVA)
involves breaking of variance into components

$$\mathsf{var}(Y) = \mathsf{E}(\mathsf{var}[Y|X]) + \mathsf{var}(\mathsf{E}[Y|X])$$

1.1 Distributions

χ_1^2 distribution
Let $Z \sim \mathcal{N}(0,1)$. $V = Z^2$ has a χ^2 distribution with 1 degree of freedom

$$f(v) = \frac{1}{\sqrt{2\pi}} v^{-1/2} e^{-v/2}$$

Gamma distribution

$$f(t) = \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t}, t \geq 0$$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

χ_n^2 distribution
Let $V_1, \ldots, V_n$ be IID χ_1^2

$$V = \sum_{i=1}^{n} V_i$$

has a χ_n^2 distribution with n degrees of freedom

t distribution
Let $Z \sim \mathcal{N}(0,1)$ and $V \sim \chi_n^2$ be independent

$$t_n = \frac{Z}{\sqrt{V/n}}$$

has a t distribution with n degrees of freedom

F distribution
Let $V \sim \chi_m^2$ and $W \sim \chi_n^2$ be independent

$$F_{m,n} = \frac{V/m}{W/n}$$

has an F distribution with (m,n) degrees of freedom
*Note: $t_n^2 = F_{1,n}$

1.2 Sample Variance

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$\bar{X}$ and S^2 are independent

$$\bar{X} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$$

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$$