CHAPTER ONE

Who Wants To Be a Millionaire?

'Do we know what the sequence of numbers is? Okay, here, we can do it in our heads . . . fifty-nine, sixty-one, sixty-seven . . . seventy-one . . . Aren't these all prime numbers?' A little buzz of excitement circulated through the control room. Ellie's own face momentarily revealed a flutter of something deeply felt, but this was quickly replaced by a sobriety, a fear of being carried away, an apprehension about appearing foolish, unscientific. Carl Sagan, *Contact*

One hot and humid morning in August 1900, David Hilbert of the University of Göttingen addressed the International Congress of Mathematicians in a packed lecture hall at the Sorbonne, Paris. Already recognised as one of the greatest mathematicians of the age, Hilbert had prepared a daring lecture. He was going to talk about what was unknown rather than what had already been proved. This went against all the accepted conventions, and the audience could hear the nervousness in Hilbert's voice as he began to lay out his vision for the future of mathematics. 'Who of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its development during future centuries?' To herald the new century, Hilbert challenged the audience with a list of twenty-three problems that he believed should set the course for the mathematical explorers of the twentieth century.

The ensuing decades saw many of the problems answered, and those who discovered the solutions make up an illustrious band of mathematicians known as 'the honours class'. It includes the likes of Kurt Gödel and Henri Poincaré, along with many other pioneers whose ideas have transformed the mathematical landscape. But there was one problem, the eighth on Hilbert's list, which looked as if it would survive the century without a champion: the Riemann Hypothesis.

Of all the challenges that Hilbert had set, the eighth had a special place in his heart. There is a German myth about Frederick Barbarossa, a much-loved German emperor who died during the Third Crusade. A legend grew that he was still alive, asleep in a cavern in the Kyffhäuser Mountains. He would awake only when Germany needed him. Somebody allegedly asked Hilbert, 'If you were to be revived like Barbarossa, after five hundred years, what would you do?' His reply: 'I would ask, "Has someone proved the Riemann Hypothesis?"'

As the twentieth century drew to a close, most mathematicians had resigned themselves to the fact that this jewel amongst all of Hilbert's problems was not only likely to outlive the century, but might still be unanswered when Hilbert awoke from his five-hundred-year slumber. He had stunned the first International Congress of the twentieth century with his revolutionary lecture full of the unknown. However, there turned out to be a surprise in store for those mathematicians who were planning to attend the last Congress of the century.

On April 7, 1997, computer screens across the mathematical world flashed up some extraordinary news. Posted on the website of the International Congress of Mathematicians that was to be held the following year in Berlin was an announcement that the Holy Grail of mathematics had finally been claimed. The Riemann Hypothesis had been proved. It was news that would have a profound effect. The Riemann Hypothesis is a problem which is central to the whole of mathematics. As they read their email, mathematicians were thrilling to the prospect of understanding one of the greatest mathematical mysteries.

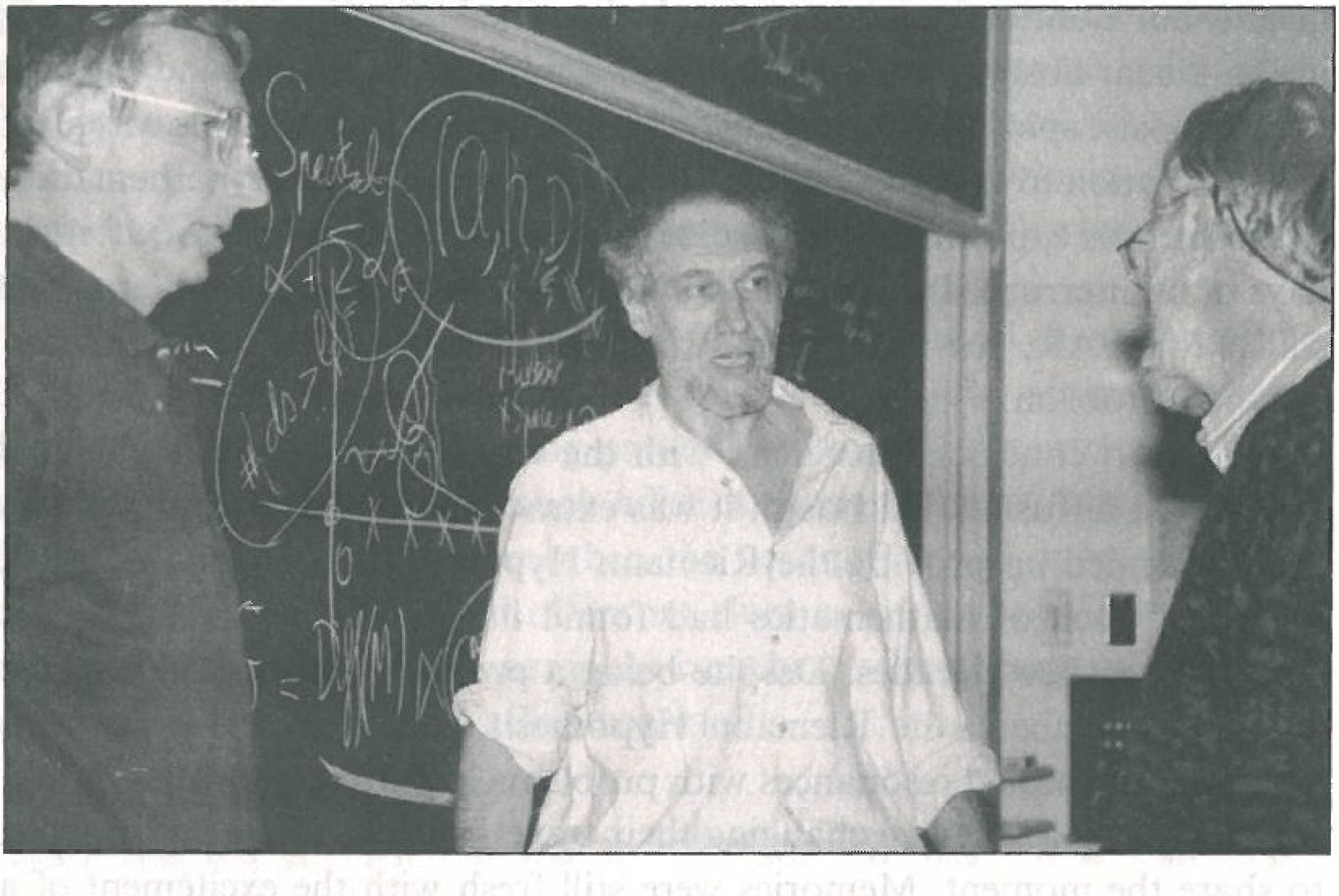
The announcement came in a letter from Professor Enrico Bombieri. One could not have asked for a better, more respected source. Bombieri is one of the guardians of the Riemann Hypothesis and is based at the prestigious Institute for Advanced Study in Princeton, once home to Einstein and Gödel. He is very softly spoken, but mathematicians always listen carefully to anything he has to say.

Bombieri grew up in Italy, where his prosperous family's vineyards gave him a taste for the good things in life. He is fondly referred to by colleagues as 'the Mathematical Aristocrat'. In his youth he always cut a dashing figure at conferences in Europe, often arriving in a fancy sports car. Indeed, he was quite happy to fuel a rumour that he'd once come sixth in a twenty-four-hour rally in Italy. His successes on the mathematical circuit were more concrete and led to an invitation in the 1970s to go to Princeton, where he has remained ever since. He has replaced his enthusiasm for rallying with a passion for painting, especially portraits.

But it is the creative art of mathematics, and in particular the challenge of the Riemann Hypothesis, that gives Bombieri the greatest buzz. The Riemann Hypothesis had been an obsession for Bombieri ever since he first read about it at the precocious age of fifteen. He had always been fascinated by properties of numbers as he browsed through the mathematics books his father, an economist, had collected in his extensive library. The Riemann Hypothesis, he discovered, was regarded as the deepest and most fundamental problem in number theory. His passion for the problem was further fuelled when his father offered to buy him a Ferrari if he ever solved it – a desperate attempt on his father's part to stop Enrico driving his own model.

According to his email, Bombieri had been beaten to his prize. 'There are fantastic developments to Alain Connes's lecture at IAS last Wednesday,' Bombieri began. Several years previously, the mathematical world had been set alight by the news that Alain Connes had turned his attention to trying to crack the Riemann Hypothesis. Connes is one of the revolutionaries of the subject, a benign Robespierre of mathematics to Bombieri's Louis XVI. He is an extraordinarily charismatic figure whose fiery style is far from the image of the staid, awkward mathematician. He has the drive of a fanatic convinced of his world-view, and his lectures are mesmerising. Amongst his followers he has almost cult status. They will happily join him on the mathematical barricades to defend their hero against any counter-offensive mounted from the *ancien régime*'s entrenched positions.

Connes is based at France's answer to the Institute in Princeton, the Institut des Hautes Études Scientifiques in Paris. Since his arrival there in



Alain Connes, professor at the Institut des Hautes Études Scientifiques and at the Collège de France.

1979, he has created a completely new language for understanding geometry. He is not afraid to take the subject to the extremes of abstraction. Even the majority of the mathematical ranks who are usually at home with their subject's highly conceptual approach to the world have balked at the abstract revolution Connes is proposing. Yet, as he has demonstrated to those who doubt the necessity for such stark theory, his new language for geometry holds many clues to the real world of quantum physics. If it has instilled terror in the hearts of the mathematical masses, then so be it.

Connes's audacious belief that his new geometry could unmask not only the world of quantum physics but explain the Riemann Hypothesis – the greatest mystery about numbers – was met with surprise and even shock. It reflected his disregard for conventional boundaries that he dare venture into the heart of number theory and confront head-on the most difficult outstanding problem in mathematics. Since his arrival on the scene in the mid-nineties, there had been an expectancy in the air that if anyone had the resources to conquer this notoriously difficult problem, it was Alain Connes.

But it was not Connes who appeared to have found the last piece in the complex jigsaw. Bombieri went on to explain that a young physicist in the audience had seen 'in a flash' how to use his bizarre world of 'supersymmetric fermionic–bosonic systems' to attack the Riemann Hypothesis. Not many mathematicians knew quite what this cocktail of buzzwords meant, but Bombieri explained that it described 'the physics corresponding to a near-absolute zero ensemble of a mixture of anyons and morons with opposite spins'. It still sounded rather obscure, but then this was after all the solution to the most difficult problem in the history of mathematics, so no one was expecting an easy solution. According to Bombieri, after six days of uninterrupted work and with the help of a new computer language called MISPAR, the young physicist had finally cracked mathematics' toughest problem.

Bombieri concluded his email with the words, 'Wow! Please give this the highest diffusion.' Although it was extraordinary that a young physicist had ended up proving the Riemann Hypothesis, it came as no great surprise. Much of mathematics had found itself entangled with physics over the past few decades. Despite being a problem with its heart in the theory of numbers, the Riemann Hypothesis had for some years been showing unexpected resonances with problems in particle physics.

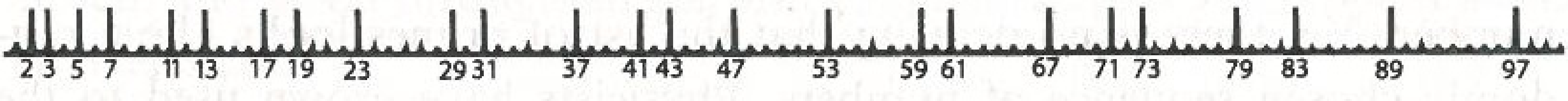
Mathematicians were changing their travel plans to fly in to Princeton to share the moment. Memories were still fresh with the excitement of a few years earlier when an English mathematician, Andrew Wiles, had announced a proof of Fermat's Last Theorem in a lecture delivered in Cambridge in June 1993. Wiles had proved that Fermat had been right in his claim that the equation $x^n + y^n = z^n$ has no solutions when n is bigger than 2. As Wiles laid down his chalk at the end of the lecture, the champagne bottles started popping and the cameras began flashing.

Mathematicians knew, however, that proving the Riemann Hypothesis would be of far greater significance for the future of mathematics than knowing that Fermat's equation has no solutions. As Bombieri had discovered at the tender age of fifteen, the Riemann Hypothesis seeks to understand the most fundamental objects in mathematics – prime numbers.

Prime numbers are the very atoms of arithmetic. The primes are those indivisible numbers that cannot be written as two smaller numbers multiplied together. The numbers 13 and 17 are prime, whilst 15 is not since it can be written as 3 times 5. The primes are the jewels studded throughout the vast expanse of the infinite universe of numbers that mathematicians have explored down the centuries. For mathematicians they instil a sense of wonder: 2, 3, 5, 7, 11, 13, 17, 19, 23, . . . – timeless numbers that exist in some world independent of our physical reality. They are Nature's gift to the mathematician.

Their importance to mathematics comes from their power to build all other numbers. Every number that is not a prime can be constructed by multiplying together these prime building blocks. Every molecule in the physical world can be built out of atoms in the periodic table of chemical elements. A list of the primes is the mathematician's own periodic table. The prime numbers 2, 3 and 5 are the hydrogen, helium and lithium in the mathematician's laboratory. Mastering these building blocks offers the mathematician the hope of discovering new ways of charting a course through the vast complexities of the mathematical world.

Yet despite their apparent simplicity and fundamental character, prime numbers remain the most mysterious objects studied by mathematicians. In a subject dedicated to finding patterns and order, the primes offer the ultimate challenge. Look through a list of prime numbers, and you'll find that it's impossible to predict when the next prime will appear. The list seems chaotic, random, and offers no clues as to how to determine the next number. The list of primes is the heartbeat of mathematics, but it is a pulse wired by a powerful caffeine cocktail:



The prime numbers up to 100 – mathematics' irregular heartbeat.

Can you find a formula that generates the numbers in this list, some magic rule that will tell you what the 100th prime number is? This question has been plaguing mathematical minds down the ages. Despite over two thousand years of endeavour, prime numbers seem to defy attempts to fit them into a straightforward pattern. Generations have sat listening to the rhythm of the prime-number drum as it beats out its sequence of numbers: two beats, followed by three beats, five, seven, eleven. As the beat goes on, it becomes easy to believe that random white noise, without any inner logic, is responsible. At the centre of mathematics, the pursuit of order, mathematicians could only hear the sound of chaos.

Mathematicians can't bear to admit that there might not be an explanation for the way Nature has picked the primes. If there were no structure to mathematics, no beautiful simplicity, it would not be worth studying. Listening to white noise has never caught on as an enjoyable pastime. As the French mathematician Henri Poincaré wrote, 'The scientist does not study Nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If Nature were not beautiful, it would not be worth knowing, and if Nature were not worth knowing, life would not be worth living.'

One might hope that the prime-number heartbeat settles down after a jumpy start. Not so – things just seem to get worse the higher you count. Here are the primes amongst the 100 numbers either side of 10,000,000. First, those below 10,000,000:

> 9,999,901 9,999,907, 9,999,929, 9,999,931, 9,999,937, 9,999,943, 9,999,971, 9,999,973, 9,999,991

But look now at how few there are in the 100 numbers above 10,000,000:

> 10,000,019, 10,000,079.

It is hard to guess at a formula that could generate this kind of pattern. In fact, this procession of primes resembles a random succession of numbers much more than it does a nice orderly pattern. Just as knowing the first 99 tosses of a coin won't help you much in guessing the result of the 100th toss, so do the primes seem to defy prediction.

Prime numbers present mathematicians with one of the strangest tensions in their subject. On the one hand a number is either prime or it isn't. No flip of a coin will suddenly make a number divisible by some smaller number. Yet there is no denying that the list of primes looks like a randomly chosen sequence of numbers. Physicists have grown used to the idea that a quantum die decides the fate of the universe, randomly choosing at each throw where scientists will find matter. But it is something of an embarrassment to have to admit that these fundamental numbers on which mathematics is based appear to have been laid out by Nature flipping a coin, deciding at each toss the fate of each number. Randomness and chaos are anathema to the mathematician.

Despite their randomness, prime numbers – more than any other part of our mathematical heritage – have a timeless, universal character. Prime numbers would be there regardless of whether we had evolved sufficiently to recognise them. As the Cambridge mathematician G.H. Hardy said in his famous book *A Mathematician's Apology*, '317 is a prime not because we think so, or because our minds are shaped in one way or another, but *because it is so*, because mathematical reality is built that way.'

Some philosophers might take issue with such a Platonist view of the world – this belief in an absolute and eternal reality beyond human existence – but to my mind that is what makes them philosophers and not mathematicians. There is a fascinating dialogue between Alain Connes, the mathematician who featured in Bombieri's email, and the neurobiologist Jean-Pierre Changeux in *Conversations on Mind, Matter and Mathematics*. The tension in this book is palpable as the mathematician argues for the existence of mathematics outside the mind, and the neurologist is determined to refute any such idea: 'Why wouldn't we see "$\pi = 3.1416$" written in gold letters in the sky or "6.02×10^{23}" appear in the reflections of a crystal ball?' Changeux declares his frustration at Connes's insistence that 'there exists, independently of the human mind, a raw and immutable mathematical reality' and at the heart of that world we find the unchanging list of primes. Mathematics, Connes declares, 'is unquestionably the only *universal* language'. One can imagine a different chemistry or biology on the other side of the universe, but prime numbers will remain prime whichever galaxy you are counting in.

In Carl Sagan's classic novel *Contact*, aliens use prime numbers to contact life on earth. Ellie Arroway, the book's heroine, has been working at SETI, the Search for Extraterrestrial Intelligence, listening to the crackle of the cosmos. One night, as the radio telescopes are turned towards Vega, they suddenly pick up strange pulses through the background noise. It takes Ellie no time to recognise the drumbeat in this radio signal. Two pulses are followed by a pause, then three pulses, five, seven, eleven, and so on through all the prime numbers up to 907. Then it starts all over again.

This cosmic drum was playing a music that earthlings couldn't fail to recognise. Ellie is convinced that only intelligent life could generate this beat: 'It's hard to imagine some radiating plasma sending out a regular set of mathematical signals like this. The prime numbers are there to attract our attention.' Had the alien culture transmitted the previous ten years of alien winning lottery numbers, Ellie couldn't have distinguished them from the background noise. Even though the list of primes looks as random a list as the lottery winnings, its universal constancy has determined the choice of each number in this alien broadcast. It is this structure that Ellie recognises as the sign of intelligent life.

Communicating using prime numbers is not just science fiction. Oliver Sacks in his book *The Man Who Mistook His Wife for a Hat* documents twenty-six-year-old twin brothers, John and Michael, whose deepest form of communication was to swap six-digit prime numbers. Sacks tells of when he first discovered them secretly exchanging numbers in the corner of a room: 'they looked, at first, like two connoisseurs wine-tasting, sharing rare tastes, rare appreciations'. At first, Sacks can't figure out what the twins are up to. But as soon as he cracks their code, he memorises some eight-digit primes which he drops surreptitiously into the conversation at their next meeting. The twins' surprise is followed by deep concentration which turns to jubilation as they recognise another prime number. Whilst Sacks had resorted to prime number tables to find his primes, how the twins were generating their primes is a tantalising puzzle. Could it be that these autistic-savants were in possession of some secret formula that generations of mathematicians had missed?

The story of the twins is a favourite of Bombieri's.

> It is hard for me to hear this story without feeling awe and astonishment at the workings of the brain. But I wonder: Do my non-mathematical friends have the same response? Do they have any inkling how bizarre, how prodigious and even other-worldly was the singular talent the twins so naturally enjoyed? Are they aware that mathematicians have been struggling for centuries to come up with a way to do what John and Michael did spontaneously: to generate and recognize prime numbers?

Before anyone could find out how they were doing it, the twins were separated at the age of thirty-seven by their doctors, who believed that their private numerological language had been hindering their development. Had they listened to the arcane conversations that can be heard in the common rooms of university maths departments, these doctors would probably have recommended closing them down too.

It's likely that the twins were using a trick based on what's called Fermat's Little Theorem to test whether a number is prime. The test is similar to the way in which autistic-savants can quickly identify that April 13, 1922, for instance, was a Thursday – a feat the twins performed regularly on TV chat shows. Both tricks depend on doing something called clock or modular arithmetic. Even if they lacked a magic formula for the primes, their skill was still extraordinary. Before they were separated they had reached twenty-digit numbers, well beyond the upper limit of Sacks's prime number tables.

Like Sagan's heroine listening to the cosmic prime number beat and Sacks eavesdropping on the prime number twins, mathematicians for centuries had been straining to hear some order in this noise. Like Western ears listening to the music of the East, nothing seemed to make sense. Then, in the middle of the nineteenth century, came a major breakthrough. Bernhard Riemann began to look at the problem in a completely new way. From his new perspective, he began to understand something of the pattern responsible for the chaos of the primes. Underlying the outward noise of the primes was a subtle and unexpected harmony. Despite this great step forward, this new music kept many of its secrets out of earshot. Riemann, the Wagner of the mathematical world, was undaunted. He made a bold prediction about the mysterious music that he had discovered. This prediction is what has become known as the Riemann Hypothesis. Whoever proves that Riemann's intuition about the nature of this music was right will have explained why the primes give such a convincing impression of randomness.

Riemann's insight followed his discovery of a mathematical looking-glass through which he could gaze at the primes. Alice's world was turned upside down when she stepped through her looking-glass. In contrast, in the strange mathematical world beyond Riemann's glass, the chaos of the primes seemed to be transformed into an ordered pattern as strong as any mathematician could hope for. He conjectured that this order would be maintained however far one stared into the never-ending world beyond the glass. His prediction of an inner harmony on the far side of the mirror would explain why outwardly the primes look so chaotic. The metamorphosis provided by Riemann's mirror, where chaos turns to order, is one which most mathematicians find almost miraculous. The challenge that Riemann left the mathematical world was to prove that the order he thought he could discern was really there.

Bombieri's email of April 7, 1997, promised the beginning of a new era. Riemann's vision had not been a mirage. The Mathematical Aristocrat had offered mathematicians the tantalising possibility of an explanation for the apparent chaos in the primes. Mathematicians were keen to loot the many other treasures they knew should be unearthed by the solution to this great problem.

A solution of the Riemann Hypothesis will have huge implications for many other mathematical problems. Prime numbers are so fundamental to the working mathematician that any breakthrough in understanding their nature will have a massive impact. The Riemann Hypothesis seems unavoidable as a problem. As mathematicians navigate their way across the mathematical terrain, it is as though all paths will necessarily lead at some point to the same awesome vista of the Riemann Hypothesis.

Many people have compared the Riemann Hypothesis to climbing Mount Everest. The longer it remains unclimbed, the more we want to conquer it. And the mathematician who finally scales Mount Riemann will certainly be remembered longer than Edmund Hillary. The conquest of Everest is marvelled at not because the top is a particularly exciting place to be, but because of the challenge it poses. In this respect the Riemann Hypothesis differs significantly from the ascent of the world's tallest peak. Riemann's peak is a place we all want to sit upon because we already know the vistas that will open up to us should we make it to the top. The person who proves the Riemann Hypothesis will have made it possible to fill in the missing gaps in thousands of theorems that rely on it being true. Many mathematicians have simply had to assume the truth of the Hypothesis in reaching their own goals.

The dependence of so many results on Riemann's challenge is why mathematicians refer to it as a hypothesis rather than a conjecture. The word 'hypothesis' has the much stronger connotation of a necessary assumption that a mathematician makes in order to build a theory. 'Conjecture', in contrast, represents simply a prediction of how mathematicians believe their world behaves. Many have had to accept their inability to solve Riemann's riddle and have simply adopted his prediction as a working hypothesis. If someone can turn the hypothesis into a theorem, all those unproven results would be validated.

By appealing to the Riemann Hypothesis, mathematicians are staking their reputations on the hope that one day someone will prove that Riemann's intuition was correct. Some go further than just adopting it as a working hypothesis. Bombieri regards it as an article of faith that the primes behave as Riemann's Hypothesis predicts. It has become virtually a cornerstone in the pursuit of mathematical truth. If, however, the Riemann Hypothesis turns out to be false, it will completely destroy the faith we have in our intuition to sniff out the way things work. So convinced have we become that Riemann was right that the alternative will require a radical revision of our view of the mathematical world. In particular, all the results that we believe exist beyond Riemann's peak would disappear in a puff of smoke.

Most significantly, a proof of the Riemann Hypothesis would mean that mathematicians could use a very fast procedure guaranteed to locate a prime number with, say, a hundred digits or any other number of digits you care to choose. You might legitimately ask, 'So what?' Unless you are a mathematician such a result looks unlikely to have a major impact on your life.

Finding hundred-digit primes sounds as pointless as counting angels on a pinhead. Although most people recognise that mathematics underlies the construction of an aeroplane or the development of electronics technology, few would expect the esoteric world of prime numbers to have much impact on their lives. Indeed, even in the 1940s G.H. Hardy was of the same mind: 'both Gauss and lesser mathematicians may be justified in rejoicing that here is one science [number theory] at any rate whose very remoteness from ordinary human activities should keep it gentle and clean'.

But a more recent turn of events has seen prime numbers take centre stage in the rough and dirty world of commerce. No longer are prime numbers confined to the mathematical citadel. In the 1970s, three scientists, Ron Rivest, Adi Shamir and Leonard Adleman, turned the pursuit of prime numbers from a casual game played in the ivory towers of academia into a serious business application. By exploiting a discovery made by Pierre de Fermat in the seventeenth century, these three found a way to use the primes to protect our credit card numbers as they travel through the electronic shopping malls of the global marketplace. When the idea was first proposed in the 1970s, no one had any idea how big e-business would turn out to be. But today, without the power of prime numbers there is no way this business could exist. Every time you place an order on a website, your computer is using the security provided by the existence of prime numbers with a hundred digits. The system is called RSA after its three inventors. So far, over a million primes have already been put to use to protect the world of electronic commerce.

Every business trading on the Internet therefore depends on prime numbers with a hundred digits to keep their business transactions secure. The expanding role of the Internet will ultimately lead to each of us being uniquely identified by our very own prime numbers. Suddenly there is a commercial interest in knowing how a proof of the Riemann Hypothesis might help in understanding how primes are distributed throughout the universe of numbers.

The extraordinary thing is that although the *construction* of this code depends on discoveries about primes made by Fermat over three hundred years ago, to *break* this code depends on a problem that we still can't answer. The security of RSA depends on our inability to answer basic questions about prime numbers. Mathematicians know enough about the primes to build these Internet codes, but not enough to break them. We can understand one half of the equation but not the other. The more we demystify the primes, however, the less secure these Internet codes are becoming. These numbers are the keys to the locks that protect the world's electronic secrets. This is why companies such as AT&T and Hewlett-Packard are ploughing money into endeavours to understand the subtleties of prime numbers and the Riemann Hypothesis. The insights gained could help to break these prime number codes, and all companies with an Internet presence want to be the first to know when their codes become insecure. And this is the reason why number theory and business have become such strange bedfellows. Business and security agencies are keeping a watchful eye on the blackboards of the pure mathematicians.

So it wasn't only the mathematicians who were getting excited about Bombieri's announcement. Was this solution of the Riemann Hypothesis going to cause a meltdown of e-business? Agents from the NSA, the US National Security Agency, were dispatched to Princeton to find out. But as mathematicians and security agents made their way to New Jersey, a number of people began to smell something fishy in Bombieri's email. Fundamental particles have been given some crazy names – gluons, cascade hyperons, charmed mesons, quarks, the last of these courtesy of James Joyce's *Finnegans Wake*. But 'morons'? Surely not! Bombieri has an unrivalled reputation for appreciating the ins and outs of the Riemann Hypothesis, but those who know him personally are also aware of his wicked sense of humour.

Fermat's Last Theorem had fallen foul of an April Fool prank that emerged just after a gap had appeared in the first proof that Andrew Wiles had proposed in Cambridge. With Bombieri's email, the mathematical community had been duped again. Eager to relive the buzz of seeing Fermat proved, they had grabbed the bait that Bombieri had thrown at them. And the delights of forwarding email meant that the first of April had disappeared from the original source as it rapidly disseminated. This, combined with the fact that the email was read in countries with no concept of April Fool's Day, made the prank far more successful than Bombieri could have imagined. He finally had to own up that his email was a joke. As the twenty-first century approached, we were still com-

Enrico Bombieri, professor at the Institute for Advanced Study, Princeton.

pletely in the dark as to the nature of the most fundamental numbers in mathematics. It was the primes that had the last laugh.

Why had mathematicians been so gullible that they believed Bombieri? It's not as though they give up their trophies lightly. The stringent tests that mathematicians require to be passed before a result can be declared proven far exceed those deemed sufficient in other subjects. As Wiles realised when a gap appeared in his first proof of Fermat's Last Theorem, completing 99 per cent of the jigsaw is not enough: it would be the person who put in the last piece who would be remembered. And the last piece can often remain hidden for years.

The search for the secret source that fed the primes had been going on for over two millennia. The yearning for this elixir had made mathematicians all too susceptible to Bombieri's ruse. For years, many had simply been too frightened to go anywhere near this notoriously difficult problem. But it was striking how, as the century drew to a close, more and more mathematicians were prepared to talk about attacking it. The proof of Fermat's Last Theorem only helped to fuel the expectation that great problems could be solved.

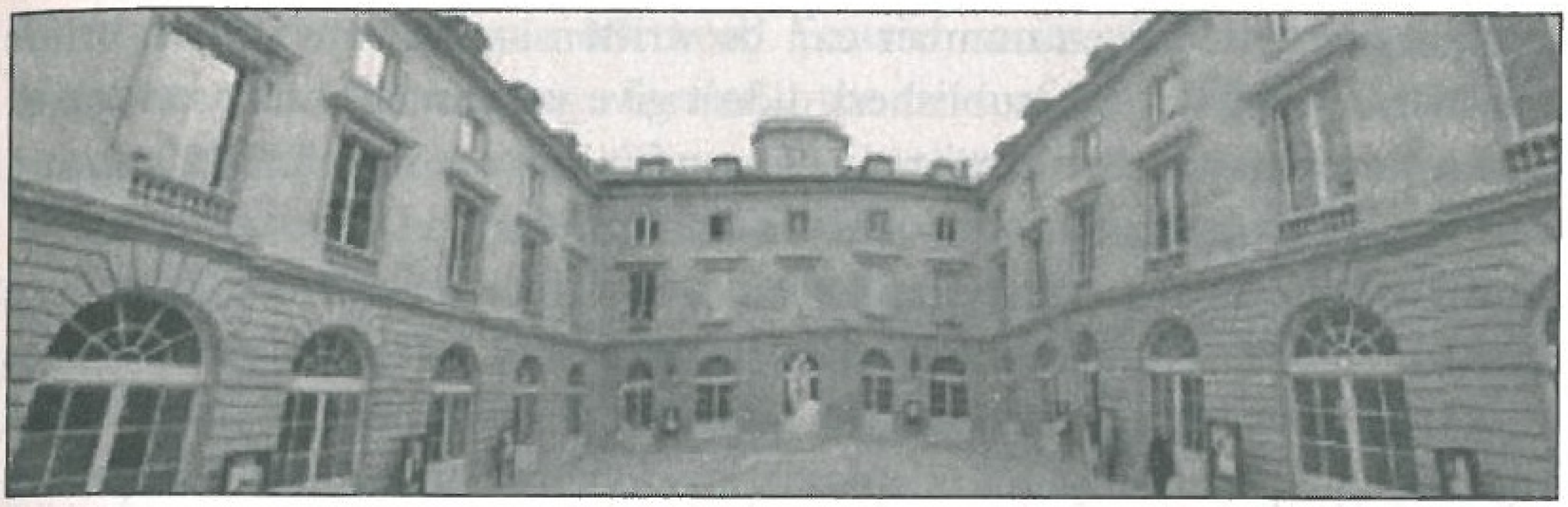
Mathematicians had enjoyed the attention that Wiles's solution to Fermat had brought them as mathematicians. This feeling undoubtedly contributed to the desire to believe Bombieri. Suddenly, Andrew Wiles was being asked to model chinos for Gap. It felt good. It felt almost sexy to be a mathematician. Mathematicians spend so much time in a world that fills them with excitement and pleasure. Yet it is a pleasure they rarely have the opportunity to share with the rest of the world. Here was a chance to flaunt a trophy, to show off the treasures that their long, lonely journeys had uncovered.

A proof of the Riemann Hypothesis would have been a fitting mathematical climax to the twentieth century. The century had opened with Hilbert's direct challenge to the world's mathematicians to crack this enigma. Of the twenty-three problems on Hilbert's list, the Riemann Hypothesis was the only problem to make it into the new century unvanquished.

On May 24, 2000, to mark the 100th anniversary of Hilbert's challenge, mathematicians and the press gathered in the Collège de France in Paris to hear the announcement of a fresh set of seven problems to challenge the mathematical community for the new millennium. They were proposed by a small group of the world's finest mathematicians, including Andrew Wiles and Alain Connes. The seven problems were new except for one that had appeared on Hilbert's list: the Riemann Hypothesis. In obeisance to the capitalist ideals that shaped the twentieth century, these challenges come with some extra spice. The Riemann Hypothesis and the other six problems now have a price tag of one million dollars apiece. Incentive indeed for Bombieri's fictional young physicist – if glory weren't enough.

The idea for the Millennium Problems was the brainchild of Landon T. Clay, a Boston businessman who made his money in trading mutual funds on a buoyant stock market. Despite dropping out of mathematics at Harvard he has a passion for the subject, a passion he wants to share. He realises that money is not the motivating force for mathematicians: 'It's the desire for truth and the response to the beauty and power and elegance of mathematics that drive mathematicians.' But Clay is not naive, and as a businessman he knows how a million dollars might inspire another Andrew Wiles to join the chase for the solutions of these great unsolved problems. Indeed, the Clay Mathematics Institute's website, where the Millennium Problems were posted, was so overwhelmed by hits the day after the announcement that it collapsed under the strain.

The seven Millennium Problems are different in spirit to the twenty-three problems chosen a century before. Hilbert had set a new agenda



The Collège de France.

for mathematicians in the twentieth century. Many of his problems were original and encouraged a significant shift in attitudes towards the subject. Rather than focusing on the particular, like Fermat's Last Theorem, Hilbert's twenty-three problems inspired the community to think more conceptually. Instead of picking over individual rocks in the mathematical landscape, Hilbert offered mathematicians the chance of a balloon flight high above their subject to encourage them to understand the overarching lay of the land. This new approach owes a lot to Riemann, who fifty years before had begun this revolutionary shift from mathematics as a subject of formulas and equations to one of ideas and abstract theory.

The choice of the seven problems for the new millennium was more conservative. They are the Turners in the mathematical gallery of problems, whereas Hilbert's questions were a more modernist, avant-garde collection. The conservatism of the new problems was partly because their solutions were expected to be sufficiently clear cut for their solvers to be awarded the million-dollar prize. The Millennium Problems are questions that mathematicians have known about for some decades, and in the case of the Riemann Hypothesis, over a century. They are a classic selection.

Clay's seven million dollars is not the first time that money has been offered for solutions to mathematical problems. In 1997 Wiles picked up 75,000 Deutschmarks for his proof of Fermat's Last Theorem, thanks to a prize offered in 1908 by Paul Wolfskehl. The story of the Wolfskehl Prize is what had brought Fermat to Wiles's attention at the impressionable age of ten. Clay believes that if he can do the same for the Riemann Hypothesis, it will be a million dollars well spent. More recently, two publishing houses, Faber & Faber in the UK and Bloomsbury in the USA, offered a million dollars for a proof of Goldbach's Conjecture as a publicity stunt to launch their publication of Apostolos Doxiadis's novel *Uncle Petros and Goldbach's Conjecture*. To earn the money you had to explain why every even number can be written as the sum of two prime numbers. However, the publishers didn't give you much time to crack it. The solution had to be submitted before midnight on March 15, 2002, and was bizarrely open only to US and UK residents.

Clay believes that mathematicians receive little reward or recognition for their labours. For example, there is no Nobel Prize for Mathematics that they can aspire to. Instead, the award of a Fields Medal is considered the ultimate prize in the mathematical world. In contrast to Nobel prizes, which tend to be awarded to scientists at the end of their careers for achievements long past, Fields Medals are restricted to mathematicians below the age of forty. This is not because of the generally held belief that mathematicians burn out at an early age. John Fields, who conceived of the idea and provided funds for the prize, wanted its award to spur on the most promising mathematicians to even greater achievements. The medals are awarded every four years on the occasion of the International Congress of Mathematicians. The first ones were awarded in Oslo in 1936.

The age limit is strictly adhered to. Despite Andrew Wiles's extraordinary achievement in proving Fermat's Last Theorem, the Fields Medal committee weren't able to award him a medal at the Congress in Berlin in 1998, the first opportunity after the final proof was accepted, for he was born in 1953. They did have a special medal struck to honour Wiles's achievement. But it still does not compare to being a member of the illustrious club of Fields Medal winners. The recipients include many of the key players in our drama: Enrico Bombieri, Alain Connes, Atle Selberg, Paul Cohen, Alexandre Grothendieck, Alan Baker, Pierre Deligne. Those names account for nearly a fifth of the medals ever awarded.

But it is not for the money that mathematicians aspire to these medals. In contrast to the big bucks behind the Nobel prizes, the purse that accompanies a Fields Medal contains a modest 15,000 Canadian dollars. So Clay's millions will help compete with the monetary kudos of the Nobel prizes. In contrast to Fields Medals and the Faber–Bloomsbury Goldbach prize, the money is there regardless of age or nationality, and with no time limits for a solution, except for the ticking clock of inflation.

However, the greatest incentive for the mathematician chasing one of the Millennium problems is not the monetary reward but the intoxicating prospect of the immortality that mathematics can bestow. Solving one of Clay's problems may earn you a million dollars, but that is nothing compared with carving your name on civilisation's intellectual map. The Riemann Hypothesis, Fermat's Last Theorem, Goldbach's Conjecture, Hilbert space, the Ramanujan tau function, Euclid's algorithm, the

Hardy–Littlewood Circle Method, Fourier series, Gödel numbering, a Siegel zero, the Selberg trace formula, the sieve of Eratosthenes, Mersenne primes, the Euler product, Gaussian integers – these discoveries have all immortalised the mathematicians who have been responsible for unearthing these treasures in our exploration of the primes. Those names will live on long after we have forgotten the likes of Aeschylus, Goethe and Shakespeare. As G.H. Hardy explained, 'languages die and mathematical ideas do not. "Immortality" may be a silly word, but probably a mathematician has the best chance of whatever it may mean.'

Those mathematicians who have laboured long and hard on this epic journey to understand the primes are more than just names set in mathematical stone. The twists and turns that the story of the primes has taken are the products of real lives, of a dramatis personae rich and varied. Historical figures from the French revolution and friends of Napoleon give way to modern-day magicians and Internet entrepreneurs. The stories of a clerk from India, a French spy spared execution and a Jewish Hungarian fleeing the persecution of Nazi Germany are bound together by an obsession with the primes. All these characters bring a unique perspective in their attempt to add their name to the mathematical roll call. The primes have united mathematicians across many national boundaries: China, France, Greece, America, Norway, Australia, Russia, India and Germany are just a few of the countries from which have come prominent members of the nomadic tribe of mathematicians. Every four years they converge to tell the stories of their travels at an International Congress.

It is not only the desire to leave a footprint in the past which motivates the mathematician. Just as Hilbert dared to look forward into the unknown, a proof of the Riemann Hypothesis would be the start of a new journey. When Wiles addressed the press conference at the announcement of the Clay prizes he was keen to stress that the problems are not the final destination:

> There is a whole new world of mathematics out there, waiting to be discovered. Imagine if you will, the Europeans in 1600. They know that across the Atlantic there is a New World. How would they have assigned prizes to aid in the discovery and development of the United States? Not a prize for inventing the airplane, not a prize for inventing the computer, not a prize for founding Chicago, not a prize for machines that would harvest areas of wheat. These things have become a part of America, but such things could not have been imagined in 1600. No, they would have given a prize for solving such problems as the problem of longitude.

The Riemann Hypothesis is the longitude of mathematics. A solution to the Riemann Hypothesis offers the prospect of charting the misty waters of the vast ocean of numbers. It represents just a beginning in our understanding of Nature's numbers. If we can only find the secret of how to navigate the primes, who knows what else lies out there, waiting for us to discover?