ST2132 Tutorial 8 Solution 2210

1. (a)

$$-\frac{1}{2}\log(2\pi) - \log\sigma - \frac{1}{2\sigma^2}(x_i-\mu)^2$$

$$\ell(\mu,\sigma) = -n\log\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

(b) $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2}$.

$$\frac{\partial\ell}{\partial\mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) = \frac{n}{\sigma^2}(\bar{x}-\mu), \qquad \frac{\partial\ell}{\partial\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$\frac{\partial\ell}{\partial\mu} = 0$ gives $\mu = \bar{x}$. Substituting μ by $\bar{x}$ yields $\frac{\partial\ell}{\partial\sigma} = -\frac{n}{\sigma} + \frac{n\hat{\sigma}^2}{\sigma^2}$, so that $\frac{\partial\ell}{\partial\sigma} = 0$ gives $\sigma = \hat{\sigma}$.

(c) At $(\bar{x}, \hat{\sigma})$,

$$\frac{\partial^2\ell}{\partial\mu^2} = -\frac{n}{\hat{\sigma}^2}, \qquad \frac{\partial^2\ell}{\partial\sigma^2} = -\frac{2n}{\hat{\sigma}^2}, \qquad \frac{\partial^2\ell}{\partial\mu\partial\sigma} = \frac{\partial^2\ell}{\partial\sigma\partial\mu} = 0$$

The determinant is

$$\left(-\frac{n}{\hat{\sigma}^2}\right) \times \left(-\frac{2n}{\hat{\sigma}^2}\right) - 0^2 = \frac{2n^2}{\hat{\sigma}^4} > 0$$

(d) All three conditions are satisfied from (c).

2. (a) Replacing σ by $\sqrt{\nu}$ in 1(a) gives the loglikelihood.

(b)

$$\frac{\partial\ell}{\partial\mu} = \frac{1}{\nu}\sum_{i=1}^{n}(x_i-\mu) = \frac{n}{\nu}(\bar{x}-\mu), \qquad \frac{\partial\ell}{\partial\nu} = -\frac{n}{2\nu} + \frac{1}{2\nu^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$\frac{\partial\ell}{\partial\mu} = 0$ gives $\mu = \bar{x}$. Subtituting μ by $\bar{x}$ in $\frac{\partial\ell}{\partial\nu}$ and setting it to 0 gives $\nu = \hat{\sigma}^2$. At $(\bar{x}, \hat{\sigma}^2)$,

$$\frac{\partial^2\ell}{\partial\mu^2} = -\frac{n}{\hat{\sigma}^2}, \qquad \frac{\partial^2\ell}{\partial\sigma^2} = -\frac{n}{2\hat{\sigma}^2}, \qquad \frac{\partial^2\ell}{\partial\mu\partial\sigma} = \frac{\partial^2\ell}{\partial\sigma\partial\mu} = 0$$

Similar as before, the determinant is positive. By the criteria in 1(d), $(\bar{x}, \hat{\sigma}^2)$ is the ML estimate of (μ, ν).

(c) The ML estimate of ν is the square of the ML estimate of σ.

3. (a) The data are realisations of $(X_1, \ldots, X_6) \sim$ Multinomial(60,$(p_1, \ldots, p_6)$).

(b) Since $\mathrm{E}(X_i) = 60p_i$, $\pi = \mathrm{E}(X_i)/60$, so $\hat{p}_i = X_i/60$.

(c) Since $\mathrm{var}(\hat{p}_i) = p_i(1-p_i)/60$, SE of the i-th estimate is $\sqrt{p_i(1-p_i)}/\sqrt{60}$.

(d) Bootstrap says to substituting p_i by its estimate in the SE formula.

(e) The interval for p_1 is (0.00, 0.13), which does not include $1/6 \approx 0.17$, suggesting that the die is unfair.

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x = c(4,6,17,16,8,9)
est = x/60
round(est, 2) # (b) (0.07,0.10,0.28,0.27,0.13,0.15)
SE = sqrt(est*(1-est)/60)
round(SE, 3) # (c) (0.032,0.039,0.058,0.057,0.044,0.046)
round(cbind(est - 1.96*SE, est + 1.96*SE), 2) # (e)
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4. (a) The counts are realisations of $(X_1, X_2, X_3, X_4) \sim$ Multinomial(3839,$\mathbf{p}$), where

$$p_1 = \frac{2+\theta}{4}, p_2 = p_3 = \frac{1-\theta}{4}, p_4 = \frac{\theta}{4}$$

(b) $0 \le 1 - r \le 1/2$, so $\Theta = [0, 1/4]$.

(c) The data are $x_1 = 1997, x_2 = 906, x_3 = 904, x_4 = 32$, summing to $n = 3839$. The log likelihood is

$$\ell(\theta) = x_1 \log(2+\theta) + (x_2 + x_3)\log(1-\theta) + x_4 \log\theta$$

Setting the derivative

$$\ell'(\theta) = \frac{x_1}{2+\theta} - \frac{x_2+x_3}{1-\theta} + \frac{x_4}{\theta}$$

to 0 and rearranging, we get

$$n\theta^2 + (-x_1 + 2x_2 + 2x_3 + x_4)\theta - 2x_4 = 0$$

which yields the given equation. The solutions are

$$\frac{-1655 \pm \sqrt{1655^2 - 4(3839)(-64)}}{2(3839)}$$

Only the larger root lies in Θ: 0.04.

$$\ell''(\theta) = -\frac{x_1}{(2+\theta)^2} - \frac{x_2+x_3}{(1-\theta)^2} - \frac{x_4}{\theta^2}$$

is negative at 0.04, so it is the ML estimate.