ST2132 Tutorial 1 Solution 2210

1. (a) Since adding a constant does not change variance, $\text{var}(aX+bY+c) = \text{var}(aX+bY)$. Let $\mu_X = \text{E}(X)$ and $\mu_Y = \text{E}(Y)$. Writing $(aX+bY)-(a\mu_X+b\mu_Y) = a(X-\mu_X)+b(Y-\mu_Y)$ gives

$$\{(aX+bY)-(a\mu_X+b\mu_Y)\}^2 = a^2(X-\mu_X)^2+b^2(Y-\mu_Y)^2+2ab(X-\mu_X)(Y-\mu_Y)$$

Now take expectation on both sides.

(b) This is a start

$$\begin{aligned} \text{cov}(aX+bY+c,Z) &= \text{E}\{(aX+bY+c)Z\} - \{a\text{E}(X)+b\text{E}(Y)+c)\}\text{E}(Z) \\ &= \cdots \end{aligned}$$

[Can you derive (a) from (b)?]

2. (a) Let X have the Bernoulli(1/2) distribution. By the Law of Large Numbers, mean and variance are roughly $\text{E}(X) = 1/2$ and $\text{var}(X) = 1/2(1-1/2) = 1/4$.

(b) Let X_1 and X_2 be IID Bernoulli(1/2) RV's. The y's are realisations of X_1+X_2, so their mean and variance are roughly $2\times 1/2 = 1$ and $2\times 1/2\times(1-1/2) = 1/2$.

[What is the distribution of X_1+X_2 called?]

(c) The z's are realisations of $2X$. Mean and variance are roughly $2\times 1/2 = 1$ and $4\times 1/4 = 1$.

[Can you write down the distribution of $2X$?]

3. (a)

$$\begin{aligned} \Pr(X_2=1) &= \frac{3}{10}\times\frac{2}{9}+\frac{7}{10}\times\frac{3}{9} = \frac{3}{10} \\ \Pr(X_1=1,X_2=1) &= \frac{3}{10}\times\frac{2}{9} = \frac{1}{15} \\ \Pr(X_1=1|X_2=1) &= \frac{1}{15}\div\frac{3}{10} = \frac{2}{9} \end{aligned}$$

(b) Select rows with first column showing 1. The mean and variance of their second column are roughly $\text{E}[X_2|1]$ and $\text{var}[X_2|1]$.

4. (a) np, $np(1-p)$, $n(1-p)$, $np(1-p)$.

(b) Since $H+T=n$, $\text{cov}(H,T) = \text{cov}(H,n-H) = \text{cov}(H,n)-\text{cov}(H,H)$. $\text{cov}(H,n) = 0$ from 1(b); $\text{cov}(H,H) = \text{var}(H)$ from lecture. Hence

$$\text{cov}(H,T) = np(1-p)$$

Another good way is to start from $0 = \text{var}(H+T)$, using 1(a).

(c)

$$\text{E} = \begin{bmatrix} np \\ n(1-p) \end{bmatrix} \qquad \text{var} = \begin{bmatrix} np(1-p) & -np(1-p) \\ -np(1-p) & np(1-p) \end{bmatrix}$$

The distribution can be obtained from the general case in lecture notes, with some change in symbols:

$$\Pr(H = h, T = t) = \binom{n}{h\ t} p^h (1-p)^t$$

where h and t are non-negative integers summing to n.

[Does the distribution look familiar?]

5. (a) Multinomial$(n, \mathbf{p})$, where $\mathbf{p} = (p_1, \ldots, p_r)$.

(b) $X_i \sim$ Binomial(n, p_i). $\text{E}(X_i) = np_i$, $\text{var}(X_i) = np_i(1-p_i)$.

(c) $X_i + X - j \sim$ Binomial$(n, p_i + p_j)$. Expand $n(p_i + p_j)(1 - p_i - p_j)$, and use 1(a) to get

$$\text{cov}(X_i, X_j) = -np_ip_j$$

[Show $(p_i + p_j)(1 - p_i - p_j) = p_i(1-p_i) - p_ip_j + p_j(1-p_j) - p_jp_i$, then continue.]

(d) $\text{E}(X_i) = np_i$, $\text{var}(X_i) = np_i(1-p_i)$, $\text{cov}(X_i, X_j) = -np_ip_j, i \neq j$. Or

$$\text{E} = \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_r \end{bmatrix} \qquad \text{var} = \begin{bmatrix} np_1(1-p_1) & -np_1p_2 & \cdots & -np_1p_r \\ -np_2p_1 & np_2(1-p_2) & \cdots & -np_2p_r \\ \vdots & \vdots & \ddots & \vdots \\ -np_rp_1 & -np_rp_2 & \cdots & np_r(1-p_r) \end{bmatrix}$$

6. (a) There are several methods. (i) Expand the right side to work to the left side. (ii) Expand

$$(x_i - z)^2 = \{(x_i - \bar{x}) - (z - \bar{x})\}^2$$

keeping the round brackets intact.

Here is a tidy method. Let $\Pr(X = x_i) = 1/n$, $1 \leq i \leq n$. Then (*) gives

$$\text{E}\{(X - z)^2\} = \text{var}(X) + \{\text{E}(X) - z\}^2$$

which is the identity divided by n.

(b) Let $z = 0$. "orthogonal". Let the two vectors be denoted by u and v. The left side is $(u+v)\cdot(u+v)$. The right side is $u\cdot u + v\cdot v$, so $u\cdot v = 0$. This is Pythagoras Theorem, in n dimensions.