ST2132 Tutorial 9 2210

1. Let $\mathbf{x} = (x_1, x_2, x_3)$ be a realisation of $\mathbf{X} = (X_1, X_2, X_3)$ with distribution fiven on slide 30 of Parameter Estimation II.

   (a) Simplify $\Pr(\mathbf{X} = \mathbf{x})$. Hence obtain the refined loglikelihood function $\ell(\theta)$.

   (b) Show that the ML estimate of θ is $\frac{x_2+2x_3}{2n}$.

   (c) Find two estimators of θ using MOM. Which method seems better, MOM or ML?

   (d) Calculate the information on θ contained in $\mathbf{X}$.

2. Let $X_1, \ldots, X_n$ be IID Normal(μ, ν) random variables.

   (a) Show that

   $$\mathcal{I}(\mu, \nu) = \begin{bmatrix} \frac{1}{\nu} & 0 \\ 0 & \frac{1}{2\nu^2} \end{bmatrix}$$

   Hence obtain the approximate variance of the ML estimators $(\hat{\mu}, \hat{\nu})$, for large n.

   (b) What is the exact variance of $(\hat{\mu}, \hat{\nu})$?

3. Let $(X_1, X_2, X_3) \sim$ Multinomial$(n, (p_1, p_2, p_3))$. Let $\hat{\mathbf{p}}$ be the ML estimator of $\mathbf{p} = (p_1, p_2, p_3)$.

   (a) Derive $\text{var}(\hat{\mathbf{p}})$ from the variance of (X_1, X_2, X_3).

   (b) Derive $\mathcal{I}$, the information on (p_1, p_2) contained in the Multinomial$(1, (p_1, p_2, p_3))$ distribution.

   (c) $\frac{\mathcal{I}^{-1}}{n}$ is a certain submatrix of $\text{var}(\hat{\mathbf{p}})$. Make a guess to obtain an expression for $\mathcal{I}^{-1}$, and verify it by multiplying with $\mathcal{I}$.

4. Use the Fisher information to estimate the SE of the ML estimate of θ in Tutorial 8 Question 4.

5. Let $X_1, \ldots, X_n$ be IID N$(0,\sigma^2)$ RV's.

   (a) Derive the MOM estimator of σ.

   (b) Derive the ML estimator of σ.

   (c) Derive $\mathcal{I}(\sigma)$.

   (d) For large n, approximately, what is the variance of the ML estimator of σ?

   (e) Show that $n\bar{X}^2/S^2 \sim F_{1,n-1}$, where

   $$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

   (f) What is the distribution of $\sqrt{n}\bar{X}/\sigma$?