ST2132 Tutorial 6 Solution 2210

1. (a)

$$\Pr\left(\bar{X} - t_{\frac{\alpha}{2},n-1}\frac{S}{\sqrt{n}} \le \mu \le \bar{X} + t_{\frac{\alpha}{2},n-1}\frac{S}{\sqrt{n}}\right) = 1-\alpha$$

(b) From slide 20 of Probability Review II, we know

$$\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$$

Hence

$$\Pr\left(-t_{\frac{\alpha}{2},n-1} \le \frac{\bar{X}-\mu}{S/\sqrt{n}} \le t_{\frac{\alpha}{2},n-1}\right) = 1-\alpha$$

and the inequalities are equivalent to those in (a).

(c) $\bar{x} = -0.21$, estimated SE ≈ 0.75, CI $= (-2.30, 1.88)$. One possible code:

```
x = c(-1.53, 2.24, 0.73, -0.72, -1.77)
xbar = mean(x) # estimate of mu
s = sd(x)
SE = s/sqrt(5) # esitmated SE
xbar + qt(0.975,4)*SE*c(-1,1) # exact 95% CI
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(d) Since $t_{\frac{\alpha}{2},n-1} \to z_{\frac{\alpha}{2}}$ as $n \to \infty$, it gets closer to the large-sample CI.

2. (a) Yes. $\mathrm{E}(\hat{\mu}_k) = \frac{1}{n}\sum_{i=1}^{n}\mathrm{E}(X_i^k) = \mu_k$.

(b) $\mathrm{var}(X_i^k) = \mathrm{E}(X_i^{2k}) - \{\mathrm{E}(X_i^k)\}^2 = \mu_{2k} - \mu_k^2$. Hence

$$\mathrm{var}(\hat{\mu}_k) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{var}(X_i^k) = \frac{\mu_{2k}-\mu_k^2}{n}$$

3. (a) $\mu_1 = \mathrm{E}(X_i) = p$. So the MOM estimator of p is $\hat{\mu}_1 = \bar{X}$.

(b) Since $\bar{X} = Y/n$, where $Y \sim \mathrm{Binomial}(n,p)$,

$$\Pr(\hat{p} = y/n) = \binom{n}{y}p^y(1-p)^{n-y}, \qquad y = 0, 1, \ldots, n.$$

(c) The MOM estimate of p is $\bar{x} = 0.3$. Exact SE is

$$\mathrm{SD}(\hat{p}) = \frac{\sqrt{p(1-p)}}{\sqrt{n}}$$

estimated as $\frac{\sqrt{0.3\times 0.7}}{\sqrt{10}} \approx 0.14$.

(d) It is unlikely to be good: Since the sample size is quite small, the normal approximation is not so accurate, so the actual confidence level is likely quite a bit less than 95%.

4. (a) $\mu_1 = \mathrm{E}(Y) = np$. The MOM estimator of p is $\hat{\mu}_1/n = Y/n$.

   (b) They are identical.

5. (a)

$$\mu_1 = \mathrm{E}(X_i) = \int_{-1}^{1} \frac{x + \alpha x^2}{2}\ \mathrm{d}x = \frac{\alpha}{2}\left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{\alpha}{3}$$

   $\int_{-1}^{1} x\ \mathrm{d}x = 0$ since x is an odd function. MOM estimator of α is $\hat{\alpha} = 3\hat{\mu}_1 = 3\bar{X}$. It is unbiased.

   (b) MOM estimate of α is $3 \times 0.04 = 0.12$. To get the SE, we need $\mathrm{var}(\hat{\alpha}) = 9\ \mathrm{var}(\bar{X}) = \frac{9}{225}\mathrm{var}(X_i)$.

$$\mu_2 = \mathrm{E}(X_i^2) = \int_{-1}^{1} \frac{x^2 + \alpha x^3}{2}\ \mathrm{d}x = \left[\frac{x^3}{6}\right]_{-1}^{1} = \frac{1}{3}$$

   so that

$$\mathrm{var}(X_i) = \mu_2 - \mu_1^2 = \frac{3 - \alpha^2}{9}$$

   Hence the SE is

$$\mathrm{SD}(\hat{\alpha}) = \frac{\sqrt{3 - \alpha^2}}{\sqrt{225}}$$

   estimated as $\sqrt{3 - 0.12^2}/15 \approx 0.12$.

   (c) An approximate 95%-CI for α is $(0.12 \pm 1.96 \times 0.12) = (-0.11, 0.35)$.