ST2132 Tutorial 7 Solution 2210

1. (a) For any $i = 1, \dots, r$, $\mathrm{E}(X_i) = np_i$. MOM estimator of p_i is $\hat{p}_i = X_i/n$. This is analogous to the binomial case in Tutorial 6 Question 4.

   (b) MOM estimates: 0.2 for p_1, 0.7 for p_2, 0.1 for p_3. Exact SE:

$$\mathrm{SD}(\hat{p}_i) = \frac{\sqrt{p_i(1-p_i)}}{\sqrt{10}}, \qquad i = 1, 2, 3$$

   estimated as

$$\frac{\sqrt{0.2 \times 0.8}}{\sqrt{10}} \approx 0.13, \qquad \frac{\sqrt{0.7 \times 0.3}}{\sqrt{10}} \approx 0.14, \qquad \frac{\sqrt{0.1 \times 0.9}}{\sqrt{10}} \approx 0.09$$

2. (a) Since both the expectation and variance of the Poisson(λ) are λ, by Tutorial 3 problem 3, $\mathrm{E}(\hat{\lambda}_1) = \mathrm{E}(\hat{\lambda}_2) = \lambda$. Both estimators are unbiased.

   (b) $\mathrm{var}(\hat{\lambda}_1) = \frac{\lambda}{n}$, so $\mathrm{SD}(\hat{\lambda}_1) = \frac{\sqrt{\lambda}}{\sqrt{n}}$.

   (c) $\mathrm{E}(\hat{\lambda}_1) \approx 0.80$, $\mathrm{E}(\hat{\lambda}_2) \approx 0.80$, $\mathrm{SD}(\hat{\lambda}_1) \approx 0.09$, consistent with (a) and (b): $\frac{\sqrt{0.8}}{\sqrt{100}} \approx 0.09$. $\mathrm{SD}(\hat{\lambda}_2) \approx 0.15$ is new.

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dat = matrix(rpois(1000000, 0.8), 1000, 100)
hl1 = apply(dat, 1, mean)
hl2 = apply(dat, 1, var)
mean(hl1)
mean(hl2)
sd(hl1)
sd(hl2)
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   [The second decimal place is not so stable. For instance, $\mathrm{SD}(\hat{\lambda}_2)$ may also be estimated as 0.14 or 0.16. Increasing the number of iterations gives more stability, but slowly.]

   (d) The simulation suggests that $\hat{\lambda}_1$ has a smaller SE.

3. (a)

$$\mu_1 = \int_0^\infty x\lambda e^{-\lambda x}\,\mathrm{d}x = \left[-xe^{-\lambda x}\right]_0^\infty + \int_0^\infty e^{-\lambda x}\,\mathrm{d}x = \left[-\frac{e^{-\lambda x}}{\lambda}\right]_0^\infty = \frac{1}{\lambda}$$

   (b) Since $\lambda = 1/\mu_1$, $\hat{\lambda} = 1/\hat{\mu}_1 = 1/\bar{X} = n/Y$.

$$\mathrm{E}(\hat{\lambda}) = \int_0^\infty \frac{n}{y}\frac{\lambda^n}{\Gamma(n)}y^{n-1}e^{-\lambda y}\mathrm{d}y = n\lambda\frac{\Gamma(n-1)}{\Gamma(n)}\int_0^\infty \frac{\lambda^{n-1}}{\Gamma(n-1)}y^{n-2}e^{-\lambda y}\mathrm{d}y = \frac{n}{n-1}\lambda$$

   [How to fix the bias? Substracting $\lambda/(n-1)$ is impossible, since λ is unknown. Mmultiplying by $(n-1)/n$ works.]

4. (a) $\Pr(X_i = x_i) = p^{x_i}(1-p)^{1-x_i}$.

(b) Joint probability is

$$\prod_{i=1}^{n} \Pr(X_i = x_i) = p^{\sum_{i=1}^{n} x_i}(1-p)^{n-\sum_{i=1}^{n} x_i} = p^y(1-p)^{n-y}$$

The logarithm is $\ell(p) = y \log p + (n-y)\log(1-p)$.

$$\ell'(p) = \frac{y}{p} - \frac{n-y}{1-p}, \qquad \ell''(p) = -\frac{y}{p^2} - \frac{n-y}{(1-p)^2}$$

Setting $\ell'(p) = 0$ gives $p = y/n$. Since $\ell''(p) < 0$ always, this is a maximum.

(c) $Y \sim \text{Binomial}(n, p)$. $\Pr(Y = y) = \binom{n}{y} p^y (1-p)^{n-y}$. The logarithm is the same as in (b) except for a constant, so it is maximum when $p = y/n$.

[Given realisations from n IID Bernoulli(p) RV's, the ML estimate of p is the proportion of 1's. Given a realisation from a binomial(n, p) RV, the ML estimate of p is also the proportion of 1's.]

5. $\mathrm{E}\{(\hat{\theta} - \theta)^2\} = \mathrm{var}(\hat{\theta} - \theta) + \{\mathrm{E}(\hat{\theta} - \theta)\}^2 = \mathrm{var}(\hat{\theta}) + \{\mathrm{E}(\hat{\theta}) - \theta\}^2$. $\mathrm{var}(\hat{\theta})$ is the square of the SE, and $\mathrm{E}(\hat{\theta}) - \theta$ is the bias.