ST2132 Tutorial 10 Solution 2210

1. (a) Using Tutorial 9 Question 2, for large n, approximately

$$\begin{bmatrix} \hat{\mu} \\ \hat{\nu} \end{bmatrix} \sim \mathrm{N}\left(\begin{bmatrix} \mu \\ \nu \end{bmatrix}, \begin{bmatrix} \frac{\nu}{n} & 0 \\ 0 & \frac{2\nu^2}{n} \end{bmatrix} \right) = \mathrm{N}\left(\begin{bmatrix} \mu \\ \sigma^2 \end{bmatrix}, \begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{bmatrix} \right)$$

(b) From (a),

$$1 - \alpha \approx \Pr\left(\hat{\sigma}^2 - z_{\frac{\alpha}{2}} \frac{\sqrt{2}\sigma^2}{\sqrt{n}} \leq \sigma^2 \leq \hat{\sigma}^2 + z_{\frac{\alpha}{2}} \frac{\sqrt{2}\sigma^2}{\sqrt{n}} \right)$$

where $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$. An approximate $(1-\alpha)$-CI for ν is

$$\left(\hat{\sigma}^2 - z_{\frac{\alpha}{2}} \frac{\sqrt{2}\hat{\sigma}^2}{\sqrt{n}}, \hat{\sigma}^2 + z_{\frac{\alpha}{2}} \frac{\sqrt{2}\hat{\sigma}^2}{\sqrt{n}} \right)$$

where $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2$.

2. (a) Let (Y_1, Y_2, Y_3, Y_4) have a similar multinomial distribution, but with 1 trial.

$$\log f(\mathbf{Y}) = Y_1 \log(2+\theta) + (Y_2 + Y_3)\log(1-\theta) + Y_4 \log\theta - \log 4$$

$$\frac{\mathrm{d}\log f(\mathbf{Y})}{\mathrm{d}\theta} = \frac{Y_1}{2+\theta} - \frac{Y_2+Y_3}{1-\theta} + \frac{Y_4}{\theta}, \qquad \frac{\mathrm{d}^2 \log f(\mathbf{Y})}{\mathrm{d}\theta^2} = -\frac{Y_1}{(2+\theta)^2} - \frac{Y_2+Y_3}{(1-\theta)^2} - \frac{Y_4}{\theta^2}$$

Hence

$$\mathcal{I}(\theta) = \frac{1}{4}\left(\frac{1}{2+\theta} + \frac{2}{1-\theta} + \frac{1}{\theta} \right) = \frac{1+2\theta}{2(2+\theta)(1-\theta)\theta}$$

For large n, approximately

$$\hat{\theta}_n \sim \mathrm{Normal}\left(\theta, \frac{2(2+\theta)(1-\theta)\theta}{(1+2\theta)n} \right)$$

(b) Substituting the ML estimate of θ, 0.04, into the asymptotic SD gives the estimated SE of 0.006. An approximately 95%-CI for θ is

$$0.04 \pm 1.96 \times 0.006 \approx (0.03, 0.05)$$

3. (a) For large n, $(\hat{p}_1, \ldots, \hat{p}_5)$ is approximately normally distributed with expectation $(p_1, \ldots, p_5)$ and variance given in slide 15 of Distribution of ML Estimators, with $r = 6$:

$$\frac{1}{n}\begin{bmatrix} p_1(1-p_1) & -p_1p_2 & -p_1p_3 & -p_1p_4 & -p_1p_5 \\ -p_2p_1 & p_2(1-p_2) & -p_2p_3 & -p_2p_4 & -p_2p_5 \\ -p_3p_1 & -p_3p_2 & p_3(1-p_3) & -p_3p_4 & -p_3p_5 \\ -p_4p_1 & -p_4p_2 & -p_4p_3 & p_4(1-p_4) & -p_4p_5 \\ -p_5p_1 & -p_5p_2 & -p_5p_3 & -p_5p_4 & p_5(1-p_5) \end{bmatrix}$$

Note that the matrix after $1/n$ is the inverse of the information matrix, and that $p_1 + cdots + p_5$ is not 1, but $1 - p_6$. The variance of $(\hat{p}_1 \ldots, \hat{p}_6)$ is not invertible.

4. The idea is to expand around a. Since $f(x) = x\log x - x\log a$,

$$f'(x) = \log x + 1 - \log a, \qquad f''(x) = 1/x$$

so that $f(a) = 0$, $f'(a) = 1$, $f''(a) = 1/a$. Hence

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 = 0 + 1 \times (x-a) + \frac{1}{2a} \times (x-a)^2$$

as to be shown.