ST2132 Tutorial 9 Solution 2210

1. (a)

$$\frac{n!}{x_1!x_2!x_3!}\{(1-\theta)^2\}^{x_1}\{2\theta(1-\theta)\}^{x_2}\{\theta^2\}^{x_3} = \frac{n!}{x_1!x_2!x_3!}2^{x_2}(1-\theta)^{2x_1+x_2}\theta^{x_2+2x_3}$$

$$\ell(\theta) = (2x_1+x_2)\log(1-\theta) + (x_2+2x_3)\log\theta$$

(b)

$$\ell'(\theta) = -\frac{2x_1+x_2}{1-\theta} + \frac{x_2+2x_3}{\theta}, \qquad \ell''(\theta) = -\frac{2x_1+x_2}{(1-\theta)^2} - \frac{x_2+2x_3}{\theta^2}$$

$\ell'(\theta) = 0$ implies $x_2 + 2x_3 = (2x_1 + 2x_2 + 2x_3)\theta = 2n\theta$. $\ell''(\theta) < 0$ always, so the ML estimate is $\frac{x_2+2x_3}{2n}$.

(c) $\mathrm{E}(X_3) = n\theta^2$, so one possible estimator is $\sqrt{\frac{X_3}{n}}$. Another one is $1 - \sqrt{\frac{X_1}{n}}$. X_2 can also be used, though the estimator is more complicated. Each of the three MOM estimators does not use the whole data, and it is unclear how they can be combined. ML is better.

(d)

$$\log f(\mathbf{X}) = c + (2X_1 + X_2)\log(1-\theta) + (X_2 + 2X_3)\log\theta$$

$$\frac{\partial \log f(\mathbf{X})}{\partial\theta} = -\frac{2X_1+X_3}{1-\theta} + \frac{X_2+2X_3}{\theta}$$

$$\frac{\partial^2 \log f(\mathbf{X})}{\partial\theta^2} = -\frac{2X_1+X_3}{(1-\theta)^2} - \frac{X_2+2X_3}{\theta^2}$$

$\mathrm{E}(X_2 + 2X_3) = 2n\theta$. Since $2X_1 + X3 = 2n - (X_2 + 2X_3)$, $\mathrm{E}(2X_1 + X_2) = 2n(1-\theta)$.

$$\mathcal{I}(\theta) = \frac{2n(1-\theta)}{(1-\theta)^2} + \frac{2n\theta}{\theta^2} = \frac{2n}{\theta(1-\theta)}$$

2. (a)

$$\log f(X) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log\nu - \frac{(X-\mu)^2}{2\nu}$$

$$\frac{\partial \log f(X)}{\partial\mu} = \frac{X-\mu}{\nu}, \qquad \frac{\partial \log f(X)}{\partial\nu} = -\frac{1}{2\nu} + \frac{(X-\mu)^2}{2\nu^2}$$

$$\frac{\partial^2 \log f(X)}{\partial\mu^2} = -\frac{1}{\nu}, \qquad \frac{\partial^2 \log f(X)}{\partial\nu^2} = \frac{1}{2\nu^2} - \frac{(X-\mu)^2}{\nu^3} : \mathrm{E} = \frac{1}{2\nu^2} - \frac{1}{\nu^2} = -\frac{1}{2\nu^2}$$

$$\frac{\partial^2 \log f(X)}{\partial\mu\partial\nu} = \frac{\partial^2 \log f(X)}{\partial\nu\partial\mu} = -\frac{X-\mu}{\nu^2} : \mathrm{E} = 0$$

For large n,

$$\mathrm{var}\begin{bmatrix}\hat\mu \\ \hat\nu\end{bmatrix} \approx \frac{\mathcal{I}(\mu,\nu)^{-1}}{n} = \begin{bmatrix}\frac{\nu}{n} & 0 \\ 0 & \frac{2\nu^2}{n}\end{bmatrix}$$

(b) $\hat{\nu} = \nu/n \times (n-1)S^2/\nu$ and $\hat{\mu}$ are independent, so their covariance is 0.

$$\text{var}(\hat{\nu}) = \frac{\nu^2}{n^2} \times 2(n-1) = \frac{n-1}{n}\frac{2\nu^2}{n}$$

$$\text{var}\left[\begin{array}{c} \hat{\mu} \\ \hat{\nu} \end{array}\right] = \left[\begin{array}{cc} \frac{\nu}{n} & 0 \\ 0 & \frac{n-1}{n}\frac{2\nu^2}{n} \end{array}\right]$$

3. (a)

$$\text{var}(\hat{\mathbf{p}}) = \frac{1}{n^2}\text{var}(X_1, X_2, X_3) = \frac{1}{n}\begin{pmatrix} p_1(1-p_1) & -p_1p_2 & -p_1p_3 \\ -p_1p_2 & p_2(1-p_2) & -p_2p_3 \\ -p_1p_3 & -p_2p_2 & p_3(1-p_3) \end{pmatrix}$$

(b) Let $(Y_1, Y_2, Y_3) \sim$ Multinomial(1,(p_1, p_2, p_3)).

$$\log f(\mathbf{Y}) = Y_1 \log p_1 + Y_2 \log p_2 + Y_3 \log(1 - p_1 - p_2)$$

$$\frac{\partial \log f(\mathbf{Y})}{\partial p_i} = \frac{Y_i}{p_i} - \frac{Y_3}{1-p_1-p_2} = \frac{Y_i}{p_i} - \frac{Y_3}{p_3}, \quad i = 1, 2$$

$$\frac{\partial^2 \log f(\mathbf{Y})}{\partial p_i^2} = -\frac{Y_i}{p_i^2} - \frac{Y_3}{p_3^2}, \quad i = 1, 2$$

$$\frac{\partial^2 \log f(\mathbf{Y})}{\partial p_i p_j} = -\frac{Y_3}{p_3^2}, \quad i \neq j$$

Since $\text{E}(Y_i) = p_i$,

$$\mathcal{I}(p_1, p_2) = \begin{pmatrix} \frac{1}{p_1} + \frac{1}{p_3} & \frac{1}{p_3} \\ \frac{1}{p_3} & \frac{1}{p_2} + \frac{1}{p_3} \end{pmatrix}$$

(c) Guess top left 2×2 submatrix:

$$\mathcal{I}^{-1} = \begin{pmatrix} p_1^2 & -p_1p_2 \\ -p_1p_2 & p_2(1-p_2) \end{pmatrix}$$

Multiplying this with $\mathcal{I}$ should give the identity matrix.

4. Probability mass function of the multinomial distribution with 1 trial:

$$f(\mathbf{y}) = \left(\frac{2+\theta}{4}\right)^{y_1} \left(\frac{1-\theta}{4}\right)^{y_2+y_3} \left(\frac{\theta}{4}\right)^{y_4}, \qquad x_i = 0, 1, \quad y_1 + y_2 + y_3 + y_4 = 1$$

$$\log f(\mathbf{Y}) = Y_1 \log(2+\theta) + (Y_2 + Y_3)\log(1-\theta) + Y_4 \log\theta - \log 4$$

$$\frac{\mathrm{d} \log f(\mathbf{Y})}{\mathrm{d}\theta} = \frac{Y_1}{2+\theta} - \frac{Y_2 + Y_3}{1-\theta} + \frac{Y_4}{\theta}$$

$$\frac{\mathrm{d}^2 \log f(\mathbf{Y})}{\mathrm{d}\theta^2} = -\frac{Y_1}{(2+\theta)^2} - \frac{Y_2+Y_3}{(1-\theta)^2} - \frac{Y_4}{\theta^2}$$

$$\mathcal{I}(\theta) = \frac{1}{4(2+\theta)} + \frac{1}{2(1-\theta)} + \frac{1}{4\theta}$$

SE in 0.04 is roughly

$$\sqrt{\frac{\mathcal{I}(0.04)^{-1}}{3839}} \approx 0.01$$

5. (a) $\mu_1 = \mathrm{E}(X_i) = 0$, $\mu_2 = \mathrm{E}(X_i^2) = \sigma^2$. So $\sigma = \sqrt{\mu_2}$, and MOM estimator is

$$\sqrt{\hat{\mu}_2} = \sqrt{\frac{1}{n}\sum_{i=1}^n X_i^2}$$

(b)

$$\ell(\sigma) = -n\log\sigma - \frac{\sum_{i=1}^n X_i^2}{2\sigma^2}$$

$$\ell'(\sigma) = -\frac{n}{\sigma} + \frac{\sum_{i=1}^n X_i^2}{\sigma^3}, \qquad \ell''(\sigma) = \frac{n}{\sigma^2} - \frac{3\sum_{i=1}^n X_i^2}{\sigma^4}$$

$\ell'(\sigma) = 0$ gives $\sigma^2 = \frac{1}{n}\sum_{i=1}^n X_i$. At this σ, $\ell'' = -2n/\sigma^2 < 0$. So the ML estimator is the same as the MOM.

(c)

$$\log f(X) = -\frac{1}{2}\log(2\pi) - \log\sigma - \frac{X^2}{2\sigma^2}$$

$$\frac{\mathrm{d}\log f(X)}{\mathrm{d}\sigma} = -\frac{1}{\sigma} + \frac{X^2}{\sigma^3}, \qquad \frac{\mathrm{d}^2\log f(X)}{\mathrm{d}\sigma^2} = \frac{1}{\sigma^2} - \frac{3X^2}{\sigma^4}$$

Hence $\mathcal{I}(\sigma) = 2/\sigma^2$.

(d) Approximately

$$\frac{\mathcal{I}(\sigma)^{-1}}{n} = \frac{\sigma^2}{2n}$$

(e) From Tutorial 6 Question 1, $\sqrt{n}\bar{X}/S \sim t_{n-1}$. From Probability Review II slide 18, its square has an $F_{1,n-1}$ distribution.

(f) Since $\bar{X} \sim \mathrm{N}(0, \sigma^2/n)$, $\bar{X}/(\sigma/\sqrt{n}) = \sqrt{n}\bar{X}/\sigma \sim \mathrm{N}(0{,}1)$.