1 Introduction

Some questions to ask before starting on a problem

- Extract out important keywords (what DS to use?)
- Edge cases? *e.g.* if `size==0` or `size==1`,
- Trivial cases? can just hardcode

Code styling

- CS2030 Code Styling Guide
- Google Java Styling Guide
- **Modularity**: use method to print answers inside main method

```
\\ print answer
ans = simulate(n,k,m);
printAns();
```

- **No global variables**

2 Java

How to throw exception?

```
public class MyException extends
    Exception {
  private int var;
  public MyException(int var) {
    this.var = var
  }
  public int getVar() {
    return this.var;
  }
}

public class Main {
  public static void main(String[] args
   ) {
    try {
      ...
      throw new MyException(errorVar);
    } catch (MyException e) {
      System.out.println(e.getVar());
    }
  }
}
```

3 Data Structures

$$O(1) < O(\log(n)) < O(n^c) \text{ where } c < 1$$

$$O(n) < O(\log(n!)) = O(n\log(n)) < O(n^2)$$

$$O(n^k)[\text{ where } k > 2] < O(k^n)[\text{ where } k \geq 1] < O(n!)$$

How to implement Data Structures?

- Composition: use well-known DS as an attribute of the implemented DS
- Inheritance: extends well-known DS

3.1 Linked List

- Motivation: implementation of list using array needs to occupy contiguous memory space (can result in memory error)
- Variants of linked list:
  - Tailed (need to maintain `head` and `tail`)
  - Circular
  - Doubly linked (`prev` and `next` attributes for `ListNode`)
- How to find cycle?
  Answer: use fast and slow pointers

```
    slow = slow.next;
    fast = fast.next.next;
```

- **[IMPT]** Drawing pictures is very important to visualize the program!
- Sometimes maintaining two pointers is good (especially for deletion)

```
Node head = new Node();
Node prev = head;
```

Java API: `ArrayList` or `LinkedList`

```
\\ constructor
ArrayList<Integer> list = new
 ArrayList<Integer>();
```

3.2 Stack

```
// to construct an array of generics
E[] arr = (E[]) new Object[size];
/*
// does not work
E[] arr = new E[size]
*/
```

Uses:

- **[IMPT]** Converting infix to postfix expression (Lecture 4 Slide 28)
- **[IMPT]** Evaluating postfix expression

3.3 Queue

Uses:

- **[IMPT]** Breadth-first traversal of trees
- Sliding Window (especially important for contiguous blocks of stuff)

Implementations:

- ArrayDeque: can remove from back and front
  - However Java API does not allow random access
  - Hence, need to implement our own (Lab 3C)

4 Recursion

[IMPT] Recipe for recursion (3 fingers)

1. General recursive case: identify simpler instances of the same problem
2. Base case: cases that we can solve without recursion
3. Be sure that we are able to reach the simplest instance so that we won't end up in infinite loop

Uses

- Insert item into sorted LinkedList
- Tower of Hanoi
- **[IMPT]** Combination (n choose k)
- Binary search

- Finding k-th smalles element (use pivot element p)
  move elements < p to the left of p
  move elements > p to the right of p
- Printing all permutations of a String

Overloading: same function name but with different parameters (useful in Java)

Backtracking

- Solving problems recursively by trying to build a solution incrementally, one piece at a time, removing those solutions that fail to satisfy the constraints of the problem at any point in time
- *e.g.* Queens Lab 4B: board is fixed
  queens can be added or removed!

```
Pair pos = new Paid(i,j);
if (isValidBoard(pos, currentQueens
    )) {
  currentQueens.add(pos);
  solveQueens(n, queensLeft-1,
   currentQueens);
  currentQueens.remove(pos);
}
```

5 Sorting

Some definitions

- **Sort key**: use particular value of an object to do comparison and sort
- **In-place**: requires only a constant amount of extra space during the sorting process
- **Stable**: relative order of elements with the same key value is preserved by the algorithm

Some ideas used in sorting:

- Internal vs external sort
- Iterative vs recursive
- Comparison vs non-comparison based
  e.g. radix sort
- Divide and conquer

Applications

- Uniqueness testing
- Deleting duplicates
- Frequency counting
- Efficient searching

	Iterative	Recursive
Comparison	Bubble, Selection, Insertion	Quick, Merge
Non-comparison		Radix

5.1 Algorithms

5.1.1 Selection Sort

Time complexity: $O(n^2)$
Limitation: Not stable

5.1.2 Bubble Sort

Time complexity: $O(n^2)$

- Using flag: $O(n)$ `isSorted`, is the input already sorted?

5.1.3 Insertion Sort

Time complexity:

- Best case: input already sorted ($O(n)$)
- Worst case: input reversely sorted ($O(n^2)$)

5.1.4 Merge Sort

Time complexity:

- `merge(arr, left, mid, right)` is $O(right-left+1)$
- `merge` is called $\log n$ times
- Hence $O(n \log n)$

Limitations:

- Need temporary array to store values during the `merge` process (not in-place)

5.1.5 Quick Sort

Time complexity:

- `partition()`
- `quicksort(a, i, p)`
- Worst case is when it is already sorted, so the first group (elements $<$ p) is empty: $O(n^2)$
- Best case: occurs when array is divided into 2 equal halves
  - Depth is $\log n$
  - Each level takes n comparisons (including swaps)
  - Hence $O(n \log n)$ which is also the average case

Limitation: Not stable

5.1.6 Radix Sort

Treat each data as a character string: no comparison needed
Trick: sort by unit digit $\rightarrow$ tenth digit $\rightarrow$ hundredth and so on...
Time complexity:

- Initialize 10 groups (queues) to group the elements
- Complexity is $O(dn)$ where d is the maximum number of digits of the n numeric strings in the array

Limitation: Not in-place

5.1.7 Bucket Sort

How it works:

- There are b buckets, and each element `arr` is inserted into bucket according to a function *e.g.* `(int) arr[j]*10`
- Similar to radix sort but b can be any number (base?)
  e.g. Tut 5 Q 3(b) where $N \leq$ `arr[i]` $\leq 3N$, we can have $3N$ buckets $1, 2, 3, \ldots, 3N$ so that each bucket contains only 1 element
- So only 1 pass is needed a.k.a. $O(3N)$ time
- Possible problem: takes up alot of space?

	Worst Case	Best Case	In-place?	Stable?
Selection	$O(n^2)$	$O(n^2)$		No
Insertion	$O(n^2)$	$O(n)$		
Bubble	$O(n^2)$	$O(n^2)$		
Bubble (Flag)	$O(n^2)$	$O(n)$		
Merge	$O(n \log n)$	$O(n \log n)$	No	
Radix	$O(dn)$	$O(dn)$	No	
Quick	$O(n^2)$	$O(n \log n)$		No

5.2 Java Sorting

For list/arrays:
- To convert arrays to list use `Arrays.asList`
- `Arrays.sort` or `Collections.sort`

For others: use `Collections.sort(list, compObj)`

```
import java.util.Comparator;
class ObjComparator implements
    Comparator<Obj> {
  public int compare(Obj o1, Obj o2) {
    // if positive, o1 > o2
    // if negative, o1 < o2
    // if zero,     o1 = o2
  }
    public boolean equals(Object obj) {
      // check to see if we have the
   same comparator object
      return this = obj;
    }
}
```

6 Hashing

Map ADT: <key, value> pairs mapping with 3 basic operations
- **Retrieval**: retrieve value using the given key
- **Insertion**: insert/replace a value using the given key
- **Deletion**: delete the <key, value> pair using the given key

Hash Table: data structure that uses a **hash function** to efficiently map keys to values, for efficient search and retrieval
Types of Tables:
- Direct Addressing Table
  - Restrictions:
    - Keys must be non-negative integer values
    - Range of keys must be small
    - Keys must be dense
- Hash Tables
  - Map large integers to smaller integers (mod?)
  - Map non-integers to integers
  - **Collision**: hash function does not guarantee two different keys go to two different slots
    two different keys have the same **hash value**
  - Criteria of good hash functions
    - Fast to compute
    - Scatter keys evenly
    - Less collisions
    - Need less slots
  - Perfect hash functions: *one-to-one* mapping between keys and hash values *i.e.* no collision occurs
  - Minimal perfect hash functions: table size is the same as the number of keywords supplied
  - **Uniform Hash functions**: distributes keys evenly in the hash table (e.g. mod, floor function)

$$hash(k) = \left\lfloor \frac{km}{X} \right\rfloor \text{ where } k = 0, 1, 2, \ldots, X-1$$

    - Division method: map into a table with m slots
      $$hash(k) = k \bmod m$$
    - Multiplication method:
      1. Multiply by a constant real number A between 0 and 1
         **Knuth recommends $A = 1/\phi = 0.618033$ to minimize collisions*
      2. Extract the fractional part
      3. Multiply by m, the hash table size
         $$hash(k) = \lfloor m(kA - \lfloor kA \rfloor) \rfloor$$
  - How to choose m?
    * Pick a prime number close to a power of two
    * power of 10 or 2 are no good cos it's the same as extracting the last few digits of decimal/binary representation
  - **Hashing of Strings**
    - Summing all the characters is no good: because strings with same letters but different orders will collide
    - How to solve: take into account order of characters!

```
hash(s):
  sum=0
  for each character c in s:
    sum = sum*31+s
  return sum%m
```

6.1 Collision Resolution

$$\alpha(\text{load factor}) = \frac{n(\text{total number of keys})}{m(\text{number of slots})}$$

α measures how full the hash table is

Criteria of good collision resolution method
- Minimize clustering
- Always find an empty slot if it exists
- Give different probe sequences when 2 initial probes are the same (secondary hash function)
- Fast

6.1.1 Separate Chaining

- Use LinkedList to store keys with the same slot location
- Ideally can sort the LinkedList based on key to aid in searching

Some problems:
- find(key) and delete(key) takes $O(n)$ time
- α is the average length of the LinkedList and will increase when n increases
  - Hence it is good to keep α bounded $\Rightarrow$ reconstruct the whole table when α exceeds a bound
- Not cache friendly

6.1.2 Linear Probing

Probe sequence:

```
hash(key)
hash(key+1)%m
hash(key+2)%m ...
```

- **Insert:**
  When we get a collision, we scan linearly for the next available slot and put the key there
- **Find:**
  Probe sequence increases linearly from hash(k) until current key is equal to the key we want to find
- **Delete**
  - **[IMPT]** Cannot simply remove a value because find() only works when contiguous cells are occupied
  - So how? Use lazy deletion (three different states of a slot)
    * Occupied
    * Occupied but mark as deleted
    * Empty

Some problems:

- **Primary clustering**: Can create many consecutive slots, increasing running time of find/insert/delete $O(n)$

Modified linear probing: (d and m are co-primes) to avoid primary clustering

```
hash(key)
hash(key+1*d)%m
hash(key+2*d)%m ...
```

6.1.3 Quadratic Probing

```
hash(key)
hash(key+1^2)%m
hash(key+2^2)%m ...
```

Theorem of Quadratic Probing

> If $\alpha < 0.5$(half-full) and m is prime, the we can always find an empty slot

Some problems

- When table is more than half-full, there can be endless looping!
  - To avoid table half-full, we can **resize** the table
  - Usually new $m = 2 \times m$
  - But also need to re-hash all existing keys (expensive operation)
- **Secondary clustering**: if two keys have the same initial position, their probe sequences are the same, but not as bad as linear probing

6.1.4 Double Hashing

Use a secondary hash function

```
hash(key)
hash(key+1*hash2(key))%m
hash(key+2*hash2(key))%m ...
```

Note that the secondary hash function must not evaluate to 0 (otherwise it's the same as separate chaining if not worse because of infinite loop)

$$\text{hash}_2(\text{key}) = p - (\text{key mod } p)$$

7 Trees

Some terminologies: ancestor, descendant, parent, sibling, child, root, leaf node

- Internal node: has one or more children, but root node is not an internal node
- Level of a node: level of root is 0, depends on how far it is from the root
- Height: maximum level of the nodes
- Size: number of nodes
- **Binary tree**: each node has at most 2 ordered children
- **Full binary tree**: all nodes at level $< h$ have two children, where h is the height of the tree
- **Complete binary tree**: full down to level $h-1$, with level h filled in from left to right

Implementations

- Reference based
- Array based

```
class BinaryTree {
  int root;
  int free; // free space
  TreeNode tree[];
}

```

  - What is free space?
    the last element where the slot is free, if there are multiple free slots, all the slots before the last will link towards the last one ⇒ last one is the pointer of the free
  - Representing complete tree using an array: use heap?

$$i_{left} = i * 2 + 1, i_{parent} = i//2$$

7.1 Traversals

7.1.1 Post-order traversal

Traverse the root after traversing the left and right subtrees

```
postorder(T) {
  if T is not empty then {
    postorder(T.left)
    postorder(T.right)
    print(T.item)
}}
```

7.1.2 Pre-order traversal

Traverse the root before traversing the left and right subtrees

```
preorder(T) {
  if T is not empty then {
    print(T.item)
    postorder(T.left)
```

```
    postorder(T.right)
}}
```

7.1.3 In-order traversal

Traverse the root in between traversing the left and right subtrees (**Do a sweep from left to right**)

```
inorder(T) {
  if T is not empty then {
    postorder(T.left)
    print(T.item)
    postorder(T.right)
}}
```

7.1.4 Level-order traversal

Traverse the tree level by level and from left to right (Queue is important)

```
levelorder(T) {
  if T is empty return
  Q = new Queue
  Q.enqueue(T)

  while Q is not empty {
    curr = Q.dequeue()
    print curr.item
    if curr.left is not empty {
      Q.enqueue(curr.left)
    } if curr.right is not empty {
        Q.enqueue(curr.right)
        }
}}
```

7.1.5 Evaluation of Expression Tree

Note that post-order, in-order, and pre-order of expression tree will produce postfix, infix, and prefix expressions **[IMPT]** Recursive procedure!

7.2 Binary Search Trees (BST)

Some interesting stuff

- We can construct an AVL Tree/BST from a sorted array in $O(N)$ time where N is the size of the array

Visualgo Hacks:

- Possible number of structurally different BSTs with n distinct elements

$$f(n) = \sum_{i=0}^{n-1} f(i) \times f(n-1-i)$$

  where $f(0) = 1$ and $f(1) = 1$
- Minimum number of vertices in an AVL Tree of a given height h

$$f(h) = f(h-1) + f(h-2) + 1$$

  where $f(1) = 2$ and $f(2) = 4$

Some operations: Usually $O(h)$, but note that it's possible that $h = n$ if it is skewed (hence need AVL Tree)

- Find min/max element
- Search for x
- Insertion
- Deletion (3 cases)
  - node to be deleted T has no children
  - T has only 1 child (left)
  - node to be deleted T has two children $\Rightarrow$ replace with **successor (smallest element in the right subtree)**
- Successor/Predecessor
  [IMPT] note that if this was to be implemented, we need a `.parent` attribute as an addition to the `.child attribute`
- Inorder traversal $\Rightarrow$ each node will be traversed not more than 3 times! wow

7.3 AVL Trees

Property:

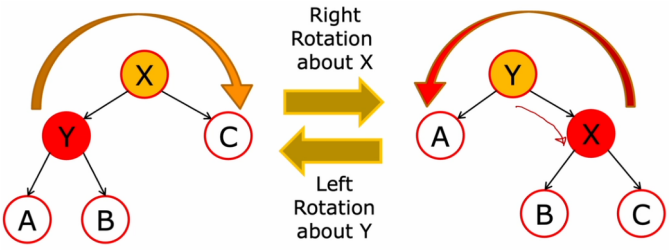
- At any node, the difference in height between left and right subtree is at most 1 (invariant)

$$|h_l - h_r| \leq 1$$

- A height balanced tree with N vertices has height $h < 2 \times \log_2(N) \Rightarrow h = O(\log(N))$
- Minimal AVL Tree of height h: having the height h and fewest possible number of nodes

Operations

- `rotateRight()/rotateLeft()`



- `insert()`: do BST to find the appropriate node to insert to, then have two cases, and also need to pass through parents to see if they are height-balanced
  - insert outside: single rotation
  - insert inside: double rotation
- Find k-th smallest item
  store `.size` attribute and do recursive stuff **[IMPT]**
  Quickselect and partition on unsorted array
  Running time:
  - BST: $O(h)$
  - Unsorted array: Best case is $O(N)$, Worst case is $O(N^2)$

7.4 TreeMap Java API

- `higherkey()`, `floorkey()`: similar to predecessor and successor
- Has sorted key in tree structure

8 Priority Queue

Property
- Insert item with a given key
- Remove the item with maximum key

Implementations
- Unsorted list: insertion takes $O(1)$ but deletion takes $O(n)$ to remove the maximum key
- Sorted list: insertion takes $O(n)$ time but deletion takes $O(n)$ time
- Heap!

8.1 Heap

VisuAlgo Hacks
- Maximum number of swaps between heap elements required to construct a max heap of n elements using the $O(n)$ `BuildHeap(arrr)`: happens when the array is sorted in ascending order
- Maximum number of comparison between heap elements required to construct a max heap of n elements using the $O(n)$ `BuildHeap(arrr)`

Properties:
- A **complete binary tree**
  - Either is empty,
  - , or satisfies the **heap property**: for every node v, the search key in v is greater or equal to those in the children of v
- Usually we talk about max heap
- **[IMPT]** Half of the items are leaves!
- Some access stuff

```
left(i)   = 2*i + 1
right(i)  = 2*i + 2
parent(i) = floor((i-1)/2)
```

Operations
- `heapRebuild(i)`: swap down from index i until it reaches a leaf and satisfies the heap property
- `heapify()`: build a heap from an unsorted array (utilises `heapRebuild`) and is used for heapSort
  - **Running time** = $O(n)$
  - Total number of nodes $= n = 2^{h+1} - 1$
  - Total number of bubbling down $= n-h-1$ which is less than the total number of edges connecting the nodes
  - Worst case is when the array is sorted in ascending order (assuming max heap)

```
heapify(arr) {
  for (int i=size/2; i>=0; i--)
    {
     heapRebuild(i);
}}
```

- `heapSort`: partition the unsorted array into two parts, the heap and sorted portion; remove the max value from heap and put it into the sorted portion so that eventually the array is in ascending order
  - In-place
  - Not stable (because of bubbling/swapping operations)
  - Complexity $= O(n \log n)$

9 Union-Find Disjoint Sets (UFDS)

Definition: collection of disjoin sets, ordering not important
- Each set is modeled as a tree
- A collection of disjoin sets forms a forest of trees
- Each set is represented by a representative item (root)
- `p[i]` records the parent of item `i`, if `p[i]==i`, then `i` is a root

Operations:
- `unionSet(i,j)`: union two disjoin sets containing `i` and `j` respectively
  - **Union-by-Rank** heuristic: make the resulting combined tree shorter by adding shorter tree to longer tree
  - If height of trees are the same, we do not use this heuristic and just add normally
  - Use another integer array `rank`
    - `rank[i]` is the upper bound of the height of subtree rooted at `i`

```
unionSet(int i, int j) {
  if (!isSameSet(i,j)) {
    int x=findSet(i);
    int y=findSet(j);
    if (rank[x] > rank[y]) {
      p[y]=x; // add y to x
    } else {
       p[x]=y; // add x to y
       if (rank[x]==rank[y]) {
         rank[y]=rank[y]+1;
       }
     }
  }
}
```

- `findSet`: which set an item belongs to, trace recursively until `p[i]==i`
  - Then we compress the nodes on the path to the root to make future find operations very fast $O(1)$
  - Complexity is $O(\log N)$, but with path compression can get to $O(1)$

```
findSet(int i) {
  if (p[i]==i) {
    return i;
  } else {
    p[i] = findSet(p[i]);
    return p[i];
  }
}
```

- `isSameSet(i,j)`: Check if two items belong to the same set

9.1 Uses of UFDS

- **[IMPT]** Detect Cycle in a graph!!!!
  - Process edges one by one, union vertices if they have the same edge
  -

10 Graphs

Properties

- Loops are possible; trees don't have loops
- Multiple paths are possible; Trees only have 1 path from A to B

Types of graphs

- Weighted/Unweighted
- Directed/Undirected
- **Complete Graph**: Simple graph with n vertices and ${}_nC_2$ edges
- Sparse/Dense: not so many edges vs many edges (arbitrary definition)
- Disconnected/Connected

Some problems

- Shortest Path Problem: What is the shortest way to travel between A and B?
- Traveling Salesman Problem: How to minimize the cost of visiting n cities such that we visit each city exactly once, and finishing at the city where we start from?
- Topological Sort: Find a sequence of modules to take that satisfy the prerequisite requirements
- 4-Colours Problem

10.1 Implementation

Some terminologies

- In/Out Degree of a vertex: for directed graph
- Cycle: only possible if there are **[IMPT]** $geqn$ edges!!
- Path: number of edges in a path
- Adjacent vertices: $adj(v)$ is a set of vertices adjacent to v
  - For directed graphs

$$\sum_v |adj(v)| = |E|$$

  - For undirected graphs

$$\sum_v |adj(v)| = 2|E|$$

Implementations

- Adjacency matrix: space complexity is $O(V^2)$
- Edge List: list all the edges (an Edge class) in a list space complexity is $O(E)$ where $E = O(V^2)$
- Adjacency List: stores all the neighbours of vertex A; space complexity is $O(V + E)$
- Vertex Map: similar to adjacency list but uses HashMap

10.2 Traversal

10.2.1 BFS

- Use Queue for ordering
- How to differentiate visited vs univisited vertices?
  - 1D array `visited` of size V, `visited[v]=0` initially, and `visited[v]=1` when v is visited
- How to memorize the path?
  - 1D array `parent` of size V, where `p[v]` denotes the predecessor of v
- The edges will form a spanning tree
- Running time is $O(E)+O(V) = O(V+E)$ if adjacency list is used (Main loop (secondary loop) + initialization inside first loop)

$$O(\sum_v adj(curr)) = O(E)$$

- Running time is $O(V^2)$ if adjacency matrix is used

Uses:

- **Building BFS Tree**

```
Q = new Queue
Q.enq(v)
mark v as visited
while Q is not empty {
  curr = Q.deq()
  print curr
  for each w in curr.adj { // an array of adjacent nodes
    if w is not visited {
      Q.enq(w)
      w.parent = curr
      mark w as visited
}}}
```

- **Building BFS Tree**

```
Q = new Queue
Q.enq(v)
mark v as visited
v.level=0
while Q is not empty {
  curr = Q.deq()
  print curr
  for each w in curr.adj { // an array of adjacent nodes
    if w is not visited {
      Q.enq(w)
      w.level = curr.level+1
      mark w as visited
}}}
```

10.2.2 DFS

- Go to each path as far as we can go
- Use a stack
- Edges used in DFS traversal will form a DFS spanning tree
- $O(V + E)$

```
S = new Stack
S.push(v)
print and mark v as visited
while S is not empty {
  curr = S.top() // top of stack
  if every vertex in adj(curr) is
   visited {
    S.pop()
  } else {
    let w be an unvisited vertex in adj
   (curr)
    S.push(w)
```

```
    print and mark w as visited
}}
```

10.2.3 Uses of BFS/DFS

- Searching for a vertex/node
- Reachability test: is v reachable from vertex u?
- Find shortest path between 2 vertices
- Identifying components (disconnected subgraphs)
  - Run BFS on one node, and then label the visited vertices if they can be reached from the first node *e.g.* 1,2,3,...
- **[IMPT]** Checking bipartite graph
- **[IMPT]** Finding Spanning Tree
  - The tree is the MST if the edges are unweighted
  - If weighted, then maybe not
- **Cycle Detection**: DFS!
  - Undirected: need to make sure that the path goes back to parent node
    - **[IMPT]** Running time is $O(V)$ if the graph is not necessarily connected; can just stop if there are more than $V-1$ edges traversed in that particular connected component
    - Possible false positive: two-nodes cycle, need to protect against this
  - Directed: 3 states: unvisited, not completed, completed (Tut 9)
  - **[IMPT]** Alternatively can also use UFDS!
- **Alternative DFS cycle detection in directed graph**
  - There is a cycle in a graph only if there is a back edge present in the graph
  - Check vertices currently in the recursion stack to detect back edge
    - Basically check if we visit a vertex where it is on the path we are currently tracing
- **Topological sort**: Module selection
  - Should be a **Directed Acyclic Graph (DAG)**: a directed graph with no cycle since otherwise there will be a cycle where mods are prerequisites of each other
  - Nice properties: if we go according to Topological Sort order, we ensure that the vertex will not be revisited again
  - **Goal**: Order the vertices, such that if there is a path from u to v, u appears before v in the output
  - **Complexity**: $O(V+E)$

```
q = new Queue()
put all vertices with in-degree
    0 into q
while q is not empty
  v = q.deq()
  print v
  remove v from G
  enqueue neighbours of v with
   in-degree 0
```

10.3 Shortest Path

- **Path**: sequence of vertices $v_1, v_2, \ldots, v_n$ where $(v_i, v_{i+1}) \in E$
- **Cost of a path**: sum of the cost of all edges in the path
- **distance(v)**: shortest distance from s to v
- **parent(v)**: previous node on the shortest path so far from s to v
- **weight(u, v)**: the weight of the edge from u to v
- BFS does not work because must track of the smallest distance so far
  - Need to be able to backtrack and undo path if better solution is found

[IMPT] Some other uses for SSSP:

- Can use Djikstra to compute minimax path (Lab 11A),
  - Initialize `maxcap = 0` for all vertices, except `maxcap[src] = INFINITY`
  - Relax operation updates the `maxcap` for each vertex (cap is the weight of edge connecting `u` and `v`)

```
// maintains the maximum value
    of all the edges connected to
     v
maxcap[v] = max(maxcap[v], min(
    maxcap[u], cap))
```

10.3.1 Dijkstra Algorithm

Some observations [IMPT]

- Set an attribute `distance` in each node, which measures (preliminary) distance from source to the node.

```
d = distance(v) + cost(v,w);
if (distance(w) > d) {
  // set distance to be d
  distance(w) = d;
  parent(w)=v
}
```

- Let `v` be a node with the minimum distance `distance(v)` out of all the unvisited nodes, then no matter how many times we call `relax`, the distance is still unchanged

```
color all vertices yellow (uncompleted)
for each vertex w:
  distance(w) = INFINITY
distance(s) = 0
while there are yellow vertices:
  v = yellow vertex with the min
   distance (distance(v))
  color v red
  for each neighbour w of v:
    relax(v,w)
```

Time complexity

- Total time complexity is $O((V+E)\log V)$
- Initialization takes $O(V)$ time
- Remember that $\sum_v adj(v) = 2|E|$ so there would be $O(E)$ relax operations in total, where hopefully `relax` is $O(\log V)$
- **[IMPT]** `decreaseKey` need to be efficient (preferably $O(\log V)$) so that `relax` can run in $O(\log V)$ time
  - Need to keep another array to keep track of where a specified node with id `w` is in the heap

  - Or alternatively can store this attribute in the Vertex class
  - **[IMPT]** since PQ is not updated automatically when an item's priority is updated, then we need to call decreaseKey to update every time we update

```
// initialization is O(V)
for each vertex w {
  distance(w) = INFINITY;
}
distance(s) = 0;
pq = new PriorityQueue(V);

// main loop is O((V+E) log V)
while pq is not empty { // O(V)
  v = pq.deleteMin(); // O(log V) heap
  // means is colored red already
  for each neighbour w of v {
    //relax(v,w) // O(adj(v) * log V) in total
    // because when distance is updated, heap is not updated, so need to update manually using heapify() or heapRebuild
    d = distance(v) + cost(v,w);
    if (distance(w) > d) {
      // set distance to be d
      pq.decreaseKey(w,d);
      parent(w)=v
    }
}}
```

10.3.2 Modified Dijkstra's Algorithm

Key idea (**Lazy Data Structure**):

- No decreaseKey operation, simply add the relaxed node to the PriorityQueue (allow duplicate nodes in the PQ)
- Since the relaxed node has lower weight anyways
- If the node is extracted already, mark other nodes with same IDs as processed already
- Optimization: Once we have finalized $|V|$ nodes, we can stop already
- Running time: $O(V+E)\log V$
  - Initialization is $O(V)$
  - The number of edges is an upper bound on the queue size (we can relax each edge maximum once)
  - Upper bound for the number of edges is $|E| = |V|^2$
  - Each enqueue/dequeue operation is $O(\log |E|) = O(\log |V|^2) = O(\log |V|)$ where $|E|$ is the max queue size
  - Same complexity as original Dijkstra, but probably a constant factor difference

```
for each vertex w {
  D[w] = INFINITY; // distance array
}
D[s] = 0;
pq = new PriorityQueue();
pq.push(s,0); // enqueue source
count = |V|; // optimization

while (pq.isEmpty() && count > 0) {
  (d,u) = pq.deleteMin();
  if (d == D[u]) {
    count--;
    foreach neighbour w of u {
      if (D[w] > D[u] + cost(u,w)) {
        D[w] = D[u] + cost(u,w); // relax
        pq.push((D[w],w)); // push to pq
        parent(w) = u;
      }
    }
  }
}
```

10.3.3 Bellman-Ford Algorithm

- Dijkstra's Algorithm does not allow negative weights as the vertex with minimum distance is already considered as finalized
  - Can deal with negative weight cycles (detection)
  - Can detect negative weight cycles
  - Is quite simple (no special data structure needed)
  - but has higher runtime complexity than Dijkstra
- Negative cycle is a special case (a cycle that sums up to negative number) since we can traverse the cycle an infinite number of times to get a $-\infty$ cost of path
- Complexity is $O(VE)$ which can reach up to $O(V^3)$ if $E \sim V^2$
- Use an edge list (don't need to be sorted)

```
class Edge {
  int src, dest, weight;
  Edge() {
    src = dest = weight = 0;
  }
}

// edge list
int V, E;
Edge[] Edge;
```

- Initialization $O(V)$: set the distance of every node to infinite except src

```
int[] dist = new int[V];
for (int i==0; i<V; ++i) {
  dist[i] = Integer.MAX_VALUE;
}
dist[src] = 0;
```

- Do the following $|V-1|$ times: relax each edge in every round, each round is $O(E)$ which means in total we have $O(VE)$

```
// relax
if (dist[v] > dist[u] + weight of edge uv) {
  // update dist[v]
```

```
    dist[v] = dist[u] + weight of
      edge uv;
}
```

- **[IMPT]** Last round (V^{th} round) is used to check **negative weight cycle**
  - If there is anymore changing edgeweight, then it has negative cycle and thus not valid

10.4 Minimum Spanning Tree (MST)

MST Problem: given connected graph G with positive edge weights, find a minimum weight set of edges that connects all of the vertices

- Note that a tree of a graph G has $|V|$ number of edges
- Minimize total costs to connect every single vertex

[IMPT] Uses of MST Algorithms

- Find the path which contains the max minimum edge to maximize capacity (Lab 11A)
- Find the most difficult cell along the easiest path (Tut 10.1)

10.4.1 Kruskal's Algorithm

Greedy algorithm!: it works because the tree will confirm have $|V| - 1$ edges

1. Sort all edges in non-decreasing order of their weight (from smallest to largest)
   - Edge List representation is useful
2. Pick the smallest edge
3. Check if it forms a cycle with the spanning tree formed so far. If cycle is not formed, include this edge, else, discard it
   - Cycle detection uses UFDS
4. Repeat step 2-3 until there are $(|V| - 1)$ edges in the spanning tree

Complexity

- Sorting of Edges takes $O(E \log E)$ time
- Iterating through all edges and apply union-find algorithm takes at most $O(\log V)$ per edge, so $O(E \log V)$
- Overall complexity is $O(E \log E + E \log V) = O(E \log V)$

10.4.2 Prim's Algorithm

Key Idea"

1. Start with an empty spanning tree
2. Maintain two sets of vertices
   - 1st set contains the vertices in the MST
   - 2nd set contains vertices not yet included
3. At every step, consider all the edges that connect the two sets, and pick the minimum weight edge from these edges
   - Conceptually similar to Dijkstra as we finalize some edges
   - **Graph cut**: a group of edges that connects two sets of vertices in a graph
   - At this step we need to find the minimum weight edge from the cut and include this vertex into the MST
4. Move the other endpoint of the edge to the set containing the MST

Procedure/Algorithm

1. Initialize an array `parent` and `weight` that keeps track of the weight and parent
   - The parent array is the output which is used to show the constructed MST
2. Create a set `mstSet` that keeps track of vertices already included in the MST
3. Assign a key value to all vertices in the graph
   - The first vertex has key value 0
   - The rest have key value ∞
4. While `mstSet` does not include all vertices, do the following:
   (a) Pick a vertex u which is not in `mstSet` and has minimum key value
   (b) Include u into `mstSet`
   (c) Update key values of all adjacent vertices of u
      - For every adjacent vertex v, if weight of edge $u \leftrightarrow v$ is less than the previous key value of v, update the key value as weight$(u \leftrightarrow v)$

Complexity

- $O(V^2)$ if the input graph is represented using an adjacency list
- With binary heap, can be reduced to $O(E \log V)$ (same as Kruskal's)

11 Java Tricks

- OOP is important (CardGame)
  - If it involves an array, OOP is useful, methods can just modify properties/attributes of the object class (*e.g.* `reversed=true; increment=4`)
    - Especially true if only need to print statement at the end
- Invariance: property that stays constant???
  Lab 5C: Pancakes: Use number of inversions if it's even or not
- Use `StringBuilder` for return statements
  - Java StringBuilder API
  - Zigzag conversion
- Tut 07 $\Rightarrow$ Contiguous cells can be represented by start and end only
- **[IMPT] Iterator**

```
LinkedList<Integer> ll = new
    LinkedList<>();
Iterator<Integer> it = ll.iterator
    ();
it.next();
// must call next first before
    removing;
it.remove();
it.next();
// must call next first
it.remove();
```

- **Mix and Match**:
  - Sometimes it's useful to combine two data structures rather than maintaining two separate DS
    - *e.g.* BST + Sorted List for easy successor and printOrder access
    - *e.g.* Queue + Doubly Linked List + BST