ST2132 Tutorial 2 2210

1. In slide 29 of Probability Review I, let $N = 4$ and $K = 1$.

   (a) If $x_1 = 1$, what predictor for X_2 has the smallest MSE, and what is the MSE?

   (b) Repeat (a), with $x_1 = 0$.

   (c) Might you find the predictor in (b) strange? Suppose $x_1 = 0$, and the predictor in (a) is used to predict X_2. What is the MSE?

   (d) Suppose $x_1 = 1$ and I use $\mathrm{E}(X_1)$ to predict X_2. What is my MSE?

   (e) Repeat (d), with $x_1 = 0$.

2. This is a continuation from slide 29 of Probability Review I. See also slide 9 of Probability Review II. Recall that

$$\mathrm{E}[X_2|X_1] = \frac{K - X_1}{N - 1}, \qquad \mathrm{var}[X_2|X_1] = \frac{(K - X_1)\{(N - K) - (1 - X_1)\}}{(N - 1)^2}$$

   (a) Calculate $\mathrm{E}(\mathrm{E}[X_2|X_1])$.

   (b) Show that

$$\mathrm{var}[X_2|X_1] = \frac{K(N - K) - K(1 - X_1) - (N - K)X_1}{(N - 1)^2}$$

   (c) Show that

$$\begin{aligned} \mathrm{var}(\mathrm{E}[X_2|X_1]) &= \frac{K(N - K)}{(N - 1)^2}\frac{1}{N^2} \\ \mathrm{E}(\mathrm{var}[X_2|X_1]) &= \frac{K(N - K)}{(N - 1)^2}\frac{N - 2}{N} \end{aligned}$$

   (d) Obtain the sum of the expressions in (c).

3. (a) Show that for any $\alpha > 0$, $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.

   (b) Let X be a Gamma(α, λ) RV. Calculate $\mathrm{E}(X)$ and $\mathrm{var}(X)$.

4. Let $Y_n \sim \chi^2_n$. According to the Central Limit Theorem, as $n \to \infty$, the distribution of

$$\frac{Y_n - a}{b}$$

   converges to N(0,1). Determine the values of a and b.

5. Let X and Y have the bivariate normal distribution with

$$\mathrm{E}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} \qquad \mathrm{var}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}$$

where $\sigma_X > 0$, $\sigma_Y > 0$ and $-1 < \rho < 1$. It can be shown that

(i) $X \sim \mathrm{N}(\mu_X, \sigma_X^2)$

(ii) Given $X = x$ (any real number),

$$Y \sim \mathrm{N}\left(\mu_Y + \frac{\rho\sigma_Y}{\sigma_X}(x - \mu_X), (1-\rho^2)\sigma_Y^2\right)$$

(a) Given x, what predictor has the smallest MSE, and what is its MSE value?

(b) What are the distributions of $\mathrm{E}[Y|X]$ and $\mathrm{var}[Y|X]$? Verify the formulae for $\mathrm{E}(Y)$ and $\mathrm{var}(Y)$.

(c) What is the mean MSE for predicting Y from X?

(d) The correlation between X and Y is

$$\frac{\mathrm{cov}(X, Y)}{\mathrm{SD}(X)\mathrm{SD}(Y)}$$

Calculate the correlation.

(e) Show that

$$\frac{\mathrm{E}[Y|x] - \mu_Y}{\sigma_Y} = \rho\,\frac{x - \mu_X}{\sigma_X}$$