

1. (a) Since adding a constant does not change variance, $\text{var}(aX + bY + c) = \text{var}(aX + bY)$. Let $\mu_X = E(X)$ and $\mu_Y = E(Y)$. Writing $(aX + bY) - (a\mu_X + b\mu_Y) = a(X - \mu_X) + b(Y - \mu_Y)$ gives

$$\{(aX + bY) - (a\mu_X + b\mu_Y)\}^2 = a^2(X - \mu_X)^2 + b^2(Y - \mu_Y)^2 + 2ab(X - \mu_X)(Y - \mu_Y)$$

Now take expectation on both sides.

- (b) This is a start

$$\begin{aligned} \text{cov}(aX + bY + c, Z) &= E\{(aX + bY + c)Z\} - \{aE(X) + bE(Y) + c\}E(Z) \\ &= \dots \end{aligned}$$

[Can you derive (a) from (b)?]

2. (a) Let X have the Bernoulli(1/2) distribution. By the Law of Large Numbers, mean and variance are roughly $E(X) = 1/2$ and $\text{var}(X) = 1/2(1 - 1/2) = 1/4$.

(b) Let X_1 and X_2 be IID Bernoulli(1/2) RV's. The y 's are realisations of $X_1 + X_2$, so their mean and variance are roughly $2 \times 1/2 = 1$ and $2 \times 1/2 \times (1 - 1/2) = 1/2$.

[What is the distribution of $X_1 + X_2$ called?]

(c) The z 's are realisations of $2X$. Mean and variance are roughly $2 \times 1/2 = 1$ and $4 \times 1/4 = 1$.

[Can you write down the distribution of $2X$?]

3. (a)

$$\begin{aligned} \Pr(X_2 = 1) &= \frac{3}{10} \times \frac{2}{9} + \frac{7}{10} \times \frac{3}{9} = \frac{3}{10} \\ \Pr(X_1 = 1, X_2 = 1) &= \frac{3}{10} \times \frac{2}{9} = \frac{1}{15} \\ \Pr(X_1 = 1 | X_2 = 1) &= \frac{1}{15} \div \frac{3}{10} = \frac{2}{9} \end{aligned}$$

(b) Select rows with first column showing 1. The mean and variance of their second column are roughly $E[X_2|1]$ and $\text{var}[X_2|1]$.

4. (a) np , $np(1 - p)$, $n(1 - p)$, $np(1 - p)$.

(b) Since $H + T = n$, $\text{cov}(H, T) = \text{cov}(H, n - H) = \text{cov}(H, n) - \text{cov}(H, H)$. $\text{cov}(H, n) = 0$ from 1(b); $\text{cov}(H, H) = \text{var}(H)$ from lecture. Hence

$$\text{cov}(H, T) = np(1 - p)$$

Another good way is to start from $0 = \text{var}(H + T)$, using 1(a).

(c)

$$E = \begin{bmatrix} np \\ n(1-p) \end{bmatrix} \quad \text{var} = \begin{bmatrix} np(1-p) & -np(1-p) \\ -np(1-p) & np(1-p) \end{bmatrix}$$

The distribution can be obtained from the general case in lecture notes, with some change in symbols:

$$\Pr(H = h, T = t) = \binom{n}{h \ t} p^h (1-p)^t$$

where h and t are non-negative integers summing to n .

[Does the distribution look familiar?]

5. (a) Multinomial(n, \mathbf{p}), where $\mathbf{p} = (p_1, \dots, p_r)$.

(b) $X_i \sim \text{Binomial}(n, p_i)$. $E(X_i) = np_i$, $\text{var}(X_i) = np_i(1-p_i)$.

(c) $X_i + X_j \sim \text{Binomial}(n, p_i + p_j)$. Expand $n(p_i + p_j)(1 - p_i - p_j)$, and use 1(a) to get

$$\text{cov}(X_i, X_j) = -np_i p_j$$

[Show $(p_i + p_j)(1 - p_i - p_j) = p_i(1 - p_i) - p_i p_j + p_j(1 - p_j) - p_j p_i$, then continue.]

(d) $E(X_i) = np_i$, $\text{var}(X_i) = np_i(1 - p_i)$, $\text{cov}(X_i, X_j) = -np_i p_j$, $i \neq j$. Or

$$E = \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_r \end{bmatrix} \quad \text{var} = \begin{bmatrix} np_1(1-p_1) & -np_1(1-p_2) & \cdots & np_1(1-p_r) \\ -np_2(1-p_1) & np_2(1-p_2) & \cdots & np_2(1-p_r) \\ \vdots & \vdots & \ddots & \vdots \\ -np_r(1-p_1) & -np_r(1-p_2) & \cdots & np_r(1-p_r) \end{bmatrix}$$

6. (a) There are several methods. (i) Expand the right side to work to the left side.

(ii) Expand

$$(x_i - z)^2 = \{(x_i - \bar{x}) - (z - \bar{x})\}^2$$

keeping the round brackets intact.

Here is a tidy method. Let $\Pr(X = x_i) = 1/n$, $1 \leq i \leq n$. Then (*) gives

$$E\{(X - z)^2\} = \text{var}(X) + \{E(X) - z\}^2$$

which is the identity divided by n .

(b) Let $z = 0$. “orthogonal”. Let the two vectors be denoted by u and v . The left side is $(u + v) \cdot (u + v)$. The right side is $u \cdot u + v \cdot v$, so $u \cdot v = 0$.