# 1 Table of Distributions

| Notation | PMF/PDF and Support | Expected Value | Variance | MGF |
|---|---|---|---|---|
| Bernoulli<br>Be($p$) | $P(X=1)=p$<br>$P(X=0)=q$ | $p$ | $pq$ | $q+pe^t$ |
| Binomial<br>Bin($n,p$) | $P(X=k)=\binom{n}{k}p^kq^{n-k}$<br>$k=0,1,2,\dots,n$ | $np$ | $npq$ | $(q+pe^t)^n$ |
| Geometric<br>Geom($p$) | $P(X=k)=pq^{k-1}$<br>$k=1,2,\dots$ | $\frac{1}{p}$ | $\frac{q}{p^2}$ | $\frac{pe^t}{1-qe^t}$,<br>$qe^t<1$ |
| Negative Binomial<br>NB($r,p$) | $P(X=k)=\binom{k-1}{r-1}p^rq^{k-r}$<br>$k=r,r+1,\dots$ | $\frac{r}{p}$ | $\frac{rq}{p^2}$ | $\left(\frac{pe^t}{1-qe^t}\right)^r$,<br>$qe^t<1$ |
| Hypergeometric<br>HGeom($n,N,m$) | $P(X=k)=\frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$<br>$k=0,1,\dots n$ | $\frac{nm}{N}$ | $\frac{nm}{N}\left[\frac{(n-1)(m-1)}{N-1}+1-\frac{nm}{N}\right]$ | messy |
| Poisson<br>Poisson($\lambda$) | $P(X=k)=\frac{e^{-\lambda}\lambda^k}{k!}$<br>$k=0,1,2,\dots$ | $\lambda$ | $\lambda$ | $e^{\lambda(e^t-1)}$ |
| Uniform<br>U($a,b$) | $f(x)=\frac{1}{b-a}$<br>$x\in(a,b)$ | $\frac{a+b}{2}$ | $\frac{(b-a)^2}{12}$ | $\frac{e^{bt}-e^{at}}{(b-a)t}$ |
| Normal<br>$\mathcal{N}(\mu,\sigma^2)$ | $f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$<br>$x\in(-\infty,\infty)$ | $\mu$ | $\sigma^2$ | $\exp\left(\mu t+\frac{\sigma^2t^2}{2}\right)$ |
| Exponential<br>Exp($\lambda$) | $f(x)=\lambda e^{-\lambda x}$<br>$x\in[0,\infty)$ | $\frac{1}{\lambda}$ | $\frac{1}{\lambda^2}$ | $\frac{\lambda}{\lambda-t}$,<br>$t<\lambda$ |
| Gamma<br>Gamma($a,\lambda$) | $f(x)=\frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}$<br>$x\in[0,\infty)$ | $\frac{\alpha}{\lambda}$ | $\frac{\alpha}{\lambda^2}$ | $\left(\frac{\lambda}{\lambda-t}\right)^a$,<br>$t<\lambda$ |
| Beta<br>Beta($a,b$) | $f(x)=\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}$<br>$x\in(0,1)$ | $\frac{a}{a+b}$ | $\frac{ab}{(a+b)^2(a+b+1)}$ | messy |
| Chi-Square<br>$\chi^2_n$ | $f(x)=\frac{1}{2^{n/2}\Gamma(n/2)}x^{n/2-1}e^{-x/2}$<br>$x\in(0,\infty)$ | $n$ | $2n$ | $(1-2t)^{-n/2}$<br>$t<1/2$ |

# 2 Useful Results

## 2.1 Probability

**Sample space** is the set of all possible outcomes of an experiment, usually denoted by $S$. For tossing two dice,

$$S=\{(1,1),(1,2),\dots,(1,6),(2,1),\dots,(6,6)\}$$
$$=\{(i,j):1\le i,j\le 6\}$$

**Event** Any subset $A$ of the sample space is an event

**Odds** The odds of an event $A$ is defined by $\frac{P(A)}{1-P(A)}$

### Increasing/Decreasing Events

If $\{E_n\}$ is an increasing sequence of events, then

$$\lim_{n\to\infty}E_n=\bigcup_{i=1}^{\infty}E_i$$

If $\{E_n\}$ is a decreasing sequence of events, then

$$\lim_{n\to\infty}E_n=\bigcap_{i=1}^{\infty}E_i$$

If $\{E_n\}$ is either an increasing or decreasing sequence of events, then

$$P(\lim_{n\to\infty}E_n)=\lim_{n\to\infty}P(E_n)$$

### Conditional Probability

$$P(A|B)=P(A|BC)P(C|B)+P(A|BC^C)P(C^C|B)$$

**Inclusion/Exclusion Principle** Let $E_1, E_2, \dots, E_n$ be any events, then

$$\begin{aligned}P(E_1 \cup E_2 \cup \cdots \cup E_n) = &\sum_{i=1}^{n} P(E_i) - \sum_{1 \le i_1 < i_2 \le n} P(E_{i_1} \cap E_{i_2}) + \cdots \\ &(-1)^{r+1} \sum_{1 \le i_1 < \cdots < i_r \le n} P(E_{i_1} \cap \cdots \cap E_{i_r}) \\ &+ \cdots + (-1)^{n+1} P(E_1 \cap \cdots \cap E_n)\end{aligned}$$

**Derangement/Matching Problem**

$$P(\text{at least one match}) = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots (-1)^{n+1}\frac{1}{n!}$$

$$P(k \text{ matches}) = \binom{n}{k} \frac{1}{n(n-1)\cdots(n-k+1)} \cdot \left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots (-1)^{n-k}\frac{1}{(n-k)!}\right)$$

## 2.2 Discrete RV

Suppose $X \sim$ Poisson$(\lambda)$ and $Y \sim$ Bin$(X, p)$, then $X - Y$ and $Y$ are independent and

$$X - Y \sim \text{Poisson}(\lambda(1-p)), Y \sim \text{Poisson}(\lambda p)$$

## 2.3 Continuous RV

**Exponential Distribution**

$$P(X > s + t | X > s) = P(X > t)$$

**Gamma Distribution**

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$$

- $\Gamma(1) = 1$ and $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$
- $\Gamma(n) = (n-1)!$ for $n = 1, 2, 3, \dots$
- If $X_i \sim$ Exp$(\lambda)$ independently, then $\sum_{i=1}^n X_i \sim$ Gamma$(n, \lambda)$

**Beta Distribution**

$$\begin{aligned}B(a,b) &= \int_0^1 t^{a-1}(1-t)^{b-1} dt \\ &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\end{aligned}$$

$$\text{Beta}(1,1) \equiv \text{U}(0,1)$$

**Chi-Square Distribution**

$$\chi_n^2 \equiv \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

**Monotonic Transformation**

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left|\frac{d}{dy} g^{-1}(y)\right| & \text{if } y \in \mathcal{R}(g) \\ 0 & \text{otherwise}\end{cases}$$

## 2.4 Joint Distribution

$$\begin{aligned}P(a_1 < X \le a_2, b_1 < Y \le b_2) &= F_{X,Y}(a_2, b_2) \\ &- F_{X,Y}(a_1, b_2) + F_{X,Y}(a_1, b_1) - F_{X,Y}(a_2, b_1)\end{aligned}$$

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

$$f_{X,Y}(x,y) = h(x)g(y) \Leftrightarrow X \text{ and } Y \text{ are independent}$$

**Convolution of Independent Distributions**

$$F_{X+Y}(a) = \int_{-\infty}^{\infty} F_X(a-y) f_Y(y) dy$$

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y) f_Y(y) dy$$

**Bivariate Normal Distribution**

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \times \exp\left(-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right)$$

**Joint Transformation**

$$f_{Y_1,Y_2,\dots,Y_n}(y_1, y_2, \dots, y_n) = f_{X_1,X_2,\dots,X_n}(x_1, x_2, \dots, x_n) \frac{1}{|J(x_1, x_2, \dots, x_n)|}$$

where

$$J(x_1, x_2, \dots, x_n) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \frac{\partial g_n}{\partial x_2} & \cdots & \frac{\partial g_n}{\partial x_n}\end{vmatrix},$$

$Y_i = g_i(X_1, X_2, \dots, X_n)$, and $x_i = h_i(y_1, y_2, \dots, y_n)$ for $i = 1, 2, \dots, n$

## 2.5 Expectations and Variance

### Expectations

- If $X$ and $Y$ are independent, then $E[g(X)h(Y)] = E[g(x)]E[h(y)]$
- $E(X) = E[E[X|Y]]$

### Covariance

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] \\ = E(XY) - E(X)E(Y)$$

$$Cov\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j Cov(X_i, Y_j)$$

$$Var\left(\sum_{k=1}^{n} X_k\right) = \sum_{k=1}^{n} Var(X_k) + 2\sum_{1\le i<j\le n} Cov(X_i, X_j)$$

$$Var(X|Y) = E[X^2|Y] - [E(X|Y)]^2$$

$$Var(X) = E[Var(X|Y)] + Var[E(X|Y)]$$

### Correlation

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

$\rho(X,Y) = \pm 1$ iff $Y = \pm aX + b$ where $a = \frac{\sigma_Y}{\sigma_X}$

### Moment Generating Functions

$$M_X(t) = E(e^{tX})$$

$$E(X^n) = \frac{d^n}{dt^n} M_X(t)\bigg|_{t=0}$$

### Taylor Series Expansion

$$g(x) = g(\mu) + \frac{g'(\mu)}{1!}(x-\mu) + \frac{g''(\mu)}{2!}(x-\mu)^2 + \dots$$

**Multiplicative Property** If $X$ and $Y$ are independent, then $M_{X+Y}(t) = M_X(t)M_Y(t)$

**Uniqueness Property** If $\exists t$ such that $M_X(t) = M_Y(t) \forall t \in (-h,h)$ then $X$ and $Y$ have the same distributions

### Joint Moment Generating Functions

$$M_{X_1,X_2,\dots,X_n}(t_1,t_2,\dots,t_n) = E[e^{t_1X_1+t_2X_2+\dots+t_nX_n}]$$

$$M_{X_i}(t) = E[e^{tX_i}] = M_{X_1,X_2,\dots,X_n}(0,\cdots,0,t,0,\cdots,0)$$

## 2.6 Inequalities

$$a \le X \le b \Rightarrow a \le E(X) \le b$$

**Monotone Property** $X \le Y \Rightarrow E(X) \le E(Y)$

**Boole's inequality**

$$P\left(\bigcup_{k=1}^{n} A_k\right) \le \sum_{k=1}^{n} P(A_k)$$

**Markov's inequality** Let $X$ be a **nonnegative** random variable. For $a > 0$, we have

$$P(X \ge a) \le \frac{E(X)}{a}$$

**Chebyshev's inequality** For $a > 0$, we have

$$P(|X-\mu| \ge a) \le \frac{Var(X)}{a^2}$$

**One-sided Chebyshev's inequality** If $X$ is a r.v. with mean $0$ and finite variance $\sigma^2$, then, for any $a > 0$,

$$P(X \ge a) \le \frac{\sigma^2}{\sigma^2 + a^2}$$

**Jensen's inequality** If $g(x)$ is a convex function, then

$$E[g(x)] \ge g[E(X)]$$

# 3 Poker Terms

**blackjack** The goal of blackjack is to beat the dealer's hand without going over 21. If you go over 21 you bust, and the dealer wins regardless of the dealer's hand. If a player's first two cards are an ace and a "ten-card" (a picture card or 10), giving a count of 21 in two cards, this is a natural or "blackjack."

**five of a kind** A hand that contains five cards of one rank, but is only possible when using one or more wild cards (3♦ 3♣ 3♥ 3♠ Jkr)

**flush** A hand that contains five cards all of the same suit, not all of sequential rank (K♣ 10♣ 7♣ 6♣ 4♣)

**four of a kind** A hand that contains four cards of one rank and one card of another rank (9♦ 9♣ 9♥ 9♠ J♥)

**full house** A hand that contains three cards of one rank and two cards of another rank (3♣ 3♠ 3♦ 6♣ 6♥)

**high card** A hand that does not fall into any other category (K♥ J♥ 8♣ 7♦ 4♠)

**one pair** A hand that contains two cards of one rank and three cards of three other ranks (4♥ 4♠ K♠ 10♦ 5♠)

**pair** See **one pair**

**rank** Individual cards are ranked, from highest to lowest: A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3 and 2. Suits are not ranked, so hands that differ by suit alone are of equal rank.

**set** See **three of a kind**

**straight** A hand that contains five cards of sequential rank, not all of the same suit (7♣ 6♠ 5♠ 4♥ 3♥)

**straight flush** A hand that contains five cards of sequential rank, all of the same suit (A♥ K♥ Q♥ J♥ 10♥ or 5♥ 4♥ 3♥ 2♥ A♥)

**suit** One of the categories into which the cards of a deck are divided: diamonds (♦), clubs (♣), hearts (♥) and spades (♠)

**three of a kind** A hand that contains three cards of one rank and two cards of two other ranks (2♦ 2♠ 2♣ K♠ 6♥)

**two pair** A hand that contains two cards of one rank, two cards of another rank and one card of a third rank (J♥ J♣ 4♣ 4♠ 9♥)

**no pair** See **high card**