1 Table of Distributions

Notation	PMF/PDF and Support	Expected Value	Variance	MGF
Bernoulli Be(p)	$P(X=1)=p$ $P(X=0)=q$	p	pq	$q+pe^t$
Binomial Bin(n,p)	$P(X=k)=\binom{n}{k}p^kq^{n-k}$ $k=0,1,2,\dots,n$	np	npq	$(q+pe^t)^n$
Geometric Geom(p)	$P(X=k)=pq^{k-1}$ $k=1,2,\dots$	$\frac{1}{p}$	$\frac{q}{p^2}$	$\frac{pe^t}{1-qe^t}$, $qe^t<1$
Negative Binomial NB(r,p)	$P(X=k)=\binom{k-1}{r-1}p^rq^{k-r}$ $k=r,r+1,\dots$	$\frac{r}{p}$	$\frac{rq}{p^2}$	$\left(\frac{pe^t}{1-qe^t}\right)^r$, $qe^t<1$
Hypergeometric HGeom(n,N,m)	$P(X=k)=\frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$ $k=0,1,\dots n$	$\frac{nm}{N}$	$\frac{nm}{N}\left[\frac{(n-1)(m-1)}{N-1}+1-\frac{nm}{N}\right]$	messy
Poisson Poisson(λ)	$P(X=k)=\frac{e^{-\lambda}\lambda^k}{k!}$ $k=0,1,2,\dots$	λ	λ	$e^{\lambda(e^t-1)}$
Uniform U(a,b)	$f(x)=\frac{1}{b-a}$ $x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t}$
Normal $\mathcal{N}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $x\in(-\infty,\infty)$	μ	σ^2	$\exp\left(\mu t+\frac{\sigma^2t^2}{2}\right)$
Exponential Exp(λ)	$f(x)=\lambda e^{-\lambda x}$ $x\in[0,\infty)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$, $t<\lambda$
Gamma Gamma(a,λ)	$f(x)=\frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}$ $x\in[0,\infty)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^a$, $t<\lambda$
Beta Beta(a,b)	$f(x)=\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}$ $x\in(0,1)$	$\frac{a}{a+b}$	$\frac{ab}{(a+b)^2(a+b+1)}$	messy
Chi-Square χ^2_n		n	$2n$	$(1-2t)^{-n/2}$

2 Useful Results

2.1 Combinatorial Analysis

Derangement/Matching Problem

$$P(\text{at least one match})=1-\frac{1}{2!}+\frac{1}{3!}-\cdots(-1)^{n+1}\frac{1}{n!}$$

$$P(k\text{ matches})=\binom{n}{k}\frac{1}{n(n-1)\cdots(n-k+1)}\cdot\left(1-1+\frac{1}{2!}-\frac{1}{3!}+\cdots(-1)^{n-k}\frac{1}{(n-k)!}\right)$$

2.2 Discrete RV

Suppose $X\sim$ Poisson(λ) and $Y\sim$ Bin(X,p), then $X-Y$ and Y are independent and

$$X-Y\sim\text{Poisson}(\lambda(1-p)),Y\sim\text{Poisson}(\lambda p)$$

2.3 Continuous RV

Exponential Distribution

$$P(X>s+t|X>s)=P(X>t)$$

Gamma Distribution

$$\Gamma(\alpha)=\int_0^\infty e^{-t}t^{\alpha-1}dt$$

- $\Gamma(1) = 1$ and $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$
- $\Gamma(n) = (n-1)!$ for $n = 1, 2, 3, \ldots$
- If $X_i \sim$ Exp(λ) independently, then $\sum_{i=1}^n X_i \sim$ Gamma(n, λ)

Beta Distribution

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$\text{Beta}(1,1) \equiv \text{U}(0,1)$$

Chi-Square Distribution

$$\chi_n^2 \equiv \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

Monotonic Transformation

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| & \text{if } y \in \mathcal{R}(g) \\ 0 & \text{otherwise} \end{cases}$$

2.4 Joint Distribution

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2,b_2) - F_{X,Y}(a_1,b_2) + F_{X,Y}(a_1,b_1) - F_{X,Y}(a_2,b_1)$$

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y}F_{X,Y}(x,y)$$

$$f_{X,Y}(x,y) = h(x)g(y) \Leftrightarrow X \text{ and } Y \text{ are independent}$$

Convolution of Independent Distributions

$$F_{X+Y}(a) = \int_{-\infty}^{\infty} F_X(a-y)f_Y(y)dy$$

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y)f_Y(y)dy$$

Bivariate Normal Distribution

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \times \exp\left(-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right)$$

Joint Transformation

$$f_{Y_1,Y_2,\ldots,Y_n}(y_1,y_2,\ldots,y_n) = f_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ldots,x_n)\frac{1}{|J(x_1,x_2,\ldots,x_n)|}$$

where

$$J(x_1,x_2,\ldots,x_n) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \frac{\partial g_n}{\partial x_2} & \cdots & \frac{\partial g_n}{\partial x_n} \end{vmatrix},$$

$Y_i = g_i(X_1, X_2, \ldots, X_n)$, and $x_i = h_i(y_1, y_2, \ldots, y_n)$ for $i = 1, 2, \ldots, n$

2.5 Expectations and Variance

Expectations

- If X and Y are independent, then $E[g(X)h(Y)] = E[g(x)]E[h(y)]$
- $E(X) = E[E[X|Y]]$

Covariance

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = E(XY) - E(X)E(Y)$$

$$Cov\left(\sum_{i=1}^n a_iX_i, \sum_{j=1}^m b_jY_j\right) = \sum_{i=1}^n\sum_{j=1}^m a_ib_jCov(X_i,Y_j)$$

$$Var\left(\sum_{k=1}^n X_k\right) = \sum_{k=1}^n Var(X_k) + 2\sum_{1\leq i<j\leq n} Cov(X_i,X_j)$$

$$Var(X|Y) = E[X^2|Y] - [E(X|Y)]^2$$

$$Var(X) = E[Var(X|Y)] + Var[E(X|Y)]$$

Correlation

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

$$\rho(X,Y) = \pm 1 \text{ iff } Y = \pm aX + b \text{ where } a = \frac{\sigma_Y}{\sigma_X}$$

Moment Generating Functions

$$M_X(t) = E(e^{tX})$$

$$E(X^n) = \frac{d^n}{dt^n}M_X(t)\bigg|_{t=0}$$

Taylor Series Expansion

$$g(x) = g(\mu) + \frac{g'(\mu)}{1!}(x-\mu) + \frac{g''(\mu)}{2!}(x-\mu)^2 + \ldots$$

Multiplicative Property If X and Y are independent, then $M_{X+Y}(t) = M_X(t)M_Y(t)$

Uniqueness Property If $\exists t$ such that $M_X(t) = M_Y(t)\forall t \in (-h,h)$ then X and Y have the same distributions

Joint Moment Generating Functions

$$M_{X_1,X_2,\dots,X_n}(t_1,t_2,\dots,t_n) = E[e^{t_1X_1+t_2X_2+\cdots+t_nX_n}]$$
$$M_{X_i}(t) = E[e^{tX_i}] = M_{X_1,X_2,\dots,X_n}(0,\cdots,0,t,0,\cdots,0)$$

2.6 Inequalities

$$a \le X \le b \Rightarrow a \le E(X) \le b$$

Monotone Property $X \le Y \Rightarrow E(X) \le E(Y)$

Boole's inequality

$$P\left(\bigcup_{k=1}^{n} A_k\right) \le \sum_{k=1}^{n} P(A_k)$$

Markov's inequality Let X be a **nonnegative** random variable. For $a > 0$, we have

$$P(X \ge a) \le \frac{E(X)}{a}$$

Chebyshev's inequality For $a > 0$, we have

$$P(|X-\mu| \ge a) \le \frac{Var(X)}{a^2}$$

One-sided Chebyshev's inequality If X is a r.v. with mean 0 and finite variance σ^2, then, for any $a > 0$,

$$P(X \ge a) \le \frac{\sigma^2}{\sigma^2 + a^2}$$

Jensen's inequality If $g(x)$ is a convex function, then

$$E[g(x)] \ge g[E(X)]$$

3 Key Terms

blackjack The goal of blackjack is to beat the dealer's hand without going over 21. If you go over 21 you bust, and the dealer wins regardless of the dealer's hand. If a player's first two cards are an ace and a "ten-card" (a picture card or 10), giving a count of 21 in two cards, this is a natural or "blackjack."

event space

five of a kind A hand that contains five cards of one rank, but is only possible when using one or more wild cards

flush

four of a kind

full house Three cards of the same rank, and two cards of a different, matching rank

high card

one pair

rank Individual cards are ranked, from highest to lowest: A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3 and 2. Suits are not ranked, so hands that differ by suit alone are of equal rank.

sample space

straight

straight flush A hand that contains five cards of sequential rank, all of the same suit

suit diamonds (♦), clubs (♣), hearts (♥) and spades (♠)

three of a kind

two pair

no pair see **high card**