1 Table of Distributions

Notation	PMF/PDF and Support	Expected Value	Variance	MGF
Bernoulli Be(p)	$P(X=1)=p$ $P(X=0)=q$	p	pq	$q+pe^t$
Binomial Bin(n,p)	$P(X=k)=\binom{n}{k}p^kq^{n-k}$ $k=0,1,2,\dots,n$	np	npq	$(q+pe^t)^n$
Geometric Geom(p)	$P(X=k)=pq^{k-1}$ $k=1,2,\dots$	$\frac{1}{p}$	$\frac{q}{p^2}$	$\frac{pe^t}{1-qe^t}$, $qe^t<1$
Negative Binomial NB(r,p)	$P(X=k)=\binom{k-1}{r-1}p^rq^{k-r}$ $k=r,r+1,\dots$	$\frac{r}{p}$	$\frac{rq}{p^2}$	$\left(\frac{pe^t}{1-qe^t}\right)^r$, $qe^t<1$
Hypergeometric HGeom(n,N,m)	$P(X=k)=\frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$ $k=0,1,\dots n$	$\frac{nm}{N}$	$\frac{nm}{N}\left[\frac{(n-1)(m-1)}{N-1}+1-\frac{nm}{N}\right]$	messy
Poisson Poisson(λ)	$P(X=k)=\frac{e^{-\lambda}\lambda^k}{k!}$ $k=0,1,2,\dots$	λ	λ	$e^{\lambda(e^t-1)}$
Uniform U(a,b)	$f(x)=\frac{1}{b-a}$ $x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t}$
Normal $\mathcal{N}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $x\in(-\infty,\infty)$	μ	σ^2	$\exp\left(\mu t+\frac{\sigma^2t^2}{2}\right)$
Exponential Exp(λ)	$f(x)=\lambda e^{-\lambda x}$ $x\in[0,\infty)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$, $t<\lambda$
Gamma Gamma(a,λ)	$f(x)=\frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}$ $x\in[0,\infty)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^a$, $t<\lambda$
Beta Beta(a,b)	$f(x)=\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}$ $x\in(0,1)$	$\frac{a}{a+b}$	$\frac{ab}{(a+b)^2(a+b+1)}$	messy
Chi-Square χ^2_n		n	$2n$	$(1-2t)^{-n/2}$

2 Useful Results

2.1 Combinatorial Analysis

Derangement/Matching Problem

$$P(\text{at least one match})=1-\frac{1}{2!}+\frac{1}{3!}-\cdots(-1)^{n+1}\frac{1}{n!}$$

$$P(k\text{ matches})=\binom{n}{k}\frac{1}{n(n-1)\cdots(n-k+1)}\cdot\left(1-1+\frac{1}{2!}-\frac{1}{3!}+\cdots(-1)^{n-k}\frac{1}{(n-k)!}\right)$$

2.2 Discrete RV

Suppose $X\sim$ Poisson(λ) and $Y\sim$ Bin(X,p), then $X-Y$ and Y are independent and

$$X-Y\sim\text{Poisson}(\lambda(1-p)),Y\sim\text{Poisson}(\lambda p)$$

2.3 Continuous RV

Exponential Distribution

$$P(X>s+t|X>s)=P(X>t)$$

Gamma Distribution

$$\Gamma(\alpha)=\int_0^\infty e^{-t}t^{\alpha-1}dt$$

- $\Gamma(1) = 1$ and $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$
- $\Gamma(n) = (n-1)!$ for $n = 1, 2, 3, \ldots$
- If $X_i \sim \text{Exp}(\lambda)$ independently, then $\sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \lambda)$

Beta Distribution

$$\begin{aligned} B(a,b) &= \int_0^1 t^{a-1}(1-t)^{b-1}dt \\ &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \end{aligned}$$

$$\text{Beta}(1,1) \equiv \text{U}(0,1)$$

Chi-Square Distribution

$$\chi_n^2 \equiv \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

Monotonic Transformation

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| & \text{if } y \in \mathcal{R}(g) \\ 0 & \text{otherwise} \end{cases}$$

2.4 Joint Distribution

$$\begin{aligned} P(a_1 < X \le a_2, b_1 < Y \le b_2) &= F_{X,Y}(a_2, b_2) \\ &- F_{X,Y}(a_1, b_2) + F_{X,Y}(a_1, b_1) - F_{X,Y}(a_2, b_1) \end{aligned}$$

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

$$f_{X,Y}(x,y) = h(x)g(y) \Leftrightarrow X \text{ and } Y \text{ are independent}$$

Convolution of Independent Distributions

$$F_{X+Y}(a) = \int_{-\infty}^{\infty} F_X(a-y) f_Y(y) dy$$

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y) f_Y(y) dy$$

Bivariate Normal Distribution

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \times \exp\left(-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right)$$

Joint Transformation

$$f_{Y_1,Y_2,\ldots,Y_n}(y_1, y_2, \ldots, y_n) = f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) \frac{1}{|J(x_1, x_2, \ldots, x_n)|}$$

where

$$J(x_1, x_2, \ldots, x_n) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \frac{\partial g_n}{\partial x_2} & \cdots & \frac{\partial g_n}{\partial x_n} \end{vmatrix},$$

$Y_i = g_i(X_1, X_2, \ldots, X_n)$, and $x_i = h_i(y_1, y_2, \ldots, y_n)$ for $i = 1, 2, \ldots, n$

2.5 Expectations and Variance

Expectations

- If X and Y are independent, then $E[g(X)h(Y)] = E[g(x)]E[h(y)]$
- $E(X) = E[E[X|Y]]$

Covariance

$$\begin{aligned} Cov(X,Y) &= E[(X-\mu_X)(Y-\mu_Y)] \\ &= E(XY) - E(X)E(Y) \end{aligned}$$

$$Cov\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j Cov(X_i, Y_j)$$

$$Var\left(\sum_{k=1}^{n} X_k\right) = \sum_{k=1}^{n} Var(X_k) + 2\sum_{1 \le i < j \le n} Cov(X_i, X_j)$$

$$Var(X|Y) = E[X^2|Y] - [E(X|Y)]^2$$

$$Var(X) = E[Var(X|Y)] + Var[E(X|Y)]$$

Correlation

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

$$\rho(X,Y) = \pm 1 \text{ iff } Y = \pm aX + b \text{ where } a = \frac{\sigma_Y}{\sigma_X}$$

Moment Generating Functions

$$M_X(t) = E(e^{tX})$$

$$E(X^n) = \left.\frac{d^n}{dt^n} M_X(t)\right|_{t=0}$$

Taylor Series Expansion

$$g(x) = g(\mu) + \frac{g'(\mu)}{1!}(x-\mu) + \frac{g''(\mu)}{2!}(x-\mu)^2 + \ldots$$

Multiplicative Property If X and Y are independent, then $M_{X+Y}(t) = M_X(t)M_Y(t)$

Uniqueness Property If $\exists t$ such that $M_X(t) = M_Y(t) \forall t \in (-h, h)$ then X and Y have the same distributions

Joint Moment Generating Functions

$$M_{X_1,X_2,\dots,X_n}(t_1,t_2,\dots,t_n) = E[e^{t_1X_1+t_2X_2+\cdots+t_nX_n}]$$
$$M_{X_i}(t) = E[e^{tX_i}] = M_{X_1,X_2,\dots,X_n}(0,\cdots,0,t,0,\cdots,0)$$

2.6 Inequalities

$$a \le X \le b \Rightarrow a \le E(X) \le b$$

Monotone Property $X \le Y \Rightarrow E(X) \le E(Y)$

Boole's inequality

$$P\left(\bigcup_{k=1}^{n} A_k\right) \le \sum_{k=1}^{n} P(A_k)$$

Markov's inequality Let X be a <u>**nonnegative**</u> random variable. For $a > 0$, we have

$$P(X \ge a) \le \frac{E(X)}{a}$$

Chebyshev's inequality For $a > 0$, we have

$$P(|X-\mu| \ge a) \le \frac{Var(X)}{a^2}$$

One-sided Chebyshev's inequality If X is a r.v. with mean 0 and finite variance σ^2, then, for any $a > 0$,

$$P(X \ge a) \le \frac{\sigma^2}{\sigma^2 + a^2}$$

Jensen's inequality If $g(x)$ is a convex function, then

$$E[g(x)] \ge g[E(X)]$$

3 Poker Terms

blackjack The goal of blackjack is to beat the dealer's hand without going over 21. If you go over 21 you bust, and the dealer wins regardless of the dealer's hand. If a player's first two cards are an ace and a "ten-card" (a picture card or 10), giving a count of 21 in two cards, this is a natural or "blackjack."

five of a kind A hand that contains five cards of one rank, but is only possible when using one or more wild cards (3♦ 3♣ 3♥ 3♠ Jkr)

flush A hand that contains five cards all of the same suit, not all of sequential rank (K♣ 10♣ 7♣ 6♣ 4♣)

four of a kind A hand that contains four cards of one rank and one card of another rank (9♦ 9♣ 9♥ 9♠ J♥)

full house A hand that contains three cards of one rank and two cards of another rank (3♣ 3♠ 3♦ 6♣ 6♥)

high card A hand that does not fall into any other category (K♥ J♥ 8♣ 7♦ 4♠)

one pair A hand that contains two cards of one rank and three cards of three other ranks (4♥ 4♠ K♠ 10♦ 5♠)

pair See **one pair**

rank Individual cards are ranked, from highest to lowest: A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3 and 2. Suits are not ranked, so hands that differ by suit alone are of equal rank.

set See **three of a kind**

straight A hand that contains five cards of sequential rank, not all of the same suit (7♣ 6♠ 5♠ 4♥ 3♥)

straight flush A hand that contains five cards of sequential rank, all of the same suit (A♥ K♥ Q♥ J♥ 10♥ or 5♥ 4♥ 3♥ 2♥ A♥)

suit One of the categories into which the cards of a deck are divided: diamonds (♦), clubs (♣), hearts (♥) and spades (♠)

three of a kind A hand that contains three cards of one rank and two cards of two other ranks (2♦ 2♠ 2♣ K♠ 6♥)

two pair A hand that contains two cards of one rank, two cards of another rank and one card of a third rank (J♥ J♣ 4♣ 4♠ 9♥)

no pair See **high card**