Optimist Racing - Math and Physics

Claude Richard

October 28, 2012

This document contains all the mathematics used (except the easy math LOL) for the program.

0.1 Conventions

This section describes the notational conventions used throughout this document. If A is a matrix (or vector), then we denote the i$^{\text{th}}$ row of A by A_i. We also denote the i$^{\text{th}}$ column of A by A^i.

1 Electric Colours

There are 3 RGB colours (red, green, blue). There are 8 electric colours. The electric colours can be thought of as subsets of $\{red, green, blue\}$. The electric colours are:

$$
\begin{aligned}
&\textbf{ElectricBlack} && = \emptyset \\
&\textbf{ElectricRed} && = \{red\} \\
&\textbf{ElectricGreen} && = \{green\} \\
&\textbf{ElectricBlue} && = \{blue\} \\
&\textbf{ElectricYellow} && = \{red, green\} \\
&\textbf{ElectricMagenta} && = \{red, blue\} \\
&\textbf{ElectricCyan} && = \{green, blue\} \\
&\textbf{ElectricWhite} && = \{red, green, blue\}
\end{aligned}
$$

In the physics engine, 2 objects of Electric colours X and Y respectively can interact with each other if and only if $X \bigcap Y \neq \emptyset$

2 Smooth Floor Grid

File: Bezier.hpp You can create a floor which is differentiable n times for an arbitrary n. Note that the complexity is at least O(n). (I didn't calculate it exactly) Specify f, fx, fy, etc. for each vertex in a grid. Then the whole grid will be n-times differentiable everywhere.

3 Lagrangian Mechanics

Let $\overrightarrow{x}$ represent the vector of generalized coordinates, of length n. Let $\overrightarrow{Q}$ represent the vector of generalized forces, also of length n. For a given system, let T be the kinetic energy and V be the potential energy of the system. Express $L = T - V$ as a function of the generalized coordinates and their derivatives with respect to time.

$$T = T(x_1, \ldots, x_n, \dot{x_1}, \ldots, \dot{x_n})$$
$$V = V(x_1, \ldots, x_n)$$
$$L = L(x_1, \ldots, x_n, \dot{x_1}, \ldots, \dot{x_n})$$

Then there are n Lagrange Equations which together govern the motion of the generalized coordinates.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x_i}} - \frac{\partial L}{\partial x_i} = Q_i$$

3.1 Matrix Form

You should be able to express L in matrix form:

$$L = \frac{1}{2}\dot{\overrightarrow{x}}^T A \dot{\overrightarrow{x}} - V$$

Where A is a symmetric nxn matrix dependent on the generalized coordinates but not their derivatives wrt time.

$$A = A(x_1, \ldots, x_n)$$
$$A = A^T$$

We will now derive a vector equation to replace the n Lagrange equations. We have the following:

$$\frac{\partial L}{\partial x_i} = \frac{1}{2}\dot{\overrightarrow{x}}^T \frac{\partial A}{\partial x_i}\dot{\overrightarrow{x}} - \frac{\partial V}{\partial x_i}$$
$$\frac{\partial L}{\partial \dot{x_i}} = A_i \dot{\overrightarrow{x}}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x_i}} = A_i \ddot{\overrightarrow{x}} + \sum_{j=1}^{n} \dot{x_j}(\frac{\partial A}{\partial x_j})_i \dot{\overrightarrow{x}}$$

The i^{th} Lagrange equation then becomes:

$$A_i \ddot{\overrightarrow{x}} + \sum_{j=1}^{n} \dot{x_j}(\frac{\partial A}{\partial x_j})_i \dot{\overrightarrow{x}} - \frac{1}{2}\dot{\overrightarrow{x}}^T \frac{\partial A}{\partial x_i}\dot{\overrightarrow{x}} + \frac{\partial V}{\partial x_i} = Q_{x_i}$$

Now for each i, let M_i be a matrix such that $(M_i)_j = (\frac{\partial A}{\partial x_j})_i$. All equations combined become the following vector equation:

$$A\ddot{\overrightarrow{x}} + \sum_{j=1}^{n} \dot{x_j}(\frac{\partial A}{\partial x_j} - \frac{1}{2}M_j)\dot{\overrightarrow{x}} + \nabla V = \overrightarrow{Q}$$

3.2 Kinetic energy relating to the center-of-mass

Let the system be composed of one object of mass m, which does not rotate, and whose position as a function of the generalized coordinates is:

$$position = \begin{bmatrix} f(\overrightarrow{x}) \\ g(\overrightarrow{x}) \\ h(\overrightarrow{x}) \end{bmatrix}$$

We will now derive the kinetic energy as a function of $\overrightarrow{x}$, $\dot{\overrightarrow{x}}$, and the three functions f, g, and h. The velocity of the object is:

$$velocity = \begin{bmatrix} (\nabla f)^T \dot{\overrightarrow{x}} \\ (\nabla g)^T \dot{\overrightarrow{x}} \\ (\nabla h)^T \dot{\overrightarrow{x}} \end{bmatrix}$$

So its kinetic energy is:

$$T = \frac{1}{2} m [\dot{\overrightarrow{x}}^T [(\nabla f)(\nabla f)^T + (\nabla g)(\nabla g)^T + (\nabla h)(\nabla h)^T] \dot{\overrightarrow{x}}]$$

Assume the potential energy is zero. Then $L = T$. We will now derive the Lagrange equations for this system.

$$\frac{\partial L}{\partial x_i} = \frac{1}{2} m \dot{\overrightarrow{x}}^T [H(f)^i(\nabla f)^T + H(g)^i(\nabla g)^T + H(h)^i(\nabla h)^T + (\nabla f)H(f)_i + (\nabla g)H(g)_i + (\nabla h)H(h)_i] \dot{\overrightarrow{x}}$$

$$\frac{\partial L}{\partial \dot{x_i}} = m[\frac{\partial f}{\partial x_i}(\nabla f)^T + \frac{\partial g}{\partial x_i}(\nabla g)^T + \frac{\partial h}{\partial x_i}(\nabla h)^T] \dot{\overrightarrow{x}}$$

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{x_i}} &= m[\frac{\partial f}{\partial x_i}(\nabla f)^T + \frac{\partial g}{\partial x_i}(\nabla g)^T + \frac{\partial h}{\partial x_i}(\nabla h)^T] \ddot{\overrightarrow{x}} \\ &+ m \dot{\overrightarrow{x}}^T [H(f)^i(\nabla f)^T + H(g)^i(\nabla g)^T + H(h)^i(\nabla h)^T] \dot{\overrightarrow{x}} \\ &+ m \dot{\overrightarrow{x}}^T [\frac{\partial f}{\partial x_i} H(f) + \frac{\partial g}{\partial x_i} H(g) + \frac{\partial h}{\partial x_i} H(h)] \dot{\overrightarrow{x}} \end{aligned}$$

The matrix form of the Lagrange equations is:

$$\begin{aligned} & m[(\nabla f)(\nabla f)^T + (\nabla g)(\nabla g)^T + (\nabla h)(\nabla h)^T] \ddot{\overrightarrow{x}} + \\ & m[(\nabla f) \dot{\overrightarrow{x}}^T H(f) + (\nabla g) \dot{\overrightarrow{x}}^T H(g) + (\nabla h) \dot{\overrightarrow{x}}^T H(h)] \dot{\overrightarrow{x}} = Q \end{aligned}$$

3.3 Kinetic energy relating to the Euler angles

Let the system be composed of one object, whose center-of-mass does not move, and whose Euler angles as a function of the generalized coordinates are α, β and γ. This means that to go from an orientation parallel to the world axes to the object's orientation, you must turn left (around the z-axis) by α, then tilt back (around the new model x-axis) by β, then turn left around the model z-axis by γ. Let the object have a moment of inertia tensor

$$I = \begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{xy} & I_y & I_{yz} \\ I_{xz} & I_{yz} & I_z \end{bmatrix}$$

Also denote

$$\alpha_1 = \alpha, \alpha_2 = \beta, \alpha_3 = \gamma$$

$$G = \begin{bmatrix} \nabla\alpha & \nabla\beta & \nabla\gamma \end{bmatrix}$$

$$E = \begin{bmatrix} I_x \sin^2\beta\cos^2\gamma & (I_y - I_x)\sin\beta\sin\gamma\cos\gamma & I_z\cos\beta \\ (I_y - I_x)\sin\beta\sin\gamma\cos\gamma & I_x\sin^2\gamma + I_y\cos^2\gamma & 0 \\ I_z\cos\beta & 0 & I_z \end{bmatrix}$$

$$\nabla_0 L = \begin{bmatrix} \frac{\partial L}{\partial x_1} \\ \vdots \\ \frac{\partial L}{\partial x_n} \end{bmatrix}$$

$$\nabla_1 L = \begin{bmatrix} \frac{\partial L}{\partial \dot{x}_1} \\ \vdots \\ \frac{\partial L}{\partial \dot{x}_n} \end{bmatrix}$$

We will now derive the matrix form of the Lagrange equations for this system. Let T represent the kinetic energy, and assume the potential energy is zero. We can derive the following lemmas:

$$\dot{\alpha}_i = \dot{\vec{x}}^T \nabla\alpha_i$$

$$(\dot{\nabla\alpha_i}) = H(\alpha_i)\dot{\vec{x}}$$

$$T = L$$

$$T = \begin{bmatrix} \dot{\alpha} & \dot{\beta} & \dot{\gamma} \end{bmatrix} E \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}$$

Therefore:

$$L = \frac{1}{2}\dot{\vec{x}}^T G E G^T \dot{\vec{x}}$$

$$\nabla_0 L = \sum_{i=1}^{3} \frac{1}{2}\nabla\alpha_i \dot{\vec{x}}^T G \frac{\partial E}{\partial \alpha_i} G^T \dot{\vec{x}} + H(\alpha_i)\dot{\vec{x}} E_i G^T \dot{\vec{x}}$$

$$\nabla_1 L = G E G^T \dot{\vec{x}}$$

Anyway, after a while we find the final vector equation:

$$G E G^T \ddot{\vec{x}} + \sum_{i=1}^{3} G E^i \dot{\vec{x}}^T H(\alpha_i)\dot{\vec{x}} + G\frac{\partial E}{\partial \alpha_i} G^T \dot{\vec{x}} \dot{\vec{x}}^T \nabla\alpha_i - \frac{1}{2}\nabla\alpha_i \dot{\vec{x}}^T G \frac{\partial E}{\partial \alpha_i} G^T \dot{\vec{x}} = \vec{Q}$$

3.4 Combining center-of-mass movement and rotation, and several objects

In the previous two sections we focused on center-mass movement, then on rotation, for only one object. What if we have several objects that have both rotation and moving centers-of-mass? Let the system have k objects, and n generalized coordinates. Let the position of object i be

$$position_i = \begin{bmatrix} f_i(\overrightarrow{x}) \\ g_i(\overrightarrow{x}) \\ h_i(\overrightarrow{x}) \end{bmatrix}$$

and let the Euler angles of object i be

$$Euler_i = \begin{bmatrix} \alpha_i(\overrightarrow{x}) \\ \beta_i(\overrightarrow{x}) \\ \gamma_i(\overrightarrow{x}) \end{bmatrix}$$

If we use the center-of-mass section above for object i, we would get the equation

$$C_i \ddot{\overrightarrow{x}} = \overrightarrow{c_i}$$

And if we use the rotation section above for object i, we would get

$$R_i \ddot{\overrightarrow{x}} = \overrightarrow{r_i}$$

Where C_i, R_i, $\overrightarrow{c_i}$ and $\overrightarrow{r_i}$ depend on $\overrightarrow{x}$ and $\dot{\overrightarrow{x}}$. For the combined system, the equation is simply

$$A\ddot{\overrightarrow{x}} = \overrightarrow{Q} - \overrightarrow{b}$$

where

$$A = \sum_{i=1}^{n} (C_i + R_i)$$

$$b = \sum_{i=1}^{n} (\overrightarrow{c_i} + \overrightarrow{r_i})$$

and $\overrightarrow{Q}$ is the generalized force on the system, which will be covered in the next sections.

3.5 Transforming forces and torques between model and world coordinate systems

This is just a preliminary section to clarify the transformations in the next sections. Say you apply a force or torque $\overrightarrow{F_w}$ in world coordinates. There is a force/torque $\overrightarrow{F_m}$ in model coordinates, which is equivalent to $\overrightarrow{F_w}$ in world coordinates. Define the following matrices:

$$R_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix}$$

$$R_\gamma = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3.6 Generalized force from 3D force on center-of-mass

You are given a system consisting of one object, which may or may not rotate. Its position is as above,

$$position = \begin{bmatrix} f(\overrightarrow{x}) \\ g(\overrightarrow{x}) \\ h(\overrightarrow{x}) \end{bmatrix}$$

You apply an external force $\overrightarrow{F_w}$ (in 3D world coordinates) on the object at its center-of-mass. The generalized force on the coordinate x_i is

$$Q_{x_i} = \frac{\partial position}{\partial x_i}^T \overrightarrow{F_w}$$

So the generalized force vector must be

$$Q = \begin{bmatrix} \nabla f & \nabla g & \nabla h \end{bmatrix} \overrightarrow{F_w}$$

3.7 Generalized force from 3D torque

Now instead of applying a force, you apply a torque $\overrightarrow{T_w}$ about the center-of-mass, in world coordinates. Alternatively, apply a torque $\overrightarrow{T_m}$ in model coordinates (i.e. rotated according to current Euler angles).

4 Lagrangian Models

The racer includes 2 objects. The bottom has position x,y,z, rotated by theta. The top is attached to a massless rigid rod, of which the other end is attached to the bottom. To get to the top: start with (x,y,z), rotate left by theta, tilt right by phi, go up by length. When you're gliding on the floor, z = f(x,y) where f describes the floor. When you're falling, z is an additional generalized coordinate.

5 Camera

TODO: write out this section to describe good camera movement around the lagrangian models