Celestial Mechanics / Computational Astrodynamics

Homework nr. 5

EXERCISE 1

The mirror configuration theorem states that if N point masses subjected only to their mutual gravitational attraction have at a certain epoch their radius vector in the synodic system (the vector connecting their position to the one of the CM) perpendicular to every velocity vector, then the orbit of each mass after that epoch is a mirror image prior to that epoch.

We identify two possible mirror configurations:

- All points lie on a plane and the velocities are perpendicular to that plane
- All particle s lie on a segment and the velocities are perpendicular to it (but not necessarily parallel to each other).

A corollary to the mirror theorem states that in presence of mirror configuration we have periodic orbits.

We will work in the synodic reference frame of our 3 body system in the xy plane: it is centered in the CM, has its x-axis along the direction that connects the two massive bodies and rotates together with the system.

We consider, for instance, in the case of a motion on a plane, as a mirror configuration a state vector of the type:

$$\boldsymbol{x} = (x, 0, 0, \dot{y}\ )^T$$

To compute the "real "vector of initial conditions that identifies a periodic orbits we proceed by steps:

1) Starting from a given initial state vector

$$\boldsymbol{x_0} = (x_0, 0, 0, \dot{y_0}\ )^T$$

we proceed integrating it numerically using the following equations of motion:

$$\ddot{x} - 2\dot{y} = x - \frac{(1-\mu)(x-x_1)}{r_1^3} - \frac{\mu(x-x_2)}{r_2^3}$$

$$\ddot{y} + 2\dot{x} = y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}$$

Where $r_i = \sqrt{(x-x_i)^2 + y^2}$.

2) When we reach the point where the position lies on the x-axis we stop iterating. At this point the position vector would be of the type:

$$\boldsymbol{x} = \left(x_f, 0, \dot{x}_f, \dot{y}_f\right)^T$$

Generally it will not have ha zero x component of the velocity, so it isn't actually the mirror configuration we want to obtain (that would be $x_f^s = \left(x_f^s, 0,0, \dot{y}_f^s\right)^T$). In fact it differs from it by an amount:

$$\delta\boldsymbol{x_f} = \boldsymbol{x_f} - \boldsymbol{x_f^s}$$

$$\delta \boldsymbol{x_f} = \frac{\partial \boldsymbol{x}(t_f, \boldsymbol{x_0})}{\partial \boldsymbol{x_0}} \delta \boldsymbol{x_0} + \frac{\partial \boldsymbol{x}(t_f, \boldsymbol{x_0})}{\partial t} \delta t_f$$

That can be written in terms of the sensitivity matrix $\boldsymbol{\Phi}$ as:

$$\delta \boldsymbol{x_f} = \boldsymbol{\Phi}(t_f, \boldsymbol{t_0}) \delta \boldsymbol{x_0} + \boldsymbol{x}(t_f, \boldsymbol{x_0}) \delta t_f$$

The sensitivity matrix is the solution of the system of differential equations:

$$\frac{d\boldsymbol{\Phi}}{dt} = \boldsymbol{A\Phi}$$

Where $A = \begin{pmatrix} 0 & \mathbb{I} \\ G & K \end{pmatrix}$, $K = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ and $G = \begin{pmatrix} \mathcal{U}^*_{xx} & \mathcal{U}^*_{xy} \\ \mathcal{U}^*_{yx} & \mathcal{U}^*_{yy} \end{pmatrix}$. $\mathcal{U}^*$ is the effective potential and its gradients are given by:

$$\mathcal{U}^*_{xx} = 1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} + 3(1-\mu)\frac{(x-x_1)^2}{r_1^5} + 3\mu\frac{(x-x_2)^2}{r_2^5}$$

$$\mathcal{U}^*_{xy} = \mathcal{U}^*_{yx} = 3y\left((1-\mu)\frac{x-x_1}{r_1^5} + \mu\frac{x-x_2}{r_2^5}\right)$$

$$\mathcal{U}^*_{yy} = 1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} + 3y^2\left(\frac{1-\mu}{r_1^5} + \frac{\mu}{r_2^5}\right)$$

From these relations we obtain the sensitivity matrix.

3) Once we solve for the sensitivity matrix we can adopt different strategies to obtain the variation in the initial conditions needed in order to have the right mirror configuration at the time t_f (that for instance is half of the period).
The variation that has to be applied to the initial values is the one such that:

$$\boldsymbol{x_f} + \delta \boldsymbol{x_f} = \left(x'_{t_f}, 0, 0, \dot{y}'_{t_f}\right)^T$$

Using the sensitivity matrix we can rewrite the variation:

$$\delta \boldsymbol{x_f} = \boldsymbol{\Phi}(t_f, t_0) \delta \boldsymbol{x_0} + \dot{\boldsymbol{x}}_{\boldsymbol{0}} \delta t_f$$

This translates into different constraint on the initial values. In fact we can choose either to put a constraint on the initial position, or to put it on the initial velocity:

- <u>Vincolated initial position</u>: gives rise to a correction in the initial velocity:

$$\delta \dot{y}_0 = -\frac{\dot{y}_{t_f} \dot{x}_{t_f}}{B_{\dot{x}}}$$

Where $B_{\dot{x}} = \Phi_{34}\dot{y}_{t_f} - \Phi_{24}\ddot{x}_{t_f}$.

- <u>Vincolated velocity:</u> gives rise to the correction on the initial position:

$$\delta x_0 = -\frac{\dot{x_{t_f}}\,\dot{y_{t_f}}}{A_{\dot{x}}}$$

Where $A_{\dot{x}} = \Phi_{31}\dot{y_{t_f}} - \Phi_{21}\ddot{x}_{t_f}$.

4) Now that we have the corrections on the initial state vector we apply them to get the new initial state vector. At this point we integrate again the equations of motion and get the second mirror configuration. If the value of the $\dot{x}_{t_f}$ coordinate is less than the accuracy we stop the process, if not we iterate again the procedure until it is.

THE CODE

First I have implemented the equations of motion of the restricted three body problem in the library “threeBody.h” and “threeBody.cpp”. The same equations of motion are also implemented in “threeBodyPhi.h” and “threeBodyPhi.cpp”, that also contain the differential equations used to solve for the Φ matrix.

I started integrating the $3B$ problem with an RK method and to check if the results made sense I have used both my RK4 fixed step and the RK4-5 adaptive step of the odeint library and plotted the orbit in the synodic system.

In order to find the half period, that is the time at which the mass crosses again the x-axis, I have overloaded my RK4 adding a check, so that it would stop when the x axis is reached for the first time. In order to have a better precision in finding the half period (since we want at least 10 significative digits in the result), and since it is computationally difficult to integrate the whole half period with a small time step, I have used a strategy: when the integration with the initial time step h stops (because it reaches the x-axis), I start again from the previous step, using a one order smaller time step and integrate again. The procedure goes on until a precision of 10^{-15} in finding the half period.

Note that the differential equations for the body and for the Φ matrix are solved simultaneously because the differential equations for Φ require the position of the body.

At this point we can use the equations above to compute the correction on the initial state vector. I have applied two strategies: vincolated initial position and vincolated initial velocity.

The correction is then added to the initial state vector, to get the new state vector of initial condition.

This whole procedure is iterated (into a do-while cicle) until the correction (δx_0 , δv_{y_0}) value has reached the prefixed tolerance (in this case I have used a tolerance of 10^{-15}).

The value of the final period of the orbit and the final initial state vector are stored into files named “period_correction.dat”.

Once the last initial condition is known, I integrate again with an RK4 to get the corrected orbit.

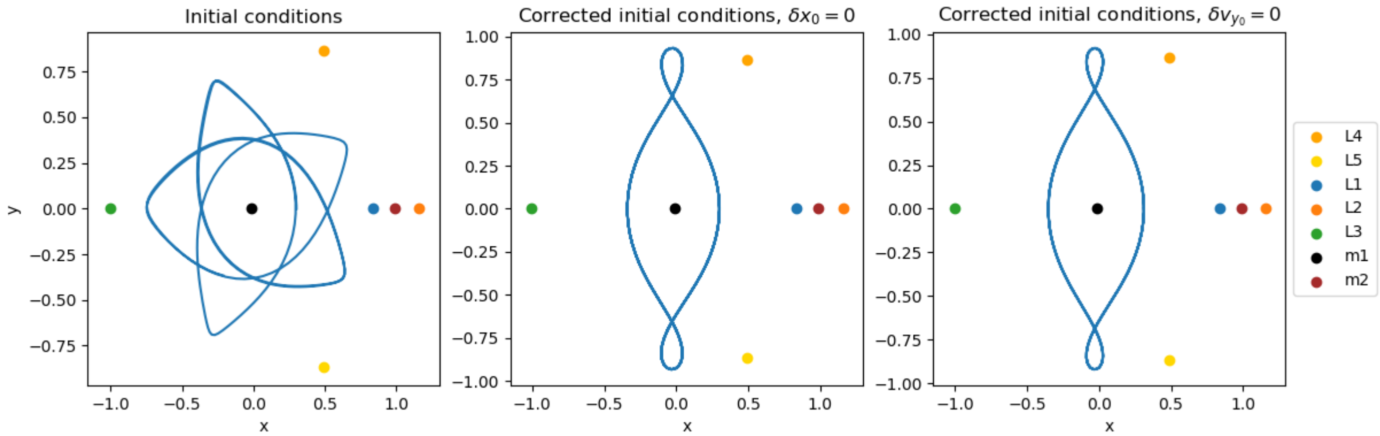
The following plots show qualitatively the orbits without corrected initial state and with the two corrected states.

STATE VECTOR 1

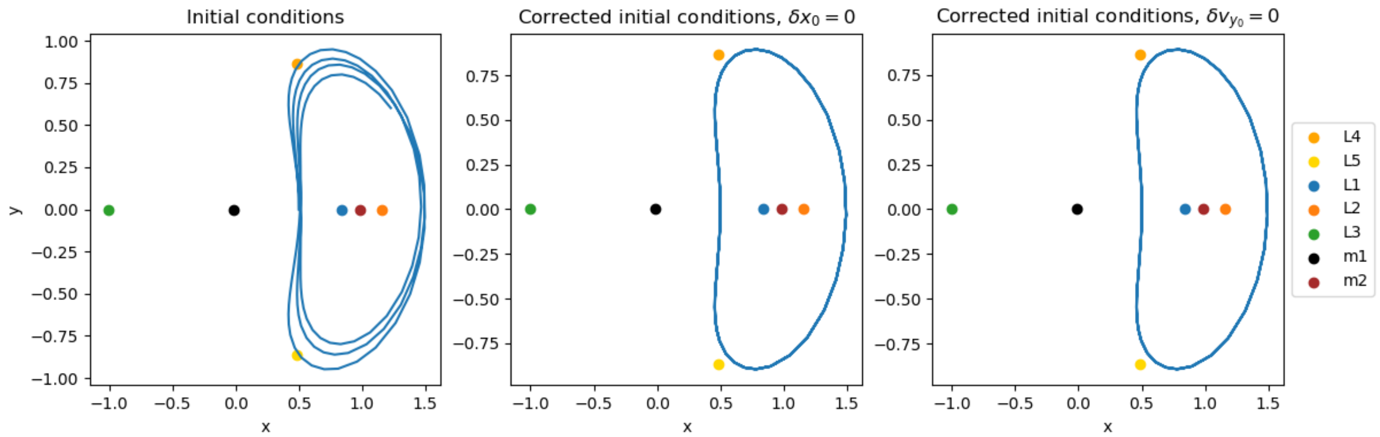
The image shows on the left the integration over a period of $t = 20$ (I chose this period to better appreciate the changes over time) done using the initial conditions as were provided.

The figure in the middle shows the integration done with the corrected initial conditions using the first method (imposing$\delta x_0 = 0$, so with a change in the v_{y_0}).

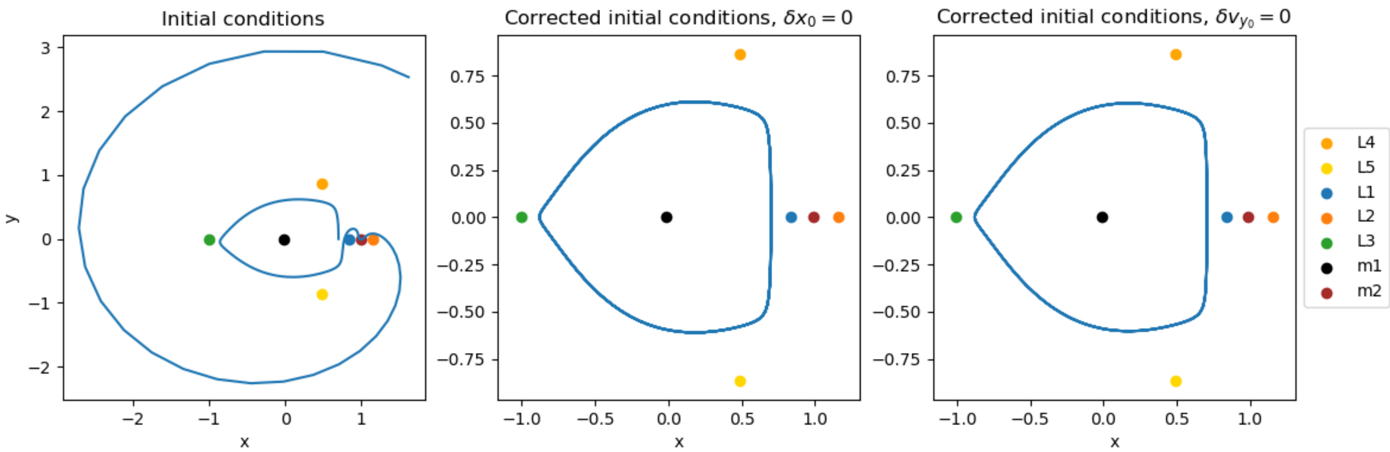
The third figure shows the integration with the corrected initial state using the second method ($\delta v_{y_0} = 0$, so with corrected initial positionδx_0).
In all the plots are shown the Lagrangian points and the two massive bodies, to give a visual reference.



STATE VECTOR 2



STATE VECTOR 3



STATE VECTOR 4

The fourth state vector is the most imperfect one. The corrected orbit is closed only if we choose a precision of 15 decimal digits and even in this case we can recover a closed orbit for just 4 periods.

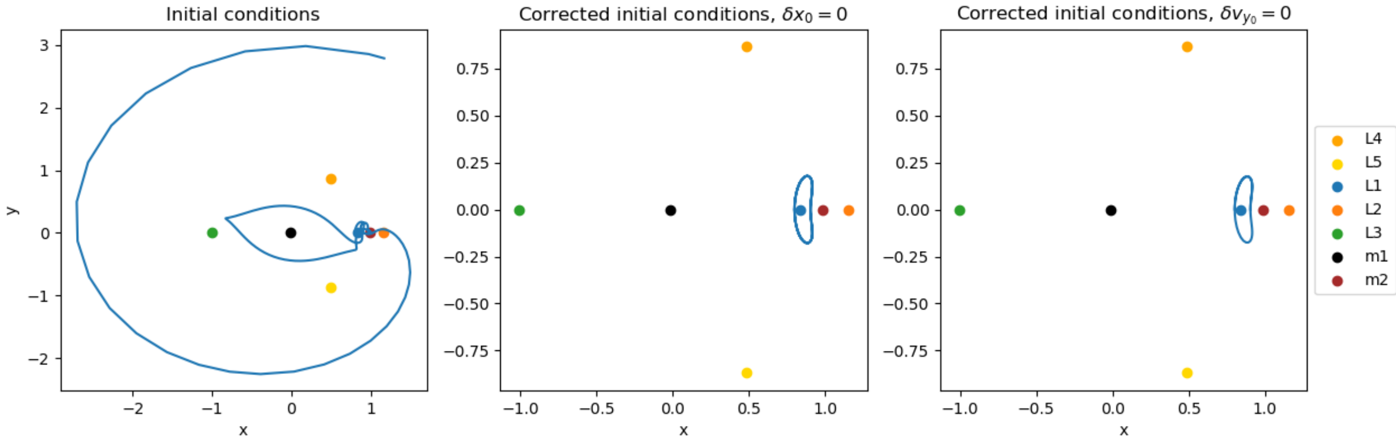


TABLE OF CORRECTIONS

Here we can see the correction on the initial states computed for the 4 dynamical states using the two correction methods and a tolerance of 15 significative digits:

Method $\delta x_0 = 0$

Dynamical state	x_0	y_0	v_{x_0}	v_{y_0}	δv_{y_0}
1	0.3	0	0	1.86646267835956	0.06646267835956
2	0.5	0	0	1.20411083266471	0.00411083266471
3	0.7	0	0	0.539978845857909	0. 039978845857909
4	0.8	0	0	0.353697960236227	0.053697960236227

Method $\delta v_{y_0} = 0$

Dynamical state	x_0	y_0	v_{x_0}	v_{y_0}	δx_0
1	0.312025930624967	0	0	1.8	0.012025930624967
2	0.501326383455158	0	0	1.2	0.001326383455158
3	0.704638440099034	0	0	0.53	0.004638440099034
4	0.800527471817359	0	0	0.35	0.000527471817359

I have noticed that the fourth dynamical state is the one that gives the worst result in terms of periodic orbit. Its corrected orbit is, in fact, periodic only for around 4 periods. This could be caused by many errors that occur due to the finite number of digits that can be stored into a double, that leads to a finite precision achievable with the RK integration.

EXERCISE 2

The orbits in the synodic reference frame (the one corotating with the two massive bodies) have already been shown in the previous figures.

The inertial reference frame differs from the synodic one because it does not rotate, so that the inertial coordinates are given, using polar coordinates, by the synodic one adding the rotation of the frame. In the inertial reference frame, calling ρ, θ the polar coordinates in the synodic rf and ρ', θ' the inertial coordinates, we have:

$$\begin{cases} \rho' = \rho \\ \theta' = \theta + \omega t \end{cases}$$

Where ω is the rotation of the synodic axes (in our units we have $\omega = 1$).

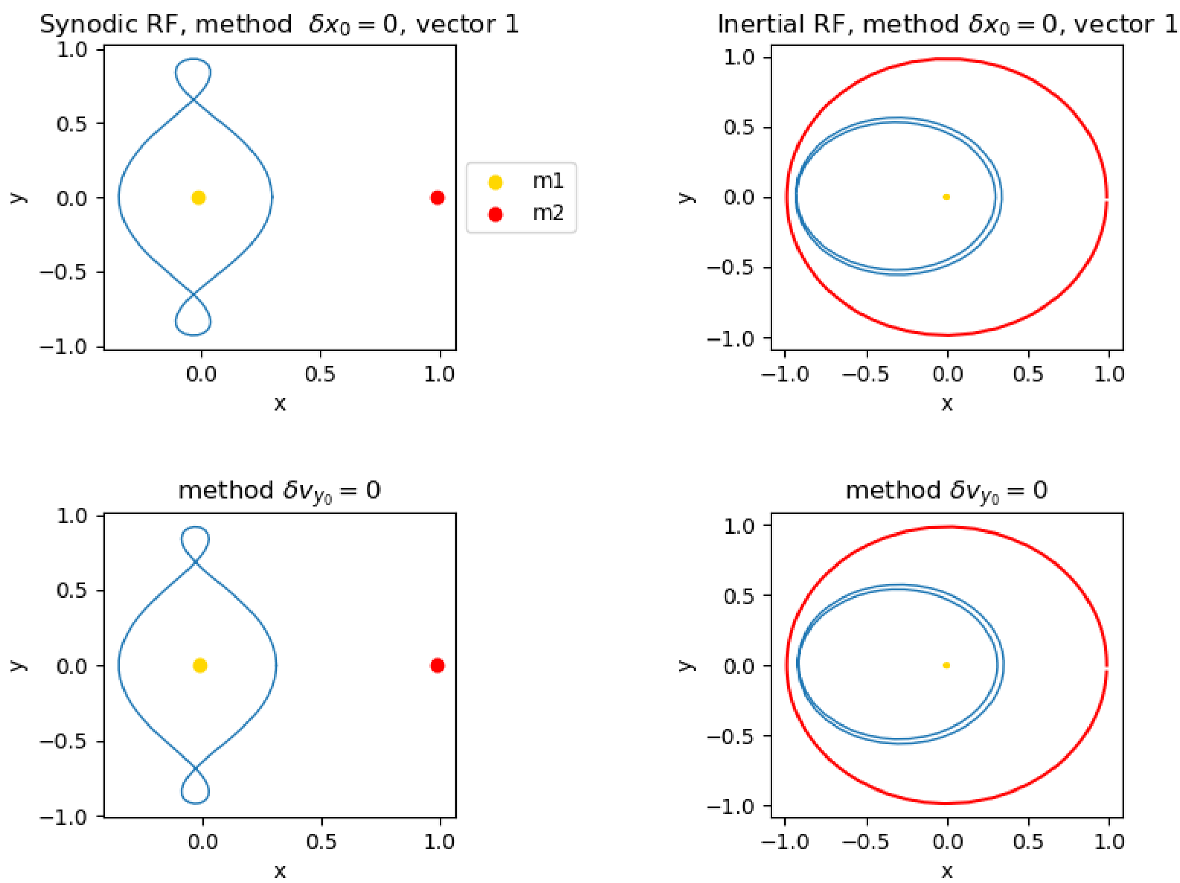
Once we transform from one frame to the other, we go back to the cartesian coordinates and plot the orbits.

I have decided to plot the orbits first using only one period and then plotting 50 periods in order to appreciate the changes with time. In the plot I have also shown the position of the two massive bodies (in the synodic RF) and their orbit around the center of mass (in the inertial RF).

ONE PERIOD

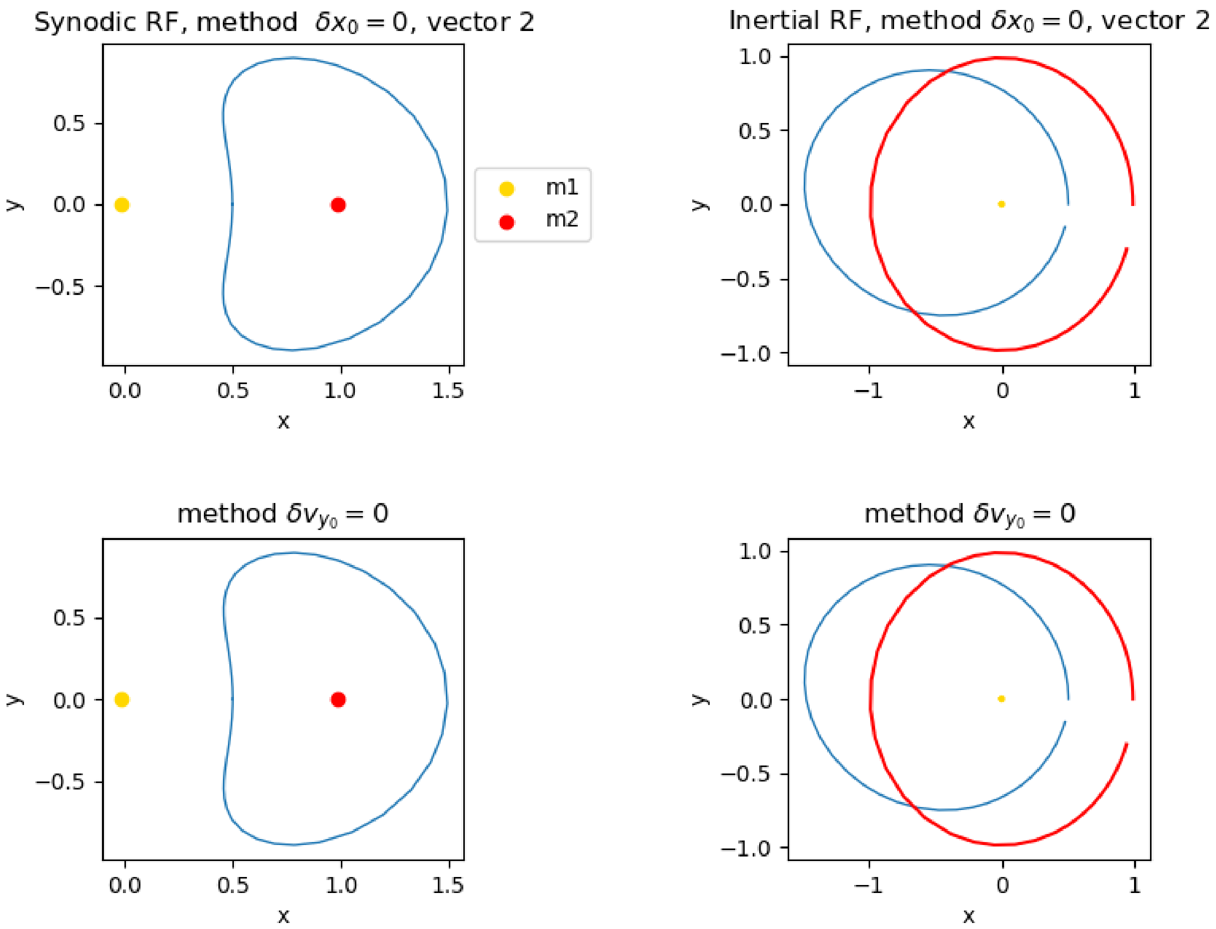
The following plots show the orbits in the synodic and inertial reference frame plotted over a time interval of 1 period.

STATE VECTOR 1



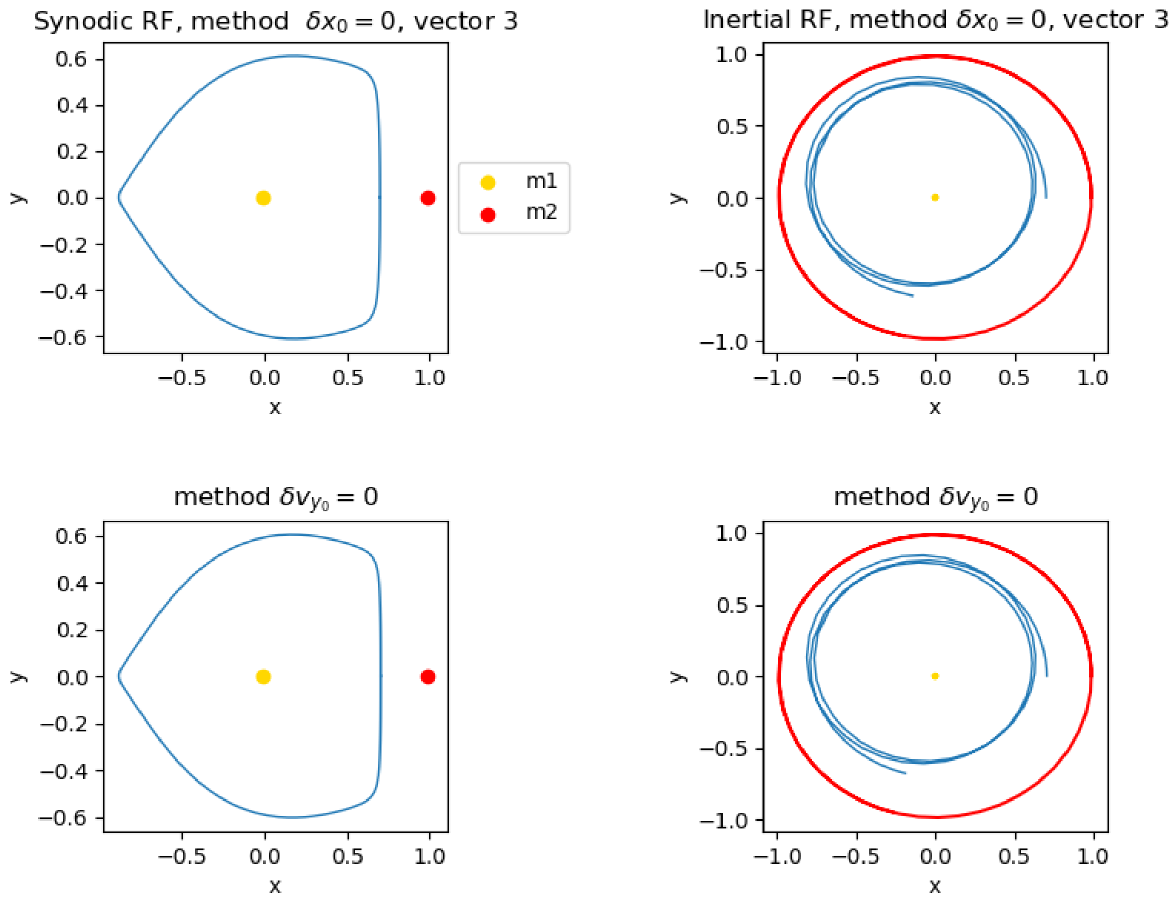
In this case the period of the massless particle in the synodic reference frame is slightly less than the period that needs m_2 to circle around the CM (we see that the orbit is not closed).

STATE VECTOR 2



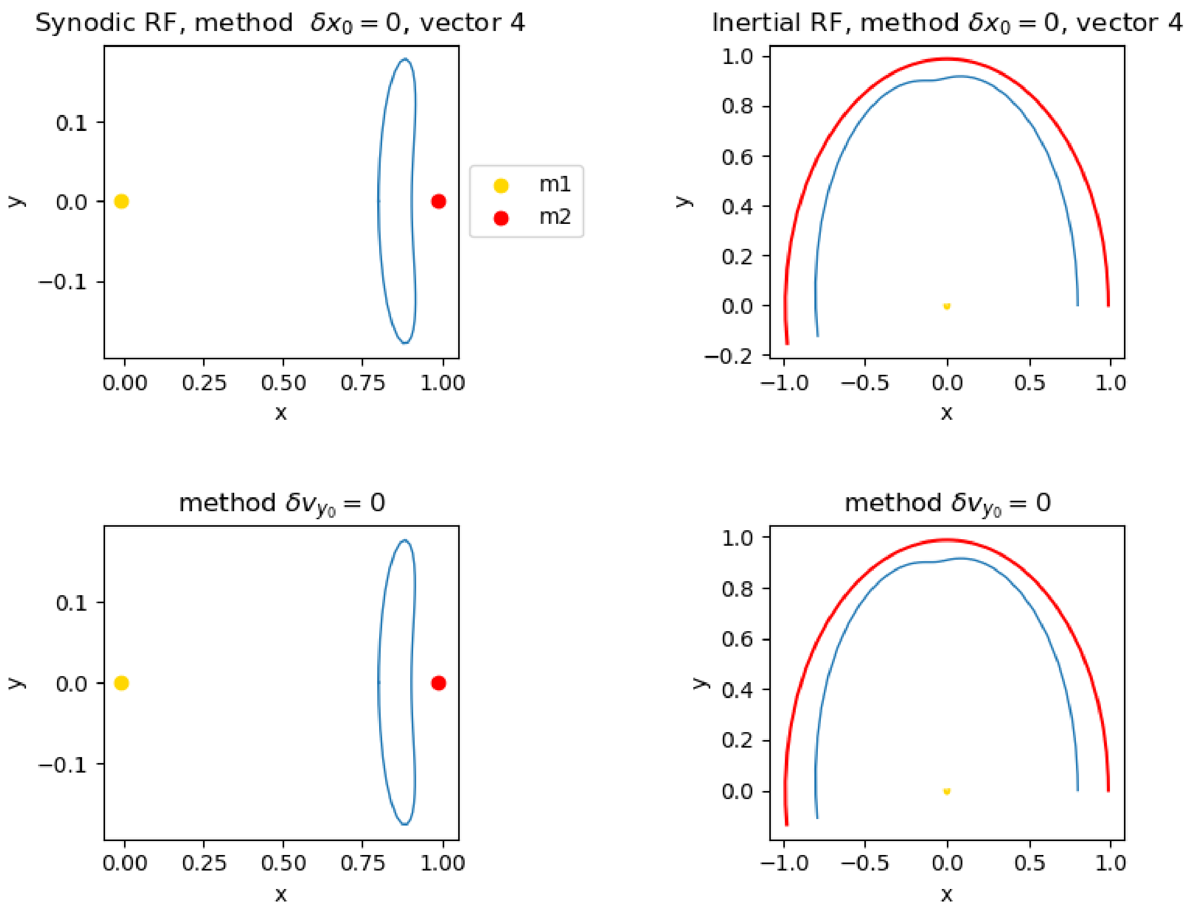
In this case it is evident that the period of the massless particle in the synodic RF is less than the period of m_2 because the orbit is visibly not closed.

STATE VECTOR 3



The third state vector, instead, has a smaller period and we can see the red orbit closed.

STATE VECTOR 4

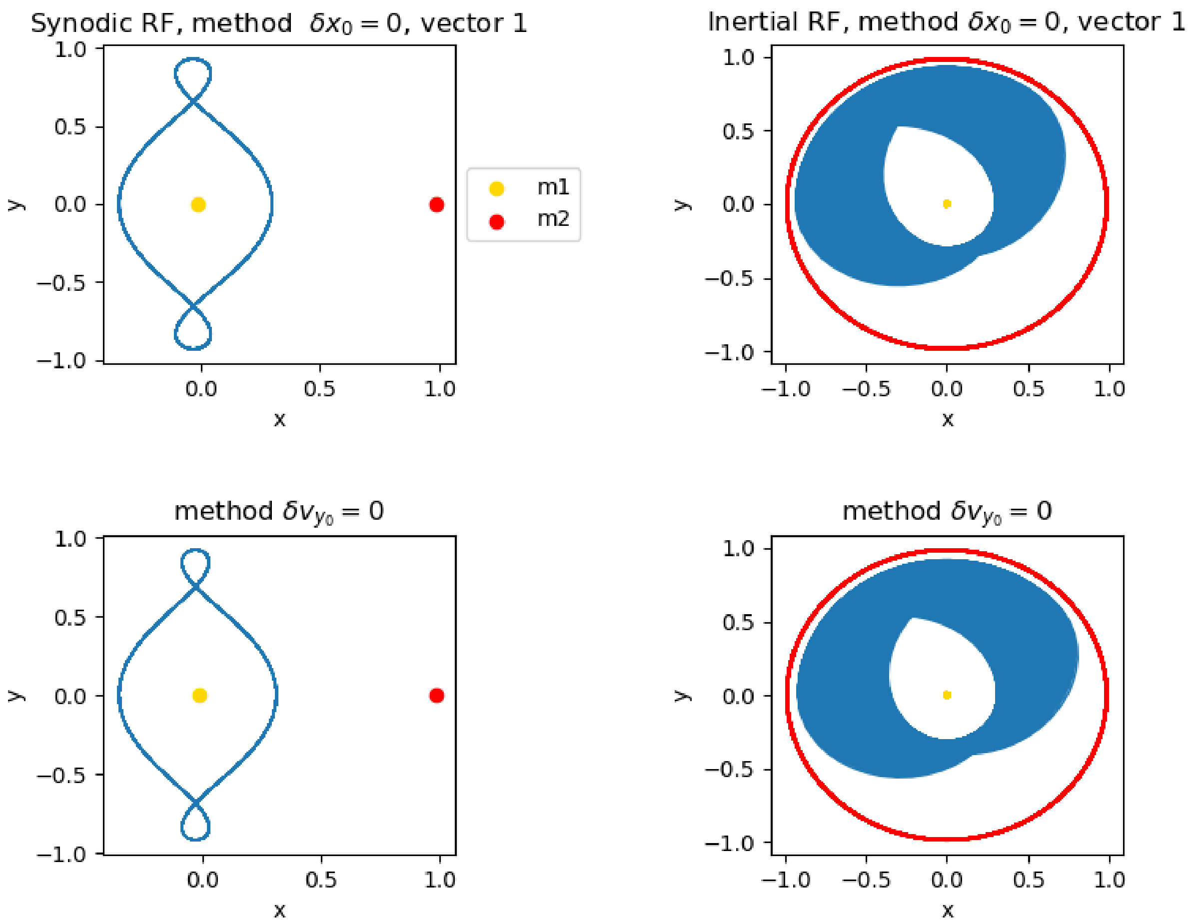


The fourth state vector is the one with the smallest period. Plotting just one orbit we can see it has a periodic orbit in the synodic reference frame.

50 PERIODS

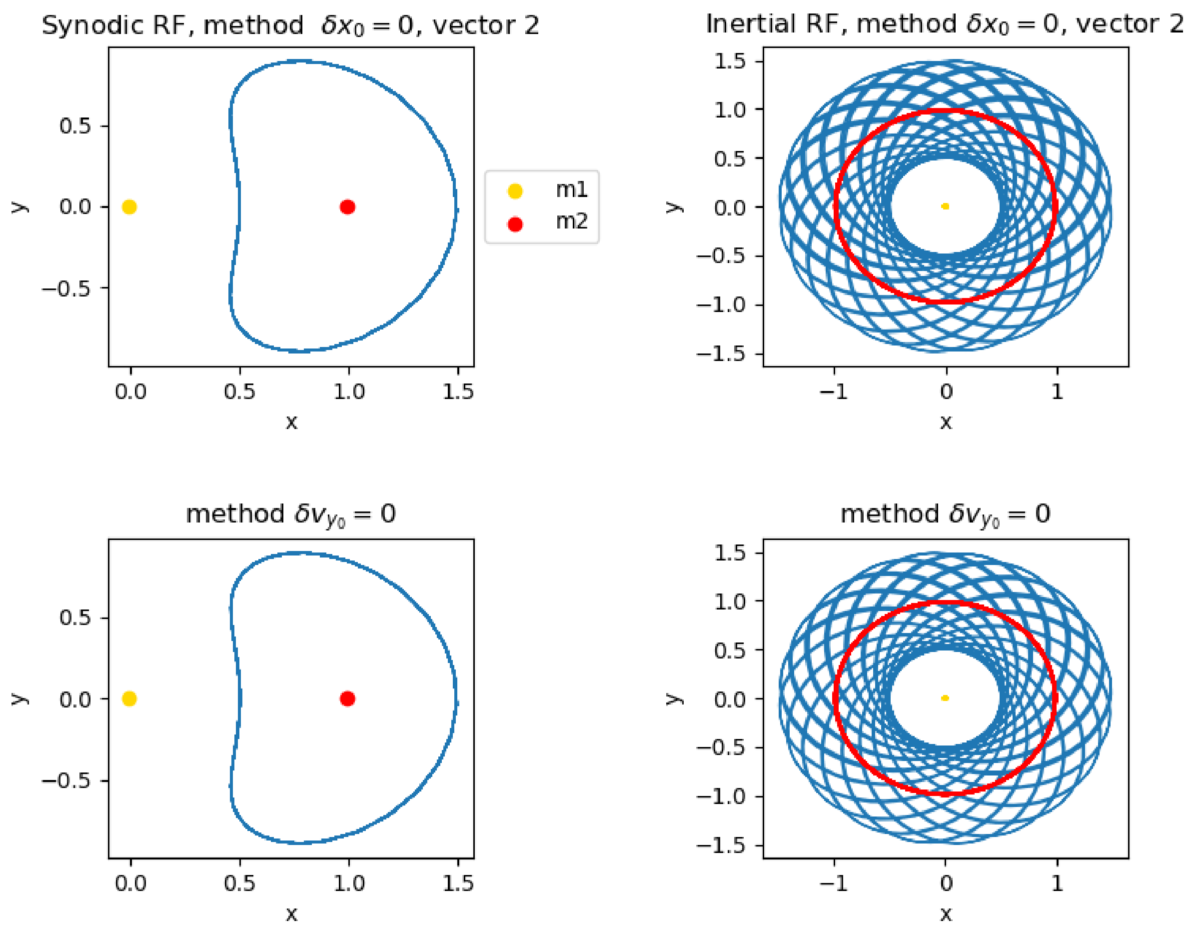
The following plots show the orbits in the two reference frames after 50 periods, using the two correction methods:

STATE VECTOR 1



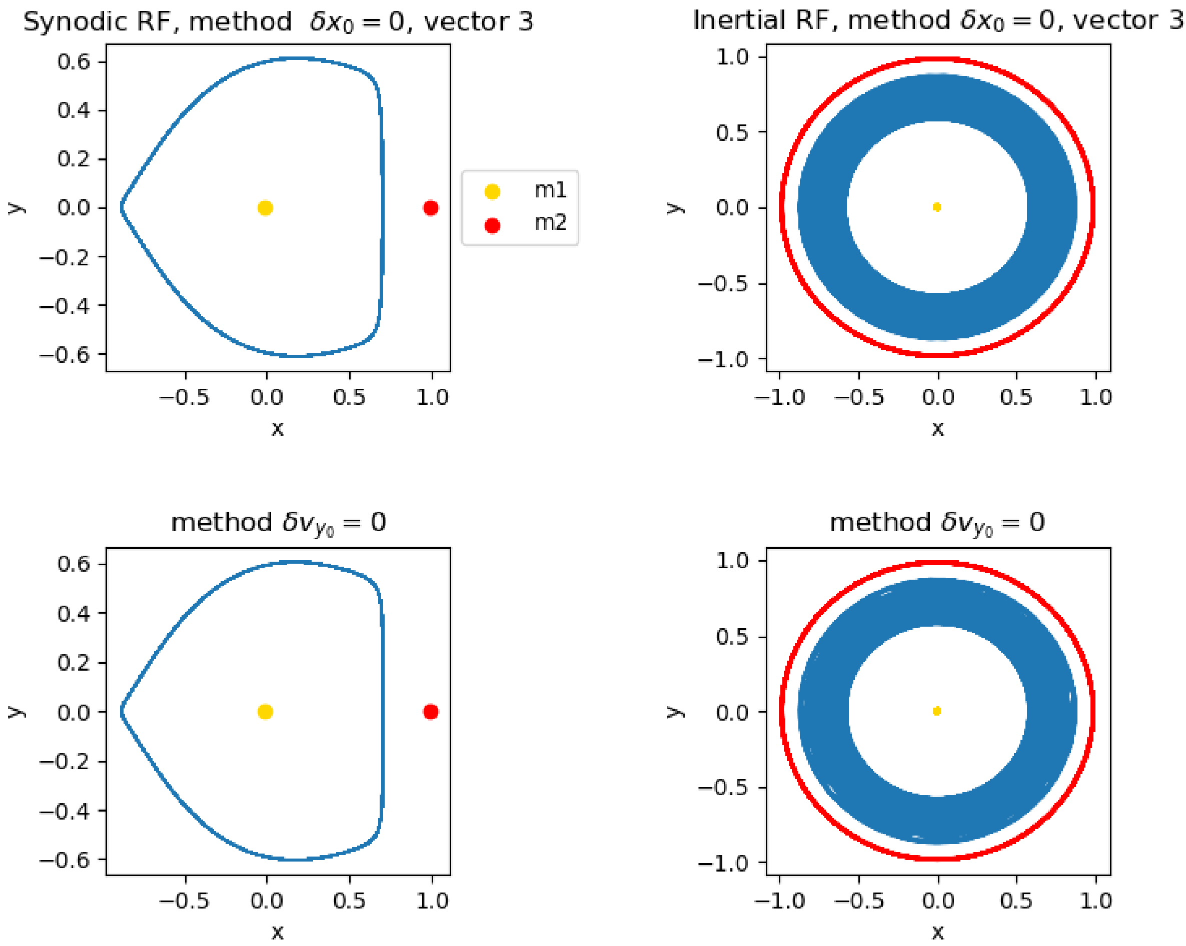
We can see that after 50 periods in the synodic RF we still have a closed and periodic orbit (this means that our correction was good enough to achieve the real initial mirror configuration). In the inertial RF we see a periodic orbit.

STATE VECTOR 2



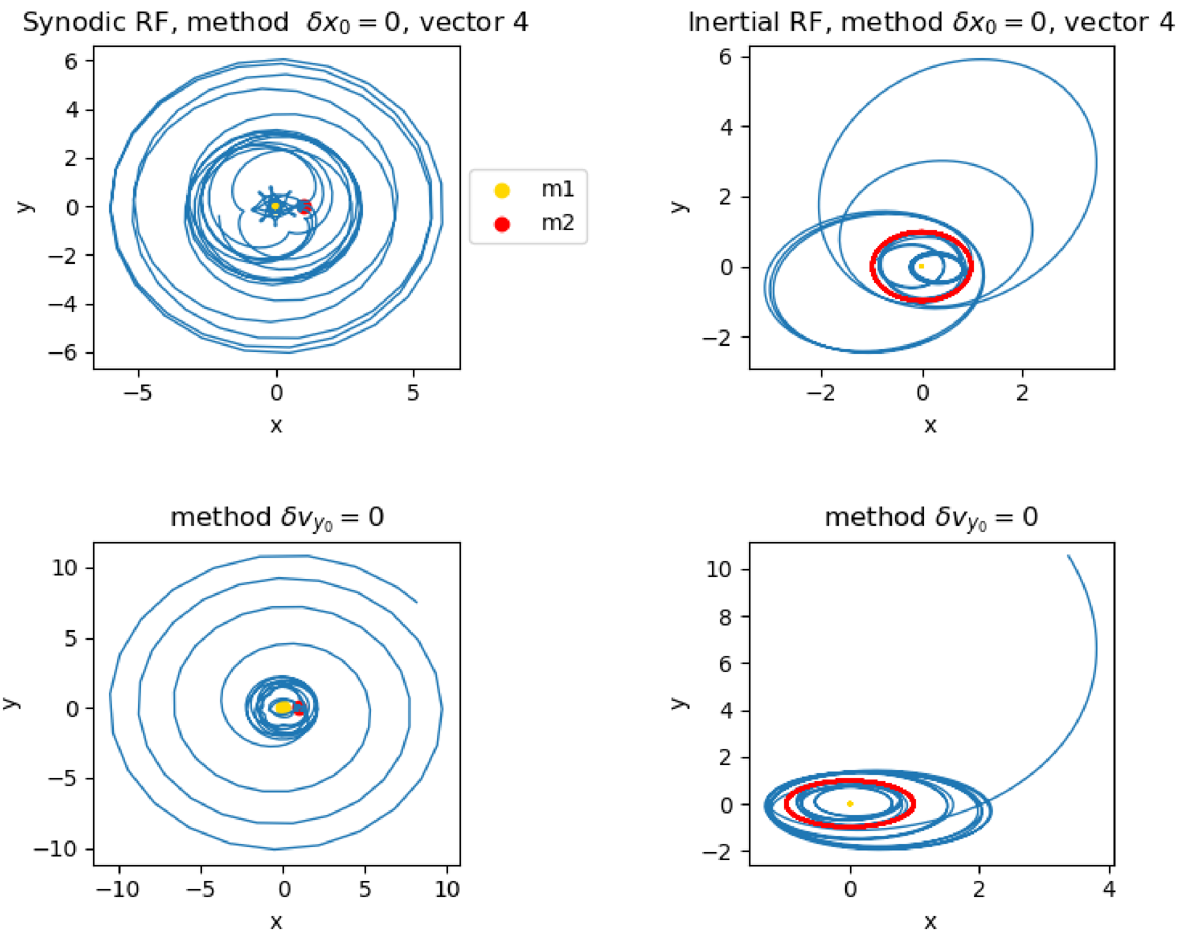
For the second state vector we still find a periodic orbit in the synodic RF and in the inertial one. Also this time we managed to achieve the mirror configuration.

STATE VECTOR 3



The same can be commented for the third state vector.

STATE VECTOR 4



In the case of the fourth state vector here we clearly see what we had already commented. After more than 4 periods the orbit in the synodic reference frame is not periodic anymore, it diverges and so does the one in the inertial RF.

This may be caused by the fact that we have a finite precision in decimal digits that we can achieve by integrating and maybe the correction required to reach the mirror configuration was higher than the 10^{-15} tolerance we work with. Requiring more significative digits may be caused by the fact that the vector is, compared to the others, really close to one of the two masses and therefore subjected to more frequent changes in the gravitational field.

EXERCISE 3

We have considered the first time that the point crosses again the x-axis to be half of the period, so the period will be twice the last value of $t = T/2$ computed using the two different approximation methods. Theses considerations lead to the following table, in which we give the values of the periods for the 4 dynamical states in the 2 different methods:

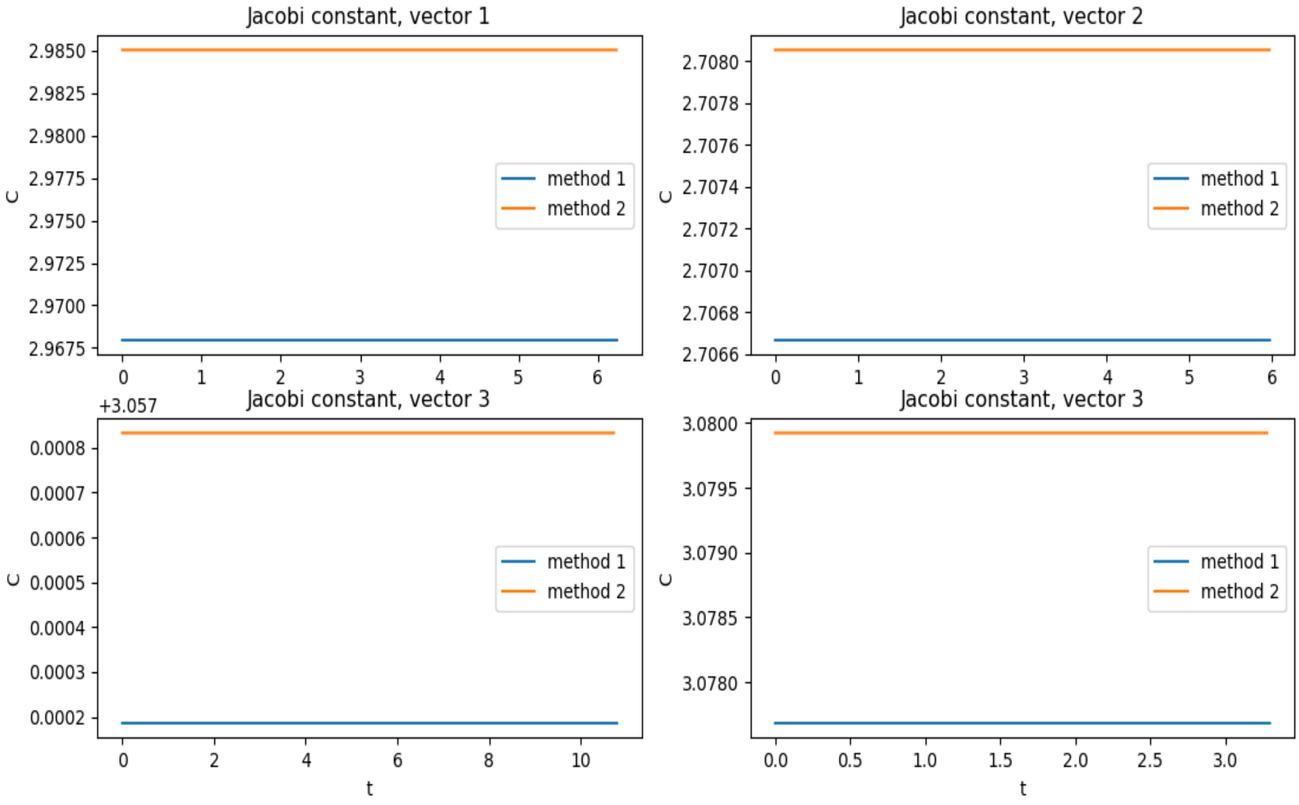
Method $\delta x_0 = 0$

Dynamical state	Period	Jacobi constant
1	6.24111432579364	2.9679520569201
2	5.97074210348859	2.7066649882761
3	10.7905070234183	3.0571859148664
4	3.29624298355991	3.0776849099201

Method $\delta v_{y_0} = 0$

Dynamical state	Period	Jacobi constant
1	6.23562738302507	2.9850502523862
2	5.967620164447	2.7080546654258
3	10.7231842971136	3.0578327473871
4	3.27755683677416	3.0799217249612

The following plot shows the values of the Jacobi constants over one period for each vector obtained with the two different methods.



We can see that the constant is very well approximated, indeed it changes its value over one period at the 15^{th} significative digit (this is the reason why I have not computed the standard deviation).

We can see that the two different methods give two different values for the Jacobi constant, that is something that we might expect since the two corrections method give rise to different initial conditions and therefore different orbital parameters.

EXERCISE 4

The Hill zero velocity curves are the one for which the velocity is zero and represent the region in which, according to its Jacobi constant, a particle is allowed to move.

Starting from the expression for the square of the velocity (that must be positive), and projecting it onto the $x - y$ plane, we get that the zero velocity curves are the one that solve:

$$f(\boldsymbol{r}, C) = x^2 + y^2 + 2\left(\frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right) - C = 0$$

In order to plot the Hill's curves I have used a contour plot, created a grid with 10000 points, evaluated their value of f and plotted only the zero ones.

The Lagrangian points correspond to the stationary points of the square velocity function. In particular, considering the case of a motion in the $x - y$ plane, the Lagrangian points must satisfy the following equations:

$$\frac{\partial f}{\partial x}|_{L_i} = \frac{\partial f}{\partial y}|_{L_i} = 0$$

Where the function f represents the square of velocity and is given by the equation above.

The Lagrangian points also represent the limit to which different Hill's zero velocity curves converge (if we lower the value of C we arrive at degenerated curves that are the Lagrangian points), and in particular there are 5 of them, 2 associated with the triangular solution and 3 to the colinear one.

The triangular one are obtained simplifying the y and multiplying bot sizes by $(x - x_i)$, so that their coordinates are:

$$\boldsymbol{x_{L_4}} = \left(\frac{1}{2} - \mu, \frac{\sqrt{3}}{2}\right)^T$$

$$\boldsymbol{x_{L_5}} = \left(\frac{1}{2} - \mu, \frac{\sqrt{3}}{2}\right)^T$$

The case of the colinear points is more complicated. These points are aligned with the two masses and y cannot be simplified. This leads to a quintic for the x coordinate of the massless particle, and since the massless particle can be found either between the masses or to their side, there are 3 possible colinear points. These are found solving the quantic:

- Point L_2 (configuration of masses 1-2-3)
$$x^5 + (4\mu - 2)x^4 + (6\mu^2 - 6\mu + 1)x^3 + (4\mu^3 - 6\mu^2 + 2\mu - 1)x^2 + (\mu^4 - 2\mu^3 + \mu^2 - 4\mu + 2)x + (-3\mu^2 + 3\mu - 1) = 0$$
- Point L_3 (configuration of masses 3-1-2)

$$x^5 + (4\mu - 2)x^4 + (6\mu^2 - 6\mu + 1)x^3 + (4\mu^3 - 6\mu^2 + 2\mu + 1)x^2 + (\mu^4 - 2\mu^3 + \mu^2 + 4\mu - 2)x + (3\mu^2 - 3\mu + 1) = 0$$

- Point L_1 (configuration of masses 1-3-2)

$$x^5 + (4\mu - 2)x^4 + (6\mu^2 - 6\mu + 1)x^3 + (4\mu^3 - 6\mu^2 + 4\mu - 1)x^2 + (\mu^4 - 2\mu^3 + 5\mu^2 - 4\mu + 2)x + (2\mu^3 - 3\mu^2 + 3\mu - 1) = 0$$

So the colinear Lagrangian points can be found either solving numerically the quintic (using for examples a Newton-Raphson method) or using an approximative method. It can be proven that the Lagrangian colinear points can be approximated as follows:

- $\boldsymbol{L_1}$

$$x_1 = 1 - \mu - \gamma_1$$

Where:

$$\gamma_1 = \alpha \left(1 - \frac{1}{3}\alpha - \frac{1}{9}\alpha^2 - \frac{23}{81}\alpha^3\right)$$

Where:

$$\alpha = \left(\frac{1}{3}\frac{\mu}{1-\mu}\right)^{1/3}$$

- $\boldsymbol{L_2}$

$$x_2 = 1 - \mu - \gamma_2$$

Where

$$\gamma_2 = \alpha \left(1 + \frac{1}{3}\alpha - \frac{1}{9}\alpha^2 - \frac{31}{81}\alpha^3\right)$$

- $\boldsymbol{L_3}$

$$x_3 = -\mu - \gamma_3$$

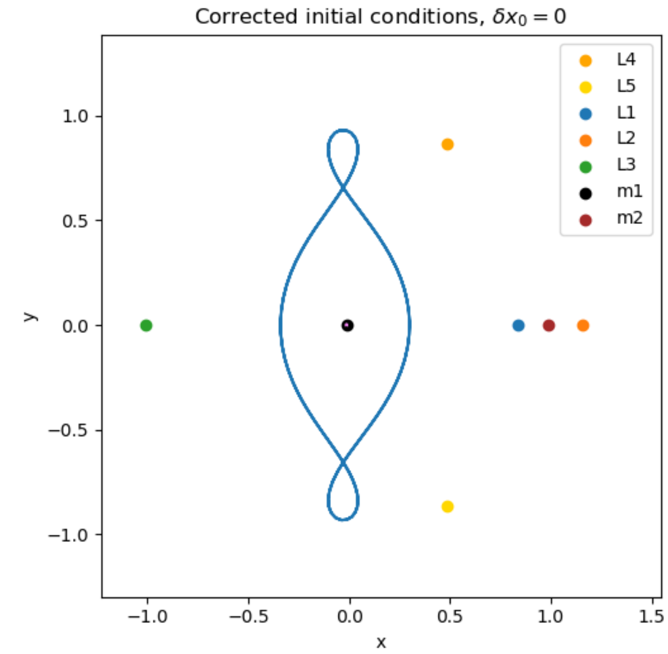
Where

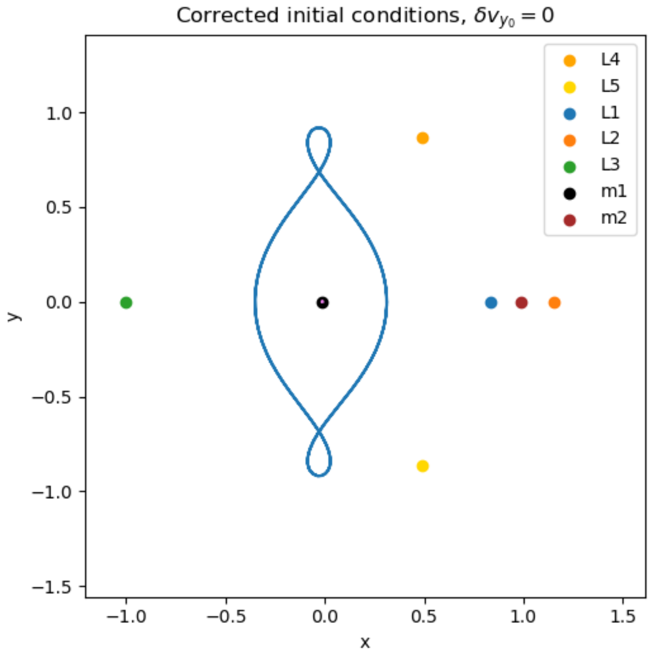
$$\gamma_3 = 1 - \frac{7}{12}\mu\left(1 + \frac{23}{84}\left(\frac{7}{12}\mu\right)^2\right)$$

In the code I chose to use the approximations to plot the Lagrangian points.

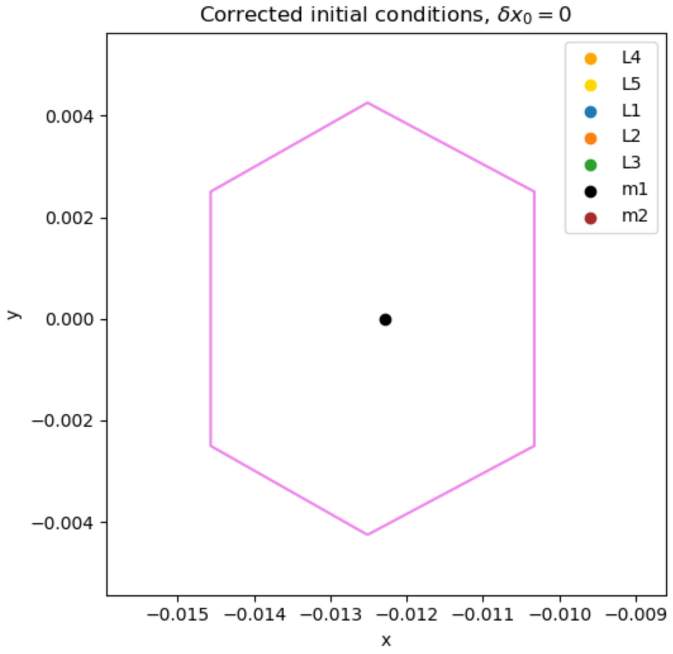
The following plots show the orbits in the synodic RF computed with the two approximation methods along with the Hille zero velocity curves (in violet):

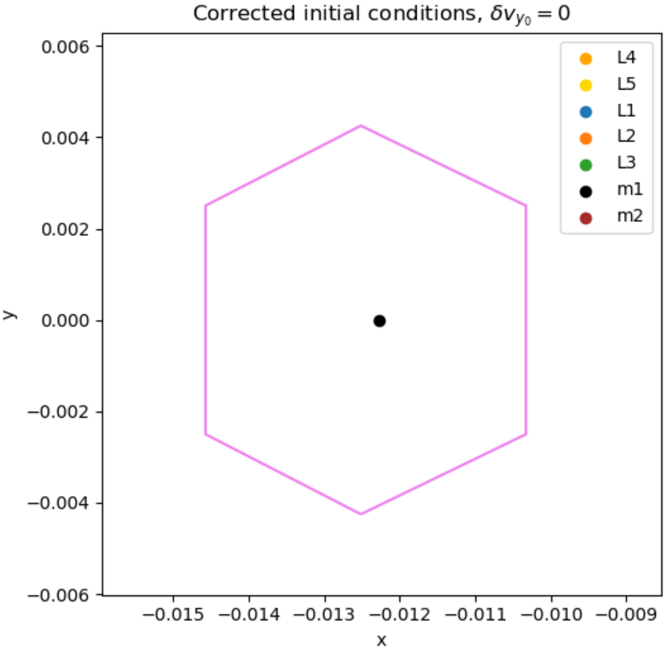
STATE VECTOR 1



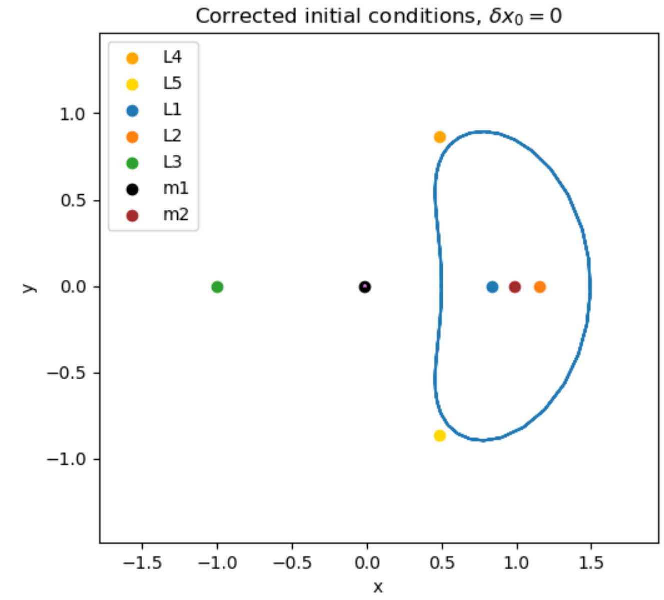


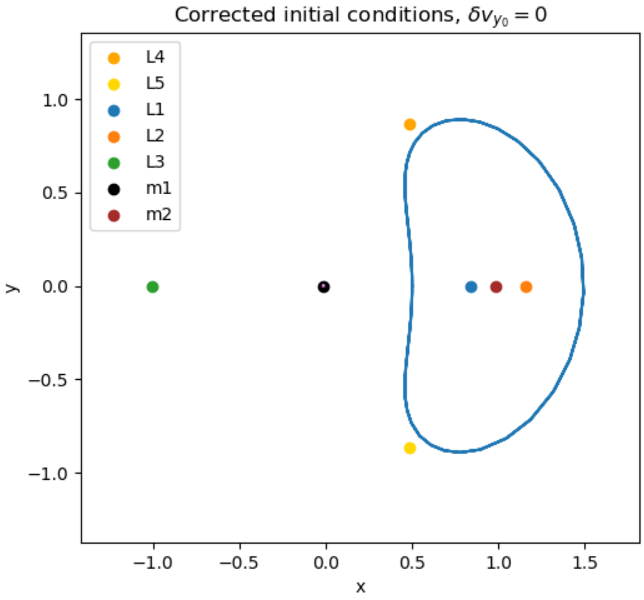
We notice that it seems not to exist the Hill's curve, but if we zoom in enough, we notice it rounds the first body:



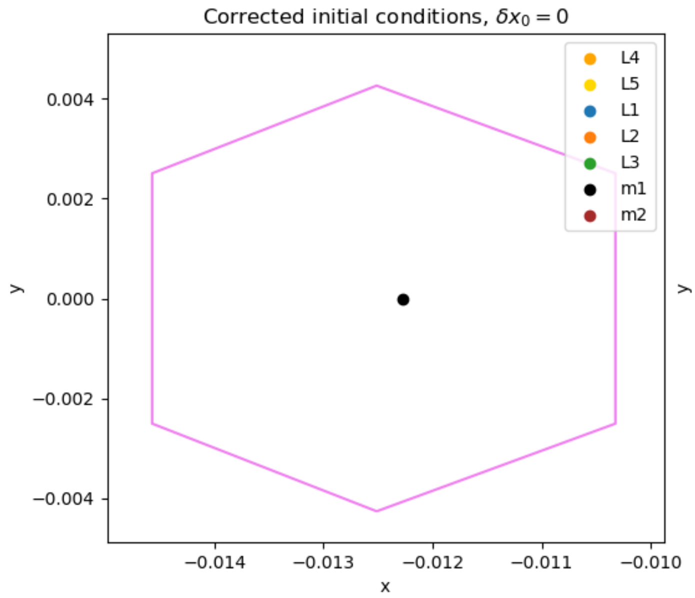


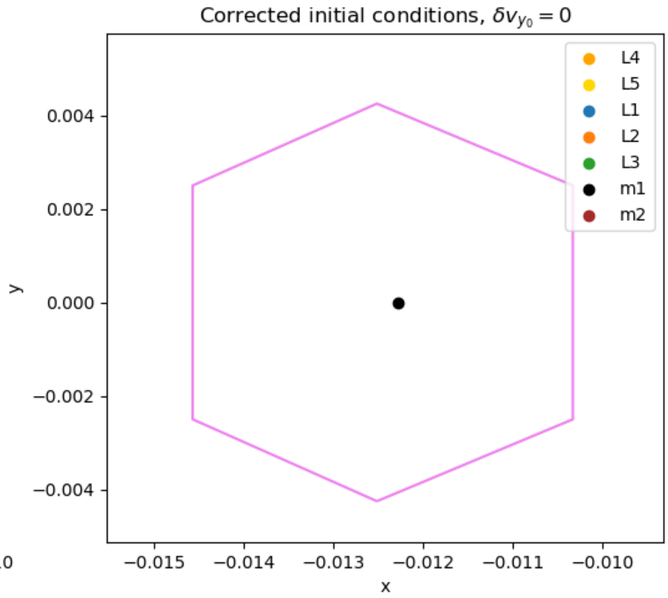
STATE VECTOR 2



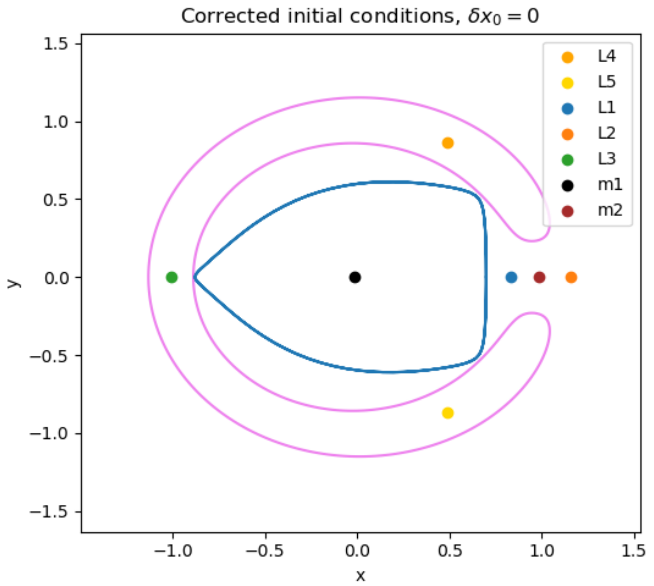


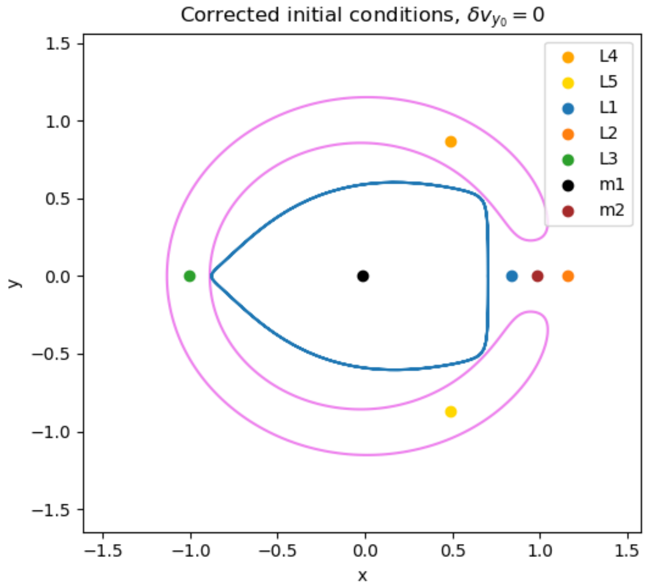
The same can be observed for the second state vector, that shows the same curve around the first body.





STATE VECTOR 3





STATE VECTOR 4