Problem Set 3
Issued: 5 October 2018
Due: 12 October 2018, 11:59 pm

FIRST Name ______________________ LAST Name ______________________

Discussion Section Time: __________ SID (All Digits): ________________

Linus Pauling and Respect for Elders

When an old and distinguished person speaks to you, listen to him carefully and with respect—but do not believe him. Never put your trust into anything but your own intellect. Your elder, no matter whether he has gray hair or has lost his hair, no matter whether he is a Nobel laureate—may be wrong. The world progresses, year by year, century by century, as the members of the younger generation find out what was wrong among the things that their elders said. So you must always be skeptical—always think for yourself."[a]

[a]Excerpts from *Linus Pauling: Scientist and Peacemaker*, Edited by Clifford Mead and Thomas Hager, Oregon State University Press, 2001, ISBN: 0870714899.

Santiago Ramón y Cajal and Respect for Intellectual Authorities

Far from humbling one's self before the great authorities of science, those beginning research must understand that—by a cruel but inevitable law—their destiny is to grow a little at the expense of the great one's reputation. It is very common for those beginning their scientific explorations with some success to do so by weakening the pedestal of a historic or contemporary hero.

By way of classic examples, recall Galileo refuting Aristotle's view of gravity, Copernicus tearing down Ptolemy's system of the universe, Lavoisier destroying Stahl's concept of phlogiston, and Virchow refuting the idea of spontaneous generation held by Schwann, Schleiden, and Robin.[a]

[a]Excerpts from *Advice for a Young Investigator*, by Santiago Ramón y Cajal, translated by Neely Swanson and Larry W. Swanson, The MIT Press, 1999, ISBN: 0-262-68150-1.

Policy Statement

- We encourage you to collaborate, but only in a group of up to *five* current EECS 120 students.
- On the solution document that you turn in for grading, you must write the names of your collaborators below your own; each teammate must submit for our evaluation a distinct, self-prepared solution document containing original contributions to the collaborative effort.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- Unless we explicitly state otherwise, you will receive full credit *only if* you explain your work succinctly, but clearly and convincingly.
- Typically, we evaluate your solutions for only a subset of the assigned problems. A priori, you do not know which subset we will grade. It is to your advantage to make a bona fide effort at tackling *every* assigned problem.
- If you are asked to provide a "sketch," it refers to a *hand-drawn* sketch, well-labeled to indicate all the salient features—not a plot generated by a computing device.
- On occasion, a problem set contains one or more problems designated as "optional." We do NOT grade such problems. Nevertheless, you are responsible for learning the subject matter within their scope.

Coverage Overview

This problem set covers aspects of the Discrete-Time Fourier Series (DTFS)—including its close link to the Discrete Fourier Transform (DFT).

It also explores the Continuous-Time Fourier Series (CTFS) in both complex-exponential and trigonometric forms.

And it addresses certain properties of the inner product of complex vectors, an understanding of which is crucial to developing a mastery of the DTFS in particular, and Fourier Analysis in general.

The scope of this problem set includes subject matter covered in lectures, discussions, and office hours up to, and including, 11 October 2018.

List of Your Collaborators

- Name: ______________________________ SID: ________________________
- Name: ______________________________ SID: ________________________
- Name: ______________________________ SID: ________________________
- Name: ______________________________ SID: ________________________

HW3.1 (Inner Product) Let $\boldsymbol{x} \in \mathbb{C}^p$ and $\boldsymbol{y} \in \mathbb{C}^p$ be two p-dimensional vectors of complex numbers, thought of as column vectors, where p is a positive integer. The *inner product* of $\boldsymbol{x}$ with $\boldsymbol{y}$, denoted $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$, is the complex number defined as

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^T \boldsymbol{y}^*.$$

Here $\boldsymbol{x}^T$ denotes the transpose of $\boldsymbol{x}$, and $\boldsymbol{y}^*$ denotes the complex conjugate of $\boldsymbol{y}$. For example, if

$$\boldsymbol{x} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{p-1} \end{bmatrix} \quad \text{and} \quad \boldsymbol{y} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{p-1} \end{bmatrix},$$

then

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{k=0}^{p-1} x_k \, y_k^*.$$

In the following, $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3} \ldots, \boldsymbol{x_k}, \boldsymbol{y_1}, \boldsymbol{y_2}$ are all vectors in $\mathbb{C}^p$.

(a) Show that $\langle \boldsymbol{x}, \boldsymbol{x} \rangle$ is a real number. Further, show that $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \geq 0$, with equality if, and only if, $\boldsymbol{x}$ is the zero vector.

(b) Show that $\langle \boldsymbol{y}, \boldsymbol{x} \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle^*$.

(c) Show that $\langle \boldsymbol{x_1} + \boldsymbol{x_2}, \boldsymbol{y} \rangle = \langle \boldsymbol{x_1}, \boldsymbol{y} \rangle + \langle \boldsymbol{x_2}, \boldsymbol{y} \rangle$.

True or False? $\langle \boldsymbol{x}, \boldsymbol{y_1} + \boldsymbol{y_2} \rangle = \langle \boldsymbol{x}, \boldsymbol{y_1} \rangle + \langle \boldsymbol{x}, \boldsymbol{y_2} \rangle$. No explanation is necessary.

(d) We know that $\langle \alpha\, \boldsymbol{x}, \boldsymbol{y} \rangle = \alpha \langle \boldsymbol{x}, \boldsymbol{y} \rangle$, where α is a complex scalar. Show that $\langle \boldsymbol{x}, \alpha\, \boldsymbol{y} \rangle = \alpha^* \langle \boldsymbol{x}, \boldsymbol{y} \rangle$.

(e) If $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$, we say that $\boldsymbol{x}$ is *orthogonal* to $\boldsymbol{y}$, and we denote this by $\boldsymbol{x} \perp \boldsymbol{y}$. According to part (b), this implies that $\langle \boldsymbol{y}, \boldsymbol{x} \rangle = 0$. Show that if $\boldsymbol{x}$ is orthogonal to $\boldsymbol{y}$, then

$$\langle \boldsymbol{x} + \boldsymbol{y}, \boldsymbol{x} + \boldsymbol{y} \rangle = \langle \boldsymbol{x}, \boldsymbol{x} \rangle + \langle \boldsymbol{y}, \boldsymbol{y} \rangle.$$

Explain how this is an articulation of Pythagoras's theorem.

(f) Suppose $\boldsymbol{x} = \alpha_1 \, \boldsymbol{x_1} + \ldots + \alpha_k \, \boldsymbol{x_k}$, for some complex scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$. Show that if

$$\langle \boldsymbol{x}, \boldsymbol{x_i} \rangle = 0 \quad \text{for all } 1 \leq i \leq k,$$

then $\boldsymbol{x}$ must be the zero vector.

HW3.2 (DTFS Expansion) Determine the complex-exponential discrete-time Fourier series (DTFS) expansion for each signal $x : \mathbb{Z} \to \mathbb{R}$ described below, or explain why no such expansion exists. For each case where a DTFS expansion exists, be sure to identify the period p and the fundamental frequency ω_0.

For parts (f) and (g), assume x denotes *exactly one period* of a periodic signal $\widetilde{x}$. Your answer would then be the DTFS expansion of the periodic signal $\widetilde{x}$ for certain values of n, and zero otherwise.

(a) $x(n) = \sin\left(\frac{2\pi}{5}n\right) + \cos\left(\frac{4\pi}{5}n\right), \quad \forall n.$

(b) $x(n) = \cos\left(\frac{3\pi}{5}n\right) + \frac{1}{3}\cos\left(\frac{4\pi}{5}n\right), \quad \forall n.$

(c) $x(n) = \cos\left(\frac{\sqrt{2}\,\pi}{5}n\right), \quad \forall n.$

(d) $x(n) = \cos\left(\frac{2\pi}{3}n\right) + (-1)^n, \quad \forall n.$

(e) $x(n) = \sum_{l=-\infty}^{+\infty} \delta(n - lp)$, where p is a positive integer.

(f) $x(n) = \delta(n+2) + 2\,\delta(n+1) + 3\,\delta(n) + 2\,\delta(n-1) + \delta(n-2).$

(g) $x(n) = \delta(n+2) + \delta(n+1) - \delta(n-1) - \delta(n-2).$

HW3.3 (A DTFS Puzzle) Consider a discrete-time signal $x : \mathbb{Z} \to \mathbb{R}$ having the following the properties:

- $x(n + 4l) = x(n), \ \forall l, n \in \mathbb{Z}$.
- $\displaystyle\sum_{n=-1}^{2} x(n) = 2$.
- $\displaystyle\sum_{n=-1}^{2} (-1)^n x(n) = 4$.
- $\displaystyle\sum_{n=-1}^{2} x(n) \cos\left(\frac{\pi}{2} n\right) = \sum_{n=-1}^{2} x(n) \sin\left(\frac{\pi}{2} n\right) = 0$.

(a) Determine the complex exponential Fourier series coefficients X_{-1}, X_0, X_1, and X_2 for the signal x. From the coefficients, determine and provide a well-labeled plot for the signal x.

You may continue your work in the blank space allocated for this purpose at the top of the next page

Part (a) Continued:

(b) Based on your results from part (a), determine the fundamental period p and fundamental frequency ω_0 of the signal x. Express x in terms of its DTFS coefficients and complex exponentials of appropriate frequencies, and identify each coefficient and its corresponding harmonic frequency.

HW3.4 (The DTFS and Its Fraternal Twin, the DFT) To represent finite-length or periodic discrete-time signals, engineers have traditionally used a complex exponential Fourier series expansion that is slightly different from the Discrete-Time Fourier Series (DTFS). The expansion of choice is called the Discrete Fourier Transform (DFT).[1]

In this problem, you will discover the simple relationship between the DTFS and the DFT. Recall that the analysis and synthesis equations of the DTFS expansion of a periodic discrete-time signal $x : \mathbb{Z} \to \mathbb{C}$ are:

$$x = \sum_{k=\langle p \rangle} X_k \, \phi_k \qquad \overset{\text{DTFS}}{\longleftrightarrow} \qquad X_k = \frac{\langle x, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle}.$$

Written in the time domain, these equations are:

$$x(n) = \sum_{k=\langle p \rangle} X_k \, \phi_k(n) \qquad \overset{\text{DTFS}}{\longleftrightarrow} \qquad X_k = \frac{1}{p} \sum_{n=\langle p \rangle} x(n) \, \phi_k^*(n),$$

where $\phi_k(n) = e^{ik\omega_0 n}$; $\langle x, \phi_k \rangle = \sum_{n=\langle p \rangle} x(n) \, \phi_k^*(n)$; and $\langle \phi_k, \phi_k \rangle = \sum_{n=\langle p \rangle} \phi_k(n) \, \phi_k^*(n) = p$.

(a) Consider the set of basis signals $\Psi = \{\psi_0, \ldots, \psi_{p-1}\}$, where $\psi_k = \dfrac{1}{p}\phi_k$. Determine $\langle \psi_k, \psi_l \rangle$ for each of the cases $k = l$ and $k \neq l$. Use what you already know about $\langle \phi_k, \phi_l \rangle$ to simplify your work.

[1] The Fast Fourier Transform (FFT) is merely a reference to a family of computationally-efficient algorithms for implementing the DFT. The FFT is not a different transform.

(b) We wish to express a p-periodic signal x in terms of the basis functions in Ψ, as follows:

$$x = \sum_{k=\langle p \rangle} X'_k \, \psi_k.$$

Determine an expression for the coefficients X'_k in terms of projections of the signal x onto the basis functions in Ψ. What is the relationship between the DFT coefficients X'_k and the DTFS coefficients X_k? Write down the analysis and synthesis equations for the DFT.

HW3.5 (LTI Processing of Periodic Signals) Consider a periodic, discrete-time signal $x : \mathbb{Z} \to \mathbb{R}$ having the discrete-time Fourier series (DTFS) expansion

$$x(n) = \sum_{k=\langle p \rangle} X_k \, e^{ik\omega_0 n} \, ,$$

where ω_0 denotes the fundamental frequency of the signal; if p is the period of x, then $\omega_0 = 2\pi/p$.

Suppose x is the input signal applied to a linear time-invariant (LTI) system characterized by the impulse response $h : \mathbb{Z} \to \mathbb{R}$ and corresponding frequency response H, where

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) \, e^{-i\omega n} \, , \quad \forall \omega.$$

Let y be the corresponding output signal.

(a) Prove that the output signal y is periodic; that is, show that if $x(n+p) = x(n)$, then $y(n+p) = y(n)$.

(b) Let the DTFS expansion of the output signal y be

$$y(n) = \sum_{k=\langle p \rangle} Y_k \, e^{ik\omega_0 n} \, .$$

(i) Express the output-signal DTFS coefficients Y_k in terms of the input signal DTFS X_k and the frequency response H.

(ii) Suppose the impulse response of the LTI system is given by $h(n) = \delta(n - n_0)$, where $n_0 \in \mathbb{Z}$. Explicitly determine the output-signal DTFS coefficients Y_k in terms of the input-signal DTFS coefficients X_k.

HW3.6 (The Output DTFS of an N-Fold Upsampler) Consider a discrete-time system whose input and output signals are denoted by $x : \mathbb{Z} \to \mathbb{R}$ and $y : \mathbb{Z} \to \mathbb{R}$, respectively. The output is obtained by upsampling the input by a factor of N, where $N \in \{2, 3, \ldots\}$. That is,

$$\forall n \in \mathbb{Z}, \quad y(n) = \begin{cases} x\left(\dfrac{n}{N}\right) & \text{if } n \bmod N = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose the input signal x is periodic with fundamental frequency $\omega_0 = 2\pi/p$, where p denotes the period, and has the discrete-time Fourier series (DTFS) expansion

$$x(n) = \sum_{k=\langle p \rangle} X_k \, e^{ik\omega_0 n} .$$

(a) Determine the period $\widehat{p}$ and the corresponding fundamental frequency $\widehat{\omega}_0$ of the periodic output signal y. Your answers must be in terms of p, ω_0, and N.

(b) Determine the DTFS coefficients $Y_k, k \in \{0, 1, \dots, \widehat{p} - 1\}$, in terms of N and the DTFS coefficients X_k of the input signal.

HW3.7 (Continuous-Time Periodic Signals) A continuous-time signal x is said to be *periodic* if there is a real number $p > 0$ such that

$$\forall t \in \mathbb{R}, \quad x(t+p) = x(t). \tag{1}$$

The smallest p for which Equation (1) holds is called the *fundamental period* of x. The *fundamental frequency* of x is denoted by ω_0 and is defined as $\omega_0 = 2\pi/p$.

Note that the fundamental period of a continuous-time signal that is constant for all t (i.e., period of a DC signal) is undefined. The only frequency component in a DC signal, however, is $\omega_0 = 0$; no other frequency is present in a signal having a constant value for all t.

This is a subtlety that does not arise in the context of discrete-time constant signals, which have a well-defined fundamental period $p = 1$ and fundamental frequency $\omega_0 = 2\pi$, which is the same as $\omega_0 = 0$ radians per sample.

A signal that is not periodic is called *non-periodic* or *aperiodic*.

For each continuous-time signal x described below, either identify the fundamental period p and the fundamental frequency ω_0, or declare the signal as aperiodic. Explain your reasoning succinctly, but clearly and convincingly.

(a) $x(t) = \cos\left(\frac{3\pi}{4}t\right) + \sin\left(\frac{2\pi}{3}t\right), \ \forall t.$

(b) $x(t) = \cos^2(t), \ \forall t.$

(c) $x(t) = \cos\left(t^2\right), \ \forall t.$

(d) $x(t) = \cos(t) + \sin\left(\frac{2\pi}{3}t\right), \ \forall t.$

(e) $x(t) = \exp\left[i\left(\frac{3\pi}{4}t + \frac{2\pi}{5}\right)\right], \ \forall t.$

HW3.8 (Continuous-Time Trigonometric Fourier Series)
A periodic, real-valued continuous-time signal x can be represented by the trigonometric Fourier series expansion

$$\begin{aligned} x(t) &= A_0 + \sum_{k=1}^{+\infty} A_k \cos \frac{2\pi k}{p} t + \sum_{\ell=1}^{+\infty} B_\ell \sin \frac{2\pi \ell}{p} t \\ &= A_0 + \sum_{k=1}^{+\infty} A_k \cos k\omega_0 t + \sum_{\ell=1}^{+\infty} B_\ell \sin \ell\omega_0 t \,, \end{aligned} \tag{2}$$

where $\omega_0 = \frac{2\pi}{p}$ is the fundamental frequency, and p the fundamental period, of x. Equation 2 is the *synthesis equation* of the trigonometric Fourier series expansion.

More compactly,

$$x = \sum_{k=0}^{+\infty} A_k \chi_k + \sum_{\ell=1}^{+\infty} B_\ell \psi_\ell,$$

which is a linear combination of functions χ_k and ψ_ℓ, where

$$\begin{aligned} \chi_k(t) &= \cos k\omega_0 t, \quad k \in \mathbb{Z}^+, \text{and} \\ \psi_\ell(t) &= \sin \ell\omega_0 t, \quad \ell \in \mathbb{N}. \end{aligned}$$

Recall that $\mathbb{Z}^+ \triangleq \{0, 1, 2, \ldots\}$ and $\mathbb{N} \triangleq \{1, 2, 3, \ldots\}$.

In this problem, you will show that

$$\{\chi_0, \chi_1, \psi_1, \chi_2, \psi_2, \chi_3, \psi_3, \ldots\}$$

is a set of mutually-orthogonal functions. You will then exploit the mutual orthogonality of these functions to determine the coefficients A_k and B_ℓ.

(a) In what follows you're given a set of inner products, and you're asked to determine a fairly simple expression for each. You may proceed in one of two ways.

In one method, you can express $\chi_k(t)$ and $\psi_\ell(t)$ in terms of the complex exponential functions

$$\phi_r(t) = e^{ir\omega_0 t}, \quad r \in \mathbb{Z},$$

and exploit the orthogonality property of the complex exponentials to arrive at the appropriate expression for each inner product that you're asked to characterize.

Alternatively, you can find each of the inner products in (i)–(iv) below by inserting the appropriate functions in the definition of the inner product. If you choose this method, the following trigonometric identities may prove helpful to you:

$$\begin{aligned}\cos\alpha\cos\beta &= \frac{1}{2}\cos(\alpha+\beta)+\frac{1}{2}\cos(\alpha-\beta)\\ \cos\alpha\sin\beta &= \frac{1}{2}\sin(\alpha+\beta)-\frac{1}{2}\sin(\alpha-\beta)\end{aligned}$$

(i) $\langle\chi_0,\chi_0\rangle$.

(ii) $\langle\chi_k,\chi_k\rangle$, where $k\neq 0$.

(iii) $\langle \chi_k, \chi_m \rangle$, where $k \neq m$.

(iv) $\langle \chi_k, \psi_\ell \rangle$, where $k \in \mathbb{Z}^+$ and $\ell \in \mathbb{N}$.

(b) Take the inner product of each side of Equation 2 with an appropriate function χ_k or ψ_ℓ, and exploit the mutual orthogonality results you obtained in part (a) to determine the following expressions for the coefficients A_k and B_ℓ, where $k \in \mathbb{Z}^+$ and $\ell \in \mathbb{N}$. These are the *analysis equations* of the trigonometric Fourier series.

$$
\begin{aligned}
A_0 &= \frac{1}{p} \int_{\langle p \rangle} x(t) dt. \\
A_k &= \frac{2}{p} \int_{\langle p \rangle} x(t) \cos k\omega_0 t \, dt, \quad 1 \leq k. \\
B_\ell &= \frac{2}{p} \int_{\langle p \rangle} x(t) \sin \ell\omega_0 t \, dt, \quad 1 \leq \ell.
\end{aligned}
$$

(c) Recall that the complex exponential Fourier series expansion of x is

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k \, e^{ik\omega_0 t},$$

and exploit the fact that x is real-valued (which means that the coefficients X_k are conjugate-symmetric, i.e., $X_k^* = X_{-k}$), to establish the following relations between the trigonometric and complex exponential Fourier series coefficients:

$$\begin{aligned} A_0 &= X_0 \\ A_k &= +2\,\mathsf{Re}(X_k), \quad 1 \leq k \\ B_\ell &= -2\,\mathsf{Im}(X_\ell), \quad 1 \leq \ell. \end{aligned}$$

HW3.9 (Continuous-Time Trigonometric Fourier Series Coefficients) In this problem, you will get some practice determining the sine-cosine Fourier series coefficients of periodic signals. You will also discover a couple of relationships between the structure of a signal and its Fourier series coefficients.

Each signal has fundamental period p, fundamental frequency $\omega_0 = 2\pi/p$, and trigonometric Fourier series expansion of the form

$$x(t) = A_0 + \sum_{k=1}^{+\infty} A_k \cos k\omega_0 t + \sum_{\ell=1}^{+\infty} B_\ell \sin \ell\omega_0 t.$$

Provide a well-labeled plot of each signal over at least *three* periods, and determine all its Fourier series coefficients A_k and B_ℓ.

(a) The signal x is a periodic sawtooth waveform characterized over one period by

$$x(t) = t, \quad -\frac{p}{2} \leq t < \frac{p}{2}.$$

(b) A triangular waveform x is characterized over one period by

$$x(t) = \begin{cases} -t & -\dfrac{p}{2} \le t < 0 \\ +t & 0 \le t < +\dfrac{p}{2}. \end{cases}$$

(c) The periodic signal x is symmetric; that is, $x(t) = x(-t)$, $\forall t \in \mathbb{R}$. Show that the sine coefficients B_ℓ are all zero.

(d) The periodic signal x is antisymmetric; that is, $x(t) = -x(-t)$, $\forall t \in \mathbb{R}$. Show that the cosine coefficients A_k are all zero, including A_0.