STRONG-WEAK DUALITY AND QUANTUM MODULARITY OF RESURGENT TOPOLOGICAL STRINGS

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MOTIVATIONS

Enumerative invariants from resurgence

Resurgent asymptotic series arise naturally as perturbative expansions in quantum theories.

The machinery of resurgence uniquely associates them with a non-trivial collection of complex numbers, known as **Stokes constants**, capturing information about the **non-perturbative sectors** of the theory.

In some remarkable cases, the Stokes constants can be (conjecturally) interpreted in terms of **enumerative invariants** based on the counting of BPS states.

- Seiberg–Witten curve of 4d $\mathcal{N} = 2$ super Yang-Mills theory
[Grassi, Gu, Mariño, 2019]

- Complex Chern–Simons theory on Seifert fibered homology spheres
[Andersen, Mistegård, 2018]

- Complex Chern–Simons theory on complements of hyperbolic knots
[Garoufalidis, Gu, Mariño, 2020]

- Standard/Nekrasov–Shatashvili topological string theory on (toric) Calabi–Yau 3-folds
[Alim, Saha, Teschner, Tulli, 2021 - Gu, Mariño, 2021 - 2022 - Rella, 2022 - Gu, Kashani-Poor, Klemm, Mariño, 2023]

- Refined topological string theory on toric Calabi–Yau 3-folds
[Alexandrov, Mariño, Pioline, 2023]

THE RESURGENCE TOOLBOX

Basic notions in resurgence — I

Let $\varphi(z)$ be a (simple) **resurgent Gevrey-1** asymptotic series of the form

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}[[z]]\,, \quad a_n \sim A^{-n} n! \quad n \gg 1\,, \quad A \in \mathbb{R}\,.$$

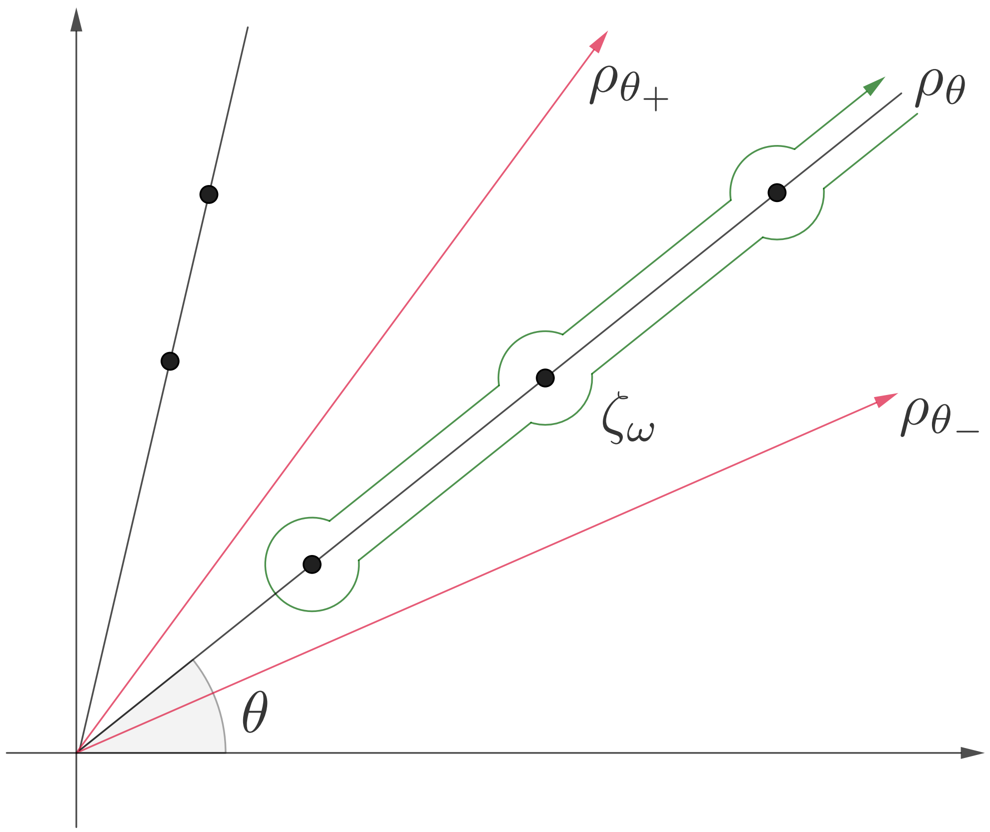
Its **Borel–Laplace resummation** is the two-step process

$$\varphi(z) \longrightarrow \underbrace{\hat{\varphi}(\zeta) = \sum_{k=0}^{\infty} \frac{a_k}{k!} \zeta^k}_{\substack{\text{locally analytic at } \zeta = 0 \\ \text{with singularities at } \zeta = \zeta_\omega}} \longrightarrow \underbrace{s_\theta(\varphi)(z) = \int_{\rho_\theta} \mathrm{e}^{-\zeta} \hat{\varphi}(\zeta z)\, d\zeta}_{\substack{\text{locally analytic in the complex } z\text{-plane} \\ \text{with discontinuities at } \arg(z) = \arg(\zeta_\omega)}}\,,$$

where $\rho_\theta = \mathrm{e}^{\mathrm{i}\theta}\mathbb{R}_+$, $\theta = \arg(\zeta)$. When $\theta = \arg(\zeta_\omega)$, the line ρ_θ is a **Stokes ray**.
If ζ_ω is a logarithmic branch point, we have

$$\hat{\varphi}(\zeta) = -\frac{S_\omega}{2\pi\mathrm{i}} \log(\zeta - \zeta_\omega) \underbrace{\hat{\varphi}_\omega(\zeta - \zeta_\omega)}_{\substack{\text{locally analytic} \\ \text{at } \zeta - \zeta_\omega = 0}} + \ldots\,, \quad S_\omega \in \mathbb{C} \quad (\text{Stokes constant}).$$

Basic notions in resurgence — II



The **discontinuity** across ρ_θ is

$$\begin{aligned}\mathrm{disc}_\theta \varphi(z) &= s_{\theta_+}(\varphi)(z) - s_{\theta_-}(\varphi)(z) = \\ &= \sum_\omega S_\omega \, \mathrm{e}^{-\zeta_\omega/z} \, s_{\theta_-}(\varphi_\omega)(z)\,.\end{aligned}$$

The **median resummation** across ρ_θ is

$$\mathcal{S}_\theta^{\mathrm{med}} \varphi(z) = \frac{s_{\theta_+}(\varphi)(z) + s_{\theta_-}(\varphi)(z)}{2}\,.$$

The **Stokes automorphism** $\mathfrak{S}_\theta$ across ρ_θ is defined by $s_{\theta_+} = s_{\theta_-} \circ \mathfrak{S}_\theta$.

Schematically, $\varphi \longrightarrow \{\varphi_\omega, S_\omega\} \longrightarrow \{\varphi_{\omega'}, S_{\omega\omega'}\}$.

Each series in this process can be promoted to **basic trans-series** as

$$\Phi_\omega(z) = \mathrm{e}^{-\zeta_\omega/z} \, \varphi_\omega(z)\,.$$

The **minimal resurgent structure** is the smallest subset of $\left\{\Phi_\omega(z)\right\}$ closed under $\mathfrak{S}$.

TOPOLOGICAL STRINGS BEYOND PERTURBATION THEORY

From topological strings to quantum operators and back

Let X be a toric Calabi–Yau 3-fold. The Weyl quantization of its mirror curve $\Sigma \subset \mathbb{C}^* \times \mathbb{C}^*$ leads to (positive-definite and trace-class) **quantum operators** $\rho_{j=1,\ldots,g_\Sigma}$ acting on $L^2(\mathbb{R})$.
[Grassi, Hatsuda, Mariño, 2014 - Codesido, Grassi, Mariño, 2015]

Their generalized Fredholm determinant defines the **fermionic spectral traces** by

$$\Xi(\vec{\kappa}, \vec{\xi}, \hbar) = \sum_{N_1 \geq 0} \cdots \sum_{N_{g_\Sigma} \geq 0} Z(\overrightarrow{N}, \vec{\xi}, \hbar)\, \kappa_1^{N_1} \cdots \kappa_{g_\Sigma}^{N_{g_\Sigma}} .$$

The **total grand potential** of the A-model topological string theory on X can be written as
[Hatsuda, Mariño, Moriyama, Okuyama, 2013]

$$J(\overrightarrow{\mu}, \vec{\xi}, \hbar) = \underbrace{J^{\mathrm{WS}}(\overrightarrow{\mu}, \vec{\xi}, \hbar)}_{\substack{\text{standard} \\ \text{topological string}}} + \underbrace{J^{\mathrm{WKB}}(\overrightarrow{\mu}, \vec{\xi}, \hbar)}_{\substack{\text{Nekrasov-Shatashvili} \\ \text{topological string}}} .$$

The **Topological Strings/Spectral Theory** correspondence states that
[Hatsuda, Moriyama, Okuyama, 2012 - Grassi, Hatsuda, Mariño, 2014 - Codesido, Grassi, Mariño, 2015]

$$Z(\overrightarrow{N}, \vec{\xi}, \hbar) = \frac{1}{(2\pi \mathrm{i})^{g_\Sigma}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\mu_1 \cdots \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\mu_{g_\Sigma}\, \mathrm{e}^{J(\vec{\mu}, \vec{\xi}, \hbar) - \overrightarrow{N} \cdot \vec{\mu}} , \quad (\kappa_j = \mathrm{e}^{\mu_j}) .$$

Resurgence in topological string theory — I

Let $Z(\overrightarrow{N}, \vec{\xi}, \hbar)$ be analytically continued to $\hbar \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Consider the **semiclassical perturbative expansion**

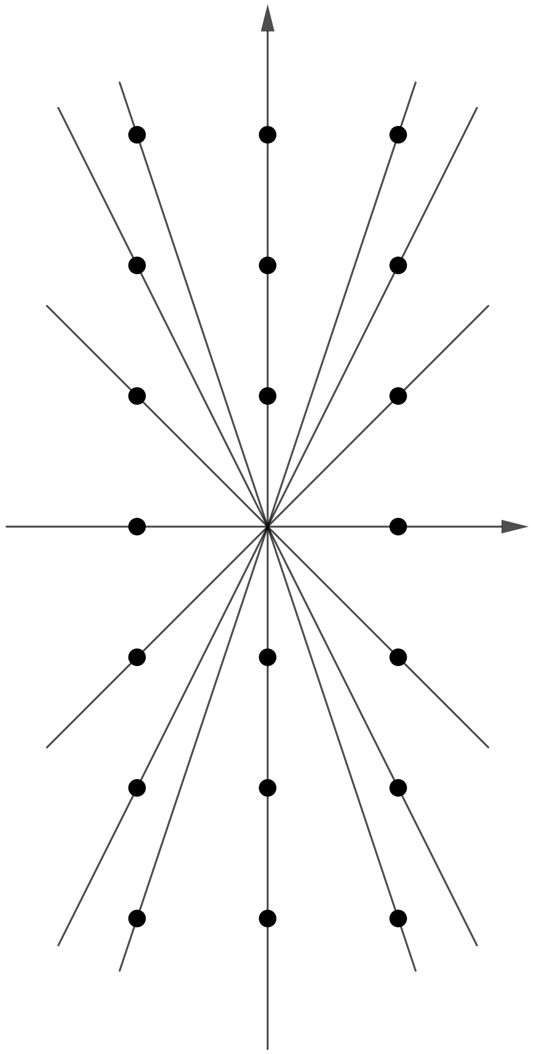
$$\phi_{\overrightarrow{N}}(\hbar) = \log Z(\overrightarrow{N}, \vec{\xi}, \hbar \to 0)\,, \quad \overrightarrow{N} \in \mathbb{N}^{g_\Sigma}\,,$$

which is a (simple) resurgent Gevrey-1 asymptotic series.

We describe a conjectural proposal for the **minimal resurgent structure** of $\phi_{\overrightarrow{N}}(\hbar)$ at fixed $\overrightarrow{N}$:

$$\Phi_{\sigma,n;\overrightarrow{N}}(\hbar) = \mathrm{e}^{-nA/\hbar}\, \phi_{\sigma;\overrightarrow{N}}(\hbar) \quad \text{(peacock patterns)},$$

where $n \in \mathbb{N}$, $\sigma \in \{0, \ldots, l\}$, $l \in \mathbb{N}_+$, and $A \in \mathbb{C}$.



The corresponding **Stokes constants** satisfy

$$S_{\sigma,\sigma',n;\overrightarrow{N}} \in \mathbb{Q} \quad \text{(enumerative invariants)}, \quad S_{\sigma,\sigma';\overrightarrow{N}}(q) = \sum_{n\in\mathbb{N}} S_{\sigma,\sigma',n;\overrightarrow{N}}\, q^n \quad \text{(q-series)}.$$

Peacock patterns are expected in theories controlled by quantum curves.
[Grassi, Gu, Mariño, 2019 - Garoufalidis, Gu, Mariño, 2020 - 2022 - Gu, Mariño, 2021 - Rella, 2022]

Resurgence in topological string theory — II

Consider the **dual weakly-coupled regime** $g_s = 4\pi^2/\hbar \to 0$ (strong-weak coupling duality).

At fixed $\overrightarrow{N}$, the (simple) resurgent Gevrey-1 asymptotic series

$$\psi_{\overrightarrow{N}}(g_s) = \log Z(\overrightarrow{N}, \vec{\xi}, \hbar \to \infty)\,, \quad \overrightarrow{N} \in \mathbb{N}^{g_\Sigma}\,,$$

is conjectured to have <u>the same resurgent structure</u> described before:

peacock pattern in the Borel plane	$\longrightarrow$	infinitely many Stokes constants in $\mathbb{Q}$	$\longrightarrow$	integer invariants of the geometry

Some remarks:

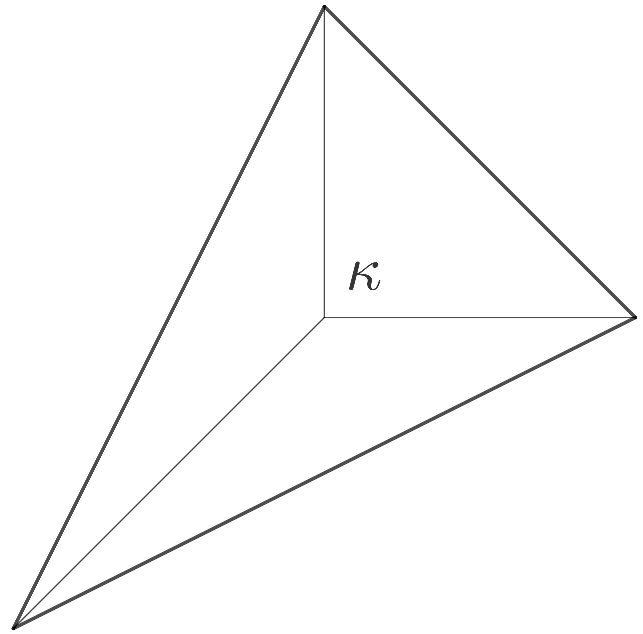
1. The asymptotic expansion $Z(\overrightarrow{N}, \vec{\xi}, \hbar \to \infty)$ has an exponential pre-factor of the form e^{-1/g_s} (*conifold volume conjecture for toric CYs*).
   [Gu, Mariño, 2021]

2. The asymptotic expansion $Z(\overrightarrow{N}, \vec{\xi}, \hbar \to 0)$ has no exponential pre-factor of the form $e^{-1/\hbar}$ (*analytic prediction of the TS/ST correspondence*).
   [Rella, 2022]

LOCAL $\mathbb{P}^2$ — A CASE STUDY

Introduction to the local $\mathbb{P}^2$ geometry

Local $\mathbb{P}^2$ is the total space of the canonical bundle over the projective surface $\mathbb{P}^2$.



It is a **toric del Pezzo CY 3-fold** with one complex modulus κ and no mass parameters.

The Weyl quantization of its mirror curve gives

$$\mathsf{O}_{\mathbb{P}^2}(\mathsf{x},\mathsf{y}) = \mathrm{e}^{\mathsf{x}} + \mathrm{e}^{\mathsf{y}} + \mathrm{e}^{-\mathsf{x}-\mathsf{y}}\,, \quad [\mathsf{x},\mathsf{y}] = \mathrm{i}\hbar \quad (\mathsf{x},\mathsf{y} \text{ self-adjoint Heisenberg operators}),$$

acting on $L^2(\mathbb{R})$. The inverse operator $\rho_{\mathbb{P}^2} = \mathsf{O}_{\mathbb{P}^2}^{-1}$ is positive-definite and of trace class.
[Grassi, Hatsuda, Mariño, 2014 - Kashaev, Mariño, 2015]

The **first spectral trace** $Z_{\mathbb{P}^2}(1,\hbar) = \mathrm{Tr}(\rho_{\mathbb{P}^2})$ is known in closed form as

$$\mathrm{Tr}(\rho_{\mathbb{P}^2}) = \frac{1}{\sqrt{3}\mathrm{b}} \mathrm{e}^{-\frac{\pi \mathrm{i}}{36}\mathrm{b}^2 + \frac{\pi \mathrm{i}}{12}\mathrm{b}^{-2} + \frac{\pi \mathrm{i}}{4}} \frac{(q^{2/3};\, q)_\infty^2}{(q^{1/3};\, q)_\infty} \frac{(w;\, \tilde{q})_\infty}{(w^{-1};\, \tilde{q})_\infty^2}\,, \quad 2\pi \mathrm{b}^2 = 3\hbar\,,$$

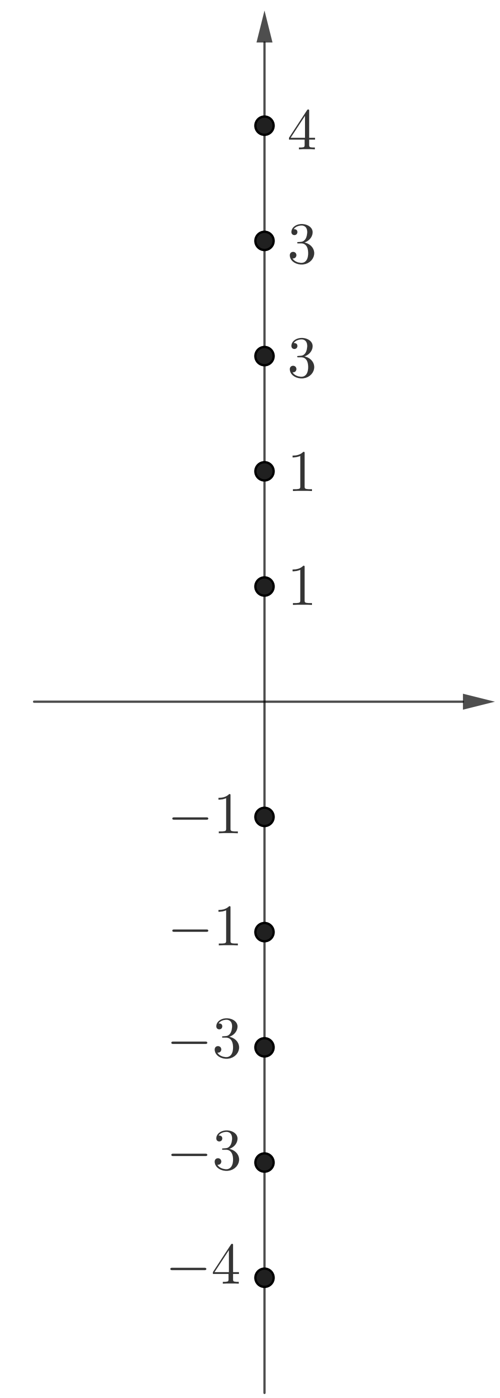
where $q = \mathrm{e}^{2\pi \mathrm{i} \mathrm{b}^2}$, $\tilde{q} = \mathrm{e}^{-2\pi \mathrm{i} \mathrm{b}^{-2}}$, and $w = \mathrm{e}^{2\pi \mathrm{i}/3}$, showing an explicit factorization into **holomorphic/anti-holomorphic blocks**.
[Kashaev, Mariño, 2015 - Mariño, Zakany, 2015 - Gu, Mariño, 2021]

Exact solution to the resurgent structure at weak coupling

We obtain an **all-orders perturbative expansion** for $\log Z_{\mathbb{P}^2}(1, \hbar \to 0)$, which gives a Gevrey-1 asymptotic series

$$\phi(\hbar) = \sum_{n=1}^{\infty} a_{2n} \hbar^{2n} \in \mathbb{Q}[[\hbar]] ,$$

$$a_{2n} \sim (-1)^n (2n)! A_0^{-2n} , \quad n \gg 1 , \quad A_0 = 4\pi^2/3 .$$



We present a **fully analytic solution** to the resurgent structure of $\phi(\hbar)$.
[Rella, 2022]

1. Exact, explicit resummation of $\hat{\phi}(\zeta)$ as a simple resurgent function.
2. Logarithmic branch points at $\zeta_n = n A_0 \mathrm{i}$, $n \in \mathbb{Z}_{\neq 0}$.
3. Local expansion at $\zeta = \zeta_n$:

$$\hat{\phi}(\zeta) = -\frac{S_n}{2\pi \mathrm{i}} \log(\zeta - \zeta_n) + \ldots \longrightarrow \hat{\phi}_n(\zeta) = 1 , \quad n \in \mathbb{Z}_{\neq 0} ,$$

$\underline{\textit{Proposition:}}$ $S_1 = 3\sqrt{3}\mathrm{i} , \quad \dfrac{S_n}{S_1} = \dfrac{\alpha_n}{n} \in \mathbb{Q}_{>0} \quad n \in \mathbb{Z}_{\neq 0,1} ,$

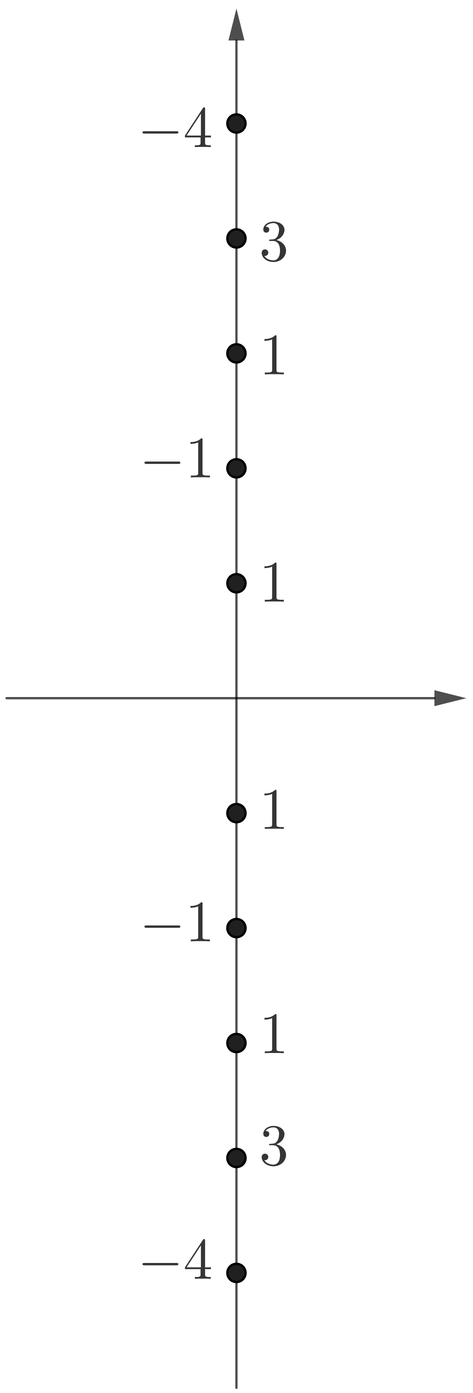
$$\alpha_n = -\alpha_{-n} , \quad \alpha_n \in \mathbb{Z}_{>0} \quad n \in \mathbb{Z}_{>0} .$$

Exact solution to the resurgent structure at strong coupling

We obtain an **all-orders perturbative expansion** for $\log Z_{\mathbb{P}^2}(1, \hbar \to \infty)$, which gives a Gevrey-1 asymptotic series

$$\psi(\tau) = \sum_{n=1}^{\infty} b_{2n} \tau^{2n-1} \in \mathbb{Q}[\pi, \sqrt{3}][\![\tau]\!]\,, \quad \tau = -\frac{A_\infty}{\hbar}\,.$$

$$b_{2n} \sim (-1)^n (2n)!A_\infty^{-2n}\,, \quad n \gg 1\,, \quad A_\infty = 2\pi/3 = A_0/2\pi\,.$$



We present a **fully analytic solution** to the resurgent structure of $\psi(\tau)$:
[Rella, 2022]

1. Exact, explicit resummation of $\hat{\psi}(\zeta)$ as a simple resurgent function.
2. Logarithmic branch points at $\zeta_n = nA_\infty \mathrm{i}$, $n \in \mathbb{Z}_{\neq 0}$.
3. Local expansion at $\zeta = \zeta_n$:

$$\hat{\psi}(\zeta) = -\frac{R_n}{2\pi\mathrm{i}} \log(\zeta - \zeta_n) + \ldots \longrightarrow \hat{\psi}_n(\zeta) = 1\,, \quad n \in \mathbb{Z}_{\neq 0}\,.$$

<u>*Proposition:*</u> $R_1 = 3\,, \quad \dfrac{R_n}{R_1} = \dfrac{\beta_n}{n} \in \mathbb{Q}_{\neq 0} \quad n \in \mathbb{Z}_{\neq 0,1}\,,$

$$\beta_n = \beta_{-n}\,, \quad \beta_n \in \mathbb{Z}_{\neq 0} \quad n \in \mathbb{Z}_{>0}\,.$$

Analytic formulae for the Stokes constants

We present **exact number-theoretic statements** on the Stokes constants $S_n, R_n, n \in \mathbb{Z}_{>0}$.
[Rella, 2022]

Proposition 1: The normalized Stokes constants are **divisor sum functions**:

$$\frac{S_n}{S_1} = \sum_{d|n} \frac{1}{d} \chi_{3,2}(d) \,, \quad \frac{R_n}{R_1} = \sum_{d|n} \frac{d}{n} \chi_{3,2}(d) \,,$$

where $\chi_{3,2}(n) = [n]_3$ is the unique non-principal Dirichlet character modulo 3. They are **multiplicative arithmetic functions.**

Proposition 2: The Stokes constants are **generated by the $q, \tilde{q}$-series appearing in the holomorphic and anti-holomorphic blocks** of the spectral trace:

$$\mathrm{disc}_{\frac{\pi}{2}} \phi(\hbar) = \sum_{n=1}^{\infty} S_n \tilde{q}^n = -\mathrm{i}\pi - 3 \log \frac{(w;\, \tilde{q})_\infty}{(w^{-1};\, \tilde{q})_\infty} \,, \quad \tilde{q} = \mathrm{e}^{-4\pi^2 \mathrm{i}/3\hbar} \,, \quad w = \mathrm{e}^{2\pi \mathrm{i}/3} \,,$$

$$\mathrm{disc}_{\frac{\pi}{2}} \psi(\tau) = \sum_{n=1}^{\infty} R_n q^{n/3} = 3 \log \frac{(q^{2/3};\, q)_\infty}{(q^{1/3};\, q)_\infty} \,, \quad q = \mathrm{e}^{-2\pi \mathrm{i}/\tau} \,,$$

giving exact expressions for the discontinuities of the series $\phi(\hbar), \psi(\tau)$.

A bridge to analytic number theory — I

Proposition 1: The perturbative coefficients $a_{2n}, b_{2n}, n \in \mathbb{Z}_{>0}$, satisfy **exact large-order relations**:

$$a_{2n} = \frac{\Gamma(2n)}{\pi \mathrm{i}\, (A_0 \mathrm{i})^{2n}} \sum_{m=1}^{\infty} \frac{S_m}{m^{2n}} \quad \text{(Dirichlet series evaluated at } 2n\text{)},$$

$$b_{2n} = \frac{\Gamma(2n-1)}{\pi \mathrm{i} (A_\infty \mathrm{i})^{2n-1}} \sum_{m=1}^{\infty} \frac{R_m}{m^{2n-1}} \quad \text{(Dirichlet series evaluated at } 2n\text{-}1\text{)}.$$

Proposition 2: The two Dirichlet series defined by the Stokes constants satisfy an **Euler product expansion** indexed by the set of prime numbers $\mathbb{P}$:

$$\sum_{m=1}^{\infty} \frac{S_m/S_1}{m^{2n}} = \prod_{p \in \mathbb{P}} \sum_{k=0}^{\infty} \frac{S_{p^k}/S_1}{p^{k(2n)}}, \quad \sum_{m=1}^{\infty} \frac{R_m/R_1}{m^{2n-1}} = \prod_{p \in \mathbb{P}} \sum_{k=0}^{\infty} \frac{R_{p^k}/R_1}{p^{k(2n-1)}}.$$

The Dirichlet series above are indeed **L-series**.

Corollary: Up to the simple prefactors above, the perturbative coefficients of the series $\phi(\hbar)$ and $\psi(\tau)$ are values of the **L-series encoding the corresponding Stokes constants evaluated at integer points**.

A bridge to analytic number theory — II

Recall that the multiplication of Dirichlet series is compatible with the **Dirichlet convolution** of arithmetic functions, that is,

$$f(m) = \big(f_1 * f_2\big)(m)\,,\; m \in \mathbb{Z}_{>0} \;\longrightarrow\; \sum_{m=1}^{\infty} \frac{f(m)}{m^s} = \sum_{m=1}^{\infty} \frac{f_1(m)}{m^s} \sum_{m=1}^{\infty} \frac{f_2(m)}{m^s}\,,\; s \in \mathbb{C}\,,\; \Re(s) > 1\,.$$

Theorem: The weak and strong coupling L-series **factorise according to the Dirichlet decomposition** of the Stokes constants into the product of two well-known **L-functions**:

$$\frac{S_m}{S_1} = \big(\chi_{3,2} F_{-1} * F_0\big)(m) \;\longrightarrow\; L_0(s) = \sum_{m=1}^{\infty} \frac{S_m/S_1}{m^s} = L(s+1, \chi_{3,2})\,\zeta(s) \qquad (\hbar \to 0),$$

$$\frac{R_m}{R_1} = \big(\chi_{3,2} F_0 * F_{-1}\big)(m) \;\longrightarrow\; L_\infty(s) = \sum_{m=1}^{\infty} \frac{R_m/R_1}{m^s} = L(s, \chi_{3,2})\,\zeta(s+1) \qquad (\hbar \to \infty),$$

where $F_\alpha(m) = m^\alpha$, $\chi_{3,2}(m) = [m]_3$.

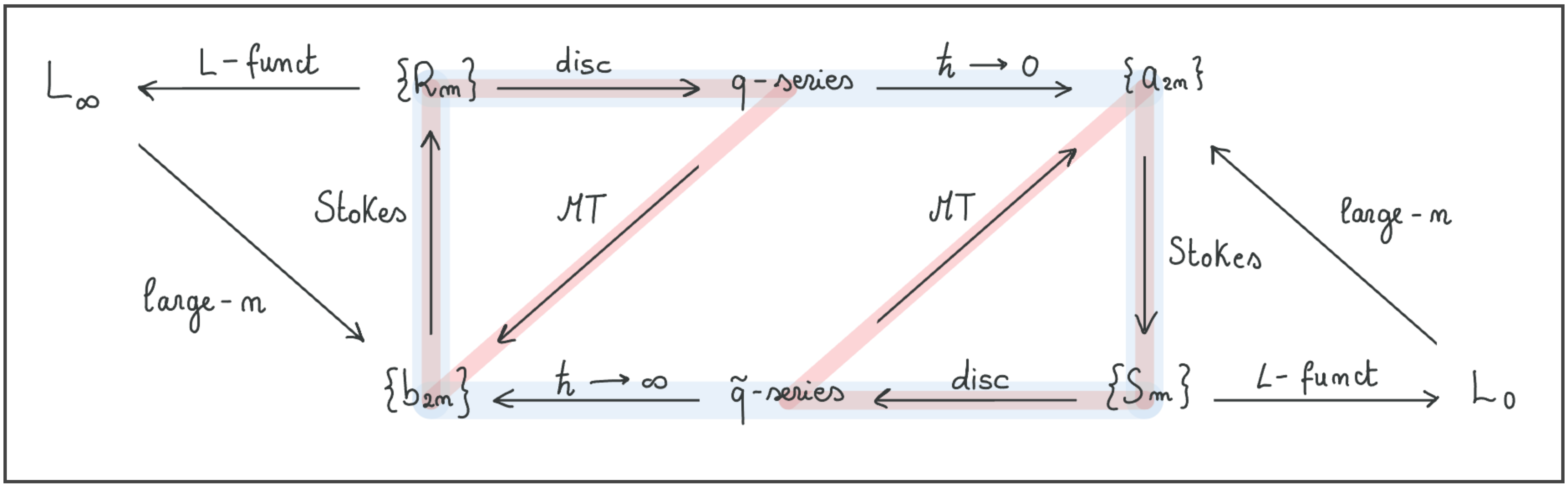
Corollary: The weak and strong coupling L-functions are related by a symmetric **unitary shift** in the arguments of the factors.

A full-fledged analytic number-theoretic symmetry

The number-theoretic properties of the exact resurgent structures of the weak and strong coupling perturbative expansions fit together to compose a full-fledged symmetry.

Corollary: The perturbative coefficients of the series $\phi(\hbar)$ and $\psi(\tau)$ are given by the **Mellin transform of the discontinuities**:

$$a_{2n} = \frac{1}{\pi \mathrm{i}} \int_0^{\infty} \hbar^{-2n-1} \mathrm{disc}_{\frac{\pi}{2}} \phi(\hbar) \, d\hbar \,, \quad b_{2n} = \frac{1}{\pi \mathrm{i}} \int_0^{\infty} \tau^{-2n} \mathrm{disc}_{\frac{\pi}{2}} \psi(\tau) \, d\tau \,.$$



STRONG-WEAK DUALITY REFORMULATED

[Fantini, Rella, to appear]

From symmetry to a global description — I

Theorem 1: The strong and weak coupling normalized Stokes constants are related by:

$$\frac{S_n}{S_1} = \frac{R_n}{R_1}\chi(n)\,, \quad \chi(n) = (f_3 * \chi_{3,2})(n)\,, \quad n \in \mathbb{Z}_{>0}\,,$$

where $f_3(n) = \prod_{p\in\mathbb{P}} f_3(p)^{k_p}$ for $n = \prod_{p\in\mathbb{P}} p^{k_p}$ and $f_3(p) = \delta_{p,1} + 3\delta_{p,3}$.

The same **arithmetic twist** relates L-functions $L_0(s)$, $L_\infty(s)$, $s \in \mathbb{C}$, $\mathfrak{R}(s) > 1$. We consider their meromorphic continuation to $s \in \mathbb{C}$ and upgrade them to the **completed L-functions**

$$\Lambda_0(s) = \frac{3^{s/2}}{4\pi^s}\Gamma\left(\frac{s}{2}\right)^2 L_0(s)\,, \quad \Lambda_\infty(s) = \frac{3^{s/2}}{4\pi^{s+1}}\Gamma\left(\frac{s+1}{2}\right)^2 L_\infty(s)\,.$$

Theorem 2: The strong and weak coupling completed L-functions are related by:

$$s\Lambda_0(s) = 2\pi\mathrm{i}\Lambda_\infty(-s)\,, \quad s \in \mathbb{C}\,.$$

This is a consequence of the remarkable factorisation of $L_0(s)$ and $L_\infty(s)$ into the products of $L(s, \chi_{3,2})$ and $\zeta(s)$ with cross-shifted arguments.

From symmetry to a global description — II

We introduce the **unified completed L-function**

$$\Lambda(s) = s\Lambda_0(s) + 2\pi \mathrm{i} \Lambda_\infty(s-1)\,, \quad s \in \mathbb{C}\,.$$

<u>*Corollary:*</u> The unified completed L-function satisfies the conventional **functional equation**:

$$\Lambda(s) = \Lambda(1-s)\,, \quad s \in \mathbb{C}\,.$$

The same does <u>not</u> hold for the individual completed L-functions $\Lambda_0(s)$ and $\Lambda_\infty(s)$.

The resurgent behaviours in the weak and strong $\hbar$-regimes descend from a **unique global number-theoretic structure** with a peculiar symmetry.

The perturbative content of one regime recovers the non-perturbative content of the other in a mathematically precise way. This is a **manifestation of the underlying physical dualities**.

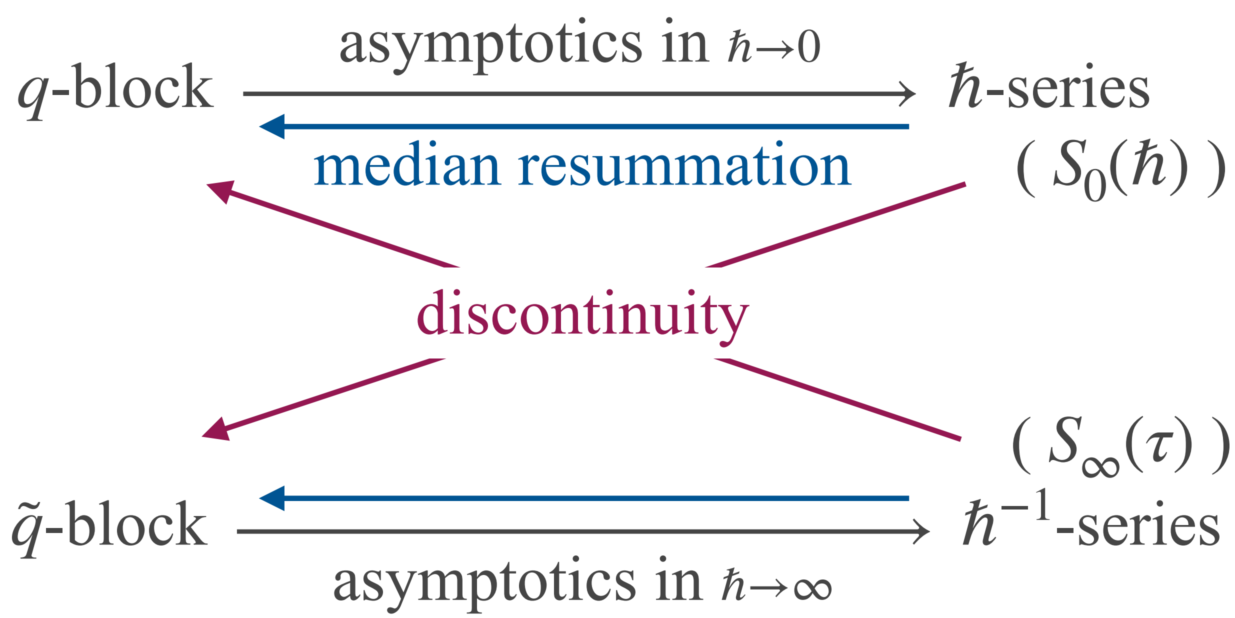
$$\begin{tikzcd}
\text{ST on } \hat{X} \text{ at } \hbar \to 0 \arrow[rr, leftrightarrow, "\text{resurgent duality}"] \arrow[d, leftrightarrow, "\text{TS/ST}"'] & & \text{ST on } \hat{X} \text{ at } \hbar \to \infty \arrow[d, leftrightarrow, "\text{TS/ST}"] \\
\text{TS on } X \text{ at } g_s \to \infty \arrow[rr, leftrightarrow, dashed] & & \text{TS on } X \text{ at } g_s \to 0
\end{tikzcd}$$

MODULAR RESURGENT STRUCTURES

[Fantini, Rella, to appear]

Median resummation of the $q, \tilde{q}$-series

The holomorphic and anti-holomorphic blocks are different functions. Yet, the holomorphic block resurges from the perturbative expansion of the anti-holomorphic block and vice versa.



Theorem: The **median resummation** of $S_0(\hbar)$ and $S_\infty(\tau)$ returns the associated $q, \tilde{q}$-series:

$$\mathcal{S}^{\mathrm{med}}_{\frac{\pi}{2}} S_0(\hbar) = \mathrm{disc}_{\frac{\pi}{2}} \psi(-2\pi/3\hbar)\,, \quad \mathcal{S}^{\mathrm{med}}_{\frac{\pi}{2}} S_\infty(\tau) = \mathrm{disc}_{\frac{\pi}{2}} \phi(-2\pi/3\tau)\,.$$

Proposition: The function $f:\ \mathbb{C}\backslash\mathbb{R}_{\leq 0} \to \mathbb{C}$ defined by

$$f(\hbar) = \mathrm{disc}_{\frac{\pi}{2}} \left[\psi(-2\pi/3\hbar) - \phi(\hbar)\right] = \mathcal{S}^{\mathrm{med}}_{\frac{\pi}{2}} \left[S_0(\hbar) - S_\infty(-2\pi/3\hbar)\right]$$

is a weight-0 **quantum modular form** for $\mathrm{SL}_2(\mathbb{Z})$. *Work in progress on* $\Gamma_1(3)$.

Resurgence and quantum modularity

<u>*Definition:*</u> Let $\varphi(z)$, $z \in \mathbb{C}$, be a (simple) resurgent Gevrey-1 asymptotic series. We say that $\varphi(z)$ has a **modular resurgent structure** when:

1. the Borel transform $\hat{\varphi}(\zeta)$ is singular at $\zeta = kA$, $k \in \Omega \subseteq \mathbb{Z}$, $A \in \mathbb{C}$;

2. the Stokes constants S_k, $k \in \Omega \subseteq \mathbb{Z}$, are the coefficients of an L-function.

<u>*Conjecture:*</u> Let $f(z) : \mathbb{H} \to \mathbb{C}$ be an analytic function which extends to $\mathbb{Q}$. If its asymptotic expansion $\varphi(z)$ has a modular resurgent structure, then (t.f.a.e.):

1. $f(z)$ is a quantum modular form for some $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$;

2. $\mathcal{S}_\theta^{\mathrm{med}} \varphi(z) = f(z)$.

Further evidence from examples of q-series ($q = \mathrm{e}^{2\pi \mathrm{i} \tau}$) from combinatorics and quantum invariants of 3-manifolds and knots.

- $f(\tau) = q^{1/24} \sum_{n \geq 0} (q;q)_n$ is weight-3/2 for $\mathrm{SL}_2(\mathbb{Z})$.
- $f(\tau) = q^{1/24} + q^{1/24} \sum_{n \geq 0} (-1)^n q^{n+1} (q;q)_n = -2q^{1/24} \sum_{n \geq 1} \dfrac{(-1)^n q^{-n^2}}{(q^{-1};q^{-2})_n}$ is weight-1 for $\Gamma_0(2)$.

CONCLUSIONS

Final remarks and open problems

The resurgence of the spectral theory dual to the topological string on a toric CY 3-fold unveils a universal mathematical structure of **non-perturbative sectors** (*peacock patterns*) and **Stokes constants** (*enumerative invariants*).

Their geometric and physical meaning is yet to be understood. Indirect evidence suggests a relation to the Donaldson–Thomas invariants.
[Alim, Saha, Teschner, Tulli, 2021 - Gu, Kashani-Poor, Klemm, Mariño, 2023 - Alexandrov, Mariño, Pioline, 2023]

The resurgence of the first spectral trace of local $\mathbb{P}^2$ displays a **global analytic number-theoretic structure** encompassing and intertwining the regimes of $\hbar \to 0$ and $\hbar \to \infty$.

The strong-weak duality between $\hbar$ and g_s (*TS/ST correspondence*) translates it into a statement on the topological string. This is a **manifestation of the underlying physical dualities**. Work in progress on the precise formalisation of this intuition.
[Rella, 2022]

The interplay of q**-series** (*median resummation*) and L**-functions** (*discontinuity*) plays a central role in the cross-relations between the weak and strong resurgent structures.

Our results fit into a broader research program linking the resurgent properties of q-series and quantum modular forms. Work in progress on the **modular resurgent structures**.
[Fantini, Rella, to appear - Fantini, Goswami, Kontsevich, Kumar, to appear]

THANK YOU!