# STRONG-WEAK DUALITY AND QUANTUM MODULARITY OF RESURGENT TOPOLOGICAL STRINGS

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# MOTIVATIONS

# Enumerative invariants from resurgence

**Resurgent asymptotic series** arise naturally as perturbative expansions in quantum theories.

The machinery of resurgence uniquely associates them with a non-trivial collection of complex numbers, known as **Stokes constants**, capturing information about the **non-perturbative sectors** of the theory.

In some remarkable cases, the Stokes constants can be (conjecturally) interpreted in terms of **enumerative invariants** based on the counting of BPS states.

- Seiberg–Witten curve of 4d $\mathcal{N} = 2$ super Yang-Mills theory
  [Grassi, Gu, Mariño, 2019]

- Complex Chern–Simons theory on Seifert fibered homology spheres
  [Andersen, Mistegård, 2018]

- Complex Chern–Simons theory on complements of hyperbolic knots
  [Garoufalidis, Gu, Mariño, 2020]

- Standard topological string theory on (toric) Calabi–Yau 3-folds for $g_s \to 0$
  [Gu, Mariño, 2021 - 2022 - Rella, 2022 - Gu, Kashani-Poor, Klemm, Mariño, 2023]

- Nekrasov–Shatashvili topological string theory on toric Calabi–Yau 3-folds for $\hbar \to 0$
  [Gu, Mariño, 2022 - Rella, 2022]

# FROM TOPOLOGICAL STRINGS TO QUANTUM OPERATORS AND BACK

# From topological strings to quantum operators

Let $X$ be a toric Calabi-Yau (CY) 3-fold.

Local mirror symmetry pairs $X$ with an algebraic curve $\Sigma \subset \mathbb{C}^* \times \mathbb{C}^*$ of genus $g_\Sigma$, whose Weyl quantization leads to **quantum-mechanical operators**

$$\rho_j\,, \quad j = 1, \ldots, g_\Sigma\ ,$$

acting on $L^2(\mathbb{R})$. They are conjectured to be positive-definite and of trace class, under some assumptions on the mass parameters $\vec{\xi}$.
[Grassi, Hatsuda, Mariño, 2014 - Codesido, Grassi, Mariño, 2015]

Their **generalized Fredholm determinant** $\Xi(\vec{\kappa}, \vec{\xi}, \hbar)$ is an entire function of the true complex deformation parameters $\kappa_j$. Its local expansion at the orbifold point $\vec{\kappa} = 0$ is

$$\Xi(\vec{\kappa}, \vec{\xi}, \hbar) = \sum_{N_1 \geq 0} \cdots \sum_{N_{g_\Sigma} \geq 0} \underbrace{Z(\vec{N}, \vec{\xi}, \hbar)}_{\substack{\text{analytic function} \\ \text{of } \hbar \in \mathbb{R}_{>0}}} \quad \kappa_1^{N_1} \cdots \kappa_{g_\Sigma}^{N_{g_\Sigma}}\,,$$

where the coefficient functions $Z(\vec{N}, \vec{\xi}, \hbar)$ are the **fermionic spectral traces**.
We will consider their analytic continuation to $\hbar \in \mathbb{C} \backslash \mathbb{R}_{\leq 0}$.

# From quantum operators to topological strings

The conjectural **Topological Strings/Spectral Theory** (TS/ST) correspondence states that

$$Z(\overrightarrow{N}, \vec{\xi}, \hbar) = \frac{1}{(2\pi \mathrm{i})^{g_\Sigma}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\mu_1 \cdots \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\mu_{g_\Sigma}\, \mathrm{e}^{J(\vec{\mu}, \vec{\xi}, \hbar) - \overrightarrow{N}\cdot\vec{\mu}} \,,$$

where the chemical potentials $\mu_j$ are defined by $\kappa_j = e^{\mu_j}$.
[Hatsuda, Moriyama, Okuyama, 2012 - Grassi, Hatsuda, Mariño, 2014 - Codesido, Grassi, Mariño, 2015]

The **total grand potential** of the A-model topological string theory on $X$ can be written as

$$J(\overrightarrow{\mu}, \vec{\xi}, \hbar) = \underbrace{J^{\mathrm{WS}}(\overrightarrow{\mu}, \vec{\xi}, \hbar)}_{\text{worldsheet grand potential}} + \underbrace{J^{\mathrm{WKB}}(\overrightarrow{\mu}, \vec{\xi}, \hbar)}_{\text{WKB grand potential}} \,,$$

where $J^{WS}$ and $J^{WKB}$ encode the contributions from the total free energies of the standard and Nekrasov-Shatashvili (NS) topological strings on $X$, respectively.
[Hatsuda, Mariño, Moriyama, Okuyama, 2013]

The string coupling constant $g_s$ is related to the quantum deformation parameter $\hbar$ by the **strong-weak coupling duality** $g_s = \dfrac{4\pi^2}{\hbar}$.

# THE RESURGENCE TOOLBOX

# Basic notions in resurgence — I

Let $\varphi(z)$ be a (simple) **resurgent Gevrey-1** asymptotic series of the form

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}[[z]]\,, \quad a_n \sim A^{-n} n! \quad n \gg 1\,, \quad A \in \mathbb{R}\,.$$

Its **Borel–Laplace resummation** is the two-step process

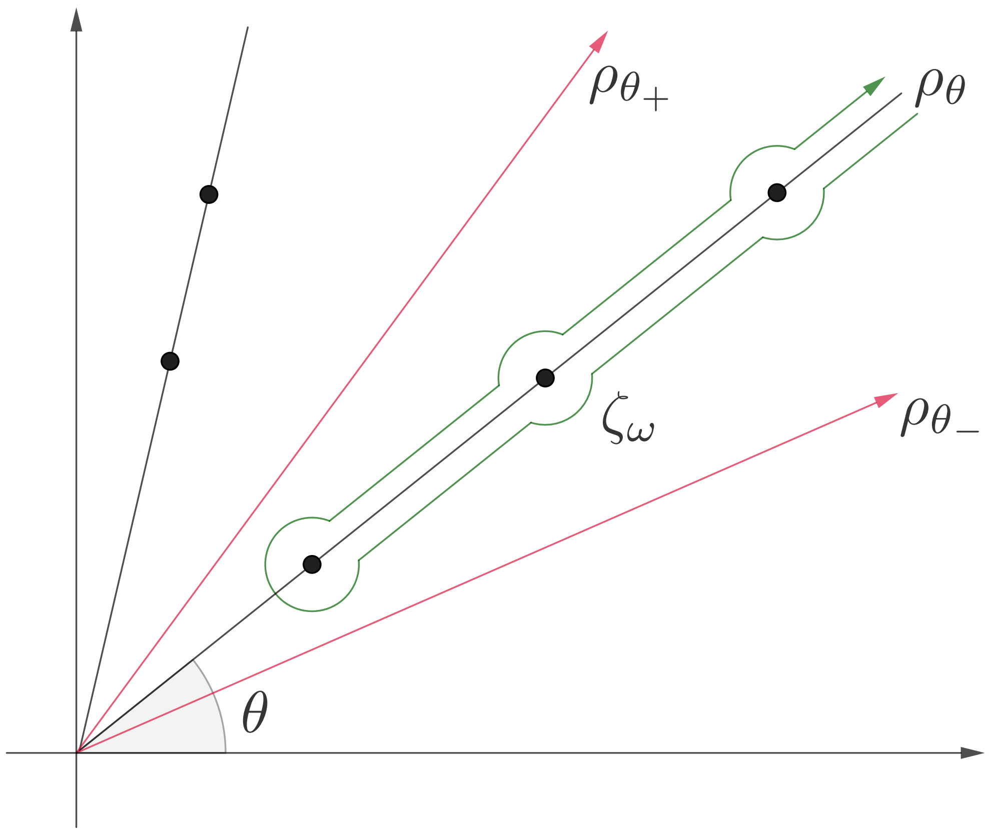
$$\varphi(z) \longrightarrow \underbrace{\hat{\varphi}(\zeta) = \sum_{k=0}^{\infty} \frac{a_k}{k!} \zeta^k}_{\substack{\text{locally analytic at } \zeta = 0 \\ \text{with singularities at } \zeta = \zeta_\omega}} \longrightarrow \underbrace{s_\theta(\varphi)(z) = \int_{\rho_\theta} \mathrm{e}^{-\zeta} \hat{\varphi}(\zeta z)\, d\zeta}_{\substack{\text{locally analytic in the complex } z\text{-plane} \\ \text{with discontinuities at } \arg(z) = \arg(\zeta_\omega)}}\,,$$

where $\rho_\theta = \mathrm{e}^{\mathrm{i}\theta}\,\mathbb{R}_+$, $\theta = \arg(\zeta)$. If $\zeta_\omega$ is a logarithmic branch point, we have

$$\hat{\varphi}(\zeta) = -\frac{S_\omega}{2\pi\mathrm{i}} \log(\zeta - \zeta_\omega) \underbrace{\hat{\varphi}_\omega(\zeta - \zeta_\omega)}_{\text{locally analytic at } \zeta - \zeta_\omega = 0} + \ldots\,,$$

where $S_\omega \in \mathbb{C}$ is the **Stokes constant**. When $\theta = \arg(\zeta_\omega)$, the line $\rho_\theta$ is a **Stokes ray**.

# Basic notions in resurgence — II




The **discontinuity** across $\rho_\theta$ is

$$\begin{aligned}\mathrm{disc}_\theta \varphi(z) &= s_{\theta_+}(\varphi)(z) - s_{\theta_-}(\varphi)(z) = \\ &= \sum_\omega S_\omega \, \mathrm{e}^{-\zeta_\omega/z} \, s_{\theta_-}(\varphi_\omega)(z) \,.\end{aligned}$$

The **median resummation** across $\rho_\theta$ is

$$\mathcal{S}_\theta^{\mathrm{med}} \varphi(z) = \frac{s_{\theta_+}(\varphi)(z) + s_{\theta_-}(\varphi)(z)}{2} \,.$$

The **Stokes automorphism** $\mathfrak{S}_\theta$ across $\rho_\theta$ is defined by $s_{\theta_+} = s_{\theta_-} \circ \mathfrak{S}_\theta$ .

Schematically, $\varphi \;\longrightarrow\; \{\varphi_\omega \,, S_\omega\} \;\longrightarrow\; \{\varphi_{\omega'} \,, S_{\omega\omega'}\}$ .

Each series in this process can be promoted to **basic trans-series** as

$$\Phi_\omega(z) = \mathrm{e}^{-\zeta_\omega/z} \, \varphi_\omega(z) \,.$$

The **minimal resurgent structure** is the smallest subset of $\left\{ \Phi_\omega(z) \right\}$ closed under $\mathfrak{S}$ .

# TOPOLOGICAL STRINGS BEYOND PERTURBATION THEORY

# Resurgence in topological string theory — I

Consider the **semiclassical perturbative expansion**

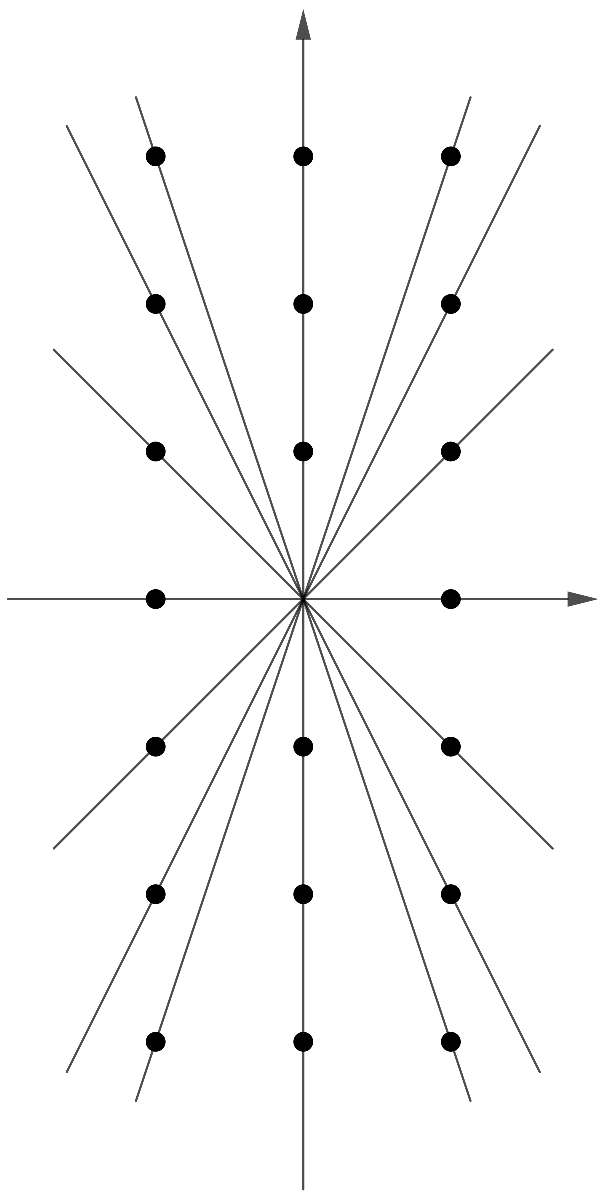
$$\phi_{\vec{N}}(\hbar) = \log Z(\vec{N}, \vec{\xi}, \hbar \to 0)\,, \quad \vec{N} \in \mathbb{N}^{g_\Sigma}\,,$$

which is a (simple) resurgent Gevrey-1 asymptotic series.

We describe a conjectural proposal for the **minimal resurgent structure** of $\phi_{\vec{N}}(\hbar)$ at fixed $\vec{N}$:

$$\Phi_{\sigma,n;\vec{N}}(\hbar) = \mathrm{e}^{-nA/\hbar}\,\phi_{\sigma;\vec{N}}(\hbar) \quad \text{(peacock patterns)},$$

where $n \in \mathbb{N}$, $\sigma \in \{0, \ldots, l\}$, $l \in \mathbb{N}_+$, and $A \in \mathbb{C}$.



The corresponding **Stokes constants** satisfy

$$S_{\sigma,\sigma',n;\vec{N}} \in \mathbb{Q}\,, \quad S_{\sigma,\sigma';\vec{N}}(q) = \sum_{n\in\mathbb{N}} S_{\sigma,\sigma',n;\vec{N}}\, q^n \quad \text{($q$-series)},$$

and they give non-trivial **enumerative invariants** of the geometry.

Peacock patterns are expected in theories controlled by quantum curves.
[Grassi, Gu, Mariño, 2019 - Garoufalidis, Gu, Mariño, 2020 - 2022 - Gu, Mariño, 2021 - Rella, 2022]

# Resurgence in topological string theory — II

Consider the **dual weakly-coupled regime** $g_s = 4\pi^2/\hbar \to 0$ of the topological string.

At fixed $\overrightarrow{N}$, the (simple) resurgent Gevrey-1 asymptotic series

$$\psi_{\overrightarrow{N}}(g_s) = \log Z(\overrightarrow{N}, \vec{\xi}, \hbar \to \infty)\,, \quad \overrightarrow{N} \in \mathbb{N}^{g_\Sigma}\,,$$

is conjectured to have <u>the same resurgent structure</u> described before:

| peacock pattern in the Borel plane | $\longrightarrow$ | infinitely many Stokes constants in $\mathbb{Q}$ | $\longrightarrow$ | integer invariants of the geometry |
|---|---|---|---|---|

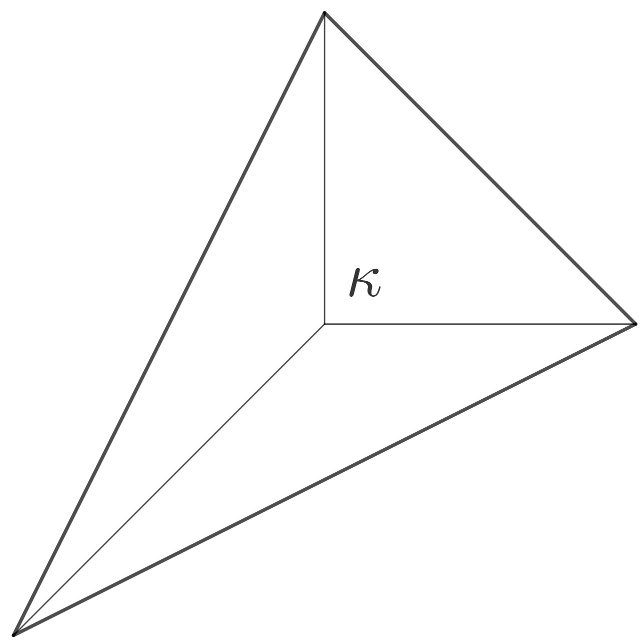
Some remarks:

1. The asymptotic expansion $Z(\overrightarrow{N}, \vec{\xi}, \hbar \to \infty)$ has an exponential pre-factor of the form $e^{-1/g_s}$ (*conifold volume conjecture for toric CYs*). Its Stokes constants are **integers**.
[Gu, Mariño, 2021]

2. The asymptotic expansion $Z(\overrightarrow{N}, \vec{\xi}, \hbar \to 0)$ has no exponential pre-factor of the form $e^{-1/\hbar}$ (*new analytic prediction of the TS/ST correspondence*). Its Stokes constants are generally **complex numbers**.
[Rella, 2022]

# LOCAL $\mathbb{P}^2$ — A CASE STUDY

# Introduction to the local $\mathbb{P}^2$ geometry




Local $\mathbb{P}^2$ is the total space of the canonical bundle over the projective surface $\mathbb{P}^2$.

It is a **toric del Pezzo CY 3-fold** with one complex modulus $\kappa$ and no mass parameters.

The Weyl quantization of its mirror curve gives

$$\mathsf{O}_{\mathbb{P}^2}(\mathsf{x},\mathsf{y}) = e^{\mathsf{x}} + e^{\mathsf{y}} + e^{-\mathsf{x}-\mathsf{y}}\,, \quad [\mathsf{x},\mathsf{y}] = i\hbar \quad (\mathsf{x},\mathsf{y} \text{ self-adjoint Heisenberg operators}),$$

acting on $L^2(\mathbb{R})$. The inverse operator $\rho_{\mathbb{P}^2} = \mathsf{O}_{\mathbb{P}^2}^{-1}$ is positive-definite and of trace class.
[Grassi, Hatsuda, Mariño, 2014 - Kashaev, Mariño, 2015]

The **first spectral trace** $Z_{\mathbb{P}^2}(1,\hbar) = \mathrm{Tr}(\rho_{\mathbb{P}^2})$ is known in closed form as

$$\mathrm{Tr}(\rho_{\mathbb{P}^2}) = \frac{1}{\sqrt{3}\mathrm{b}} e^{-\frac{\pi i}{36}\mathrm{b}^2 + \frac{\pi i}{12}\mathrm{b}^{-2} + \frac{\pi i}{4}} \frac{(q^{2/3};\, q)_\infty^2}{(q^{1/3};\, q)_\infty} \frac{(w;\, \tilde{q})_\infty}{(w^{-1};\, \tilde{q})_\infty^2}\,, \quad 2\pi\mathrm{b}^2 = 3\hbar\,,$$

where $q = e^{2\pi i \mathrm{b}^2}$, $\tilde{q} = e^{-2\pi i \mathrm{b}^{-2}}$, and $w = e^{2\pi i/3}$, showing an explicit factorization into **holomorphic/anti-holomorphic blocks**.
[Kashaev, Mariño, 2015 - Mariño, Zakany, 2015 - Gu, Mariño, 2021]

# Exact solution to the resurgent structure at weak coupling

We obtain an **all-orders perturbative expansion** for $\log Z_{\mathbb{P}^2}(1, \hbar \to 0)$, which gives a Gevrey-1 asymptotic series

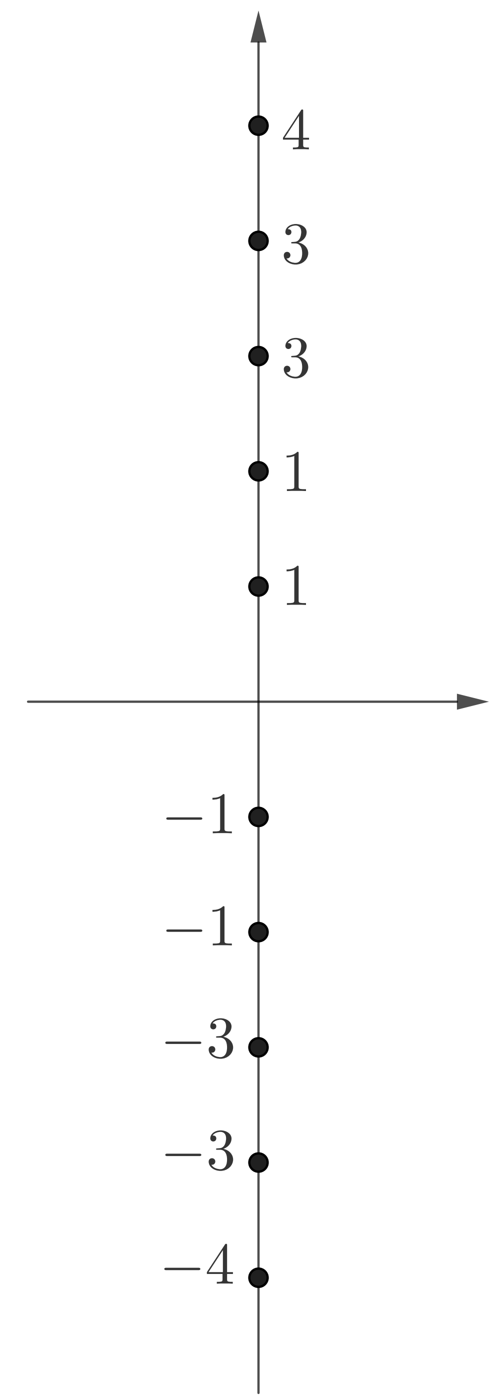
$$\phi(\hbar) = \sum_{n=1}^{\infty} a_{2n} \hbar^{2n} \in \mathbb{Q}[[\hbar]] \,,$$

$$a_{2n} \sim (-1)^n (2n)! A_0^{-2n} \,, \quad n \gg 1 \,, \quad A_0 = 4\pi^2/3 \,.$$

We present a **fully analytic solution** to the resurgent structure of $\phi(\hbar)$.
[Rella, 2022]

1. Exact, explicit resummation of $\hat{\phi}(\zeta)$ as a simple resurgent function.
2. Logarithmic branch points at $\zeta_n = n A_0 \mathrm{i}$, $n \in \mathbb{Z}_{\neq 0}$.
3. Local expansion at $\zeta = \zeta_n$:

$$\hat{\phi}(\zeta) = -\frac{S_n}{2\pi \mathrm{i}} \log(\zeta - \zeta_n) + \ldots \longrightarrow \hat{\phi}_n(\zeta) = 1 \,, \quad n \in \mathbb{Z}_{\neq 0} \,,$$




*Proposition:* $S_1 = 3\sqrt{3}\mathrm{i} \,, \quad \dfrac{S_n}{S_1} = \dfrac{\alpha_n}{n} \in \mathbb{Q}_{>0} \quad n \in \mathbb{Z}_{\neq 0,1} \,,$

$$\alpha_n = -\alpha_{-n} \,, \quad \alpha_n \in \mathbb{Z}_{>0} \quad n \in \mathbb{Z}_{>0} \,.$$

# Exact solution to the resurgent structure at strong coupling

We obtain an **all-orders perturbative expansion** for $\log Z_{\mathbb{P}^2}(1, \hbar \to \infty)$, which gives a Gevrey-1 asymptotic series

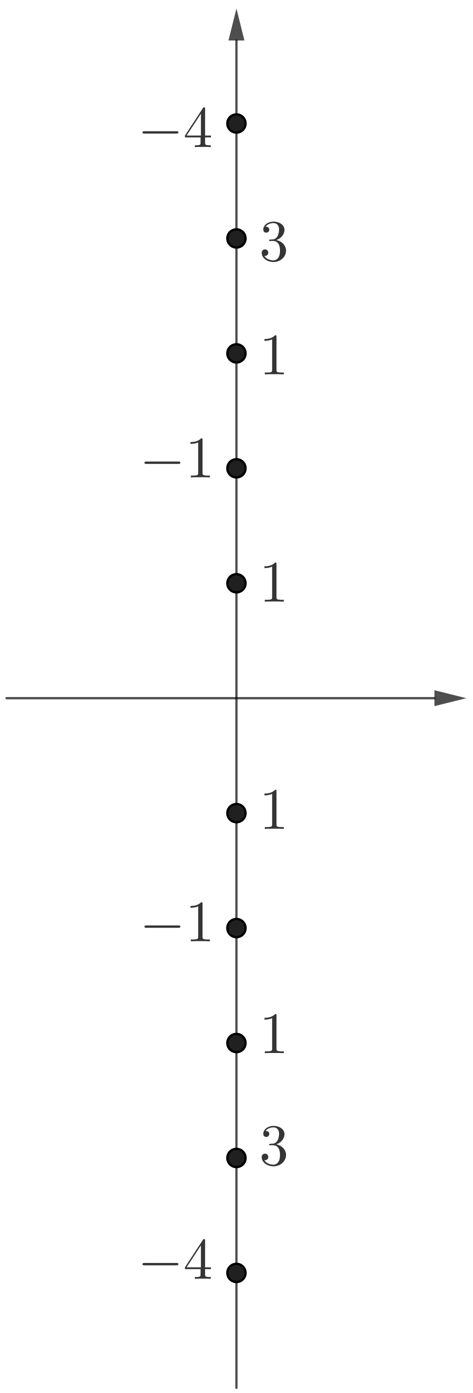
$$\psi(\tau) = \sum_{n=1}^{\infty} b_{2n} \tau^{2n-1} \in \mathbb{Q}[\pi, \sqrt{3}][[\tau]] , \quad \tau = -\frac{A_\infty}{\hbar} .$$

$$b_{2n} \sim (-1)^n (2n)! A_\infty^{-2n} , \quad n \gg 1 , \quad A_\infty = 2\pi/3 = A_0/2\pi .$$

We present a **fully analytic solution** to the resurgent structure of $\psi(\tau)$:
[Rella, 2022]

1. Exact, explicit resummation of $\hat{\psi}(\zeta)$ as a simple resurgent function.
2. Logarithmic branch points at $\zeta_n = n A_\infty \mathrm{i}$, $n \in \mathbb{Z}_{\neq 0}$.
3. Local expansion at $\zeta = \zeta_n$:

$$\hat{\psi}(\zeta) = -\frac{R_n}{2\pi \mathrm{i}} \log(\zeta - \zeta_n) + \ldots \longrightarrow \hat{\psi}_n(\zeta) = 1 , \quad n \in \mathbb{Z}_{\neq 0} .$$




*Proposition:* $R_1 = 3 , \quad \dfrac{R_n}{R_1} = \dfrac{\beta_n}{n} \in \mathbb{Q}_{\neq 0} \quad n \in \mathbb{Z}_{\neq 0,1} ,$

$$\beta_n = \beta_{-n} , \quad \beta_n \in \mathbb{Z}_{\neq 0} \quad n \in \mathbb{Z}_{>0} .$$

# Analytic formulae for the Stokes constants

We present **exact number-theoretic statements** on the Stokes constants $S_n, R_n$, $n \in \mathbb{Z}_{>0}$.
[Rella, 2022]

*Proposition 1:* The normalized Stokes constants are **divisor sum functions**:

$$\frac{S_n}{S_1} = \sum_{d|n} \frac{1}{d} \chi_{3,2}(d)\,, \quad \frac{R_n}{R_1} = \sum_{d|n} \frac{d}{n} \chi_{3,2}(d)\,,$$

where $\chi_{3,2}(n) = [n]_3$ is the unique non-principal Dirichlet character modulo 3. They are **multiplicative arithmetic functions.**

*Proposition 2:* The Stokes constants are **generated by the $q, \tilde{q}$-series appearing in the holomorphic and anti-holomorphic blocks** of the spectral trace:

$$\mathrm{disc}_{\frac{\pi}{2}} \phi(\hbar) = \sum_{n=1}^{\infty} S_n \tilde{q}^n = -\mathrm{i}\pi - 3 \log \frac{(w;\, \tilde{q})_\infty}{(w^{-1};\, \tilde{q})_\infty}\,, \quad \tilde{q} = \mathrm{e}^{-4\pi^2 \mathrm{i}/3\hbar}\,, \quad w = \mathrm{e}^{2\pi \mathrm{i}/3}\,,$$

$$\mathrm{disc}_{\frac{\pi}{2}} \psi(\tau) = \sum_{n=1}^{\infty} R_n q^{n/3} = 3 \log \frac{(q^{2/3};\, q)_\infty}{(q^{1/3};\, q)_\infty}\,, \quad q = \mathrm{e}^{-2\pi \mathrm{i}/\tau}\,,$$

giving exact expressions for the discontinuities of the series $\phi(\hbar), \psi(\tau)$.

# A bridge to analytic number theory — I

*Proposition 1:* The perturbative coefficients $a_{2n}, b_{2n}$, $n \in \mathbb{Z}_{>0}$, satisfy **exact large-order relations**:

$$a_{2n} = \frac{\Gamma(2n)}{\pi \mathrm{i}\,(A_0 \mathrm{i})^{2n}} \sum_{m=1}^{\infty} \frac{S_m}{m^{2n}} \quad \text{(Dirichlet series evaluated at } 2n\text{)},$$

$$b_{2n} = \frac{\Gamma(2n-1)}{\pi \mathrm{i}(A_\infty \mathrm{i})^{2n-1}} \sum_{m=1}^{\infty} \frac{R_m}{m^{2n-1}} \quad \text{(Dirichlet series evaluated at } 2n\text{-}1\text{)}.$$

*Proposition 2:* The two Dirichlet series defined by the Stokes constants satisfy an **Euler product expansion** indexed by the set of prime numbers $\mathbb{P}$ :

$$\sum_{m=1}^{\infty} \frac{S_m/S_1}{m^{2n}} = \prod_{p \in \mathbb{P}} \sum_{k=0}^{\infty} \frac{S_{p^k}/S_1}{p^{k(2n)}}, \quad \sum_{m=1}^{\infty} \frac{R_m/R_1}{m^{2n-1}} = \prod_{p \in \mathbb{P}} \sum_{k=0}^{\infty} \frac{R_{p^k}/R_1}{p^{k(2n-1)}}.$$

The Dirichlet series above are indeed **L-series**.

*Corollary:* Up to the simple prefactors above, the perturbative coefficients of the series $\phi(\hbar)$ and $\psi(\tau)$ are values of the **L-series encoding the corresponding Stokes constants evaluated at integer points**.

# A bridge to analytic number theory — II

Recall that the multiplication of Dirichlet series is compatible with the **Dirichlet convolution** of arithmetic functions, that is,

$$f(m) = \big(f_1 * f_2\big)(m)\,,\; m \in \mathbb{Z}_{>0} \longrightarrow \sum_{m=1}^{\infty} \frac{f(m)}{m^s} = \sum_{m=1}^{\infty} \frac{f_1(m)}{m^s} \sum_{m=1}^{\infty} \frac{f_2(m)}{m^s}\,,\; s \in \mathbb{C}\,,\; \Re(s) > 1\,.$$

*Theorem:* The weak and strong coupling L-series **factorise according to the Dirichlet decomposition** of the Stokes constants into the product of two well-known **L-functions**:

$$\frac{S_m}{S_1} = \big(\chi_{3,2}F_{-1} * F_0\big)(m) \longrightarrow L_0(s) = \sum_{m=1}^{\infty} \frac{S_m/S_1}{m^s} = L(s+1, \chi_{3,2})\,\zeta(s) \qquad (\hbar \to 0),$$

$$\frac{R_m}{R_1} = \big(\chi_{3,2}F_0 * F_{-1}\big)(m) \longrightarrow L_\infty(s) = \sum_{m=1}^{\infty} \frac{R_m/R_1}{m^s} = L(s, \chi_{3,2})\,\zeta(s+1) \qquad (\hbar \to \infty),$$

where $F_\alpha(m) = m^\alpha$, $\chi_{3,2}(m) = [m]_3$.

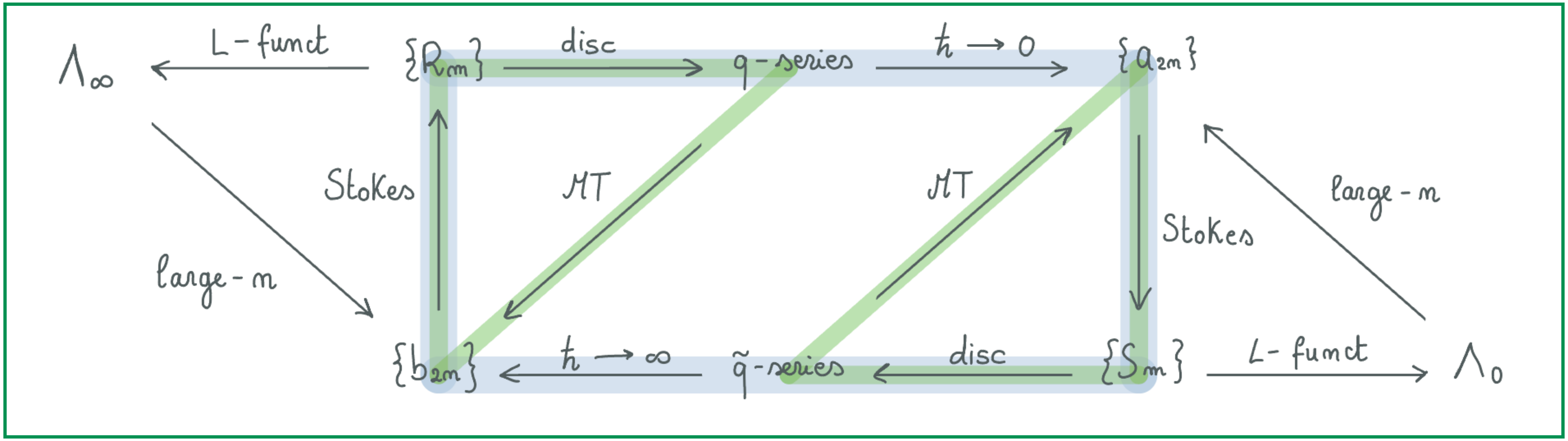
*Corollary:* The weak and strong coupling L-functions are related by a symmetric **unitary shift** in the arguments of the factors.

# A full-fledged analytic number-theoretic symmetry

The number-theoretic properties of the exact resurgent structures of the weak and strong coupling perturbative expansions fit together to compose a full-fledged symmetry.

*Corollary:* The perturbative coefficients of the series $\phi(\hbar)$ and $\psi(\tau)$ are given by the **Mellin transform of the discontinuities**:

$$a_{2n} = \frac{1}{\pi \mathrm{i}} \int_0^\infty \hbar^{-2n-1} \mathrm{disc}_{\frac{\pi}{2}} \phi(\hbar) \, d\hbar \,, \quad b_{2n} = \frac{1}{\pi \mathrm{i}} \int_0^\infty \tau^{-2n} \mathrm{disc}_{\frac{\pi}{2}} \psi(\tau) \, d\tau \,.$$



# STRONG-WEAK DUALITY REFORMULATED

[Fantini, Rella, to appear]

## From symmetry to unification — I

*Theorem 1:* The strong and weak coupling normalized Stokes constants are related by:

$$\frac{S_n}{S_1} = \frac{R_n}{R_1}\chi(n)\,, \quad \chi(n) = (f_3 * \chi_{3,2})(n)\,, \quad n \in \mathbb{Z}_{>0}\,,$$

where $f_3(n) = \prod_{p\in\mathbb{P}} f_3(p)^{k_p}$ for $n = \prod_{p\in\mathbb{P}} p^{k_p}$ and $f_3(p) = \delta_{p,1} + 3\delta_{p,3}$.

The same **arithmetic twist** relates the strong and weak coupling L-functions $L_0(s)$, $L_\infty(s)$, $s \in \mathbb{C}\,,\ \Re(s) > 1$. We consider their meromorphic continuation to $s \in \mathbb{C}$ and upgrade them to the dual **completed L-functions**

$$\Lambda_0(s) = \frac{3^{s/2}}{4\pi^s}\Gamma\left(\frac{s}{2}\right)^2 L_0(s)\,, \quad \Lambda_\infty(s) = \frac{3^{s/2}}{4\pi^{s+1}}\Gamma\left(\frac{s+1}{2}\right)^2 L_\infty(s)\,.$$

*Theorem 2:* The strong and weak coupling completed L-functions are related by:

$$s\Lambda_0(s) = 2\pi\mathrm{i}\Lambda_\infty(-s)\,, \quad s \in \mathbb{C}\,.$$

This is a consequence of the remarkable factorisation of the L-functions $L_0(s)$ and $L_\infty(s)$ into the products of $L(s, \chi_{3,2})$ and $\zeta(s)$ with cross-shifted arguments.

# From symmetry to unification — II

We introduce the **unified completed L-function**

$$\Lambda(s) = s\Lambda_0(s) + 2\pi \mathrm{i}\Lambda_\infty(s-1)\,, \quad s \in \mathbb{C}\,.$$

<u>*Corollary:*</u> The unified completed L-function satisfies the conventional **functional equation**:

$$\Lambda(s) = \Lambda(1-s)\,, \quad s \in \mathbb{C}\,.$$

The same does <u>not</u> hold for the individual completed L-functions $\Lambda_0(s)$ and $\Lambda_\infty(s)$.

The resurgent behaviours in the weak and strong $\hbar$-regimes descend from a **unique global number-theoretic structure** with a peculiar symmetry.

The perturbative information content of one regime recovers the non-perturbative content of the other in a mathematically precise way. This is a notable **manifestation of the underlying physical dualities**.

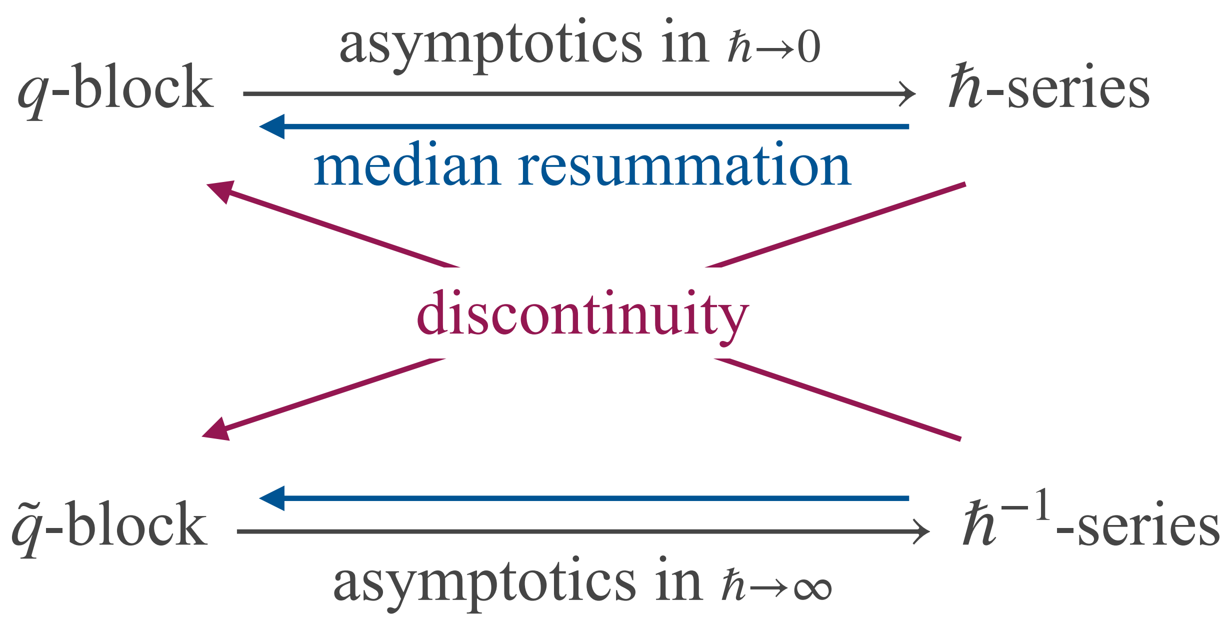
$$\begin{tikzcd}
\text{ST on } X \text{ at } \hbar \to 0 \arrow[r, leftrightarrow, "\text{number-theoretic}", "\text{resurgent duality}"'] \arrow[d, leftrightarrow, "\text{TS/ST}"'] & \text{ST on } X \text{ at } \hbar \to \infty \arrow[d, leftrightarrow, "\text{TS/ST}"] \\
\text{TS on } \hat{X} \text{ at } g_s \to \infty \arrow[r, leftrightarrow, "\textit{mirror symmetry, S-duality ?}"] & \text{TS on } \hat{X} \text{ at } g_s \to 0
\end{tikzcd}$$

# TOWARDS A UNIVERSAL PICTURE

[Fantini, Rella, to appear]

# Median resummation of the $q, \tilde{q}$-series

The holomorphic and anti-holomorphic blocks are different functions. Yet, the holomorphic block resurges from the perturbative expansion of the anti-holomorphic block and vice versa.




*Theorem:* The **median resummation** of the perturbative series $S_0(\hbar)$ and $S_\infty(\tau)$ obtained from the asymptotic expansions of the $q, \tilde{q}$-series generating the Stokes constants returns the same $q, \tilde{q}$-series:

$$\mathcal{S}^{\mathrm{med}}_{\frac{\pi}{2}} S_0(\hbar) = 3 \log \frac{(q^{2/3};\, q)_\infty}{(q^{1/3};\, q)_\infty} = \mathrm{disc}_{\frac{\pi}{2}} \psi(-2\pi/3\hbar)\,,$$

$$\mathcal{S}^{\mathrm{med}}_{\frac{\pi}{2}} S_\infty(\tau) = -\pi \mathrm{i} - 3 \log \frac{(w;\, \tilde{q})_\infty}{(w^{-1};\, \tilde{q})_\infty} = \mathrm{disc}_{\frac{\pi}{2}} \phi(-2\pi/3\tau)\,.$$

# Resurgence and quantum modularity

*Proposition:* The function $f: \mathbb{C}\backslash\mathbb{R}_{\leq 0} \to \mathbb{C}$ defined by

$$f(\hbar) = \mathrm{disc}_{\frac{\pi}{2}}\left[\psi(-2\pi/3\hbar) - \phi(\hbar)\right] = \mathcal{S}^{\mathrm{med}}_{\frac{\pi}{2}}\left[S_0(\hbar) - S_\infty(-2\pi/3\hbar)\right]$$

is a weight-0 **quantum modular form** for $\mathrm{SL}_2(\mathbb{Z})$.

Work in progress on an underlying quantum modularity for the congruence subgroup $\Gamma_1(3)$.

*Definition:* Let $\varphi(z)$, $z \in \mathbb{C}$, be a (simple) resurgent Gevrey-1 asymptotic series. We say that $\varphi$ has a **modular resurgent structure** when:

1. the Borel transform $\hat{\varphi}(\zeta)$ is singular at $\zeta = kA$, $k \in \Omega \subseteq \mathbb{Z}$, $A \in \mathbb{C}$;
2. the Stokes constants $S_k$, $k \in \Omega \subseteq \mathbb{Z}$, are the coefficients of an L-function.

*Conjecture:* Let $f(z): \mathbb{H} \to \mathbb{C}$ be an analytic function which extends to $\mathbb{Q}$. If its asymptotic expansion $\varphi(z)$ has a modular resurgent structure, then (t.f.a.e.):

1. $f(z)$ is a quantum modular form for some $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$;
2. $\mathcal{S}^{\mathrm{med}}_{\theta}\varphi(z) = f(z)$.

Further evidence from examples of $q$-series related to knot invariants.

# CONCLUSIONS

# Final remarks and open problems

The resurgence of the topological string on a toric CY 3-fold unveils a universal mathematical structure of **non-perturbative sectors** (*peacock patterns*) and **Stokes constants** (*enumerative invariants*).

Their geometric and physical meaning is yet to be understood. Evidence suggests a relation to the Donaldson–Thomas invariants.
[Alim, Saha, Teschner, Tulli, 2021 - Gu, Kashani-Poor, Klemm, Mariño, 2023]

The resurgence of the first spectral trace of local $\mathbb{P}^2$ displays a **global analytic number-theoretic structure** encompassing and intertwining the regimes of $\hbar \to 0$ and $\hbar \to \infty$.

The weak-strong duality between $\hbar$ and $g_s$ (*TS/ST correspondence*) translates it into a statement on the topological string theory. This is a **manifestation of the underlying physical dualities**. Work in progress on the precise formalisation of this intuition.

The interplay of $q$**-series** (*median resummation*) and $L$**-functions** (*discontinuity*) plays a central role in the cross-relations between the weak and strong resurgent structures.

Our results fit into a broader research program linking the resurgent properties of $q$-series and quantum modular forms. Work in progress on the **modular resurgent structures**.
[Fantini, Rella, to appear - Fantini, Goswami, Kontsevich, Kumar, to appear]

# THANK YOU!