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Market Microstructure
Problem Set 3

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Exercise 1. Similarly to what we did in Problem Set 2 for the Glosten-Milgrom model, we would like here to relax the dichotomy informed/noise traders in Kyle's model. More precisely, we assume that informed traders are in fact traders who utilise a signal s to infer the true value ν of the traded asset. Their signal is written as $s = \nu + \eta$, where η is normally distributed with zero mean and variance σ_η^2, and it is independent of ν and of u, the latter being the size of noise traders' orders. In other words, η represents the error in the signal. Traders who follow the signal s are referred to as *partially informed*.

1. Assume that partially informed traders send market orders of size

$$X(s) = \beta(s - \mu), \tag{1}$$

so that, if the trading price is p, their profit is $(\nu - p)X(s)$. Compute the partially informed traders' expected profit conditional on the knowledge of the signal s. Comment on the meaning of σ_η.

2. Assume that the dealers are perfectly competitive and that their quotes $p(q)$ for a given order size q are expressed as $p(q) = \mu + \lambda q$. Furthermore, let the size of partially informed traders' market orders be as in equation (1). Following the line of the classical Kyle's model, find the coefficients λ and β at equilibrium.

Exercise 2. Consider informed investors in the classical Kyle's model presented in Lecture slides. We extend the description of their behaviour by saying that, after acquiring the position (long or short) given by $X(\nu)$, they will in a successive trading day unwind this position. They model the price ν_l at which they will be able to unwind their position by the formula

$$\nu_l = \nu + \epsilon, \tag{2}$$

where ϵ is centred, normally distributed, and independent of any other variable in the model. Their assessment of trading opportunity is therefore

$$\Big(\nu - p - \sigma_\epsilon^2 X(\nu)\Big)X(\nu) = \underbrace{(\nu - p)X(\nu)}_{\text{profit from trading}} - \underbrace{\mathrm{Var}\big[(\nu_l - \nu)X(\nu)|X(\nu)\big]}_{\text{aversion against risk of unwinding}}, \tag{3}$$

where σ_ϵ^2 is the variance of ϵ. In a Nash equilibrium argument along the lines of that presented in class, compute the value of the coefficient β in the informed traders' trading strategy $X(\nu) = \beta(\nu - \mu)$, where their assessment of trading opportunity is given by equation (3) rather that by plain expected profit.

Solutions

Solution to Exercise 1. Recall that η is normally distributed with mean zero and variance σ_η^2, that ν is normally distributed with mean μ and variance σ_ν^2, and that they are independent. Therefore, s is normally distributed with mean μ and variance $\sigma_\eta^2 + \sigma_\nu^2$ and

$$\mathbb{E}[\nu|s] = \frac{\sigma_\nu^2}{\sigma_\nu^2 + \sigma_\eta^2}s + \frac{\sigma_\eta^2}{\sigma_\nu^2 + \sigma_\eta^2}\mu.$$

Recall also that $p = \mu + \alpha q$, and thus the transaction price can be written as

$$p = \alpha\beta s + (1 - \alpha\beta)\mu + \alpha u.$$

Hence,

$$\begin{aligned}\mathbb{E}[(\nu - p)X(\nu)|s] =& \beta(s-\mu)\mathbb{E}[\nu - p|s] \\ =& \beta(s-\mu)\Big(\mathbb{E}[\nu|s] - \alpha\beta s - (1-\alpha\beta)\mu\Big) \\ =& \beta\left(\frac{\sigma_\nu^2}{\sigma_\nu^2+\sigma_\eta^2} - \alpha\beta\right)(s-\mu)^2.\end{aligned} \tag{4}$$

This answers part 1. As for the second part, observe on the one hand that the optimiser $\hat{\beta}$ of equation (4) is

$$\hat{\beta} = \frac{1}{2\alpha}\frac{\sigma_\nu^2}{\sigma_\nu^2+\sigma_\eta^2}.$$

On the other hand, in perfectly competitive markets, dealers' quotes are $p = \mathbb{E}[\nu|q]$, which is explicitly computed as

$$p = \mu + \frac{\beta\sigma_\nu^2}{\beta^2(\sigma_\nu^2+\sigma_\eta^2)+\sigma_u^2}\, q.$$

Hence, by solving the system

$$\begin{cases}\hat{\alpha} = \dfrac{\hat{\beta}\sigma_\nu^2}{\hat{\beta}^2(\sigma_\nu^2+\sigma_\eta^2)+\sigma_u^2} \\ \hat{\beta} = \dfrac{1}{2\hat{\alpha}}\dfrac{\sigma_\nu^2}{\sigma_\nu^2+\sigma_\eta^2},\end{cases}$$

we get the equilibrium levels $\hat{\alpha}$ and $\hat{\beta}$ that were required in part 2.

Solution to Exercise 2. Informed traders' expectation of the quantity in equation (3), conditional on their trade size $X(\nu) = x$, is $(\nu - \mu - \alpha x - \sigma_\epsilon^2 x)x$. Hence,

$$\hat{x} = \frac{\nu - \mu}{2(\alpha + \sigma_\epsilon^2)}$$

is the size of the order that maximises their objective. By solving the system

$$\begin{cases}\hat{\alpha} = \dfrac{\hat{\beta}\sigma_\nu^2}{\hat{\beta}^2\sigma_\nu^2+\sigma_u^2} \\ \hat{\beta} = \dfrac{1}{2(\hat{\alpha}+\sigma_\epsilon^2)}\end{cases}$$

we get the required equilibrium levels $\hat{\alpha}$ and $\hat{\beta}$.