ATML - TUTORIAL 3

PROBLEM SET 3 — SOLUTIONS

EXERCISE 1.1

It suffices to observe that

$$-\int_0^T \dot{X}_t S_t \, dt = x_0 S_0 + \int_0^T X_t \, dS_t$$

and thus

$$\mathbb{E}\left[-\int_0^t \dot{X}_t S_t \, dt\right] = x_0 S_0$$

does not depend on $X \in \mathcal{A}$.

EXERCISE 1.2

$\forall X \in \mathcal{A}$ $\forall t < \tau < T$ using equation (2) we have

$$C_t^*(X_t) \leq \mathbb{E}_t\left[\int_t^\tau \left(\dot{X}_u^2 + \lambda X_u S_u\right) du + C_\tau^*(X_\tau)\right]$$

$$= \mathbb{E}_t\left[C_\tau^*(X_\tau) + \int_0^\tau \left(\dot{X}_u^2 + \lambda X_u S_u\right) du\right] - \int_0^t \left(\dot{X}_u^2 + \lambda X_u S_u\right) du$$

Hence,

$$C_t^*(X_t) + \int_0^t \left(\dot{X}_u^2 + \lambda X_u S_u\right) du$$

$$\leq \mathbb{E}_t\left[C_\tau^*(X_\tau) + \int_0^\tau \left(\dot{X}_u^2 + \lambda X_u S_u\right) du\right]$$

EXERCISE 1.3

Define $\varphi(v,b) := bv + v^2$ and observe that

$$\varphi(v,b) \geq -\frac{1}{2}b^2 \quad \forall v \; \forall b \in \mathbb{R}.$$

Define

$$f(s,x,v,a,b,\tau) := \tau + \frac{\sigma^2}{2}s^2 a + \varphi(v,b) + \lambda x s.$$

The drift of dM_t is

$$\partial_t G + \frac{\sigma^2}{2}S_t^2 \partial_{ss}^2 G + \dot{X}_t \partial_x G + \dot{X}_t^2 + \lambda X_t S_t$$

$$= f(S_t, X_t, \dot{X}_t, \partial_{ss}^2 G, \partial_x G, \partial_t G)$$

$$\geq \partial_t G + \frac{\sigma^2}{2}S_t^2 \partial_{ss}^2 G - \frac{1}{2}(\partial_x G)^2 + \lambda X_t S_t.$$

Therefore if the RHS is non-negative, the drift of dM_t is non-negative.

EXERCISE 1.4

(a) Straightforward computation.

(b) We only need to argue that

$$\lim_{t \uparrow T} \mathbb{E}\left[\frac{X_t^2}{T-t}\right] = 0 \quad \forall X \in A.$$

By Jensen inequality we have that

$$X_t^2 = (X_T - X_t)^2 = \left(\int_t^T \dot{X}_u \, du\right)^2$$

$$\leq (T-t)^2 \fint_t^T \dot{X}_u^2 \, du.$$

Therefore,

$$\frac{1}{T-t} \mathbb{E}[X_t^2] \leq (T-t) \, \mathbb{E}\left[\fint_t^T \dot{X}_u^2 \, du\right] \longrightarrow 0.$$

EXERCISE 1.5

Using Ito formula and equation (3) we have

$$G^*(t, X_t, S_t) = G^*(0, X_0, S_0) + \int_0^t \left(\partial_t G^* + \dot{X}_u \partial_x G^* + \frac{\sigma^2}{2} S_u^2 \partial_{ss}^2 G^* \right) du + \int_0^t \partial_s G^* \, dS_u$$

$$= \int_0^t \left(\frac{1}{4} (\partial_x G^*)^2 - \lambda X_u S_u + \dot{X}_u \partial_x G^* \right) du + \int_0^t \partial_s G^* \, dS_u .$$

Since $\forall v, b \in \mathbb{R}$ it holds $\frac{1}{4} b^2 + bv \geq -v^2$, it follows

$$G^*(t, X_t, S_t) \geq G^*(0, X_0, S_0) + \int_0^t \left(-\dot{X}_u^2 - \lambda X_u S_u \right) du + \int_0^t \partial_s G^* \, dS_u .$$

We conclude by taking expectation on both sides and using EXERCISE 1.4 (b).

EXERCISE 1.6

It suffices to observe that $X^* \in \mathcal{A}$ and

$$G^*(0, X_0, S_0) = \mathbb{E}\left[\int_0^T \left((\dot{X}_t^*)^2 + \lambda X_t^* S_t \right) dt \right] .$$