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BERTSIMAS & LO EXECUTION PROBLEM

GENERAL SETUP

0: start time of execution

T: execution horizon

Time is discrete $t = 0 \longrightarrow T$.

X_t : inventory at time t.

$-\Delta X_{t+1} = -X_{t+1} + X_t$ quantity to execute in $[t, t+1]$.

$F = F(t, s, \{X_u\}_{u=t}^{T})$ expected execution cost at time t if the asset price at time t is $S_t = s$ and the inventory trajectory is $\{X_u\}_{u=t}^{T}$. This is defined as

$$F(t, s, \{X_u\}_{u=t}^{T}) = \mathbb{E}_t\left[\sum_{u=t}^{T-1} (-\Delta X_{u+1}) S_{u+1} \,\middle|\, S_t = s\right].$$

$C_t^* = C_t^*(x, s)$ value function at time t, i.e. the optimal execution cost. It depends on the current value x of the inventory at time t, and on the asset price s at time t. This is defined as

$$C_t^*(x, s) = \inf\left\{F(t, s, \{X_u\}_{u=t}^{T}) : \begin{array}{l} X_{u+1} \in \mathcal{F}_u \ \forall u = t \_\_ T, \\ X_t = x, \ X_T = 0 \end{array}\right\}.$$

BELLMAN EQUATION

$\forall\ 0 \le t \le T-1$ it must hold

$$C_t^*(x,s) = \inf \left\{ \mathbb{E}_t \left[\left(-\Delta X_{t+1}\right) S_{t+1} + C_{t+1}^*\left(X_{t+1}, S_{t+1}\right) \right] : \begin{array}{l} X_{t+1} \hat{\in} \mathcal{F}_t \\ X_t = x,\ X_T = 0 \end{array} \right\}.$$

LEMMA

Let $G_t = G_t(x,s,v)$ be defined as

$$G_t(x,s,v) = \mathbb{E}_t\left[S_{t+1} \middle| \Delta X_{t+1} = -v, S_t = s\right] \cdot v + \mathbb{E}_t\left[C_{t+1}^*(x-v, S_{t+1}) \middle| \Delta X_{t+1} = -v, S_t = s\right].$$

Define

$$G_t^*(x,s) = \inf\left\{G_t(x,s,v) : v \in \mathbb{R}\right\}.$$

Then,

$$C_t^*(x,s) = G_t^*(x,s).$$

PROOF.
The inequality "$\le$" follows from the fact that, on the one hand, X_{t+1} is $\mathcal{F}_t$-measurable and, on the other hand, given S_t, S_{t+1} depends on $\{X_u\}_{u=0}^{T-1}$ only via ΔX_{t+1}.
The inequality "$\ge$" is clear.

$\square$

PERMANENT IMPACT WITH NO - INFORMATION

The asset price S_t is modelled as

$$S_{t+1} = S_t - \gamma \Delta X_{t+1} + \sigma S_0 \Delta W_{t+1}.$$

PROPOSITION 1

It holds

$$C^*_t(x,s) = sx + \frac{\gamma}{2} \frac{T-t+1}{T-t} x^2$$

and the optimal execution strategy is given by

$$-\Delta X^*_{t+1} = \frac{X_0}{T}.$$

PROOF.

We show by induction over $K = 1 \_\_ T$ that

$$C^*_{T-K}(x,s) = sx + \frac{\gamma}{2} \frac{K+1}{K} x^2.$$

For $K=1$, $C^*_{T-1}(x,s)$ is actually an infimum over the singleton $\Delta X_T = -x$. Hence,

$$\begin{aligned} C^*_{T-1}(x,s) &= F(T-1, s, \{x, 0\}) \\ &= \mathbb{E}_{T-1}\left[S_T \middle| S_{T-1} = s, \Delta X_{T-1} = -x \right] \cdot x \\ &= sx + \gamma x^2. \end{aligned}$$

Hence we assume that the claim holds up to $K < T$ and we argue for the induction step.

We can compute

$$G_{T-K-1}(x,s,v) = (s+\gamma v)v + \mathbb{E}_{T-K-1}\left[(x-v)S_{T-K} + \frac{\gamma}{2}\frac{K+1}{K}(x-v)^2 \,\middle|\, \Delta X_{T-K} = -v,\ S_{T-K-1} = s\right]$$

$$= (s+\gamma v)v + (s+\gamma v)(x-v) + \frac{\gamma}{2}\frac{K+1}{K}(x-v)^2$$

$$= (s+\gamma v)x + \frac{\gamma}{2}\frac{K+1}{K}(x-v)^2.$$

Thus,

$$\operatorname*{argmin}_{v\in\mathbb{R}} G_{T-K-1}(x,s,v) = \frac{x}{K+1}$$

and

$$G^*_{T-K-1}(x,s) = G_{T-K-1}\left(x,s,\frac{x}{K+1}\right)$$

$$= sx + \gamma\frac{x^2}{K+1} + \frac{\gamma}{2}\frac{K+1}{K}\left(\frac{Kx}{K+1}\right)^2$$

$$= sx + \frac{\gamma}{2}\frac{K+2}{K+1}x^2,$$

proving the induction step.

We conclude by noticing that

$$F\left(0, S_0, \left\{\frac{T-t}{T}X_0\right\}_{t=0}^{T}\right) = \frac{X_0}{T}\sum_{t=0}^{T-1}\left(S_0 + (t+1)\gamma\frac{X_0}{T}\right)$$

$$= \frac{X_0}{T}\left(TS_0 + \gamma\frac{X_0}{T}\frac{(T+1)T}{2}\right)$$

$$= X_0S_0 + \frac{\gamma}{2}\frac{T+1}{T}X_0^2$$

$$= C^*_0(X_0,S_0).$$

$\square$

PERMANENT IMPACT WITH INFORMATION

The asset price is modelled as

$$\begin{cases} S_{t+1} = S_t + \gamma A_t - \theta \Delta X_{t+1} + \sigma S_0 \Delta W_{t+1} \\ A_{t+1} = \rho A_t + \eta \Delta \tilde{W}_{t+1} . \end{cases}$$

The AR(1) process $\{A_t\}_{t=0}^{T}$ represents information.

The value function accommodates conditioning with respect to the value $A_t = a$:

$$\begin{aligned} C_t^* &= C_t^*(x, s, a) \\ &= \inf \left\{ F(t, s, a, \{X_u\}_{u=t}^{T}) : \begin{matrix} X_{u+1} \hat{\in} \mathcal{F}_u \ \forall t \le u \le T-1 \\ X_t = x, \ X_T = 0 \end{matrix} \right\}, \end{aligned}$$

where

$$\begin{aligned} &F(t, s, a, \{X_u\}_{u=t}^{T}) \\ &\qquad = \mathbb{E}_t \left[\sum_{u=t}^{T-1} (-\Delta X_{u+1}) S_{u+1} \,\middle|\, S_t = s, A_t = a \right]. \end{aligned}$$

PROPOSITION 2

It holds

$$C_t^*(x,s,a) = sx + c_{1,T-t-1}x^2 + c_{2,T-t-1}ax + c_{3,T-t-1}a^2 + c_{4,T-t-1}$$

where for $K = 1 \text{—} T$

$$c_{1,K} = \frac{\gamma}{2}\,\frac{K+2}{K+1}$$

$$c_{2,K} = \gamma + \frac{\gamma\rho\, c_{2,K-1}}{2\, c_{1,K-1}}, \qquad c_{2,0} = \gamma$$

$$c_{3,K} = \rho^2 c_{3,K-1} - \frac{\rho^2}{4}\,\frac{c_{2,K-1}^2}{c_{1,K-1}}, \qquad c_{3,0} = 0$$

$$c_{4,K} = c_{4,K-1} + c_{3,K-1}\eta^2 \qquad c_{4,0} = 0.$$

Moreover the optimal execution strategy is given by

$$-\Delta X_{t+1}^* = \frac{1}{T-t}X_t + \frac{\rho}{2}\,\frac{c_{2,T-t-2}}{c_{1,T-t-2}}A_t.$$

PROOF.

We show by induction over $k = 0 \_\_\_ T-1$ that

$$C^*_{T-k-1}(x, s, a) = sx + c_{1,k} x^2 + c_{2,k} ax + c_{3,k} a^2 + c_{4,k}.$$

For $k = 0$, $C^*_{T-1}(x, s, a)$ is actually an infimum over the singleton $\Delta X_T = -x$. Hence,

$$\begin{aligned} C^*_{T-1}(x, s, a) &= F(T-1, s, a, \{x, 0\}) \\ &= \mathbb{E}_{T-1}\left[x S_T \mid S_{T-1} = s, A_{T-1} = a, X_{T-1} = x \right] \\ &= sx + \gamma ax + \theta x^2. \end{aligned}$$

This gives

$$c_{1,0} = \theta$$
$$c_{2,0} = \gamma$$
$$c_{3,0} = 0$$
$$c_{4,0} = 0$$

as stated.

Hence, we assume that the claim holds up to $k-1 < T-2$ and we argue for the induction step.

$$G_{T-K-1}(x, s, a, v) = \mathbb{E}_{T-K-1}\left[S_{T-K} \middle| \Delta X_{T-K} = -v, S_{T-K-1} = s, A_{T-K-1} = a\right] \cdot v$$

$$+ \mathbb{E}_{T-K-1}\left[C^*_{T-K}(x-v, S_{T-K}, A_{T-K}) \middle| \Delta X_{T-K} = -v, S_{T-K-1} = s, A_{T-K-1} = a\right]$$

$$= (s + \gamma a + \theta v) v$$

$$+ \mathbb{E}_{T-K-1}\Big[(x-v) S_{T-K} + c_{1,K-1}(x-v)^2$$

$$+ c_{2,K-1} A_{T-K}(x-v)$$

$$+ c_{3,K-1} A^2_{T-K} + c_{4,K-1} \Big| \Delta X_{T-K} = -v, S_{T-K-1} = s, A_{T-K-1} = a\Big]$$

$$= (s + \gamma a + \theta v) x$$

$$+ c_{1,K-1}(x-v)^2 + c_{2,K-1}(x-v)\rho a$$

$$+ c_{3,K-1}(\rho^2 a^2 + \eta^2) + c_{4,K-1} .$$

thus,

$$\underset{v \in \mathbb{R}}{\operatorname{argmin}}\ G_{T-K-1}(x, s, a, v) = \left(1 - \frac{\theta}{2c_{1,K-1}}\right) x + \frac{\rho c_{2,K-1}}{2c_{1,K-1}} a$$

and

$$G^*_{T-K-1}(x, s, a) = sx + \theta\left(1 - \frac{\theta}{4c_{1,K-1}}\right) x^2$$

$$+ \left(\gamma + \frac{\theta \rho\, c_{2,K-1}}{2c_{1,K-1}}\right) a x + \left(\rho^2 c_{3,K-1} - \frac{\rho^2 c^2_{2,K-1}}{4c_{1,K-1}}\right) a^2$$

$$+ c_{4,K-1} + c_{3,K-1}\eta^2 .$$

Finally, the optimal execution strategy is given by

$$-\Delta X^*_{T-K} = \operatorname*{argmin}_{v \in \mathbb{R}} G_{T-K-1}\left(X_{T-K-1}, S_{T-K-1}, A_{T-K-1}, v\right)$$

$$= \left(1 - \frac{\gamma}{2c_{1,K-1}}\right) X_{T-K-1} + \frac{\rho\, c_{2,K-1}}{2\, c_{1,K-1}} A_{T-K-1}$$

$$= \frac{1}{K+1} X_{T-K-1} + \frac{\rho}{2} \frac{c_{2,K-1}}{c_{1,K-1}} A_{T-K-1}.$$

$\square$

PERMANENT IMPACT WITH RISK ADJUSTMENT

The asset price is modelled as

$$S_{t+1} = S_t - \theta \Delta X_{t+1} + \sigma S_0 \Delta W_{t+1}.$$

We define the "unimpacted" price as

$$\tilde{S}_{t+1} = \tilde{S}_t + \tilde{\sigma} \Delta W_{t+1}, \quad \tilde{S}_0 = S_0$$

where $\tilde{\sigma} = \sigma S_0$.

We consider the following risk-adjusted expected execution cost

$$F(t, s, \{X_u\}_{u=t}^{T})$$
$$= \mathbb{E}_t \left[\sum_{u=t}^{T-1} (-\Delta X_{u+1}) S_{u+1} + 2\alpha \Delta \tilde{S}_{u+1}^2 X_{u+1} \,\middle|\, S_t = s \right].$$

PROPOSITION 3

It holds

$$C_t^*(x, s) = sx + \frac{\theta}{2} \frac{T-t+1}{T-t} x^2 + \alpha \sigma^2 (T-t-1) x$$

and the optimal execution strategy is given by

$$-\Delta X_{t+1}^* = \frac{X_t^*}{T-t} + (T-t-1) \alpha \frac{\sigma^2}{\theta}.$$

PROOF.

We show by induction over $K = 1 \text{ --- } T$ that

$$C^*_{T-K}(x, s) = sx + \frac{\theta}{2}\frac{K+1}{K}x^2 + \alpha\sigma^2(K-1)x.$$

For $K=1$, $C^*_{T-1}(x,s)$ is actually an infimum over the singleton $-\Delta X_T = x$. Hence,

$$C^*_{T-1}(x,s) = F(T-1, s\{x,0\})$$

$$= \mathbb{E}_{T-1}\left[xS_T + 2\alpha\tilde{\sigma}^2\Delta W_T^2 \cdot 0 \,\middle|\, S_{T-1}=s, X_{T-1}=x\right]$$

$$= sx + \theta x^2.$$

Hence we assume that the claim holds up to $K<T$ and we argue for the induction step.

$$G_{T-K-1}(x,s,v) = \mathbb{E}_{T-K-1}\Big[vS_{T-K} + 2\alpha\tilde{\sigma}^2\Delta W_{T-K}^2(x-v) \,\Big|\, S_{T-K-1}=s, X_{T-K-1}=x, -\Delta X_{T-K}=v\Big]$$

$$+ \mathbb{E}_{T-K-1}\left[C^*_{T-K}(x-v, S_{T-K}) \,\middle|\, \Delta X_{T-K}=-v, S_{T-K-1}=s\right]$$

$$= (s+\theta v)v + 2\alpha\tilde{\sigma}^2(x-v)$$

$$+ \mathbb{E}_{T-K-1}\left[(x-v)S_{T-K} + \frac{\theta}{2}\frac{K+1}{K}(x-v)^2 + \alpha\sigma^2(K-1)(x-v)\right]$$

$$= (s+\theta v)x + \frac{\theta}{2}\frac{k+1}{k}(x-v)^2 + \alpha\tilde{\sigma}^2(k+1)(x-v).$$

Therefore,

$$\operatorname*{argmin}_{v\in\mathbb{R}} G_{T-k-1}(x,s,v) = \frac{x}{k+1} + \alpha\frac{\tilde{\sigma}^2}{\theta}k$$

and

$$G^*_{T-k-1}(x,s) = sx + \frac{\theta}{2}\frac{k+2}{k+1}x^2 + \alpha\tilde{\sigma}^2 kx,$$

proving the induction step.

Finally

$$-\Delta X^*_{T-k} = \operatorname*{argmin}_{v\in\mathbb{R}} G_{T-k-1}(X^*_{T-k-1}, S_{T-k-1}, v)$$

$$= \frac{X^*_{T-k-1}}{k+1} + \alpha\frac{\tilde{\sigma}^2}{\theta}k.$$

$\square$

PROBLEM SET 1 — SOLUTIONS

EXERCISE 1.1.

From PROPOSITION 1 above we know that

$$\mathbb{E}\left[\sum_{t=0}^{T-1}(-\Delta X_{t+1})S_{t+1}\right] = S_0 X_0 + \frac{\theta}{2}\left(1+\frac{1}{T}\right)X_0^2.$$

Moreover, we can compute

$$2\alpha\tilde{\sigma}^2\sum_{t=0}^{T-1}X_{t+1} = 2\alpha\tilde{\sigma}\frac{X_0}{T}\sum_{t=1}^{T}(T-t) = \alpha\tilde{\sigma}^2 X_0(T-1).$$

Hence the optimal horizon is given by the minimizer of

$$T \longmapsto \frac{\theta}{2}X_0^2\cdot\frac{1}{T} + \alpha\tilde{\sigma}^2 X_0 T.$$

This is

$$T^* = \sqrt{\frac{\theta}{2}\frac{X_0}{\alpha\tilde{\sigma}}}.$$

EXERCISE 1.2

This is PROPOSITION 3 above.

EXERCISE 2

This is PROPOSITION 2 above.