ATML-TUTORIAL

1) UNDERSTAND LIQUIDITY PROVISION AND MM ARGUMENT IN OW12

2) DYNAMIC PROGR AND INDUCTION ARG AGAIN

3) PYTHON IMPLEM OF INDUCTION ARGUMENT

Imperial College London

Algorithmic Trading and Machine Learning
Problem Set 2

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Exercise 1. Consider the following trade execution setting. Time t is discrete, and it ranges from 0, start of the execution, to T, execution horizon. Let X_t denote the time-t quantity left to execute by time T. Let S_t denote the time-t price of the asset being traded. We assume both permanent and transient price impact and we let S_t be

$$S_t = s_0 + \sigma s_0 W_t - \theta(X_t - x_0) - \gamma \sum_{u=1}^{t} \Delta X_u e^{-\rho(t-u)}, \tag{1}$$

where s_0 is the initial value of the asset price, $-\Delta X_{t+1} = -X_{t+1} + X_t$ is the quantity executed in the subinterval $[t, t+1]$, and W is a standard one-dimensional Brownian motion. We let $\mathfrak{F}$ be the filtration generated by W.

1.1 Obizhaeva&Wang (2012). Consider the following expected execution cost

$$F(t, s, \{X_u\}_{u=t}^{T}) = \mathbb{E}_t \left[\sum_{u=t}^{T-1} (-\Delta X_{u+1}) \left(S_u - \alpha \Delta X_{u+1}\right) \Big| S_t = s \right], \tag{2}$$

where S_t is as in equation (1), and $\alpha \geq 0$. Find the optimal execution strategy

$$\{X_t^\star\}_{t=0}^{T} = \operatorname{argmin} \left\{ F(0, s_0, \{X_t\}_{t=0}^{T}) : X_{t+1} \hat{\in} \mathfrak{F}_t \forall t,\ X_0 = x_0,\ X_T = 0 \right\}$$

Exercise 2. Consider the following trade execution setting. Time t is continuous and it ranges from 0, start of the execution, to 1, execution horizon. Let X_t denote the time-t quantity left to execute by time 1. Let S_t denote the time-t price of the asset being traded. Let $\mathfrak{F}$ be the market information filtration. Given a finite collection of trading times $\{t_n\}$ in the unit interval $[0,1]$, define

$$\begin{aligned}\mathcal{A}(\{t_n\}) := \Big\{ X : [0,1] \to \mathbb{R} :\ & X \text{ is piecewise constant, right-continuous, } X \text{ is } \mathfrak{F}\text{-predictable}, \\ & X \text{ has discontinuities at } \{t_u\},\ X_0 = x_0,\ X_1 = 0 \Big\}.\end{aligned}$$

This is the set $\mathcal{A}(\{t_n\})$ of all piecewise constant right-continuous predictable processes with jump times among the specified $\{t_n\}$. For a generic X in $\mathcal{A}(\{t_n\})$ we set $dX_t := X_t - X_{t-} = \lim_{\epsilon \downarrow 0}(X_t - X_{t-\epsilon})$, so that

$$dX_t = \sum_n x_n \mathbb{1}\{t = t_n\},$$

where $x_{n+1} := X_{t_{n+1}} - X_{t_n}$ for $n = 0, \ldots$. For any bounded $\mathbb{R}$-valued function $\kappa = \kappa(s,t)$ defined on $[0,1] \times [0,1]$ we have

$$\int_0^t \kappa(s,t) dX_s = \sum_{t_n \leq t} \kappa(t_n, t) x_n.$$

Let N be a positive integer. Define $t_n^{(N)} := n/N$, and $\mathcal{A}^{(N)} := \mathcal{A}(\{t_n^{(N)}\})$.

2.1 Find

$$X^{(N),\star} = \operatorname{argmin} \left\{ \mathbb{E}\left[-\int_0^1 S_t^{(N)} dX_t^{(N)} \right] : X^{(N)} \in \mathcal{A}^{(N)} \right\},$$

where

$$S_t^{(N)} = s_0 - \int_0^t \Big(\theta + \kappa(s,t)\Big) dX_s^{(N)} + \sigma s_0 W_t,$$

where $\kappa(s,t) = \gamma \exp(-\rho(t-s))$, and W is a standard one-dimensional Brownian motion.

2.2 Does $X^{(N),\star}$ have a pointwise limit as $N \to \infty$? If it does, establish the limit.

PROBLEM SET 2 – SOLUTIONS

EXERCISE 1

Observe that we can write

$$S_{t+1} = \left(1-e^{-\rho}\right)\left(s_0 + \sigma s_0 W_t - \theta(X_t - x_0)\right) + e^{-\rho} S_t - (\theta+\gamma)\Delta X_{t+1} + \sigma s_0 \Delta W_{t+1}.$$

We can compute

$$\mathbb{E}_t\left[S_{t+1} \mid S_t = s, X_t = x, \Delta X_{t+1} = -v\right] = \left(1-e^{-\rho}\right)\varphi(x) + e^{-\rho} s + (\theta+\gamma)v$$

where $\varphi(x) := s_0 - \theta x + \theta x_0$.

Similarly we can compute

$$\mathbb{E}_t\left[S_{t+1}^2 \mid S_t = s, X_t = x, \Delta X_{t+1} = -v\right] = \sigma^2 s_0^2 + \left(\left(1-e^{-\rho}\right)\varphi(x) + e^{-\rho}s + (\theta+\gamma)v\right)^2.$$

Define

$$C_t^*(s,x) = \inf\left\{F\left(t, s, \{X_u\}_{u=t}^T\right) : X_{u+1} \in \mathcal{F}_u\ \forall u,\ X_u = x,\ X_T = 0\right\}$$

and assume the functional form

$$C_t^*(s,x) = c_{1,t}x^2 + c_{2,t}s^2 + c_{3,t}sx + c_{4,t}x + c_{5,t}s + c_{6,t}$$

for some time-dependent coefficients $c_{1,t}, c_{2,t}, c_{3,t}, c_{4,t}, c_{5,t}, c_{6,t}$ yet to be established.

Define

$$G_t(s, x, v) := \mathbb{E}_t\left[(-\Delta X_{t+1})(S_t - \alpha \Delta X_{t+1}) \middle| S_t = s, X_t = x, \Delta X_{t+1} = -v\right]$$

$$+ \mathbb{E}_t\left[C^*_{t+1}(x - v, S_{t+1}) \middle| S_t = s, X_t = x, \Delta X_{t+1} = -v\right].$$

Given the assumed functional form of C^* we can express for $t < T-1$

$$\begin{aligned} G_t(s, x, v) = \; & v(s + \alpha v) \\ & + c_{1,t+1}(x-v)^2 \\ & + c_{2,t+1}\mathbb{E}_t\left[S^2_{t+1} \middle| S_t = s, X_t = x, \Delta X_{t+1} = -v\right] \\ & + c_{3,t+1}(x-v)\mathbb{E}_t\left[S_{t+1} \middle| S_t = s, X_t = x, \Delta X_{t+1} = -v\right] \\ & + c_{4,t+1}(x-v) \\ & + c_{5,t+1}\mathbb{E}_t\left[S_{t+1} \middle| S_t = s, X_t = x, \Delta X_{t+1} = -v\right] \\ & + c_{6,t+1}. \end{aligned}$$

On the other hand, given the constraint $X_T = 0$, we have

$$C^*_{T-1}(s, x) = G_{T-1}(s, x, x) = x(s + \alpha x),$$

which gives

$$c_{1,T-1} = \alpha, \; c_{2,T-1} = 0, \; c_{3,T-1} = 1, \; c_{4,T-1} = 0, \; c_{5,T-1} = 0, \; c_{6,T-1} = 0.$$

The induction argument proceeds as follows:

$$-\Delta X^*_{t+1} = \operatorname{argmin}\left\{ G_t(s, x, v) : v \in \mathbb{R} \right\}$$

$$G^*_t(s, x) = G_t(s, x, -\Delta X^*_{t+1})$$

Equating $G^*_t(s,x) = C^*_t(s,x)$ we get (backward) recursive relations for the coefficients $c_{1,t}$, $c_{2,t}$, $c_{3,t}$, $c_{4,t}$, $c_{5,t}$, $c_{6,t}$.

This concludes.

$\square$

EXERCISE 2.1

Notice that

$$S^{(N)}_{t_{n+1}} = S^{(N)}_{t_n} - (\theta + \gamma) x_{n+1}$$

and thus

$$-\int_0^1 S^{(N)}_t dX^{(N)}_t = -\sum_{n<N} S^{(N)}_{t_{n+1}} x_{n+1}$$

$$= -\sum_{n<N} \left(S^{(N)}_{t_n} + (\theta + \gamma) x_{n+1} \right) x_{n+1} .$$

Therefore $X^{(N),*}$ is the solution to EXERCISE 1 with the following replacements:

$$T \leftarrow N$$

$$\sigma \leftarrow \sigma / \sqrt{N} .$$

EXERCISE 2.2

See PROPOSITION 2 in OBIZHAEVA & WANG (2012)

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