Imperial College London

Algorithmic Trading and Machine Learning
Problem Set 2

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Exercise 1. Consider the following trade execution setting. Time t is discrete, and it ranges from 0, start of the execution, to T, execution horizon. Let X_t denote the time-t quantity left to execute by time T. Let S_t denote the time-t price of the asset being traded. We assume both permanent and transient price impact and we let S_t be

$$S_t = s_0 + \sigma s_0 W_t - \theta(X_t - x_0) - \gamma \sum_{u=1}^{t} \Delta X_u e^{-\rho(t-u)}, \tag{1}$$

where s_0 is the initial value of the asset price, $-\Delta X_{t+1} = -X_{t+1} + X_t$ is the quantity executed in the subinterval $[t, t+1]$, and W is a standard one-dimensional Brownian motion. We let $\mathfrak{F}$ be the filtration generated by W.

1.1 Obizhaeva&Wang (2012). Consider the following expected execution cost

$$F(t, s, \{X_u\}_{u=t}^{T}) = \mathbb{E}_t \left[\sum_{u=t}^{T-1} (-\Delta X_{u+1}) \left(S_u - \alpha \Delta X_{u+1}\right) \Big| S_t = s \right], \tag{2}$$

where S_t is as in equation (1), and $\alpha \geq 0$. Find the optimal execution strategy

$$\{X_t^\star\}_{t=0}^{T} = \text{argmin} \left\{ F(0, s_0, \{X_t\}_{t=0}^{T}) : X_{t+1} \hat{\in} \mathfrak{F}_t \forall t,\ X_0 = x_0,\ X_T = 0 \right\}$$

Exercise 2. Consider the following trade execution setting. Time t is continuous and it ranges from 0, start of the execution, to 1, execution horizon. Let X_t denote the time-t quantity left to execute by time 1. Let S_t denote the time-t price of the asset being traded. Let $\mathfrak{F}$ be the market information filtration. Given a finite collection of trading times $\{t_n\}$ in the unit interval $[0, 1]$, define

$$\begin{aligned}\mathcal{A}(\{t_n\}) := \Big\{ X : [0,1] \to \mathbb{R} :\ & X \text{ is piecewise constant, right-continuous, } X \text{ is } \mathfrak{F}\text{-predictable,} \\ & X \text{ has discontinuities at } \{t_u\},\ X_0 = x_0,\ X_1 = 0 \Big\}.\end{aligned}$$

This is the set $\mathcal{A}(\{t_n\})$ of all piecewise constant right-continuous predictable processes with jump times among the specified $\{t_n\}$. For a generic X in $\mathcal{A}(\{t_n\})$ we set $dX_t := X_t - X_{t-} = \lim_{\epsilon \downarrow 0}(X_t - X_{t-\epsilon})$, so that

$$dX_t = \sum_n x_n \mathbb{1}\{t = t_n\},$$

where $x_{n+1} := X_{t_{n+1}} - X_{t_n}$ for $n = 0, \ldots$. For any bounded $\mathbb{R}$-valued function $\kappa = \kappa(s,t)$ defined on $[0,1] \times [0,1]$ we have

$$\int_0^t \kappa(s,t) dX_s = \sum_{t_n \leq t} \kappa(t_n, t) x_n.$$

Let N be a positive integer. Define $t_n^{(N)} := n/N$, and $\mathcal{A}^{(N)} := \mathcal{A}(\{t_n^{(N)}\})$.

2.1 Find

$$X^{(N),\star} = \text{argmin} \left\{ \mathbb{E}\left[-\int_0^1 S_t^{(N)} dX_t^{(N)} \right] : X^{(N)} \in \mathcal{A}^{(N)} \right\},$$

where

$$S_t^{(N)} = s_0 - \int_0^t \Big(\theta + \kappa(s,t)\Big) dX_s^{(N)} + \sigma s_0 W_t,$$

where $\kappa(s,t) = \gamma \exp(-\rho(t-s))$, and W is a standard one-dimensional Brownian motion.

2.2 Does $X^{(N),\star}$ have a pointwise limit as $N \to \infty$? If it does, establish the limit.