Imperial College London

Algorithmic Trading and Machine Learning
Problem Set 3

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Exercise 1. Consider the following trade execution setting. Time t is continuous, and it ranges from 0, start of the execution, to T, execution horizon. Let S_t denote the time-t fundamental price of the asset being traded. 'Fundamental price' means that S_t does not incorporate any impact; the mid-price can be thought of as a proxy for the fundamental price of an asset. We assume that S_t evolves as a geometric Brownian motion, namely that $dS_t = \sigma S_t dW_t$, for some constant σ and some standard one-dimensional Brownian motion W. We let $\mathfrak{F}$ be the filtration generated by W. The execution price at time t of the amount v is $S_t + \theta v$, for some coefficient of temporary impact θ.

Let x_0 denote the overall quantity to execute. Let X_t denote the time-t quantity left to execute by time T. We say that a stochastic process $\{X_t(\omega) : 0 \leq t \leq T, \omega \in \Omega\}$ is an admissible inventory trajectory if: 1. $X_0 = x_0$ and $X_T = 0$; 2. for every ω the path $t \mapsto X_t(\omega)$ is differentiable; 3. the derivative process $\{\dot{X}_t(\omega) = dX_t(\omega)/dt : 0 \leq t \leq T, \omega \in \Omega\}$ is progressively measurable with respect to $\mathfrak{F}$; 4. the following squared-integrability holds: $\mathbb{E}[\int_0^T \dot{X}_t^2 dt] < \infty$. We let $\mathfrak{A}$ denote the class of admissible inventory trajectories. Every admissible inventory trajectory represents a trading strategy and viceversa, so we use the two terms interchangeably.

We formulate the problem of optimal trade execution following Gatheral and Schied (2011). More precisely, the overall execution cost associated with the admissible trading strategy X in $\mathfrak{A}$ is

$$\int_0^T (-\dot{X}_t) \left(S_t - \theta \dot{X}_t \right) dt,$$

and its overall risk is

$$\int_0^T X_t S_t dt.$$

Therefore, the expected risk-adjusted execution cost is

$$\mathbb{E}\left[\int_0^T (-\dot{X}_t) \left(S_t - \theta \dot{X}_t \right) dt + \tilde{\lambda} \int_0^T X_t S_t dt \right],$$

for some coefficient $\tilde{\lambda}$ of risk aversion.

1.1 Show that minimising risk-adjusted execution costs over the set $\mathfrak{A}$ of admissible inventory trajectories is equivalent to the following minimisation problem:

$$\inf \left\{ \mathbb{E}\left[\int_0^T \left(\dot{X}_t^2 + \lambda X_t S_t \right) dt \right] : X \in \mathfrak{A} \right\}, \tag{1}$$

where $\lambda = \tilde{\lambda}/\theta$.

1.2 Let $C_t^\star(x)$ be the value function for the minimisation in (1), namely $C_t^\star(x)$ is the minimal trading cost that a trader may incur into from time t to time T if the inventory at time t is equal to x. We cast the Bellman principle in our setting by stating that for all X in $\mathfrak{A}$ and all $t < \tau < T$ it holds

$$C_t^\star(x) = \inf \left\{ \mathbb{E}_t \left[\int_t^\tau \left(\dot{X}_u^2 + \lambda X_u S_u \right) du + C_\tau^\star(X_\tau) \right] : \quad X \in \mathfrak{A}, X_t = x \right\} \tag{2}$$

Show that this implies that

$$C_t^\star(X_t) + \int_0^t (\dot{X}_u^2 + \lambda X_u S_u) du$$

is a local sub-martingale for all X in $\mathfrak{A}$.

1.3 We take for a fact that if we can find a smooth function $G^\star = G^\star(t,s,x)$ such that $G^\star(t,S_t,X_t)+\int_0^t(\dot{X}_u^2+\lambda X_u S_u)du$ is a local sub-martingale for all X in $\mathfrak{A}$, then $C_t^\star(x) = G^\star(t,S_t,x)$ for all $0 \le t \le T$ and all $x \in \mathbb{R}$.

Let $G = G(t,s,x)$ be a function in $C^{1,2,1}([0,T]\times\mathbb{R}\times\mathbb{R})$. For an arbitrary X in $\mathfrak{A}$, define

$$M_t := G(t,S_t,X_t) + \int_0^t (\dot{X}_u^2 + \lambda X_u S_u)du.$$

Show that M is a local sub-martingale if G satisfies the following partial differential equation:

$$\partial_t G + \frac{\sigma^2}{2}s^2\partial_{ss}^2 G - \frac{1}{4}(\partial_x G)^2 + \lambda xs \ge 0 \tag{3}$$

for all $0 \le t \le T$ and all s, x in $\mathbb{R}$.

1.4 Define the function

$$\begin{aligned} G^\star(t,s,x) :=& \frac{x^2}{T-t} + \frac{\lambda sx}{2}(T-t) \\ &- \frac{\lambda^2 s^2}{8\sigma^6}\left(\exp(\sigma^2(T-t)) - 1 - \sigma^2(T-t) - \frac{\sigma^4}{2}(T-t)^2\right) \end{aligned} \tag{4}$$

(a) Show that $G^\star$ solves equation (3) with equality;

(b) Show that for all X in $\mathfrak{A}$ it holds $\mathbb{E}[G^\star(t,S_t,X_t)] \to 0$ as $t \uparrow T$.

1.5 Show that for all X in $\mathfrak{A}$ it holds

$$G^\star(0,x_0,s_0) \le \mathbb{E}\left[\int_0^T \left(\dot{X}_t^2 + \lambda X_t S_t\right)dt\right], \tag{5}$$

where $G^\star$ is as in equation (4).

1.6 Conclude that

$$X_t^\star = \frac{T-t}{T}x_0 - \frac{\lambda}{4}(T-t)\int_0^t S_u du \tag{6}$$

is the optimal trading strategy.