Imperial College London

Algorithmic Trading and Machine Learning
Problem Set 1

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Exercise 1 ("Risk-adjustment in Bertsimas&Lo permanent price impact execution setting")**.** Consider the following Bertsimas&Lo execution setting. Time t is discrete, and it ranges from 0, start of the execution, to T, execution horizon. Let X_t denote the time-t quantity left to execute by time T. Let S_t denote the time-t price of the asset being traded. We assume permanent price impact and we let S_t evolve according to the following dynamics

$$S_{t+1} = S_t - \theta \Delta X_{t+1} + \sigma S_0 \Delta W_{t+1},$$

where $-\Delta X_{t+1} = -X_{t+1} + X_t$ is the quantity executed in the subinterval $[t, t+1]$, and $\Delta W_{t+1} = W_{t+1} - W_t$ is the increment of the standard one-dimensional Brownian motion W. We let $\mathfrak{F}$ be the filtration generated by W.

Define the "unimpacted" price

$$\tilde{S}_{t+1} = \tilde{S}_t + \tilde{\sigma}\Delta W_{t+1}, \qquad \tilde{S}_0 = S_0,$$

where $\tilde{\sigma} = \sigma S_0$, and consider the following risk-adjusted expected execution cost

$$F(t, s, \{X_u\}_{u=t}^T) = \mathbb{E}_t \left[\sum_{u=t}^{T-1} S_{u+1}(-\Delta X_{u+1}) + 2\alpha^2 \Delta \tilde{S}_{u+1}^2 X_{u+1} \Big| S_t = s \right], \tag{1}$$

where α is a coefficient of risk aversion.

1.1 Horizon setting for linear executions. Let x_0 be a positive integer representing the number of contracts to execute. Let X be the optimal strategy for x_0 in the case of no risk aversion, i.e. $\alpha = 0$. In other words, let X be

$$\{X_t\}_{t=0}^T = \operatorname{argmin} \left\{ \mathbb{E}\left[\sum_{t=0}^{T-1} S_{t+1}(-\Delta X_{t+1}) \right] : \ X_0 = x_0,\ X_T = 0,\ X_{t+1} \hat{\in} \mathfrak{F}_t \, \forall t \right\},$$

where $X_{t+1} \hat{\in} \mathfrak{F}_t \, \forall t$ means that X_{t+1} is measurable with respect to $\mathfrak{F}_t$ for all $t = 0, \dots, T-1$, or, in other words, that the amount to be executed in the time subinterval $[t, t+1]$ is decided at time t. Find the optimal hotizon $T^\star$ for this execution strategy with respect to the risk-adjusted expected execution cost of equation (1). This means finding

$$T^\star = \operatorname{argmin} \left\{ \mathbb{E}\left[\sum_{t=0}^{T-1} S_{t+1}(-\Delta X_{t+1}) + 2\alpha^2 \Delta \tilde{S}_{t+1}^2 X_{t+1} \right] : \ T \in \mathbb{N} \right\}.$$

1.2 Optimal risk-adjusted execution. Find the optimal execution strategy $X^\star$ for the risk-adjusted expected execution cost. This means finding

$$\{X_t^\star\}_{t=0}^T = \operatorname{argmin} \left\{ F(0, S_0, \{X_t\}_{t=0}^T) : \ X_0 = x_0,\ X_T = 0,\ X_{t+1} \hat{\in} \mathfrak{F}_t \, \forall t \right\},$$

where F is as in equation (1).

Exercise 2 ("Permanent price impact with information")**.** Consider the following Bertsimas&Lo execution setting. Time t is discrete, and it ranges from 0, start of the execution, to T, execution horizon. Let X_t denote the time-t quantity left to execute by time T. Let S_t denote the time-t price of the asset being traded. We assume permanent price impact and we let S_t evolve according to the following dynamics

$$\begin{aligned} S_{t+1} &= S_t + \gamma A_t - \theta \Delta X_{t+1} + \sigma S_0 \Delta W_{t+1} \\ A_{t+1} &= \rho A_t + \eta \Delta \tilde{W}_{t+1}, \end{aligned}$$

where $-\Delta X_{t+1} = -X_{t+1} + X_t$ is the quantity executed in the subinterval $[t, t+1]$, and $\Delta W_{t+1} = W_{t+1} - W_t$, $\Delta \tilde{W} t + 1 = \tilde{W}_{t+1} - \tilde{W}_t$ are the increments of the standard one-dimensional Brownian motions W and $\tilde{W}$, independent one from each other. We let $\mathfrak{F}$ be the minimal filtration to which both W and $\tilde{W}$ are adapted. The AR(1) process $\{A_t\}_{t=0}^T$ represents information.

Define the expected execution cost F as follows

$$F(t, s, a, \{X_u\}_{u=t}^T) = \mathbb{E}_t \left[\sum_{u=t}^{T-1} S_{u+1}(-\Delta X_{u+1}) \middle| S_t = s,\ A_t = a \right].$$

Show that the optimal execution strategy $X^\star$, defined as the minimiser

$$\{X_t^\star\}_{t=0}^T = \operatorname{argmin} \left\{ F(0, S_0, A_0, \{X_t\}_{t=0}^T) : \ \ X_0 = x_0,\ X_T = 0,\ X_{t+1} \hat{\in} \mathfrak{F}_t\ \forall t \right\},$$

is given by

$$X_{t+1}^\star = \frac{T-t-1}{T-t} X_t^\star + \frac{\rho c_{2,T-t-2}}{2 c_{1,T-t-2}} A_t, \qquad X_0^\star = x_0,$$

where $c_{1,-1} = 1$, $c_{1,0} = \theta$, $c_{2,-1} = 0$, $c_{2,0} = \gamma$ and for $k = 1, \ldots, T$

$$c_{1,k} = \frac{\theta}{2} \frac{k+2}{k+1}, \qquad c_{2,k} = \gamma + \frac{\theta}{2} \frac{\rho c_{2,k-1}}{c_{1,k-1}}.$$