

Mechanics of good trade execution in the framework of linear temporary market impact

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Abstract. We define the concept of good trade execution and we construct explicit adapted good trade execution strategies in the framework of linear temporary market impact. Good trade execution strategies are dynamic, in the sense that they react to the actual realisation of the traded asset price path over the trading period; this is paramount in volatile regimes, where price trajectories can considerably deviate from their expected value. Remarkably however, the implementation of our strategies does not require the full specification of an SDE evolution for the traded asset price, making them robust across different models. Moreover, rather than minimising the *expected* trading cost, good trade execution strategies minimise trading costs in a *pathwise* sense, a point of view not yet considered in the literature. The mathematical apparatus for such a pathwise minimisation hinges on certain random Young differential equations that correspond to the Euler-Lagrange equations of the classical Calculus of Variations. These Young differential equations characterise our good trade execution strategies in terms of an initial value problem that allows for easy implementations.

1 Introduction

Executions of large trades can affect the price of the traded asset, a phenomenon known as *market impact*. The price is affected in the direction unfavourable to the trade: while selling, the market impact decreases the price; while buying, the market impact increases the price. Therefore, a trader who wishes to minimise her trading costs has to split her order into a sequence of smaller sub-orders which are executed over a finite time horizon. How to optimally split a large order is a question that naturally arises.

Academically, the literature discussing such an optimal split was initiated by the seminal papers by Almgren and Chriss [AC00] and by Bertsimas and Lo [BL98]. Both papers deal with the trading process of one large market participant who would like to buy or sell a large amount of shares or contracts during a specified duration. The optimisation problem is formulated as a trade-off between two pressures. On the one hand, market impact demands to trade slowly in order to minimise the unfavourable impact that the execution itself has on the price. On the other hand, traders have an incentive to trade rapidly, because they do not want to carry the risk of adverse price movements away from their decision price. Such a trade-off between market impact and market risk is usually translated into a stochastic control problem where the trader's strategy (i.e. the control) is the trading speed. The class of admissible strategies defines the set over which the risk-cost functional is optimised.

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In the design of mathematical models for optimal trade execution we identify two phases. The first phase is the description of trading costs. This refers to the choice of a function F that depends on time, asset price, quantity to execute and trading speed, and models the instantaneous cost of trading. The overall cost during the time window $[0, T]$ is then expressed as the time integral

$$J(q) = \int_0^T F(t, x_t, q_t, \dot{q}_t) dt,$$

where the path $t \mapsto x_t$ is the evolution of the asset price during the trading period. The letter q stands for quantity of the asset and the trajectory $t \mapsto q(t)$, $[0, T] \rightarrow \mathbb{R}$, is referred to as inventory trajectory. Its time derivative \dot{q} is the rate of execution and it represents the control variable that a trader modulates while executing the trade.

The minimisation of the trading cost J faces the challenge that the price path $t \mapsto x_t$ is not known at the beginning of the trading period. Hence, in order to gain some predictive power, a stochastic model for the evolution of the asset price is introduced. This is the second phase in the design of mathematical models for trade execution. Concretely, it means that a stochastic process $\{S_t : 0 \leq t \leq T\}$ is introduced and the actual price trajectory (x_t) is thought of as a realisation of this stochastic process. Then, the mathematical optimisation focuses on the expected trading cost

$$\mathbb{E} \left[\int_0^T F(t, S_t, q_t, \dot{q}_t) dt \right]. \quad (1.1)$$

Notice that this entails a considerable degree of model dependency, in that the optimisation is based on the distributional assumptions on the price process.

Two alternatives exist for the minimisations of the expected trading cost in equation (1.1). These alternatives are static minimisation (giving rise to static trading strategies) and dynamic minimisation (giving rise to dynamic trading strategies).

Static strategies are completely decided at the beginning of the trading period; they are based only on the information available at the initial time of the trade. Mathematically, this is formulated by considering q as a deterministic path. In this case it is often observed that, by interchanging expectation and time integral in equation (1.1), the actual realisation of the price process disappears from the formulas, replaced by its expected trajectory. When the expected price path is the only feature of the price process that enters the formulas (as in [AC00]), the static strategy does not take into account the volatility of asset prices, whose role however is paramount in financial markets. A visual representation of the relevance of volatility in the context of trade execution is provided by Figure 1.

In Figure 1 Almgren and Chriss's framework is adopted. The price process is a standard one-dimensional Brownian motion and two price paths are considered, one with low volatility and the other with high volatility. Notwithstanding the remarkable difference between the two, they have the same expected path (dashed blue line in the first quadrant) and, as a consequence, the static liquidation strategy is the same for both price paths (dashed blue line in the second quadrant). The simplicity of the model is such that it compromises on the possibility to distinguish rather different market regimes. This is made clear by comparing the static optimal solution with the a-posteriori one.

The a-posteriori solution is the minimiser q of the cost functional J given the actual price trajectory x . This is not implementable in real trading because it is anticipative, in that it assumes that the entire price trajectory is known at the beginning of the trading period. However, since it is independent of the choice of the price process, the a-posteriori solution constitutes a useful term of comparison for the stochastic model. In the example of Figure 1, we observe how different the two a-posteriori solutions corresponding to the two market regimes are. In the case of low volatility, the a-posteriori solution is close to the static one, because the price path does not depart significantly from its expected trajectory. Instead, in the case of high volatility, the a-posteriori solution deviates from the static one: the inventory

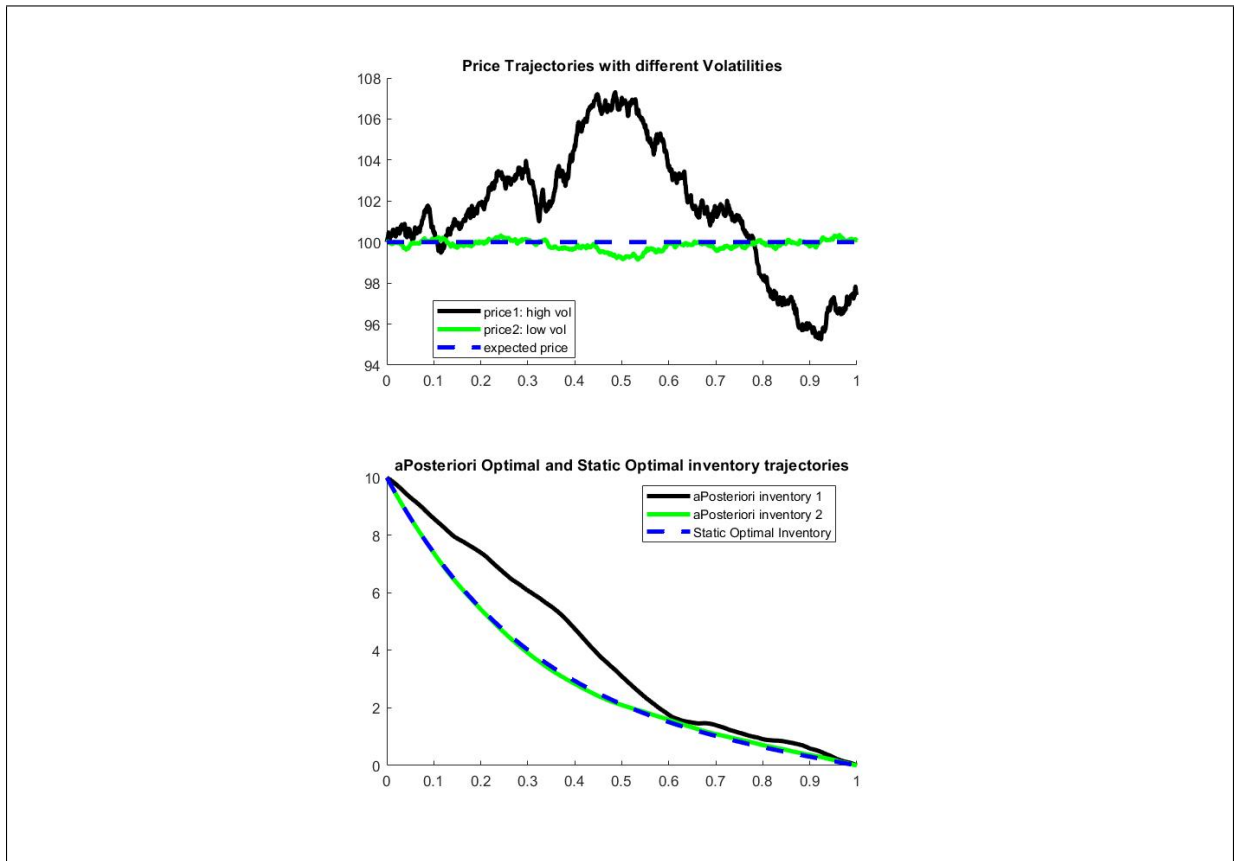


Figure 1: A-posteriori optimal and static optimal inventories in two different volatility regimes

trajectory is considerably steeper where the price is above its expected value, and it is almost flat when the price is below its expected value.

In order to take into account more features of the price process (such as its volatility), the literature on optimal trade execution has utilised the mathematical techniques of stochastic optimal control. This has produced the second alternative the minimisation of the expected cost in equation (1.1), and dynamic trading strategies proliferated since Bertsimas and Lo [BL98] (discrete time) and Gatheral and Schied [GS11] (continuous time). An excellent presentation of the techniques of stochastic optimal control applied to trade execution is contained in the textbook by Cartea et al. [CJP15].

Dynamic trading strategies take fully into account the distributional features of the price process because they are obtained via the Hamilton-Jacobi-Bellman equation, in which the generator of the diffusion that models the price enters.¹ Furthermore, dynamic strategies are random when seen from the initial time, in that they depend on the information that is revealed to the trader during the trading period. Mathematically, this means that dynamic strategies are stochastic processes adapted to the relevant market information filtration. Since deterministic strategies are in particular adapted stochastic processes, the class of static strategies is a subset of the class of dynamic strategies. Therefore, the minimisation over the class of dynamic strategies is expected to improve the result obtained when minimising over the smaller class of static strategies.

This however is not always confirmed in the models. Indeed, despite the mathematical sophistication, cases exist in which optimal trading strategies, although sought among dynamic ones, are in fact static. One of such cases is for example the “Liquidation without penalties only temporary impact” in [CJP15, Section 6.3], an other is the “Optimal acquisition with terminal penalty and temporary impact” in [CJP15, Section 6.4]. This reduction to static optimal solutions clashes with the intuition for which trading strategies should take into account actual realisations of price paths, as the a-posteriori solutions in Figure 1 suggest.

A second drawback of applying the technique of HJB equation to the problem of optimal trade execution is the heavy model dependence. Optimality of the trading strategies holds under the assumption that the price follows some specified dynamics, and this invests of considerable importance the second phase in the design of mathematical models.

In this paper, we propose a new alternative for the minimisation of trading costs. This new alternative considers the pathwise optimisation of the cost functional J without taking expectation. We observe that the reason for the anticipativeness of a-posteriori solutions is the imposition of the constraint that the liquidation terminates exactly at the (arbitrarily fixed) trading horizon. Relaxing this constraint enables to produce adapted pathwise solutions that display two remarkable features. On the one hand, they avoid the degeneracy to static trajectories even in the cases where the techniques of HJB equation do not produce genuinely dynamic strategies; on the other hand, their model dependence is moderate and confined to the expected trajectory of the price path, as was the case for static strategies, rather than to the full law of the price process.

Our trading strategies give rise to inventory trajectories that are obtained in closed-form formulas. Moreover, we can characterise these trajectories as solutions to certain random Young differential equations, inspired by the second-order Euler-Lagrange equations in the classical Calculus of Variations. Such a characterisation allows to implement the inventory trajectories via an easily-simulated initial value problem.

The relaxation of the exact termination time also links our pathwise approach to the discussion about trading schedules with flexible horizons. Unlike most of the literature on optimal trade execution, some authors (e.g. Bechler and Ludkovski [BL15], Easley and de Prado and O’Hara [EPOH15]) make

¹In the case of linear temporary market impact and quadratic inventory cost, a recent work by Belak and Muhle-Karbe and Ou [BMKO18] actually discusses techniques that can be more generally applied to the case of general semimartingales. In this case there is no HJB equation; instead the authors rely on forward-backward stochastic differential equations. In Section 2, we will review this general solution.

the execution horizon T an endogenous variable in the optimisation problem. This results in trading schedules whereby the termination time of the execution is not known a priori; rather, it is adapted to the market conditions in which the trader operates. Our relaxation of the rigid terminal constraint is inscribed in the same order of ideas. It hinges on the unbiasedness of liquidation errors, a property not considered so far in the literature about dynamic trading strategies.

The rest of the paper is organised as follows. Section 2 describes in detail the mathematical framework in which the problem of optimal trade execution is formulated. Our descriptions examines in particular three aspects of the mathematical models. The first aspect is the reduction to static optimal inventories that happens in the context of stochastic optimal control of the expected quantity in equation (1.1). Proposition 2.3 examines such a reduction, listing its causes. This is novel in the literature and answers the questions raised in [BDG14], [BP18] and [BBDN18] about the comparison between static and dynamic solutions to the problem of optimal trade execution. The second aspect is the unbiasedness of liquidation errors (sub-Section 2.2). The third aspect is the degree of model dependency of optimal trade executions. With regards to this third aspect, we formulate a concept of robustness for execution strategies in Definition 2.1 that captures the distinction hinted above between the model dependency of static strategies and the model dependency of dynamic strategies. Currently in the literature there is no framework of optimal trade execution with non-static inventory trajectories, unbiased liquidation errors and robust strategies – see Figure 2. This motivates our proposal in Section 3.

Section 3 presents the concept of good trade execution and states a closed-form formula for good trade executions when risk aversion is expressed as quadratic inventory cost. The characterisation of good trade executions under quadratic inventory costs is contained in sub-Section 3.2. Here, the closed-form formula of the previous paragraph is characterised as a solution to an initial value problem expressed in terms of a random Young differential equation, and uniqueness of the good trade execution is established. Applications are given in sub-Section 3.3.

Section 4 presents good trade executions with risk criteria other than the quadratic inventory cost. In particular, sub-Section 4.2 presents good trade executions when the risk criterion is inspired by the value-at-risk adopted in Gatheral and Schied [GS11].

Finally, Section 5 concludes the paper, and Appendix A presents the mathematical apparatus on which the characterisation of good trade executions is based.

2 Framework

We adopt the perspective of liquidation; the case of acquisition is *mutatis mutandis* the same. Let q_0 denote initial inventory, and let $q_T = 0$ be the liquidation target. The letter q stands for quantity of the asset and the trajectory $t \mapsto q(t)$, $[0, T] \rightarrow \mathbb{R}$, shall be referred to as inventory trajectory. Its time derivative \dot{q} is the rate of execution and it represents the control variable that a trader modulates while executing the trade. Without yet referring to any probabilistic structure, let us introduce the space of such inventory trajectories:

$$\mathcal{Q}_{\text{pw}}^{0, q_0} := \left\{ q : [0, T] \rightarrow \mathbb{R}, \quad q \text{ absolutely continuous,} \right. \\ \left. q(0) = q_0, q(T) = q_T \right\}.$$

The subscript “pw” stands for “pathwise” and emphasises the non-probabilistic perspective.

We will use the term price process to refer to the stochastic process $\{S_t : 0 \leq t \leq T\}$ used to model the time evolution of the asset fundamental price, defined on some probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. A price process will always be assumed to be such that: 1. for all $0 \leq t \leq T$ the second moment of S_t is finite; 2. the maps $t \mapsto \mathbb{E}S_t$ and $t \mapsto \mathbb{E}S_t^2$ are in $L^1[0, T]$; 3. there exists some $p \geq 1$ such that all the paths of

S are of finite p -variation, i.e. for all ω in Ω ,

$$\|S(\omega)\|_{p\text{-var},[0,T]} < \infty.$$

Notice that the paths of the price process are not necessarily assumed to be continuous.

Given a price process $\{S_t : 0 \leq t \leq T\}$, we let $\{\mathfrak{F}_t : 0 \leq t \leq T\}$ be the minimal \mathbb{P} -completed right-continuous filtration generated by S . It is always assumed that \mathfrak{F}_0 is trivial.

If the price process is a semimartingale, we additionally introduce the following terminology. We say that the semimartingale S is a totally square integrable special semimartingale if the following two conditions hold:

1. the semimartingale S is a special semimartingale, i.e. it admits a canonical decomposition

$$S = A + M,$$

where A is a predictable bounded variation process, and M is a local martingale with zero mean;

2. the following integrability holds:

$$\mathbb{E}[\langle M \rangle_T] + \mathbb{E}[\|A\|_{2,[0,T]}^2] < \infty,$$

where $\langle M \rangle$ denotes the quadratic variation of the local martingale M , and $\|A\|_{2,[0,T]}$ denotes the 2-variation of the path A on the time interval $[0, T]$.

Execution rates are progressively measurable square-integrable processes; more precisely, we define the space of execution rates as

$$\begin{aligned} \mathcal{R} := \Big\{ r \in L^2\big([0, T] \times \Omega, \mathcal{B}[0, T] \otimes \mathfrak{F}, dt \otimes \mathbb{P}\big) : \\ r \text{ is } (\mathfrak{F}_t)_t\text{-progressively measurable} \Big\}. \end{aligned} \quad (2.1)$$

Notice that the measurability depends on the filtration of the price process.

Admissible inventory trajectories are first integrals of execution rates with initial value \mathbf{q}_\circ . More precisely, we define the space $\mathcal{Q}^{0, \mathbf{q}_\circ}$ of admissible inventory trajectories as

$$\mathcal{Q}^{0, \mathbf{q}_\circ} := \left\{ q_t = \mathbf{q}_\circ + \int_0^t r_u du : r \in \mathcal{R} \right\}. \quad (2.2)$$

Among admissible inventory trajectories we distinguish those that are fuel-constrained, namely such that their terminal value is $\mathbf{q}_T = 0$. Thus, a fuel-constrained admissible inventory trajectory is an $(\mathfrak{F}_t)_t$ -adapted process with absolutely continuous paths, with deterministic initial value \mathbf{q}_\circ , terminal value $\mathbf{q}_T = 0$, and such that its derivative is in \mathcal{R} . More precisely, we define the space $\mathcal{Q}_{\text{fuel}}^{0, \mathbf{q}_\circ}$ of fuel-constrained admissible inventory trajectories as

$$\mathcal{Q}_{\text{fuel}}^{0, \mathbf{q}_\circ} := \left\{ q_t = \mathbf{q}_\circ + \int_0^t r_u du : r \in \mathcal{R} \text{ and } q_T = \mathbf{q}_T \right\}.$$

Notice that every realisation of a generic q in $\mathcal{Q}_{\text{fuel}}^{0, \mathbf{q}_\circ}$ is a path in $\mathcal{Q}_{\text{pw}}^{0, \mathbf{q}_\circ}$, namely for all q in $\mathcal{Q}_{\text{fuel}}^{0, \mathbf{q}_\circ}$ and all ω in Ω it holds

$$(q_t(\omega))_{0 \leq t \leq T} \in \mathcal{Q}_{\text{pw}}^{0, \mathbf{q}_\circ}.$$

In the space of fuel-constrained inventory trajectories we isolate the subspace of static trajectories, given by

$$\mathcal{Q}_{\text{static}}^{0, \mathbf{q}_\circ} = \left\{ q \in \mathcal{Q}_{\text{fuel}}^{0, \mathbf{q}_\circ} : q_t \in \mathfrak{F}_0 \text{ for all } t \geq 0 \right\}.$$

These are the execution strategies whose entire trajectories are \mathfrak{F}_0 -measurable, namely deterministic. We say that the admissible inventory trajectories not in $\mathcal{Q}_{\text{static}}^{0, q_0}$ are non-static (or dynamic): therefore, the admissible inventory trajectory q is non-static if q is in $\mathcal{Q}^{0, q_0} \setminus \mathcal{Q}_{\text{static}}^{0, q_0}$.

In the literature, the mathematical background for optimal trade execution is often stochastic control. Hence, in view of Bellman's principle, it is convenient to extend the definitions of the spaces of inventory trajectories to the case where the initial time is not zero. The symbols $\mathcal{Q}_{\text{pw}}^{t, q_t}$, \mathcal{Q}^{t, q_t} , $\mathcal{Q}_{\text{fuel}}^{t, q_t}$ and $\mathcal{Q}_{\text{static}}^{t, q_t}$ will denote the straightforward generalisations of the definitions above to the case where the initial time is t in $[0, T)$ and the trajectories are pinned to the value q_t at time t .

The distinction between static and non-static inventory trajectories has been expressed in terms of measurability with respect to the sigma algebra \mathfrak{F}_0 . Alternatively, we can express this distinction emphasising its meaning in the practical implementation of trading strategies.

Assume that q_t is the optimal inventory to be held at time t as per a mathematical model of trade execution. The formula for q_t will in particular depend on the one hand on the price path $\{S_s : 0 \leq s \leq t\}$ up to the present time t ; on the other hand it will depend on the projection of the price trajectory from the present time t to the time horizon T . Such projection is based on the distributional assumptions on the stochastic process used to model the price; hence we write $\{\text{Law}_t(S_r) : t \leq r \leq T\}$ for such a projection, which stands for the probability law of the price process S conditioned on \mathfrak{F}_t .

In general, the inventory q_t can also depend on aspects of the model other than $\{S_s : 0 \leq s \leq t\}$ and $\{\text{Law}_t(S_r) : t \leq r \leq T\}$. In particular, the formula for the inventory q_t can also depend on: 1. the time horizon T ; 2. the probability laws $\{\text{Law}_s(S_r) : s \leq t, r \geq s\}$ conditioned at times s smaller than t ; 3. the laws not just of the conditional marginals $\text{Law}_s(S_r)$ but of higher-dimensional cylinder functions of the paths of the price process. The dependence 1. on the time horizon will always be present in the following, but we suppress it from the notation. Secondly, in view of the dependence 2. on marginal laws conditioned before the current time t , we extend the class $\{\text{Law}_t(S_r) : t \leq r \leq T\}$ to $\{\text{Law}_s(S_r) : 0 \leq s \leq t, s \leq r \leq T\}$. Thirdly, we neglect the possible dependence 3. because in fact such a dependence will never occur in the formulas below. Overall, we can write the formula for q_t as the function

$$q_t = q\left(t, \{S_s : 0 \leq s \leq t\}, \left\{\text{Law}_s(S_r) : \begin{smallmatrix} 0 \leq s \leq t \\ s \leq r \leq T \end{smallmatrix}\right\}\right). \quad (2.3)$$

The inventory trajectory is static if in fact there is no dependence on $\{S_s : 0 \leq s \leq t\}$; it is non-static if there is explicit dependence on $\{S_s : 0 \leq s \leq t\}$.

Let $\{x_t : 0 \leq t \leq T\}$ be the actual realisation of the price path during the execution period. The practical implementation of the trading strategy entails to compute at every time t the inventory

$$q_t^x(S) := q\left(t, \{x_s : 0 \leq s \leq t\}, \left\{\text{Law}_s(S_r) : \begin{smallmatrix} 0 \leq s \leq t \\ s \leq r \leq T \end{smallmatrix}\right\}\right). \quad (2.4)$$

The inventory trajectory $t \mapsto q_t^x(S)$ is called implemented inventory trajectory given the observation x and the price process S . In the case of static inventory trajectory, the implemented inventory trajectory does not take x into account; it is therefore fully based on the distributional assumptions on the price process only.

We remark that $\{x_t : 0 \leq t \leq T\}$ is not a stochastic process, but it is the single path of the fundamental asset price that is plugged in the theoretical formula. Since one single path is observed in the execution period $[0, T]$, the distributional assumptions on the stochastic process S cannot really be investigated. Only pathwise properties of x can be measured. One of such properties is the quadratic variation of x , on which the calibration of our dynamic models of trade execution will be based.

Recently, the academic literature has worked on incorporating short-term price predictors in the classical framework of optimal trade execution: Cartea and Jaimungal [CJ16] described executions that incorporate the effects of order-flow imbalances, modelled as Markov processes; Lehalle and Neuman [LN19] discussed in the same spirit markovian short-term price predictors; Casgrain and Jaimungal [CJ19b] developed execution models where predictors are driven by latent factors. In all these works,

order-flow imbalances and other short-term price predictors are incorporated as drifts of the price process. This in particular is reflected in the expected trajectories $t \mapsto \mathbb{E}_s[S_t]$ at time s . We focus on the expected trajectory at the initial time $s = 0$ of the execution, and we formulate the concept of robustness that will be used in our work.

The expected trajectory $t \mapsto \mathbb{E}[S_t]$ influences the trading strategy and it is susceptible to interpretation. Indeed, assume that the liquidator expects the price to increase during the liquidation period. Then, the liquidator would be prone to execute the largest part of her trade close to the liquidation horizon T . On the contrary, if the price is expected to plunge, the liquidator would be prone to execute most of her trade close to the initial time of the liquidation.

The expected trajectory $t \mapsto \mathbb{E}[S_t]$ is the path of the means of the marginals of S . Because of the interpretability of $t \mapsto \mathbb{E}[S_t]$ among the distributional assumptions on the price, we isolate it from the rest and we hold it fixed when examining different stochastic models for the fundamental price. This leads to the following concept of robustness.

Definition 2.1 (“Marginal robustness”). *Let q be an admissible inventory trajectory and consider it as a function of time, of price path and of the law of the price process as done in equations (2.3) and (2.4). Let x be the observed price during the execution. The inventory trajectory q is called marginally robust if for every pair of price processes S and \tilde{S} such that $\mathbb{E}[S_t] = \mathbb{E}[\tilde{S}_t]$ for all t , the implemented inventory trajectories $q^x(S)$ and $q^x(\tilde{S})$ are the same.*

Proposition 2.2. *Consider an inventory trajectory q and write it as a function of time, of the price path and of the marginal laws of the price process as in equation (2.3). Assume that in fact q_t depends on $\{Law_s(S_r) : \substack{0 \leq s \leq t \\ s \leq r \leq T}\}$ only through the marginal means $\{\mathbb{E}[S_r] : 0 \leq r \leq T\}$, namely*

$$q_t = q(t, \{S_s : 0 \leq s \leq t\}, \{\mathbb{E}[S_r] : 0 \leq r \leq T\}). \quad (2.5)$$

Then, the inventory trajectory q is marginally robust.

The classical optimal trade execution proposed by Almgren and Chriss [AC00] was originally formulated with a strong assumption on the price process, i.e. under the assumption that the price process is an arithmetic Brownian motion. However, Almgren and Chriss inventory trajectory is static and thus their classical formula for the optimal inventory does not actually depend on S being an arithmetic Brownian motion. The actual dependence is merely on S being a martingale. Indeed, one can check that Almgren and Chriss’s execution is marginally robust in the sense of Definition 2.1. However, a marginally robust execution is not necessarily static.

2.1 Temporary market impact

Temporary market impact is a widely spread description of transaction costs used in optimal trade execution. Let S_t denote the price process at time t . We say that the liquidator exerts a temporary market impact on S_t if for some continuous function g in $C(\mathbb{R}^2)$ the execution price of her order at time t is

$$g(S_t, \dot{q}_t),$$

where $t \mapsto q_t$ is the liquidator’s inventory trajectory, and \dot{q}_t denotes its time derivative at time t . A well-known example of temporary market impact is given by $g(S, r) = S + c_1^2 r$, for some coefficient $c_1 > 0$ of market impact. In this case, the execution cost is a linear function of the rate of execution \dot{q} ; since q is decreasing, the steepest the inventory trajectory is at time t , the smaller the execution price is at time t . The classical formulation by Almgren and Chriss [AC00] utilises linear temporary market impact.

Classically in the literature, the optimal trade execution problem is formulated as a stochastic control problem.

The stochastic control problem is formulated by introducing the state variable $X = (S, q)$ whose dynamics is controlled by an execution rate \dot{q} in \mathcal{R} . In order to emphasise such dependence we can write $X = X^r$, where r is the control in the space \mathcal{R} . For q in $\mathcal{Q}_{\text{fuel}}^{0, q_0}$, the objective function $H = H^{\dot{q}}$ is given by

$$H^{\dot{q}}(t, x_1, x_2) := \mathbb{E}_{t, S_t=x_1, q_t=x_2} \left[\int_t^T F(s, X_s^{\dot{q}}, \dot{q}_s) ds \right], \quad (2.6)$$

where $F = F(t, X, r) = F(t, S, q, r)$ is a lagrangian that describes risk-adjusted execution-impacted costs from trade. The stochastic optimisation problem for fuel-constrained inventory trajectories is formulated as

$$\inf \left\{ H^{\dot{q}}(0, S_0, q_0) : q \in \mathcal{Q}_{\text{fuel}}^{0, q_0} \right\}. \quad (2.7)$$

An important component of the lagrangian F is the description of the impact function, namely the function g above. Indeed, the optimisation problem concerns revenues from trade, which in the infinitesimal time dt are

$$-g(S_t, \dot{q}_t) \dot{q}_t dt.$$

In the linear case this formula becomes

$$(-S_t \dot{q}_t - c_1^2 \dot{q}_t^2) dt.$$

Here we observe that the revenues decompose in a first summand $S_t \dot{q}_t$ where the price process appears, and a second summand $c_1^2 \dot{q}_t^2$ that does not comprise the price process. Clearly, such a decomposition holds in more general situations than the one of linear market impact. If this decomposition holds for the whole lagrangian F and if the bounded variation component A of the price process S is deterministic, then we observe the reduction of optimal dynamic solutions to optimal static ones. This happens in some cases studied in the literature (cf. [CJP15, Sections 6.3 and 6.4]), where the optimal inventory trajectory, although sought dynamic, is eventually found to be static. This means that the optimiser of (2.7) is in the space $\mathcal{Q}_{\text{static}}^{0, q_0}$ of static inventory trajectories. The following proposition explains this phenomenon, pointing out those aspects of the model that cause the reduction to static trade executions.

Proposition 2.3 (“Reduction to static optimal trade executions”). *Assume that*

$$F(t, X, r) = rS + L(t, q, r), \quad (2.8)$$

for some Caratheodory function² L that does not depend on S . Assume that there exist an integrable function α on $[0, T]$ and a constant $\beta \geq 0$ such that

$$|L(t, q, r)| \leq \alpha(t) + \beta (q^2 + r^2).$$

Let the price process S be a totally square integrable canonical semimartingale with canonical decomposition

$$S_t = A_t + M_t. \quad (2.9)$$

Assume that A is \mathfrak{F}_0 -measurable, namely that the drift of the price process is deterministic. Then, for all $0 \leq t \leq T$ it holds

$$\inf \left\{ H^{\dot{q}}(t, S_t, q_t) : q \in \mathcal{Q}_{\text{static}}^{t, q_t} \right\} = \inf \left\{ H^{\dot{q}}(t, S_t, q_t) : q \in \mathcal{Q}_{\text{fuel}}^{t, q_t} \right\}.$$

²Let U be an open subset of \mathbb{R}^n and let $F : U \times \mathbb{R}^d \rightarrow \mathbb{R}$. Then F is said to be a Caratheodory function if

1. for Lebesgue-almost every x in U the map $\xi \mapsto F(x, \xi)$ is continuous;
2. for every ξ in \mathbb{R}^d the function $x \mapsto F(x, \xi)$ is Lebesgue-measurable.

The function L in the statement of Proposition 2.3 is assumed to be a Caratheodory function with the choices: 1. the open interval $(0, T)$ as the subset U of \mathbb{R}^n ; 2. the two-dimensional variable (q, r) as the variable ξ in the definition.

Proof. Let q be in $\mathcal{Q}_{\text{fuel}}^{t, q_t}$. Let X be the state variable $X = (S, q)$ and let Y be the two dimensional path $Y = (q, A)$. Let φ be the function $\varphi(x_1, x_2) = x_1 x_2$. Notice that

$$\varphi(X_r^{\dot{q}}) - \varphi(X_t^{\dot{q}}) - \int_t^r X_s^{\dot{q}} dY_s, \quad t \leq r,$$

is a centred martingale. Hence,

$$\begin{aligned} H^{\dot{q}}(t, S_t, \mathbf{q}_t) &= \mathbb{E}_t \left[\int_t^T F(s, X_s^{\dot{q}}, \dot{q}_s) ds \right. \\ &\quad \left. + \varphi(X_T^{\dot{q}}) - \varphi(X_t^{\dot{q}}) - \int_t^T X_r^{\dot{q}} dY_r \right] \\ &= -\mathbf{q}_t S_t + \mathbb{E}_t \left[\int_t^T (-q_r dA_s + L(r, q_r, \dot{q}_r) dr) \right] \end{aligned}$$

It holds

$$\begin{aligned} \inf_{q \in \mathcal{Q}_{\text{fuel}}^{t, q_t}} H^{\dot{q}}(t, S_t, \mathbf{q}_t) \\ \geq -\mathbf{q}_t S_t + \mathbb{E}_t \left[\inf_{q \in \mathcal{Q}_{\text{pw}}^{t, q_t}} \int_t^T (-q_r dA_s + L(s, q_s, \dot{q}_s) ds) \right], \end{aligned}$$

where the infimum on the right hand side is taken in a pathwise sense for each realisation of the price S . In fact, the integrand does not depend on such a realisation (i.e. it does not depend on ω in Ω) because A is non-random. Therefore,

$$\begin{aligned} \inf_{q \in \mathcal{Q}_{\text{fuel}}^{t, q_t}} H^{\dot{q}}(t, S_t, \mathbf{q}_t) \\ \geq -\mathbf{q}_t S_t + \inf_{q \in \mathcal{Q}_{\text{static}}^{t, q_t}} \mathbb{E}_t \left[\int_t^T (-q_r dA_s + L(s, q_s, \dot{q}_s) ds) \right] \\ = \inf_{q \in \mathcal{Q}_{\text{static}}^{t, q_t}} H^{\dot{q}}(t, S_t, \mathbf{q}_t). \end{aligned}$$

This concludes the proof. \square

Remark 2.4. A simple case where the optimal trading strategy is non-static is discussed by Gatheral and Schied in [GS11]. This means that the optimal inventory trajectory obtained from the stochastic control problem in [GS11] is in the space $\mathcal{Q}_{\text{fuel}}^{0, q_0} \setminus \mathcal{Q}_{\text{static}}^{0, q_0}$. In view of Proposition 2.3, we understand the dynamism of their solution by noticing the following. The risk measure adopted by those authors (see [GS11, Section 2.1]) is the value at risk of the position $q_t S_t$, and this has the consequence of disrupting the assumption that the lagrangian F can be decomposed as in equation (2.8). Indeed, Gatheral and Schied consider the optimisation

$$\inf \left\{ \mathbb{E} \left[\int_0^T (\dot{q}_t^2 + \lambda q_t S_t) dt \right] : q \in \mathcal{Q}_{\text{fuel}}^{0, q_0} \right\}, \quad (2.10)$$

where the price process $S_t = \exp(\sigma W_t - \sigma^2 t/2)$ is the exponential martingale of σW , where W denotes the standard one-dimensional Brownian motion, and where λ and σ are positive coefficients. Equation (2.10)

is [GS11, Equation (2.7)]. Alternatively, it can be noticed that the same minimisation as in equation (2.10) is produced by choosing $F(t, X, r) = rS + r^2$ and $dS = \lambda S dt + \sigma S dW$. Indeed, the expected cost

$$\mathbb{E} \left[\int_0^T (\dot{q}_t^2 + \dot{q}_t S_t) dt \right], \quad (2.11)$$

$$\text{with } dS_t = \lambda S_t dt + \sigma S_t dW_t,$$

differs from the expected cost in equation (2.10) (where the price process is the exponential martingale) only by a constant. With the modelling choices in equation (2.11), the lagrangian F does not incorporate any risk criterion and thus it satisfies the assumptions of Proposition 2.3, but the price process S has a position dependent drift coefficient, violating the assumption that A in equation (2.9) is deterministic.

Remark 2.5. In view of Proposition 2.3, we understand why incorporating signals (i.e. short-term price predictors) in the framework of optimal trade execution leads to dynamic optimal strategies (cf. [CJ16], [LN19]). Indeed, signals are incorporated by modelling the price evolution as

$$dS_t = I_t dt + \sigma(t, S_t) dW_t,$$

where I_t is a Markov process that represents the signal. The stochasticity of I disrupts the assumption on the drift A in Proposition 2.3.

Corollary 2.6. *Assume the setting of Proposition 2.3. Assume that the price process is modelled as the diffusion*

$$dS_t = \mu(t) dt + \sigma(t, S_t) dW_t, \quad (2.12)$$

for some measurable Lipschitz coefficients μ and σ with linear growth. The drift coefficient μ is taken to be a deterministic function of time only. Assume that

1. *for all t the map $(q, r) \mapsto -\mu(t)q + L(t, q, r)$ is strictly convex;*
2. *there exist exponents $p > m \geq 1$ and coefficients $\alpha_1 > 0$, $\alpha_2, \alpha_3 \in \mathbb{R}$ such that*

$$-\mu(t)q + L(t, q, r) \geq \alpha_1 |r|^p + \alpha_2 |q|^m + \alpha_3,$$

for all t , q and r .

Then, the infimum in equation (2.7) is attained for some optimal deterministic q in $\mathcal{Q}_{static}^{0, q_0} \cap W^{1,p}[0, T]$, where $W^{1,p}[0, T]$ denotes the Sobolev space of absolutely continuous function such that their p -th power and the p -th power of their derivative are integrable on the time interval $[0, T]$.

Remark 2.7. The assumptions on F in Corollary 2.6 are satisfied in particular by the classical choice

$$F(t, S, q, r) = rS + c_1^2 r^2 + c_2^2 (q - q_T)^2, \quad (2.13)$$

where $c_1 > 0$ is a coefficient of temporary market impact and $c_2 \geq 0$ is a coefficient of risk aversion (or of inventory cost). Therefore, Corollary 2.6 explains why in [CJP15, Section 6.3] the optimal solution is sought dynamic and eventually found to be static. This also says that, although in [AC00] the optimal trade execution was sought only over the class \mathcal{Q}_{static} for tractability, this was in fact without loss of generality.

Proof of Corollary 2.6. The fact that the infimum over $\mathcal{Q}_{fuel}^{0, q_0}$ is actually the same as the infimum over $\mathcal{Q}_{static}^{0, q_0}$ follows from Proposition 2.3. Existence, uniqueness and p -integrability of the minimiser follow from the two assumptions on the function $(t, q, r) \mapsto -\mu(t)q + L(t, q, r)$; see [Dac08, Theorem 4.1]. \square

Linear permanent market impact does not affect minimisation

In this article, we will be focusing on temporary market impact. It is here shown that incorporating linear permanent market impact does not modify the minimisation problem arising from several models.

Let F be a lagrangian that does not account for permanent market impact but only for temporary one. Assume that it can be written in the form

$$F(t, X, r) = r\hat{S} + \tilde{F}(t, q, r), \quad (2.14)$$

where $\hat{S} = g(S, r)$ represents the temporarily impacted price. Incorporating linear permanent market impact would entail to replace \hat{S} with

$$\hat{S} + c_4^2(q - q_\circ),$$

for some coefficient $c_4 \geq 0$ of permanent market impact. This says that at time t the price is also impacted proportionally to the amount³ $q_t - q_\circ$ executed in the time interval $[0, t]$. The lagrangian would then read

$$G(t, X, r) = r\left[\hat{S} + c_4^2(q - q_\circ)\right] + \tilde{F}(t, q, r). \quad (2.15)$$

Proposition 2.8. *Let q and η be in $\mathcal{Q}_{\text{fuel}}^{0, q_\circ}$. Then,*

$$\begin{aligned} \int_0^T G(t, X_t^\eta, \dot{\eta}_t) dt - \int_0^T G(t, X_t^q, \dot{q}_t) dt \\ = \int_0^T F(t, X_t^\eta, \dot{\eta}_t) dt - \int_0^T F(t, X_t^q, \dot{q}_t) dt, \end{aligned}$$

where the lagrangian F is as in equation (2.14) and the lagrangian G is as in equation (2.15).

Remark 2.9. Proposition 2.8 implies that if q is a minimiser for (2.7) with the lagrangian F given as in equation (2.14), then it is also a minimiser for the problem

$$\inf \left\{ \mathbb{E}_{t=0, S_0, q_\circ} \left[\int_0^T G(t, X_t^\eta, \dot{\eta}_t) dt \right] : \eta \in \mathcal{Q}_{\text{fuel}}^{0, q_\circ} \right\}.$$

with lagrangian G given as in equation (2.15). Notice however that the statement in Proposition 2.8 is pathwise.

Proof of Proposition 2.8. Let q be as in the statement and let e be absolutely continuous over $[0, T]$ with $e(0) = e(T) = 0$. The general η in $\mathcal{Q}_{\text{fuel}}^{0, q_\circ}$ is written as $\eta = q + e$. Then, it suffices to notice that

$$\begin{aligned} \int_0^T [(q - q_\circ + e)(\dot{q} + \dot{e}) - (q - q_\circ)\dot{q}] dt \\ = \int_0^T [(q - q_\circ)\dot{e} + e(\dot{q} + \dot{e})] dt \\ = - \int_0^T e\dot{q} dt + \int_0^T e(\dot{q} + \dot{e}) dt \\ = 0. \end{aligned}$$

□

³Such an amount is negative in the case of liquidation, so that the impacted price is lower than the fundamental price; whereas $q_t - q_\circ$ is positive in the case of acquisition so that the impacted price is higher. As for the rest of the paper our perspective is that of liquidation.

2.2 Errors of liquidation

The space $\mathcal{Q}_{\text{fuel}}^{0, \mathbf{q}_0}$ of fuel-constrained admissible inventory trajectories has been isolated from the space $\mathcal{Q}^{0, \mathbf{q}_0}$ of first integrals of execution rates. An inventory trajectory q in $\mathcal{Q}^{0, \mathbf{q}_0} \setminus \mathcal{Q}_{\text{fuel}}^{0, \mathbf{q}_0}$ is said to commit a liquidation error, because with positive probability $q_T \neq \mathbf{q}_T$. Liquidation errors are common among dynamic solutions to optimal trade execution problems. This is because the mathematical techniques used for dynamic solutions are not well-suited to simultaneously impose the two constraints $q_0 = \mathbf{q}_0$ and $q_T = \mathbf{q}_T$. Clearly, the constraint $q_0 = \mathbf{q}_0$ has the priority and hence the constraint $q_T = \mathbf{q}_T$ is relaxed. The usual relaxation entails to introduce a terminal penalisation for the outstanding inventory at final time. Hence, if F is the lagrangian describing risk-adjusted cost of trade, it is custom to relax the minimisation in equation (2.7) and consider instead the problem

$$\inf \left\{ \mathbb{E} \left[\int_0^T F(t, X_t^{\dot{q}}, \dot{q}_t) dt + c_5^2 (q_T - \mathbf{q}_T)^2 \right] : q \in \mathcal{Q}^{0, \mathbf{q}_0} \right\}, \quad (2.16)$$

where $c_5 \geq 0$ is a coefficient of penalisation for outstanding terminal inventory. Notice that the minimisation is performed over the broad class $\mathcal{Q}^{0, \mathbf{q}_0}$ of first integrals of execution rates. Notice also that the objective function in equation (2.16) can be expressed in the general form discussed so far because

$$\begin{aligned} \mathbb{E} \left[\int_0^T F(t, X_t^{\dot{q}}, \dot{q}_t) dt + c_5^2 (q_T - \mathbf{q}_T)^2 \right] \\ = \mathbb{E} \left[\int_0^T G(t, X_t^{\dot{q}}, \dot{q}_t) dt \right], \end{aligned}$$

where $G(t, X_t^{\dot{q}}, \dot{q}_t) = F(t, X_t^{\dot{q}}, \dot{q}_t) + 2c_5^2 \dot{q}_t q_t$.

In the case of linear temporary market impact, the paper by Belak and Muhle-Karbe and Ou [BMKO18] is the one in the literature that achieves the greatest generality with respect to the distributional assumptions on the price process S . Indeed, the price process is only assumed to be a totally square integrable special semimartingale. In particular, this weakens the Markovianity assumption in Lehalle and Neumann [LN19], which already generalised the classical setting of Almgren and Chriss [AC00] (price process as arithmetic Brownian motion) and of Gatheral and Schied [GS11] (price process as geometric Brownian motion). The motivation for the increased generality in Belak and Muhle-Karbe and Ou [BMKO18] is provided by the problem of optimal liquidation in target zone models; these are models in which asset prices have reflecting boundaries enforced by regulatory interventions. The drift of the price process therefore describes such enforcements and typically disrupt the Markovianity. In Lehalle and Neumann [LN19] instead, the increased generality was motivated by the willingness to incorporate price signals (i.e. short-term price predictors) in the framework of trade execution. In this paragraph we refer to Belak and Muhle-Karbe and Ou [BMKO18] to retain the greatest generality, but it remains understood that the scope of such a generality is not limited to handle the liquidation in target zone models: in particular, price processes that incorporate signals can be thought of as exemplifications of the discussion to follow.

The minimisation problem in Belak and Muhle-Karbe and Ou [BMKO18] is

$$\inf \left\{ \mathbb{E} \left[\int_0^T F(t, S_t, q_t, \dot{q}_t) dt + c_5^2 (q_T - \mathbf{q}_T)^2 - q_T S_T \right] : q \in \mathcal{Q}^{0, \mathbf{q}_0} \right\}, \quad (2.17)$$

where the lagrangian F is $F(t, S_t, q_t, \dot{q}_t) = \dot{q}_t S_t + c_1^2 \dot{q}_t^2 + c_2^2 q_t^2$, namely the one considered in equation (2.13). Equation (2.17) is [BMKO18, Equation (2.1)]. Such an optimisation is the same considered in [LN19, Equation (3.3)], but in [BMKO18, Equation (2.1)] the price process is only assumed to be a

totally square integrable special semimartingale, whereas in [LN19, Equation (3.3)] the price process is in addition assumed to be a continuous Markov process. Inside the expectation in equation (2.17), the first summand represents risk-adjusted revenues from trade along the inventory trajectory, the second summand represents the penalisation of liquidation error and the third summand represents the terminal asset position, i.e. the value at time T of the portfolio holding q_T units of the asset with price S_T . The first summand is the primal descriptor of the liquidation problem, whereas the second and the third summands are adjustments that account for the relaxation of the fuel-constraint. For all q in $\mathcal{Q}_{\text{fuel}}^{0, q_0}$ the objective function reduces to the objective function in Remark 2.7, to which Corollary 2.6 applies.

Let c_3 be the ratio $c_3 = c_2/c_1$ between the coefficient c_2 of risk aversion and the coefficient c_1 of linear temporary market impact. Let c_6 be the ratio $c_6 = c_5/c_1$ between the coefficient c_5 of penalisation of outstanding inventory at time T and the coefficient c_1 of linear temporary market impact. Define the functions φ and Φ as follows:

$$\begin{aligned}\varphi(t) &:= c_3 \cosh(c_3 t) + c_6^2 \sinh(c_3 t), & t \geq 0, \\ \Phi(s, t) &:= \frac{\varphi(T-t)}{\varphi(T-s)}, & 0 \leq s \leq t \leq T.\end{aligned}\tag{2.18}$$

Let $v(t)$ be the following conditional expectation at time t :

$$\begin{aligned}v(t) &:= \mathbb{E}_t \left[\frac{1}{2c_1^2} \int_t^T \Phi(t, r) dS_r \right] \\ &= \mathbb{E}_t \left[\frac{1}{2c_1^2} \int_t^T \Phi(t, r) dA_r \right],\end{aligned}\tag{2.19}$$

where S is the price process and it is assumed to be a totally square integrable special semimartingale with canonical decomposition $S = A + M$. Then, [BMKO18, Theorem 3.1] proves that the optimal inventory trajectory that solves the minimisation problem in equation (2.17) is

$$q_t = \Phi(0, t) q_0 + \int_0^t \Phi(s, t) v(s) ds.\tag{2.20}$$

Remark 2.10. In the spirit of Proposition 2.3, we remark that the solution q to the minimisation problem in equation (2.17) is static if the drift of the price process is deterministic, in particular if the price process is a martingale.

The function v defined in equation (2.19) can be rewritten as

$$\begin{aligned}v(t) &= \frac{1}{2c_1^2} \Phi(t, T) \mathbb{E}_t[S_T] - S_t \\ &\quad - \frac{1}{2c_1^2} c_3 \int_t^T \mathbb{E}_t[S_r] \partial_r \Phi(t, r) dr.\end{aligned}\tag{2.21}$$

This expression only utilises integration in time and highlights the fact that at time t the function $v(t)$ is a linear function of the current price S_t and of the expected trajectory $r \mapsto \mathbb{E}_t[S_r]$. Consider this in relation to the discussion in equations (2.3) and (2.4). By using equation (2.21) inside equation (2.20), we can rephrase the solution to the minimisation (2.17) as

$$\begin{aligned}q_t &= \Phi(0, t) q_0 + \frac{1}{2c_1^2} \int_0^t \Phi(s, t) \left\{ \Phi(s, T) \mathbb{E}_s[S_T] - S_s \right\} ds \\ &\quad - \frac{1}{2c_1^2} \int_0^t \Phi(s, t) \int_s^T \mathbb{E}_s[S_r] \partial_r \Phi(s, r) dr ds,\end{aligned}\tag{2.22}$$

where indeed we see the abstract structure of equation (2.3). Notice in particular that the dependence on $\{\text{Law}_t(S_r) : r \geq t\}$ happens to be only through the marginal means $\{\mathbb{E}_t[S_r] : r \geq t\}$.

Here we are interested in pointing out an aspect of the solution in equation (2.20) which we call bias of the liquidation error. We say that the inventory trajectory q in $\mathcal{Q}^{0, \mathbf{q}_0}$ has a biased liquidation error if $\mathbb{E}q_T \neq \mathbf{q}_T$, i.e. if the expected terminal value of the trajectory is not the liquidation target $\mathbf{q}_T = 0$. By extension, an inventory trajectory with biased liquidation error is also referred to as biased inventory trajectory.

The liquidation error of the inventory trajectory in equation (2.20) is not biased only if

$$\frac{\varphi(0)}{\varphi(T)} \mathbf{q}_0 = \frac{1}{2c_1^2} \int_0^T \Phi(t, T) \mathbb{E} \left[\int_t^T \Phi(t, r) dA_r \right] dt.$$

This condition is not robust with respect to the choice of the coefficients of market impact, risk aversion, and terminal penalisation of outstanding inventory. In other words, the optimal inventory trajectory of equation (2.20) is in general biased. In particular, in the case in which the price process is a martingale, the optimal inventory trajectory of equation (2.20) is such that the terminal value is

$$q_T = \frac{c_3 \mathbf{q}_0}{c_3 \cosh(c_3 T) + c_6^2 \sinh(c_3 T)}.$$

In this case then, the optimal inventory trajectory will always finish with a positive inventory left to liquidate after the initially fixed time horizon T of the liquidation.

A robustly unbiased inventory trajectory is obtained from equation (2.20) only in the limit as $c_5 \uparrow \infty$, which yields the fuel-constrained solution

$$\begin{aligned} q_t = & R(0, t) \mathbf{q}_0 - \frac{1}{2c_1^2} \int_0^t R(s, t) S_s ds \\ & - \frac{1}{2c_1^2} \int_0^t R(s, t) \int_s^T \mathbb{E}_s[S_r] \partial_r R(s, r) dr ds, \end{aligned}$$

where $R(s, t) = \sinh(c_3(T - t)) / \sinh(c_3(T - s))$. This says that the inventory trajectory in equation (2.20) has unbiased liquidation error only in the degenerate case of deterministic terminal inventory.

In Section 3 we will propose an alternative to the inventory trajectory in equation (2.20). This proposed alternative will not be fuel-constrained and yet it will have unbiased liquidation error. The combination of these two features will allow us to connect the design of the inventory trajectory to the choice of the liquidation horizon: in particular, our proposed inventory trajectory will unbiasedly terminate the liquidation sooner if the market condition are favourable, whereas it will delay part of the execution after the initially chosen horizon in the case of unfavourable market conditions. The termination time of the liquidation will be the random time

$$\tau = \inf \{t \geq 0 : q_t = \mathbf{q}_T\}. \quad (2.23)$$

Exemplifications of this will be provided in Section 3.3.

2.3 Model classes with linear temporary market impact

In this paragraph, the models of optimal trade execution with linear temporary market impact are organised in classes. These classes will depend on the constraint imposed on terminal inventory and on the assumptions on the price process. We will discuss the reciprocal positions of the model classes with regards to the following three properties: marginal robustness, non-static inventories, and unbiasedness.

None of the models will simultaneously enjoy all these three properties, motivating our proposal in Section 3.

Consider the following minimisation problem

$$\inf \left\{ \mathbb{E} \left[\int_0^T (\dot{q}_t S_t + c_1^2 \dot{q}_t^2 + c_2^2 (q_t - q_T)^2) dt \right] : q \in \mathcal{Q}_{\text{fuel}}^{0, q_0} \right\}. \quad (2.24)$$

This is the minimisation problem of equation (2.7) with the lagrangian F as in equation (2.13). In equation (2.24) the structural formulation of the trade execution problem comprises the following three choices: 1. linear temporary market impact; 2. risk-adjustment expressed in terms of inventory cost $c_2^2 \int_0^T (q_t - q_T)^2 dt$; 3. fuel-constraint of the inventory trajectories. Once the dynamics for the price process is specified, the model is fully determined. For every choice of the dynamics of the price process we have a model and all such models will be grouped in the class denoted with the symbol **LTIF**. The symbol **LTIF** stands for “Linear Temporary Market Impact Fuel-constraint”; we omit choice number 2 about the risk-adjustment because, until further notice (i.e. Section 4), it will always be the same. If S is a fixed dynamics (for example, arithmetical Brownian motion), then we write **LTIF**(S) to refer to the problem formulation in equation (2.24) where the price process is assumed to follow the dynamics S (in the example, the arithmetic Brownian motion).

From the model class **LTIF**, we isolate the following subclass

$$\text{mart-LTIF} := \{\text{LTIF}(S) : S \text{ is a martingale}\},$$

namely the subclass of all minimisation problem of equation (2.24) where the price process is assumed to be a martingale. In particular, the classical Almgren and Chriss model is an instance in **mart-LTIF**, where the price process is assumed to be an arithmetic Brownian motion.

In the model class **mart-LTIF**, the optimal solution is static – see Proposition 2.3. This clashes with the intuition that a liquidator should be able to better execute her trade if she adjusts her strategy according to the market conditions during the liquidation. Such an intuition is not captured by models in **mart-LTIF**. Instead, optimal inventory trajectories are not reduced to be static in the case of semimartingales with random drift.

In the works by Cartea and Jaimungal [CJ16], by Lehalle and Neuman [LN19] and by Belak and Muhle-Karbe and Ou [BMKO18], dynamic solutions to the optimal execution problem with linear temporary market impact are obtained. As explained in the previous paragraph, this is done in particular by relaxing the fuel constraint. Hence, the structural formulation of the trade execution problem is the one in equation (2.17), which we denote with the symbol **LTIP**. The letter **P** stands for terminal Penalisation of outstanding inventory. The fuel-constrained model class **LTIF** is retrieved from **LTIP** in the limit as the coefficient c_5 of penalisation of terminal inventory goes to ∞ . Belak and Muhle-Karbe and Ou [BMKO18] consider the models **LTIP**(S), where S is a totally square integrable canonical semimartingale; we denote this model class with the symbol **semimart-LTIP**. When the fuel-constraint is imposed, we use the symbol **semimart-LTIF**. Similarly, Lehalle and Neuman [LN19] consider the models **LTIP**(S), where S is a totally square integrable special semimartingale and a continuous Markov process. In the case of terminal penalisation of outstanding inventory, we denote this class with the symbol **markov-LTIP**; in the case of fuel-constraint, we denote this class with **markov-LTIF**. Notice that **markov-LTIP** is a subclass of **semimart-LTIP**, and **markov-LTIF** is a subclass of **semimart-LTIF**.

We use the symbol **LTI** in order to denote a model class of trade execution problems with linear temporary market impact where however we do not specify if there are restrictions to the price process and we do not specify whether the inventory trajectories are fuel-constrained (as in **LTIF**), or terminally penalised (as in **LTIP**), or otherwise handled at time horizon T .

With specific reference to trade execution problems with linear temporary market impact, we re-express the concept of marginal robustness introduced in Definition 2.1 as follows.

Definition 2.11. Let LTI be a class of trade execution problems with linear temporary market impact. Let x be the actual price path observed during the liquidation period. We say that the model class LTI is marginally robust if for every pair (S, \tilde{S}) of price processes admissible in LTI and such that

$$\mathbb{E}[S_t] = \mathbb{E}[\tilde{S}_t], \quad 0 \leq t \leq T,$$

the following holds:

$$q_t^x = \tilde{q}_t^x, \quad 0 \leq t \leq T,$$

where q^x is the implemented inventory derived from the optimal inventory in the model LTI(S), and \tilde{q}^x is the implemented inventory derived from the optimal inventory in the model LTI(\tilde{S}).

The model class semimart-LTIP is sufficiently general to avoid the reduction to optimal static inventory trajectory. This is true for all price processes that are totally square integrable canonical semimartingales with random drift. The subclass of semimart-LTIP where only deterministic drifts are considered is

$$\text{det-LTIP} := \{\text{semimart-LTIP}(S) : S = A + M, \text{ and } A \in \mathfrak{F}_0\}.$$

In particular, martingale dynamics fall in the model class det-LTIP. We let random-LTIP := semimart-LTIP \setminus det-LTIP be the collection of models in semimart-LTIP with random drift. Similarly, we let random-LTIF := semimart-LTIF \setminus det-LTIF be the collection of fuel-constrained models in semimart-LTIF with random drift.

In det-LTIP the optimal inventory reduces to be static, and the dependence on the probability law of the price process S is only through the first moment of the marginals of S . Hence, we have that the model class det-LTIP is marginally robust, in the sense of Definition 2.11. Indeed, from equation (2.21) we see that in the case of deterministic drift the optimal solution to the optimisation in (2.17) is

$$\begin{aligned} q_t = & \Phi(0, t) \mathbf{q}_0 + \frac{1}{2c_1^2} \int_0^t \Phi(s, t) \Phi(s, T) \left(\mathbb{E}[S_T] - \mathbb{E}[S_s] \right) ds \\ & - \frac{1}{2c_1^2} \int_0^t \Phi(s, t) \left[\int_s^T \mathbb{E}[S_r] \partial_r \Phi(s, r) dr - \mathbb{E}[S_s] (\Phi(s, T) - 1) \right] ds. \end{aligned}$$

In particular, mart-LTIF is marginally robust. However, in general, the model class semimart-LTIP is not marginally robust as the following example shows.

Example 2.12 (“semimart-LTIP is not marginally robust”). Let μ be a real number and let $H > 0$ be a positive real number. Let $0 < \sigma < 1$. Define the function $u = u(s, t, H)$ as follows

$$u(s, t, H) := \exp \left\{ \left(\mu - \frac{\sigma^2}{2} \right) (t - s) + \frac{\sigma^2}{2} \frac{t^{2H} - s^{2H}}{t^{2H-1}} \right\}, \quad 0 \leq s < t. \quad (2.25)$$

Let W be a standard one-dimensional Brownian motion. Define the process $\zeta(t, H)$ as follows

$$\zeta(t, H) := \int_0^t \frac{s^{H-\frac{1}{2}}}{t^{H-\frac{1}{2}}} dW_s.$$

Let S be the geometric Brownian motion

$$S_t = s_0 \exp \left\{ \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right\}.$$

Let \tilde{S} be the semimartingale

$$\tilde{S}_t = s_0 \exp \left\{ \left(\mu - \frac{\sigma^2}{2} \right) t + \zeta(t, \frac{1}{2\sigma^2}) \right\}.$$

For $0 \leq s < t$ it holds

$$\mathbb{E}_s[S_t] = S_s \cdot u(s, t, \frac{1}{2}) = S_s \exp\{\mu(t-s)\}, \quad (2.26)$$

and

$$\mathbb{E}_s[\tilde{S}_t] = \tilde{S}_s \cdot u(s, t, \frac{1}{2\sigma^2}), \quad (2.27)$$

where u is as in equation (2.25). Hence, S and \tilde{S} are marginally identical, because for all $t > 0$ it holds

$$\mathbb{E}_{s=0}[S_t] = \mathbb{E}_{s=0}[\tilde{S}_t].$$

However, for strictly positive $s > 0$, the conditional expectations $\mathbb{E}_s[S_t]$ and $\mathbb{E}_s[\tilde{S}_t]$ are different.

Let q be the optimal inventory trajectory in the model semimart-LTIP(S) and let \tilde{q} be the optimal inventory trajectory in the model semimart-LTIP(\tilde{S}). Let x be the actual realisation of the price path during the liquidation period. Deploying the liquidation strategy q would entail to follow the inventory trajectory

$$\begin{aligned} q_t^x = & \Phi(0, t)q_0 + \frac{1}{2c_1^2} \int_0^t \Phi(s, t) \left\{ \Phi(s, T)x_s \cdot e^{\mu(T-s)} - x_s \right\} ds \\ & - \frac{1}{2c_1^2} \int_0^t \Phi(s, t)x_s \int_s^T e^{\mu(r-s)} \partial_r \Phi(s, r) dr ds, \end{aligned}$$

where Φ is as in equation (2.18). This is the implemented inventory derived from formula (2.22) with the conditional expectation of the price process as in equation (2.26). Instead, deploying the liquidation strategy \tilde{q} would entail to follow the inventory trajectory

$$\begin{aligned} \tilde{q}_t^x = & \Phi(0, t)q_0 + \frac{1}{2c_1^2} \int_0^t \Phi(s, t) \left\{ \Phi(s, T)x_s \cdot u(s, T, \frac{1}{2\sigma^2}) - x_s \right\} ds \\ & - \frac{1}{2c_1^2} \int_0^t \Phi(s, t)x_s \int_s^T u(s, r, \frac{1}{2\sigma^2}) \partial_r \Phi(s, r) dr ds, \end{aligned}$$

where Φ is as in equation (2.18). This is the implemented inventory derived from formula (2.22) with the conditional expectation of the price process as in equation (2.27).

Therefore $q_t^x \neq \tilde{q}_t^x$, despite the fact that the distributional assumptions on the price processes in semimart-LTIP(S) and semimart-LTIP(\tilde{S}) are such that $\mathbb{E}S_t = \mathbb{E}\tilde{S}_t$ for every t .

Figure 2 summarises how the model classes are positioned with regards to the three properties examined: marginal robustness, non-static inventories, and unbiasedness of liquidation errors. We observe the following two incompatibilities: deterministic drifts are incompatible with non-static solutions, and unbiased trajectories cannot but be fuel-constrained. In particular, the intersection of all three properties is empty. Filling this intersection is one of the motivations behind our proposal in the next sections. In particular, in Section 3 we propose a problem formulation for trade execution with linear temporary market impact and inventory costs that produces solutions that are simultaneously marginally robust, non-static and unbiased. This will be the case also for martingale price processes and for non-fuel-constrained trajectories.

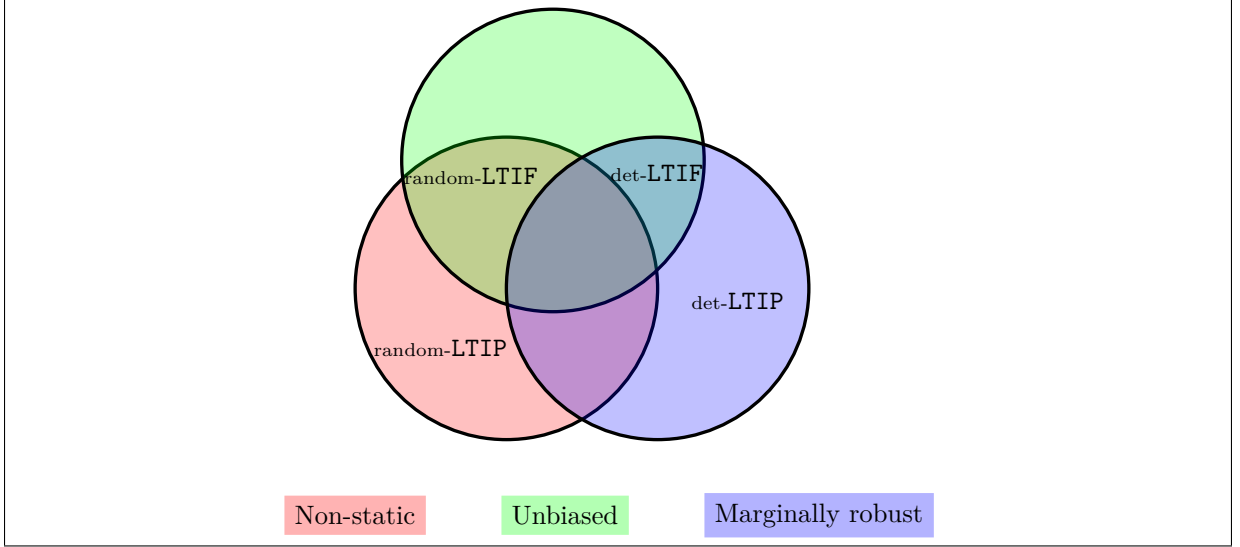


Figure 2: Model classes and their properties

3 Good trade executions

Recall that a price process is a stochastic process S on some probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ such that: 1. for all $0 \leq t \leq T$ the second moment of S_t is finite; 2. the maps $t \mapsto \mathbb{E}S_t$ and $t \mapsto \mathbb{E}S_t^2$ are in $L^1[0, T]$; 3. there exists some $p \geq 1$ such that all the paths of S are of finite p -variation. It is not necessarily assumed that S is continuous. Let $\{\mathfrak{F}_t : 0 \leq t \leq T\}$ be the minimal \mathbb{P} -completed right-continuous filtration generated by S and introduce the class \mathcal{R} of inventory rates and the class \mathcal{Q}^{0, q_0} of admissible inventory trajectories as in equations (2.1) and (2.2) respectively. From the class \mathcal{Q}^{0, q_0} of admissible inventory trajectories we isolate the class of unbiased admissible inventory trajectories. An unbiased admissible inventory trajectory is defined as an $(\mathfrak{F}_t)_t$ -adapted process with absolutely continuous paths, with deterministic initial value q_0 , expected terminal value q_T , and such that its derivative is in \mathcal{R} . More precisely, we define the space \mathcal{U}^{0, q_0} of unbiased admissible inventory trajectories as

$$\mathcal{U}^{0, q_0} := \left\{ q_t = q_0 + \int_0^t r_u du : r \in \mathcal{R} \text{ and } \mathbb{E}[q_T] = q_T \right\}.$$

The constraint $\mathbb{E}[q_T] = q_T$ relaxes the fuel constraint $q_T = q_T$ used in the definition of $\mathcal{Q}_{\text{fuel}}^{0, q_0}$. Without loss of generality the liquidation target q_T can be taken equal to 0; however, we do not suppress it from our equations because this makes the formulas easier to interpret.

We consider linear temporary market impact (already introduced in Section 2.1), which means that the revenues from trade are modelled as

$$\text{rev}(q) := - \int_0^T \left(S_t + c_1^2 \dot{q}_t \right) \dot{q}_t dt, \quad (3.1)$$

where $c_1 > 0$ is a coefficient of market impact. Notice that the execution price $S_t + c_1^2 \dot{q}_t$ and the infinitesimal revenue $-(S_t + c_1^2 \dot{q}_t) \dot{q}_t dt$ are respectively obtained as limit of

$$S_t + c_1^2 \frac{q_t - q_s}{t - s}$$

and

$$\left(S_t + c_1^2 \frac{q_t - q_s}{t - s} \right) [q_s - q_t]$$

as $s \uparrow t$.

In this section, we consider inventory cost as the risk-adjustment to the revenues in equation (3.1). This aligns to the modelling choices in Lehalle and Neuman [LN19] and in Belak and Muhle-Karbe and Ou [BMKO18] discussed in Section 2. Alternative risk criteria will be discussed in Section 4. Hence, we add to the functional \mathbf{rev} in equation (3.1) the penalisation

$$-c_2^2 \int_0^T (q_t - q_T)^2 dt, \quad (3.2)$$

which penalises trajectories that keep the inventory far from the liquidation target. Here, $c_2 \geq 0$ is a coefficient of risk aversion. Consequently, once adjusted for risk aversion, the optimisation of risk-adjusted revenues from trade offers the following minimisation problem:

$$\inf \left\{ \int_0^T F(t, S_t, q_t, \dot{q}_t) dt : q \in \mathcal{U}^{0, q_0} \right\}, \quad (3.3)$$

where the Lagrangian F is

$$F(t, S, q, r) := rS + c_1^2 r^2 + c_2^2 (q - q_T)^2. \quad (3.4)$$

We use the symbol J to denote the map $q \mapsto \int_0^T F(t, S_t, q_t, \dot{q}_t) dt$, for q in \mathcal{U}^{0, q_0} . Notice that the lagrangian F in equation (3.4) is the same as the lagrangian in equation (2.13). However, the optimisation in equation (3.3) is pathwise and hence it differs from the classical optimisation of expected risk-adjusted revenues used in equation (2.7). For this reason, the martingale cancellation exploited in the proof of Proposition 2.3 is not applicable to the present case: we will be able to produce a non-static solution also in the case where the price process has deterministic drift (in particular, where the price process is a martingale).

Compared to the modelling choices in [LN19] and in [BMKO18], our relaxation of the fuel constraint entails that the inventory is sought in \mathcal{U}^{0, q_0} ; hence we do not modify the objective function by introducing the terms of terminal asset position $q_T S_T$ and of terminal inventory cost $c_2^2 (q_T - q_T)^2$ that instead appeared in equation (2.17).

We remark that for each t the function $(q, r) \mapsto F(t, S_t, q, r)$ is convex. This is crucial for the approach that we are developing.

Referring to the coefficients c_1 and c_2 of market impact and of risk aversion, we define the following pathwise norm

$$|\eta|_{c_1, c_2}^2 := \int_0^T (c_2^2 \eta_t^2 + c_1^2 \dot{\eta}_t^2) dt, \quad (3.5)$$

for paths η in $W^{1,2}[0, T]$. Given an admissible inventory trajectory q , the $L^2(\mathbb{P})$ norm of the random variable $|q|_{c_1, c_2}$ defines the following norm on \mathcal{U}^{0, q_0} :

$$\|q\|_{c_1, c_2}^2 := \mathbb{E} [|q|_{c_1, c_2}^2]. \quad (3.6)$$

Lastly, for future reference, we define c_3 as the ratio of the coefficients of risk aversion and of market impact, namely $c_3 := c_2/c_1$.

3.1 Definition of the concept of good trade execution and statement of the solution

Every square-integrable random variable ξ with $\mathbb{E}[\xi] = q_T$ identifies a subclass of trajectories in \mathcal{U}^{0, q_0} with specified terminal (random) variable. More precisely, for every ξ in $L^2(\Omega, \mathfrak{F}, \mathbb{P})$ with $\mathbb{E}[\xi] = q_T$ we define

$$\mathcal{U}^{0, q_0}(\xi) := \{q \in \mathcal{U}^{0, q_0} : \mathbb{P}(q_T = \xi) = 1\}.$$

The class $\mathcal{U}^{0,q^\circ}(\xi)$ with $\xi \equiv \mathbf{q}_T$ is the class of inventory trajectories that commit no liquidation error, namely $\mathcal{U}^{0,q^\circ}(\mathbf{q}_T) = \mathcal{Q}_{\text{fuel}}^{0,q^\circ}$.

Definition 3.1 (“Optimal execution of terminal variable ξ ”). *Let ξ be in $L^2(\Omega, \mathfrak{F}, \mathbb{P})$ with $\mathbb{E}[\xi] = \mathbf{q}_T$. We say that q in \mathcal{U}^{0,q° is the optimal execution of terminal variable ξ if q minimises $\eta \mapsto J(\eta)$ on $\mathcal{U}^{0,q^\circ}(\xi)$, namely if $\mathbb{P}(q_T = \xi) = 1$ and for all $\eta \in \mathcal{U}^{0,q^\circ}(\xi)$ it holds*

$$\begin{aligned} \int_0^T F(t, S_t, \eta_t, \dot{\eta}_t) dt \\ \geq \int_0^T F(t, S_t, q_t, \dot{q}_t) dt \end{aligned}$$

with probability 1.

For every $q \in \mathcal{U}^{0,q^\circ}$ we trivially have that $q \in \mathcal{U}^{0,q^\circ}(q_T)$. We can define a “tubular” neighbourhood of $\mathcal{U}^{0,q^\circ}(q_T)$ by looking at those trajectories η in \mathcal{U}^{0,q° such that the difference $\eta_T - q_T$ of the terminal values is controlled by the distance between the processes η and q as measured by the norm $\|\cdot\|_{c_1, c_2}$. More precisely, for q in \mathcal{U}^{0,q° and $C \geq 0$ we set

$$\mathcal{U}^{0,q^\circ}(q, C) := \left\{ \eta \in \mathcal{U}^{0,q^\circ} : \|\eta_T - q_T\|_{L^2(\mathbb{P})} \leq C \|\eta - q\|_{c_1, c_2}^2 \right\}. \quad (3.7)$$

This captures the idea of η_T not being too far from the terminal value q_T given that the trajectory η has kept close to q on the time window $0 \leq t < T$.

Also, we define a pathwise analogous to the tubular neighbourhood introduced in equation (3.7). Given a non-negative ξ in $L^2(\mathbb{P})$ we define

$$\mathcal{U}_{\text{pw}}^{0,q^\circ}(q, \xi) := \left\{ \eta \in \mathcal{U}^{0,q^\circ} : |\eta_T - q_T| \leq \xi \|\eta - q\|_{c_1, c_2}^2 \right\}. \quad (3.8)$$

With the notation introduced so far, we will now formulate the concept of good trade execution. Later, in Proposition 3.4 we will give a closed-form formula for a good trade execution in the case of the minimisation in equation (3.3) with the Lagrangian F as in equation (3.4).

Definition 3.2 (“Good trade execution”). *We say that q in \mathcal{U}^{0,q° is a (C, ξ) -good trade execution for the minimisation in equation (3.3) if there exist $\xi \in L_+^2(\mathbb{P})$ and $C > 0$ such that*

1. *for all η in $\mathcal{U}^{0,q^\circ}(q, C)$ it holds*

$$\begin{aligned} \mathbb{E} \left[\int_0^T F(t, S_t, \eta_t, \dot{\eta}_t) dt \right] \\ \geq \mathbb{E} \left[\int_0^T F(t, S_t, q_t, \dot{q}_t) dt \right]; \end{aligned}$$

2. *for all η in $\mathcal{U}_{\text{pw}}^{0,q^\circ}(q, \xi)$ it holds*

$$\begin{aligned} \int_0^T F(t, S_t, \eta_t, \dot{\eta}_t) dt \\ \geq \int_0^T F(t, S_t, q_t, \dot{q}_t) dt, \end{aligned}$$

with probability one.

When we emphasise the path $t \mapsto q_t$ of a good trade execution, we use interchangeably the term *good inventory trajectory*.

Remark 3.3. A good trade execution is in particular an optimal execution of its own terminal variable: if q is as in Definition 3.2, then q is an optimal execution of terminal variable q_T as defined in Definition 3.1. In other words, a (C, ξ) -good trade execution is in particular a $(0, 0)$ -good trade execution.

Proposition 3.4 (“Closed-form formula for the good trade execution under quadratic inventory cost”). *Let α be the function $\alpha(t) = 1 - \sinh(c_3(T - t)) / \sinh(c_3T)$, and let K be the constant*

$$K = \frac{1}{2c_1^2 \sinh(c_3T)} \int_0^T \cosh(c_3(T - u)) \mathbb{E}[S_u] du.$$

Then, the inventory trajectory

$$\begin{aligned} q_t := & (1 - \alpha(t)) \mathbf{q}_0 + \alpha(t) \mathbf{q}_T \\ & - \frac{1}{2c_1^2} \int_0^t \cosh(c_3(t - u)) S_u du \\ & + K \sinh(c_3t), \end{aligned} \tag{3.9}$$

is a (C, ξ) -good trade execution for the minimisation in equation (3.3) with the lagrangian F given in equation (3.4). The constant C is explicitly given by the formula

$$C^{-1} = c_3 \int_0^T \sinh(c_3(T - u)) \text{Var}^{\frac{1}{2}}(S_u) du;$$

the random variable ξ is explicitly given by the formula

$$\begin{aligned} \xi^{-1} = & \left| 2c_1c_2 \frac{\mathbf{q}_T - \mathbf{q}_0}{\sinh(c_3T)} \right. \\ & - c_3 \int_0^T \sinh(c_3(T - t)) S_t dt \\ & \left. + c_3 \frac{\cosh(c_3T)}{\sinh(c_3T)} \int_0^T \cosh(c_3(T - t)) \mathbb{E}[S_t] dt \right|. \end{aligned}$$

Remark 3.5. The structure of the solution q in equation (3.9) is threefold: a time-dependent convex combination between initial inventory \mathbf{q}_0 and liquidation target \mathbf{q}_T appears on the first line; a dynamic response to the actual price trajectory appears on the second line; an adjustment for the terminal constraint $\mathbb{E}[q_T] = \mathbf{q}_T$ appears on the third line. In particular, the good trade execution in equation (3.9) is an instance of the formula in equation (2.5); hence, by Proposition 2.2 the good inventory trajectory of equation (3.9) is marginally robust.

If in the integral appearing on the second line of equation (3.9) we replace the fundamental price S_u with its expected value $\mathbb{E}[S_u]$, then the inventory trajectory q is turned into the optimal static one, i.e. into the minimiser of $\mathbb{E}[J(\eta)]$ over all η in $\mathcal{Q}_{\text{static}}^{0, \mathbf{q}_0}$.

Instead, if in the definition of the constant K appearing on the third line of equation (3.9) we replace $\mathbb{E}[S_u]$ with S_u , then the inventory trajectory q is turned into the optimal a-posteriori one, i.e. into the minimiser of $J(\eta)$ over all η in $\mathcal{Q}_{\text{pw}}^{0, \mathbf{q}_0}$.

Proof of Proposition 3.4. Let q be as in equation (3.9). The fact that q is in $\mathcal{U}^{0, \mathbf{q}_0}$ is apparent. Let $f_t := 2c_1^2 \dot{q}_t + S_t$ and notice that f is absolutely continuous with derivative

$$\dot{f}_t = 2c_1^2 (q_t - \mathbf{q}_T). \tag{3.10}$$

Let η be in $\mathcal{U}^{0, \mathfrak{q}_0}$. We write e for the difference $e := \eta - q$, and we observe that $e_0 = 0$ and $\mathbb{E}e_T = 0$. Then, we have

$$\begin{aligned} J(\eta) - J(q) &= \int_0^T \left[f_t \dot{e}_t + 2c_1^2(q_t - \mathfrak{q}_T)e_t \right] dt \\ &\quad + \int_0^T (c_1^2 \dot{e}^2 + c_2^2 e^2) dt. \end{aligned}$$

The second integral on the right hand side is $|e|_{c_1, c_2}^2$. Using integration by parts we see that in fact $J(\eta) - J(q) = f_T e_T + |e|_{c_1, c_2}^2$, because of equation (3.10). Therefore, the difference $J(\eta) - J(q)$ is non-negative if

$$|e_T| \leq \frac{|e|_{c_1, c_2}^2}{|f_T|}.$$

This gives $\xi = 1/|f_T|$.

Secondly, consider the expected difference $\mathbb{E}J(\eta) - \mathbb{E}J(q) = \mathbb{E}[f_T e_T] + \|e\|_{c_1, c_2}^2$. We can estimate

$$\mathbb{E}[f_T e_T] \leq \text{Var}^{\frac{1}{2}}(f_T) \|e_T\|_{L^2(\mathbb{P})},$$

because $\mathbb{E}e_T = 0$. Moreover,

$$\text{Var}^{\frac{1}{2}}(f_T) \leq c_3 \int_0^T \sinh(c_3(T-t)) \text{Var}^{\frac{1}{2}}(S_t) dt.$$

Therefore, the expected difference $\mathbb{E}J(\eta) - \mathbb{E}J(q)$ is non-negative if

$$\|e_T\|_{L^2(\mathbb{P})} \leq \frac{\|e\|_{c_1, c_2}^2}{c_3 \int_0^T \sinh(c_3(T-t)) \text{Var}^{\frac{1}{2}}(S_t) dt}.$$

This gives the constant C in the statement and concludes the proof. \square

Remark 3.6. We stress the fact that the good inventory trajectory of equation (3.9) is written without assuming a particular SDE dynamics for the price evolution. In particular, Proposition 3.4 applies to the case in which the price process is modelled as a fractional Brownian motion, or as the sum of a possibly discontinuous semimartingale and a fractional Brownian motion.

Remark 3.7. Of the good trade execution q in (3.9) we can compute

$$\begin{aligned} \|q_T - \mathfrak{q}_T\|_{L^2(\mathbb{P})} &= \frac{1}{2c_1^2} \mathbb{E}^{\frac{1}{2}} \left[\left(\int_0^T \cosh(c_3(T-t)) (S_t - \mathbb{E}[S_t]) dt \right)^2 \right] \end{aligned}$$

and estimate

$$\begin{aligned} \text{Var}(q_T) &\leq \frac{T}{4c_1^4} \int_0^T \cosh^2(c_3(T-t)) \text{Var}(S_t) dt. \end{aligned}$$

We therefore remark the following two facts:

1. The smaller $\int \text{Var}(S_t) dt$ is, the more precise the good execution q of Proposition 3.4 is.

2. The square of the coefficient c_1 of linear market impact is inversely proportional to the standard deviation of $q(T)$. This says that the precision with which the good execution q of equation (3.9) gets to its liquidation target \mathbf{q}_T increases when the strategy itself can exert more influence on the execution price.

Remark 3.8. Admissible inventory trajectories q have been absolutely continuous stochastic processes on $[0, T]$ such that $\mathbb{E}[q_T] = \mathbf{q}_T$. This has meant that the constant K in equation (3.9) has been chosen to minimise $\mathbb{E}[(q_T - \mathbf{q}_T)^2]$. We can give two alternatives to this minimisation:

1. Choose K in such a way to minimise

$$\mathbb{E} \left[\int_{t_0}^T (q_t - \mathbf{q}_T)^2 dt \right],$$

for some $0 \leq t_0 < T$. The symbol $\int_{t_0}^T dt$ stands for the mean $\frac{1}{T-t_0} \int_{t_0}^T dt$. This yields

$$K = \frac{1}{2c_1^2 \int_{t_0}^T \sinh^2(c_3 t) dt} \int_{t_0}^T \sinh(c_3 t) \psi(t) dt,$$

where $\psi(t) = \int_0^t \cosh(c_3(t-u)) \mathbb{E}[S_u] du - (1 - \alpha(t))(\mathbf{q}_0 - \mathbf{q}_T)$. Notice that this tends to the former choice when $t_0 \uparrow T$.

2. Choose K in such a way to minimise

$$\mathbb{E} \left[\left(\int_{t_0}^T q_t dt - \mathbf{q}_T \right)^2 \right],$$

for some $0 \leq t_0 < T$. This yields

$$K = \frac{c_3(T-t_0)}{2c_1^2 (\cosh(c_3 T) - \cosh(c_3 t_0))} \int_{t_0}^T \psi(t) dt,$$

with ψ as above.

Notice however that the alternative choices for the constant K make the corresponding inventory trajectory fall out of the set \mathcal{U} .

3.2 Euler-Lagrange equation and the characterisation of the good trade execution

The differential equation in (3.10) is the linchpin on which the derivation of the good inventory trajectory is based. This equation is the Euler-Lagrange equation associated with the functional J . Equation (3.10) is a random ordinary differential equation, where differentiation is possible because the price process is cancelled out in the sum $2c_1^2 \dot{q}_t + S_t$. Such a cancellation allows to circumvent the need of an integration with respect to the price process. However this integration is possible; Appendix A presents the theory of the Euler-Lagrange equation in the presence of a (rough) price path. The motivation for this theory comes from the fact that equation (3.10) can be rewritten as

$$\begin{cases} dq_t = r_t dt \\ dr_t = c_3^2 (q_t - \mathbf{q}_T) dt - dS_t / 2c_1^2. \end{cases} \quad (3.11)$$

We interpret the system in equation (3.11) as a random Young differential equation. This equation is useful in simulations because it avoids the computation of the integrals in equation (3.9) and the evaluation of hyperbolic functions. Moreover, since we use Young integration to integrate with respect to the price process S , the system in equation (3.11) has a pathwise meaning and thus it also makes sense in the practical implementation of the trading strategy, where the price process S is replaced by the single price path x observed during the liquidation (cf. Section 2.3).

In this paragraph, we apply the theory of Appendix A in order to characterise the good trade execution of Proposition 3.4 in terms of an initial value problem for the dynamics in (3.11). We start by applying Definition A.7 to the case of equation (3.11).

Definition 3.9. *Let q be in $\mathcal{U}^{0, \mathfrak{q}_0}$. We say that q solves equation (3.11) if for all ω in Ω , all η in $C_0^\infty(0, T)$ and all $0 \leq s \leq t \leq T$ the following holds:*

$$\begin{aligned} \int_s^t \dot{\eta}_u dq_u(\omega) &= \eta_t \dot{q}_t(\omega) - \eta_s \dot{q}_s(\omega) \\ &\quad - c_3^2 \int_s^t \eta_u (q_u(\omega) - \mathfrak{q}_T) du + \frac{1}{2c_1^2} \int_s^t \eta_u dS_u(\omega). \end{aligned} \quad (3.12)$$

We remark that Definition 3.9 is pathwise: a scenario ω in Ω could be fixed and the definition would still makes sense. The integral on the left hand side of equation (3.12) is well defined because for all ω in Ω the path $\dot{q}(\omega)$ is in $L^2[0, T]$. Similarly, the first integral on the right hand side has a pathwise meaning and it is well defined because $q(\omega)$ is in $L^2[0, T]$ for all ω in Ω . Thirdly, the integral $\int \eta dS$ on the right hand side of equation (3.12) is the Young integral introduced in Lemma A.6.

Lemma 3.10. *Let q and \tilde{q} be two solutions to equation (3.11). If $c_3 \neq 0$, then there exist constants K_1 and K_2 such that*

$$q_t - \tilde{q}_t = K_1 e^{c_3 t} + K_2 e^{-c_3 t};$$

if $c_3 = 0$, then there exist constants K_1 and K_2 such that

$$q_t - \tilde{q}_t = K_1 + K_2 t.$$

In particular, in both cases the difference between q and \tilde{q} is deterministic.

Proof. Let η be arbitrary in $C_0^\infty(0, T)$. By Definition 3.9 we have that

$$\begin{aligned} \int_0^t \dot{\eta}_u dq_u &= \eta_t \dot{q}_t - c_3^2 \int_0^t \eta_u (q_u - \mathfrak{q}_T) du + \frac{1}{2c_1^2} \int_0^t \eta_u dS_u; \\ \int_0^t \dot{\eta}_u d\tilde{q}_u &= \eta_t \dot{\tilde{q}}_t - c_3^2 \int_0^t \eta_u (\tilde{q}_u - \mathfrak{q}_T) du + \frac{1}{2c_1^2} \int_0^t \eta_u dS_u. \end{aligned}$$

Let $\epsilon_t := q_t - \tilde{q}_t$. Subtract one line from the other and obtain that the function

$$t \mapsto \int_0^t \dot{\eta}_u \epsilon_u du + c_3^2 \int_0^t \eta_u \epsilon_u du - \eta_t \dot{\epsilon}_t$$

is constantly null. The first two summands are differentiable in t and hence the third summand $\eta_t \dot{\epsilon}_t$ is differentiable too. Differentiating we obtain

$$\dot{\eta}_t \dot{\epsilon}_t + c_3^2 \eta_t \epsilon_t - \dot{\eta}_t \dot{\epsilon}_t - \eta_t \ddot{\epsilon}_t = 0.$$

Hence $\ddot{\epsilon}_t = c_3^2 \epsilon_t$, proving the lemma. \square

Lemma 3.11. *The solution q to equation (3.11) with constraint*

$$\begin{cases} q_0 = \mathfrak{q}_0 \\ \mathbb{E}q_T = \mathfrak{q}_T \end{cases} \quad (3.13)$$

is unique.

Proof. Let q and \tilde{q} be two solutions to equation (3.11) satisfying the constraints in (3.13). Assume $c_3 \neq 0$. Then, by Lemma 3.10 it must be

$$q_t - \tilde{q}_t = K_1 e^{c_3 t} + K_2 e^{-c_3 t},$$

for constants K_1 and K_2 . Therefore

$$\begin{cases} q_0 - \tilde{q}_0 = K_1 + K_2 = 0 \\ \mathbb{E}q_T - \mathbb{E}\tilde{q}_T = K_1 e^{c_3 T} + K_2 e^{-c_3 T} = 0 \end{cases}$$

Solving for K_1 and K_2 we find $K_1 = K_2 = 0$.

The case $c_3 = 0$ is analogous. □

Having established uniqueness of the solution to equation (3.11) with constraint (3.13), we link equation (3.11) to the good trade execution of Proposition 3.4. This link is established as an application of Lemma A.8 in the Appendix.

Lemma 3.12. *Let q be in $\mathcal{U}^{0, \mathfrak{q}_0}$. Then, the following are equivalent*

1. *the inventory trajectory q solves the Euler-Lagrange equation (3.11);*
2. *the function $f_t := 2c_1^2 \dot{q}_t + S_t$ is absolutely continuous with derivative*

$$\dot{f}_t = 2c_2^2 q_t.$$

In particular, the (C, ξ) -good trade execution in Proposition 3.4 solves the Euler-Lagrange equation (3.11).

We are finally in the position to prove the main result of this paragraph.

Proposition 3.13 (“Characterisation of good trade execution via Euler-Lagrange equation”). *The good trade execution for the minimisation of (3.3) with the Lagrangian F as in equation (3.4) is characterised as the solution to the Euler-Lagrange equation in (3.11) with initialisation*

$$\begin{cases} q_0 = \mathfrak{q}_0 \\ r_0 = -S_0/(2c_1^2) + c_3 \sinh^{-1}(c_3 T) \left[(\mathfrak{q}_T - \mathfrak{q}_0) \cosh(c_3 T) + \tilde{K} \right], \end{cases} \quad (3.14)$$

where

$$\begin{aligned} \tilde{K} = & \frac{\cosh(c_3 T)}{2c_1^2} \int_0^T \cosh(c_3 t) \mathbb{E}[S_t] dt \\ & - \frac{\sinh(c_3 T)}{2c_1^2} \int_0^T \sinh(c_3 t) \mathbb{E}[S_t] dt. \end{aligned}$$

In particular, the good trade execution in Proposition 3.4 is the only good trade execution for the minimisation of (3.3) with the Lagrangian F as in equation (3.4).

Remark 3.14. Proposition 3.4 gives a characterisation of the good trade execution in terms of an initial value problem that is easily simulated. This is the practical relevance of the characterisation. We will rely on the initial value problem (3.11) with initial conditions (3.14) in our numerical experiments in Section 3.3.

Proof. First we examine the following two implications.

1. *A good trade execution solves equation (3.11).* Let \tilde{q} be a good trade execution for the minimisation of (3.3) with the Lagrangian F as in equation (3.4). Then in particular, for every ω in Ω it holds

$$\tilde{q}(\omega) = \operatorname{argmin} \left\{ J(\eta) : \eta \in \tilde{q}(\omega) + W_0^{1,2}(0, T) \right\}.$$

Therefore, by Proposition A.9, we have that $\tilde{q}(\omega)$ solves the equation

$$\begin{cases} d\tilde{q}_t(\omega) = \tilde{r}_t(\omega)dt \\ d\tilde{r}_t(\omega) = c_3^2 \tilde{q}_t(\omega)dt - dS_t(\omega)/2c_1^2. \end{cases}$$

2. *A solution to equation (3.11) is a good trade execution.* Assume that q solves the Euler-Lagrange equation in (3.11). Then, by Lemma 3.12 we have that $f_t(\omega) := 2c_1^2 \dot{q}_t(\omega) + S_t(\omega)$ is absolutely continuous in t for all ω in Ω and its time derivative is $\dot{f}_t(\omega) = 2c_2^2 q_t(\omega)$. Hence, for all ω in Ω and all η in $q(\omega) + W_0^{1,2}(0, T)$ we have

$$\begin{aligned} J(\eta) - J(q(\omega)) &= \int_0^T (2c_2^2 q_t(\omega) - \dot{f}_t(\omega)) e_t dt \\ &\quad + \int_0^T (c_1^2 \dot{e}_t^2 + c_2^2 e_t^2) dt, \end{aligned}$$

where $e = \eta - q(\omega)$. The first summand on the right hand side is null and thus $J(\eta) \geq J(q(\omega))$. This shows that q is a $(0, 0)$ -good trade execution.

In view of these two implications and of Lemma 3.11, it only remains to show that the initialisations in equation (3.14) are equivalent to the constraints in (3.13).

Consider an equation of the form

$$dY_t = (AY_t + B)dt + DdS_t,$$

where the unknown Y is in \mathbb{R}^2 , the matrix A is in $\mathbb{R}^{2 \times 2}$ and B and D are two-dimensional real vectors. Equation (3.11) is of this form with the choice $Y = (q, r)^\top$ and

$$A = \begin{pmatrix} 0 & 1 \\ c_3^2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ -c_3^2 q_T \end{pmatrix} = 0, \quad D = \begin{pmatrix} 0 \\ -\frac{1}{2c_1^2} \end{pmatrix}.$$

Assume first that $t \mapsto \mathbb{E}[S_t]$ is differentiable. Then, if we set $\mu_t = \mathbb{E}[Y_t]$ we have that μ solves the ordinary differential equation

$$\dot{\mu}_t = A\mu_t + \varphi_t D, \tag{3.15}$$

where $\varphi_t = d\mathbb{E}[S_t]/dt$. This ordinary differential equation has a two-dimensional space of solutions; hence we can exploit these two degrees of freedom to adjust for the constraints $q_0 = \mu_0 = q_0$ and $\mu_T = q_T = 0$.

Define the function $e_1 = e_1(t)$ as

$$\begin{aligned} e_1(t) &= e_1(0) + \mathbb{E}[S_t] \sinh(c_3 t) / (2c_1 c_2) \\ &\quad - \frac{1}{2c_1^2} \int_0^t \cosh(c_3 u) \mathbb{E}[S_u] du. \end{aligned}$$

Define the function $e_2 = e_2(t)$ as

$$e_2(t) = e_2(0) + \left(S_0 - \cosh(c_3 t) \mathbb{E}[S_t] \right) / (2c_1 c_2) \\ + \frac{1}{2c_1^2} \int_0^t \sinh(c_3 u) \mathbb{E}[S_u] du.$$

The general solution to equation (3.15) is

$$\begin{cases} \mu_t^{(1)} = e_1(t) \cosh(c_3 t) + e_2(t) \sinh(c_3 t) \\ \mu_t^{(2)} = c_3 e_1(t) \sinh(c_3 t) + c_3 e_2(t) \cosh(c_3 t). \end{cases}$$

The constraints $q_0 = \mu_0 = \mathbf{q}_\circ$ and $\mu_T = \mathbf{q}_\tau = 0$ impose the choices

$$\begin{cases} e_1(0) = \mathbf{q}_\circ \\ e_2(0) = R + \sinh^{-1}(c_3 T) \left[\mathbf{q}_\tau - e_1(T) \cosh(c_3 T) \right], \end{cases}$$

where

$$R = \frac{\cosh(c_3 T) \mathbb{E}[S_T] - S_0}{2c_1 c_2} - \frac{1}{2c_1^2} \int_0^T \sinh(c_3 u) \mathbb{E}[S_u] du.$$

Hence, the constraints $q_0 = \mu_0 = \mathbf{q}_\circ$ and $\mu_T = \mathbf{q}_\tau = 0$ are translated into the initialisation in the statement.

The case where the map $t \mapsto \mathbb{E}[S_t]$ is not differentiable is handled via a standard approximation argument. \square

3.3 Applications

3.3.1 Intraday stock price: brownian motion

In this paragraph, we give an application of the trading schedule proposed in Section 3 to the case of liquidation of shares traded on a limit order book. As an example, we consider shares of Apple Inc. (AAPL) traded on the NASDAQ order book. In this example the price process is the arithmetic Brownian motion; this is the benchmark for optimal trade execution since the seminal paper by Almgren and Chriss [AC00]. However we would like to emphasise that the trading schedule proposed in Section 3 does not rely on the price process being an arithmetic Brownian motion, nor a semimartingale. More sophisticated models for the price signal can be used, in particular those encompassing rough trajectories.

The data to which our model will be calibrated are summarised in Table 1. This table reports statistics of the mid price of AAPL stock on the NASDAQ order book, as observed in the month of January 2019. For each trading day, the time window 10am-3pm is considered and the statistics on the price are based on observations in this time window. Following the conventions of the LOBSTER dataset,⁴ time is measured in seconds after midnight, and the unit of price is $1.0e - 04\$$. A visual representation of the mid price and of the best bid volumes observed on January 29th is provided by Figure 3.

Let W be a standard one-dimensional Brownian motion and let the price process S be $S_t = S_0 + \sigma W_t$, where S_0 is the mid price observed at 10am (reported in the second column of Table 1), and t denotes seconds after 10am. The volatility parameter σ is estimated according to the formula

$$[S]_t = \sigma^2 t,$$

where $[S]_t$ denotes the quadratic variation of the price process. The second last column in Table 1 reports this measurement of the volatility σ where the quadratic variation is estimated at 3pm. Finally, the last

⁴<https://lobsterdata.com/info/WhatIsLOBSTER.php>

date	initial.price	avg.price	std.price	max.price	min.price	volatility	bidVolume
2019-01-02	1.551528e+06	1.573641e+06	10609.885630	1588400	1545700	123.400320	886.197329
2019-01-03	1.442540e+06	1.438782e+06	8009.646338	1457150	1420850	145.819047	1188.751986
2019-01-04	1.453756e+06	1.471987e+06	7925.790647	1485350	1451850	109.428439	1291.683663
2019-01-07	1.472455e+06	1.472358e+06	6105.759311	1488250	1459100	127.799213	1139.538298
2019-01-08	1.490168e+06	1.500335e+06	8381.548706	1518150	1485350	117.70729	1075.23755
2019-01-10	1.513191e+06	1.526785e+06	5087.848872	1536500	1511150	107.649766	1046.712111
2019-01-11	1.530947e+06	1.522908e+06	4101.365889	1534950	1515150	90.781331	1317.150110
2019-01-14	1.500257e+06	1.499643e+06	2456.131227	1507950	1492200	86.306946	1347.534461
2019-01-15	1.515814e+06	1.523678e+06	5251.548539	1533850	1513200	95.682490	1164.402792
2019-01-16	1.552520e+06	1.549901e+06	2568.906492	1557500	1543300	86.815577	1206.761646
2019-01-17	1.537375e+06	1.545890e+06	9385.791843	1576450	1534600	113.691224	1193.393924
2019-01-18	1.564948e+06	1.571147e+06	2749.642228	1578750	1562200	101.692620	1017.907481
2019-01-22	1.553784e+06	1.547023e+06	6167.822659	1556750	1528750	103.605824	997.719024
2019-01-23	1.546358e+06	1.530565e+06	7178.892096	1549850	1517050	103.526701	1208.373604
2019-01-24	1.530342e+06	1.524633e+06	4399.450324	1536450	1517350	91.531719	1286.708610
2019-01-25	1.563176e+06	1.570534e+06	4102.321246	1581350	1562250	86.868355	1235.438517
2019-01-28	1.545477e+06	1.551820e+06	4811.143834	1562850	1542500	90.959393	1287.208898
2019-01-29	1.573303e+06	1.561697e+06	6169.511164	1579750	1546250	92.804304	1243.846938
2019-01-30	1.607729e+06	1.629211e+06	12145.157242	1661400	1602400	142.917517	1201.775358
2019-01-31	1.665671e+06	1.677437e+06	6230.453616	1689950	1661900	106.829303	958.527024

Table 1: Statistics of AAPL in January 2019. Source: LOBSTER

column in Table 1 reports the sum of the average best bid volume and the average second best bid volume in the time window 10am-3pm.

The setting of our simulated liquidation is as follows. The inventory q_0 to be liquidated corresponds to ten times the maximum bid volume reported in Table 1. The decision price x_0 that initiates the liquidation is taken to be the midprice at 10am on the last trading day in January (2019-01-31). The time horizon of the liquidation is 1 hour, which we adopt as unit of time. The price trajectories are then simulated according to the model

$$S_t = x_0 + \sigma W_t, \quad (3.16)$$

where the volatility parameters is the average of the volatilities measured in January as reported in Table 1.

Two aspects of the model are worth noticing. Firstly, we notice that the model in equation (3.16) implies that the decision price is also the expected price during the liquidation period. In other words, the model presupposes that the liquidator expects the price to oscillates around the decision price. This might not necessarily correspond to the liquidator's belief, in which case a drift should be introduced in order to accommodate the liquidator's expectation. As pointed out at the beginning of the paragraph however, for simplicity here we consider only the arithmetic Brownian motion, which is the benchmark since [AC00]. Secondly, we notice that, since the unit of time for the measurements of volatilities in Table 1 was one second and the unit of time in our simulated liquidation is one hour, a conversion factor of 60 exists between the volatilities of the two settings.

Figure 4 shows the implementation of the good trade execution of Proposition 3.4. Four scenarios are represented, corresponding to four different samples of the price process. For each sample path, the corresponding inventory trajectory is obtained by implementing the dynamics of equation (3.11) with initialisation (3.14). The inventory trajectories are stopped when they reach the liquidation target $q_T = 0$. This can happen before or after the execution horizon $T = 1.0$: it happens before T if the error of liquidation at time T as per the dynamics of equation (3.11) is negative; it happens after T if the error of liquidation at time T as per the dynamics of equation (3.11) is positive. In this way, the relaxation of the fuel constraint provides a link with the choice of the execution horizon. More formally, the termination time of the liquidation is

$$\tau = \inf \{t \geq 0 : q_t = q_T\}.$$

The realised inventory trajectories dynamically adjust their termination time depending on the market conditions. Take for example inventory #1, which corresponds to price trajectory #1. The realised brownian path stays above its expected value (which coincides with the decision price) and consequently

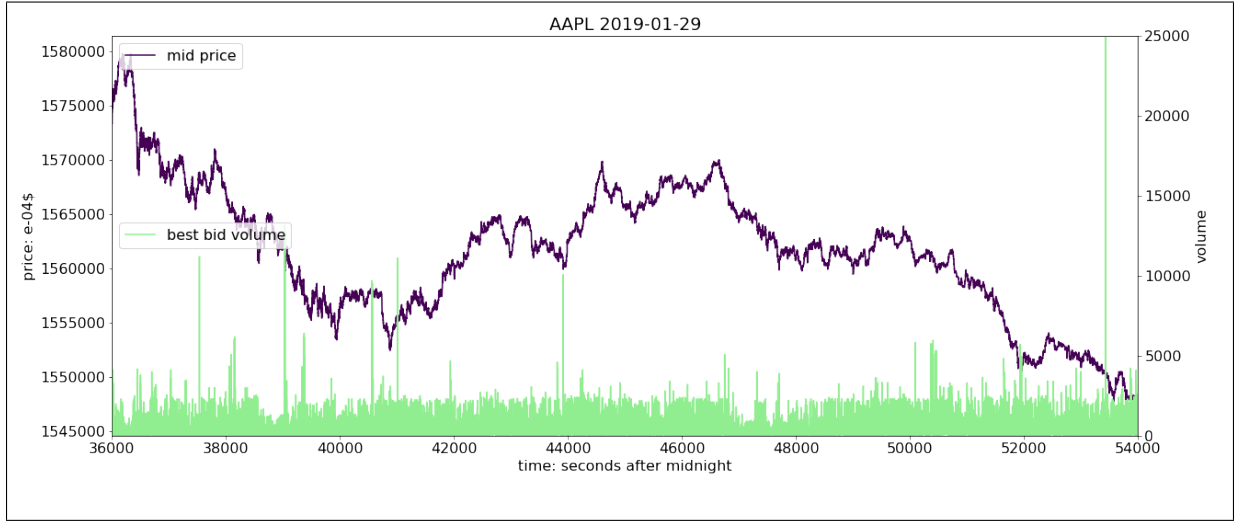


Figure 3: mid price and best bid volume of AAPL on 29 Jan 2019. source: LOBSTER

inventory #1 liquidates faster than the static solution, reducing the exposure to volatility: the trade-off between a fast execution and a parsimonious rate is resolved in favour of the former, because market conditions are favourable to the liquidation.

A different situation is exhibited in scenario #4. In this case the brownian path stays below its expected value, oscillating below the decision price. A fast execution in this case would make the price plunge even further, negatively affecting the liquidator's revenues. Because of this, inventory #4 liquidates at a slower pace than the static solution, reducing the liquidator's impact on the price: the balance between short exposure to the market and parsimonious rate of execution is in this case resolved in favour of the latter.

The reciprocal positions of the good inventory trajectories #1 and #4 are characteristic of the type of reaction to observed prices that good trade executions permit. We emphasise that, despite being dynamic solutions, good inventory trajectories do not rely on a full SDE specification for the evolution of the asset price. This is a model robustness feature.

As observed in Remark 3.7, the variance of the liquidation error depends on the variance of the fundamental price S and on the parameters c_1 and c_2 of market impact and of risk aversion. Since the liquidation error influences the termination time of the implemented inventory trajectory, these coefficients can be tuned to adjust for the desired degree of departure from the static inventory trajectory. Notice that such a tuning relates to the trustfulness attributed to the expected price. Indeed, while the realised price unfolds, the liquidator observes it being above or below its expected value; having reasons to believe that the price will eventually oscillate around such an expected value, the liquidation is either executed more promptly or more parsimoniously depending on the direction of departure from the expectation.

It thus becomes apparent the paramount role of the expected trajectory $t \mapsto \mathbb{E}[S_t]$; this reinforced our focus on the concept of robustness introduced in Definition 2.1.

Finally, Figure 5 represents liquidation errors and termination times of good trade execution corresponding to 10000 samples of the price process. In the table of Figure 5, liquidation errors are reported as a percentage of the initial inventory q_0 , whereas in the histogram liquidation errors are reported in

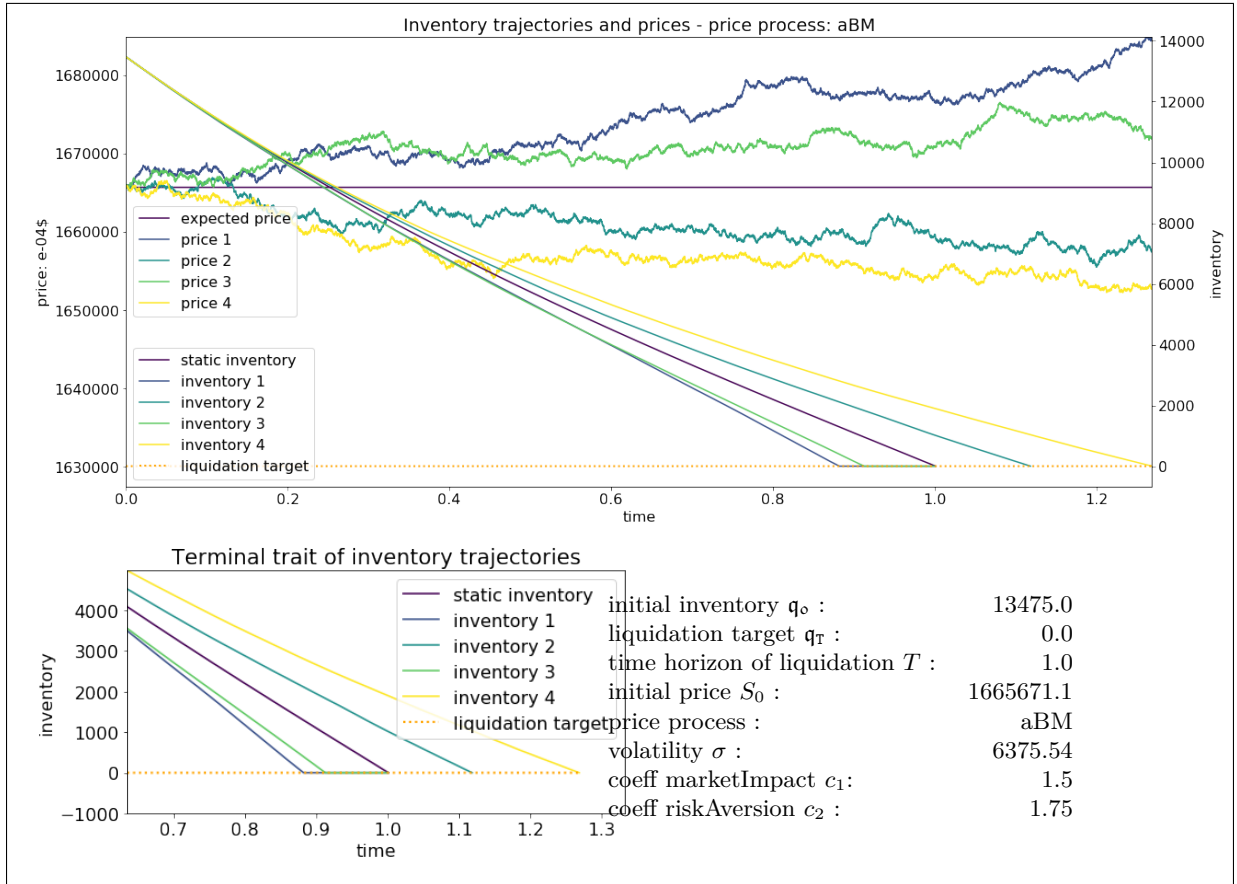


Figure 4: Simulation of price and inventory trajectories

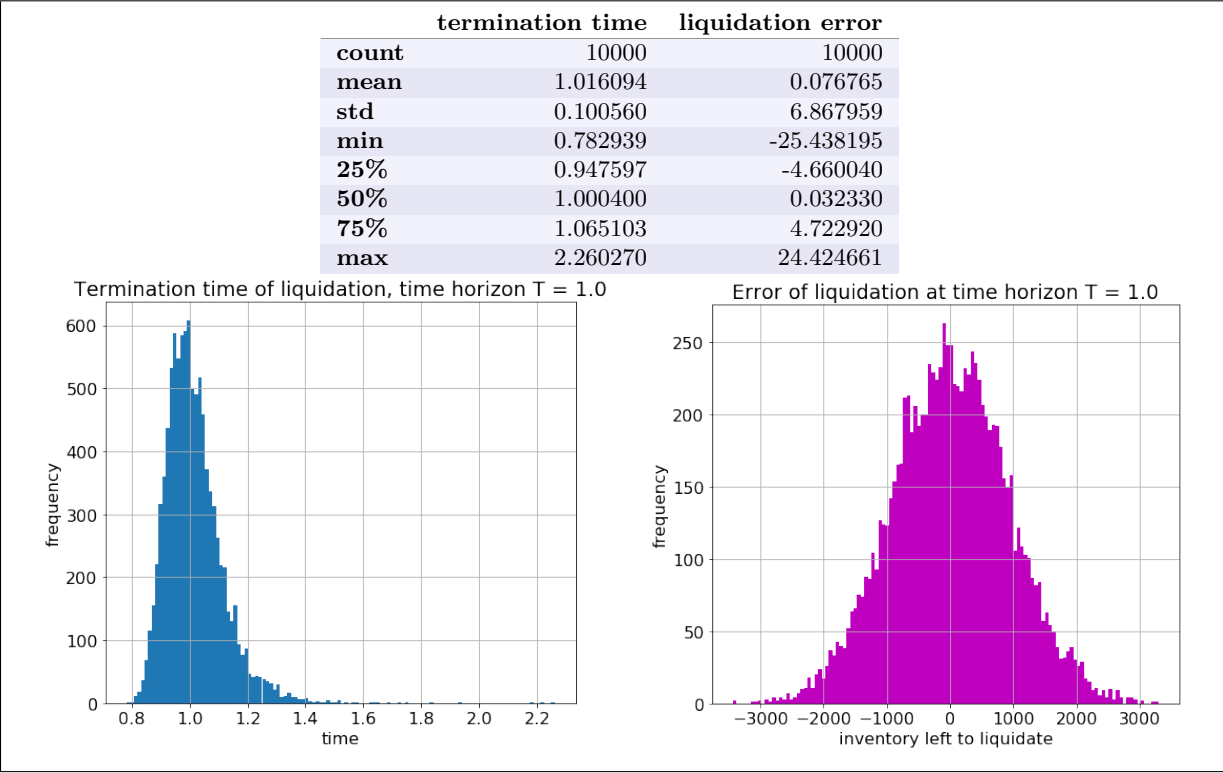


Figure 5: Termination times and liquidation errors for the simulation in Figure 4

actual units of inventory. The histogram of liquidation errors makes visually clear that such errors are unbiased, with symmetric distribution around the liquidation target $q_T = 0$.

3.3.2 5Y government bonds: brownian bridge

The second application that we consider has an entirely different time scale from the one of intraday AAPL stock price, i.e. from the time scale at which models of trade executions are usually applied. In this paragraph, we consider the sale of 5Y government bonds motivated by market conditions whereby bond yields are negative. Historically, this was observed in 2016 for German bunds and such a phenomenon reappeared in March 2019. Four examples are reported in Figure 6, where we show the prices of 5Y German bunds with maturities October 2019, April 2020, October 2020, and April 2021.

The reason for negative yields was twofold. On the one hand, central banks launched programmes to stimulate the economy by cutting interest rates and by injecting liquidity via the so-called quantitative easing. On the other hand, in volatile economic regimes risk-averse investors tend to move their capital to safe investments as government bonds (a phenomenon known as flight-to-quality), and this exerted upward pressure on bond prices.

We consider an investor who wishes to unwind a long position on a negative yielding bond before incurring into the sure loss at the bond's maturity. The sale is done gradually in time, on the one hand because the investor does not want to suddenly loose all the liquidity associated with the bonds, and on the other hand because the investor is mindful of the market impact that an abrupt sale would incur into.

In Figure 6 we observe that, after the period of overprice and as the maturity approaches, the trajectories converge on a downward slope towards the face value, which pins the trajectories at maturity. We base our modelling choices on this observation and, in line with a tradition that dates back to the Seventies ([Boy70]), we adopt a Brownian bridge as the price process. More precisely, the price process S is modelled as

$$dS_t = \frac{V - S_t}{T - t} dt + \sigma dW_t, \quad S_0 = x_0, \quad (3.17)$$

where V is the face value of the bond, W is a standard one-dimensional Brownian motion, and σ is the volatility coefficient. In this model for the price process, the expected price path is the line segment from the decision price x_0 at time $t = 0$ to the face value at maturity. Hence, contrary to what happened for the arithmetic Brownian motion, in this case the decision price and the expected price do not coincide. Instead, the quadratic variation of the price process in equation (3.17) is the same as the quadratic variation of the price process in the previous paragraph; hence, we estimate the volatility coefficient σ as done for the intraday AAPL stock price. For the current example of the German bund however, we adopt 300 days as time unit. The estimated volatilities are reported in the last column of Table 2, together with other statistics of the bond prices.

The setting of our simulated liquidation is as follows. We let the decision price x_0 be the highest of the bond prices on February 24th 2017 (reported in Table 2). We suppose that the liquidation starts 600 trading days before the maturity of the bond and we let the liquidation horizon $T = 1.0$ be the middle point in between the start of the liquidation and the maturity. The volatility of the simulated trajectories is taken to be the average of the volatilities reported in Table 2. The tuning parameters of the models c_1 and c_2 are chosen two orders of magnitude smaller than those in Figure 4; this allows for greater variability of the inventory trajectories.

Figure 7 shows good trade executions in this setting. The inventory trajectories are simulated following the dynamics in equation (3.11) with initialisations (3.14).

The way in which the good trade executions react to market scenarios is analogous to what was observed in Figure 4. In particular, in scenarios #1 and #3 the liquidation is faster than the static solution; in this way it exploits favourable market conditions. Notice that this is advantageous especially

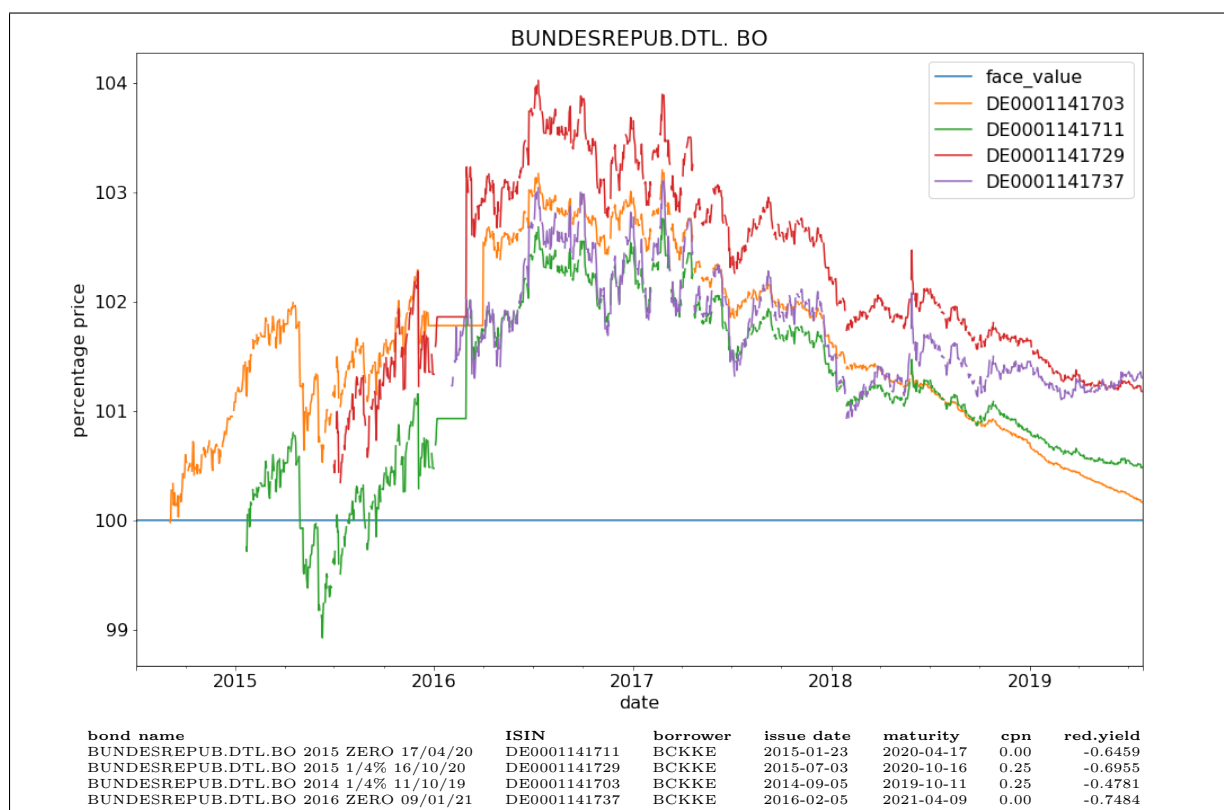


Figure 6: date: 29/07/2019, source: Thomson Reuters Datastream

bond	price on 24/02/2017	avg	std	vol
DE0001141711	102.758	101.202385	0.774095	0.940906
DE0001141729	103.893	102.246960	0.830204	1.162145
DE0001141703	103.201	101.593300	0.806122	1.363008
DE0001141737	103.097	101.757502	0.505060	1.190914

Table 2: Statistics of DE00011417*

in scenario #3, where a faster liquidation means that the liquidator concentrates her sale before the price plunges below its expected trend. On the contrary, in scenario #4, the good inventory trajectory is less steep than the static solution, because unfavourable market conditions recommend to parsimoniously impact the price.

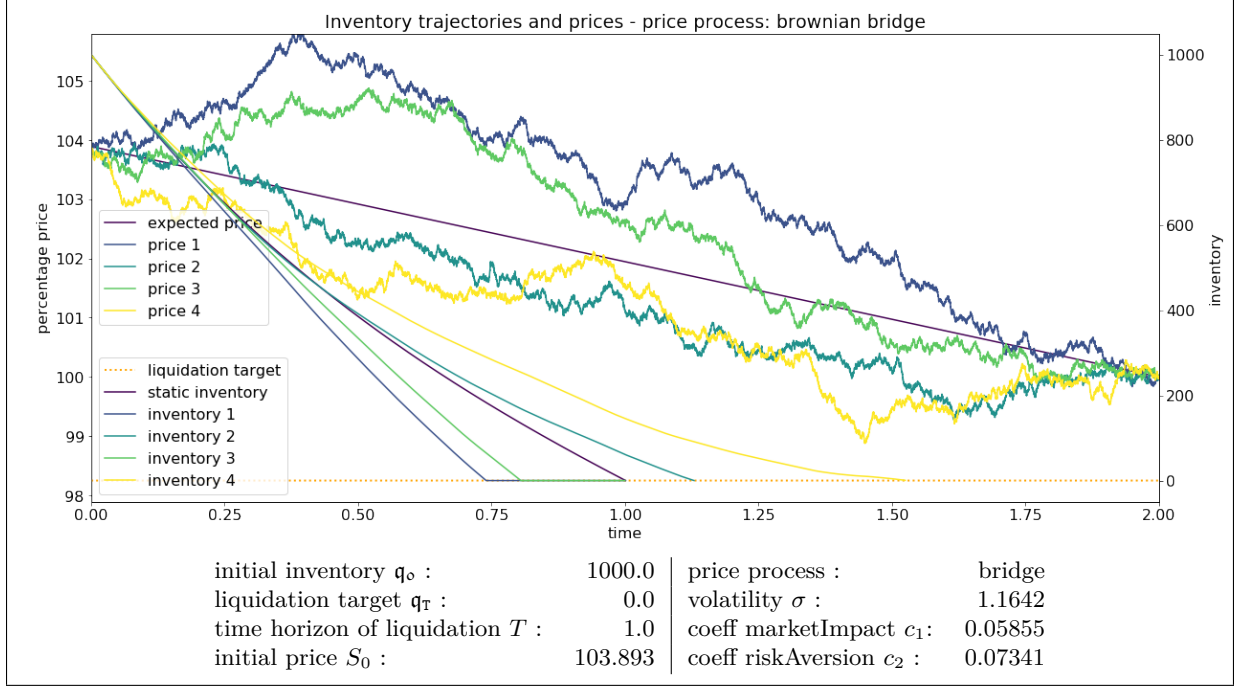


Figure 7: Simulation of bond prices and corresponding inventory trajectories

4 Alternative risk criteria

We present two alternatives to the risk aversion formulated in equation (3.2). The first alternative preserves the same structure but increases the weight of the coefficient c_2 of risk aversion linearly in time. The second alternative is instead inspired by the value at risk for geometric Brownian motion used in [GS11].

4.1 Linearly time-dependent coefficient of risk aversion

The third summand in the Lagrangian F of equation (3.4) accounts for the risk aversion. So far this term has been taken constant in the time variable t . We now propose a linear t -dependence, with higher risk aversion for t closer to the liquidation horizon T . More precisely, we consider the Lagrangian

$$F(t, S, q, r) := rS + c_1^2 r^2 + c_2^2 t (q - q_T)^2. \quad (4.1)$$

and we compute a good inventory trajectory for the minimisation of

$$J(\eta) = \int_0^T F(t, S_t, \eta_t, \dot{\eta}_t) dt, \quad \eta \in \mathcal{U}^{0, q_0}.$$

The lagrangian F in equation (4.1) satisfies the decomposition in equation (A.1) and for every t the function $(q, r) \mapsto F(t, S_t, q, r)$ is convex.

Recall that $c_1 > 0$ is the coefficient of instantaneous market impact, $c_2 \geq 0$ is the coefficient of risk aversion, and $c_3 = c_2/c_1$. The proposed time-dependency of the risk aversion is reflected in the following modifications of the norms in equations (3.5) and (3.6): for η in $W^{1,2}[0, T]$ we define

$$|\eta|_{c_1, c_2 \sqrt{t}}^2 := \int_0^T (c_2^2 t \eta_t^2 + c_1^2 \dot{\eta}_t^2) dt; \quad (4.2)$$

for q in $\mathcal{U}^{0, \mathbf{q}_0}$ we define

$$\|q\|_{c_1, c_2 \sqrt{t}}^2 := \mathbb{E} \left[|q|_{c_1, c_2 \sqrt{t}}^2 \right]. \quad (4.3)$$

Consequently, this section's counterpart to the static tubular neighbourhood in equation (3.7) is

$$\mathcal{U}^{0, \mathbf{q}_0}(q, C) := \left\{ \eta \in \mathcal{U}^{0, \mathbf{q}_0} : \|\eta_T - q_T\|_{L^2(\mathbb{P})} \leq C \|\eta - q\|_{c_1, c_2 \sqrt{t}}^2 \right\}, \quad (4.4)$$

and this section's counterpart to the pathwise tubular neighbourhood in equation (3.8) is

$$\mathcal{U}_{\text{pw}}^{0, \mathbf{q}_0}(q, \xi) := \left\{ \eta \in \mathcal{Q} : |\eta_T - q_T| \leq \xi \|\eta - q\|_{c_1, c_2 \sqrt{t}}^2 \right\}. \quad (4.5)$$

The definition of the good trade execution q in $\mathcal{U}^{0, \mathbf{q}_0}$ for the minimisation of the cost functional J in equation (4.1) with the lagrangian F as in equation (4.1) is as in Definition 3.2, where however the neighbourhoods $\mathcal{U}^{0, \mathbf{q}_0}(q_T, C)$ and $\mathcal{U}_{\text{pw}}^{0, \mathbf{q}_0}(q_T, \xi)$ are those in equations (4.4) and (4.5) respectively, rather than those in equations (3.7) and (3.8).

Having formulated the concept of good trade execution in the present case of time-dependent coefficient of risk-aversion, we now proceed as done in Sections 3.1 and 3.2, namely: in Proposition 4.1 we give a close form formula for a good trade execution for the minimisation of the cost functional J in (4.1); then in Lemma 4.2 and in Proposition 4.3 we show that in fact such a good trade execution is unique and we characterise it as the solution of a random Young differential equation. In order to shorten the notation we take $\mathbf{q}_T = 0$ and we omit this from the formulas.

Proposition 4.1 (“Closed-form formula for the good trade execution under linearly time dependent risk criterion”). *Let \mathbf{Ai} and \mathbf{Bi} be the first and the second Airy's functions, namely the two independent solutions to the second order linear ordinary differential equation $u''(t) - tu(t) = 0$. Moreover, set*

$$\begin{aligned} \alpha(t) &= \mathbf{Ai}(c_3^{2/3}t), \\ \beta(t) &= \mathbf{Bi}(c_3^{2/3}t), \\ \phi(t) &= \frac{1}{2c_1^2} \int_0^t \alpha^{-2}(s) \int_0^s \alpha(u) dS_u ds, \end{aligned}$$

where the innermost integral in the definition of ϕ is the Young integral introduced in Appendix A. Define the constants c_A and c_B as follows

$$\begin{aligned} c_A &= \frac{\beta(T)\mathbf{q}_0 - \alpha(T)\beta(0)\mathbb{E}\phi(T)}{\alpha(0)\beta(T) - \alpha(T)\beta(0)}, \\ c_B &= \frac{\alpha(T)}{\alpha(0)\beta(T) - \alpha(T)\beta(0)} \left(\alpha(0)\mathbb{E}\phi(T) - \mathbf{q}_0 \right). \end{aligned}$$

Then, the inventory trajectory

$$q_t = c_A \alpha(t) + c_B \beta(t) - \alpha(t) \phi(t) \quad (4.6)$$

is a (C, ξ) -good trade execution, where

$$C = 1/\|S_T - 2c_1^2 \dot{\alpha}\phi(T)\|_{L^2(\mathbb{P})},$$

$$\xi = 1/|S_T + 2c_1^2 c_A \dot{\alpha}(T) + 2c_1^2 c_B \dot{\beta}(T) - 2c_1^2 \dot{\alpha}\phi(T)|,$$

where the symbol $\dot{\alpha}\phi$ denotes the time derivative of the product function $t \mapsto \alpha(t)\phi(t)$.

Proof. The proof is conducted along the same lines of the proof of Proposition 3.4. Let q be as in equation (4.6). It is straightforward to check that indeed q is in $\mathcal{U}^{0, \mathbf{q}_\circ}$. Let $f_t := 2c_1^2 \dot{q}_t + S_t$ and notice that f is absolutely continuous with derivative

$$\dot{f}_t = 2c_1^2 t q_t. \quad (4.7)$$

Let η be in $\mathcal{U}^{0, \mathbf{q}_\circ}$. We write e for the difference $e := \eta - q$, and we observe that $e_0 = 0$ and $\mathbb{E}e_T = 0$. Then, we have

$$\begin{aligned} J(\eta) - J(q) &= \int_0^T \left[f_t \dot{e}_t + 2c_1^2 t q_t e_t \right] dt \\ &\quad + \int_0^T (c_1^2 \dot{e}_t^2 + c_2^2 t e_t^2) dt. \end{aligned}$$

The second integral on the right hand side is $|e|_{c_1, c_2 \sqrt{t}}^2$. Using integration by parts we see that in fact $J(\eta) - J(q) = f_T e_T + |e|_{c_1, c_2 \sqrt{t}}^2$, because of equation (4.7). Therefore, the difference $J(\eta) - J(q)$ is non-negative if

$$|e_T| \leq \frac{|e|_{c_1, c_2}^2}{|f_T|}.$$

This gives $\xi = 1/|f_T|$.

Secondly, consider the expected difference $\mathbb{E}J(\eta) - \mathbb{E}J(q) = \mathbb{E}[f_T e_T] + \|e\|_{c_1, c_2}^2$. We can estimate

$$\mathbb{E}[f_T e_T] \leq \text{Var}^{\frac{1}{2}}(f_T) \|e_T\|_{L^2(\mathbb{P})},$$

because $\mathbb{E}e_T = 0$. We have

$$\text{Var}(f_T) = \|S_T - 2c_1^2 \dot{\alpha}\phi(T)\|_{L^2(\mathbb{P})}^2.$$

Therefore, the expected difference $\mathbb{E}J(\eta) - \mathbb{E}J(q)$ is non-negative if

$$\|e_T\|_{L^2(\mathbb{P})} \leq \frac{\|e\|_{c_1, c_2 \sqrt{t}}^2}{\|S_T - 2c_1^2 \dot{\alpha}\phi(T)\|_{L^2(\mathbb{P})}}.$$

This gives the constant C in the statement and concludes the proof. \square

The Lagrangian F in equation (4.1) satisfies the assumptions in Proposition A.9; equation (A.11) in the present case reads

$$\begin{cases} dq_t = r_t dt \\ dr_t = c_3^2 t q_t dt - dS_t / 2c_1^2 \end{cases} \quad (4.8)$$

We now characterise the good trade execution in Proposition 4.1 as the unique solution to the random Young differential equation (4.8).

Lemma 4.2. *Let q and \tilde{q} be two solutions to equation (4.8). Then, the difference $q - \tilde{q}$ is deterministic. As a consequence, the solution to the equation (4.8) with constraint*

$$\begin{cases} q_0 = \mathbf{q}_\circ \\ \mathbb{E}q_T = \mathbf{q}_T = 0 \end{cases} \quad (4.9)$$

is unique.

Proof. The proof is conducted along the same lines as the proofs of Lemma 3.10 and Lemma 3.11. \square

Proposition 4.3 (“Characterisation of good trade execution via Euler-Lagrange equation”). *The good trade execution of Proposition 4.1 is characterised as the unique solution to the random Young differential equation (4.8) with initialisation*

$$\begin{cases} q_0 = \mathbf{q}_0 \\ r_0 = e_A(0)\alpha_0 + e_B(0)\beta_0, \end{cases}$$

where α and β are as in Proposition 4.1, and

$$\begin{aligned} e_A(0) &= \frac{\beta_T \mathbf{q}_0 - \beta_0 \tilde{K}}{\alpha_0 \beta_T - \alpha_T \beta_0} \\ e_B(0) &= \frac{\alpha_0 \tilde{K} - \alpha_T \mathbf{q}_0}{\alpha_0 \beta_T - \alpha_T \beta_0} \\ \tilde{K} &= \frac{S_0}{2c_1^2 W_0} (\alpha_T \beta_0 - \alpha_0 \beta_T) \\ &\quad + \frac{1}{2c_1^2} \int_0^T \mathbb{E}[S_u] \frac{d}{du} \left(\frac{\alpha_T \beta_u - \beta_T \alpha_u}{W_u} \right) du \\ W_t &= \alpha_t \dot{\beta}_t - \dot{\alpha}_t \beta_t. \end{aligned}$$

In particular, the good trade execution in Proposition 4.1 is the only good trade execution for the minimisation of (4.1) with the Lagrangian F as in equation (4.1).

Proof. The proof is conducted along the same lines of the proof of Proposition 3.13, relying on Lemma 4.2 above and on Proposition A.9 in the Appendix. In particular, if we set $Y = (q, r)^\top$ and $\mu = \mathbb{E}[Y]$, then μ solves the ordinary differential equation

$$\begin{cases} \dot{\mu}_t^{(1)} = \mu_t^{(2)} \\ \dot{\mu}_t^{(2)} = c_3^2 t \mu_t^{(1)} - \varphi_t / 2c_1^2, \end{cases} \quad (4.10)$$

where $\varphi_t = d\mathbb{E}[S_t]/dt$. The map $t \mapsto \mathbb{E}[S_t]$ is temporarily assumed to be differentiable; at the end of the argument a standard approximation argument can be used to extend the findings to the general case. The general solution to equation (4.10) is

$$\begin{cases} \mu_t^{(1)} = e_A(t)\alpha_t + e_B(t)\beta_t \\ \mu_t^{(2)} = e_A(t)\dot{\alpha}_t + e_B(t)\dot{\beta}_t, \end{cases}$$

where α and β are as in Proposition 4.1 and

$$\begin{aligned} e_A(t) &= e_A(0) + \frac{1}{2c_1^2} \left(\frac{\beta_t}{W_t} \mathbb{E}[S_t] - \frac{\beta_0}{W_0} S_0 \right) \\ &\quad - \frac{1}{2c_1^2} \int_0^t \mathbb{E}[S_u] \frac{d}{du} \left(\frac{\beta_u}{W_u} \right) du \\ e_B(t) &= e_B(0) - \frac{1}{2c_1^2} \left(\frac{\alpha_t}{W_t} \mathbb{E}[S_t] - \frac{\alpha_0}{W_0} S_0 \right) \\ &\quad + \frac{1}{2c_1^2} \int_0^t \mathbb{E}[S_u] \frac{d}{du} \left(\frac{\alpha_u}{W_u} \right) du, \end{aligned}$$

where W is the Wronskian $W_t = \alpha_t \dot{\beta}_t - \dot{\alpha}_t \beta_t$. Choosing the constants $e_A(0)$ and $e_B(0)$ as in the statement guarantees that the constraints in equation (4.9) are satisfied. \square

4.2 VaR-inspired risk criterion

In [GS11], the authors model the fundamental price S as a geometric Brownian motion and they adopt the value at risk as measure of risk aversion. This means penalising the revenues from trade of equation (3.1) by subtracting a term proportional to $q_t S_t$ at every time t . Inspired by their modelling choices, we now discuss the minimisation of

$$J(q) = \int_0^T \left(\dot{q}_t S_t + c_1^2 \dot{q}_t^2 + c_2^2 q_t S_t \right) dt, \quad (4.11)$$

over q in \mathcal{U}^{0, q_0} . However, we do not assume any SDE dynamics for the price process, which retains the generality of the previous sections. Compared to the former lagrangians in equations (3.4) and (4.1), the lagrangian $F(t, S, q, r) = rS + c_1^2 r^2 + c_2^2 qS$ in the functional J of equation (4.11) no longer has the quadratic term in q . This is reflected in the following modifications of the norms in equations (3.5) and (3.6): for η in $W^{1,2}[0, T]$ we define

$$|\eta|_{c_1}^2 := \int_0^T c_1^2 \dot{\eta}_t^2 dt; \quad (4.12)$$

for q in \mathcal{U}^{0, q_0} we define

$$\|q\|_{c_1}^2 := \mathbb{E} [|q|_{c_1}^2]. \quad (4.13)$$

Consequently, this section's counterpart to the static tubular neighbourhood in equation (3.7) is

$$\mathcal{U}^{0, q_0}(q, C) := \left\{ \eta \in \mathcal{U}^{0, q_0} : \|\eta_T - q_T\|_{L^2(\mathbb{P})} \leq C \|\eta - q\|_{c_1}^2 \right\}, \quad (4.14)$$

and this section's counterpart to the pathwise tubular neighbourhood in equation (3.8) is

$$\mathcal{U}_{pw}^{0, q_0}(q, \xi) := \left\{ \eta \in \mathcal{Q} : |\eta_T - q_T| \leq \xi |\eta - q|_{c_1}^2 \right\}. \quad (4.15)$$

The definition of the good trade execution q in \mathcal{U}^{0, q_0} for the minimisation of the cost functional J in equation (4.11) is as in Definition 3.2, where however the neighbourhoods $\mathcal{U}^{0, q_0}(q_T, C)$ and $\mathcal{U}_{pw}^{0, q_0}(q_T, \xi)$ are those in equations (4.14) and (4.15) respectively, rather than those in equations (3.7) and (3.8).

Having formulated the concept of good trade execution in the present case of VaR-inspired risk-aversion, we now proceed as done in Sections 3.1 and 3.2, namely: in Proposition 4.4 we give a close form formula for a good trade execution for the minimisation of the cost functional J in (4.11); then in Lemma 4.7 and in Proposition 4.8 we show that in fact such a good trade execution is unique and we characterise it as the solution of a random Young differential equation.

Proposition 4.4 (“Closed-form formula for the good trade execution under VaR-inspired risk criterion”).
Let K be the constant

$$K = \frac{1}{2c_1^2 T} \int_0^T \left(\mathbb{E}[S_s] - c_2^2 \int_0^s \mathbb{E}[S_u] du \right) ds.$$

Then, the inventory trajectory

$$\begin{aligned} q_t = & \left(1 - \frac{t}{T}\right) q_0 + \frac{t}{T} q_T \\ & - \frac{1}{2c_1^2} \int_0^t \left(S_s - c_2^2 \int_0^s S_u du \right) ds \\ & + Kt \end{aligned} \quad (4.16)$$

is a (C, ξ) -good trade execution for the minimisation of J in equation (4.11). The random variable ξ is explicitly given by the formula

$$\xi^{-1} = \left| \frac{2c_1^2}{T}(\mathbf{q}_T - \mathbf{q}_0) + 2c_1^2 K + c_2^2 \int_0^T S_t dt \right|$$

and the constant C is explicitly given by the formula $C^{-1} = \|\xi^{-1}\|_{L^2(\mathbb{P})}$.

Proof. The proof is conducted along the same lines of the proof of Proposition 3.4. Let q be as in equation (4.16). It is straightforward to check that indeed q is in $\mathcal{U}^{0, \mathbf{q}_0}$. As done in the proof of Proposition 3.4 and in the proof of Proposition 4.1, we let $f_t := 2c_1^2 \dot{q}_t + S_t$ and we observe that that f is absolutely continuous. In the present case the derivative of f is

$$\dot{f}_t = c_1^2 S_t. \quad (4.17)$$

Let η be in $\mathcal{U}^{0, \mathbf{q}_0}$. We write e for the difference $e := \eta - q$, and we observe that

$$\begin{aligned} J(\eta) - J(q) &= \int_0^T \left[f_t \dot{e}_t + c_1^2 S_t e_t \right] dt \\ &\quad + \int_0^T c_1^2 \dot{e}_t^2 dt. \end{aligned}$$

The second integral on the right hand side is $|e|_{c_1}^2$. Using integration by parts we see that in fact $J(\eta) - J(q) = f_T e_T + |e|_{c_1}^2$, because of equation (4.17). Therefore the conclusion is reached as done in the proof of Proposition 3.4 and in the proof of Proposition 4.1. \square

Remark 4.5. The good trade execution in equation (4.16) has the same structure of the one in equation (3.9), namely: a time-dependent convex combination of \mathbf{q}_0 and \mathbf{q}_T (first line), a dynamic response to the realisation of the price path (second line), and an adjustment for the constraint $\mathbb{E}q_T = \mathbf{q}_T$ (third line). Moreover, notice that

$$\lim_{c_2 \downarrow 0} \frac{\sinh\left(\frac{c_2}{c_1}(T-t)\right)}{\sinh\left(\frac{c_2}{c_1}T\right)} = 1 - \lim_{c_2 \downarrow 0} \frac{\sinh\left(\frac{c_2}{c_1}t\right)}{\sinh\left(\frac{c_2}{c_1}T\right)} = 1 - \frac{t}{T},$$

so that the good trade execution in equation (4.16) and the good trade execution in equation (3.9) agree when the risk aversion vanishes, i.e. in the limit as $c_2 \downarrow 0$.

Remark 4.6. Of the good trade execution in equation (4.16), we can compute

$$\|q_T - \mathbf{q}_T\|_{L^2(\mathbb{P})} = \frac{1}{2c_1^2} \mathbb{E}^{\frac{1}{2}} \left[\left(\int_0^T \left\{ c_2^2 \int_0^t (S_u - \mathbb{E}[S_u]) du - (S_t - \mathbb{E}[S_t]) \right\} dt \right)^2 \right]$$

and estimate

$$\text{Var}(q_T) \leq \frac{T}{2c_1^4} \int_0^T \left(c_2^4 t \int_0^t \text{Var}(S_u) du + \text{Var}(S_t) \right) dt.$$

Therefore, the two facts presented in Remark 3.7 also holds for the good trade execution of Proposition 4.4.

Let $F(t, S, q, r) = rS + c_1^2 r^2 + c_2^2 qS$ be the lagrangian in the cost functional of equation (4.11). Then, F satisfies the assumptions in Proposition A.9; equation (A.11) in the present case reads

$$\begin{cases} dq_t = r_t dt \\ dr_t = c_3^2 t S_t dt / 2 - dS_t / 2c_1^2 \end{cases} \quad (4.18)$$

We now characterise the good trade execution in Proposition 4.4 as the unique solution to the random Young differential equation (4.18). Observe that equation (4.18) is solved by direct integration and this simplifies several calculations compared to the cases of Sections 3.2 and 4.1.

Lemma 4.7. *Let q and \tilde{q} be two solutions to equation (4.18). Then, the difference $q - \tilde{q}$ is deterministic. As a consequence, the solution to the equation (4.18) with constraint*

$$\begin{cases} q_0 = \mathbf{q}_0 \\ \mathbb{E}q_T = \mathbf{q}_T = 0 \end{cases} \quad (4.19)$$

is unique.

Proof. The proof is conducted along the same lines as the proofs of Lemma 3.10 and Lemma 3.11. \square

Proposition 4.8 (“Characterisation of good trade execution via Euler-Lagrange equation”). *The good trade execution of Proposition 4.4 is characterised as the unique solution to the random Young differential equation (4.18) with initialisation*

$$\begin{cases} q_0 = \mathbf{q}_0 \\ r_0 = \frac{\mathbf{q}_T - \mathbf{q}_0}{T} + \frac{1}{2c_1^2 T} \int_0^T \left(\mathbb{E}[S_t] - S_0 - c_2^2 \int_0^t \mathbb{E}[S_u] du \right) dt. \end{cases}$$

In particular, the good trade execution in Proposition 4.4 is the only good trade execution for the minimisation of (4.11).

Proof. The proof is conducted along the same lines of the proof of Proposition 3.13, relying on Lemma 4.7 above and on Proposition A.9 in the Appendix. For the determination of the initial conditions to the Young differential equation (4.18), it suffices to notice that, if we set $Y = (q, r)^\top$ and $\mu = \mathbb{E}[Y]$, then we have

$$\begin{cases} \mu_t^{(1)} = \mathbf{q}_0 + \int_0^t \mu_s^{(2)} ds \\ \mu_t^{(2)} = r_0 - \frac{1}{2c_1^2} \left(\mathbb{E}[S_t] - S_0 - c_2^2 \int_0^t \mathbb{E}[S_u] du \right). \end{cases}$$

\square

5 Conclusions

In this paper, we examined the mathematical models of optimal trade execution with respect to three properties: non-static trajectories, unbiased liquidation errors and marginal robustness. Non-static trajectories are those that react to the actual realisation of the price path during the execution, rather than being based only on assumed distributional properties of this price. Secondly, a liquidation error is said to be unbiased if its expectation is zero, entailing that the expected value of the terminal inventory coincides with the execution target. Thirdly, a model is said to be marginally robust if the only distributional property of the price that it relies upon is the expected trajectory as seen from the initial time of execution.

At present in the literature there is no model that produces execution strategies with all these three properties. Hence, we introduced our proposal for execution strategies, which instead enjoy all these. In particular, in order to have non-static solutions even when the fundamental price is modelled as a martingale, we considered the minimisation of trading costs from a *pathwise* perspective, rather than the minimisation of *expected* trading costs.

We considered three risk criteria. The first criterion is the classical quadratic inventory cost; the second is a time-dependent modification of the first; the third was inspired by the value-at-risk employed in Gatheral and Schied [GS11]. For all of them, we derived explicit closed-form formulas of our inventory trajectories. Furthermore, we characterised them through initial value problems that allow to easily implement our strategies in practice. We demonstrated this through two applications, one on the liquidation of AAPL shares, the other on the liquidation of German bunds.

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Appendix A Euler-Lagrange equations in the presence of a (rough) price path

All the statements in this appendix are pathwise, hence we leave probability out of the picture. In particular, the symbol S will denote a (possibly discontinuous) single path of finite p -variation, for some $p \geq 1$. We consider the following functions spaces. The space $C_0^\infty(0, T)$ is the space of smooth functions with compact support in the open interval $(0, T)$. The Sobolev space $W^{1,2}[0, T]$ is the space of absolutely continuous functions u on the closed interval $[0, T]$ such that u and its derivative \dot{u} are square integrable over $[0, T]$. The Sobolev space $W^{1,2}[0, T]$ is endowed with the norm

$$\|u\|_{W^{1,2}}^2 = \int_0^T (u_t^2 + \dot{u}_t^2) dt.$$

The space $W_0^{1,2}(0, T)$ is defined as the closure of $C_0^\infty(0, T)$ in $W^{1,2}[0, T]$.

Let U be an open set in \mathbb{R}^n . The function $F : U \times \mathbb{R}^d \rightarrow \mathbb{R}$ is called Caratheodory function if

1. for Lebesgue-almost every t in U the map $x \mapsto F(t, x)$ is continuous;
2. for every x in \mathbb{R}^d the function $t \mapsto F(t, x)$ is Lebesgue-measurable.

Definition A.1. Let U be an open subset of \mathbb{R}_+ . Let $F : U \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a Caratheodory function. We say that F is space-differentiable if for all t in U the map $x \mapsto F(t, x)$ is in $C^1(\mathbb{R}^d)$.

In this appendix we consider lagrangians $F : (0, T) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ for which the following decomposition holds:

$$F(t, x_1, x_2, x_3) = x_1 x_3 + L(t, x_1, x_2, x_3), \tag{A.1}$$

where L is a space-differentiable Caratheodory function on $(0, T) \times \mathbb{R}^3$ such that there exist a function α in $L^1(0, T)$ and a constant $\beta \geq 0$ such that

$$\begin{aligned} |L(t, x_1, x_2, x_3)|, |\partial_{x_2} L(t, x_1, x_2, x_3)|, |\partial_{x_3} L(t, x_1, x_2, x_3)| \\ \leq \alpha(t) + \beta(x_1^2 + x_2^2 + x_3^2), \end{aligned} \tag{A.2}$$

for all t in $(0, T)$ and all x_1, x_2, x_3 in \mathbb{R} . In the sections above, the space variables were taken to be $(x_1, x_2, x_3) = (S, q, r)$. Hence, we interpret x_1 as the placeholder for the variable S denoting the fundamental price, we interpret x_2 as the placeholder for the variable q denoting the inventory, and we interpret x_3 as the placeholder for the variable \dot{q} denoting the rate of execution. In the following, we adopt these letters for the space variables; hence, in particular $\partial_q L$ denotes the derivative of L with respect to the variable x_2 , and $\partial_r L$ denotes the derivative of L with respect to the variable x_3 .

Remark A.2. When L does not depend on x_1 , the decomposition in equation (A.1) has the form of the decomposition in equation (2.8). However, we do not restrict to this case in this appendix in order to encompass the case of the functional used by Gatheral and Schied in [GS11], which we discuss in Section 4.2.

Let $S = (S_t)_t$ be a path on $[0, T]$ and assume that S is of finite p -variation for some $p \geq 1$. We consider the functional

$$J(\eta) := \int_0^T F(t, S_t, \eta_t, \dot{\eta}_t) dt, \quad (A.3)$$

$$\eta \in W^{1,2}[0, T],$$

where the lagrangian F satisfies the decomposition in equation (A.1). Let q_0 be in $W^{1,2}[0, T]$. We consider the minimisation problem

$$\inf \left\{ J(\eta) : \eta \in q_0 + W_0^{1,2}(0, T) \right\}, \quad (A.4)$$

where the functional J is as in equation (A.3), and

$$q_0 + W_0^{1,2}(0, T) = \left\{ q_0 + u : u \in W_0^{1,2}(0, T) \right\}.$$

Proposition A.3 (“Weak form of Euler-Lagrange equation”). *Assume that q is a minimiser for (A.4). Then, for all ψ in $C_0^\infty(0, T)$ it holds*

$$\int_0^T (S_t + \partial_r L(t, S_t, q_t, \dot{q}_t)) \dot{\psi}_t dt = - \int_0^T \psi_t \partial_q L(t, S_t, q_t, \dot{q}_t) dt. \quad (A.5)$$

Proof. For ψ in $C_0^\infty(0, T)$ it holds

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_0^T [L(t, S_t, q_t + \epsilon \psi_t, \dot{q}_t + \epsilon \dot{\psi}_t) - L(t, S_t, q_t, \dot{q}_t)] dt \\ = \int_0^T [\psi_t \partial_q L(t, S_t, q_t, \dot{q}_t) + \dot{\psi}_t \partial_r L(t, S_t, q_t, \dot{q}_t)] dt, \end{aligned}$$

owing to assumption (A.2) on the growth of L and its space derivatives. Therefore, it suffices to notice that for all ψ in $C_0^\infty(0, T)$ and all $\epsilon > 0$ we have that

$$J(q + \epsilon \psi) \geq J(q).$$

□

Equation (A.5) is the weak form of the Euler-Lagrange equation. In the next paragraph, we introduce the pathwise integration with respect to the path S . This will allow to move from the condition in equation (A.5) to the stronger formulation of the Euler-Lagrange equation.

Young integration and integration-by-parts formula

Let $[0, T]$ be the closed time interval from time zero to the time horizon $T > 0$. We consider partitions π of $[0, T]$. A partition π is simultaneously considered as the finite collection of points and as the finite collection of adjacent subintervals. Given a partition π of $[0, T]$ and a time instant t in $[0, T]$, we adopt the following notational convention:

$$t' := \inf\{u \in \pi : u > t\}. \quad (\text{A.6})$$

The meshsize $|\pi|$ of the partition π is defined as

$$|\pi| := \sup\{|u' - u| : u \in \pi\}.$$

Let f be a function on $[0, T]$. If s and t are in $[0, T]$, we let $f_{s,t} := f_t - f_s$ denote the increment of f from s to t . The resulting two-parameter function $f = f_{s,t}$ is additive, in that for all $s \leq u \leq t$ it holds $f_{s,u} + f_{u,t} = f_{s,t}$. For more general two-parameter functions defined on $\{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$ we can relax the additivity and consider the following property.

Definition A.4. Let $f = f(s, t)$ be a function on $\{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$. We say that f is super-additive if for all $s \leq u \leq t$ it holds

$$f(s, u) + f(u, t) \leq f(s, t).$$

Lemma A.5. Let f be a super-additive function on $\{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$ such that it is null and uniformly continuous on the diagonal, i.e. $f(t, t) \equiv 0$ and

$$\limsup_{\epsilon \downarrow 0} \{f(s, t) : |t - s| \leq \epsilon\} = 0. \quad (\text{A.7})$$

Let g be a super-additive function on $\{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$. Let $0 < a \leq 1$ and $0 < b < 1$ be exponents such that $a + b > 1$. Then,

$$\lim_{|\pi| \downarrow 0} \sum_{u \in \pi} f^a g^b(u, u') = 0,$$

where the limit is taken along arbitrary sequences of partitions of $[0, T]$ with meshsize tending to zero.

Proof. Let $0 < \epsilon < a$ be such that $a - \epsilon + b = 1$. Then,

$$\begin{aligned} \sum_{u \in \pi} f^a g^b(u, u') &\leq \left(\sup_{u \in \pi} f^\epsilon(u, u') \right) \cdot \sum_{u \in \pi} f^{a-\epsilon} g^b(u, u') \\ &\leq \left(\sup_{u \in \pi} f^\epsilon(u, u') \right) \cdot \left(\sum_{u \in \pi} f(u, u') \right)^{a-\epsilon} \left(\sum_{u \in \pi} g(u, u') \right)^b \\ &\leq \left(\sup_{u \in \pi} f^\epsilon(u, u') \right) \cdot f^{a-\epsilon}(0, T) g^b(0, T). \end{aligned}$$

On the second line we have applied Hölder inequality and on the third line we have used super-additivity. We conclude by recalling the uniform continuity of equation (A.7). \square

Lemma A.6. Let η be an absolutely continuous path on the closed interval $[0, T]$. Let S be a path on $[0, T]$ of finite p -variation for some $p \geq 1$. Then, for all $0 \leq s \leq t \leq T$ the limit

$$\lim_{|\pi| \downarrow 0} \sum_{u \in \pi} \eta_u S_{u, u'}$$

exists and is the same along any sequence of partitions of $[s, t]$. Such a limit defines the Young integral

$$\int_s^t \eta_u dS_u.$$

Moreover, the following integration-by-parts formula holds

$$\int_s^t \eta_u dS_u + \int_s^t S_u d\eta_u = \eta_t S_t - \eta_s S_s,$$

where the second integral on the left hand side is the Stieltjes integral of S against η .

Proof. Let π be an arbitrary partition of $[s, t]$. For all u in π we have

$$\eta_u S_{u,u'} + S_u \eta_{u,u'} = \eta_{u'} S_{u'} - \eta_u S_u - \eta_{u,u'} S_{u,u'}. \quad (\text{A.8})$$

Consider first the right hand side of equation (A.8). Consider the first two summands on the right hand side of equation (A.8). If we sum over u in π we have the telescopic sum

$$\sum_{u \in \pi} (\eta_{u'} S_{u'} - \eta_u S_u) = \eta_t S_t - \eta_s S_s,$$

and this does not depend on the partition π of $[s, t]$. The third summand on the right hand side of equation (A.8) can be bound as follows:

$$|\eta_{u,u'} S_{u,u'}| \leq \|\eta\|_{1\text{-var},[u,u']} \|S\|_{p\text{-var},[u,u']}.$$

Hence, by Lemma A.5, we have

$$\lim_{|\pi| \downarrow 0} \sum_{u \in \pi} \eta_{u,u'} S_{u,u'} = 0.$$

Here, we have applied Lemma A.5 with $f(s, t) = \|\eta\|_{1\text{-var},[s,t]}$, $g(s, t) = \|S\|_{p\text{-var},[s,t]}^p$, $a = 1$, and $b = 1/p$. Therefore, if we apply the operator $\lim_{|\pi| \downarrow 0} \sum_{u \in \pi}$ to the right hand side of equation (A.8), we obtain

$$\lim_{|\pi| \downarrow 0} \sum_{u \in \pi} (\eta_{u'} S_{u'} - \eta_u S_u - \eta_{u,u'} S_{u,u'}) = \eta_t S_t - \eta_s S_s.$$

Consider now the left hand side of equation (A.8). The limit

$$\lim_{|\pi| \downarrow 0} \sum_{u \in \pi} S_u \eta_{u,u'}$$

is the Stieltjes integral $\int_s^t S_u d\eta_u$ and thus converges and is the same along any sequence of partitions with vanishing meshsize. Therefore, if we apply the operator $\lim_{|\pi| \downarrow 0} \sum_{u \in \pi}$ to both side of equation (A.8) we must have that the limit in equation (A.6) exists and is equal to

$$\eta_t S_t - \eta_s S_s - \int_s^t S_u d\eta_u.$$

This concludes the proof of the lemma. \square

Strong form of Euler-Lagrange equation

Relying on the Young integral introduced in the previous paragraph, we now formulate the strong form of the Euler-Lagrange equation associated with the minimisation problem in equation (A.4).

Definition A.7. Let c be a constant. Let S be a path on $[0, T]$ and assume that S is of finite p -variation for some $p \geq 1$. Let $b = b(t, x)$ be a Caratheodory function on $(0, T) \times \mathbb{R}^3$ such that

$$|b(t, x)| \leq \alpha(t) + \beta |x|^2,$$

for some integrable function α in $L^1(0, T)$ and some constant $\beta \geq 0$. We say that the function q in $W^{1,2}[0, T]$ solves the equation

$$\begin{cases} dq_t = r_t dt \\ dr_t = b(t, S_t, q_t, \dot{q}_t) dt - c dS, \end{cases} \quad (\text{A.9})$$

if for all η in $C_0^\infty(0, T)$ and all $0 \leq s \leq t \leq T$ it holds

$$\int_s^t \dot{\eta}_u dq_u = \eta_t \dot{q}_t - \eta_s \dot{q}_s - \int_s^t \eta_u b(u, q_u, \dot{q}_u) du + c \int_s^t \eta_u dS_u, \quad (\text{A.10})$$

where the last integral on the right hand side is the Young integral introduced in Lemma A.6.

Lemma A.8. Assume the setting of Definition A.7. Then, q solves equation (A.9) if and only if the function $f_t := \dot{q}_t + cS_t$ is absolutely continuous with derivative

$$\dot{f}_t = b(t, S_t, q_t, \dot{q}_t).$$

Proof. Apply the integration by parts established in Lemma A.6 to formula (A.10), and observe that – for all η in $C_0^\infty(0, T)$ – formula (A.10) is equivalent to

$$\int_0^T (\dot{q}_t + cS_t) \dot{\eta}_t dt = - \int_0^T \eta_t b(t, S_t, q_t, \dot{q}_t) dt.$$

□

Proposition A.9 (“Strong form of Euler-Lagrange equation”). Let S be a path on $[0, T]$ and assume that S is of finite p -variation for some $p \geq 1$. Consider a lagrangian F for which the decomposition in equation (A.1) holds, and assume that the space-differentiable Caratheodory function L is in $C^2([0, T] \times \mathbb{R}^3)$ and such that

$$\partial_r L = \frac{1}{c} r + \ell(t, q),$$

for some non-zero constant c in \mathbb{R} and some function ℓ in $C^1([0, T] \times \mathbb{R})$.⁵ Assume that q is a minimiser for (A.4). Then, q solves the Euler-Lagrange equation

$$\begin{cases} dq_t = r_t dt \\ dr_t = \left(\partial_q L(t, S_t, q_t, \dot{q}_t) - \partial_{t,r}^2 L(t, S_t, q_t, \dot{q}_t) - \dot{q}_t \partial_{q,r}^2 L(t, S_t, q_t, \dot{q}_t) \right) c dt - c dS_t, \end{cases} \quad (\text{A.11})$$

in the sense of Definition A.7.

⁵This assumption on the form of the partial derivative $\partial_r L$ is satisfied in all three examples that we consider in the paper, namely by the lagrangians $F(t, S, q, r) = rS + L(t, S, q, r)$ in equations (3.4), (4.1) and (4.11).

Proof. Under the stated assumptions on the form of the partial derivative $\partial_r L$, the condition in equation (A.5) reads

$$\int_0^T \left[S_t + \frac{1}{c} \dot{q}_t + \ell(t, q_t) \right] \dot{\psi}_t dt = - \int_0^T \psi_t \partial_q L(t, S_t, q_t, \dot{q}_t) dt.$$

This is equivalent to

$$\int_0^T \left[c S_t + \dot{q}_t \right] \dot{\psi}_t dt = -c \int_0^T \left[\partial_q L(t, S_t, q_t, \dot{q}_t) - \partial_t \ell(t, q_t) - \dot{q}_t \partial_q \ell(t, q_t) \right] \psi_t dt.$$

Notice that $\partial_t \ell = \partial_{t,r}^2 L$ and $\partial_q \ell = \partial_{q,r}^2 L$. Hence, Lemma A.8 concludes the proof. \square