ONE-SIDED IMPACT PROFILE IN THE CASE OF POISSONIAN LIQUIDATION

PROPOSITION 0

Assume that the liquidator control is "FRACTION_OF_INVENTORY" and that

$$\lambda_0(t) = \nu_0 \mathbb{1}_{[0,T)}(t),$$

where $T := \inf \left\{ t \,;\, N_0(t) \geq \left\lceil \frac{Q_0}{r} \right\rceil \right\}$.

Then,

$$\mathcal{G}^{+}_t = \bigvee_{\substack{e \geq 1 \\ x \in \mathcal{X}}} \sigma \left\{ \tilde{N}_{ex}(s) : s \leq t \right\}$$

and

$$\Lambda^{>0}(t) - \int_0^t \mathbb{E}\left[\lambda^{>0}(s) \mid \mathcal{G}^{+}_s \right] ds$$

$$= \left[\text{TURN PAGE FOR FORMULA} \right]$$

Therefore,

$$\Lambda^{\to 0}(t) - \int_0^t \mathbb{E}\left[\lambda^{\to 0}(s) \,\middle|\, \mathcal{G}_s^{\perp}\right] ds$$

$$= \nu_0 \int_0^t \left(\mathbb{1}_{[0,T)}(s) - \mathbb{P}(T > s) \right) ds$$

$$+ \sum_{x \in \mathcal{X}^{\to 0}} \sum_{e \geq 1} \int_0^t ds \, \phi_e(X(s), x) \sum_{x'} \int_{[0,s)} \kappa_{1,e}(s-u, x') \left\{ d\tilde{N}_{0,x'}(u) - \nu_0 \phi_0(X(u), x') \mathbb{P}(T > u) du \right\}$$

PROOF.

the first claim follows from the independence between N_0 and $\tilde{N}_{ex}$ for $e \geq 1$ and x in $\mathcal{X}$.

As for the second claim, notice that under the stated assumptions we have

$$\lambda^{\neg o}(t) = \nu_0 \mathbb{1}_{[0,T)}(t)$$

$$+ \sum_{x \in \mathcal{X}^{\neg o}} \sum_{e \geq 1} \phi_e(X(t), x) \sum_{x'} \int_{[0,t)} \kappa_{1e}(t-s, x') \, d\tilde{N}_{0x'} + R^{\mathcal{G}^+}_t$$

where $R^{\mathcal{G}^+}_t$ is $\mathcal{G}^+_t$ measurable and equal to

$$+ \sum_{x \in \mathcal{X}^{\neg o}} \sum_{e \geq 1} \phi_e(X(t), x) \left(\nu_e + \sum_{e' \geq 1} \sum_{x'} \int_{[0,t)} \kappa_{e'e}(t-s, x') \, d\tilde{N}_{e'x'}(s) \right).$$

therefore

$$\mathbb{E}\left[\lambda^{\neg o}(t) \mid \mathcal{G}^+_t\right] = \nu_0 \mathbb{P}(T > t)$$

$$+ \sum_{x \in \mathcal{X}^{\neg o}} \sum_{e \geq 1} \phi_e(X(t), x) \sum_{x'} \int_{[0,t)} \kappa_{1e}(t-s, x') \phi_0(X(s), x') \nu_0 \mathbb{P}(T > s) \, ds$$

$$+ R^{\mathcal{G}^+_t}$$

which, plugged into the numerator of the one-sided impact profile gives the stated expression. $\square$

PROPOSITION 1

Assume the setting of PROPOSITION 0. Then

$$\mathbb{P}(T>t) = \sum_{n=0}^{\lceil Q_0/r \rceil - 1} \frac{(\nu_0 t)^n}{n!} e^{-\nu_0 t}$$

and

$$\mathbb{E}[T \wedge t] = \frac{\lceil Q_0/r \rceil}{\nu_0} - \frac{1}{\nu_0} \sum_{n=0}^{\lceil Q_0/r \rceil - 1} \left(\left\lceil \frac{Q_0}{r} \right\rceil - n \right) \frac{(\nu_0 t)^n}{n!} e^{-\nu_0 t} .$$

PROOF.

The first claim follows from the fact that T is the arrival time of the $\left\lceil \frac{Q_0}{r} \right\rceil$-th point of a Poisson process of intensity ν_0.

As for the second claim, write

$$\mathbb{E}[T \wedge t] = \int_0^t \mathbb{P}(T>s)\,ds$$

and use lemma 1 below.

$\square$

LEMMA 1 For $0 \le y_1 < y_2$ it holds

$$\int_{y_1}^{y_2} e^{-x} \sum_{m=0}^{N} \frac{x^m}{m!}\,dx = e^{-y_1} \sum_{m=0}^{N} (N-m+1) \frac{y_1^m}{m!}$$

$$- e^{-y_2} \sum_{m=0}^{N} (N-m+1) \frac{y_2^m}{m!}$$

$$= \sum_{m=0}^{N} (N-m+1) \left(\frac{y_1^m}{m!} e^{-y_1} - \frac{y_2^m}{m!} e^{-y_2} \right)$$

PROOF OF LEMMA 1

First we show by induction that

$$\int_{y_1}^{y_2} \frac{x^m}{m!} e^{-x}\,dx = e^{-y_1} \sum_{k=0}^{m} \frac{y_1^k}{k!} - e^{-y_2} \sum_{k=0}^{m} \frac{y_2^k}{k!}.$$

The case $m=0$ is apparent. Hence for $m \ge 0$ observe that

$$\frac{d}{dx}\left(\frac{x^{m+1}}{(m+1)!} e^{-x} \right) = \frac{x^m}{m!} e^{-x} - \frac{x^{m+1}}{(m+1)!} e^{-x}$$

whence

$$\frac{y_2^{m+1}}{(m+1)!} e^{-y_2} - \frac{y_1^{m+1}}{(m+1)!} e^{-y_1} = \int_{y_1}^{y_2} \left(\frac{x^m}{m!} - \frac{x^{m+1}}{(m+1)!} \right) e^{-x}\,dx$$

proving the induction step.

We conclude the proof of the Lemma by computing

$$\int_{y_1}^{y_2} e^{-x} \sum_{n=0}^{N} \frac{x^n}{n!}\, dx = e^{-y_1} \sum_{n=0}^{N} \sum_{k=0}^{n} \frac{y_1^k}{k!} - e^{-y_2} \sum_{n=0}^{N} \sum_{k=0}^{n} \frac{y_2^k}{k!}$$

$$= \sum_{k=0}^{N} \sum_{n=k}^{N} \left(\frac{y_1^k}{k!} e^{-y_1} - \frac{y_2^k}{k!} e^{-y_2} \right)$$

$$= \sum_{k=0}^{N} (N-k+1) \left(\frac{y_1^k}{k!} e^{-y_1} - \frac{y_2^k}{k!} e^{-y_2} \right).$$

$\square$

Proposition 2

The following is an equivalent expression for $\Lambda^{-,0}(t) - \int_0^t \mathbb{E}[\lambda^{-,0}(s) \mid \mathcal{G}_s^+]\,ds$:

$$\nu_d(T \wedge t) - \left\lceil \frac{Q_0}{r} \right\rceil + \sum_{m=0}^{\lceil Q_0/r \rceil - 1} \left(\left\lceil \frac{Q_0}{r} \right\rceil - m \right) \frac{(\nu_d t)^m}{m!} e^{-\nu_d t}$$

$$+ \sum_{x \in \mathcal{X}^{>0}} \sum_{e \geq 1} \sum_{x'} \frac{\alpha_{1x'e}}{\beta_{1x'e} - 1} \sum_{0 \leq T_m < t} \phi_e(X(T_m), x) \sum_{0 \leq T_j^{0x'} < T_m} \left\{ \left(T_{m+1} \wedge t - T_j^{0x'} + 1\right)^{1-\beta_{1x'e}} - \left(T_m - T_j^{0x'} + 1\right)^{1-\beta_{1x'e}} \right\}$$

$$+ \sum_{x \in \mathcal{X}^{>0}} \sum_{e \geq 1} \sum_{x'} \frac{\alpha_{1x'e}}{\beta_{1x'e} - 1} \sum_{0 \leq T_m < t} \phi_e(X(T_m), x) \sum_{0 \leq T_j < T_m} \phi_0(X(T_j), x') \int_{T_j}^{T_{j+1}} du\, P(T > u) \cdot \left\{ \left(T_{m+1} \wedge t - u + 1\right)^{1-\beta_{1x'e}} - \left(T_m - u + 1\right)^{1-\beta_{1x'e}} \right\}$$

$$+ \sum_{x \in \mathcal{X}^{>0}} \sum_{e \geq 1} \sum_{x'} \frac{\alpha_{1x'e}}{\beta_{1x'e} - 1} \sum_{0 \leq T_m < t} \phi_0(X(T_m), x') \phi_e(X(T_{m-1}), x) \cdot \int_{T_m}^{T_{m+1} \wedge t} du\, P(T > u) \left\{ \left(T_{m+1} - u + 1\right)^{1-\beta_{1x'e}} - 1 \right\},$$

which we write as

$$\nu_0(T\wedge t) - \left\lceil\frac{Q_0}{r}\right\rceil + \sum_{m=0}^{\left\lceil\frac{Q_0}{r}\right\rceil - 1}\left(\left\lceil\frac{Q_0}{r}\right\rceil - m\right)\frac{(\nu_0 t)^m}{m!}e^{-\nu_0 t}$$

$$+ \sum_{x\in\mathcal{X}^{\neq 0}}\sum_{e\geq 1}\sum_{x'}\frac{\alpha_{ix'e}}{\beta_{ix'e}-1}\sum_{0\leq T_m<t} f^{(m)}_{x'ex}(t)$$

$$+ \sum_{x\in\mathcal{X}^{\neq 0}}\sum_{e\geq 1}\sum_{x'}\frac{\alpha_{ix'e}}{\beta_{ix'e}-1}\sum_{0<T_m<t} g^{(m)}_{x'ex}(t)$$

where

$$f^{(m)}_{x'ex}(t)$$

$$= \phi_e\left(X(T_m), x\right) \Bigg\{ \sum_{0 \le T^{0x'}_{j} < T_m} \left[\left(T_{m+1} \wedge t - T^{0x'}_{j+1}\right)^{1-\beta_{1x'e}} - \left(T_m - T^{0x'}_{j+1}\right)^{1-\beta_{1x'e}} \right]$$

$$+ \sum_{0 \le T_j < T_m} \phi_o\left(X(T_j), x'\right) \int_{T_j}^{T_{j+1}} du\, P(T > u) \left[\left(T_{m+1} \wedge t - u + 1\right)^{1-\beta_{1x'e}} - \left(T_m - u + 1\right)^{1-\beta_{1x'e}} \right] \Bigg\}$$

and

$$g^{(m)}_{x'ex}(t) = \phi_o\left(X(T_m), x'\right) \phi_e\left(X(T_{m-1}), x\right) \cdot \int_{T_m}^{T_{m+1} \wedge t} du\, P(T > u) \left[\left(T_{m+1} - u + 1\right)^{1-\beta_{1x'e}} - 1 \right].$$

PROOF OF PROPOSITION 2

The proof is organised as follows: the next page (PAGE) repeats the starting part for our computation. We then explicitly compute the inner-most integral in the s-variable (if needed we apply Fubini to make this the inner-most integral) and we then reach the stated expression

Numerator of one-sided impact profile

$$\Lambda^{-,0}(t) - \int_0^t \mathbb{E}\left[\lambda^{-,0}(s) \,|\, \mathcal{G}_s^t\right] ds$$

$$= \left| t \wedge T - \mathbb{E}[t \wedge T] \right| \vartheta_0$$

$$+ \sum_{x \in \mathcal{X}^{-,0}} \sum_{e \geq 1} \sum_{x'} \int_0^t d\tilde{N}_{0,x'}(u) \int_u^t ds \; \phi_e(X(s)_{-x}) \, \kappa_{1e}(s-u, x')$$

$$- \vartheta_0 \sum_{x \in \mathcal{X}^{-,0}} \sum_{e \geq 1} \sum_{x'} \int_0^t ds \; \phi_e(X(s)_{-x}) \int_{[0,s)} \kappa_{1e}(s-u, x') \, \phi_0(X(u)_{-x'}) \, P(T > u) \, du$$

Notice that we can explicitly compute

$$\int_u^t ds\; \phi_e\big(X(s)_-, x\big)\, \kappa_{1e}(s-u, x')$$

$$= \sum_{u \le T_m < t} \int_{T_m}^{T_{m+1}\wedge t} ds\;\; \phi_e\big(X(T_m), x\big)\kappa_{1e}(s-u, x')\,ds + \phi_e\big(X(u-), x\big)\int_u^{\text{first}(T_m)} \kappa_{1e}(s-u, x')\,ds$$

$$= \sum_{u \le T_m < t} \phi_e\big(X(T_m), x\big)\int_{T_m}^{T_{m+1}\wedge t} ds\;\; \alpha_{1x'e}(s-u+1)^{-\beta_{1x'e}}\,ds$$

$$+ \;\; \phi_e\big(X(u-), x\big)\int_u^{\text{first}(T_m \ge u)} ds\;\; \alpha_{1x'e}(s-u+1)^{-\beta_{1x'e}}\,ds$$

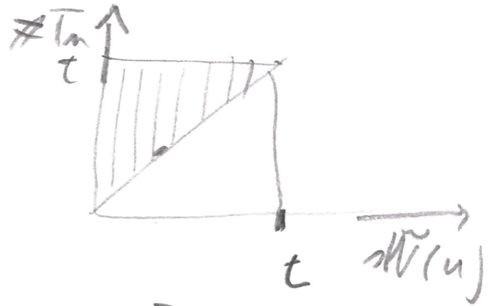
$$= \sum_{u \le T_m < t} \phi_e\big(X(T_m)_-, x\big)\,\frac{\alpha_{1x'e}}{\beta_{1x'e}-1}\left(\big(T_{m+1}\wedge t - u + 1\big)^{1-\beta_{1x'e}} - \big(T_m - u + 1\big)^{1-\beta_{1x'e}}\right)$$

$$+ \;\phi_e\big(X(u-), x\big)\,\frac{\alpha_{1x'e}}{\beta_{1x'e}-1}\left(\big(\text{first}(T_m) - u + 1\big)^{1-\beta_{1x'e}} - 1\right)$$

Hence the summands on the second line of the equation for the numerator of one-sided impact profile (PAGE) is

$$\frac{\alpha_{ix'e}}{\beta_{ix'e}-1}\int_0^t d\tilde{N}_{0x'}(u)\sum_{u\le T_n<t}\phi_e(X(T_n),x)\left(\left(T_{n+1}\wedge t-u+1\right)^{1-\beta_{ix'e}}-\left(T_n-u+1\right)^{1-\beta_{ix'e}}\right)$$

where the second component of the integral (the one from u to the first jump time $T_n\ge u$) is neglected because the integration w.r.t $d\tilde{N}_{0x'}$ will annihilate its contribution. By applying Fubini we have



$$\frac{\alpha_{ix'e}}{\beta_{ix'e}-1}\sum_{0\le T_n<t}\int_{[0,T_n)} d\tilde{N}_{0x'}(u)\,\phi_e(X(T_n),x)\left(\left(T_{n+1}\wedge t-u+1\right)^{1-\beta_{ix'e}}-\left(T_n-u+1\right)^{1-\beta_{ix'e}}\right)$$

$$=\frac{\alpha_{ix'e}}{\beta_{ix'e}-1}\sum_{0\le T_n<t}\phi_e(X(T_n),x)\sum_{0\le T_j^{0x'}<T_n}\left\{\left(T_{n+1}\wedge t-T_j^{0x'}+1\right)^{1-\beta_{ix'e}}-\left(T_n-T_j^{0x'}+1\right)^{1-\beta_{ix'e}}\right\}$$

Instead, the summands in the third line of the equation for the numerator of one-sided impact profile (PAGE) is

$$\frac{\alpha_{1x'e}}{\beta_{1x'e}-1}\int_0^t du\, \phi_e(X(u-),x')\, P(T>u)\cdot\Big\{\sum_{u\le T_m<t}\phi_e(X(T_m),x)\cdot\Big((T_{m+1}\wedge t-u+1)^{1-\beta_{1x'e}}-(T_m-u+1)^{1-\beta_{1x'e}}\Big)$$

$$+\,\phi_e(X(u-),x)\Big((T_m^u-u+1)^{1-\beta_{1x'e}}-1\Big)\Big\},$$

where $T_m^u = \inf\{T_m > u : m=1,2\dots\}$.

We continue the computation by writing

$$\frac{\alpha_{1x'e}}{\beta_{1x'e}-1}\sum_{0\le T_n<t}\int_{[0,T_n)} du\ \phi_0(X(u),x')\,P(T>u)\,\phi_e(X(T_n),x)\cdot\left((T_{n+1}\wedge t-u+1)^{1-\beta_{1x'e}}-(T_n-u+1)^{1-\beta_{1x'e}}\right)$$

$$+\frac{\alpha_{1x'e}}{\beta_{1x'e}-1}\int_0^t du\ \phi_0(X(u),x')\,P(T>u)\,\phi_e(X(u-),x)\cdot\left((\text{first } T_n \text{ after } u-u+1)^{1-\beta_{1x'e}}-1\right)$$

$$=\frac{\alpha_{1x'e}}{\beta_{1x'e}-1}\sum_{0\le T_n<t}\phi_e(X(T_n),x)\sum_{0\le T_j<T_n}\phi_0(X(T_j),x')\int_{T_j}^{T_{j+1}} du\ P(T>u)\cdot\left((T_{n+1}\wedge t-u+1)^{1-\beta_{1x'e}}-(T_n-u+1)^{1-\beta_{1x'e}}\right)$$

$$+\frac{\alpha_{1x'e}}{\beta_{1x'e}-1}\sum_{0<T_n<t}\phi_0(X(T_n),x')\,\phi_e(X(T_{n+1}),x)\cdot\int_{T_n}^{T_{n+1}\wedge t} du\ P(T>u)\left((T_{n+1}-u+1)^{1-\beta_{1x'e}}-1\right)$$

$\square$