1 INTRODUCTION

Beijing National Olympic Stadium—Bird's Nest
(Eastimages / Shutterstock)

Structural analysis, which is an integral part of any structural engineering project, *is the process of predicting the performance of a given structure under a prescribed loading condition.* The performance characteristics usually of interest in structural design are: (a) stresses or stress resultants (i.e., axial forces, shears, and bending moments); (b) deflections; and (c) support reactions. Thus, the analysis of a structure typically involves the determination of these quantities as caused by the given loads and/or other external effects (such as support displacements and temperature changes). This text is devoted to the analysis of *framed structures*—that is, structures composed of long straight members. Many commonly used structures such as beams, and plane and space trusses and rigid frames, are classified as framed structures (also referred to as *skeletal structures*).

In most design offices today, the analysis of framed structures is routinely performed on computers, using software based on the matrix methods of structural analysis. It is therefore essential that structural engineers understand the basic principles of matrix analysis, so that they can develop their own computer programs and/or properly use commercially available software—and appreciate the physical significance of the analytical results. The objective of this text is to present the theory and computer implementation of matrix methods for the analysis of framed structures in static equilibrium.

This chapter provides a general introduction to the subject of matrix computer analysis of structures. We start with a brief historical background in Section 1.1, followed by a discussion of how matrix methods differ from classical and finite-element methods of structural analysis (Section 1.2). Flexibility and stiffness methods of matrix analysis are described in Section 1.3; the six types of framed structures considered in this text (namely, plane trusses, beams, plane frames, space trusses, grids, and space frames) are discussed in Section 1.4; and the development of simplified models of structures for the purpose of analysis is considered in Section 1.5. The basic concepts of structural analysis necessary for formulating the matrix methods, as presented in this text, are reviewed in Section 1.6; and the roles and limitations of linear and nonlinear types of structural analysis are discussed in Section 1.7. Finally, we conclude the chapter with a brief note on the computer software that is provided on the publisher's website for this book (Section 1.8). (www.cengage.com/engineering)

1.1 HISTORICAL BACKGROUND

The theoretical foundation for matrix methods of structural analysis was laid by James C. Maxwell, who introduced the method of consistent deformations in 1864; and George A. Maney, who developed the slope-deflection method in 1915. These classical methods are considered to be the precursors of the matrix flexibility and stiffness methods, respectively. In the precomputer era, the main disadvantage of these earlier methods was that they required direct solution of simultaneous algebraic equations—a formidable task by hand calculations in cases of more than a few unknowns.

The invention of computers in the late 1940s revolutionized structural analysis. As computers could solve large systems of simultaneous equations, the analysis methods yielding solutions in that form were no longer at a disadvantage, but in fact were preferred, because simultaneous equations could be expressed in matrix form and conveniently programmed for solution on computers.

S. Levy is generally considered to have been the first to introduce the flexibility method in 1947, by generalizing the classical method of consistent deformations. Among the subsequent researchers who extended the flexibility method and expressed it in matrix form in the early 1950s were H. Falkenheimer, B. Langefors, and P. H. Denke. The matrix stiffness method was developed by R. K. Livesley in 1954. In the same year, J. H. Argyris and S. Kelsey presented a formulation of matrix methods based on energy principles. In 1956, M. T. Turner, R. W. Clough, H. C. Martin, and L. J. Topp derived stiffness matrices for the members of trusses and frames using the finite-element approach, and introduced the now popular *direct stiffness method* for generating the structure stiffness matrix. In the same year, Livesley presented a nonlinear formulation of the stiffness method for stability analysis of frames.

Since the mid-1950s, the development of matrix methods has continued at a tremendous pace, with research efforts in recent years directed mainly toward formulating procedures for the dynamic and nonlinear analysis of structures, and developing efficient computational techniques for analyzing large structures. Recent advances in these areas can be attributed to S. S. Archer, C. Birnstiel, R. H. Gallagher, J. Padlog, J. S. Przemieniecki, C. K. Wang, and E. L. Wilson, among others.

1.2 CLASSICAL, MATRIX, AND FINITE-ELEMENT METHODS OF STRUCTURAL ANALYSIS

Classical versus Matrix Methods

As we develop matrix methods in subsequent chapters of this book, readers who are familiar with classical methods of structural analysis will realize that both matrix and classical methods are based on the same fundamental principles—but that the fundamental relationships of equilibrium, compatibility, and member stiffness are now expressed in the form of matrix equations, so that the numerical computations can be efficiently performed on a computer.

Most classical methods were developed to analyze particular types of structures, and since they were intended for hand calculations, they often involve certain assumptions (that are unnecessary in matrix methods) to reduce the amount of computational effort required for analysis. The application of these methods usually requires an understanding on the part of the analyst of the structural behavior. Consider, for example, the moment-distribution method. This classical method can be used to analyze only beams and plane frames undergoing bending deformations. Deformations due to axial forces in the frames are ignored to reduce the number of independent joint translations. While this assumption significantly reduces the computational effort, it complicates the analysis by requiring the analyst to draw a deflected shape of the frame corresponding to each degree of freedom of sidesway (independent joint translation), to estimate the relative magnitudes of member fixed-end moments: a difficult task even in the case of a few degrees of freedom of sidesway if the frame has inclined members. Because of their specialized and intricate nature, classical methods are generally not considered suitable for computer programming.

In contrast to classical methods, matrix methods were specifically developed for computer implementation; they are *systematic* (so that they can be conveniently programmed), and *general* (in the sense that the same overall format of the analytical procedure can be applied to the various types of framed structures). It will become clear as we study matrix methods that, because of the latter characteristic, a computer program developed to analyze one type of structure (e.g., plane trusses) can be modified with relative ease to analyze another type of structure (e.g., space trusses or frames).

As the analysis of large and highly redundant structures by classical methods can be quite time consuming, matrix methods are commonly used. However, classical methods are still preferred by many engineers for analyzing smaller structures, because they provide a better insight into the behavior of structures. Classical methods may also be used for preliminary designs, for checking the results of computerized analyses, and for deriving the member force–displacement relations needed in the matrix analysis. Furthermore, a study of classical methods is considered to be essential for developing an understanding of structural behavior.

Matrix versus Finite Element Methods

Matrix methods can be used to analyze framed structures only. Finite-element analysis, which originated as an extension of matrix analysis to surface structures (e.g., plates and shells), has now developed to the extent that it can be applied to structures and solids of practically any shape or form. From a theoretical viewpoint, the basic difference between the two is that, in matrix methods, the member force–displacement relationships are based on the exact solutions of the underlying differential equations, whereas in finite-element methods, such relations are generally derived by work-energy principles from assumed displacement or stress functions.

Because of the approximate nature of its force–displacement relations, finite-element analysis generally yields approximate results. However, as will be shown in Chapters 3 and 5, in the case of linear analysis of framed structures composed of prismatic (uniform) members, both matrix and finite-element approaches yield identical results.

1.3 FLEXIBILITY AND STIFFNESS METHODS

Two different methods can be used for the matrix analysis of structures: the *flexibility* method, and the *stiffness* method. The flexibility method, which is also referred to as the *force* or *compatibility* method, is essentially a generalization in matrix form of the classical method of consistent deformations. In this approach, the primary unknowns are the redundant forces, which are calculated first by solving the structure's compatibility equations. Once the redundant forces are known, the displacements can be evaluated by applying the equations of equilibrium and the appropriate member force–displacement relations.

The stiffness method, which originated from the classical slope-deflection method, is also called the *displacement* or *equilibrium* method. In this approach, the primary unknowns are the joint displacements, which are determined first by solving the structure's equations of equilibrium. With the joint displacements known, the unknown forces are obtained through compatibility considerations and the member force–displacement relations.

Although either method can be used to analyze framed structures, the flexibility method is generally convenient for analyzing small structures with a few redundants. This method may also be used to establish member force-displacement relations needed to develop the stiffness method. The stiffness method is more systematic and can be implemented more easily on computers; therefore, it is preferred for the analysis of large and highly redundant structures. Most of the commercially available software for structural analysis is based on the stiffness method. In this text, we focus our attention mainly on the stiffness method, with emphasis on a particular version known as the *direct stiffness method*, which is currently used in professional practice. The fundamental concepts of the flexibility method are presented in Appendix B.

1.4 CLASSIFICATION OF FRAMED STRUCTURES

Framed structures are composed of straight members whose lengths are significantly larger than their cross-sectional dimensions. Common framed structures can be classified into six basic categories based on the arrangement of their members, and the types of primary stresses that may develop in their members under major design loads.

Plane Trusses

A *truss* is defined as an assemblage of straight members connected at their ends by flexible connections, and subjected to loads and reactions only at the joints (connections). The members of such an ideal truss develop only axial forces when the truss is loaded. In real trusses, such as those commonly used for supporting roofs and bridges, the members are connected by bolted or welded connections that are not perfectly flexible, and the dead weights of the members are distributed along their lengths. Because of these and other deviations from idealized conditions, truss members are subjected to some bending and shear. However, in most trusses, these secondary bending moments and shears are small in comparison to the primary axial forces, and are usually not considered in their designs. If large bending moments and shears are anticipated, then the truss should be treated as a rigid frame (discussed subsequently) for analysis and design.

If all the members of a truss as well as the applied loads lie in a single plane, the truss is classified as a *plane truss* (Fig. 1.1). The members of plane trusses are assumed to be connected by frictionless hinges. The analysis of plane trusses is considerably simpler than the analysis of space (or three-dimensional) trusses. Fortunately, many commonly used trusses, such as bridge and roof trusses, can be treated as plane trusses for analysis (Fig. 1.2).

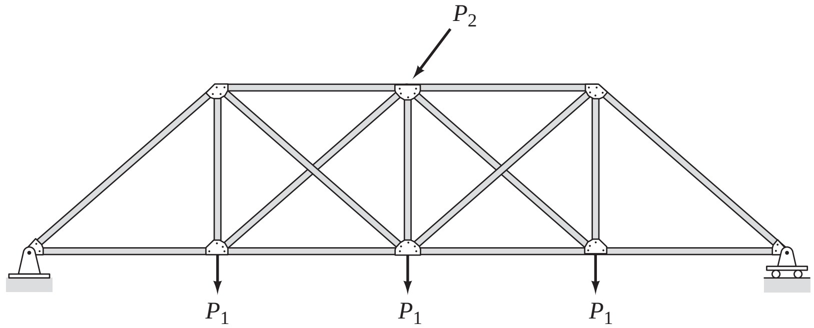


Fig. 1.1 *Plane Truss*

Fig. 1.2 *Roof Truss*
(Photo courtesy of Bethlehem Steel Corporation)

Beams

A *beam* is defined as a long straight structure that is loaded perpendicular to its longitudinal axis (Fig. 1.3). Loads are usually applied in a plane of symmetry of the beam's cross-section, causing its members to be subjected only to bending moments and shear forces.

Plane Frames

Frames, also referred to as *rigid frames,* are composed of straight members connected by rigid (moment resisting) and/or flexible connections (Fig. 1.4). Unlike trusses, which are subjected to external loads only at the joints, loads on frames may be applied on the joints as well as on the members.

If all the members of a frame and the applied loads lie in a single plane, the frame is called a *plane frame* (Fig. 1.5). The members of a plane frame are, in

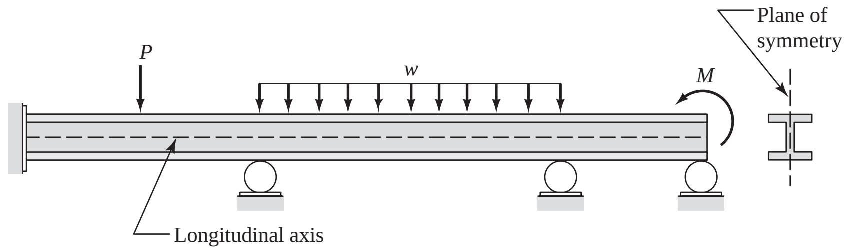


Fig. 1.3 *Beam*

Fig. 1.4 *Skeleton of a Structural Steel Frame Building*
(Joe Gough / Shutterstock)

general, subjected to bending moments, shears, and axial forces under the action of external loads. Many actual three-dimensional building frames can be subdivided into plane frames for analysis.

Space Trusses

Some trusses (such as lattice domes, transmission towers, and certain aerospace structures (Fig. 1.6)) cannot be treated as plane trusses because of the arrangement of their members or applied loading. Such trusses, referred to as *space trusses*, are analyzed as three-dimensional structures subjected to three-dimensional force systems. The members of space trusses are assumed to be connected by frictionless ball-and-socket joints, and the trusses are subjected to loads and reactions only at the joints. Like plane trusses, the members of space trusses develop only axial forces.

Grids

A *grid*, like a plane frame, is composed of straight members connected together by rigid and/or flexible connections to form a plane framework. The

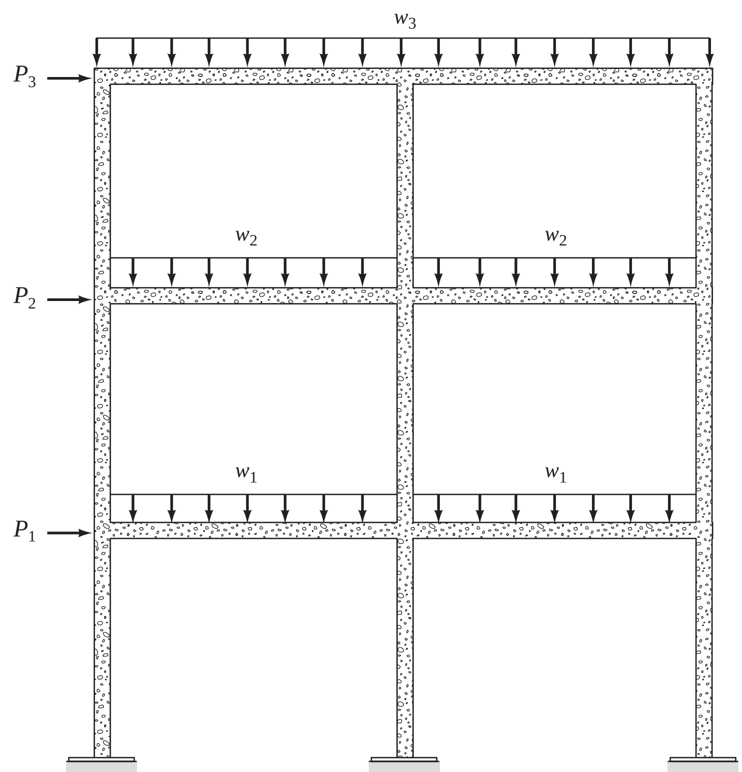


Fig. 1.5 *Plane Frame*

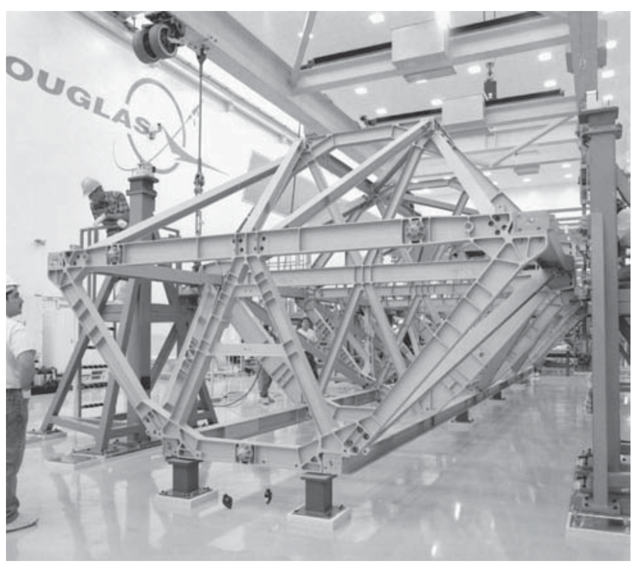


Fig. 1.6 *A Segment of the Integrated Truss Structure which Forms the Backbone of the International Space Station*

(Photo Courtesy of National Aeronautics and Space Administration 98-05165)

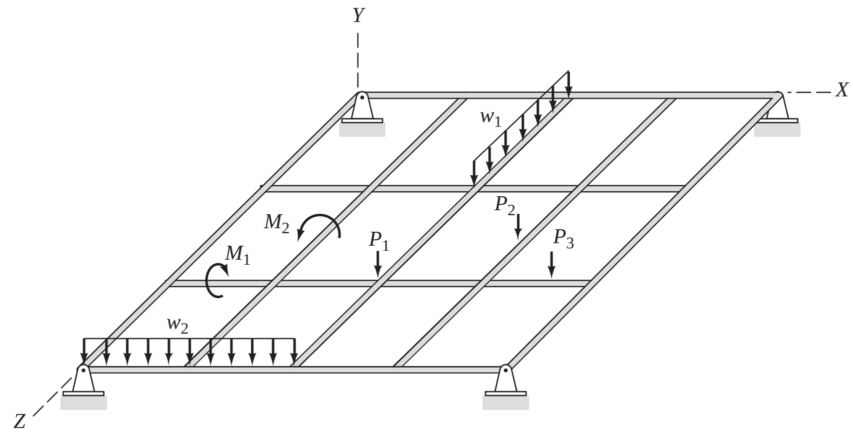


Fig. 1.7 *Grid*

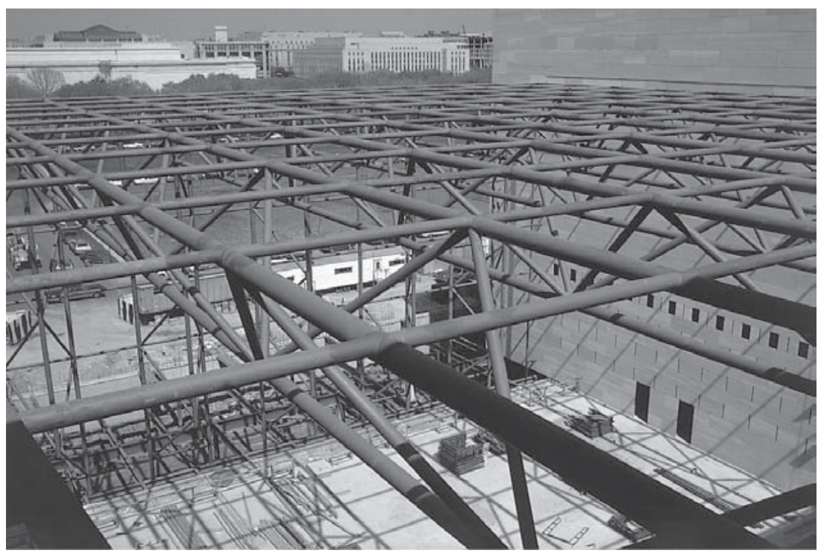


Fig. 1.8 *National Air and Space Museum, Washington, DC (under construction)*
(Photo courtesy of Bethlehem Steel Corporation)

main difference between the two types of structures is that plane frames are loaded in the plane of the structure, whereas the loads on grids are applied in the direction perpendicular to the structure's plane (Fig. 1.7). Members of grids may, therefore, be subjected to torsional moments, in addition to the bending moments and corresponding shears that cause the members to bend out of the plane of the structure. Grids are commonly used for supporting roofs covering large column-free areas in such structures as sports arenas, auditoriums, and aircraft hangars (Fig. 1.8).

Fig. 1.9 *Space Frame*
(© MNTravel / Alamy)

Space Frames

Space frames constitute the most general category of framed structures. Members of space frames may be arranged in any arbitrary directions, and connected by rigid and/or flexible connections. Loads in any directions may be applied on members as well as on joints. The members of a space frame may, in general, be subjected to bending moments about both principal axes, shears in both principal directions, torsional moments, and axial forces (Fig. 1.9).

1.5 ANALYTICAL MODELS

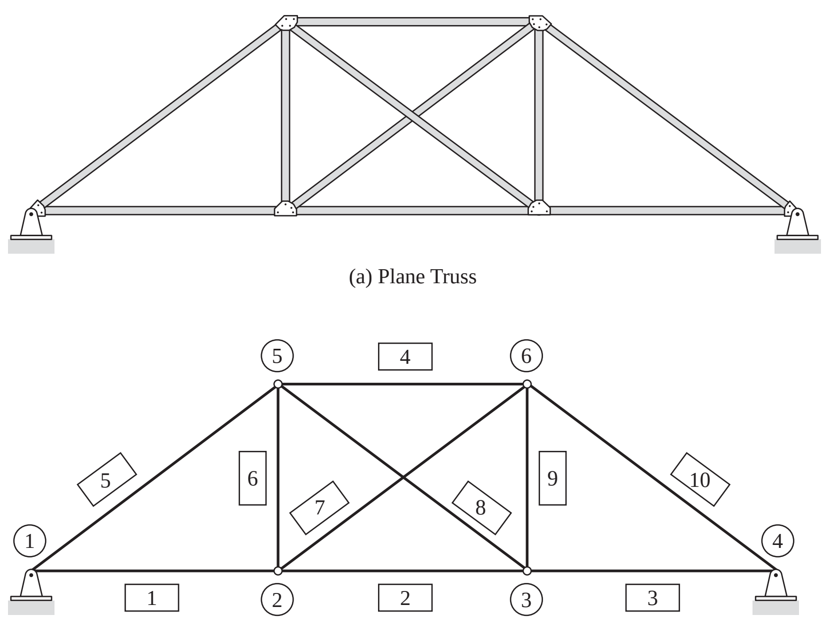
The first (and perhaps most important) step in the analysis of a structure is to develop its analytical model. An analytical model is an idealized representation of a real structure for the purpose of analysis. Its objective is to simplify the analysis of a complicated structure by discarding much of the detail (about connections, members, etc.) that is likely to have little effect on the structure's behavioral characteristics of interest, while representing, as accurately as practically possible, the desired characteristics. It is important to note that the structural response predicted from an analysis is valid only to the extent that the analytical model represents the actual structure. For framed structures, the establishment of analytical models generally involves consideration of issues such as whether the actual three-dimensional structure can be subdivided into plane structures for analysis, and whether to idealize the actual bolted or welded connections as hinged, rigid, or semirigid joints. Thus, the development of accurate analytical models requires not only a thorough understanding of structural behavior and methods of analysis, but also experience and knowledge of design and construction practices.

In matrix methods of analysis, a structure is modeled as an assemblage of straight members connected at their ends to joints. A *member* is defined as *a part of the structure for which the member force-displacement relationships to be used in the analysis are valid.* The member force-displacement relationships for the various types of framed structures will be derived in subsequent chapters. A *joint* is defined as *a structural part of infinitesimal size to which the ends of the members are connected*. In finite-element terminology, the members and joints of structures are generally referred to as *elements* and *nodes*, respectively.

Supports for framed structures are commonly idealized as fixed supports, which do not allow any displacement; hinged supports, which allow rotation but prevent translation; or, roller or link supports, which prevent translation in only one direction. Other types of restraints, such as those which prevent rotation but permit translation in one or more directions, can also be considered in an analysis, as discussed in subsequent chapters.

Line Diagrams

The analytical model of a structure is represented by a *line diagram*, on which each member is depicted by a line coinciding with its centroidal axis. The member dimensions and the size of connections are not shown. Rigid joints are usually represented by points, and hinged joints by small circles, at the intersections of members. Each joint and member of the structure is identified by a number. For example, the analytical model of the plane truss of Fig. 1.10(a) is shown in Fig. 1.10(b), in which the joint numbers are enclosed within circles to distinguish them from the member numbers enclosed within rectangles.



(a) Plane Truss

(b) Analytical Model

Fig. 1.10

1.6 FUNDAMENTAL RELATIONSHIPS FOR STRUCTURAL ANALYSIS

Structural analysis, in general, involves the use of three types of relationships:

- Equilibrium equations,
- compatibility conditions, and
- constitutive relations.

Equilibrium Equations

A structure is considered to be in equilibrium if, initially at rest, it remains at rest when subjected to a system of forces and couples. If a structure is in equilibrium, then all of its members and joints must also be in equilibrium.

Recall from *statics* that for a plane (two-dimensional) structure lying in the XY plane and subjected to a coplanar system of forces and couples (Fig. 1.11), the necessary and sufficient conditions for equilibrium can be expressed in Cartesian (XY) coordinates as

$$\boxed{\sum F_X = 0 \qquad \sum F_Y = 0 \qquad \sum M = 0} \tag{1.1}$$

These equations are referred to as the *equations of equilibrium* for plane structures.

For a space (three-dimensional) structure subjected to a general three-dimensional system of forces and couples (Fig. 1.12), the equations of equilibrium are expressed as

$$\boxed{\begin{array}{lll} \sum F_X = 0 & \sum F_Y = 0 & \sum F_Z = 0 \\ \sum M_X = 0 & \sum M_Y = 0 & \sum M_Z = 0 \end{array}} \tag{1.2}$$

For a structure subjected to static loading, the equilibrium equations must be satisfied for the entire structure as well as for each of its members and joints. In structural analysis, equations of equilibrium are used to relate the forces (including couples) acting on the structure or one of its members or joints.

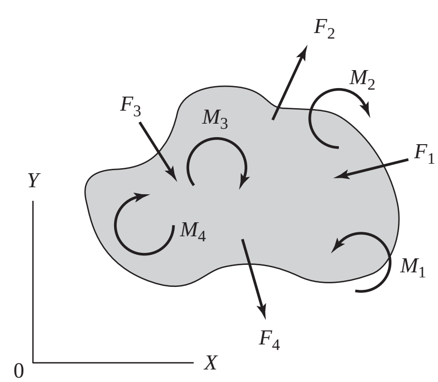


Fig. 1.11

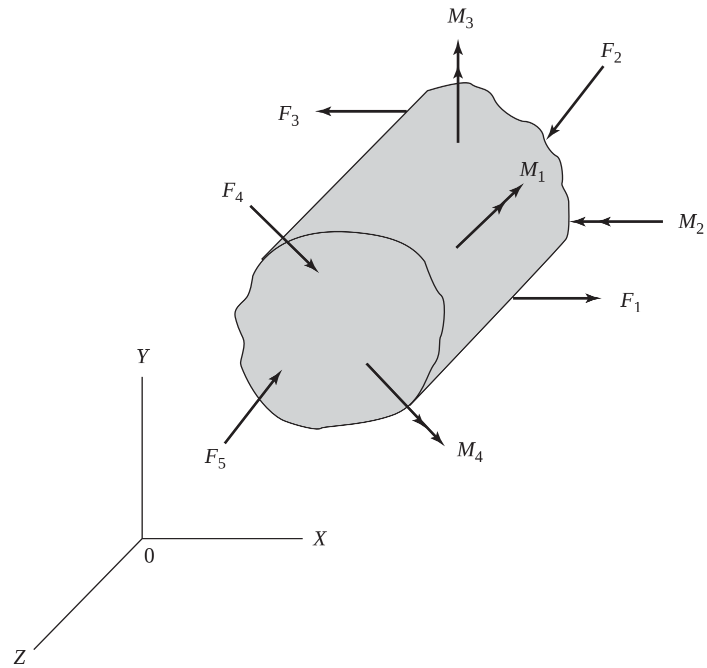


Fig. 1.12

Compatibility Conditions

The *compatibility conditions* relate the deformations of a structure so that its various parts (members, joints, and supports) fit together without any gaps or overlaps. These conditions (also referred to as the *continuity conditions*) ensure that the deformed shape of the structure is continuous (except at the locations of any internal hinges or rollers), and is consistent with the support conditions.

Consider, for example, the two-member plane frame shown in Fig. 1.13. The deformed shape of the frame due to an arbitrary loading is also depicted, using an exaggerated scale. When analyzing a structure, the compatibility conditions are used to relate member end displacements to joint displacements which, in turn, are related to the support conditions. For example, because joint 1 of the frame in Fig. 1.13 is attached to a roller support that cannot translate in the vertical direction, the vertical displacement of this joint must be zero. Similarly, because joint 3 is attached to a fixed support that can neither rotate nor translate in any direction, the rotation and the horizontal and vertical displacements of joint 3 must be zero.

The displacements of the ends of members are related to the joint displacements by the compatibility requirement that the displacements of a member's end must be the same as the displacements of the joint to which the member end is connected. Thus, as shown in Fig. 1.13, because joint 1 of the example frame displaces to the right by a distance d_1 and rotates clockwise by an angle θ_1, the left end of the horizontal member (member 1) that is attached to joint 1

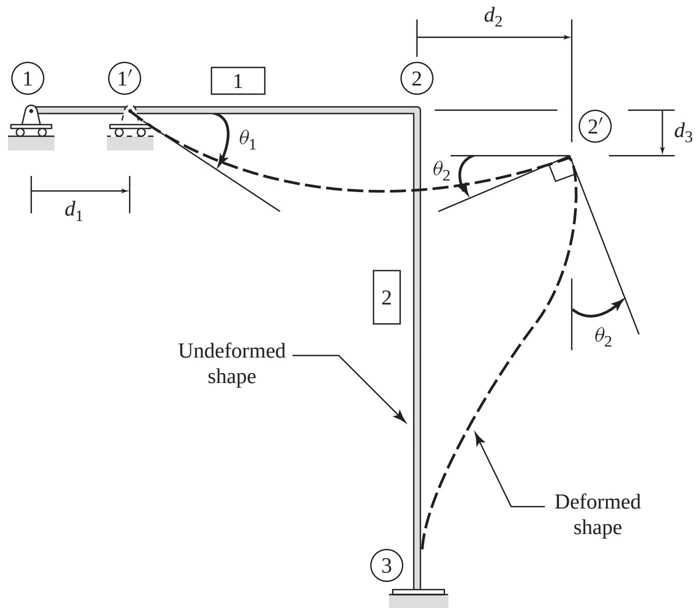


Fig. 1.13

must also translate to the right by distance d_1 and rotate clockwise by angle θ_1. Similarly, because the displacements of joint 2 consist of the translations d_2 to the right and d_3 downward and the counterclockwise rotation θ_2, the right end of the horizontal member and the top end of the vertical member that are connected to joint 2 must also undergo the same displacements (i.e., d_2, d_3, and θ_2). The bottom end of the vertical member, however, is not subjected to any displacements, because joint 3, to which this particular member end is attached, can neither rotate nor translate in any direction.

Finally, compatibility requires that the deflected shapes of the members of a structure be continuous (except at any internal hinges or rollers) and be consistent with the displacements at the corresponding ends of the members.

Constitutive Relations

The *constitutive relations* (also referred to as the *stress-strain relations*) describe the relationships between the stresses and strains of a structure in accordance with the stress-strain properties of the structural material. As discussed previously, the equilibrium equations provide relationships between the forces, whereas the compatibility conditions involve only deformations. The constitutive relations provide the link between the equilibrium equations and compatibility conditions that is necessary to establish the load-deformation relationships for a structure or a member.

In the analysis of framed structures, the basic stress-strain relations are first used, along with the member equilibrium and compatibility equations, to establish relationships between the forces and displacements at the ends of a member. The member force-displacement relations thus obtained are then treated as the constitutive relations for the entire structure, and are used to link the structure's equilibrium and compatibility equations, thereby yielding the load-deformation relationships for the entire structure. These load-deformation relations can then be solved to determine the deformations of the structure due to a given loading.

In the case of statically determinate structures, the equilibrium equations can be solved independently of the compatibility and constitutive relations to obtain the reactions and member forces. The deformations of the structure, if desired, can then be determined by employing the compatibility and constitutive relations. In the analysis of statically indeterminate structures, however, the equilibrium equations alone are not sufficient for determining the reactions and member forces. Therefore, it becomes necessary to satisfy simultaneously the three types of fundamental relationships (i.e., equilibrium, compatibility, and constitutive relations) to determine the structural response.

Matrix methods of structural analysis are usually formulated by direct application of the three fundamental relationships as described in general terms in the preceding paragraphs. (Details of the formulations are presented in subsequent chapters.) However, matrix methods can also be formulated by using work-energy principles that satisfy the three fundamental relationships indirectly. Work-energy principles are generally preferred in the formulation of finite-element methods, because they can be more conveniently applied to derive the approximate force-displacement relations for the elements of surface structures and solids.

The matrix methods presented in this text are formulated by the direct application of the equilibrium, compatibility, and constitutive relationships. However, to introduce readers to the finite-element method, and to familiarize them with the application of the work-energy principles, we also derive the member force-displacement relations for plane structures by a finite-element approach that involves a work-energy principle known as the *principle of virtual work*. In the following paragraphs, we review two statements of this principle pertaining to rigid bodies and deformable bodies, for future reference.

Principle of Virtual Work for Rigid Bodies

The *principle of virtual work for rigid bodies* (also known as the *principle of virtual displacements for rigid bodies*) can be stated as follows.

> *If a rigid body, which is in equilibrium under a system of forces (and couples), is subjected to any small virtual rigid-body displacement, the virtual work done by the external forces (and couples) is zero.*

In the foregoing statement, the term virtual simply means imaginary, not real. Consider, for example, the cantilever beam shown in Fig. 1.14(a). The free-body diagram of the beam is shown in Fig. 1.14(b), in which P_X, and P_Y are the components of the external load P in the X and Y directions, respectively, and R_1, R_2, and R_3 represent the reactions at the fixed support 1. Note that the beam is in equilibrium under the action of the forces P_X, P_Y, R_1, and R_2, and the couple R_3. Now, imagine that the beam is given an arbitrary, small virtual rigid-body displacement from its initial equilibrium position 1–2 to another position 1′–2′, as shown in Fig. 1.14(c). As this figure indicates, the total virtual

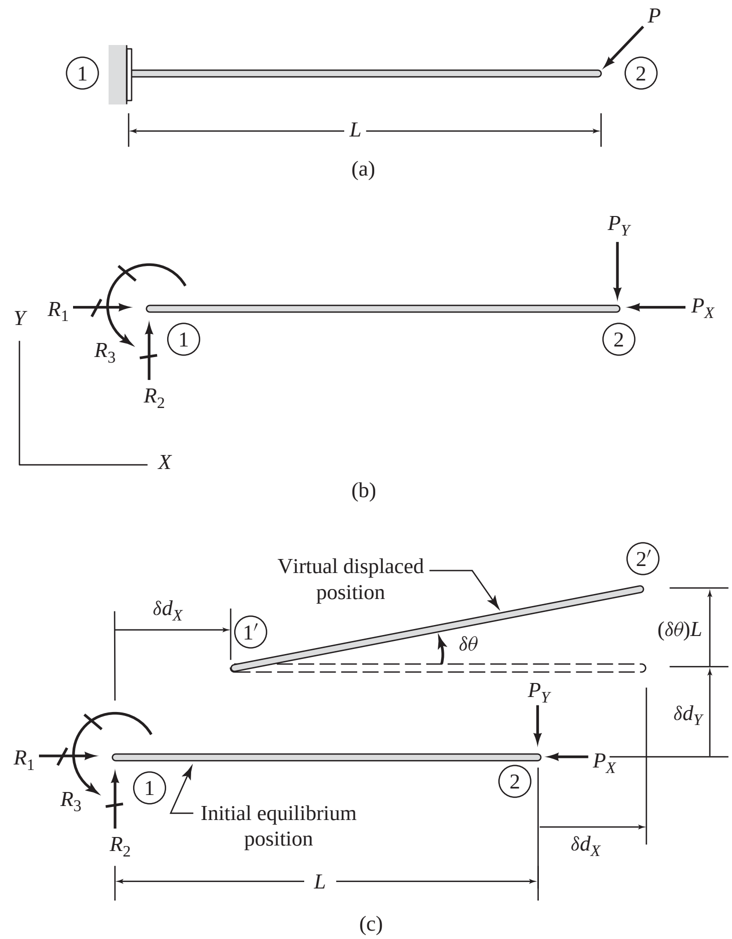


Fig. 1.14

displacement of the beam can be decomposed into rigid-body translations δd_X and δd_Y in the X and Y directions, respectively, and a rigid-body rotation $\delta\theta$ about point 1. Note that the symbol δ is used here to identify the virtual quantities. As the beam undergoes the virtual displacement from position 1–2 to position 1′–2′, the forces and the couple acting on it perform work, which is referred to as the *virtual work*. The total virtual work, δW_e, can be expressed as the algebraic sum of the virtual work δW_X and δW_Y, performed during translations in the X and Y directions, respectively, and the virtual work δW_R, done during the rotation; that is,

$$\delta W_e = \delta W_X + \delta W_Y + \delta W_R \tag{1.3}$$

During the virtual translation δd_X of the beam, the virtual work performed by the forces can be expressed as follows (Fig 1.14c).

$$\delta W_X = R_1 \delta d_X - P_X \delta d_X = (R_1 - P_X)\,\delta d_X = \left(\sum F_X\right)\delta d_X \tag{1.4}$$

Similarly, the virtual work done during the virtual translation δd_Y is given by

$$\delta W_Y = R_2 \delta d_Y - P_Y \delta d_Y = (R_2 - P_Y)\, \delta d_Y = \left(\sum F_Y\right) \delta d_Y \tag{1.5}$$

and the virtual work done by the forces and the couple during the small virtual rotation $\delta\theta$ can be expressed as follows (Fig. 1.14c).

$$\delta W_R = R_3 \delta\theta - P_Y\, (L\delta\theta) = (R_3 - P_Y L)\, \delta\theta = \left(\sum M_{①}\right) \delta\theta \tag{1.6}$$

The expression for the total virtual work can now be obtained by substituting Eqs. (1.4–1.6) into Eq. (1.3). Thus,

$$\delta W_e = \left(\sum F_X\right) \delta d_X + \left(\sum F_Y\right) \delta d_Y + \left(\sum M_{①}\right) \delta\theta \tag{1.7}$$

However, because the beam is in equilibrium, $\sum F_X = 0$, $\sum F_Y = 0$, and $\sum M_{①} = 0$; therefore, Eq. (1.7) becomes

$$\boxed{\delta W_e = 0} \tag{1.8}$$

which is the mathematical statement of the principle of virtual work for rigid bodies.

Principle of Virtual Work for Deformable Bodies

The *principle of virtual work for deformable bodies* (also called the *principle of virtual displacements for deformable bodies*) can be stated as follows.

> *If a deformable structure, which is in equilibrium under a system of forces (and couples), is subjected to any small virtual displacement consistent with the support and continuity conditions of the structure, then the virtual external work done by the real external forces (and couples) acting through the virtual external displacements (and rotations) is equal to the virtual strain energy stored in the structure.*

To demonstrate the validity of this principle, consider the two-member truss of Fig. 1.15(a), which is in equilibrium under the action of an external load P. The free-body diagram of joint 3 of the truss is shown in Fig. 1.15(b). Since joint 3 is in equilibrium, the external and internal forces acting on it must satisfy the following two equations of equilibrium:

$$\begin{aligned} &+\rightarrow \sum F_X = 0 \qquad && -F_1 \sin\theta_1 + F_2 \sin\theta_2 = 0 \\ &+\uparrow \sum F_Y = 0 \qquad && F_1 \cos\theta_1 + F_2 \cos\theta_2 - P = 0 \end{aligned} \tag{1.9}$$

in which F_1 and F_2 denote the internal (axial) forces in members 1 and 2, respectively; and θ_1 and θ_2 are, respectively, the angles of inclination of these members with respect to the vertical as shown in the figure.

Now, imagine that joint 3 is given a small virtual compatible displacement, δd, in the downward direction, as shown in Fig. 1.15(a). It should be noted that this virtual displacement is consistent with the support conditions of the truss in the sense that joints 1 and 2, which are attached to supports, are not displaced. Because the reaction forces at joints 1 and 2 do not perform any work,

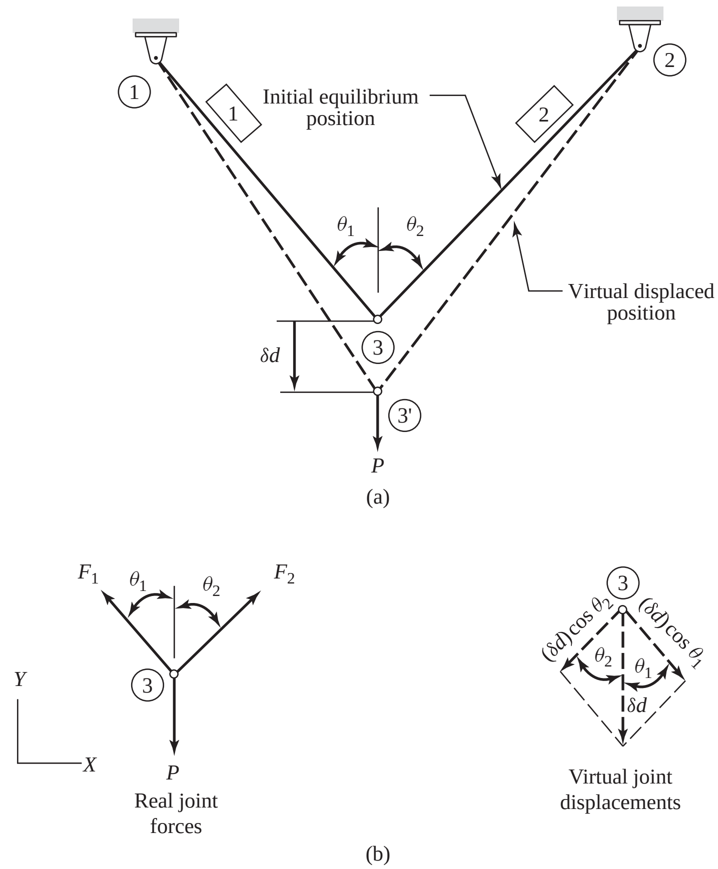


Fig. 1.15

the total virtual work for the truss, δW, is equal to the algebraic sum of the virtual work of the forces acting at joint 3. Thus, from Fig. 1.15(b),

$$\delta W = P\delta d - F_1(\delta d\cos\theta_1) - F_2(\delta d\cos\theta_2)$$

which can be rewritten as

$$\delta W = (P - F_1\cos\theta_1 - F_2\cos\theta_2)\,\delta d \tag{1.10}$$

As indicated by Eq. (1.9), the term in parentheses on the right-hand side of Eq. (1.10) is zero. Therefore, the total virtual work, δW, is zero. By substituting $\delta W = 0$ into Eq. (1.10) and rearranging terms, we write

$$P(\delta d) = F_1(\delta d\cos\theta_1) + F_2(\delta d\cos\theta_2) \tag{1.11}$$

in which the quantity on the left-hand side represents the virtual external work, δW_e, performed by the real external force P acting through the virtual external displacement δd. Furthermore, because the terms $(\delta d)\cos\theta_1$ and $(\delta d)\cos\theta_2$ are equal to the virtual internal displacements (elongations) of members 1 and 2, respectively, we can conclude that the right-hand side of Eq. (1.11) represents the virtual internal work, δW_{i}, done by the real internal forces acting through the corresponding virtual internal displacements; that is,

$$\delta W_{\mathrm{e}} = \delta W_{\mathrm{i}} \tag{1.12}$$

Realizing that the internal work is also referred to as the strain energy, U, we can express Eq. (1.12) as

$$\boxed{\delta W_{\mathrm{e}} = \delta U} \tag{1.13}$$

in which δU denotes the virtual strain energy. Note that Eq. (1.13) is the mathematical statement of the principle of virtual work for deformable bodies.

For computational purposes, it is usually convenient to express Eq. (1.13) in terms of the stresses and strains in the members of the structure. For that purpose, let us consider a differential element of a member of an arbitrary structure subjected to a general loading (Fig. 1.16). The element is in equilibrium under a general three-dimensional stress condition, due to the real forces acting on the structure. Now, as the structure is subjected to a virtual displacement, virtual strains develop in the element and the internal forces due to the real stresses perform virtual internal work as they move through the internal displacements caused by the virtual strains. For example, the virtual internal work done by the real force due to the stress σ_x as it moves through the virtual displacement caused by the virtual strain $\delta\varepsilon_x$ can be determined as follows.

$$\begin{aligned} \text{real force} = \text{stress} \times \text{area} &= \sigma_x\,(dy\,dz) \\ \text{virtual displacement} = \text{strain} \times \text{length} &= (\delta\varepsilon_x)\,dx \end{aligned}$$

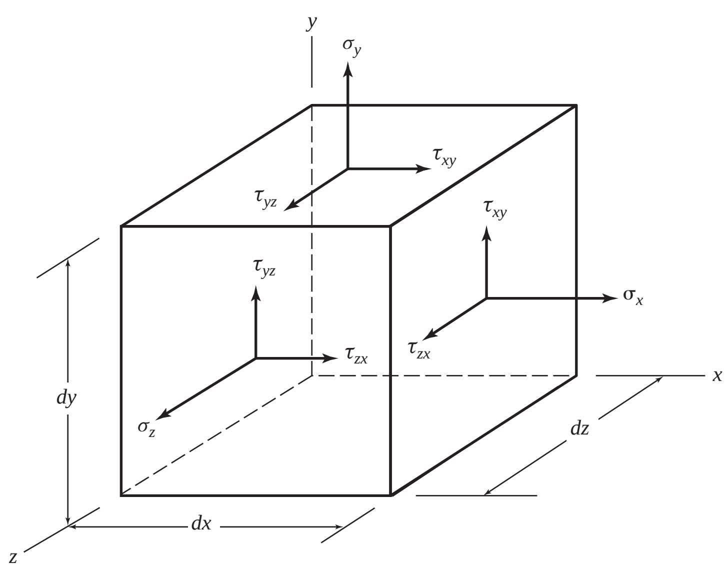


Fig. 1.16

Therefore,

$$\begin{aligned}\text{virtual internal work} &= \text{real force} \times \text{virtual displacement}\\ &= (\sigma_x\, dy\, dz)\,(\delta\varepsilon_x\, dx)\\ &= (\delta\varepsilon_x\, \sigma_x)\, dV\end{aligned}$$

in which $dV = dx\,dy\,dz$ is the volume of the differential element. Thus, the virtual internal work due to all six stress components is given by

$$\text{virtual internal work in element } dV$$
$$= (\delta\varepsilon_x\sigma_x + \delta\varepsilon_y\sigma_y + \delta\varepsilon_z\sigma_z + \delta\gamma_{xy}\tau_{xy} + \delta\gamma_{yz}\tau_{yz} + \delta\gamma_{zx}\tau_{zx})\, dV \quad \textbf{(1.14)}$$

In Eq. (1.14), $\delta\varepsilon_x, \delta\varepsilon_y, \delta\varepsilon_z, \delta\gamma_{xy}, \delta\gamma_{yz},$ and $\delta\gamma_{zx}$ denote, respectively, the virtual strains corresponding to the real stresses $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz},$ and τ_{zx}, shown in Fig. 1.16.

The total virtual internal work, or the virtual strain energy stored in the entire structure, can be obtained by integrating Eq. (1.14) over the volume V of the structure. Thus,

$$\delta U = \int_V \left(\delta\varepsilon_x\sigma_x + \delta\varepsilon_y\sigma_y + \delta\varepsilon_z\sigma_z + \delta\gamma_{xy}\tau_{xy} + \delta\gamma_{yz}\tau_{yz} + \delta\gamma_{zx}\tau_{zx}\right) dV \quad \textbf{(1.15)}$$

Finally, by substituting Eq. (1.15) into Eq. (1.13), we obtain the statement of the principle of virtual work for deformable bodies in terms of the stresses and strains of the structure.

$$\boxed{\delta W_e = \int_V \left(\delta\varepsilon_x\sigma_x + \delta\varepsilon_y\sigma_y + \delta\varepsilon_z\sigma_z + \delta\gamma_{xy}\tau_{xy} + \delta\gamma_{yz}\tau_{yz} + \delta\gamma_{zx}\tau_{zx}\right) dV} \quad \textbf{(1.16)}$$

1.7 LINEAR VERSUS NONLINEAR ANALYSIS

In this text, we focus our attention mainly on *linear analysis* of structures. Linear analysis of structures is based on the following two fundamental assumptions:

1. The structures are composed of linearly elastic material; that is, the stress-strain relationship for the structural material follows Hooke's law.
2. The deformations of the structures are so small that the squares and higher powers of member slopes, (chord) rotations, and axial strains are negligible in comparison with unity, and the equations of equilibrium can be based on the undeformed geometry of the structure.

The reason for making these assumptions is to obtain linear relationships between applied loads and the resulting structural deformations. An important advantage of linear force-deformation relations is that the *principle of* *superposition* can be used in the analysis. This principle states essentially that *the combined effect of several loads acting simultaneously on a structure equals the algebraic sum of the effects of each load acting individually on the structure*.

Engineering structures are usually designed so that under service loads they undergo small deformations, with stresses within the initial linear portions of the stress-strain curves of their materials. Thus, linear analysis generally proves adequate for predicting the performance of most common types of structures under service loading conditions. However, at higher load levels, the accuracy of linear analysis generally deteriorates as the deformations of the structure increase and/or its material is strained beyond the yield point. Because of its inherent limitations, linear analysis cannot be used to predict the ultimate load capacities and instability characteristics (e.g., buckling loads) of structures.

With the recent introduction of design specifications based on the ultimate strengths of structures, the use of *nonlinear analysis* in structural design is increasing. In a nonlinear analysis, the restrictions of linear analysis are removed by formulating the equations of equilibrium on the deformed geometry of the structure that is not known in advance, and/or taking into account the effects of inelasticity of the structural material. The load-deformation relationships thus obtained for the structure are nonlinear, and are usually solved using iterative techniques. An introduction to this still-evolving field of nonlinear structural analysis is presented in Chapter 10.

1.8 SOFTWARE

Software for the analysis of framed structures using the matrix stiffness method is provided on the publisher's website for this book, www.cengage.com/engineering. The software can be used by readers to verify the correctness of various subroutines and programs that they will develop during the course of study of this text, as well as to check the answers to the problems given at the end of each chapter. A description of the software, and information on how to install and use it, is presented in Appendix A.

SUMMARY

In this chapter, we discussed the topics summarized in the following list.

1. *Structural analysis* is the prediction of the performance of a given structure under prescribed loads and/or other external effects.

2. Both matrix and classical methods of structural analysis are based on the same fundamental principles. However, classical methods were developed to analyze particular types of structures, whereas matrix methods are more general and systematic so that they can be conveniently programmed on computers.

3. Two different methods can be used for matrix analysis of structures; namely, the *flexibility* and *stiffness* methods. The stiffness method is more systematic and can be implemented more easily on computers, and is therefore currently preferred in professional practice.

4. *Framed structures* are composed of straight members whose lengths are significantly larger than their cross-sectional dimensions. Framed structures can be classified into six basic categories: plane trusses, beams, plane frames, space trusses, grids, and space frames.

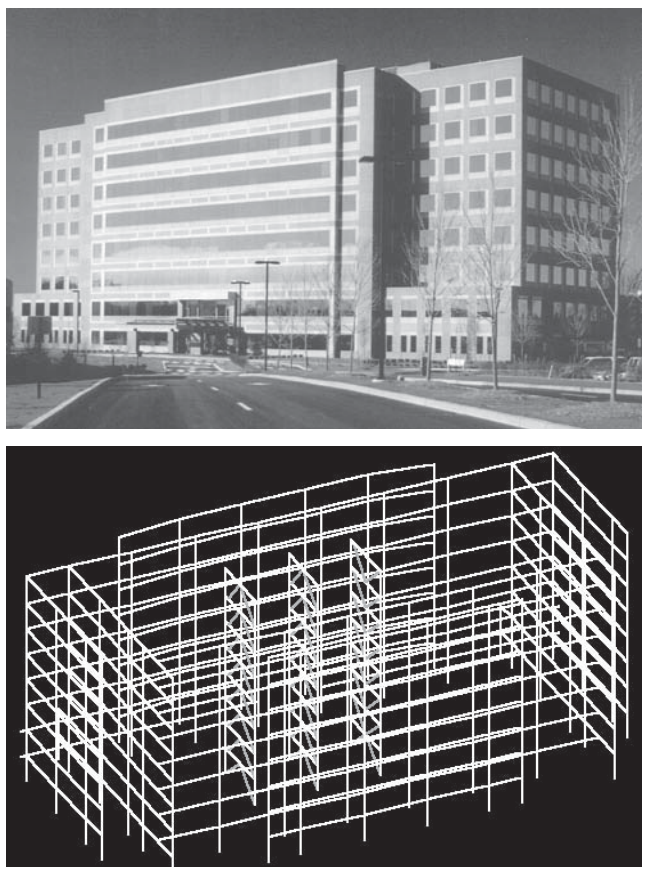
5. An *analytical model* is a simplified (idealized) representation of a real structure for the purpose of analysis. Framed structures are modeled as assemblages of straight members connected at their ends to joints, and these analytical models are represented by *line diagrams*.

6. The analysis of structures involves three fundamental relationships: equilibrium equations, compatibility conditions, and constitutive relations.

7. The principle of virtual work for deformable bodies states that if a deformable structure, which is in equilibrium, is subjected to a small compatible virtual displacement, then the virtual external work is equal to the virtual strain energy stored in the structure.

8. Linear structural analysis is based on two fundamental assumptions: the stress-strain relationship for the structural material is linearly elastic, and the structure's deformations are so small that the equilibrium equations can be based on the undeformed geometry of the structure.

2 MATRIX ALGEBRA



Somerset Corporate Center Office Building, New ersey, and its Analytical Model
(Photo courtesy of Ram International. Structural Engineer: The Cantor Seinuk Group, P.C.)

In matrix methods of structural analysis, the fundamental relationships of equilibrium, compatibility, and member force–displacement relations are expressed in the form of matrix equations, and the analytical procedures are formulated by applying various matrix operations. Therefore, familiarity with the basic concepts of matrix algebra is a prerequisite to understanding matrix structural analysis. The objective of this chapter is to concisely present the basic concepts of matrix algebra necessary for formulating the methods of structural analysis covered in the text. A general procedure for solving simultaneous linear equations, the *Gauss ordan method,* is also discussed.

We begin with the basic definition of a matrix in Section 2.1, followed by brief descriptions of the various types of matrices in Section 2.2. The matrix operations of equality, addition and subtraction, multiplication, transposition, differentiation and integration, inversion, and partitioning are defined in Section 2.3; we conclude the chapter with a discussion of the Gauss–Jordan elimination method for solving simultaneous equations (Section 2.4).

2.1 DEFINITION OF A MATRIX

A *matrix* is defined as *a rectangular array of quantities arranged in rows and columns.* A matrix with m rows and n columns can be expressed as follows.

$$\mathbf{A} = A = \begin{array}{c} \left[\begin{array}{ccccccc} A_{11} & A_{12} & A_{13} & \cdots & & \cdots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \cdots & & \cdots & A_{2n} \\ A_{31} & A_{32} & A_{33} & \cdots & & \cdots & A_{3n} \\ \cdots & \cdots & \cdots & \cdots & A_{ij} & \cdots & \\ A_{m1} & A_{m2} & A_{m3} & \cdots & & \cdots & A_{mn} \end{array}\right] \begin{array}{l} \\ \\ \\ i\text{th row} \\ \\ \end{array} \\ j\text{th column} \qquad\quad m \times n \end{array} \tag{2.1}$$

As shown in Eq. (2.1), matrices are denoted either by boldface letters (**A**) or by italic letters enclosed within brackets (A). The quantities forming a matrix are referred to as its *elements.* The elements of a matrix are usually numbers, but they can be symbols, equations, or even other matrices (called submatrices). Each element of a matrix is represented by a double-subscripted letter, with the first subscript identifying the row and the second subscript identifying the column in which the element is located. Thus, in Eq. (2.1), A_{23} represents the element located in the second row and third column of matrix **A**. In general, A_{ij} refers to an element located in the ith row and jth column of matrix **A**.

The size of a matrix is measured by the number of its rows and columns and is referred to as the *order* of the matrix. Thus, matrix **A** in Eq. (2.1), which has m rows and n columns, is considered to be of order $m \times n$ (m by n). As an example, consider a matrix **D** given by

$$\mathbf{D} = \begin{bmatrix} 3 & 5 & 37 \\ 8 & -6 & 0 \\ 12 & 23 & 2 \\ 7 & -9 & -1 \end{bmatrix}$$

The order of this matrix is 4×3, and its elements are symbolically denoted by D_{ij} with $i = 1$ to 4 and $j = 1$ to 3; for example, $D_{13} = 37$, $D_{31} = 12$, $D_{42} = -9$, etc.

2.2 TYPES OF MATRICES

We describe some of the common types of matrices in the following paragraphs.

Column Matrix (Vector)

If all the elements of a matrix are arranged in a single column (i.e., $n = 1$), it is called a *column matrix*. Column matrices are usually referred to as *vectors,* and are sometimes denoted by italic letters enclosed within braces. An example of a column matrix or vector is given by

$$\mathbf{B} = \{B\} = \begin{bmatrix} 35 \\ 9 \\ 12 \\ 3 \\ 26 \end{bmatrix}$$

Row Matrix

A matrix with all of its elements arranged in a single row (i.e., $m = 1$) is referred to as a *row matrix.* For example,

$$\mathbf{C} = 9 \quad 35 \quad -12 \quad 7 \quad 22$$

Square Matrix

If a matrix has the same number of rows and columns (i.e., $m = n$), it is called a *square matrix*. An example of a 4×4 square matrix is given by

$$\mathbf{A} = \begin{bmatrix} 6 & 12 & 0 & 20 \\ 15 & -9 & -37 & 3 \\ -24 & 13 & 8 & 1 \\ 40 & 0 & 11 & -5 \end{bmatrix} \qquad \textbf{(2.2)}$$

main diagonal

As shown in Eq. (2.2), the *main diagonal* of a square matrix extends from the upper left corner to the lower right corner, and it contains elements with matching subscripts—that is, $A_{11}, A_{22}, A_{33}, \ldots, A_{nn}$. The elements forming the main diagonal are referred to as the *diagonal elements* the remaining elements of a square matrix are called the *off-diagonal elements*.

Symmetric Matrix

When the elements of a square matrix are symmetric about its main diagonal (i.e., $A_{ij} = A_{ji}$), it is termed a *symmetric matrix*. For example,

$$\mathbf{A} = \begin{bmatrix} 6 & 15 & -24 & 40 \\ 15 & -9 & 13 & 0 \\ -24 & 13 & 8 & 11 \\ 40 & 0 & 11 & -5 \end{bmatrix}$$

Lower Triangular Matrix

If all the elements of a square matrix above its main diagonal are zero, (i.e., $A_{ij} = 0$ for $j > i$), it is referred to as a *lower triangular matrix*. An example of a 4×4 lower triangular matrix is given by

$$\mathbf{A} = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 12 & -9 & 0 & 0 \\ 33 & 17 & 6 & 0 \\ -2 & 5 & 15 & 3 \end{bmatrix}$$

Upper Triangular Matrix

When all the elements of a square matrix below its main diagonal are zero (i.e., $A_{ij} = 0$ for $j < i$), it is called an *upper triangular matrix*. An example of a 3×3 upper triangular matrix is given by

$$\mathbf{A} = \begin{bmatrix} -7 & 6 & 17 \\ 0 & 12 & 11 \\ 0 & 0 & 20 \end{bmatrix}$$

Diagonal Matrix

A square matrix with all of its off-diagonal elements equal to zero (i.e., $A_{ij} = 0$ for $i \neq j$), is called a *diagonal matrix*. For example,

$$\mathbf{A} = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 11 & 0 \\ 0 & 0 & 0 & 27 \end{bmatrix}$$

Unit or Identity Matrix

If all the diagonal elements of a diagonal matrix are equal to 1 (i.e., $I_{ij} = 1$ and $I_{ij} = 0$ for $i \neq j$), it is referred to as a *unit* (or *identity*) *matrix*. Unit matrices are commonly denoted by **I** or I. An example of a 3×3 unit matrix is given by

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Null Matrix

If all the elements of a matrix are zero (i.e., $O_{ij} = 0$), it is termed a *null matrix*. Null matrices are usually denoted by **O** or O. An example of a 3×4 null matrix is given by

$$\mathbf{O} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2.3 MATRIX OPERATIONS

Equality

Matrices **A** and **B** are considered to be equal if they are of the same order and if their corresponding elements are identical (i.e., $A_{ij} = B_{ij}$). Consider, for example, matrices

$$\mathbf{A} = \begin{bmatrix} 6 & 2 \\ -7 & 8 \\ 3 & -9 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 6 & 2 \\ -7 & 8 \\ 3 & -9 \end{bmatrix}$$

Since both **A** and **B** are of order 3×2, and since each element of **A** is equal to the corresponding element of **B**, the matrices **A** and **B** are equal to each other; that is, $\mathbf{A} = \mathbf{B}$.

Addition and Subtraction

Matrices can be added (or subtracted) only if they are of the same order. The addition (or subtraction) of two matrices **A** and **B** is carried out by adding (or subtracting) the corresponding elements of the two matrices. Thus, if $\mathbf{A} + \mathbf{B} = \mathbf{C}$, then $C_{ij} = A_{ij} + B_{ij}$; and if $\mathbf{A} - \mathbf{B} = \mathbf{D}$, then $D_{ij} = A_{ij} - B_{ij}$. The matrices **C** and **D** have the same order as matrices **A** and **B**.

EXAMPLE 2.1 Calculate the matrices $\mathbf{C} = \mathbf{A} + \mathbf{B}$ and $\mathbf{D} = \mathbf{A} - \mathbf{B}$ if

$$\mathbf{A} = \begin{bmatrix} 6 & 0 \\ -2 & 9 \\ 5 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 3 \\ 7 & 5 \\ -12 & -1 \end{bmatrix}$$

SOLUTION

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \begin{bmatrix} (6+2) & (0+3) \\ (-2+7) & (9+5) \\ (5-12) & (1-1) \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ 5 & 14 \\ -7 & 0 \end{bmatrix} \qquad \textbf{Ans}$$

$$\mathbf{D} = \mathbf{A} - \mathbf{B} = \begin{bmatrix} (6-2) & (0-3) \\ (-2-7) & (9-5) \\ (5+12) & (1+1) \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -9 & 4 \\ 17 & 2 \end{bmatrix} \qquad \textbf{Ans}$$

Multiplication by a Scalar

The product of a scalar c and a matrix $\mathbf{A}$ is obtained by multiplying each element of the matrix $\mathbf{A}$ by the scalar c. Thus, if $c\mathbf{A} = \mathbf{B}$, then $B_{ij} = cA_{ij}$.

EXAMPLE 2.2 Calculate the matrix $\mathbf{B} = c\mathbf{A}$ if $c = -6$ and

$$\mathbf{A} = \begin{bmatrix} 3 & 7 & -2 \\ 0 & 8 & 1 \\ 12 & -4 & 10 \end{bmatrix}$$

SOLUTION

$$\mathbf{B} = c\mathbf{A} = \begin{bmatrix} -6(3) & -6(7) & -6(-2) \\ -6(0) & -6(8) & -6(1) \\ -6(12) & -6(-4) & -6(10) \end{bmatrix} = \begin{bmatrix} -18 & -42 & 12 \\ 0 & -48 & -6 \\ -72 & 24 & -60 \end{bmatrix} \qquad \textbf{Ans}$$

Multiplication of Matrices

Two matrices can be multiplied only if the number of columns of the first matrix equals the number of rows of the second matrix. Such matrices are said to be *conformable* for multiplication. Consider, for example, the matrices

$$\mathbf{A} = \underset{3 \times 2}{\begin{bmatrix} 1 & 8 \\ 4 & -2 \\ -5 & 3 \end{bmatrix}} \quad \text{and} \quad \mathbf{B} = \underset{2 \times 2}{\begin{bmatrix} 6 & -7 \\ -1 & 2 \end{bmatrix}} \qquad \textbf{(2.3)}$$

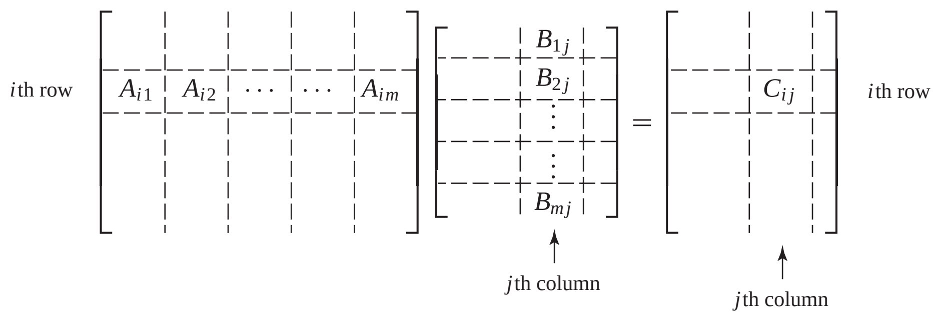
The product **AB** of these matrices is defined because the first matrix, **A**, of the sequence **AB** has two columns and the second matrix, **B**, has two rows. However, if the sequence of the matrices is reversed, then the product **BA** does not exist, because now the first matrix, **B**, has two columns and the second matrix, **A**, has three rows. The product **AB** is referred to either as **A** *postmultiplied* by **B**, or as **B** *premultiplied* by **A**. Conversely, the product **BA** is referred to either as **B** postmultiplied by **A**, or as **A** premultiplied by **B**.

When two conformable matrices are multiplied, the product matrix thus obtained has the number of rows of the first matrix and the number of columns of the second matrix. Thus, if a matrix **A** of order $l \times m$ is postmultiplied by a matrix **B** of order $m \times n$, then the product matrix $\mathbf{C} = \mathbf{AB}$ has the order $l \times n$; that is,

$$\underset{(l \times m)}{\mathbf{A}}\ \underset{(m \times n)}{\mathbf{B}} = \underset{(l \times n)}{\mathbf{C}}$$

equal

(2.4)



Any element C_{ij} of the product matrix **C** can be determined by multiplying each element of the ith row of **A** by the corresponding element of the jth column of **B** (see Eq. 2.4), and by algebraically summing the products; that is,

$$C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{im}B_{mj} \tag{2.5}$$

Eq. (2.5) can be expressed as

$$\boxed{C_{ij} = \sum_{k=1}^{m} A_{ik}B_{kj}} \tag{2.6}$$

in which m represents the number of columns of **A**, or the number of rows of **B**. Equation (2.6) can be used to determine all elements of the product matrix $\mathbf{C} = \mathbf{AB}$.

EXAMPLE 2.3 Calculate the product $\mathbf{C} = \mathbf{AB}$ of the matrices **A** and **B** given in Eq. (2.3).

SOLUTION

$$\mathbf{C} = \mathbf{AB} = \underset{(3 \times 2)}{\begin{bmatrix} 1 & 8 \\ 4 & -2 \\ -5 & 3 \end{bmatrix}} \underset{(2 \times 2)}{\begin{bmatrix} 6 & -7 \\ -1 & 2 \end{bmatrix}} = \underset{(3 \times 2)}{\begin{bmatrix} -2 & 9 \\ 26 & -32 \\ -33 & 41 \end{bmatrix}} \qquad \textbf{Ans}$$

The element C_{11} of the product matrix **C** is determined by multiplying each element of the first row of **A** by the corresponding element of the first column of **B** and summing the resulting products; that is,

$$C_{11} = 1(6) + 8(-1) = -2$$

Similarly, the element C_{12} is obtained by multiplying the elements of the first row of **A** by the corresponding elements of the second column of **B** and adding the resulting products; that is,

$$C_{12} = 1(-7) + 8(2) = 9$$

The remaining elements of **C** are computed in a similar manner:

$$C_{21} = 4(6) + (-2)(-1) = 26$$
$$C_{22} = 4(-7) - 2(2) = -32$$
$$C_{31} = -5(6) + 3(-1) = -33$$
$$C_{32} = -5(-7) + 3(2) = 41$$

A flowchart for programming the matrix multiplication procedure on a computer is given in Fig. 2.1. Any programming language (such as FORTRAN, BASIC, or C, among others) can be used for this purpose. The reader is encouraged to write this program in a general form (e.g., as a subroutine), so that it can be included in the structural analysis computer programs to be developed in later chapters.

An important application of matrix multiplication is to express simultaneous equations in compact matrix form. Consider the following system of linear simultaneous equations.

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + A_{14}x_4 &= P_1 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + A_{24}x_4 &= P_2 \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 + A_{34}x_4 &= P_3 \\ A_{41}x_1 + A_{42}x_2 + A_{43}x_3 + A_{44}x_4 &= P_4 \end{aligned} \qquad \textbf{(2.7)}$$

in which xs are the unknowns and As and Ps represent the coefficients and constants, respectively. By using the definition of multiplication of matrices, this system of equations can be expressed in matrix form as

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} \qquad \textbf{(2.8)}$$

or, symbolically, as

$$\mathbf{Ax} = \mathbf{P} \qquad \textbf{(2.9)}$$

Matrix multiplication is generally not commutative that is,

$$\boxed{\mathbf{AB} \neq \mathbf{BA}} \qquad \textbf{(2.10)}$$

Even when the orders of two matrices **A** and **B** are such that both products **AB** and **BA** are defined and are of the same order, the two products, in general, will

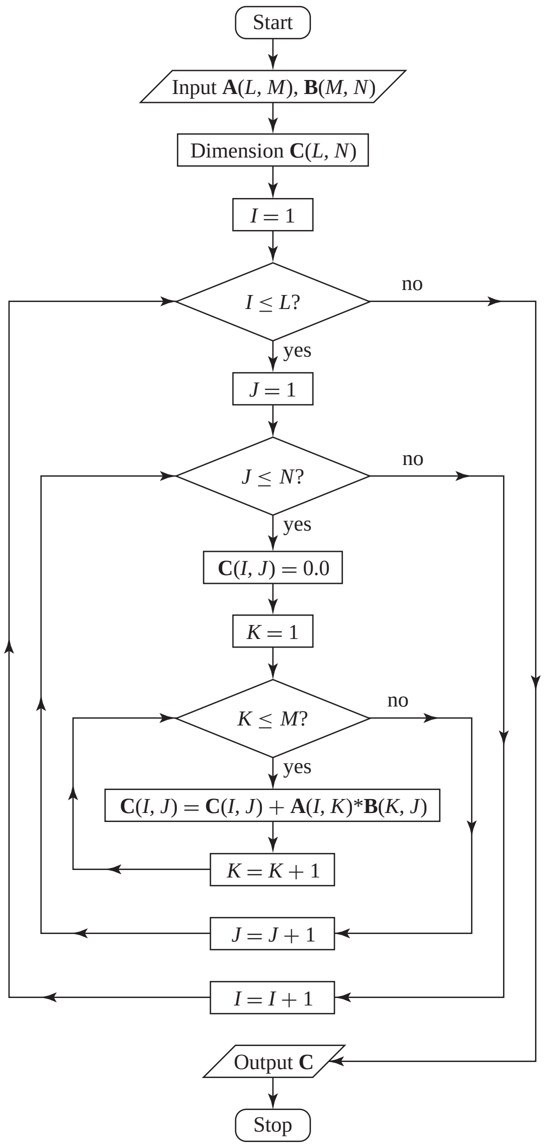


Fig. 2.1 *Flowchart for Matrix Multiplication*

not be equal. It is essential, therefore, to maintain the proper sequential order of matrices when evaluating matrix products.

EXAMPLE 2.4 Calculate the products **AB** and **BA** if

$$\mathbf{A} = \begin{bmatrix} 1 & -8 \\ -7 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 6 & -3 \\ 4 & -5 \end{bmatrix}$$

Are the products **AB** and **BA** equal

SOLUTION

$$\mathbf{AB} = \begin{bmatrix} 1 & -8 \\ -7 & 2 \end{bmatrix} \begin{bmatrix} 6 & -3 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} -26 & 37 \\ -34 & 11 \end{bmatrix} \qquad \textbf{Ans}$$

$$\mathbf{BA} = \begin{bmatrix} 6 & -3 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 & -8 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 27 & -54 \\ 39 & -42 \end{bmatrix} \qquad \textbf{Ans}$$

Comparing products **AB** and **BA**, we can see that $\mathbf{AB} \neq \mathbf{BA}$. **Ans**

Matrix multiplication is associative and distributive, provided that the sequential order in which the matrices are to be multiplied is maintained. Thus,

$$\mathbf{ABC} = (\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) \tag{2.11}$$

and

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC} \tag{2.12}$$

The product of any matrix **A** and a conformable null matrix **O** equals a null matrix; that is,

$$\mathbf{AO} = \mathbf{O} \qquad \text{and} \qquad \mathbf{OA} = \mathbf{O} \tag{2.13}$$

For example,

$$\begin{bmatrix} 2 & -4 \\ -6 & 8 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The product of any matrix **A** and a conformable unit matrix **I** equals the original matrix **A**; thus,

$$\mathbf{AI} = \mathbf{A} \quad \text{and} \quad \mathbf{IA} = \mathbf{A} \tag{2.14}$$

For example,

$$\begin{bmatrix} 2 & -4 \\ -6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -6 & 8 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ -6 & 8 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -6 & 8 \end{bmatrix}$$

We can see from Eqs. (2.13) and (2.14) that the null and unit matrices serve purposes in matrix algebra that are similar to those of the numbers 0 and 1, respectively, in scalar algebra.

Transpose of a Matrix

The *transpose* of a matrix is obtained by interchanging its corresponding rows and columns. The transposed matrix is commonly identified by placing a superscript T on the symbol of the original matrix. Consider, for example, a 3×2 matrix

$$\mathbf{B} = \underset{3 \times 2}{\begin{bmatrix} 2 & -4 \\ -5 & 8 \\ 1 & 3 \end{bmatrix}}$$

The transpose of **B** is given by

$$\mathbf{B}^T = \underset{2 \times 3}{\begin{bmatrix} 2 & -5 & 1 \\ -4 & 8 & 3 \end{bmatrix}}$$

Note that the first row of **B** becomes the first column of $\mathbf{B}^T$. Similarly, the second and third rows of **B** become, respectively, the second and third columns of $\mathbf{B}^T$. The order of $\mathbf{B}^T$ thus obtained is 2×3.

As another example, consider the matrix

$$\mathbf{C} = \begin{bmatrix} 2 & -1 & 6 \\ -1 & 7 & -9 \\ 6 & -9 & 5 \end{bmatrix}$$

Because the elements of **C** are symmetric about its main diagonal (i.e., $C_{ij} = C_{ji}$ for $i \neq j$), interchanging the rows and columns of this matrix produces a matrix $\mathbf{C}^T$ that is identical to **C** itself; that is, $\mathbf{C}^T = \mathbf{C}$. Thus, *the transpose of a symmetric matrix equals the original matrix.*

Another useful property of matrix transposition is that *the transpose of a product of matrices equals the product of the transposed matrices in reverse order*. Thus,

$$\boxed{(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T} \qquad \textbf{(2.15)}$$

Similarly,

$$(\mathbf{ABC})^T = \mathbf{C}^T\mathbf{B}^T\mathbf{A}^T \qquad \textbf{(2.16)}$$

EXAMPLE 2.5 Show that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ if

$$\mathbf{A} = \begin{bmatrix} 9 & -5 \\ 2 & 1 \\ -3 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 6 & -1 & 10 \\ -2 & 7 & 5 \end{bmatrix}$$

SOLUTION

$$\mathbf{AB} = \begin{bmatrix} 9 & -5 \\ 2 & 1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 6 & -1 & 10 \\ -2 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 64 & -44 & 65 \\ 10 & 5 & 25 \\ -26 & 31 & -10 \end{bmatrix}$$

$$(\mathbf{AB})^T = \begin{bmatrix} 64 & 10 & -26 \\ -44 & 5 & 31 \\ 65 & 25 & -10 \end{bmatrix} \qquad \textbf{(1)}$$

$$\mathbf{B}^T\mathbf{A}^T = \begin{bmatrix} 6 & -2 \\ -1 & 7 \\ 10 & 5 \end{bmatrix} \begin{bmatrix} 9 & 2 & -3 \\ -5 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 64 & 10 & -26 \\ -44 & 5 & 31 \\ 65 & 25 & -10 \end{bmatrix} \qquad \textbf{(2)}$$

By comparing Eqs. (1) and (2), we can see that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$. **Ans**

Differentiation and Integration

A matrix can be differentiated (or integrated) by differentiating (or integrating) each of its elements.

EXAMPLE 2.6 Determine the derivative $d\mathbf{A}/dx$ if

$$\mathbf{A} = \begin{bmatrix} x^2 & 3\sin x & -x^4 \\ 3\sin x & -x & \cos^2 x \\ -x^4 & \cos^2 x & 7x^3 \end{bmatrix}$$

SOLUTION By differentiating the elements of **A**, we obtain

$$\begin{aligned}
&A_{11} = x^2 && \frac{dA_{11}}{dx} = 2x \\
&A_{21} = A_{12} = 3\sin x && \frac{dA_{21}}{dx} = \frac{dA_{12}}{dx} = 3\cos x \\
&A_{31} = A_{13} = -x^4 && \frac{dA_{31}}{dx} = \frac{dA_{13}}{dx} = -4x^3 \\
&A_{22} = -x && \frac{dA_{22}}{dx} = -1 \\
&A_{32} = A_{23} = \cos^2 x && \frac{dA_{32}}{dx} = \frac{dA_{23}}{dx} = -2\cos x \sin x \\
&A_{33} = 7x^3 && \frac{dA_{33}}{dx} = 21x^2
\end{aligned}$$

Thus, the derivative $d\mathbf{A}/dx$ is given by

$$\frac{d\mathbf{A}}{dx} = \begin{bmatrix} 2x & 3\cos x & -4x^3 \\ 3\cos x & -1 & -2\cos x \sin x \\ -4x^3 & -2\cos x \sin x & 21x^2 \end{bmatrix}$$ **Ans**

EXAMPLE 2.7 Determine the partial derivative $\partial\mathbf{B}/\partial y$ if

$$\mathbf{B} = \begin{bmatrix} 2y^3 & -yz & -2xz \\ 3xy^2 & yz & -z^2 \\ 2x^2 & -2xz & 3xy^2 \end{bmatrix}$$

SOLUTION We determine the partial derivative, $\partial B_{ij}/\partial y$, of each element of **B** to obtain

$$\frac{\partial\mathbf{B}}{\partial y} = \begin{bmatrix} 6y^2 & -z & 0 \\ 6xy & z & 0 \\ 0 & 0 & 6xy \end{bmatrix}$$ **Ans**

EXAMPLE 2.8 Calculate the integral $\int_0^L \mathbf{A}\mathbf{A}^T dx$ if

$$\mathbf{A} = \begin{bmatrix} 1 - \dfrac{x}{L} \\ \dfrac{x}{L} \end{bmatrix}$$

SOLUTION First, we calculate the matrix product $\mathbf{AA}^T$ as

$$\mathbf{B} = \mathbf{AA}^T = \begin{bmatrix} \left(1 - \frac{x}{L}\right) \\ \frac{x}{L} \end{bmatrix} \left[\left(1 - \frac{x}{L}\right) \; \frac{x}{L}\right] = \begin{bmatrix} \left(1 - \frac{x}{L}\right)^2 & \frac{x}{L}\left(1 - \frac{x}{L}\right) \\ \frac{x}{L}\left(1 - \frac{x}{L}\right) & \frac{x^2}{L^2} \end{bmatrix}$$

Next, we integrate the elements of $\mathbf{B}$ to obtain

$$\int_0^L B_{11} dx = \int_0^L \left(1 - \frac{x}{L}\right)^2 dx = \int_0^L \left(1 - \frac{2x}{L} + \frac{x^2}{L^2}\right) dx$$

$$= \left(x - \frac{x^2}{L} + \frac{x^3}{3L^2}\right)_0^L = \frac{L}{3}$$

$$\int_0^L B_{21} dx = \int_0^L B_{12} dx = \int_0^L \frac{x}{L}\left(1 - \frac{x}{L}\right) dx = \int_0^L \left(\frac{x}{L} - \frac{x^2}{L^2}\right) dx$$

$$= \left(\frac{x^2}{2L} - \frac{x^3}{3L^2}\right)_0^L = \frac{L}{2} - \frac{L}{3} = \frac{L}{6}$$

$$\int_0^L B_{22} dx = \int_0^L \left(\frac{x^2}{L^2}\right) dx = \left(\frac{x^3}{3L^2}\right)_0^L = \frac{L}{3}$$

Thus,

$$\int_0^L \mathbf{AA}^T dx = \begin{bmatrix} \frac{L}{3} & \frac{L}{6} \\ \frac{L}{6} & \frac{L}{3} \end{bmatrix} = \frac{L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$ **Ans**

Inverse of a Square Matrix

The inverse of a square matrix $\mathbf{A}$ is defined as a matrix $\mathbf{A}^{-1}$ with elements of such magnitudes that the product of the original matrix $\mathbf{A}$ and its inverse $\mathbf{A}^{-1}$ equals a unit matrix $\mathbf{I}$; that is,

$$\boxed{\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}} \tag{2.17}$$

The operation of inversion is defined only for square matrices, with the inverse of such a matrix also being a square matrix of the same order as the original matrix. A procedure for determining inverses of matrices will be presented in the next section.

EXAMPLE 2.9 Check whether or not matrix $\mathbf{B}$ is the inverse of matrix $\mathbf{A}$, if

$$\mathbf{A} = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0.5 & -1 \\ 1.5 & -2 \end{bmatrix}$$

SOLUTION

$$\mathbf{AB} = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & -1 \\ 1.5 & -2 \end{bmatrix} = \begin{bmatrix} (-2+3) & (4-4) \\ (-1.5+1.5) & (3-2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Also,

$$\mathbf{BA} = \begin{bmatrix} 0.5 & -1 \\ 1.5 & -2 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} (-2+3) & (1-1) \\ (-6+6) & (3-2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, $\mathbf{B}$ is the inverse of $\mathbf{A}$; that is,

$$\mathbf{B} = \mathbf{A}^{-1}$$ **Ans**

The operation of matrix inversion serves a purpose analogous to the operation of division in scalar algebra. Consider a system of simultaneous linear equations expressed in matrix form as

$$\mathbf{Ax} = \mathbf{P}$$

in which $\mathbf{A}$ is the square matrix of known coefficients; $\mathbf{x}$ is the vector of the unknowns; and $\mathbf{P}$ is the vector of the constants. As the operation of division is not defined in matrix algebra, the equation cannot be solved for $\mathbf{x}$ by dividing $\mathbf{P}$ by $\mathbf{A}$ (i.e., $\mathbf{x} = \mathbf{P}/\mathbf{A}$). However, we can determine $\mathbf{x}$ by premultiplying both sides of the equation by $\mathbf{A}^{-1}$, to obtain

$$\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{P}$$

As $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and $\mathbf{Ix} = \mathbf{x}$, we can write

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{P}$$

which shows that a system of simultaneous linear equations can be solved by premultiplying the vector of constants by the inverse of the coefficient matrix.

An important property of matrix inversion is that *the inverse of a symmetric matrix is also a symmetric matrix.*

Orthogonal Matrix

If the inverse of a matrix is equal to its transpose, the matrix is referred to as an *orthogonal matrix.* In other words, a matrix $\mathbf{A}$ is orthogonal if

$$\mathbf{A}^{-1} = \mathbf{A}^{T}$$

EXAMPLE 2.10 Determine whether matrix $\mathbf{A}$ given below is an orthogonal matrix.

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.6 & 0 & 0 \\ -0.6 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 0.6 \\ 0 & 0 & -0.6 & 0.8 \end{bmatrix}$$

SOLUTION

$$\mathbf{AA}^T = \begin{bmatrix} 0.8 & 0.6 & 0 & 0 \\ -0.6 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 0.6 \\ 0 & 0 & -0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 0.8 & -0.6 & 0 & 0 \\ 0.6 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & -0.6 \\ 0 & 0 & 0.6 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} (0.64 + 0.36) & (-0.48 + 0.48) & 0 & 0 \\ (-0.48 + 0.48) & (0.36 + 0.64) & 0 & 0 \\ 0 & 0 & (0.64 + 0.36) & (-0.48 + 0.48) \\ 0 & 0 & (-0.48 + 0.48) & (0.36 + 0.64) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which shows that $\mathbf{AA}^T = \mathbf{I}$. Thus,

$$\mathbf{A}^{-1} = \mathbf{A}^T$$

Therefore, matrix **A** is orthogonal. **Ans**

Partitioning of Matrices

In many applications, it becomes necessary to subdivide a matrix into a number of smaller matrices called *submatrices*. The process of subdividing a matrix into submatrices is referred to as *partitioning*. For example, a 4×3 matrix **B** is partitioned into four submatrices by drawing horizontal and vertical dashed partition lines:

$$\mathbf{B} = \left[\begin{array}{rr:r} 2 & -4 & -1 \\ -5 & 7 & 3 \\ 8 & -9 & 6 \\ \hdashline 1 & 3 & 8 \end{array}\right] = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} \tag{2.18}$$

in which the submatrices are

$$\mathbf{B}_{11} = \begin{bmatrix} 2 & -4 \\ -5 & 7 \\ 8 & -9 \end{bmatrix} \qquad \mathbf{B}_{12} = \begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix}$$

$$\mathbf{B}_{21} = \begin{bmatrix} 1 & 3 \end{bmatrix} \qquad \mathbf{B}_{22} = \begin{bmatrix} 8 \end{bmatrix}$$

Matrix operations (such as addition, subtraction, and multiplication) can be performed on partitioned matrices in the same manner as discussed previously by treating the submatrices as elements—provided that the matrices are partitioned in such a way that their submatrices are conformable for the particular operation. For example, suppose that the 4×3 matrix **B** of Eq. (2.18) is to be postmultiplied by a 3×2 matrix **C**, which is partitioned into two submatrices:

$$\mathbf{C} = \left[\begin{array}{rr} 9 & -6 \\ 4 & 2 \\ \hdashline -3 & 1 \end{array}\right] = \begin{bmatrix} \mathbf{C}_{11} \\ \mathbf{C}_{21} \end{bmatrix} \tag{2.19}$$

The product **BC** is expressed in terms of submatrices as

$$\mathbf{BC} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{C}_{11} \\ \mathbf{C}_{21} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{11}\mathbf{C}_{11} + \mathbf{B}_{12}\mathbf{C}_{21} \\ \mathbf{B}_{21}\mathbf{C}_{11} + \mathbf{B}_{22}\mathbf{C}_{21} \end{bmatrix} \tag{2.20}$$

It is important to realize that matrices **B** and **C** have been partitioned in such a way that their corresponding submatrices are conformable for multiplication; that is, the orders of the submatrices are such that the products $\mathbf{B}_{11}\mathbf{C}_{11}$, $\mathbf{B}_{12}\mathbf{C}_{21}$, $\mathbf{B}_{21}\mathbf{C}_{11}$, and $\mathbf{B}_{22}\mathbf{C}_{21}$ are defined. It can be seen from Eqs. (2.18) and (2.19) that this is achieved by partitioning the rows of the second matrix **C** of the product **BC** in the same way that the columns of the first matrix **B** are partitioned. The products of the submatrices are:

$$\mathbf{B}_{11}\mathbf{C}_{11} = \begin{bmatrix} 2 & -4 \\ -5 & 7 \\ 8 & -9 \end{bmatrix} \begin{bmatrix} 9 & -6 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -20 \\ -17 & 44 \\ 36 & -66 \end{bmatrix}$$

$$\mathbf{B}_{12}\mathbf{C}_{21} = \begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix} \quad -3 \quad 1 \quad = \begin{bmatrix} 3 & -1 \\ -9 & 3 \\ -18 & 6 \end{bmatrix}$$

$$\mathbf{B}_{21}\mathbf{C}_{11} = \; 1 \quad 3 \; \begin{bmatrix} 9 & -6 \\ 4 & 2 \end{bmatrix} = \; 21 \quad 0$$

$$\mathbf{B}_{22}\mathbf{C}_{21} = \; 8 \quad -3 \quad 1 \; = \; -24 \quad 8$$

By substituting the numerical values of the products of submatrices into Eq. (2.20), we obtain

$$\mathbf{BC} = \begin{bmatrix} \begin{bmatrix} 2 & -20 \\ -17 & 44 \\ 36 & -66 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -9 & 3 \\ -18 & 6 \end{bmatrix} \\ 21 \qquad 0 \quad + \quad -24 \qquad 8 \end{bmatrix} = \begin{bmatrix} 5 & -21 \\ -26 & 47 \\ 18 & -60 \\ -3 & 8 \end{bmatrix}$$

2.4 GAUSS–JORDAN ELIMINATION METHOD

The *Gauss ordan elimination method* is one of the most commonly used procedures for solving simultaneous linear equations, and for determining inverses of matrices.

Solution of Simultaneous Equations

To illustrate the Gauss–Jordan method for solving simultaneous equations, consider the following system of three linear algebraic equations:

$$\begin{aligned} 5x_1 + 6x_2 - 3x_3 &= 66 \\ 9x_1 - x_2 + 2x_3 &= 8 \\ 8x_1 - 7x_2 + 4x_3 &= -39 \end{aligned} \tag{2.21a}$$

To determine the unknowns x_1, x_2, and x_3, we begin by dividing the first equation by the coefficient of its x_1 term to obtain

$$\begin{aligned} x_1 + 1.2x_2 - 0.6x_3 &= 13.2 \\ 9x_1 - x_2 + 2x_3 &= 8 \\ 8x_1 - 7x_2 + 4x_3 &= -39 \end{aligned} \qquad \textbf{(2.21b)}$$

Next, we eliminate the unknown x_1 from the second and third equations by successively subtracting from each equation the product of the coefficient of its x_1 term and the first equation. Thus, to eliminate x_1 from the second equation, we multiply the first equation by 9 and subtract it from the second equation. Similarly, we eliminate x_1 from the third equation by multiplying the first equation by 8 and subtracting it from the third equation. This yields the system of equations

$$\begin{aligned} x_1 + 1.2x_2 - 0.6x_3 &= 13.2 \\ -11.8x_2 + 7.4x_3 &= -110.8 \\ -16.6x_2 + 8.8x_3 &= -144.6 \end{aligned} \qquad \textbf{(2.21c)}$$

With x_1 eliminated from all but the first equation, we now divide the second equation by the coefficient of its x_2 term to obtain

$$\begin{aligned} x_1 + 1.2x_2 - 0.6x_3 &= 13.2 \\ x_2 - 0.6271x_3 &= 9.39 \\ -16.6x_2 + 8.8x_3 &= -144.6 \end{aligned} \qquad \textbf{(2.21d)}$$

Next, the unknown x_2 is eliminated from the first and the third equations, successively, by multiplying the second equation by 1.2 and subtracting it from the first equation, and then by multiplying the second equation by -16.6 and subtracting it from the third equation. The system of equations thus obtained is

$$\begin{aligned} x_1 + 0.1525x_3 &= 1.932 \\ x_2 - 0.6271x_3 &= 9.39 \\ -1.61x_3 &= 11.27 \end{aligned} \qquad \textbf{(2.21e)}$$

Focusing our attention now on the unknown x_3, we divide the third equation by the coefficient of its x_3 term (which is -1.61) to obtain

$$\begin{aligned} x_1 + 0.1525x_3 &= 1.932 \\ x_2 - 0.6271x_3 &= 9.39 \\ x_3 &= -7 \end{aligned} \qquad \textbf{(2.21f)}$$

Finally, we eliminate x_3 from the first and the second equations, successively, by multiplying the third equation by 0.1525 and subtracting it from the first equation, and then by multiplying the third equation by -0.6271 and subtracting it from the second equation. This yields the solution of the given system of equations:

$$\begin{aligned} x_1 &= 3 \\ x_2 &= 5 \\ x_3 &= -7 \end{aligned} \qquad \textbf{(2.21g)}$$

or, equivalently,

$$x_1 = 3; \quad x_2 = 5; \quad x_3 = -7 \tag{2.21h}$$

To check that this solution is correct, we substitute the numerical values of x_1, x_2, and x_3 back into the original equations (Eq. 2.21(a)):

$$\begin{aligned} 5(3) + 6(5) - 3(-7) &= 66 && \text{Checks} \\ 9(3) - 5 \quad + 2(-7) &= 8 && \text{Checks} \\ 8(3) - 7(5) + 4(-7) &= -39 && \text{Checks} \end{aligned}$$

As the foregoing example illustrates, the Gauss–Jordan method basically involves eliminating, in order, each unknown from all but one of the equations of the system by applying the following operations: dividing an equation by a scalar; and multiplying an equation by a scalar and subtracting the resulting equation from another equation. These operations (called the *elementary operations*) when applied to a system of equations yield another system of equations that has the same solution as the original system. In the Gauss–-Jordan method, the elementary operations are performed repeatedly until a system with each equation containing only one unknown is obtained.

The Gauss–Jordan elimination method can be performed more conveniently by using the matrix form of the simultaneous equations ($\mathbf{Ax} = \mathbf{P}$). In this approach, the coefficient matrix $\mathbf{A}$ and the vector of constants $\mathbf{P}$ are treated as submatrices of a partitioned *augmented matrix*,

$$\underset{n \times (n+1)}{\mathbf{G}} = \underset{n \times n}{\mathbf{A}} \;\vdots\; \underset{n \times 1}{\mathbf{P}} \tag{2.22}$$

where n represents the number of equations. The elementary operations are then applied to the rows of the augmented matrix, until the coefficient matrix is reduced to a unit matrix. The elements of the vector, which initially contained the constant terms of the original equations, now represent the solution of the original system of equations; that is,

$$\mathbf{G} = \left\{ \begin{array}{c} \mathbf{A} \;\vdots\; \mathbf{P} \\ \Big\downarrow \text{elementary operations} \\ \mathbf{I} \;\vdots\; \mathbf{x} \end{array} \right. \tag{2.23}$$

This procedure is illustrated by the following example.

EXAMPLE 2.11 Solve the system of simultaneous equations given in Eq. 2.21(a) by the Gauss–Jordan method.

SOLUTION The given system of equations can be written in matrix form as

$$\mathbf{Ax} = \mathbf{P}$$

$$\begin{bmatrix} 5 & 6 & -3 \\ 9 & -1 & 2 \\ 8 & -7 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 66 \\ 8 \\ -39 \end{bmatrix} \tag{1}$$

from which we form the augmented matrix

$$\mathbf{G} = \mathbf{A} \,\vdots\, \mathbf{P} = \left[\begin{array}{rrr|r} 5 & 6 & -3 & 66 \\ 9 & -1 & 2 & 8 \\ 8 & -7 & 4 & -39 \end{array}\right] \tag{2}$$

We begin Gauss–Jordan elimination by dividing row 1 of the augmented matrix by $G_{11} = 5$ to obtain

$$\mathbf{G} = \left[\begin{array}{rrr|r} 1 & 1.2 & -0.6 & 13.2 \\ 9 & -1 & 2 & 8 \\ 8 & -7 & 4 & -39 \end{array}\right] \tag{3}$$

Next, we multiply row 1 by $G_{21} = 9$ and subtract it from row 2; then multiply row 1 by $G_{31} = 8$ and subtract it from row 3. This yields

$$\mathbf{G} = \left[\begin{array}{rrr|r} 1 & 1.2 & -0.6 & 13.2 \\ 0 & -11.8 & 7.4 & -110.8 \\ 0 & -16.6 & 8.8 & -144.6 \end{array}\right] \tag{4}$$

We now divide row 2 by $G_{22} = -11.8$ to obtain

$$\mathbf{G} = \left[\begin{array}{rrr|r} 1 & 1.2 & -0.6 & 13.2 \\ 0 & 1 & -0.6271 & 9.39 \\ 0 & -16.6 & 8.8 & -144.6 \end{array}\right] \tag{5}$$

Next, we multiply row 2 by $G_{12} = 1.2$ and subtract it from row 1, and then multiply row 2 by $G_{32} = -16.6$ and subtract it from row 3. Thus,

$$\mathbf{G} = \left[\begin{array}{rrr|r} 1 & 0 & 0.1525 & 1.932 \\ 0 & 1 & -0.6271 & 9.39 \\ 0 & 0 & -1.61 & 11.27 \end{array}\right] \tag{6}$$

By dividing row 3 by $G_{33} = -1.61$, we obtain

$$\mathbf{G} = \left[\begin{array}{rrr|r} 1 & 0 & 0.1525 & 1.932 \\ 0 & 1 & -0.6271 & 9.39 \\ 0 & 0 & 1 & -7 \end{array}\right] \tag{7}$$

Finally, we multiply row 3 by $G_{13} = 0.1525$ and subtract it from row 1; then multiply row 3 by $G_{23} = -0.6271$ and subtract it from row 2 to obtain

$$\mathbf{G} = \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -7 \end{array}\right] \tag{8}$$

Thus, the solution of the given system of equations is

$$\mathbf{x} = \begin{bmatrix} 3 \\ 5 \\ -7 \end{bmatrix} \qquad \textbf{Ans}$$

To check our solution, we substitute the numerical value of $\mathbf{x}$ back into Eq. (1). This yields

$$\begin{bmatrix} 5 & 6 & -3 \\ 9 & -1 & 2 \\ 8 & -7 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ -7 \end{bmatrix} = \begin{bmatrix} 15 + 30 + 21 = 66 \\ 27 - 5 - 14 = 8 \\ 24 - 35 - 28 = -39 \end{bmatrix} \qquad \textbf{Checks}$$

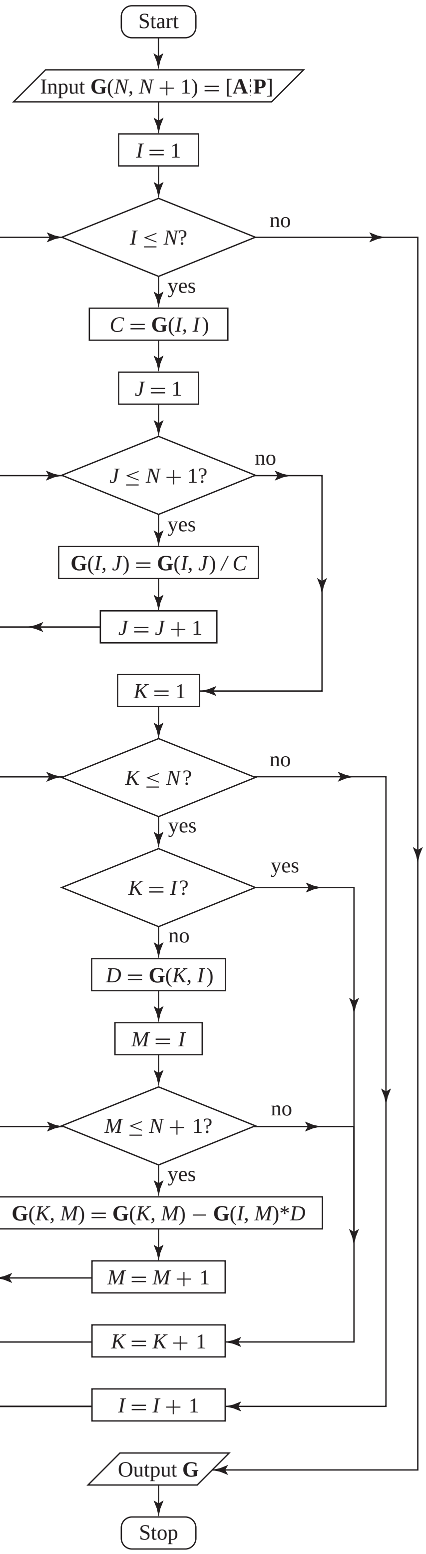


Fig. 2.2 *Flowchart for Solution of Simultaneous quations by Gauss ordan Method*

The solution of large systems of simultaneous equations by the Gauss–Jordan method is usually carried out by computer, and a flowchart for programming this procedure is given in Fig. 2.2. The reader should write this program in a general form (e.g., as a subroutine), so that it can be conveniently included in the structural analysis computer programs to be developed in later chapters.

It should be noted that the Gauss–Jordan method as described in the preceding paragraphs breaks down if a diagonal element of the coefficient matrix **A** becomes zero during the elimination process. This situation can be remedied by interchanging the row of the augmented matrix containing the zero diagonal element with another row, to place a nonzero element on the diagonal; the elimination process is then continued. However, when solving the systems of equations encountered in structural analysis, the condition of a zero diagonal element should not arise; the occurrence of such a condition would indicate that the structure being analyzed is unstable [2].

Matrix Inversion

The procedure for determining inverses of matrices by the Gauss–Jordan method is similar to that described previously for solving simultaneous equations. The procedure involves forming an augmented matrix **G** composed of the matrix **A** that is to be inverted and a unit matrix **I** of the same order as **A**; that is,

$$\underset{n \times 2n}{\mathbf{G}} = \underset{n \times n}{\mathbf{A}} \;\vdots\; \underset{n \times n}{\mathbf{I}} \tag{2.24}$$

Elementary operations are then applied to the rows of the augmented matrix to reduce **A** to a unit matrix. Matrix **I**, which was initially the unit matrix, now represents the inverse matrix $\mathbf{A}^{-1}$; thus,

$$\mathbf{G} = \left\{ \begin{array}{c:c} \mathbf{A} & \mathbf{I} \\ \big\downarrow & \text{elementary operations} \\ \mathbf{I} & \mathbf{A}^{-1} \end{array} \right. \tag{2.25}$$

EXAMPLE 2.12 Determine the inverse of the matrix shown using the Gauss–Jordan method.

$$\mathbf{A} = \begin{bmatrix} 13 & -6 & 6 \\ -6 & 12 & -1 \\ 6 & -1 & 9 \end{bmatrix}$$

SOLUTION The augmented matrix is given by

$$\mathbf{G} = \mathbf{A} \;\vdots\; \mathbf{I} = \left[\begin{array}{rrr:rrr} 13 & -6 & 6 & 1 & 0 & 0 \\ -6 & 12 & -1 & 0 & 1 & 0 \\ 6 & -1 & 9 & 0 & 0 & 1 \end{array}\right] \tag{1}$$

Numbers in brackets refer to items listed in the bibliography.

We begin the Gauss–Jordan elimination process by dividing row 1 of the augmented matrix by $G_{11} = 13$:

$$\mathbf{G} = \left[\begin{array}{ccc:ccc} 1 & -0.4615 & 0.4615 & 0.07692 & 0 & 0 \\ -6 & 12 & -1 & 0 & 1 & 0 \\ 6 & -1 & 9 & 0 & 0 & 1 \end{array}\right] \tag{2}$$

Next, we multiply row 1 by $G_{21} = -6$ and subtract it from row 2, and then multiply row 1 by $G_{31} = 6$ and subtract it from row 3. This yields

$$\mathbf{G} = \left[\begin{array}{ccc:ccc} 1 & -0.4615 & 0.4615 & 0.07692 & 0 & 0 \\ 0 & 9.231 & 1.769 & 0.4615 & 1 & 0 \\ 0 & 1.769 & 6.231 & -0.4615 & 0 & 1 \end{array}\right] \tag{3}$$

Dividing row 2 by $G_{22} = 9.231$, we obtain

$$\mathbf{G} = \left[\begin{array}{ccc:ccc} 1 & -0.4615 & 0.4615 & 0.07692 & 0 & 0 \\ 0 & 1 & 0.1916 & 0.04999 & 0.1083 & 0 \\ 0 & 1.769 & 6.231 & -0.4615 & 0 & 1 \end{array}\right] \tag{4}$$

Next, we multiply row 2 by $G_{12} = -0.4615$ and subtract it from row 1; then multiply row 2 by $G_{32} = 1.769$ and subtract it from row 3. This yields

$$\mathbf{G} = \left[\begin{array}{ccc:ccc} 1 & 0 & 0.5499 & 0.09999 & 0.04998 & 0 \\ 0 & 1 & 0.1916 & 0.04999 & 0.1083 & 0 \\ 0 & 0 & 5.892 & -0.5499 & -0.1916 & 1 \end{array}\right] \tag{5}$$

Divide row 3 by $G_{33} = 5.892$:

$$\mathbf{G} = \left[\begin{array}{ccc:ccc} 1 & 0 & 0.5499 & 0.09999 & 0.04998 & 0 \\ 0 & 1 & 0.1916 & 0.04999 & 0.1083 & 0 \\ 0 & 0 & 1 & -0.09333 & -0.03252 & 0.1697 \end{array}\right] \tag{6}$$

Multiply row 3 by $G_{13} = 0.5499$ and subtract it from row 1; then multiply row 3 by $G_{23} = 0.1916$ and subtract it from row 2 to obtain

$$\mathbf{G} = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0.1513 & 0.06787 & -0.09333 \\ 0 & 1 & 0 & 0.06787 & 0.1145 & -0.03252 \\ 0 & 0 & 1 & -0.09333 & -0.03252 & 0.1697 \end{array}\right] \tag{7}$$

Thus, the inverse of the given matrix **A** is

$$\mathbf{A}^{-1} = \begin{bmatrix} 0.1513 & 0.06787 & -0.09333 \\ 0.06787 & 0.1145 & -0.03252 \\ -0.09333 & -0.03252 & 0.1697 \end{bmatrix} \qquad \textbf{Ans}$$

Finally, we check our computations by using the relationship $\mathbf{AA}^{-1} = \mathbf{I}$:

$$\mathbf{AA}^{-1} = \begin{bmatrix} 13 & -6 & 6 \\ -6 & 12 & -1 \\ 6 & -1 & 9 \end{bmatrix} \begin{bmatrix} 0.1513 & 0.06787 & -0.09333 \\ 0.06787 & 0.1145 & -0.03252 \\ -0.09333 & -0.03252 & 0.1697 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9997 & 0.0002 & 0 \\ 0 & 0.9993 & 0 \\ 0 & 0 & 0.9998 \end{bmatrix} \approx \mathbf{I} \qquad \textbf{Checks}$$

SUMMARY

In this chapter, we discussed the basic concepts of matrix algebra that are necessary for formulating the matrix methods of structural analysis:

1. A matrix is defined as a rectangular array of quantities (elements) arranged in rows and columns. The size of a matrix is measured by its number of rows and columns, and is referred to as its order.
2. Two matrices are considered to be equal if they are of the same order, and if their corresponding elements are identical.
3. Two matrices of the same order can be added (or subtracted) by adding (or subtracting) their corresponding elements.
4. The matrix multiplication $\mathbf{AB} = \mathbf{C}$ is defined only if the number of columns of the first matrix $\mathbf{A}$ equals the number of rows of the second matrix $\mathbf{B}$. Any element C_{ij} of the product matrix $\mathbf{C}$ can be evaluated by using the relationship

$$C_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj} \tag{2.6}$$

   where m is the number of columns of $\mathbf{A}$, or the number of rows of $\mathbf{B}$. Matrix multiplication is generally not commutative; that is, $\mathbf{AB} \neq \mathbf{BA}$.
5. The transpose of a matrix is obtained by interchanging its corresponding rows and columns. If $\mathbf{C}$ is a symmetric matrix, then $\mathbf{C}^T = \mathbf{C}$. Another useful property of matrix transposition is that

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \tag{2.15}$$

6. A matrix can be differentiated (or integrated) by differentiating (or integrating) each of its elements.
7. The inverse of a square matrix $\mathbf{A}$ is defined as a matrix $\mathbf{A}^{-1}$ which satisfies the relationship:

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \tag{2.17}$$

8. If the inverse of a matrix equals its transpose, the matrix is called an orthogonal matrix.
9. The Gauss–Jordan method of solving simultaneous equations essentially involves successively eliminating each unknown from all but one of the equations of the system by performing the following operations: dividing an equation by a scalar; and multiplying an equation by a scalar and subtracting the resulting equation from another equation. These elementary operations are applied repeatedly until a system with each equation containing only one unknown is obtained.

PROBLEMS

Section 2.3

2.1 Determine the matrices $\mathbf{C} = \mathbf{A} + \mathbf{B}$ and $\mathbf{D} = \mathbf{A} - \mathbf{B}$ if

$$\mathbf{A} = \begin{bmatrix} 3 & 8 & -1 \\ 8 & -7 & -4 \\ -1 & -4 & 5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 5 & 9 & -2 \\ -9 & 6 & 3 \\ 2 & -3 & -4 \end{bmatrix}$$

2.2 Determine the matrices $\mathbf{C} = 2\mathbf{A} + \mathbf{B}$ and $\mathbf{D} = \mathbf{A} - 3\mathbf{B}$ if

$$\mathbf{A} = \begin{bmatrix} 8 & -6 & -3 \\ 1 & -2 & 0 \\ -6 & 5 & -1 \\ -2 & 8 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 & 2 & -3 \\ -4 & 3 & 0 \\ 2 & -8 & 6 \\ -1 & 4 & -7 \end{bmatrix}$$

2.3 Determine the products $\mathbf{C} = \mathbf{AB}$ and $\mathbf{D} = \mathbf{BA}$ if

$$\mathbf{A} = \begin{bmatrix} 4 & -6 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}$$

2.4 Determine the products $\mathbf{C} = \mathbf{AB}$ and $\mathbf{D} = \mathbf{BA}$ if

$$\mathbf{A} = \begin{bmatrix} 4 & 6 \\ -7 & -5 \\ 1 & -9 \\ -3 & 11 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -1 & 3 & -5 & 2 \\ -13 & -4 & 7 & 6 \end{bmatrix}$$

2.5 Determine the products $\mathbf{C} = \mathbf{AB}$ and $\mathbf{D} = \mathbf{BA}$ if

$$\mathbf{A} = \begin{bmatrix} 4 & -6 & 1 \\ -6 & 5 & 7 \\ 1 & 7 & 8 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 & 5 & 0 \\ 5 & 7 & -2 \\ 0 & -2 & 9 \end{bmatrix}$$

2.6 Determine the products $\mathbf{C} = \mathbf{AB}$ if

$$\mathbf{A} = \begin{bmatrix} 12 & -11 & 10 \\ 0 & 2 & -4 \\ -7 & 9 & 8 \\ 6 & 15 & -5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 13 & -1 & 5 \\ 16 & -9 & 0 \\ -3 & 20 & -7 \end{bmatrix}$$

2.7 Develop a computer program to determine the matrix product $\mathbf{C} = \mathbf{AB}$ of two conformable matrices $\mathbf{A}$ and $\mathbf{B}$ of any order. Check the program by solving Problems 2.4–2.6 and comparing the computer-generated results to those determined by hand calculations.

2.8 Show that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ by using the following matrices

$$\mathbf{A} = \begin{bmatrix} 21 & 10 & 16 \\ -15 & 11 & 0 \\ 13 & 20 & -9 \\ 7 & -17 & 14 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 7 & -4 \\ -1 & 9 \\ 3 & -6 \end{bmatrix}$$

2.9 Show that $(\mathbf{ABC})^T = \mathbf{C}^T\mathbf{B}^T\mathbf{A}^T$ by using the following matrices

$$\mathbf{A} = \begin{bmatrix} -9 & 0 \\ 13 & 20 \\ 8 & -3 \\ -11 & -5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 15 & -1 & -4 \\ 6 & 16 & 9 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} -7 & 10 & 6 & 0 \\ -1 & 2 & -8 & -2 \\ 16 & 12 & 2 & 8 \end{bmatrix}$$

2.10 Determine the matrix triple product $\mathbf{C} = \mathbf{B}^T\mathbf{AB}$ if

$$\mathbf{A} = \begin{bmatrix} 40 & -10 & -25 \\ -10 & 15 & 12 \\ -25 & 12 & 30 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 5 & 7 & -3 \\ -7 & 8 & 4 \\ 3 & -4 & 9 \end{bmatrix}$$

2.11 Determine the matrix triple product $\mathbf{C} = \mathbf{B}^T\mathbf{AB}$ if

$$\mathbf{A} = \begin{bmatrix} 300 & -100 \\ -100 & 200 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0.6 & 0.8 & -0.6 & -0.8 \\ -0.8 & 0.6 & 0.8 & -0.6 \end{bmatrix}$$

2.12 Develop a computer program to determine the matrix triple product $\mathbf{C} = \mathbf{B}^T\mathbf{AB}$, where $\mathbf{A}$ is a square matrix of any order. Check the program by solving Problems 2.10 and 2.11 and comparing the results to those determined by hand calculations.

2.13 Determine the derivative $d\mathbf{A}/dx$ if

$$\mathbf{A} = \begin{bmatrix} -2x^2 & 3\sin x & -7x \\ 3\sin x & \cos^2 x & -3x^3 \\ -7x & -3x^3 & 3\sin^2 x \end{bmatrix}$$

2.14 Determine the derivative $d(\mathbf{A} + \mathbf{B})/dx$ if

$$\mathbf{A} = \begin{bmatrix} -3x & 5 \\ 4x^2 & -x^3 \\ -7 & 5x \\ 2x^3 & -x^2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2x^2 & -x \\ -12x & 8 \\ 2x^3 & -3x^2 \\ -1 & 6x \end{bmatrix}$$

2.15 Determine the derivative $d(\mathbf{AB})/dx$ if

$$\mathbf{A} = \begin{bmatrix} 4x & 2 & -5x^2 \\ 2 & -3x^3 & -x \\ -5x^2 & -x & 7 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -5x^3 & -x \\ 6 & -3x^2 \\ 2x^2 & 4x \end{bmatrix}$$

2.16 Determine the partial derivatives $\partial\mathbf{A}/\partial x$, $\partial\mathbf{A}/\partial y$, and $\partial\mathbf{A}/\partial z$, if

$$\mathbf{A} = \begin{bmatrix} x^2 & -y^2 & 2z^2 \\ -y^2 & 3xy & -yz \\ 2z^2 & -yz & 4xz \end{bmatrix}$$

2.17 Calculate the integral $\int_0^L \mathbf{A}\, dx$ if

$$\mathbf{A} = \begin{bmatrix} -5 & -3x^2 \\ 4x & -x^3 \\ 2x^4 & 6 \\ 5x^2 & -x \end{bmatrix}$$

2.18 Calculate the integral $\int_0^L \mathbf{A}\, dx$ if

$$\mathbf{A} = \begin{bmatrix} 2x & -\sin x & 2\cos^2 x \\ -\sin x & 5 & -4x^3 \\ 2\cos^2 x & -4x^3 & (1-x^2) \end{bmatrix}$$

2.19 Calculate the integral $\int_0^L \mathbf{AB}\, dx$ if

$$\mathbf{A} = \begin{bmatrix} -x^3 & 2x^2 & 3 \\ 2x & -x^2 & 2x^3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -2x & x^2 \\ 5 & -2x \\ 3x^3 & -3 \end{bmatrix}$$

2.20 Determine whether the matrices **A** and **B** given below are orthogonal matrices.

$$\mathbf{A} = \begin{bmatrix} -0.28 & -0.96 & 0 & 0 \\ 0.96 & -0.28 & 0 & 0 \\ 0 & 0 & -0.28 & -0.96 \\ 0 & 0 & 0.96 & -0.28 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} -0.28 & 0.96 & 0 & 0 \\ 0.96 & -0.28 & 0 & 0 \\ 0 & 0 & -0.28 & 0.96 \\ 0 & 0 & 0.96 & -0.28 \end{bmatrix}$$

Section 2.4

2.21 through 2.25 Solve the following systems of simultaneous equations by the Gauss–Jordan method.

2.21
$$\begin{aligned} 2x_1 - 3x_2 + x_3 &= -18 \\ -9x_1 + 5x_2 + 3x_3 &= 18 \\ 4x_1 + 7x_2 - 8x_3 &= 53 \end{aligned}$$

2.22
$$\begin{aligned} 20x_1 - 9x_2 + 15x_3 &= 354 \\ -9x_1 + 16x_2 - 5x_3 &= -275 \\ 15x_1 - 5x_2 + 18x_3 &= 307 \end{aligned}$$

2.23
$$\begin{aligned} 4x_1 - 2x_2 + 3x_3 &= 37.2 \\ 3x_1 + 5x_2 - x_3 &= -7.2 \\ x_1 - 4x_2 + 2x_3 &= 30.3 \end{aligned}$$

2.24
$$\begin{aligned} 6x_1 + 15x_2 - 24x_3 + 40x_4 &= 190.9 \\ 15x_1 + 9x_2 - 13x_3 &= 69.8 \\ -24x_1 - 13x_2 + 8x_3 - 11x_4 &= -96.3 \\ 40x_1 - 11x_3 + 5x_4 &= 119.35 \end{aligned}$$

2.25
$$\begin{aligned} 2x_1 - 5x_2 + 8x_3 + 11x_4 &= 39 \\ 10x_1 + 7x_2 + 4x_3 - x_4 &= 127 \\ -3x_1 + 9x_2 + 5x_3 - 6x_4 &= 58 \\ x_1 - 4x_2 - 2x_3 + 9x_4 &= -14 \end{aligned}$$

2.26 Develop a computer program to solve a system of simultaneous equations of any size by the Gauss–Jordan method. Check the program by solving Problems 2.21 through 2.25 and comparing the computer-generated results to those determined by hand calculations.

2.27 through 2.30 Determine the inverse of the matrices shown by the Gauss–Jordan method.

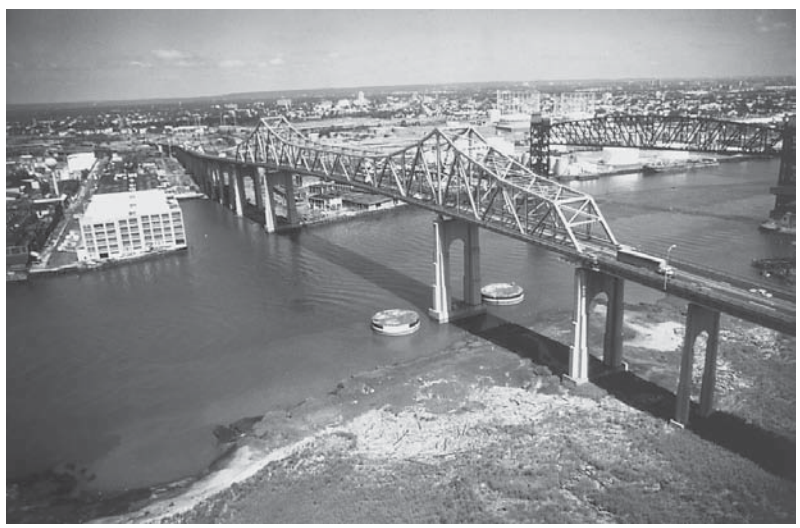
2.27 $\mathbf{A} = \begin{bmatrix} 5 & 3 & -4 \\ 3 & 8 & -2 \\ -4 & -2 & 7 \end{bmatrix}$

2.28 $\mathbf{A} = \begin{bmatrix} 6 & -4 & 1 \\ -1 & 9 & 3 \\ 4 & 2 & 5 \end{bmatrix}$

2.29 $\mathbf{A} = \begin{bmatrix} 7 & -6 & 3 & -2 \\ -6 & 4 & -1 & 5 \\ 3 & -1 & 8 & 9 \\ -2 & 5 & 9 & 2 \end{bmatrix}$

2.30 $\mathbf{A} = \begin{bmatrix} 5 & -7 & -3 & 11 \\ 10 & -6 & -13 & 2 \\ -1 & 12 & 8 & -4 \\ -9 & 7 & -5 & 6 \end{bmatrix}$

3 PLANE TRUSSES



Goethals Bridge, a Cantilever Truss Bridge between Staten Island, NY, and lizabeth, N .

(Photo courtesy of Port Authority of New ork and New Jersey)

A *plane truss* is defined as *a two-dimensional framework of straight prismatic members connected at their ends by frictionless hinged joints, and subjected to loads and reactions that act only at the joints and lie in the plane of the structure.* The members of a plane truss are subjected to axial compressive or tensile forces only.

The objective of this chapter is to develop the analysis of plane trusses based on the matrix stiffness method. This method of analysis is general, in the sense that it can be applied to statically determinate, as well as indeterminate, plane trusses of any size and shape. We begin the chapter with the definitions of the global and local coordinate systems to be used in the analysis. The concept of degrees of freedom is introduced in Section 3.2; and the member force–displacement relations are established in the local coordinate system, using the equilibrium equations and the principles of *mechanics of materials*, in Section 3.3. The finite-element formulation of member stiffness relations using the principle of virtual work is presented in Section 3.4; and transformation of member forces and displacements from a local to a global coordinate system, and vice versa, is considered in Section 3.5. Member stiffness relations in the global coordinate system are derived in Section 3.6; the formulation of the stiffness relations for the entire truss, by combining the member stiffness relations, is discussed in Section 3.7; and a step-by-step procedure for the analysis of plane trusses subjected to joint loads is developed in Section 3.8.

3.1 GLOBAL AND LOCAL COORDINATE SYSTEMS

In the matrix stiffness method, two types of coordinate systems are employed to specify the structural and loading data and to establish the necessary force–displacement relations. These are referred to as the *global* (or *structural*) and the *local* (or *member*) coordinate systems.

Global Coordinate System

The overall geometry and the load–deformation relationships for an entire structure are described with reference to a Cartesian or rectangular global coordinate system.

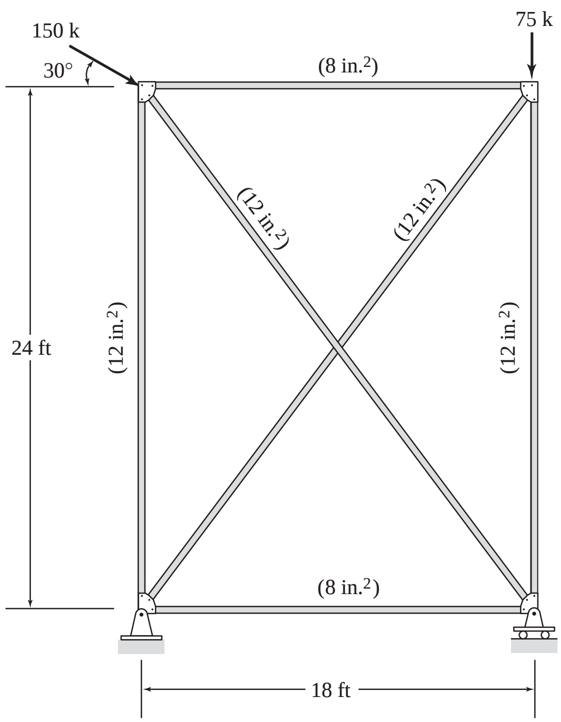
> *The global coordinate system used in this text is a right-handed XYZ coordinate system with the plane structure lying in the XY plane.*

When analyzing a plane (two-dimensional) structure, the origin of the global *XY* coordinate system can be located at any point in the plane of the structure, with the *X* and *Y* axes oriented in any mutually perpendicular directions in the structure's plane. However, it is usually convenient to locate the origin at a lower left joint of the structure, with the X and Y axes oriented in the horizontal (positive to the right) and vertical (positive upward) directions, respectively, so that the X and Y coordinates of most of the joints are positive.

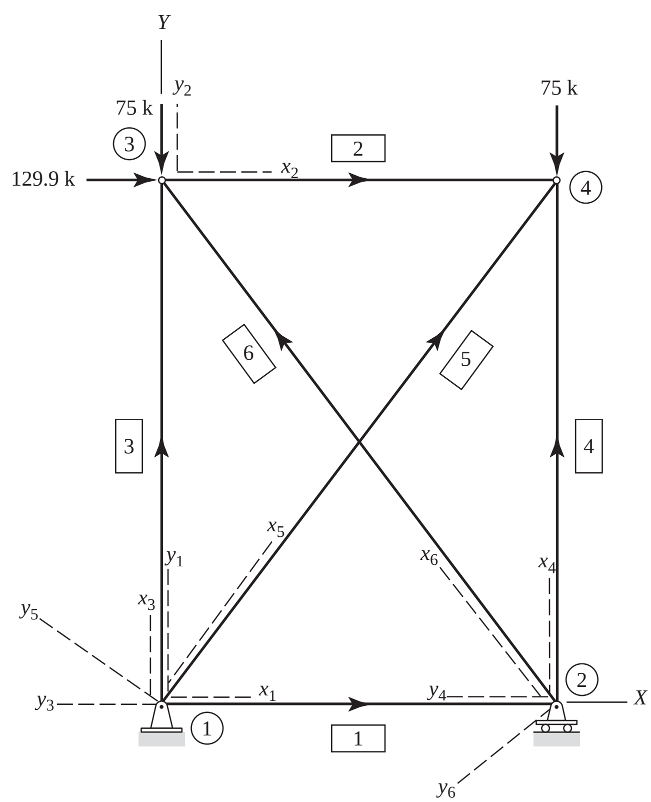
Consider, for example, the truss shown in Fig. 3.1(a), which is composed of six members and four joints. Figure 3.1(b) shows the analytical model of the truss as represented by a line diagram, on which all the joints and members are identified by numbers that have been assigned arbitrarily. The global coordinate system chosen for analysis is usually drawn on the line diagram of the structure as shown in Fig. 3.1(b). Note that the origin of the global XY coordinate system is located at joint 1.

Local Coordinate System

Since it is convenient to derive the basic member force–displacement relationships in terms of the forces and displacements in the directions along and perpendicular to members, a local coordinate system is defined for each member of the structure.

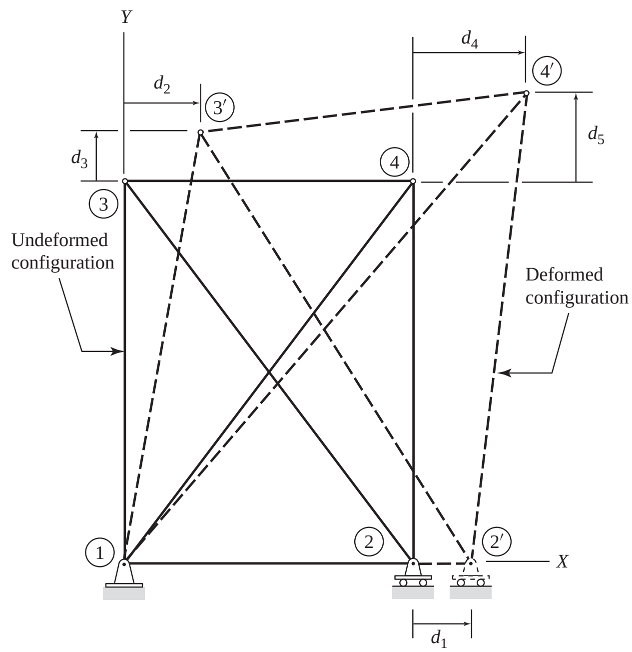


(a) Actual Truss

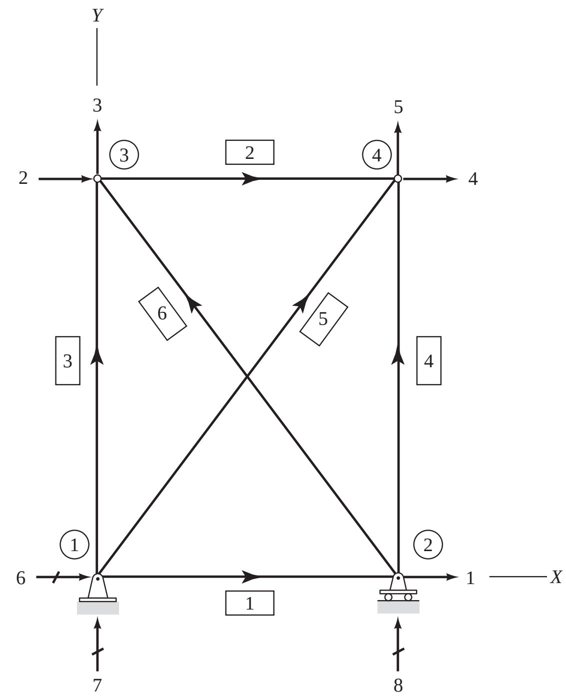


(b) Analytical Model Showing Global and Local Coordinate Systems

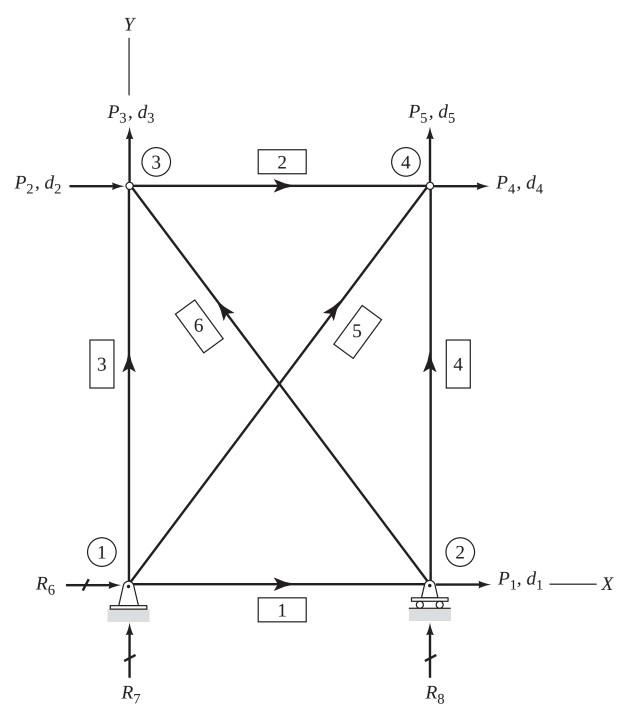
Fig. 3.1



(c) Degrees of Freedom



(d) Degrees of Freedom and Restrained Coordinate Numbers



(e) Degrees of Freedom, Joint Loads, and Reactions

Fig. 3.1 (*continued*)

> *The origin of the local xyz coordinate system for a member may be arbitrarily located at one of the ends of the member in its undeformed state, with the x axis directed along the member's centroidal axis in the undeformed state. The positive direction of the y axis is defined so that the coordinate system is right-handed, with the local z axis pointing in the positive direction of the global Z axis.*

On the line diagram of the structure, the positive direction of the x axis for each member is indicated by drawing an arrow along each member, as shown in Fig. 3.1(b). For example, this figure shows the origin of the local coordinate system for member 1 located at its end connected to joint 1, with the x_1 axis directed from joint 1 to joint 2; the origin of the local coordinate system for member 4 located at its end connected to joint 2, with the x_4 axis directed from joint 2 to joint 4, etcetera. The joint to which the member end with the origin of the local coordinate system is connected is termed the *beginning joint* for the member, and the joint adjacent to the opposite end of the member is referred to as the *end joint*. For example, in Fig. 3.1(b), member 1 begins at joint 1 and ends at joint 2, member 4 begins at joint 2 and ends at joint 4, and so on. Once the local x axis is specified for a member, its y axis can be established by applying the right-hand rule. The local y axes thus obtained for all six members of the truss are depicted in Fig. 3.1(b). It can be seen that, for each member, if we curl the fingers of our right hand from the direction of the x axis toward the direction of the corresponding y axis, then the extended thumb points out of the plane of the page, which is the positive direction of the global Z axis.

3.2 DEGREES OF FREEDOM

The *degrees of freedom* of a structure, in general, are defined as *the independent joint displacements (translations and rotations) that are necessary to specify the deformed shape of the structure when subjected to an arbitrary loading.* Since the joints of trusses are assumed to be frictionless hinges, they are not subjected to moments and, therefore, their rotations are zero. Thus, only joint translations must be considered in establishing the degrees of freedom of trusses.

Consider again the plane truss of Fig. 3.1(a). The deformed shape of the truss, for an arbitrary loading, is depicted in Fig. 3.1(c) using an exaggerated scale. From this figure, we can see that joint 1, which is attached to the hinged support, cannot translate in any direction; therefore, it has no degrees of freedom. Because joint 2 is attached to the roller support, it can translate in the X direction, but not in the Y direction. Thus, joint 2 has only one degree of freedom, which is designated d_1 in the figure. As joint 3 is not attached to a support, two displacements (namely, the translations d_2 and d_3 in the X and Y directions, respectively) are needed to completely specify its deformed position $3'$. Thus, joint 3 has two degrees of freedom. Similarly, joint 4, which is also a *free* joint, has two degrees of freedom, designated d_4 and d_5. Thus, the entire truss has a total of five degrees of freedom. As shown in Fig. 3.1(c), the joint displacements are defined relative to the global coordinate system, and are considered to be positive when in the positive directions of the X and Y axes. Note that all the joint displacements are shown in the positive sense in Fig. 3.1(c). The five joint displacement of the truss can be collectively written in matrix form as

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix}$$

in which **d** is called the *joint displacement vector*, with the number of rows equal to the number of degrees of freedom of the structure.

It is important to note that the five joint displacements d_1 through d_5 are necessary and sufficient to uniquely define the deformed shape of the truss under any arbitrary loading condition. Furthermore, the five joint displacements are independent, in the sense that each displacement can be varied arbitrarily and independently of the others.

As the foregoing example illustrates, the degrees of freedom of all types of framed structures, in general, are the same as the actual joint displacements. Thus, the number of degrees of freedom of a framed structure can be determined by subtracting the number of joint displacements restrained by supports from the total number of joint displacements of the unsupported structure; that is,

$$\begin{pmatrix} \text{number of} \\ \text{degrees of} \\ \text{freedom} \end{pmatrix} = \begin{pmatrix} \text{number of joint} \\ \text{displacements of} \\ \text{the unsupported} \\ \text{structure} \end{pmatrix} - \begin{pmatrix} \text{number of joint} \\ \text{displacements} \\ \text{restrained by} \\ \text{supports} \end{pmatrix} \qquad \textbf{(3.1)}$$

As the number of displacements of an unsupported structure equals the product of the number of degrees of freedom of a free joint of the structure and the total number of joints of the structure, we can express Eq. (3.1) as

$$NDOF = NC\ T\,(N\ \ ) - NR \qquad \textbf{(3.2)}$$

in which $NDOF$ represents the number of degrees of freedom of the structure (sometimes referred to as the *degree of kinematic indeterminacy* of the structure); $NC\ T$ represents the number of degrees of freedom of a free joint (also called the number of structure coordinates per joint); N is the number of joints; and NR denotes the number of joint displacements restrained by supports.

Since a free joint of a plane truss has two degrees of freedom, which are translations in the X and Y directions, we can specialize Eq. (3.2) for the case of plane trusses:

$$\left.\begin{array}{r} NC\ T = 2 \\ NDOF = 2(N\ \ ) - NR \end{array}\right\} \quad \text{for plane trusses} \qquad \textbf{(3.3)}$$

Let us apply Eq. (3.3) to the truss of Fig. 3.1(a). The truss has four joints (i.e., $N\ \ = 4$), and the hinged support at joint 1 restrains two joint displacements, namely, the translations of joint 1 in the X and Y directions; whereas the roller support at joint 2 restrains one joint displacement, which is the translation of joint 2 in the Y direction. Thus, the total number of joint displacements that are restrained by all supports of the truss is 3 (i.e., $NR = 3$). Substituting the numerical values of N and NR into Eq. (3.3), we obtain

$$NDOF = 2(4) - 3 = 5$$

which is the same as the number of degrees of freedom of the truss obtained previously.

The degrees of freedom (or joint displacements) of a structure are also termed the structure's *free coordinates* the joint displacements restrained by supports are commonly called the *restrained coordinates* of the structure. The free and restrained coordinates are referred to collectively as simply the *structure coordinates*. It should be noted that each structure coordinate represents an unknown quantity to be determined by the analysis, with a free coordinate representing an unknown joint displacement, and a restrained coordinate representing an unknown support reaction. Realizing that $NC\ T$ (i.e., the number of structure coordinates per joint) equals the number of unknown joint displacements and/or support reactions per joint of the structure, the total number of unknown joint displacements and reactions for a structure can be expressed as

$$\begin{pmatrix} \text{number of unknown} \\ \text{joint displacements} \\ \text{and support reactions} \end{pmatrix} = NDOF + NR = NC\ T(N\ )$$

Numbering of Degrees of Freedom and Restrained Coordinates

When analyzing a structure, it is not necessary to draw the structure's deformed shape, as shown in Fig. 3.1(c), to identify its degrees of freedom. Instead, the degrees of freedom can be directly specified on the line diagram of the structure by assigning numbers to the arrows drawn at the joints in the directions of the joint displacements, as shown in Fig. 3.1(d). The restrained coordinates are identified in a similar manner. However, the arrows representing the restrained coordinates are usually drawn with a slash ($\nrightarrow$) to distinguish them from the arrows identifying the degrees of freedom.

The degrees of freedom of a plane truss are numbered starting at the lowest-numbered joint that has a degree of freedom, and proceeding sequentially to the highest-numbered joint. In the case of more than one degree of freedom at a joint, the translation in the X direction is numbered first, followed by the translation in the Y direction. The first degree of freedom is assigned the number one, and the last degree of freedom is assigned a number equal to $NDOF$.

Once all the degrees of freedom of the structure have been numbered, we number the restrained coordinates in a similar manner, but begin with a number equal to $NDOF + 1$. We start at the lowest-numbered joint that is attached to a support, and proceed sequentially to the highest-numbered joint. In the case of more than one restrained coordinate at a joint, the coordinate in the

X direction is numbered first, followed by the coordinate in the Y direction. Note that this procedure will always result in the last restrained coordinate of the structure being assigned a number equal to 2(N).

The degrees of freedom and the restrained coordinates of the truss in Fig. 3.1(d) have been numbered using the foregoing procedure. We start numbering the degrees of freedom by examining joint 1. Since the displacements of joint 1 in both the X and Y directions are restrained, this joint does not have any degrees of freedom; therefore, at this point, we do not assign any numbers to the two arrows representing its restrained coordinates, and move on to the next joint. Focusing our attention on joint 2, we realize that this joint is free to displace in the X direction, but not in the Y direction. Therefore, we assign the number 1 to the horizontal arrow indicating that the X displacement of joint 2 will be denoted by d_1. Note that, at this point, we do not assign any number to the vertical arrow at joint 2, and change our focus to the next joint. Joint 3 is free to displace in both the X and Y directions; we number the X displacement first by assigning the number 2 to the horizontal arrow, and then number the Y displacement by assigning the number 3 to the vertical arrow. This indicates that the X and Y displacements of joint 3 will be denoted by d_2 and d_3, respectively. Next, we focus our attention on joint 4, which is also free to displace in both the X and Y directions; we assign numbers 4 and 5, respectively, to its displacements in the X and Y directions, as shown in Fig. 3.1(d). Again, the arrow that is numbered 4 indicates the location and direction of the joint displacement d_4; the arrow numbered 5 shows the location and direction of d_5.

Having numbered all the degrees of freedom of the truss, we now return to joint 1, and start numbering the restrained coordinates of the structure. As previously discussed, joint 1 has two restrained coordinates; we first assign the number 6 (i.e., $NDOF + 1 = 5 + 1 = 6$) to the X coordinate (horizontal arrow), and then assign the number 7 to the Y coordinate (vertical arrow). Finally, we consider joint 2, and assign the number 8 to the vertical arrow representing the restrained coordinate in the Y direction at that joint. We realize that the displacements corresponding to the restrained coordinates 6 through 8 are zero (i.e., $d_6 = d_7 = d_8 = 0$). However, we use these restrained coordinate numbers to specify the reactions at supports of the structure, as discussed subsequently in this section.

Joint Load Vector

External loads applied to the joints of trusses are specified as force components in the global X and Y directions. These load components are considered positive when acting in the positive directions of the X and Y axes, and vice versa. Any loads initially given in inclined directions are resolved into their X and Y components, before proceeding with an analysis. For example, the 150 k inclined load acting on a joint of the truss in Fig. 3.1(a) is resolved into its rectangular components as

$$\text{load component in } X \text{ direction} = 150 \cos 30^\circ = 129.9 \text{ k} \rightarrow$$

$$\text{load component in } Y \text{ direction} = 150 \sin 30^\circ = 75 \text{ k} \downarrow$$

These components are applied at joint 3 of the line diagram of the truss shown in Fig. 3.1(b).

In general, a load can be applied to a structure at the location and in the direction of each of its degrees of freedom. For example, a five-degree-of-freedom truss can be subjected to a maximum of five loads, P_1 through P_5, as shown in Fig. 3.1(e). As indicated there, the numbers assigned to the degrees of freedom are also used to identify the joint loads. In other words, a load corresponding to a degree of freedom d_i is denoted by the symbol P_i. The five joint loads of the truss can be collectively written in matrix form as

$$\mathbf{P} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 129.9 \\ -75 \\ 0 \\ -75 \end{bmatrix} \text{k}$$

in which **P** is called the *joint load vector* of the truss. The numerical form of **P** is obtained by comparing Figs. 3.1(b) and 3.1(e). This comparison shows that: $P_1 = 0$; $P_2 = 129.9$ k; $P_3 = -75$ k; $P_4 = 0$; and $P_5 = -75$ k. The negative signs assigned to the magnitudes of P_3 and P_5 indicate that these loads act in the negative Y (i.e., downward) direction. The numerical values of P_1 through P_5 are then stored in the appropriate rows of the joint load vector **P**, as shown in the foregoing equation. It should be noted that the number of rows of **P** equals the number of degrees of freedom (*NDOF*) of the structure.

Reaction Vector

A support that prevents the translation of a joint of a structure in a particular direction exerts a reaction force on the joint in that direction. Thus, when a truss is subjected to external loads, a reaction force component can develop at the location and in the direction of each of its restrained coordinates. For example, a truss with three restrained coordinates can develop up to three reactions, as shown in Fig. 3.1(e). As indicated there, the numbers assigned to the restrained coordinates are used to identify the support reactions. In other words, a reaction corresponding to an ith restrained coordinate is denoted by the symbol R_i. The three support reactions of the truss can be collectively expressed in matrix form as

$$= \begin{bmatrix} R_6 \\ R_7 \\ R_8 \end{bmatrix}$$

in which is referred to as the *reaction vector* of the structure. Note that the number of rows of equals the number of restrained coordinates (*NR*) of the structure.

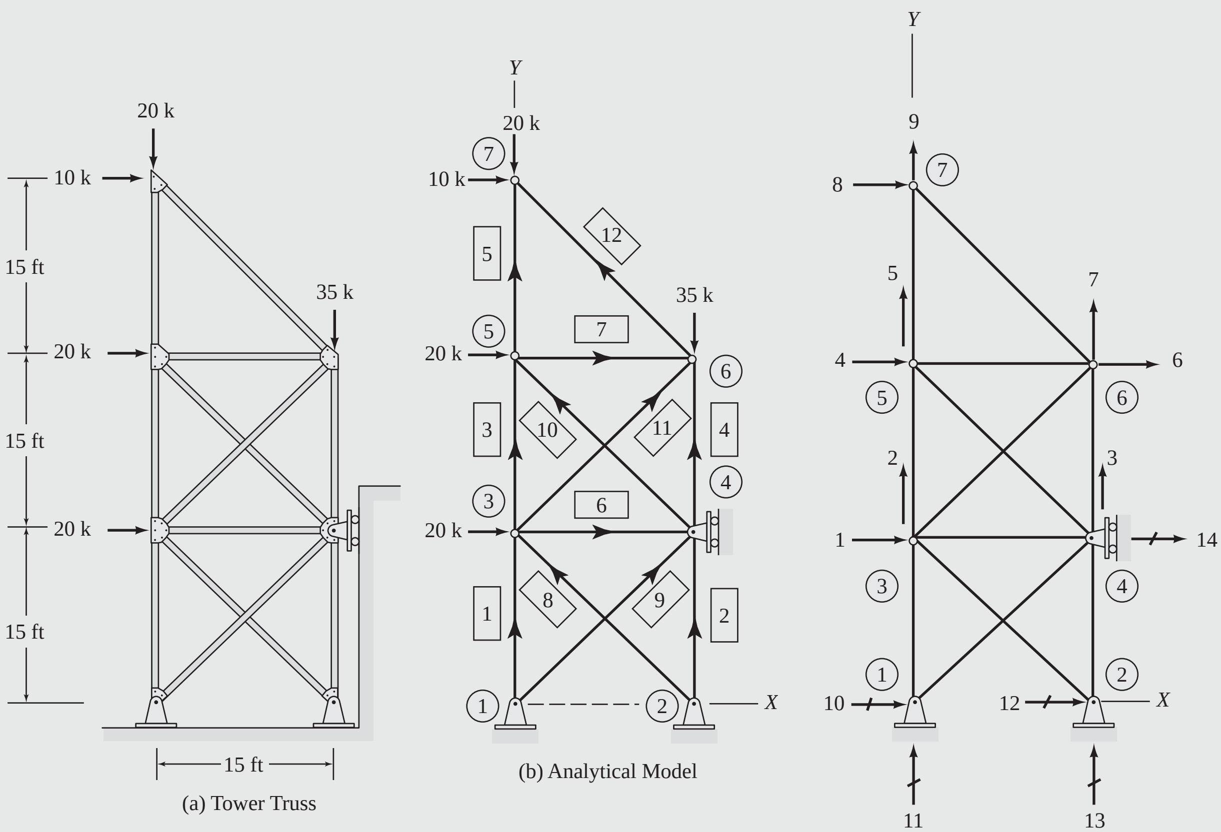
The procedure presented in this section for numerically identifying the degrees of freedom, joint loads, and reactions of a structure considerably simplifies the task of programming an analysis on a computer, as will become apparent in Chapter 4.

EXAMPLE 3.1 Identify numerically the degrees of freedom and restrained coordinates of the tower truss shown in Fig. 3.2(a). Also, form the joint load vector **P** for the truss.

SOLUTION The truss has nine degrees of freedom, which are identified by the numbers 1 through 9 in Fig. 3.2(c). The five restrained coordinates of the truss are identified by the numbers 10 through 14 in the same figure. **Ans**

By comparing Figs. 3.2(b) and (c), we express the joint load vector as

$$\mathbf{P} = \begin{bmatrix} 20 \\ 0 \\ 0 \\ 20 \\ 0 \\ 0 \\ -35 \\ 10 \\ -20 \end{bmatrix} \text{k}$$ **Ans**



(a) Tower Truss

(b) Analytical Model

(c) Degrees of Freedom and Restrained Coordinates ($NDOF = 9$, $NR = 5$)

Fig. 3.2

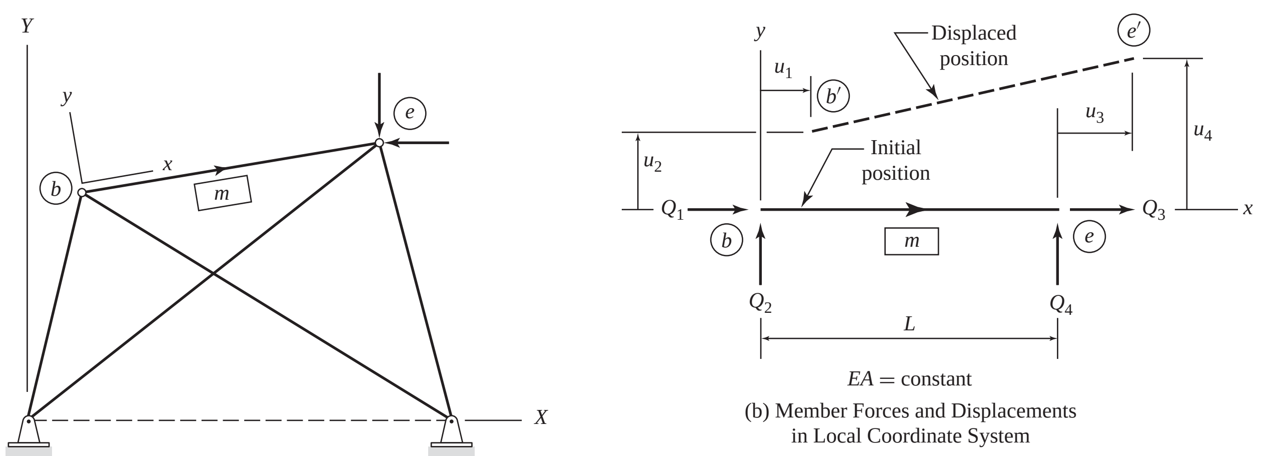
3.3 MEMBER STIFFNESS RELATIONS IN THE LOCAL COORDINATE SYSTEM

In the stiffness method of analysis, the joint displacements, **d**, of a structure due to an external loading, **P**, are determined by solving a system of simultaneous equations, expressed in the form

$$\mathbf{P} = \quad \mathbf{d} \tag{3.4}$$

in which   is called the *structure stiffness matrix*. It will be shown subsequently that the stiffness matrix for the entire structure, , is formed by assembling the stiffness matrices for its individual members. *The stiffness matrix for a member expresses the forces at the ends of the member as functions of the displacements of those ends*. In this section, we derive the stiffness matrix for the members of plane trusses in the local coordinate system.

To establish the member stiffness relations, let us focus our attention on an arbitrary member *m* of the plane truss shown in Fig. 3.3(a). When the truss is subjected to external loads, *m* deforms and internal forces are induced at its ends. The initial and displaced positions of *m* are shown in Fig. 3.3(b), where *L*, , and *A* denote, respectively, the length, oung's modulus of elasticity, and the cross-sectional area of *m*. The member is prismatic in the sense that its axial rigidity, *A*, is constant. As Fig. 3.3(b) indicates, two displacements—translations in the *x* and *y* directions—are needed to completely specify the displaced position of each end of *m*. Thus, *m* has a total of four end displacements or degrees of freedom. As shown in Fig. 3.3(b), the member end displacements are denoted by u_1 through u_4, and the corresponding member end forces are denoted by $_1$ through $_4$. Note that these end displacements and forces are defined relative to the local coordinate system of the member, and are considered positive when in the positive directions of the local *x* and *y* axes. As indicated in



(a) Plane Truss

(b) Member Forces and Displacements in Local Coordinate System

Fig. 3.3

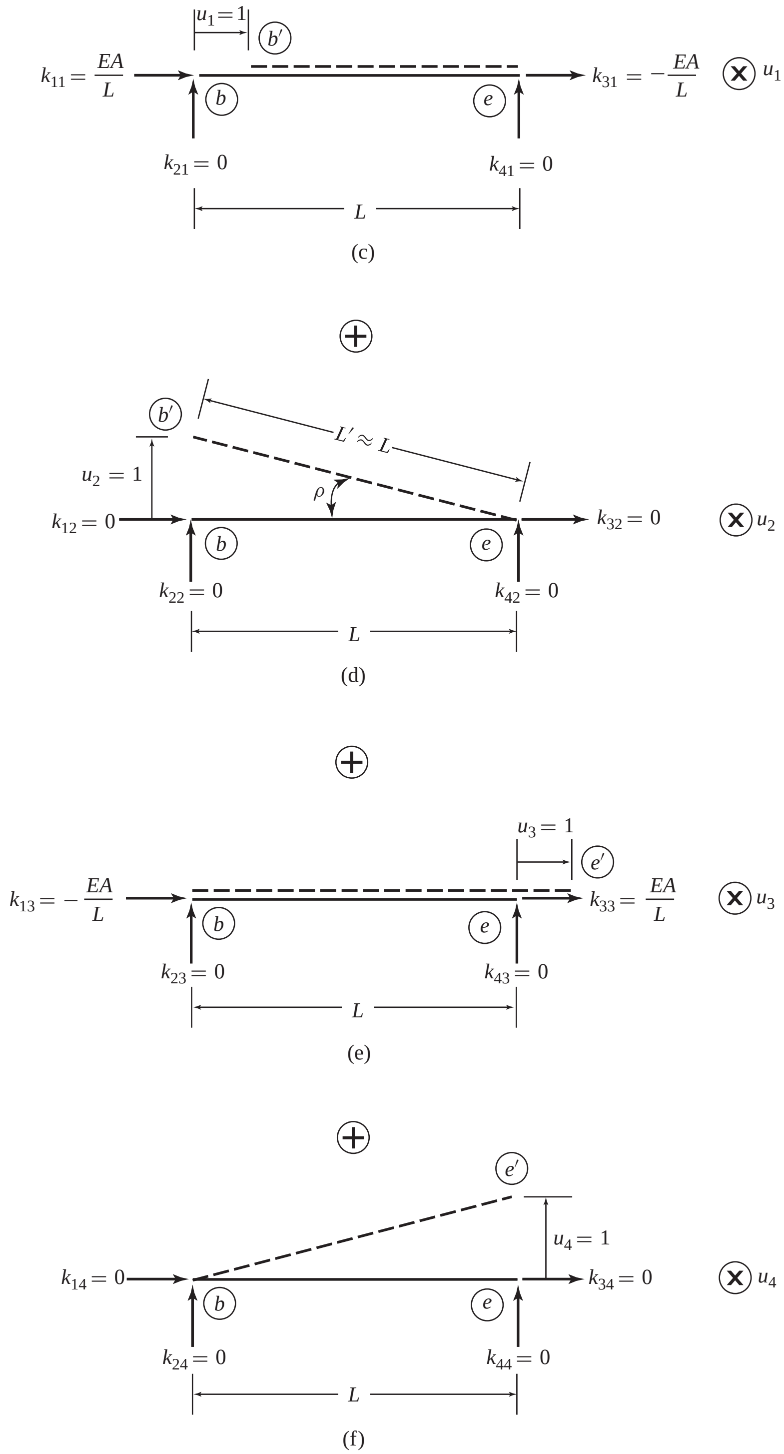


Fig. 3.3 (*continued*)

Fig. 3.3(b), the numbering scheme used to identify the member end displacements and forces is as follows:

> *Member end displacements and forces are numbered by beginning at the end of the member designated b , where the origin of the local coordinate system is located, with the translation and force in the x direction numbered first, followed by the translation and force in the y direction. The displacements and forces at the opposite end of the member, designated e, are then numbered in the same sequential order.*

It should be remembered that our objective here is to determine the relationships between member end forces and end displacements. Such relationships can be conveniently established by subjecting the member, separately, to each of the four end displacements as shown in Figs. 3.3(c) through (f); and by expressing the total member end forces as the algebraic sums of the end forces required to cause the individual end displacements. Thus, from Figs. 3.3(b) through (f), we can see that

$$_1 = k_{11}u_1 + k_{12}u_2 + k_{13}u_3 + k_{14}u_4 \tag{3.5a}$$

$$_2 = k_{21}u_1 + k_{22}u_2 + k_{23}u_3 + k_{24}u_4 \tag{3.5b}$$

$$_3 = k_{31}u_1 + k_{32}u_2 + k_{33}u_3 + k_{34}u_4 \tag{3.5c}$$

$$_4 = k_{41}u_1 + k_{42}u_2 + k_{43}u_3 + k_{44}u_4 \tag{3.5d}$$

in which k_{ij} *represents the force at the location and in the direction of* $_i$ *required, along with other end forces, to cause a unit value of displacement* u_j, *while all other end displacements are zero*. These forces per unit displacement are called *stiffness coefficients*. It should be noted that a double-subscript notation is used for stiffness coefficients, with the first subscript identifying the force and the second subscript identifying the displacement.

By using the definition of matrix multiplication, Eqs. (3.5) can be expressed in matrix form as

$$\begin{bmatrix} _1 \\ _2 \\ _3 \\ _4 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \tag{3.6}$$

or, symbolically, as

$$\boxed{\quad = \mathbf{u}} \tag{3.7}$$

in which and **u** are the member end force and member end displacement vectors, respectively, in the local coordinate system; and is called the *member stiffness matrix in the local coordinate system*.

The stiffness coefficients k_{ij} can be evaluated by subjecting the member, separately, to unit values of each of the four end displacements. The member end forces required to cause the individual unit displacements are then determined by applying the equations of equilibrium, and by using the principles of *mechanics of materials*. The member end forces thus obtained represent the stiffness coefficients for the member.

Let us determine the stiffness coefficients corresponding to a unit value of the displacement u_1 at end b of m, as shown in Fig. 3.3(c). Note that all other displacements of m are zero (i.e., $u_2 = u_3 = u_4 = 0$). Since m is in equilibrium, the end forces k_{11}, k_{21}, k_{31}, and k_{41} acting on it must satisfy the three equilibrium equations: $\sum F_x = 0$, $\sum F_y = 0$, and $\sum M = 0$. Applying the equations of equilibrium, we write

$$+ \rightarrow \sum F_x = 0 \qquad k_{11} + k_{31} = 0$$

$$k_{31} = -k_{11} \tag{3.8}$$

$$+ \uparrow \sum F_y = 0 \qquad k_{21} + k_{41} = 0 \tag{3.9}$$

$$+ \circlearrowleft \sum M_e = 0 \qquad -k_{21}(L) = 0$$

Since L is not zero, k_{21} must be zero; that is

$$k_{21} = 0 \tag{3.10}$$

By substituting Eq. (3.10) into Eq. (3.9), we obtain

$$k_{41} = 0 \tag{3.11}$$

Equations (3.8), (3.10), and (3.11) indicate that m is in equilibrium under the action of two axial forces, of equal magnitude but with opposite senses, applied at its ends. Furthermore, since the displacement $u_1 = 1$ results in the shortening of the member's length, the two axial forces causing this displacement must be compressive; that is, k_{11} must act in the positive direction of the local x axis, and k_{31} (with a magnitude equal to k_{11}) must act in the negative direction of the x axis.

To relate the axial force k_{11} to the unit axial deformation ($u_1 = 1$) of m, we use the principles of the *mechanics of materials*. Recall that in a prismatic member subjected to axial tension or compression, the normal stress σ is given by

$$\sigma = \frac{\text{axial force}}{\text{cross-sectional area}} = \frac{k_{11}}{A} \tag{3.12}$$

and the normal strain, ε, is expressed as

$$\varepsilon = \frac{\text{change in length}}{\text{original length}} = \frac{1}{L} \tag{3.13}$$

For linearly elastic materials, the stress–strain relationship is given by Hooke's law as

$$\sigma = \quad \varepsilon \tag{3.14}$$

Substitution of Eqs. (3.12) and (3.13) into Eq. (3.14) yields

$$\frac{k_{11}}{A} = \quad \left(\frac{1}{L}\right)$$

from which we obtain the expression for the stiffness coefficient k_{11},

$$k_{11} = \frac{A}{L} \tag{3.15}$$

The expression for k_{31} can now be obtained from Eq. (3.8) as

$$k_{31} = -k_{11} = -\frac{A}{L} \tag{3.16}$$

in which the negative sign indicates that this force acts in the negative x direction. Figure 3.3(c) shows the expressions for the four stiffness coefficients required to cause the end displacement $u_1 = 1$ of m.

By using a similar approach, it can be shown that the stiffness coefficients required to cause the axial displacement $u_3 = 1$ at end e of m are as follows (Fig. 3.3e).

$$k_{13} = -\frac{A}{L} \qquad k_{23} = 0 \qquad k_{33} = \frac{A}{L} \qquad k_{43} = 0 \tag{3.17}$$

The deformed shape of m due to a unit value of displacement u_2, while all other displacements are zero, is shown in Fig. 3.3(d). Applying the equilibrium equations, we write

$$+ \rightarrow \sum F_x = 0 \qquad k_{12} + k_{32} = 0$$

$$k_{32} = -k_{12} \tag{3.18}$$

$$+ \uparrow \sum F_y = 0 \qquad k_{22} + k_{42} = 0 \tag{3.19}$$

$$+ \circlearrowleft \sum M_e = 0 \qquad -k_{22}(L) = 0$$

from which we obtain

$$k_{22} = 0 \tag{3.20}$$

Substitution of Eq. (3.20) into Eq. (3.19) yields

$$k_{42} = 0 \tag{3.21}$$

Thus, the forces k_{22} and k_{42}, which act perpendicular to the longitudinal axis of m, are both zero.

As for the axial forces k_{12} and k_{32}, Eq. (3.18) indicates that they must be of equal magnitude but with opposite senses. From Fig. 3.3(d), we can see that the deformed length of the member, L', can be expressed in terms of its undeformed length L as

$$L' = \frac{L}{\cos \rho} \tag{3.22}$$

in which the angle ρ denotes the rotation of the member due to the end displacement $u_2 = 1$. Since the displacements are assumed to be small, $\cos \rho \approx 1$ and Eq. (3.22) reduces to

$$L' \approx L \tag{3.23}$$

which can be rewritten as

$$L' - L \approx 0 \tag{3.24}$$

As Eq. (3.24) indicates, the change in the length of m (or its axial deformation) is negligibly small and, therefore, no axial forces develop at the ends of m; that is,

$$k_{12} = k_{32} = 0 \tag{3.25}$$

Thus, as shown in Fig. 3.3(d), no end forces are required to produce the displacement $u_2 = 1$ of m.

Similarly, the stiffness coefficients required to cause the small end displacement $u_4 = 1$, in the direction perpendicular to the longitudinal axis of m, are also all zero, as shown in Fig. 3.3(f). Thus,

$$k_{14} = k_{24} = k_{34} = k_{44} = 0 \tag{3.26}$$

By substituting the foregoing values of the stiffness coefficients into Eq. (3.6), we obtain the following stiffness matrix for the members of plane trusses in their local coordinate systems.

$$= \begin{bmatrix} \frac{A}{L} & 0 & -\frac{A}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{A}{L} & 0 & \frac{A}{L} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \frac{A}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.27}$$

Note that the ith column of the member stiffness matrix consists of the end forces required to cause a unit value of the end displacement u_i, while all other displacements are zero. For example, the third column of consists of the four end forces required to cause the displacement $u_3 = 1$, as shown in Fig. 3.3(e), and so on. The units of the stiffness coefficients are expressed in terms of force divided by length (e.g., k/in or kN/m). When evaluating a stiffness matrix for analysis, it is important to use a consistent set of units. For example, if we wish to use the units of kips and feet, then the modulus of elasticity ( ) must be expressed in k/ft^2, area of cross-section (A) in ft^2, and the member length (L) in ft.

From Eq. (3.27), we can see that the stiffness matrix is symmetric; that is, $k_{ij} = k_{ji}$. As shown in Section 3.4, the stiffness matrices for linear elastic structures are always symmetric.

EXAMPLE 3.2

Determine the local stiffness matrices for the members of the truss shown in Fig. 3.4.

SOLUTION

e ber 1 a d 2 $= 29{,}000$ ksi, $A = 8$ in.^{2}, $L = 18$ ft $= 216$ in.

$$\frac{A}{L} = \frac{29{,}000(8)}{216} = 1{,}074.1 \text{ k/in.}$$

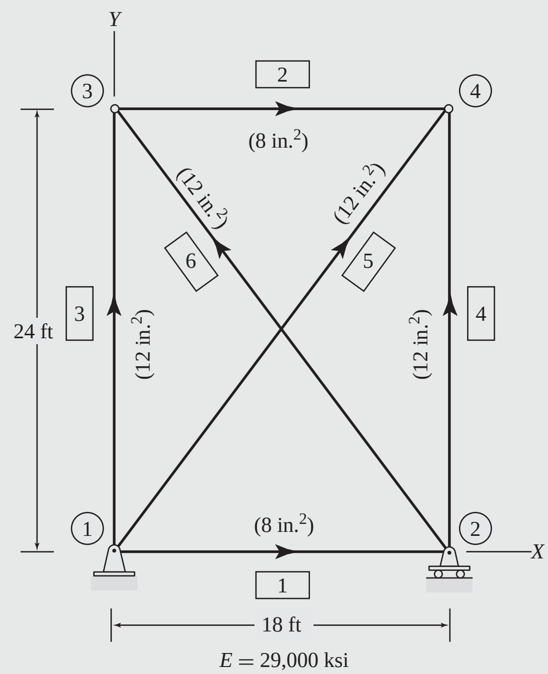


Fig. 3.4

Substitution into Eq. (3.27) yields

$$_1 = \ _2 = \begin{bmatrix} 1{,}074.1 & 0 & -1{,}074.1 & 0 \\ 0 & 0 & 0 & 0 \\ -1{,}074.1 & 0 & 1{,}074.1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{k/in.} \qquad \textbf{Ans}$$

e ber 3 a d 4 $= 29{,}000$ ksi, $A = 12$ in.^{2}, $L = 24$ ft $= 288$ in.

$$\frac{A}{L} = \frac{29{,}000(12)}{288} = 1{,}208.3 \text{ k/in.}$$

Thus, from Eq. (3.27),

$$_3 = \ _4 = \begin{bmatrix} 1{,}208.3 & 0 & -1{,}208.3 & 0 \\ 0 & 0 & 0 & 0 \\ -1{,}208.3 & 0 & 1{,}208.3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{k/in.} \qquad \textbf{Ans}$$

e ber 5 a d 6 $= 29{,}000$ ksi, $A = 12$ in.^{2},

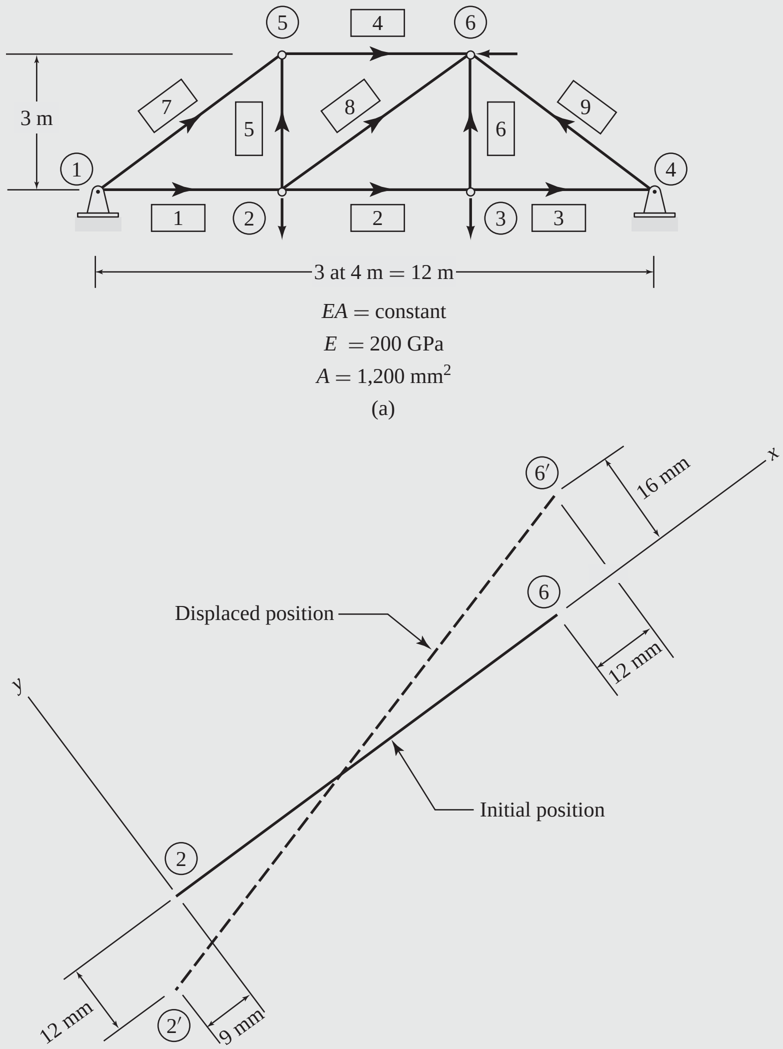
$$L = \sqrt{(18)^2 + (24)^2} = 30 \text{ ft} = 360 \text{ in.}$$

$$\frac{A}{L} = \frac{29{,}000(12)}{360} = 966.67 \text{ k/in.}$$

Thus,

$$_{5} = {}_{6} = \begin{bmatrix} 966.67 & 0 & -966.67 & 0 \\ 0 & 0 & 0 & 0 \\ -966.67 & 0 & 966.67 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ k/in.}$$ **Ans**

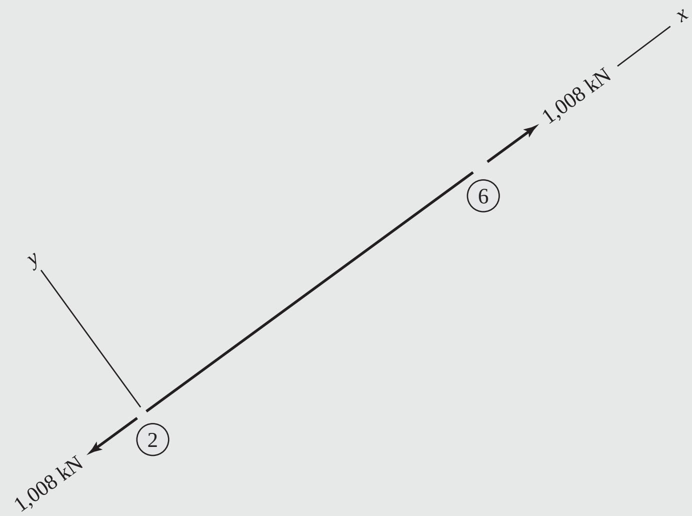
EXAMPLE 3.3 The displaced position of member 8 of the truss in Fig. 3.5(a) is given in Fig. 3.5(b). Calculate the axial force in this member.



(a)

(b) Displaced Position of Member 8

Fig. 3.5



(c) Member End Forces in Local Coordinate System

Fig. 3.5 (*continued*)

SOLUTION *Member Stiffness Matrix in the Local Coordinate System* From Fig. 3.5(a), we can see that $= 200$ GPa $= 200(10^6)$ kN/m^2; $A = 1{,}200$ mm$^2 = 0.0012$ m^2; and $L = \sqrt{(4)^2 + (3)^2} = 5$ m. Thus,

$$\frac{A}{L} = \frac{200(10^6)(0.0012)}{5} = 48{,}000 \text{ kN/m}$$

From Eq. (3.27), we obtain

$$_8 = 48{,}000 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{kN/m}$$

Member nd Displacements in the Local Coordinate System From Fig. 3.5(b), we can see that the beginning end, 2, of the member displaces 9 mm in the negative x direction and 12 mm in the negative y direction. Thus, $u_1 = -9$ mm $= -0.009$ m and $u_2 = -12$ mm $= -0.012$ m. Similarly, the opposite end, 6, of the member displaces 12 mm and 16 mm, respectively, in the x and y directions; that is, $u_3 = 12$ mm $= 0.012$ m and $u_4 = 16$ mm $= 0.016$ m. Thus, the member end displacement vector in the local coordinate system is given by

$$\mathbf{u}_8 = \begin{bmatrix} -0.009 \\ -0.012 \\ 0.012 \\ 0.016 \end{bmatrix} \text{m}$$

Member nd Forces in the Local Coordinate System We calculate the member end forces by applying Eq. (3.7). Thus,

$$= \mathbf{u}$$

$$_8 = \begin{bmatrix} _1 \\ _2 \\ _3 \\ _4 \end{bmatrix} = 48{,}000 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -0.009 \\ -0.012 \\ 0.012 \\ 0.016 \end{bmatrix} = \begin{bmatrix} -1{,}008 \\ 0 \\ 1{,}008 \\ 0 \end{bmatrix} \text{kN}$$

The member end forces, , are depicted on the free-body diagram of the member in Fig. 3.5(c), from which we can see that, since the end force $_1$ is negative, but $_3$ is positive, member 8 is subjected to a tensile axial force, $_a$, of magnitude 1,008 kN; that is,

$$_{a8} = 1{,}008 \text{ kN (T)}$$ **Ans**

3.4 FINITE-ELEMENT FORMULATION USING VIRTUAL WORK*

In this section, we present an alternate formulation of the member stiffness matrix in the local coordinate system. This approach, which is commonly used in the finite-element method, essentially involves expressing the strains and stresses at points within the member in terms of its end displacements **u**, and applying the principle of virtual work for deformable bodies as delineated by Eq. (1.16) in Section 1.6. Before proceeding with the derivation of , let us rewrite Eq. (1.16) in a more convenient matrix form as

$$\delta W_e = \int_V \delta\varepsilon^T \sigma \, dV \tag{3.28}$$

in which δW_e denotes virtual external work; V represents member volume; and $\delta\varepsilon$ and σ denote, respectively, the virtual strain and real stress vectors, which for a general three-dimensional stress condition can be expressed as follows (see Fig. 1.16).

$$\delta\varepsilon = \begin{bmatrix} \delta\varepsilon_x \\ \delta\varepsilon_y \\ \delta\varepsilon_z \\ \delta\gamma_{xy} \\ \delta\gamma_{yz} \\ \delta\gamma_{zx} \end{bmatrix} \qquad \sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} \tag{3.29}$$

Displacement Functions

In the finite-element method, member stiffness relations are based on *assumed* variations of displacements within members. Such displacement variations are

---

This section can be omitted without loss of continuity.

referred to as the *displacement* or *interpolation functions.* A displacement function describes *the variation of a displacement component along the centroidal axis of a member in terms of its end displacements.*

Consider a prismatic member of a plane truss, subjected to end displacements u_1 through u_4, as shown in Fig. 3.6(a). Since the member displaces in both the x and y directions, we need to define two displacement functions,

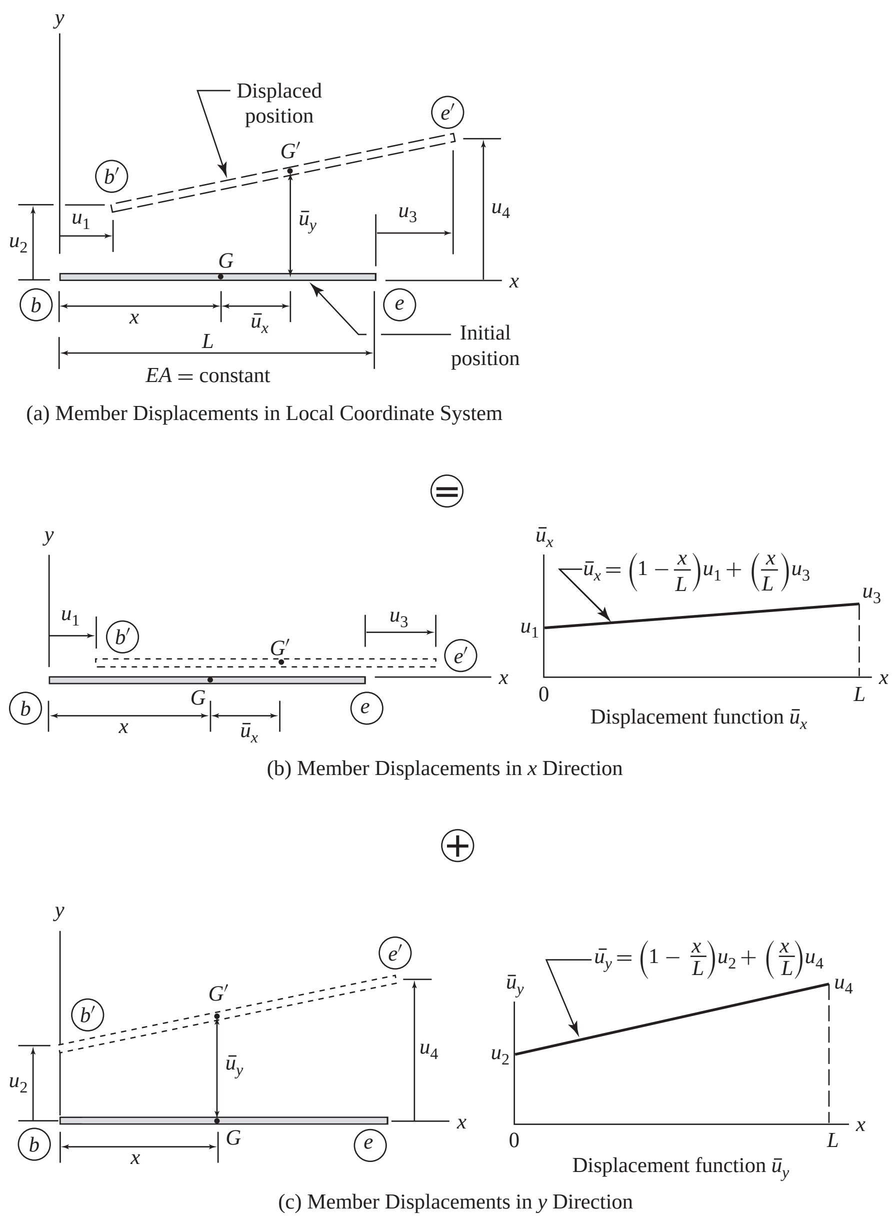


Fig. 3.6

$\bar{u}_x$ and $\bar{u}_y$, for the displacements in the x and y directions, respectively. In Fig. 3.6(a), the displacement functions $\bar{u}_x$ and $\bar{u}_y$ are depicted as the displacements of an arbitrary point G located on the member's centroidal axis at a distance x from end b (left end).

The total displacement of the member (due to u_1 through u_4) can be decomposed into the displacements in the x and y directions, as shown in Figs. 3.6(b) and (c), respectively. Note that Fig. 3.6(b) shows the member subjected to the two end displacements, u_1 and u_3, in the x direction (with $u_2 = u_4 = 0$); Fig. 3.6(c) depicts the displacement of the member due to the two end displacements, u_2 and u_4, in the y direction (with $u_1 = u_3 = 0$).

The displacement functions assumed in the finite-element method are usually in the form of *complete* polynomials,

$$\bar{u}(x) = \sum_{i=0}^{n} a_i x^i \qquad \text{with } a_i \neq 0 \tag{3.30}$$

in which n is the degree of the polynomial. The polynomial used for a particular displacement function should be of such a degree that all of its coefficients can be evaluated by applying the available boundary conditions; that is,

$$n = n_{bc} - 1 \tag{3.31}$$

with n_{bc} = number of boundary conditions.

Thus, the displacement function $\bar{u}_x$ for the truss member (Fig. 3.6b) is assumed in the form of a linear polynomial as

$$\bar{u}_x = a_0 + a_1 x \tag{3.32}$$

in which a_0 and a_1 are the constants that can be determined by applying the following two boundary conditions:

$$\text{at } x = 0 \qquad \bar{u}_x = u_1$$
$$\text{at } x = L \qquad \bar{u}_x = u_3$$

By applying the first boundary condition—that is, by setting $x = 0$ and $\bar{u}_x = u_1$ in Eq. (3.32)—we obtain

$$a_0 = u_1 \tag{3.33}$$

Next, by using the second boundary condition—that is, by setting $x = L$ and $\bar{u}_x = u_3$—we obtain

$$u_3 = u_1 + a_1 L$$

from which follows

$$a_1 = \frac{u_3 - u_1}{L} \tag{3.34}$$

By substituting Eqs. (3.33) and (3.34) into Eq. (3.32), we obtain the expression for $\bar{u}_x$,

$$\bar{u}_x = u_1 + \left(\frac{u_3 - u_1}{L}\right) x$$

or

$$\boxed{\bar{u}_x = \left(1 - \frac{x}{L}\right)u_1 + \left(\frac{x}{L}\right)u_3} \tag{3.35}$$

The displacement function $\bar{u}_y$, for the member displacement in the y direction (Fig. 3.6(c)), can be determined in a similar manner; that is, using a linear polynomial and evaluating its coefficients by applying the boundary conditions. The result is

$$\boxed{\bar{u}_y = \left(1 - \frac{x}{L}\right)u_2 + \left(\frac{x}{L}\right)u_4} \tag{3.36}$$

The plots of the displacement functions $\bar{u}_x$ and $\bar{u}_y$ are shown in Figs. 3.6(b) and (c), respectively.

It is important to realize that the displacement functions as given by Eqs. (3.35) and (3.36) have been *assumed*, as is usually done in the finite-element method. There is no guarantee that an assumed displacement function defines the actual displacements of the member, except at its ends. In general, the displacement functions used in the finite-element method only approximate the actual displacements within members (or elements), because they represent approximate solutions of the underlying differential equations. For this reason, the finite-element method is generally considered to be an approximate method of analysis. However, for the prismatic members of framed structures, the displacement functions in the form of complete polynomials do happen to describe exactly the actual member displacements and, therefore, such functions yield exact member stiffness matrices for prismatic members.

From Fig. 3.6(c), we observe that the graph of the displacement function $\bar{u}_y$ exactly matches the displaced shape of the member's centroidal axis due to the end displacements u_2 and u_4. As this displaced shape defines the actual displacements in the y direction of all points along the member's length, we can conclude that the function $\bar{u}_y$, as given by Eq. (3.36), is exact.

To demonstrate that Eq. (3.35) describes the actual displacements in the x direction of all points along the truss member's centroidal axis, consider again the member subjected to end displacements, u_1 and u_3, in the x direction as shown in Fig. 3.7(a). Since the member is subjected to forces only at its ends, the axial force, $_{\mathrm{a}}$, is constant throughout the member's length. Thus, the axial stress, σ, at point G of the member is

$$\sigma = \frac{_{\mathrm{a}}}{A} \tag{3.37}$$

in which A represents the cross-sectional area of the member at point G. Note that the axial stress is distributed uniformly over the cross-sectional area A. By substituting the linear stress–strain relationship $\varepsilon = \sigma/$ into Eq. (3.37), we obtain the strain at point G as

$$\varepsilon = \frac{\text{a}}{A} = \text{constant} = a_1 \tag{3.38}$$

in which a_1 is a constant. As this equation indicates, since the member is prismatic (i.e., A = constant), the axial strain is constant throughout the member length.

To relate the strain ε to the displacement $\bar{u}_x$, we focus our attention on the differential element G of length dx (Fig. 3.7(a)). The undeformed and deformed positions of the element are shown in Fig. 3.7(b), in which $\bar{u}_x$ and $\bar{u}_x + d\bar{u}_x$ denote, respectively, the displacements of the ends G and of the element in the x direction. From this figure, we can see that

$$\begin{aligned} \text{deformed length of element} &= dx + (\bar{u}_x + d\bar{u}_x) - \bar{u}_x \\ &= dx + d\bar{u}_x \end{aligned}$$

Therefore, the strain in the element is given by

$$\varepsilon = \frac{\text{deformed length} - \text{initial length}}{\text{initial length}} = \frac{(dx + d\bar{u}_x) - dx}{dx}$$

or

$$\boxed{\varepsilon = \frac{d\bar{u}_x}{dx}} \tag{3.39}$$

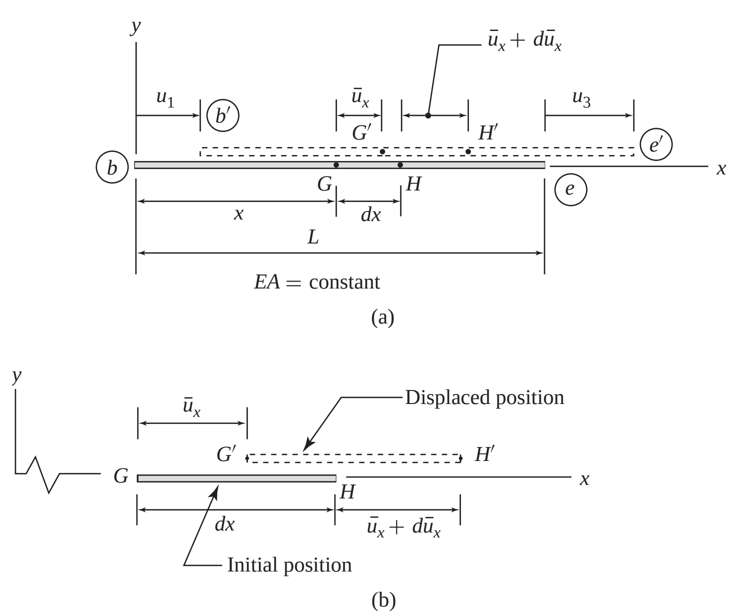


Fig. 3.7

By equating the two expressions for strain as given by Eqs. (3.38) and (3.39), we obtain

$$\frac{d\bar{u}_x}{dx} = a_1 \tag{3.40}$$

which can be rewritten as

$$d\bar{u}_x = a_1 dx \tag{3.41}$$

By integrating Eq. (3.41), we obtain

$$\bar{u}_x = a_1 x + a_0 \tag{3.42}$$

in which a_0 is the constant of integration. Note that Eq. (3.42), obtained by integrating the actual strain–displacement relationship, indicates that the linear polynomial form assumed for $\bar{u}_x$ in Eq. (3.32) was indeed correct. Furthermore, if we evaluate the two constants in Eq. (3.42) by applying the boundary conditions, we obtain an equation which is identical to Eq. (3.35), indicating that our assumed displacement function $\bar{u}_x$ (as given by Eq. (3.35)) does indeed describe the actual member displacements in the x direction.

Shape Functions

The displacement functions, as given by Eqs. (3.35) and (3.36), can be expressed alternatively as

$$\bar{u}_x = N_1 u_1 + N_3 u_3 \tag{3.43a}$$

$$\bar{u}_y = N_2 u_2 + N_4 u_4 \tag{3.43b}$$

with

$$N_1 = N_2 = 1 - \frac{x}{L} \tag{3.44a}$$

$$N_3 = N_4 = \frac{x}{L} \tag{3.44b}$$

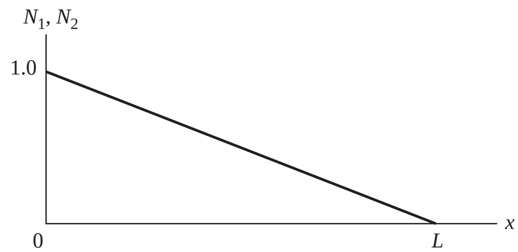
in which N_i (with $i = 1,4$) are called the *shape functions*. The plots of the four shape functions for a plane truss member are given in Fig. 3.8. We can see from this figure that *a shape function N_i describes the displacement variation along a member's centroidal axis due to a unit value of the end displacement u_i, while all other end displacements are zero.*

Equations (3.43) can be written in matrix form as

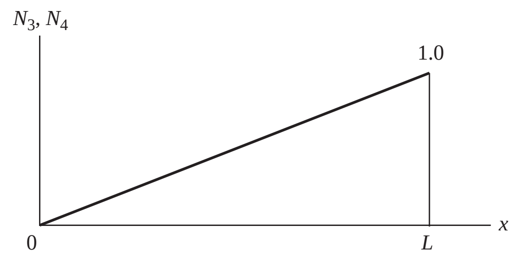
$$\begin{bmatrix} \bar{u}_x \\ \bar{u}_y \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_3 & 0 \\ 0 & N_2 & 0 & N_4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \tag{3.45}$$

or, symbolically, as

$$\boxed{\mathbf{u} = \quad \mathbf{u}} \tag{3.46}$$



(a) Shape Functions N_1 ($u_1 = 1$, $u_2 = u_3 = u_4 = 0$)
and N_2 ($u_2 = 1$, $u_1 = u_3 = u_4 = 0$)



(b) Shape Functions N_3 ($u_3 = 1$, $u_1 = u_2 = u_4 = 0$)
and N_4 ($u_4 = 1$, $u_1 = u_2 = u_3 = 0$)

Fig. 3.8 *Shape Functions for Plane Truss Member*

in which **u** is the *member displacement function vector,* and is called the *member shape function matrix*.

Strain–Displacement Relationship

As discussed previously, the relationship between the axial strain, ε, and the displacement, $\bar{u}_x$, is given by $\varepsilon = d\bar{u}_x/dx$ (see Eq. (3.39)). This strain–displacement relationship can be expressed in matrix form as

$$\varepsilon = \begin{bmatrix} \frac{d}{dx} & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_x \\ \bar{u}_y \end{bmatrix} = \mathbf{D}\bar{\mathbf{u}} \tag{3.47}$$

in which **D** is known as the *differential operator matrix*. To relate the strain, ε, to the member end displacements, **u**, we substitute Eq. (3.46) into Eq. (3.47) to obtain

$$\varepsilon = \mathbf{D}(\quad \mathbf{u}) \tag{3.48}$$

Since the end displacement vector **u** does not depend on x, it can be treated as a constant in the differentiation indicated by Eq. (3.48). In other words, the differentiation applies to , but not to **u**. Thus, Eq. (3.48) can be rewritten as

$$\boxed{\varepsilon = (\mathbf{D} \quad)\mathbf{u} = \mathbf{B}\mathbf{u}} \tag{3.49}$$

in which, $\mathbf{B} = \mathbf{D}$ is called the *member strain displacement matrix*. To determine $\mathbf{B}$, we write

$$\mathbf{B} = \mathbf{D} \quad = \begin{bmatrix} \dfrac{d}{dx} & 0 \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_3 & 0 \\ 0 & N_2 & 0 & N_4 \end{bmatrix}$$

By multiplying matrices $\mathbf{D}$ and ,

$$\mathbf{B} = \begin{bmatrix} \dfrac{dN_1}{dx} & 0 & \dfrac{dN_3}{dx} & 0 \end{bmatrix}$$

Next, we substitute the expressions for N_1 and N_3 from Eqs. (3.44) into the preceding equation to obtain

$$\mathbf{B} = \begin{bmatrix} \dfrac{d}{dx}\left(1 - \dfrac{x}{L}\right) & 0 & \dfrac{d}{dx}\left(\dfrac{x}{L}\right) & 0 \end{bmatrix}$$

Finally, by performing the necessary differentiations, we determine the strain–displacement matrix $\mathbf{B}$:

$$\mathbf{B} = \begin{bmatrix} -\dfrac{1}{L} & 0 & \dfrac{1}{L} & 0 \end{bmatrix} = \frac{1}{L} \quad -1 \quad 0 \quad 1 \quad 0 \tag{3.50}$$

Stress–Displacement Relationship

The relationship between member axial stress and member end displacements can now be established by substituting Eq. (3.49) into the stress–strain relationship, $\sigma = \quad \varepsilon$. Thus,

$$\boxed{\sigma = \quad \mathbf{Bu}} \tag{3.51}$$

Member Stiffness Matrix, k

With both member strain and stress expressed in terms of end displacements, we can now establish the relationship between member end forces and end displacements $\mathbf{u}$, by applying the principle of virtual work for deformable bodies. Consider an arbitrary member of a plane truss that is in equilibrium under the action of end forces $_1$ through $_4$. Assume that the member is given small virtual end displacements δu_1 through δu_4, as shown in Fig. 3.9. The virtual

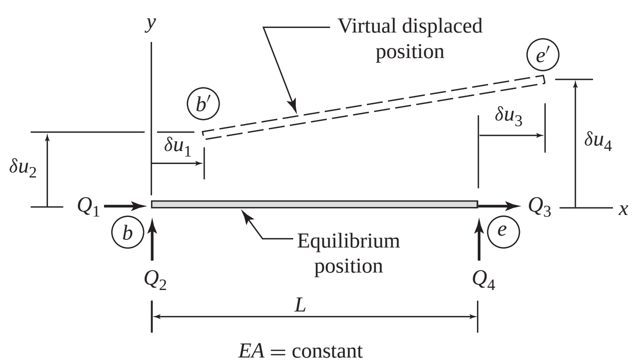


Fig. 3.9

external work done by the real member end forces $_1$ through $_4$ as they move through the corresponding virtual end displacements δu_1 through δu_4 is

$$\delta W_{\mathrm{e}} = \ _1\,\delta u_1 + \ _2\,\delta u_2 + \ _3\,\delta u_3 + \ _4\,\delta u_4$$

which can be expressed in matrix form as

$$\delta W_{\mathrm{e}} = \begin{matrix} \delta u_1 & \delta u_2 & \delta u_3 & \delta u_4 \end{matrix} \begin{bmatrix} _1 \\ _2 \\ _3 \\ _4 \end{bmatrix}$$

or

$$\delta W_{\mathrm{e}} = \delta \mathbf{u}^T \qquad \textbf{(3.52)}$$

By substituting Eq. (3.52) into the expression for the principle of virtual work for deformable bodies, as given by Eq. (3.28), we obtain

$$\delta \mathbf{u}^T \quad = \int_V \delta \varepsilon^T \sigma \, dV \qquad \textbf{(3.53)}$$

Recall that the right-hand side of Eq. (3.53) represents the virtual strain energy stored in the member. Substitution of Eqs. (3.49) and (3.51) into Eq. (3.53) yields

$$\delta \mathbf{u}^T \quad = \int_V (\mathbf{B}\,\delta \mathbf{u})^T \quad \mathbf{B}\, dV \mathbf{u}$$

Since $(\mathbf{B}\,\delta\mathbf{u})^T = \delta\mathbf{u}^T\mathbf{B}^T$, we can write the preceding equation as

$$\delta \mathbf{u}^T \quad = \delta \mathbf{u}^T \int_V \mathbf{B}^T \quad \mathbf{B}\, dV \mathbf{u}$$

or

$$\delta \mathbf{u}^T \left(\quad - \int_V \mathbf{B}^T \quad \mathbf{B}\, dV \mathbf{u} \right) = 0$$

Since $\delta\mathbf{u}^T$ can be arbitrarily chosen and is not zero, the quantity in the parentheses must be zero. Thus,

$$= \left(\int_V \mathbf{B}^T \quad \mathbf{B}\, dV \right) \mathbf{u} = \quad \mathbf{u} \qquad \textbf{(3.54)}$$

in which

$$\boxed{\quad = \int_V \mathbf{B}^T \quad \mathbf{B}\, dV} \qquad \textbf{(3.55)}$$

is the member stiffness matrix in the local coordinate system. To determine the explicit form of , we substitute Eq. (3.50) for **B** into Eq. (3.55) to obtain

$$= \frac{\quad}{L^2} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{matrix} -1 & 0 & 1 & 0 \end{matrix} \int_V dV = \frac{\quad}{L^2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \int_V dV$$

Since $\int_V dV = V = AL$, the member stiffness matrix, , becomes

$$= \frac{A}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that the preceding expression for is identical to that derived in Section 3.3 (Eq. (3.27)) using the direct equilibrium approach.

Symmetry of the Member Stiffness Matrix

The expression for the stiffness matrix as given by Eq. (3.55) is general, in the sense that the stiffness matrices for members of other types of framed structures, as well as for elements of surface structures and solids, can also be expressed in the integral form of this equation. We can deduce from Eq. (3.55) that *for linear elastic structures, the member stiffness matrices are symmetric*.

Transposing both sides of Eq. (3.55), we write

$$^T = \int_V (\mathbf{B}^T \quad \mathbf{B})^T dV$$

Now, recall from Section 2.3 that the transpose of a product of matrices equals the product of the transposed matrices in reverse order; that is, $(\mathbf{ABC})^T = \mathbf{C}^T\mathbf{B}^T\mathbf{A}^T$. Thus, the preceding equation becomes

$$^T = \int_V \mathbf{B}^T \quad {}^T(\mathbf{B}^T)^T dV$$

For linear elastic structures, is either a scalar (in the case of framed structures), or a symmetric matrix (for surface structures and solids). Therefore, $^T =$. Furthermore, by realizing that $(\mathbf{B}^T)^T = \mathbf{B}$, we can express the preceding equation as

$$^T = \int_V \mathbf{B}^T \quad \mathbf{B} dV \tag{3.56}$$

Finally, a comparison of Eqs. (3.55) and (3.56) yields

$$\boxed{\quad ^T = \quad} \tag{3.57}$$

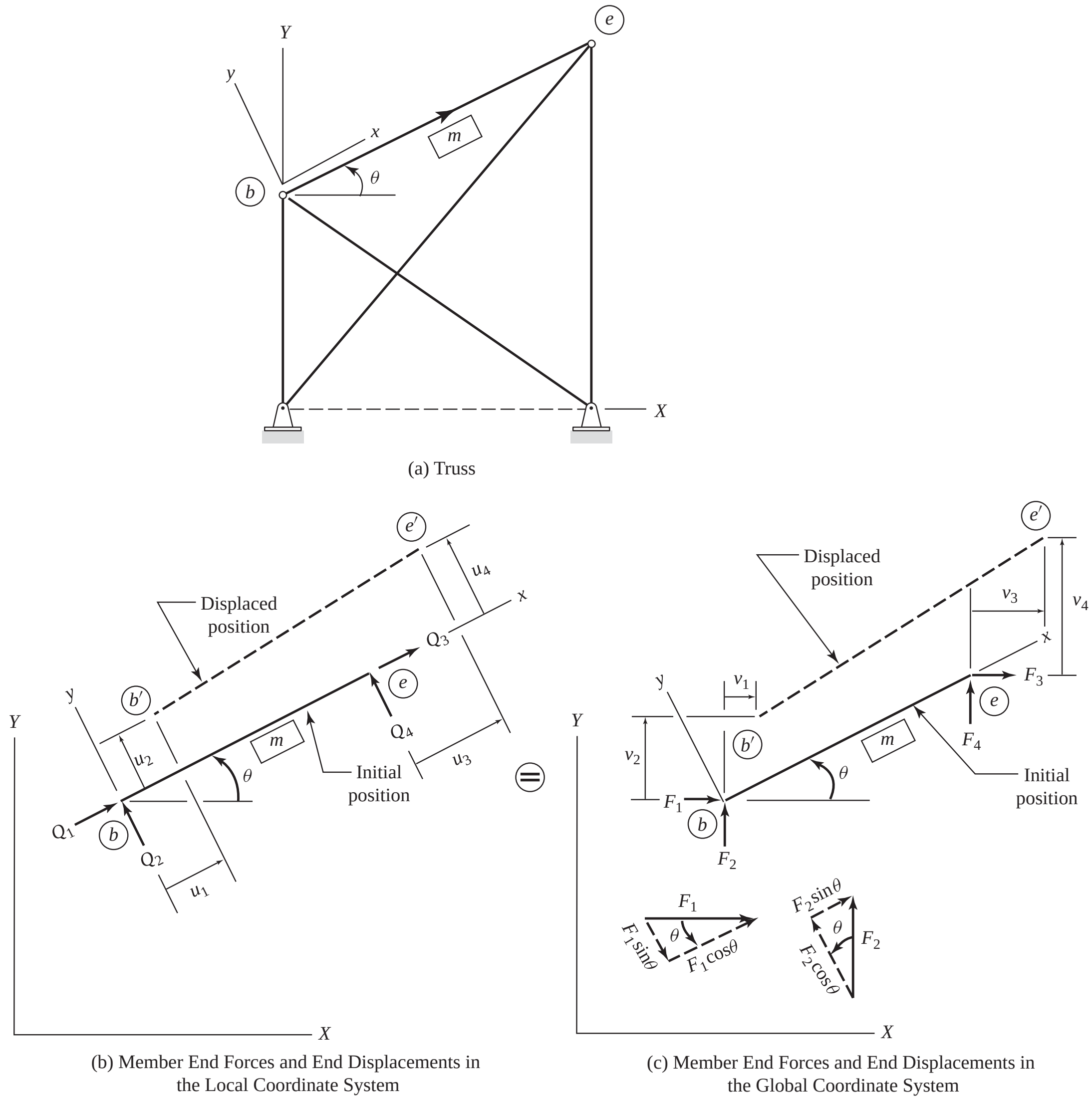
which shows that is a symmetric matrix.

3.5 COORDINATE TRANSFORMATIONS

When members of a structure are oriented in different directions, it becomes necessary to transform the stiffness relations for each member from its local coordinate system to a single global coordinate system selected for the entire structure. The member stiffness relations as expressed in the global coordinate system are then combined to establish the stiffness relations for the whole structure. In this section, we consider the transformation of member end forces and end displacements from local to global coordinate systems, and vice versa, for members of plane trusses. The transformation of the stiffness matrices is discussed in Section 3.6.

Transformation from Global to Local Coordinate Systems

Consider an arbitrary member m of a plane truss (Fig. 3.10(a)). As shown in this figure, the orientation of m relative to the global XY coordinate system is defined by an angle θ, measured counterclockwise from the positive direction of the global X axis to the positive direction of the local x axis. Recall that the stiffness matrix derived in the preceding sections relates member end forces and end displacement **u** described with reference to the local xy coordinate system of the member, as shown in Fig. 3.10(b).



(a) Truss

(b) Member End Forces and End Displacements in the Local Coordinate System

(c) Member End Forces and End Displacements in the Global Coordinate System

Fig. 3.10

Now, suppose that the member end forces and end displacements are specified with reference to the global *XY* coordinate system (Fig. 3.10(c)), and we wish to determine the equivalent system of end forces and end displacements, in the local *xy* coordinates, which has the same effect on *m*. As indicated in Fig. 3.10(c), the member end forces in the global coordinate system are denoted by F_1 through F_4, and the corresponding end displacements are denoted by v_1 through v_4. These global member end forces and end displacements are numbered beginning at member end *b*, with the force and translation in the *X* direction numbered first, followed by the force and translation in the *Y* direction. The forces and displacements at the member's opposite end *e* are then numbered in the same sequential order.

By comparing Figs. 3.10(b) and (c), we observe that at end *b* of *m*, the local force $_1$ must be equal to the algebraic sum of the components of the global forces F_1 and F_2 in the direction of the local *x* axis; that is,

$$_1 = F_1 \cos\theta + F_2 \sin\theta \tag{3.58a}$$

Similarly, the local force $_2$ equals the algebraic sum of the components of F_1 and F_2 in the direction of the local *y* axis. Thus,

$$_2 = -F_1 \sin\theta + F_2 \cos\theta \tag{3.58b}$$

By using a similar reasoning at end *e*, we express the local forces in terms of the global forces as

$$_3 = F_3 \cos\theta + F_4 \sin\theta \tag{3.58c}$$

$$_4 = -F_3 \sin\theta + F_4 \cos\theta \tag{3.58d}$$

Equations 3.58(a) through (d) can be written in matrix form as

$$\begin{bmatrix} _1 \\ _2 \\ _3 \\ _4 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \tag{3.59}$$

or, symbolically, as

$$= \tag{3.60}$$

with

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{3.61}$$

in which is referred to as the *transformation matrix*. The direction cosines of the member, necessary for the evaluation of , can be conveniently determined by using the following relationships:

$$\cos\theta = \frac{X_e - X_b}{L} = \frac{X_e - X_b}{\sqrt{(X_e - X_b)^2 + (Y_e - Y_b)^2}} \tag{3.62a}$$

$$\sin\theta = \frac{Y_e - Y_b}{L} = \frac{Y_e - Y_b}{\sqrt{(X_e - X_b)^2 + (Y_e - Y_b)^2}} \tag{3.62b}$$

in which X_b and Y_b denote the global coordinates of the beginning joint b for the member, and X_e and Y_e represent the global coordinates of the end joint e.

The member end displacements, like end forces, are vectors, which are defined in the same directions as the corresponding forces. Therefore, the transformation matrix (Eq. (3.61)), developed for transforming end forces, can also be used to transform member end displacements from the global to local coordinate system; that is,

$$\boxed{\mathbf{u} = } \tag{3.63}$$

Transformation from Local to Global Coordinate Systems

Next, let us consider the transformation of member end forces and end displacements from local to global coordinate systems. A comparison of Figs. 3.10(b) and (c) indicates that at end b of m, the global force F_1 must be equal to the algebraic sum of the components of the local forces ${}_1$ and ${}_2$ in the direction of the global X axis; that is,

$$F_1 = {}_1 \cos\theta - {}_2 \sin\theta \tag{3.64a}$$

In a similar manner, the global force F_2 equals the algebraic sum of the components of ${}_1$ and ${}_2$ in the direction of the global Y axis. Thus,

$$F_2 = {}_1 \sin\theta + {}_2 \cos\theta \tag{3.64b}$$

By using a similar reasoning at end e, we express the global forces in terms of the local forces as

$$F_3 = {}_3 \cos\theta - {}_4 \sin\theta \tag{3.64c}$$

$$F_4 = {}_3 \sin\theta + {}_4 \cos\theta \tag{3.64d}$$

We can write Eqs. 3.64(a) through (d) in matrix form as

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} {}_1 \\ {}_2 \\ {}_3 \\ {}_4 \end{bmatrix} \tag{3.65}$$

By comparing Eqs. (3.59) and (3.65), we observe that the transformation matrix in Eq. (3.65), which transforms the forces from the local to the global coordinate system, is the transpose of the transformation matrix in Eq. (3.59), which transforms the forces from the global to the local coordinate system. Therefore, Eq. (3.65) can be expressed as

$$\boxed{\ = \ ^T} \tag{3.66}$$

Furthermore, a comparison of Eqs. (3.60) and (3.66) indicates that the inverse of the transformation matrix must be equal to its transpose; that is,

$$^{-1} = \ ^T \tag{3.67}$$

which indicates that the transformation matrix is orthogonal.

As discussed previously, because the member end displacements are also vectors, which are defined in the same directions as the corresponding forces, the matrix ^{T} also defines the transformation of member end displacements from the local to the global coordinate system; that is,

$$= {}^{T}\mathbf{u} \tag{3.68}$$

EXAMPLE 3.4 Determine the transformation matrices for the members of the truss shown in Fig. 3.11.

SOLUTION **e ber 1** From Fig. 3.11, we can see that joint 1 is the beginning joint and joint 2 is the end joint for member 1. By applying Eqs. (3.62), we determine

$$\cos\theta = \frac{X_2 - X_1}{L} = \frac{6-0}{6} = 1$$

$$\sin\theta = \frac{Y_2 - Y_1}{L} = \frac{0-0}{6} = 0$$

The transformation matrix for member 1 can now be obtained by using Eq. (3.61)

$$_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$ **Ans**

As the preceding result indicates, for any member with the positive directions of its local x and y axes oriented in the positive directions of the global X and Y axes, respectively, the transformation matrix always equals a unit matrix, $\mathbf{I}$.

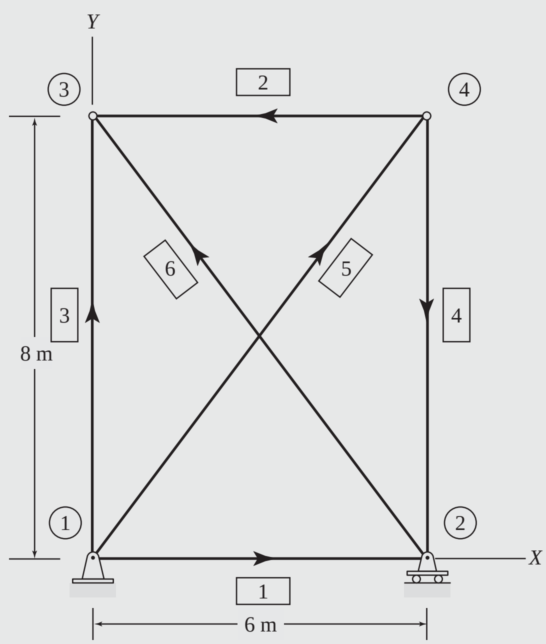


Fig. 3.11

e ber 2

$$\cos\theta = \frac{X_3 - X_4}{L} = \frac{0-6}{6} = -1$$

$$\sin\theta = \frac{Y_3 - Y_4}{L} = \frac{8-8}{6} = 0$$

Thus, from Eq. (3.61)

$${}_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$ **Ans**

e ber 3

$$\cos\theta = \frac{X_3 - X_1}{L} = \frac{0-0}{8} = 0$$

$$\sin\theta = \frac{Y_3 - Y_1}{L} = \frac{8-0}{8} = 1$$

Thus,

$${}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$ **Ans**

e ber 4

$$\cos\theta = \frac{X_2 - X_4}{L} = \frac{6-6}{8} = 0$$

$$\sin\theta = \frac{Y_2 - Y_4}{L} = \frac{0-8}{8} = -1$$

Thus,

$${}_4 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ **Ans**

e ber 5

$$L = \sqrt{(X_4 - X_1)^2 + (Y_4 - Y_1)^2} = \sqrt{(6-0)^2 + (8-0)^2} = 10 \text{ m}$$

$$\cos\theta = \frac{X_4 - X_1}{L} = \frac{6-0}{10} = 0.6$$

$$\sin\theta = \frac{Y_4 - Y_1}{L} = \frac{8-0}{10} = 0.8$$

$${}_5 = \begin{bmatrix} 0.6 & 0.8 & 0 & 0 \\ -0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0.8 \\ 0 & 0 & -0.8 & 0.6 \end{bmatrix}$$ **Ans**

e ber 6

$$L = \sqrt{(X_3 - X_2)^2 + (Y_3 - Y_2)^2} = \sqrt{(0-6)^2 + (8-0)^2} = 10 \text{ m}$$

$$\cos\theta = \frac{X_3 - X_2}{L} = \frac{0-6}{10} = -0.6$$

$$\sin\theta = \frac{Y_3 - Y_2}{L} = \frac{8-0}{10} = 0.8$$

$$_6 = \begin{bmatrix} -0.6 & 0.8 & 0 & 0 \\ -0.8 & -0.6 & 0 & 0 \\ 0 & 0 & -0.6 & 0.8 \\ 0 & 0 & -0.8 & -0.6 \end{bmatrix}$$ **Ans**

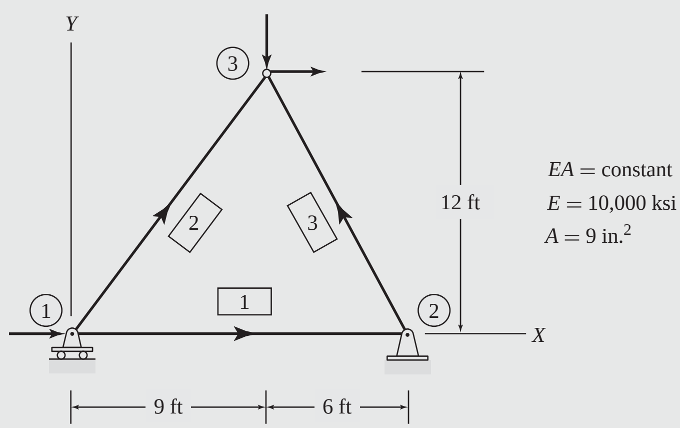
EXAMPLE 3.5 For the truss shown in Fig. 3.12(a), the end displacements of member 2 in the global coordinate system are (Fig. 3.12(b)):

$$_2 = \begin{bmatrix} 0.75 \\ 0 \\ 1.5 \\ -2 \end{bmatrix} \text{ in.}$$

Calculate the end forces for this member in the global coordinate system. Is the member in equilibrium under these forces

SOLUTION *Member Stiffness Matrix in the Local Coordinate System* $= 10{,}000$ ksi, $A = 9$ in.^{2}, $L = \sqrt{(9)^2 + (12)^2} = 15$ ft $= 180$ in.

$$\frac{A}{L} = \frac{10{,}000(9)}{180} = 500 \text{ k/in.}$$



(a) Truss

Fig. 3.12

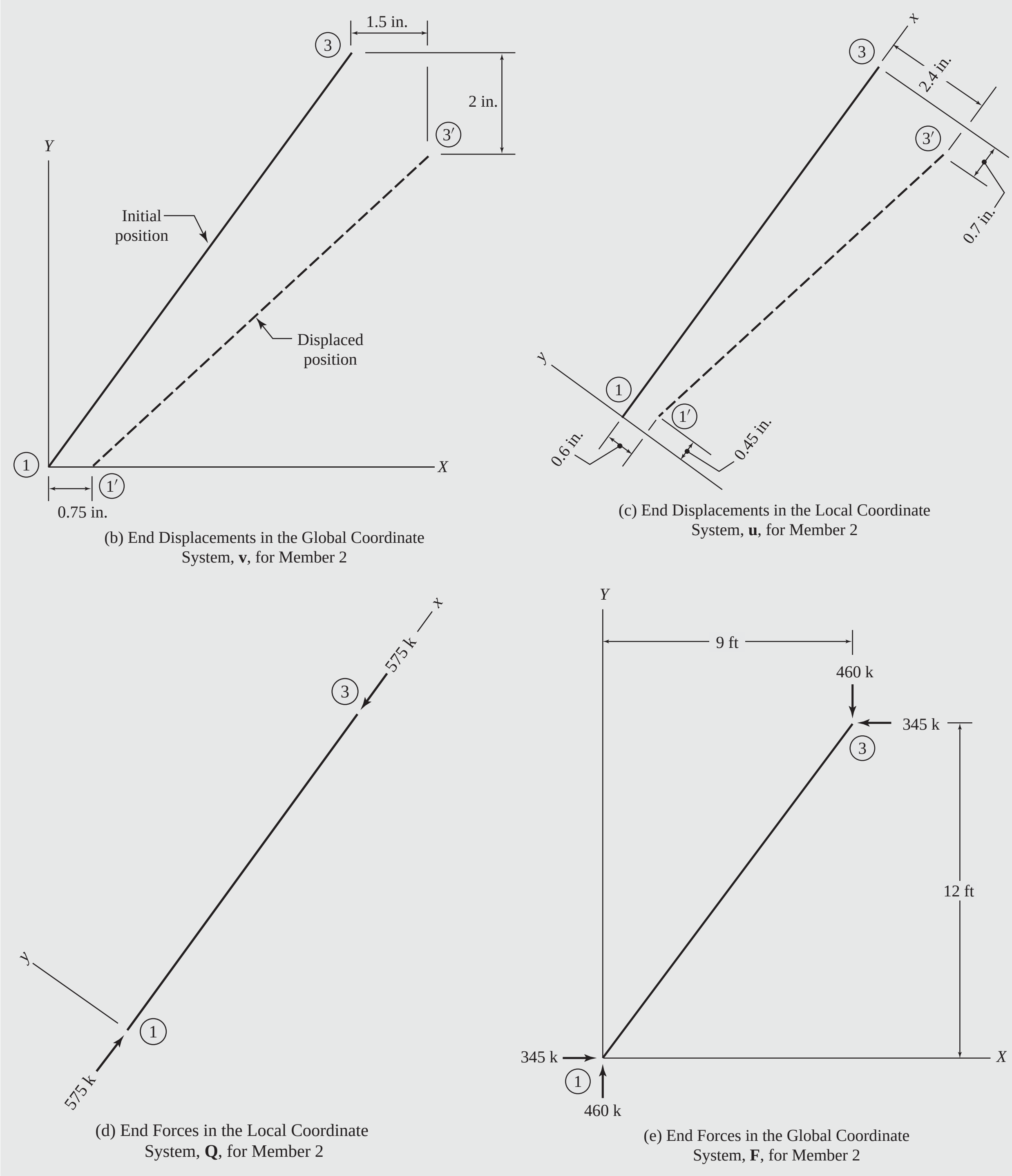


Fig. 3.12 (*continued*)

Thus, from Eq. (3.27),

$$_2 = 500 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ k/in.}$$

Transformation Matrix From Fig. 3.12(a), we can see that joint 1 is the beginning joint and joint 3 is the end joint for member 2. By applying Eqs. (3.62), we determine

$$\cos\theta = \frac{X_3 - X_1}{L} = \frac{9 - 0}{15} = 0.6$$

$$\sin\theta = \frac{Y_3 - Y_1}{L} = \frac{12 - 0}{15} = 0.8$$

The transformation matrix for member 2 can now be evaluated by using Eq. (3.61):

$$_2 = \begin{bmatrix} 0.6 & 0.8 & 0 & 0 \\ -0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0.8 \\ 0 & 0 & -0.8 & 0.6 \end{bmatrix}$$

Member nd Displacements in the Local Coordinate System To determine the member global end forces, first we calculate member end displacements in the local coordinate system by using the relationship $\mathbf{u} =$ (Eq. (3.63)). Thus,

$$\mathbf{u}_2 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.8 & 0 & 0 \\ -0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0.8 \\ 0 & 0 & -0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 0.75 \\ 0 \\ 1.5 \\ -2 \end{bmatrix} = \begin{bmatrix} 0.45 \\ -0.6 \\ -0.7 \\ -2.4 \end{bmatrix} \text{ in.}$$

These end displacements are depicted in Fig. 3.12(c).

Member nd Forces in the Local Coordinate System Next, by using the expression $= \mathbf{u}$ (Eq. (3.7)), we compute the member local end forces as

$$_2 = \begin{bmatrix} _1 \\ _2 \\ _3 \\ _4 \end{bmatrix} = 500 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.45 \\ -0.6 \\ -0.7 \\ -2.4 \end{bmatrix} = \begin{bmatrix} 575 \\ 0 \\ -575 \\ 0 \end{bmatrix} \text{ k}$$

Note that, as shown in Fig. 3.12(d), the member is in compression with an axial force of magnitude 575 k.

Member nd Forces in the Global Coordinate System Finally, we determine the desired member end forces by applying the relationship $= {}^T$ as given in Eq. (3.66). Thus,

$$_2 = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.8 & 0 & 0 \\ 0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & -0.8 \\ 0 & 0 & 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 575 \\ 0 \\ -575 \\ 0 \end{bmatrix} = \begin{bmatrix} 345 \\ 460 \\ -345 \\ -460 \end{bmatrix} \text{ k} \qquad \textbf{Ans}$$

The member end forces in the global coordinate system are shown in Fig. 3.12(e).

quilibrium Check To check whether or not the member is in equilibrium, we apply the three equations of equilibrium, as follows.

$$+\rightarrow \sum F_X = 0 \qquad 345 - 345 = 0$$ **Checks**

$$+\uparrow \sum F_Y = 0 \qquad 460 - 460 = 0$$ **Checks**

$$+ \circlearrowleft \sum M_{①} = 0 \qquad 345(12) - 460(9) = 0$$ **Checks**

Therefore, the member is in equilibrium. **Ans**

3.6 MEMBER STIFFNESS RELATIONS IN THE GLOBAL COORDINATE SYSTEM

By using the member stiffness relations in the local coordinate system from Sections 3.3 and 3.4, and the transformation relations from Section 3.5, we can now establish the stiffness relations for members in the global coordinate system.

First, we substitute the local stiffness relations $= \mathbf{u}$ (Eq. (3.7)) into the force transformation relations $= {}^{T}$ (Eq. (3.66)) to obtain

$$= {}^{T} = {}^{T}\,\mathbf{u} \tag{3.69}$$

Then, by substituting the displacement transformation relations $\mathbf{u} =$ (Eq. (3.63)) into Eq. (3.69), we determine that the desired relationship between the member end forces and end displacements , in the global coordinate system, is

$$= {}^{T} \tag{3.70}$$

Equation (3.70) can be conveniently expressed as

$$\boxed{\;=\;} \tag{3.71}$$

with

$$\boxed{\;= {}^{T}\;} \tag{3.72}$$

in which the matrix is called the *member stiffness matrix in the global coordinate system*. The explicit form of can be determined by substituting Eqs. (3.27) and (3.61) into Eq. (3.72), as

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \frac{A}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

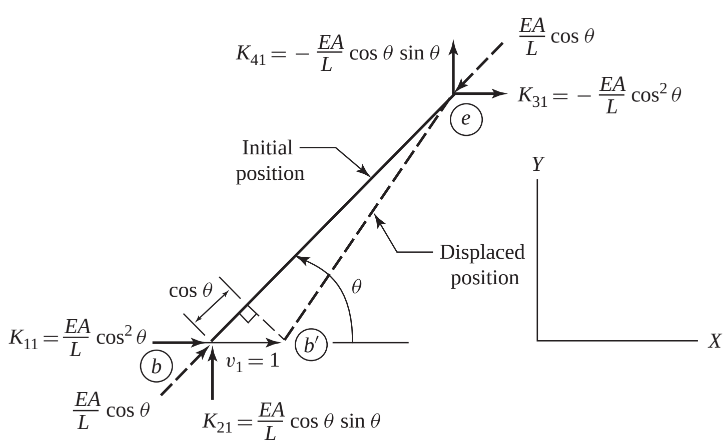
Performing the matrix multiplications, we obtain

$$= \frac{A}{L}\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\cos\theta\sin\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \tag{3.73}$$

Note that, like the member local stiffness matrix, *the member global stiffness matrix, , is symmetric*. The physical interpretation of the member global stiffness matrix is similar to that of the member local stiffness matrix; that is, *a stiffness coefficient $_{ij}$ represents the force at the location and in the direction of F_i required, along with other end forces, to cause a unit value of displacement v_j, while all other end displacements are zero*. Thus, the jth column of matrix consists of the end forces in the global coordinate system required to cause a unit value of the end displacement v_j, while all other end displacements are zero.

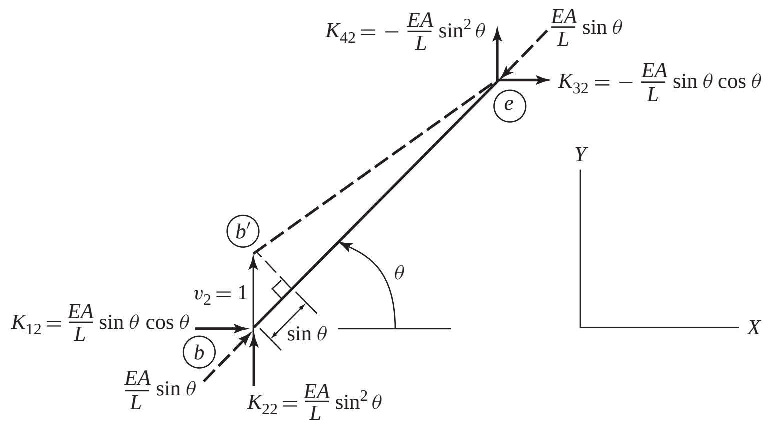
As the preceding interpretation indicates, the member global stiffness matrix can alternately be derived by subjecting an inclined truss member, separately, to unit values of each of the four end displacements in the global coordinate system as shown in Fig. 3.13, and by evaluating the end forces in the global coordinate system required to cause the individual unit displacements. Let us verify the expression for given in Eq. (3.73), using this alternative approach. Consider a prismatic plane truss member inclined at an angle θ relative to the global X axis, as shown in Fig. 3.13(a). When end b of the member is given a unit displacement $v_1 = 1$, while the other end displacements are held at zero, the member shortens and an axial compressive force develops in it. In the case of small displacements (as assumed herein), the axial deformation u_a of the member due to v_1 is equal to the component of $v_1 = 1$ in the undeformed direction of the member; that is (Fig. 3.13(a)),

$$u_a = v_1\cos\theta = 1\cos\theta = \cos\theta$$

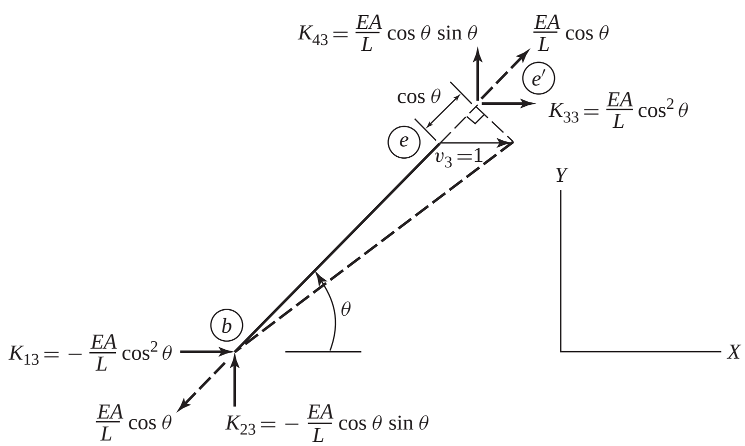


(a) First Column of **K** ($v_1 = 1$, $v_2 = v_3 = v_4 = 0$)

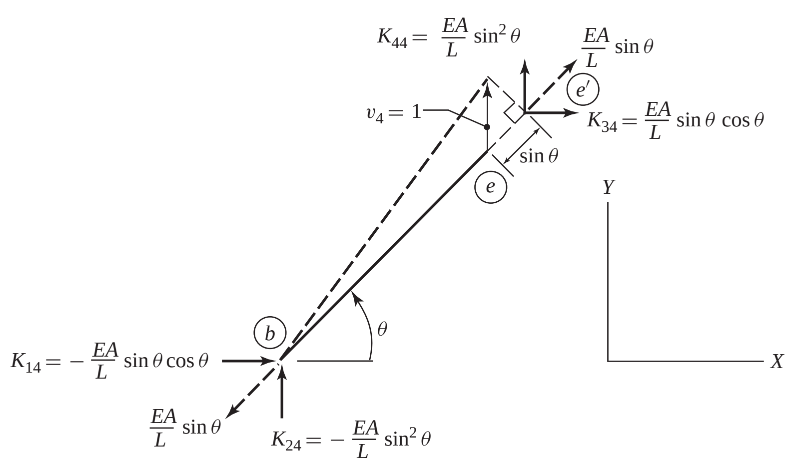
Fig. 3.13



(b) Second Column of **K** ($v_2 = 1,\ v_1 = v_3 = v_4 = 0$)



(c) Third Column of **K** ($v_3 = 1,\ v_1 = v_2 = v_4 = 0$)



(d) Fourth Column of **K** ($v_4 = 1,\ v_1 = v_2 = v_3 = 0$)

Fig. 3.13 (*continued*)

The axial compressive force $_{\mathrm{a}}$ in the member caused by the axial deformation u_{a} can be expressed as

$$_{\mathrm{a}} = \left(\frac{A}{L}\right) u_{\mathrm{a}} = \left(\frac{A}{L}\right) \cos\theta$$

From Fig. 3.13(a), we can see that the stiffness coefficients must be equal to the components of the member axial force $_{\mathrm{a}}$ in the directions of the global X and Y axes. Thus, at end b,

$$_{11} = \ _{\mathrm{a}} \cos\theta = \left(\frac{A}{L}\right) \cos^2\theta \tag{3.74a}$$

$$_{21} = \ _{\mathrm{a}} \sin\theta = \left(\frac{A}{L}\right) \cos\theta \sin\theta \tag{3.74b}$$

Similarly, at end e,

$$_{31} = - \ _{\mathrm{a}} \cos\theta = -\left(\frac{A}{L}\right) \cos^2\theta \tag{3.74c}$$

$$_{41} = - \ _{\mathrm{a}} \sin\theta = -\left(\frac{A}{L}\right) \cos\theta \sin\theta \tag{3.74d}$$

in which the negative signs for $_{31}$ and $_{41}$ indicate that these forces act in the negative directions of the X and Y axes, respectively. Note that the member must be in equilibrium under the action of the four end forces, $_{11}$, $_{21}$, $_{31}$, and $_{41}$. Also, note that the expressions for these stiffness coefficients (Eqs. (3.74)) are identical to those given in the first column of the matrix in Eq. (3.73).

The stiffness coefficients corresponding to the unit values of the remaining end displacements v_2, v_3, and v_4 can be evaluated in a similar manner, and are given in Figs. 3.13 (b) through (d), respectively. As expected, these stiffness coefficients are the same as those previously obtained by transforming the stiffness relations from the local to the global coordinate system (Eq. (3.73)).

EXAMPLE 3.6 Solve Example 3.5 by using the member stiffness relationship in the global coordinate system, $=$.

SOLUTION *Member Stiffness Matrix in the Global Coordinate System* It was shown in Example 3.5 that for member 2,

$$\frac{A}{L} = 500 \text{ k/in.}, \cos\theta = 0.6, \sin\theta = 0.8$$

Thus, from Eq. (3.73):

$$_2 = \begin{bmatrix} 180 & 240 & -180 & -240 \\ 240 & 320 & -240 & -320 \\ -180 & -240 & 180 & 240 \\ -240 & -320 & 240 & 320 \end{bmatrix} \text{ k/in.}$$

Member nd Forces in the Global Coordinate System By applying the relationship $=$ as given in Eq. (3.71), we obtain

$$
= \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} 180 & 240 & -180 & -240 \\ 240 & 320 & -240 & -320 \\ -180 & -240 & 180 & 240 \\ -240 & -320 & 240 & 320 \end{bmatrix} \begin{bmatrix} 0.75 \\ 0 \\ 1.5 \\ -2 \end{bmatrix} = \begin{bmatrix} 345 \\ 460 \\ -345 \\ -460 \end{bmatrix} \text{k}
$$

Ans

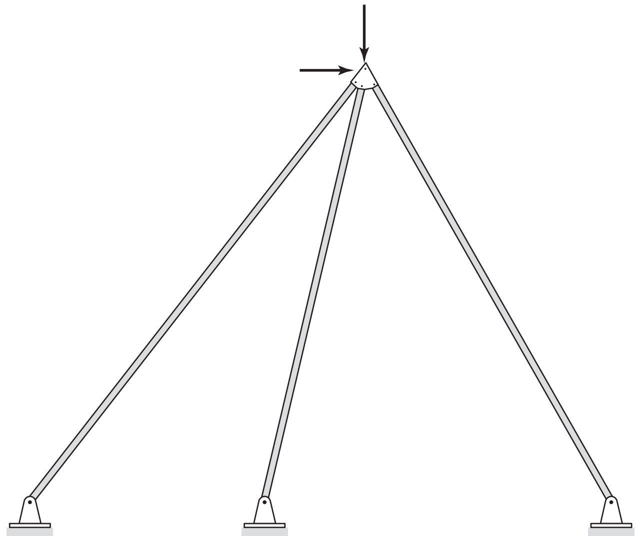
quilibrium check See Example 3.5.

3.7 STRUCTURE STIFFNESS RELATIONS

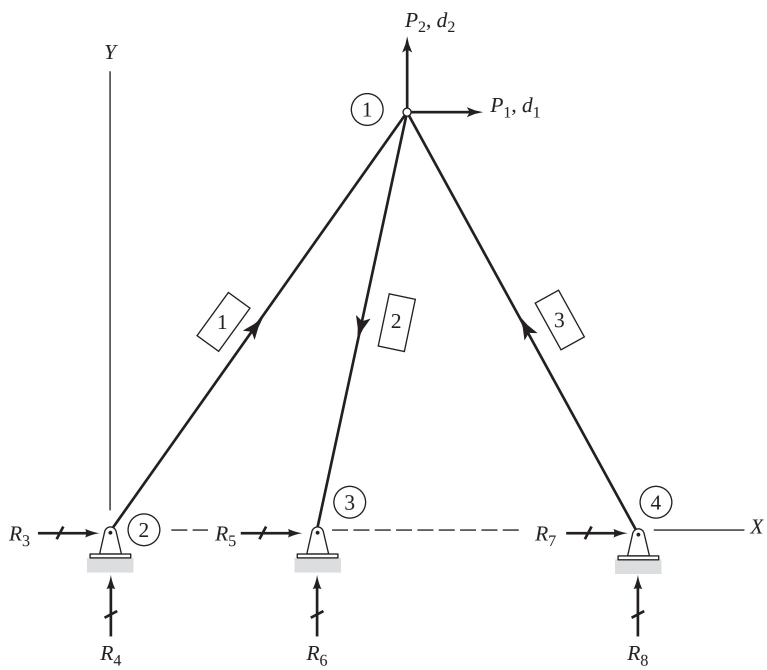
Having determined the member force–displacement relationships in the global coordinate system, we are now ready to establish the stiffness relations for the entire structure. The structure stiffness relations express the external loads **P** acting at the joints of the structure, as functions of the joint displacements **d**. Such relationships can be established as follows:

1. The joint loads **P** are first expressed in terms of the member end forces in the global coordinate system, , by applying the equations of equilibrium for the joints of the structure.
2. The joint displacements **d** are then related to the member end displacements in the global coordinate system, , by using the compatibility conditions that the displacements of the member ends must be the same as the corresponding joint displacements.
3. Next, the compatibility equations are substituted into the member force–displacement relations, $=$, to express the member global end forces in terms of the joint displacements **d**. The –**d** relations thus obtained are then substituted into the joint equilibrium equations to establish the desired structure stiffness relationships between the joint loads **P** and the joint displacements **d**.

Consider, for example, an arbitrary plane truss as shown in Fig. 3.14(a). The analytical model of the truss is given in Fig. 3.14(b), which indicates that the structure has two degrees of freedom, d_1 and d_2. The joint loads corresponding to these degrees of freedom are designated P_1 and P_2, respectively. The global end forces and end displacements for the three members of the truss are shown in Fig. 3.14(c), in which the superscript (i) denotes the member number. Note that for members 1 and 3, the bottom joints (i.e., joints 2 and 4, respectively) have been defined as the beginning joints; whereas, for member 2, the top joint 1 is the beginning joint. As stated previously, our objective is to express the joint loads **P** as functions of the joint displacement **d**.

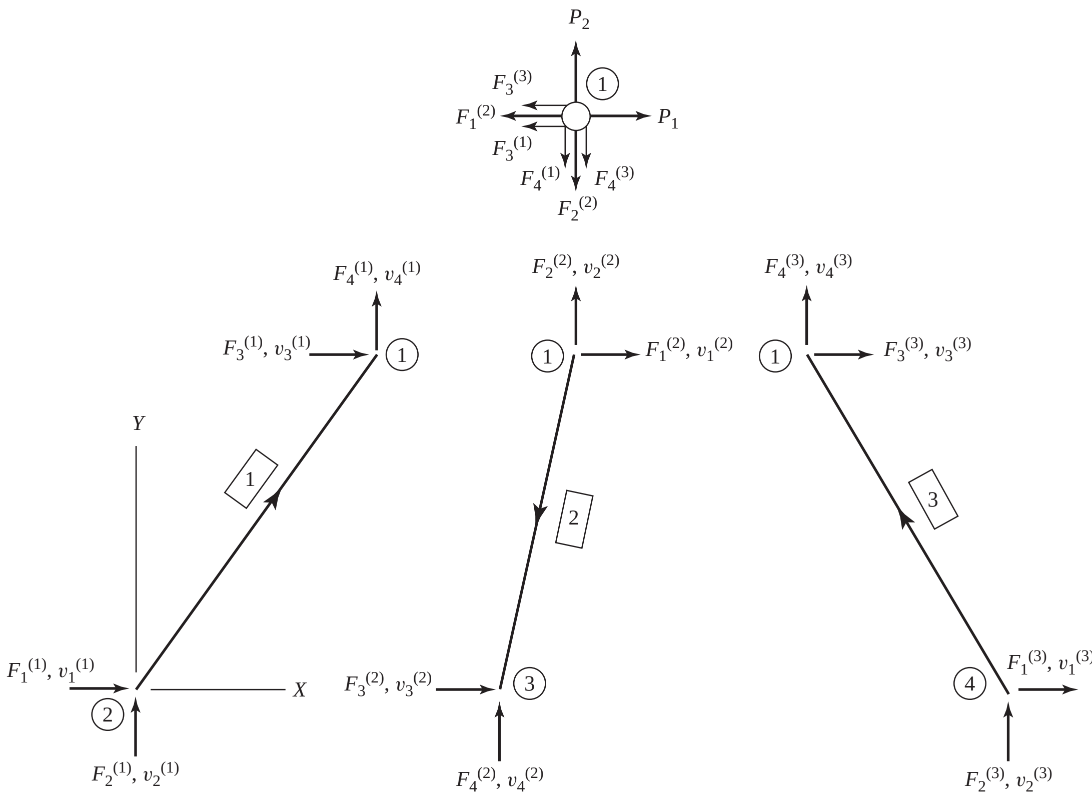


(a) Truss



(b) Analytical Model

Fig. 3.14



(c)

Fig. 3.14 (*continued*)

Equilibrium Equations

To relate the external joint loads **P** to the internal member end forces , we apply the two equations of equilibrium, $\sum F_X = 0$ and $\sum F_Y = 0$, to the free body of joint 1 shown in Fig. 3.14(c). This yields the equilibrium equations,

$$P_1 = F_3^{(1)} + F_1^{(2)} + F_3^{(3)} \tag{3.75a}$$

$$P_2 = F_4^{(1)} + F_2^{(2)} + F_4^{(3)} \tag{3.75b}$$

Compatibility Equations

By comparing Figs. 3.14(b) and (c), we observe that since the lower end 2 of member 1 is connected to the hinged support 2, which cannot translate in any direction, the two displacements of end 2 of the member must be zero. Similarly, since end 1 of this member is connected to joint 1, the displacements of end 1 must be the same as the displacements of joint 1. Thus, the compatibility conditions for member 1 are

$$v_1^{(1)} = v_2^{(1)} = 0 \qquad v_3^{(1)} = d_1 \qquad v_4^{(1)} = d_2 \tag{3.76}$$

In a similar manner, the compatibility conditions for members 2 and 3, respectively, are found to be

$$v_1^{(2)} = d_1 \qquad v_2^{(2)} = d_2 \qquad v_3^{(2)} = v_4^{(2)} = 0 \tag{3.77}$$

$$v_1^{(3)} = v_2^{(3)} = 0 \qquad v_3^{(3)} = d_1 \qquad v_4^{(3)} = d_2 \tag{3.78}$$

Member Stiffness Relations

Of the two types of relationships established thus far, the equilibrium equations (Eqs. (3.75)) express joint loads in terms of member end forces, whereas the compatibility equations (Eqs. (3.76) through (3.78)) relate joint displacements to member end displacements. Now, we will link the two types of relationships by employing the member stiffness relationship in the global coordinate system derived in the preceding section.

We can write the member global stiffness relation $=$ (Eq. (3.71)) in expanded form for member 1 as

$$\begin{bmatrix} F_1^{(1)} \\ F_2^{(1)} \\ F_3^{(1)} \\ F_4^{(1)} \end{bmatrix} = \begin{bmatrix} {}^{(1)}_{11} & {}^{(1)}_{12} & {}^{(1)}_{13} & {}^{(1)}_{14} \\ {}^{(1)}_{21} & {}^{(1)}_{22} & {}^{(1)}_{23} & {}^{(1)}_{24} \\ {}^{(1)}_{31} & {}^{(1)}_{32} & {}^{(1)}_{33} & {}^{(1)}_{34} \\ {}^{(1)}_{41} & {}^{(1)}_{42} & {}^{(1)}_{43} & {}^{(1)}_{44} \end{bmatrix} \begin{bmatrix} v_1^{(1)} \\ v_2^{(1)} \\ v_3^{(1)} \\ v_4^{(1)} \end{bmatrix} \tag{3.79}$$

from which we obtain the expressions for forces at end 1 of the member:

$$F_3^{(1)} = {}^{(1)}_{31} v_1^{(1)} + {}^{(1)}_{32} v_2^{(1)} + {}^{(1)}_{33} v_3^{(1)} + {}^{(1)}_{34} v_4^{(1)} \tag{3.80a}$$

$$F_4^{(1)} = {}^{(1)}_{41} v_1^{(1)} + {}^{(1)}_{42} v_2^{(1)} + {}^{(1)}_{43} v_3^{(1)} + {}^{(1)}_{44} v_4^{(1)} \tag{3.80b}$$

In a similar manner, we write the stiffness relations for member 2 as

$$\begin{bmatrix} F_1^{(2)} \\ F_2^{(2)} \\ F_3^{(2)} \\ F_4^{(2)} \end{bmatrix} = \begin{bmatrix} {}^{(2)}_{11} & {}^{(2)}_{12} & {}^{(2)}_{13} & {}^{(2)}_{14} \\ {}^{(2)}_{21} & {}^{(2)}_{22} & {}^{(2)}_{23} & {}^{(2)}_{24} \\ {}^{(2)}_{31} & {}^{(2)}_{32} & {}^{(2)}_{33} & {}^{(2)}_{34} \\ {}^{(2)}_{41} & {}^{(2)}_{42} & {}^{(2)}_{43} & {}^{(2)}_{44} \end{bmatrix} \begin{bmatrix} v_1^{(2)} \\ v_2^{(2)} \\ v_3^{(2)} \\ v_4^{(2)} \end{bmatrix} \tag{3.81}$$

from which we obtain the forces at end 1 of the member:

$$F_1^{(2)} = {}^{(2)}_{11} v_1^{(2)} + {}^{(2)}_{12} v_2^{(2)} + {}^{(2)}_{13} v_3^{(2)} + {}^{(2)}_{14} v_4^{(2)} \tag{3.82a}$$

$$F_2^{(2)} = {}^{(2)}_{21} v_1^{(2)} + {}^{(2)}_{22} v_2^{(2)} + {}^{(2)}_{23} v_3^{(2)} + {}^{(2)}_{24} v_4^{(2)} \tag{3.82b}$$

Similarly, for member 3, the stiffness relations are written as

$$\begin{bmatrix} F_1^{(3)} \\ F_2^{(3)} \\ F_3^{(3)} \\ F_4^{(3)} \end{bmatrix} = \begin{bmatrix} {}^{(3)}_{11} & {}^{(3)}_{12} & {}^{(3)}_{13} & {}^{(3)}_{14} \\ {}^{(3)}_{21} & {}^{(3)}_{22} & {}^{(3)}_{23} & {}^{(3)}_{24} \\ {}^{(3)}_{31} & {}^{(3)}_{32} & {}^{(3)}_{33} & {}^{(3)}_{34} \\ {}^{(3)}_{41} & {}^{(3)}_{42} & {}^{(3)}_{43} & {}^{(3)}_{44} \end{bmatrix} \begin{bmatrix} v_1^{(3)} \\ v_2^{(3)} \\ v_3^{(3)} \\ v_4^{(3)} \end{bmatrix} \tag{3.83}$$

and the forces at end 1 of the member are given by

$$F_3^{(3)} = {}^{(3)}_{31} v_1^{(3)} + {}^{(3)}_{32} v_2^{(3)} + {}^{(3)}_{33} v_3^{(3)} + {}^{(3)}_{34} v_4^{(3)} \tag{3.84a}$$

$$F_4^{(3)} = {}^{(3)}_{41} v_1^{(3)} + {}^{(3)}_{42} v_2^{(3)} + {}^{(3)}_{43} v_3^{(3)} + {}^{(3)}_{44} v_4^{(3)} \tag{3.84b}$$

Note that Eqs. (3.80), (3.82), and (3.84) express the six member end forces that appear in the joint equilibrium equations (Eqs. (3.75)), in terms of member end displacements.

To relate the joint displacements **d** to the member end forces , we substitute the compatibility equations into the foregoing member force–displacement relations. Thus, by substituting the compatibility equations for member 1 (Eqs. (3.76)) into its force–displacement relations as given by Eqs. (3.80), we express the member end forces ${}^{(1)}$ in terms of the joint displacements **d** as

$$F_3^{(1)} = {}^{(1)}_{33} d_1 + {}^{(1)}_{34} d_2 \tag{3.85a}$$

$$F_4^{(1)} = {}^{(1)}_{43} d_1 + {}^{(1)}_{44} d_2 \tag{3.85b}$$

In a similar manner, for member 2, by substituting Eqs. (3.77) into Eqs. (3.82), we obtain

$$F_1^{(2)} = {}^{(2)}_{11} d_1 + {}^{(2)}_{12} d_2 \tag{3.86a}$$

$$F_2^{(2)} = {}^{(2)}_{21} d_1 + {}^{(2)}_{22} d_2 \tag{3.86b}$$

Similarly, for member 3, substitution of Eqs. (3.78) into Eqs. (3.84) yields

$$F_3^{(3)} = {}^{(3)}_{33} d_1 + {}^{(3)}_{34} d_2 \tag{3.87a}$$

$$F_4^{(3)} = {}^{(3)}_{43} d_1 + {}^{(3)}_{44} d_2 \tag{3.87b}$$

Structure Stiffness Relations

Finally, by substituting Eqs. (3.85) through (3.87) into the joint equilibrium equations (Eqs. (3.75)), we establish the desired relationships between the joint loads **P** and the joint displacements **d** of the truss:

$$P_1 = \left({}^{(1)}_{33} + {}^{(2)}_{11} + {}^{(3)}_{33} \right) d_1 + \left({}^{(1)}_{34} + {}^{(2)}_{12} + {}^{(3)}_{34} \right) d_2 \tag{3.88a}$$

$$P_2 = \left({}^{(1)}_{43} + {}^{(2)}_{21} + {}^{(3)}_{43} \right) d_1 + \left({}^{(1)}_{44} + {}^{(2)}_{22} + {}^{(3)}_{44} \right) d_2 \tag{3.88b}$$

Equations (3.88) can be conveniently expressed in condensed matrix form as

$$\boxed{\mathbf{P} = \ \mathbf{d}} \tag{3.89}$$

in which

$$= \begin{bmatrix} {}^{(1)}_{33} + {}^{(2)}_{11} + {}^{(3)}_{33} & {}^{(1)}_{34} + {}^{(2)}_{12} + {}^{(3)}_{34} \\ {}^{(1)}_{43} + {}^{(2)}_{21} + {}^{(3)}_{43} & {}^{(1)}_{44} + {}^{(2)}_{22} + {}^{(3)}_{44} \end{bmatrix} \tag{3.90}$$

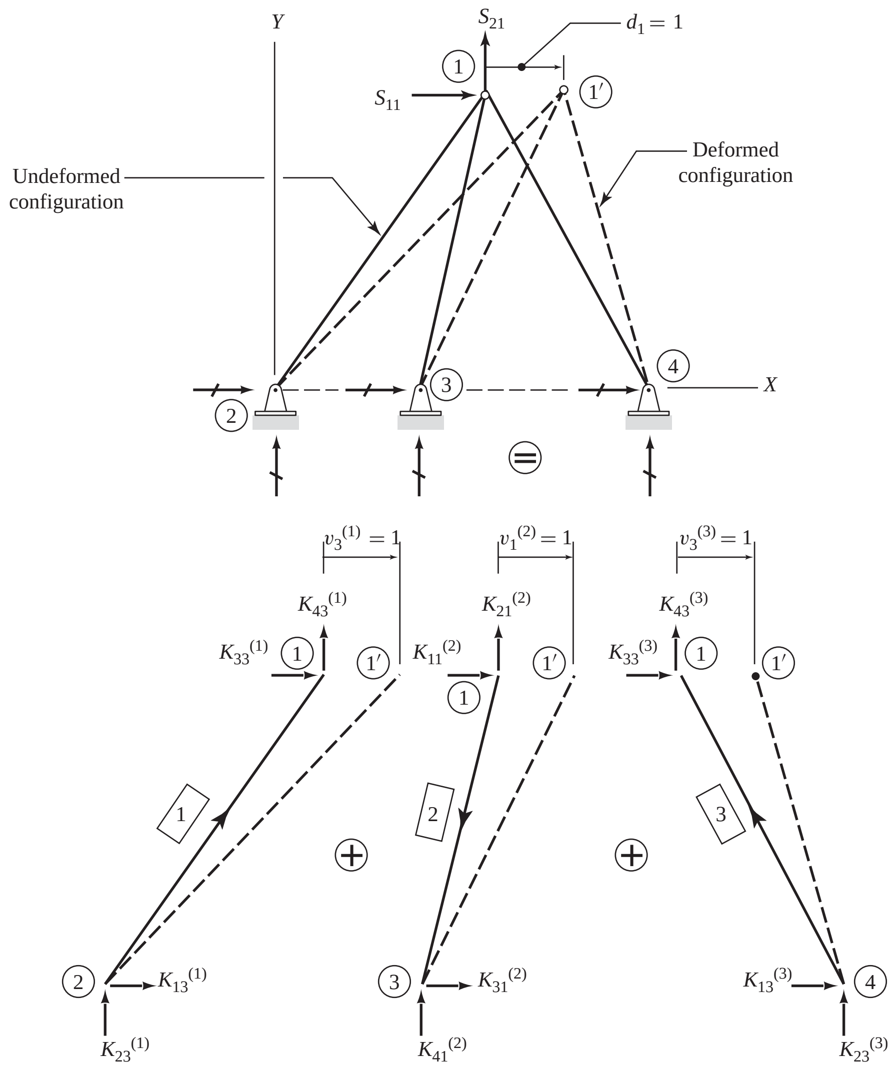
The matrix , which is a square matrix with the number of rows and columns equal to the degrees of freedom (*NDOF*), is called the *structure stiffness matrix*. The preceding method of determining the structure stiffness relationships by combining the member stiffness relations is commonly referred to as the *direct stiffness method* 48 .

Like member stiffness matrices, *structure stiffness matrices of linear elastic structures are always symmetric*. Note that in Eq. (3.90) the two off-diagonal elements of are equal to each other, because ${}^{(1)}_{34} = {}^{(1)}_{43}$, ${}^{(2)}_{12} = {}^{(2)}_{21}$, and ${}^{(3)}_{34} = {}^{(3)}_{43}$; thereby making a symmetric matrix.

Physical Interpretation of Structure Stiffness Matrix

The structure stiffness matrix can be interpreted in a manner analogous to the member stiffness matrix. *A structure stiffness coefficient S_{ij} represents the force at the location and in the direction of P_i required, along with other joint forces, to cause a unit value of the displacement d_j, while all other joint displacements are zero.* Thus, the *j*th column of the structure stiffness matrix consists of the joint loads required, at the locations and in the directions of all the degrees of freedom of the structure, to cause a unit value of the displacement d_j while all other displacements are zero. This interpretation of the structure stiffness matrix indicates that such a matrix can, alternatively, be determined by subjecting the structure, separately, to unit values of each of its joint displacements, and by evaluating the joint loads required to cause the individual displacements.

To illustrate this approach, consider again the three-member truss of Fig. 3.14. To determine its structure stiffness matrix , we subject the truss to the joint displacements $d_1 = 1$ (with $d_2 = 0$), and $d_2 = 1$ (with $d_1 = 0$), as shown in Figs. 3.15(a) and (b), respectively. As depicted in Fig. 3.15(a), the stiffness coefficients S_{11} and S_{21} (elements of the first column of) represent the horizontal and vertical forces at joint 1 required to cause a unit displacement of the joint in the horizontal direction ($d_1 = 1$), while holding it in place vertically ($d_2 = 0$). The unit horizontal displacement of joint 1 induces unit displacements, in the same direction, at the top ends of the three members connected to the joint. The member stiffness coefficients (or end forces) necessary to cause these unit end displacements of the individual members are shown in Fig. 3.15(a). Note that these stiffness coefficients are labeled in accordance with the notation for member end forces adopted in Section 3.5. (Also, recall



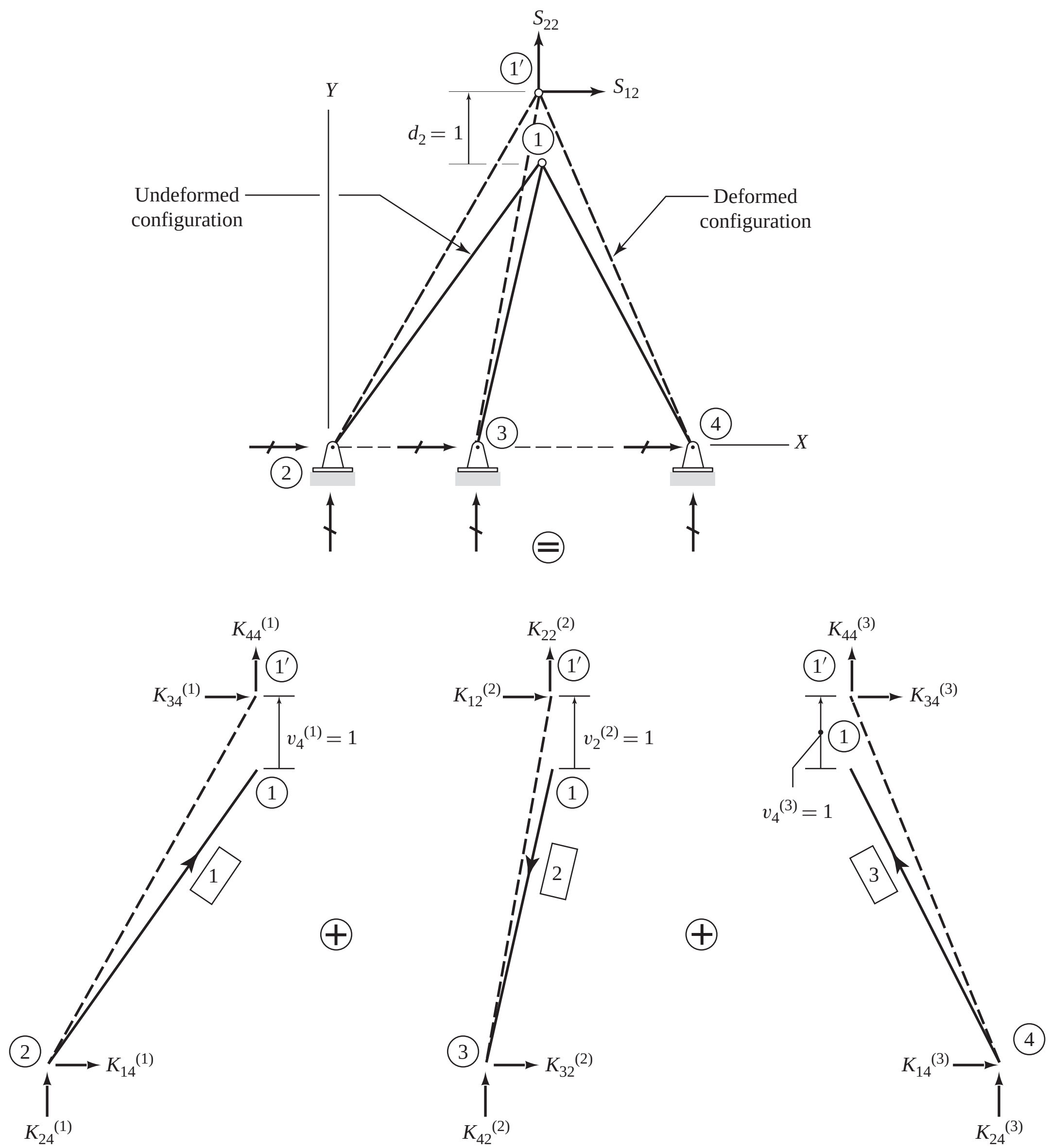
(a) First Column of **S** ($d_1 = 1$, $d_2 = 0$)

Fig. 3.15

that the explicit expressions for member stiffness coefficients, in terms of , A, L, and θ of a member, were derived in Section 3.6.)

From Fig. 3.15(a), we realize that the total horizontal force S_{11} at joint 1, required to cause the joint displacement $d_1 = 1$ (with $d_2 = 0$), must be equal to the algebraic sum of the horizontal forces at the top ends of the three members connected to the joint; that is,

$$S_{11} = {}_{33}^{(1)} + {}_{11}^{(2)} + {}_{33}^{(3)} \tag{3.91a}$$



(b) Second Column of **S** ($d_2 = 1$, $d_1 = 0$)

Fig. 3.15 (*continued*)

Similarly, the total vertical force S_{21} at joint 1 must be equal to the algebraic sum of the vertical forces at the top ends of all the members connected to the joint. Thus (Fig. 3.15a),

$$S_{21} = \,^{(1)}_{43} + \,^{(2)}_{21} + \,^{(3)}_{43} \tag{3.91b}$$

Note that the expressions for S_{11} and S_{21}, as given in Eqs. 3.91(a) and (b), are identical to those listed in the first column of the matrix in Eq. (3.90).

The stiffness coefficients in the second column of the matrix can be determined in a similar manner. As depicted in Fig. 3.15(b), the structure stiffness coefficients S_{12} and S_{22} represent the horizontal and vertical forces at joint 1 required to cause a unit displacement of the joint in the vertical direction (d_2), while holding it in place horizontally ($d_1 = 0$). The joint displacement $d_2 = 1$ induces unit vertical displacements at the top ends of the three members; these, in turn, cause the forces (member stiffness coefficients) to develop at the ends of the members. From Fig. 3.15(b), we can see that the stiffness coefficient S_{12} of joint 1, in the horizontal direction, must be equal to the algebraic sum of the member stiffness coefficients, in the same direction, at the top ends of all the members connected to the joint; that is,

$$S_{12} = {}^{(1)}_{34} + {}^{(2)}_{12} + {}^{(3)}_{34} \qquad \textbf{(3.91c)}$$

Similarly, the structure stiffness coefficient S_{22}, in the vertical direction, equals the algebraic sum of the vertical member stiffness coefficients at the top ends of the three members connected to joint 1. Thus (Fig. 3.15b),

$$S_{22} = {}^{(1)}_{44} + {}^{(2)}_{22} + {}^{(3)}_{44} \qquad \textbf{(3.91d)}$$

Again, the expressions for S_{12} and S_{22}, as given in Eqs. 3.91(c) and (d), are the same as those listed in the second column of the matrix in Eq. (3.90).

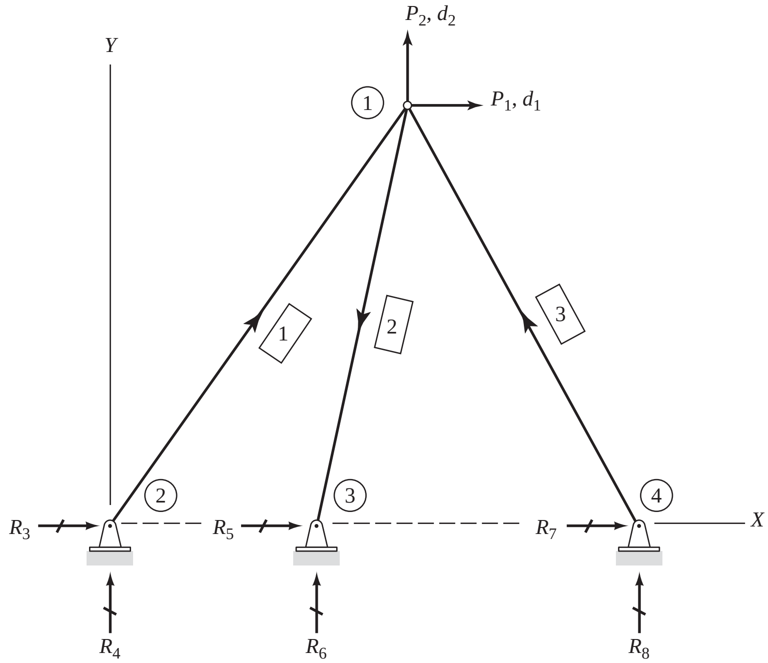
Assembly of the Structure Stiffness Matrix Using Member Code Numbers

In the preceding paragraphs of this section, we have studied two procedures for determining the structure stiffness matrix . Although a study of the foregoing procedures is essential for developing an understanding of the concept of the stiffness of multiple-degrees-of-freedom structures, these procedures cannot be implemented easily on computers and, therefore, are seldom used in practice.

From Eqs. (3.91), we observe that the structure stiffness coefficient of a joint in a direction equals the algebraic sum of the member stiffness coefficients, in that direction, at all the member ends connected to the joint. This fact indicates that the structure stiffness matrix can be formulated directly by adding the elements of the member stiffness matrices into their proper positions in the structure matrix. This technique of directly forming a structure stiffness matrix by assembling the elements of the member global stiffness matrices can be programmed conveniently on computers. The technique was introduced by S. S. Tezcan in 1963 44 , and is sometimes referred to as the *code number technique*.

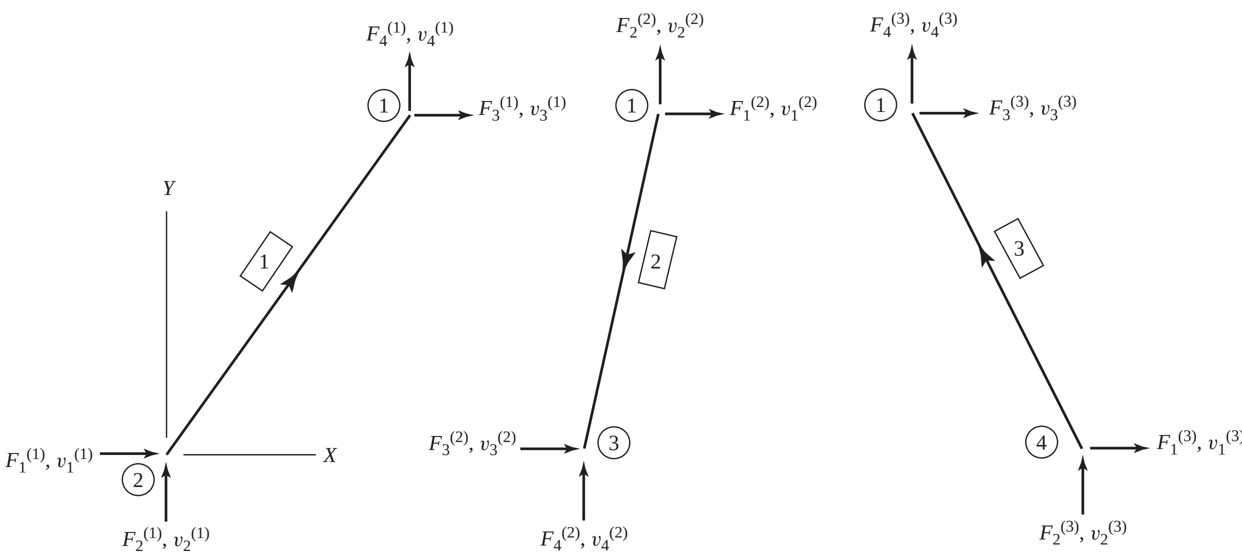
To illustrate this technique, consider again the three-member truss of Fig. 3.14. The analytical model of the truss is redrawn in Fig. 3.16(a), which shows that the structure has two degrees of freedom (numbered 1 and 2), and six restrained coordinates (numbered from 3 to 8). The stiffness matrices in the global coordinate system for members 1, 2, and 3 of the truss are designated $_1$, $_2$, and $_3$, respectively (Fig. 3.16(c)). Our objective is to form the structure stiffness matrix by assembling the elements of $_1$, $_2$, and $_3$.

To determine the positions of the elements of a member matrix in the structure matrix , we identify the number of the structure's degree of freedom or restrained coordinate, at the location and in the direction of each of the



Two degrees of freedom (1 and 2);
six restrained coordinates (3 through 8)

(a) Analytical Model



(b) Member End Forces and End Displacements in the Global Coordinate System

Fig. 3.16

$$\mathbf{K}_1 = \begin{array}{c} \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \left[\begin{array}{cccc} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} & K_{34}^{(1)} \\ K_{41}^{(1)} & K_{42}^{(1)} & K_{43}^{(1)} & K_{44}^{(1)} \end{array}\right] \end{array} \begin{array}{c} \\ 3 \\ 4 \\ 1 \\ 2 \end{array} \qquad \mathbf{K}_2 = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \left[\begin{array}{cccc} K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} & K_{14}^{(2)} \\ K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} & K_{24}^{(2)} \\ K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} & K_{34}^{(2)} \\ K_{41}^{(2)} & K_{42}^{(2)} & K_{43}^{(2)} & K_{44}^{(2)} \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \end{array} \qquad \mathbf{K}_3 = \begin{array}{c} \begin{array}{cccc} 7 & 8 & 1 & 2 \end{array} \\ \left[\begin{array}{cccc} K_{11}^{(3)} & K_{12}^{(3)} & K_{13}^{(3)} & K_{14}^{(3)} \\ K_{21}^{(3)} & K_{22}^{(3)} & K_{23}^{(3)} & K_{24}^{(3)} \\ K_{31}^{(3)} & K_{32}^{(3)} & K_{33}^{(3)} & K_{34}^{(3)} \\ K_{41}^{(3)} & K_{42}^{(3)} & K_{43}^{(3)} & K_{44}^{(3)} \end{array}\right] \end{array} \begin{array}{c} \\ 7 \\ 8 \\ 1 \\ 2 \end{array}$$

$$\mathbf{S} = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left[\begin{array}{cc} K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)} & K_{34}^{(1)} + K_{12}^{(2)} + K_{34}^{(3)} \\ K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)} & K_{44}^{(1)} + K_{22}^{(2)} + K_{44}^{(3)} \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array}$$

(c) Assembling of Structure Stiffness Matrix **S**

$$\mathbf{F}_1 = \left[\begin{array}{c} F_1^{(1)} \\ F_2^{(1)} \\ F_3^{(1)} \\ F_4^{(1)} \end{array}\right] \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array}$$

$$\mathbf{R} = \left[\begin{array}{c} R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \end{array}\right] = \left[\begin{array}{c} F_1^{(1)} \\ F_2^{(1)} \\ F_3^{(2)} \\ F_4^{(2)} \\ F_1^{(3)} \\ F_2^{(3)} \end{array}\right] \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \qquad \mathbf{F}_2 = \left[\begin{array}{c} F_1^{(2)} \\ F_2^{(2)} \\ F_3^{(2)} \\ F_4^{(2)} \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array}$$

$$\mathbf{F}_3 = \left[\begin{array}{c} F_1^{(3)} \\ F_2^{(3)} \\ F_3^{(3)} \\ F_4^{(3)} \end{array}\right] \begin{array}{c} 7 \\ 8 \\ 1 \\ 2 \end{array}$$

(d) Assembly of Support Reaction Vector **R**

Fig. 3.16 (*continued*)

member's global end displacements, . Such structure degrees of freedom and restrained coordinate numbers for a member, when arranged in the same order as the member's end displacements, are referred to as the member's *code numbers*. In accordance with the notation for member end displacements adopted in Section 3.5, the first two end displacements, v_1 and v_2, are always specified in the X and Y directions, respectively, at the beginning of the member; and the last two end displacements, v_3 and v_4, are always in the X and Y directions, respectively, at the end of the member. Therefore, the first two code numbers for a member are always the numbers of the structure degrees of freedom and/or restrained coordinates in the X and Y directions, respectively, at the beginning joint for the member; and the third and fourth member code numbers are always the numbers of the structure degrees of freedom and/or restrained coordinates in the X and Y directions, respectively, at the end joint for the member.

From Fig. 3.16(a), we can see that for member 1 of the truss, the beginning and the end joints are 2 and 1, respectively. At the beginning joint 2, the restrained coordinate numbers are 3 and 4 in the X and Y directions, respectively; whereas, at the end joint 1, the structure degree of freedom numbers, in the X and Y directions, are 1 and 2, respectively. Thus, the code numbers for member 1 are 3, 4, 1, 2. Similarly, since the beginning and end joints for member 2 are 1 and 3, respectively, the code numbers for this member are 1, 2, 5, 6. In a similar manner, the code numbers for member 3 are found to be 7, 8, 1, 2. The code numbers for the three members of the truss can be verified by comparing the member global end displacements shown in Fig. 3.16(b) with the structure degrees of freedom and restrained coordinates given in Fig. 3.16(a).

The code numbers for a member define the compatibility equations for the member. For example, the code numbers 3, 4, 1, 2 imply the following compatibility equations for member 1:

$$v_1^{(1)} = d_3 \qquad v_2^{(1)} = d_4 \qquad v_3^{(1)} = d_1 \qquad v_4^{(1)} = d_2$$

Since the displacements corresponding to the restrained coordinates 3 and 4 are zero (i.e., $d_3 = d_4 = 0$), the compatibility equations for member 1 become

$$v_1^{(1)} = v_2^{(1)} = 0 \qquad v_3^{(1)} = d_1 \qquad v_4^{(1)} = d_2$$

which are identical to those given in Eqs. (3.76).

The member code numbers can also be used to formulate the joint equilibrium equations for a structure (such as those given in Eqs. (3.75)). The equilibrium equation corresponding to an ith degree of freedom (or restrained coordinate) can be obtained by equating the joint load P_i (or the reaction R_i) to the algebraic sum of the member end forces, with the code number i, of all the members of the structure. For example, to obtain the equilibrium equations for the truss of Fig. 3.16(a), we write the code numbers for its three members by the side of their respective end force vectors, as

$$_1 = \begin{bmatrix} F_1^{(1)} \\ F_2^{(1)} \\ F_3^{(1)} \\ F_4^{(1)} \end{bmatrix}\begin{matrix} 3 \\ 4 \\ 1 \\ 2 \end{matrix} \qquad _2 = \begin{bmatrix} F_1^{(2)} \\ F_2^{(2)} \\ F_3^{(2)} \\ F_4^{(2)} \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 5 \\ 6 \end{matrix} \qquad _3 = \begin{bmatrix} F_1^{(3)} \\ F_2^{(3)} \\ F_3^{(3)} \\ F_4^{(3)} \end{bmatrix}\begin{matrix} 7 \\ 8 \\ 1 \\ 2 \end{matrix} \tag{3.92}$$

From Eq. (3.92), we can see that the member end forces with the code number 1 are: $F_3^{(1)}$ of member 1, $F_1^{(2)}$ of member 2, and $F_3^{(3)}$ of member 3. Thus, the equilibrium equation corresponding to degree of freedom 1 is given by

$$P_1 = F_3^{(1)} + F_1^{(2)} + F_3^{(3)}$$

which is identical to Eq. 3.75(a). Similarly, the equilibrium equation corresponding to degree of freedom 2 can be obtained by equating P_2 to the sum of the end forces, with code number 2, of the three members. Thus, from Eq. (3.92)

$$P_2 = F_4^{(1)} + F_2^{(2)} + F_4^{(3)}$$

which is the same as Eq. (3.75(b)).

To establish the structure stiffness matrix , we write the code numbers of each member on the right side and at the top of its stiffness matrix , as shown in Fig. 3.16(c). These code numbers now define the positions of the elements of the member stiffness matrices in the structure stiffness matrix . In other words, the code numbers on the right side of a matrix represent the row numbers of the matrix, and the code numbers at the top represent the column numbers of . Furthermore, since the number of rows and columns of equal the number of degrees of freedom (*NDOF*) of the structure, only those elements of a matrix with both row and column code numbers less than or equal to *NDOF* belong in . For example, since the truss of Fig. 3.16(a) has two degrees of freedom, only the bottom-right quarters of the member matrices $_1$ and $_3$, and the top-left quarter of $_2$, belong in (see Fig. 3.16(c)).

The structure stiffness matrix is established by algebraically adding the pertinent elements of the matrices of all the members, in their proper positions, in the matrix. For example, to assemble for the truss of Fig. 3.16(a), we start by storing the pertinent elements of $_1$ in (see Fig. 3.16(c)). Thus, the element $^{(1)}_{33}$ is stored in row 1 and column 1 of , the element $^{(1)}_{43}$ is stored in row 2 and column 1 of , the element $^{(1)}_{34}$ is stored in row 1 and column 2 of (see Fig. 3.16(c)), and the element $^{(1)}_{44}$ is stored in row 2 and column 2 of . Note that only those elements of $_1$ whose row and column code numbers are either 1 or 2 are stored in . The same procedure is then repeated for members 2 and 3. When two or more member stiffness coefficients are stored in the same position in , then the coefficients must be algebraically added. The completed structure stiffness matrix for the truss is shown in Fig. 3.16(c). Note that this matrix is identical to the one determined previously by substituting the member compatibility equations and stiffness relations into the joint equilibrium equations (Eq. (3.90)).

Once has been determined, the structure stiffness relations, $\mathbf{P} = \ \mathbf{d}$ (Eq. (3.89)), which now represent a system of simultaneous linear algebraic equations, can be solved for the unknown joint displacements **d**. With **d** known, the end displacements for each member can be obtained by applying the compatibility equations defined by its code numbers; then the corresponding end displacements **u** and end forces and can be computed by using the member's transformation and stiffness relations. Finally, the support reactions can be determined from the member end forces , by considering the equilibrium of the support joints in the directions of the restrained coordinates, as discussed in the following paragraphs.

Assembly of the Support Reaction Vector Using Member Code Numbers

The support reactions of a structure can be expressed in terms of the member global end forces , using the equilibrium requirement that the reaction in a direction at a joint must be equal to the algebraic sum of all the forces, in that direction, at all the member ends connected to the joint. Because the code numbers of a member specify the locations and directions of its global end forces with respect to the structure's degrees of freedom and/or restrained coordinates, the reaction corresponding to a restrained coordinate can be evaluated by algebraically summing those elements of the vectors of all the members whose code numbers are the same as the restrained coordinate.

As the foregoing discussion suggests, the reaction vector can be assembled from the member end force vectors , using a procedure similar to that for forming the structure stiffness matrix. To determine the reactions, we write the restrained coordinate numbers (*NDOF* + 1 through 2(*N*)) on the right side of vector , as shown in Fig. 3.16(d). Next, the code numbers of each member are written on the right side of its end force vector (Fig. 3.16(d)). Any member code number that is greater than the number of degrees of freedom of the structure (*NDOF*) now represents the restrained coordinate number of the row of in which the corresponding member force is to be stored. The reaction vector is obtained by algebraically adding the pertinent elements of the vectors of all the members in their proper positions in .

For example, to assemble for the truss of Fig. 3.16(a), we begin by storing the pertinent elements of $_1$ in . Thus, as shown in Fig. 3.16(d), the element $F_1^{(1)}$ with code number 3 is stored in row 1 of , which has the restrained coordinate number 3 by its side. Similarly, the element $F_2^{(1)}$ (with code number 4) is stored in row 2 (with restrained coordinate number 4) of . Note that only those elements of $_1$ whose code numbers are greater than 2 (= *NDOF*) are stored in . The same procedure is then repeated for members 2 and 3. The completed support reaction vector for the truss is shown in Fig. 3.16(d).

EXAMPLE 3.7 Determine the structure stiffness matrix for the truss shown in Fig. 3.17(a).

SOLUTION *Analytical Model* The analytical model of the truss is shown in Fig. 3.17(b). The structure has three degrees of freedom—the translation in the *X* direction of joint 1, and the translations in the *X* and *Y* directions of joint 4. These degrees of freedom are identified by numbers 1 through 3; and the five restrained coordinates of the truss are identified by numbers 4 through 8, as shown in Fig. 3.17(b).

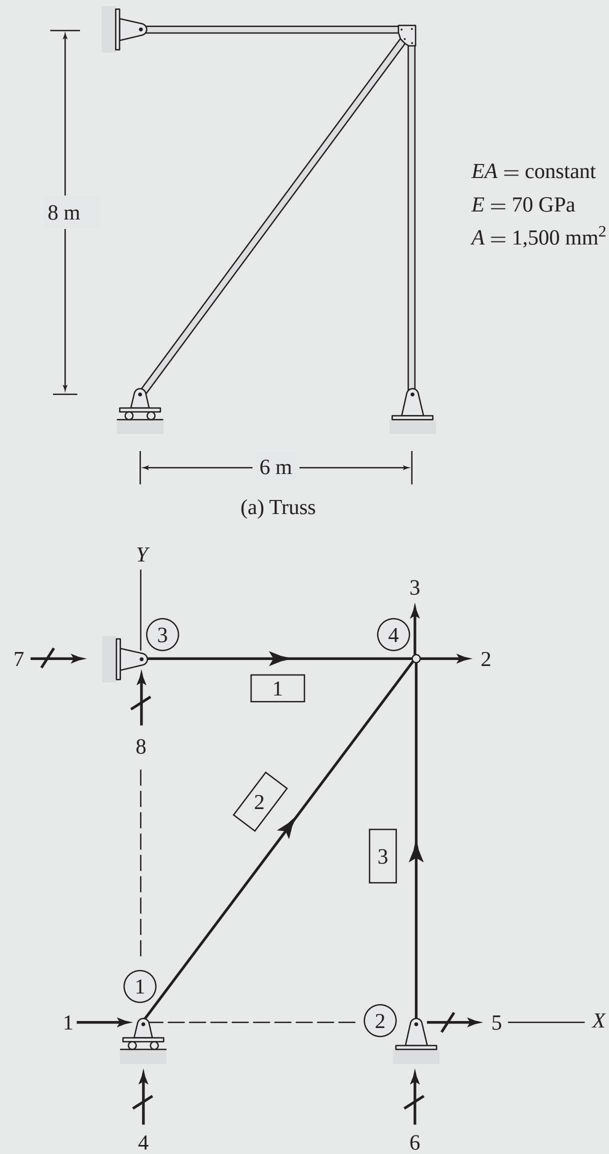
Structure Stiffness Matrix To generate the 3 × 3 structure stiffness matrix , we will determine, for each member, the global stiffness matrix and store its pertinent elements in their proper positions in by using the member's code numbers.

e ber 1 $L = 6$ m, $\cos\theta = 1$, $\sin\theta = 0$

$$\frac{A}{L} = \frac{70(10^6)\,(0.0015)}{6} = 17{,}500 \text{ kN/m}$$

The member stiffness matrix in global coordinates can now be evaluated by using Eq. (3.73).

$$_1 = \begin{array}{c} \begin{array}{cccc} 7 & 8 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} 17{,}500 & 0 & -17{,}500 & 0 \\ 0 & 0 & 0 & 0 \\ -17{,}500 & 0 & 17{,}500 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} 7 \\ 8 \\ 2 \\ 3 \end{array} \text{ kN/m} \qquad \textbf{(1)}$$



(a) Truss

(b) Analytical Model

Fig. 3.17

From Fig. 3.17(b), we observe that joint 3 has been selected as the beginning joint, and joint 4 as the end joint, for member 1. Thus, the code numbers for this member are 7, 8, 2, 3. These numbers are written on the right side and at the top of $_1$ (see Eq. (1)) to indicate the rows and columns, respectively, of the structure stiffness matrix , where the elements of $_1$ must be stored. Note that the elements of $_1$ that correspond to the restrained coordinate numbers 7 and 8 are simply disregarded. Thus, the element in row 3 and column 3 of $_1$ is stored in row 2 and column 2 of , as

$$= \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 17{,}500 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array} \qquad \mathbf{(2)}$$

e ber 2 As shown in Fig. 3.17(b), joint 1 is the beginning joint, and joint 4 is the end joint, for member 2. By applying Eqs. (3.62), we determine

$$L = \sqrt{(X_4 - X_1)^2 + (Y_4 - Y_1)^2} = \sqrt{(6-0)^2 + (8-0)^2} = 10 \text{ m}$$

$$\cos\theta = \frac{X_4 - X_1}{L} = \frac{6-0}{10} = 0.6$$

$$\sin\theta = \frac{Y_4 - Y_1}{L} = \frac{8-0}{10} = 0.8$$

$$\frac{A}{L} = \frac{70(10^6)(0.0015)}{10} = 10{,}500 \text{ kN/m}$$

By using the expression for given in Eq. (3.73), we obtain

$$_2 = \begin{array}{cc} \begin{array}{cccc} 1 & 4 & 2 & 3 \end{array} & \\ \begin{bmatrix} 3{,}780 & 5{,}040 & -3{,}780 & -5{,}040 \\ 5{,}040 & 6{,}720 & -5{,}040 & -6{,}720 \\ -3{,}780 & -5{,}040 & 3{,}780 & 5{,}040 \\ -5{,}040 & -6{,}720 & 5{,}040 & 6{,}720 \end{bmatrix} & \begin{array}{c} 1 \\ 4 \\ 2 \\ 3 \end{array} \end{array} \text{ kN/m} \qquad \mathbf{(3)}$$

From Fig. 3.17(b), we can see that the code numbers for this member are 1, 4, 2, 3. These numbers are used to add the pertinent elements of $_2$ in their proper positions in , as given in Eq. (2). Thus, now becomes

$$= \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 3{,}780 & -3{,}780 & -5{,}040 \\ -3{,}780 & 17{,}500 + 3{,}780 & 5{,}040 \\ -5{,}040 & 5{,}040 & 6{,}720 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array} \qquad \mathbf{(4)}$$

e ber 3 $L = 8$ m, $\cos\theta = 0$, $\sin\theta = 1$

$$\frac{A}{L} = \frac{70(10^6)(0.0015)}{8} = 13{,}125 \text{ kN/m}$$

By using Eq. (3.73),

$$_3 = \begin{array}{cc} \begin{array}{cccc} 5 & 6 & 2 & 3 \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 13{,}125 & 0 & -13{,}125 \\ 0 & 0 & 0 & 0 \\ 0 & -13{,}125 & 0 & 13{,}125 \end{bmatrix} & \begin{array}{c} 5 \\ 6 \\ 2 \\ 3 \end{array} \end{array} \text{ kN/m} \qquad \mathbf{(5)}$$

The code numbers for this member are 5, 6, 2, 3. By using these code numbers, the pertinent elements of $_3$ are added in (as given in Eq. (4)), yielding

$$= \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 3{,}780 & -3{,}780 & -5{,}040 \\ -3{,}780 & 17{,}500 + 3{,}780 & 5{,}040 \\ -5{,}040 & 5{,}040 & 6{,}720 + 13{,}125 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array} \text{ kN/m}$$

Since the stiffnesses of all three members of the truss have now been stored in , the structure stiffness matrix for the given truss is

$$
\begin{array}{c}
\begin{array}{ccc} 1 & 2 & 3 \end{array} \\
= \left[\begin{array}{rrr} 3{,}780 & -3{,}780 & -5{,}040 \\ -3{,}780 & 21{,}280 & 5{,}040 \\ -5{,}040 & 5{,}040 & 19{,}845 \end{array}\right]
\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \text{ kN/m}
\end{array}
\qquad \textbf{Ans}
$$

Note that the structure stiffness matrix , obtained by assembling the stiffness coefficients of the three members, is symmetric.

3.8 PROCEDURE FOR ANALYSIS

Based on the discussion presented in the previous sections, the following step-by-step procedure can be developed for the analysis of plane trusses subjected to joint loads.

1. Prepare an analytical model of the truss as follows.

   a. Draw a line diagram of the structure, on which each joint and member is identified by a number.

   b. Establish a global *XY* coordinate system, with the *X* and *Y* axes oriented in the horizontal (positive to the right) and vertical (positive upward) directions, respectively. It is usually convenient to locate the origin of the global coordinate system at a lower left joint of the structure, so that the *X* and *Y* coordinates of most of the joints are positive.

   c. For each member, establish a local *xy* coordinate system by selecting one of the joints at its ends as the beginning joint and the other as the end joint. On the structure's line diagram, indicate the positive direction of the local *x* axis for each member by drawing an arrow along the member pointing toward its end joint. For horizontal members, the coordinate transformations can be avoided by selecting the joint at the member's left end as the beginning joint.

   d. Identify the degrees of freedom (or joint displacements) and the restrained coordinates of the structure. These quantities are specified on the line diagram by assigning numbers to the arrows drawn at the joints in the *X* and *Y* directions. The degrees of freedom are numbered first, starting at the lowest-numbered joint and proceeding sequentially to the highest. In the case of more than one degree of freedom at a joint, the *X*-displacement is numbered first, followed by the *Y*-displacement. After all the degrees of freedom have been numbered, the restrained coordinates are numbered, beginning with a number equal to $NDOF + 1$. Starting at the lowest-numbered joint and proceeding sequentially to the highest, all of the restrained coordinates of the structure are numbered. In the case of more than one restrained coordinate at a joint, the *X*-coordinate is numbered first, followed by the *Y*-coordinate.

2. Evaluate the structure stiffness matrix . The number of rows and columns of   must be equal to the degrees of freedom (*NDOF*) of the structure. For each member of the truss, perform the following operations.

- **a.** Calculate its length and direction cosines. (The expressions for $\cos\theta$ and $\sin\theta$ are given in Eqs. (3.62).)
- **b.** Compute the member stiffness matrix in the global coordinate system, , using Eq. (3.73).
- **c.** Identify its code numbers, and store the pertinent elements of in their proper positions in , using the procedure described in Section 3.7.

The complete structure stiffness matrix, obtained by assembling the stiffness coefficients of all the members of the truss, must be a symmetric matrix.

3. Form the $NDOF \times 1$ joint load vector **P**.

4. Determine the joint displacements **d**. Substitute **P** and into the structure stiffness relations, $\mathbf{P} =\ \mathbf{d}$ (Eq. (3.89)), and solve the resulting system of simultaneous equations for the unknown joint displacements **d**. To check that the solution of simultaneous equations has been carried out correctly, substitute the numerical values of **d** back into the structure stiffness relations, $\mathbf{P} =\ \mathbf{d}$. If the solution is correct, then the stiffness relations should be satisfied. Note that joint displacements are considered positive when in the positive directions of the global *X* and *Y* axes; similarly, the displacements are negative in the negative directions.

5. Compute member end displacements and end forces, and support reactions. For each member of the truss, do the following.

- **a.** Obtain member end displacements in the global coordinate system, , from the joint displacements, **d**, using the member's code numbers.
- **b.** Calculate the member's transformation matrix by using Eq. (3.61), and determine member end displacements in the local coordinate system, **u**, using the transformation relationship $\mathbf{u} =$ (Eq. (3.63)). For horizontal members with local *x* axis positive to the right (i.e., in the same direction as the global *X* axis), member end displacements in the global and local coordinate systems are the same; that is, $\mathbf{u} =$. Member axial deformation, u_a, if desired, can be obtained from the relationship $u_a = u_1 - u_3$, in which u_1 and u_3 are the first and third elements, respectively, of vector **u**. A positive value of u_a indicates shortening (or contraction) of the member in the axial direction, and a negative value indicates elongation.
- **c.** Determine the member stiffness matrix in the local coordinate system, , using Eq. (3.27); then calculate member end forces in the local coordinate system by using the stiffness relationship $= \mathbf{u}$ (Eq. (3.7)). The member axial force, $_a$, equals the first element, $_1$, of the vector (i.e., $_a = {}_1$); a positive value of $_a$ indicates that the axial force is compressive, and a negative value indicates that the axial force is tensile.
- **d.** Compute member end forces in the global coordinate system, , by using the transformation relationship $= {}^T$ (Eq. (3.66)). For horizontal members with the local *x* axis positive to the right, the member

end forces in the local and global coordinate systems are the same; that is, $=$.

e. By using member code numbers, store the pertinent elements of in their proper positions in the support reaction vector , as discussed in Section 3.7.

6. To check the calculation of member end forces and support reactions, apply the three equations of equilibrium ($\sum F_X = 0$, $\sum F_Y = 0$, and $\sum M = 0$) to the free body of the entire truss. If the calculations have been carried out correctly, then the equilibrium equations should be satisfied.

Instead of following steps 5c and d, the member end forces can be determined alternatively by first evaluating the global forces , using the global stiffness relationship $=$ (Eq. (3.71)), and then obtaining the local forces from the transformation relationship $=$ (Eq. (3.60)).

EXAMPLE 3.8

Determine the joint displacements, member axial forces, and support reactions for the truss shown in Fig. 3.18(a) by the matrix stiffness method.

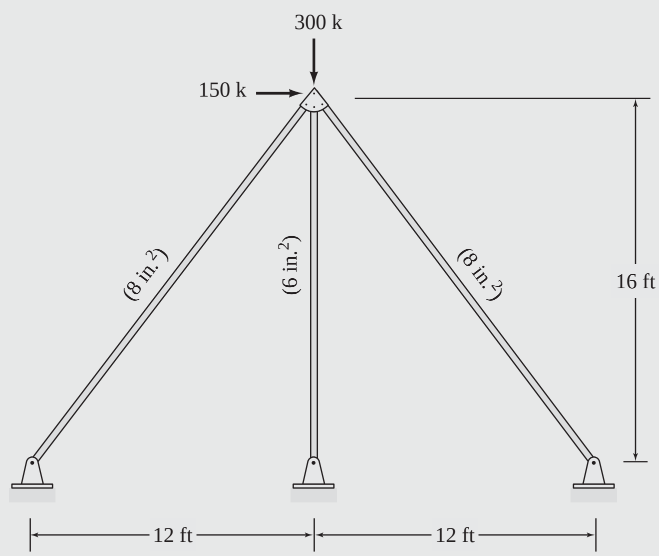
SOLUTION

Analytical Model The analytical model of the truss is shown in Fig. 3.18(b). The truss has two degrees of freedom, which are the translations of joint 1 in the *X* and *Y* directions. These are numbered as 1 and 2, respectively. The six restrained coordinates of the truss are identified by numbers 3 through 8.

Structure Stiffness Matrix

e ber 1 As shown in Fig. 3.18(b), we have selected joint 2 as the beginning joint, and joint 1 as the end joint, for member 1. By applying Eqs. (3.62), we determine

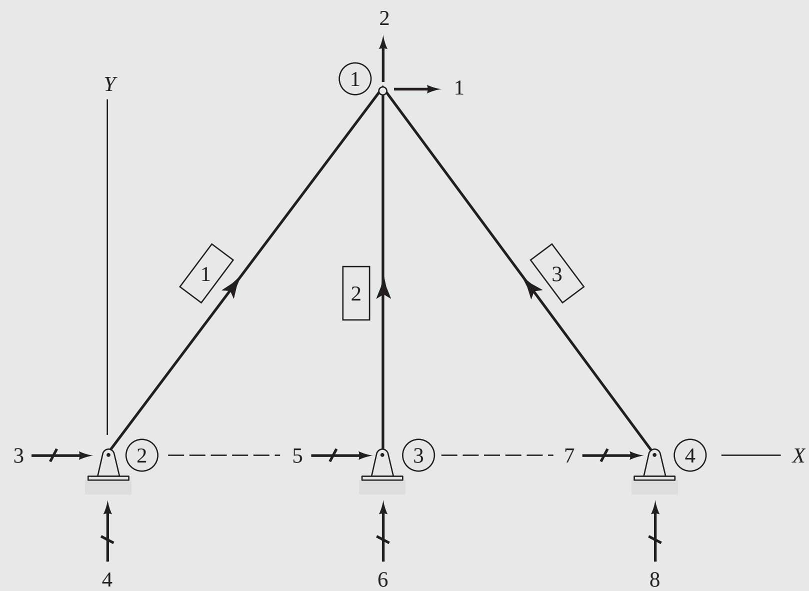
$$L = \sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2} = \sqrt{(12 - 0)^2 + (16 - 0)^2} = 20 \text{ ft}$$



$E = 29{,}000$ ksi

(a) Truss

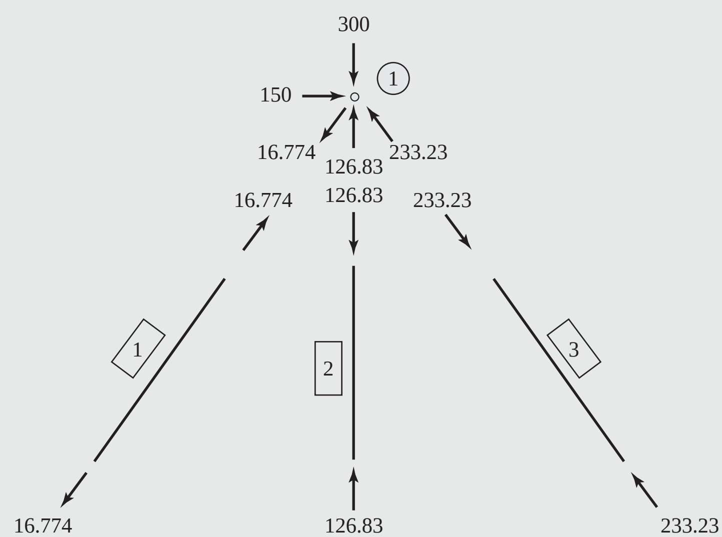
Fig. 3.18



(b) Analytical Model

$$
\mathbf{S} = \begin{array}{c}
\begin{array}{cc} 1 & 2 \end{array} \\
\left[\begin{array}{cc}
(348 + 0 + 348) & (464 + 0 - 464) \\
(464 + 0 - 464) & (618.67 + 906.25 + 618.67)
\end{array}\right]
\end{array}
\begin{array}{c} \\ 1 \\ 2 \end{array}
= \begin{array}{c}
\begin{array}{cc} 1 & 2 \end{array} \\
\left[\begin{array}{cc}
696 & 0 \\
0 & 2{,}143.6
\end{array}\right]
\end{array}
\begin{array}{c} \\ 1 \\ 2 \end{array}
\ \text{k/in.}
$$

(c) Structure Stiffness Matrix

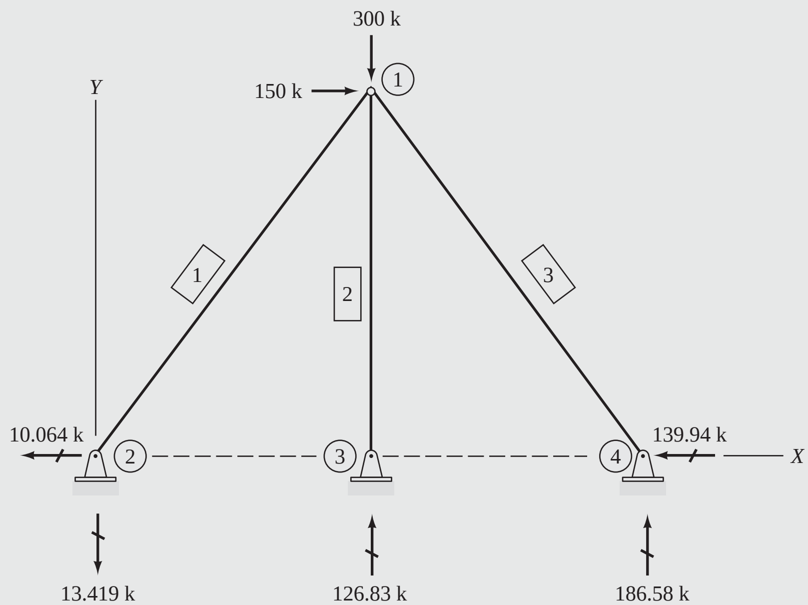


(d) Member End Forces in Local Coordinate Systems

Fig. 3.18 (*continued*)

$$\mathbf{R} = \begin{array}{c} \begin{bmatrix} -10.064 \\ -13.419 \\ 0 \\ 126.83 \\ -139.94 \\ 186.58 \end{bmatrix} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \end{array} \text{ k}$$

(e) Support Reaction Vector



(f) Support Reactions

Fig. 3.18 (*continued*)

$$\cos\theta = \frac{X_1 - X_2}{L} = \frac{12 - 0}{20} = 0.6$$

$$\sin\theta = \frac{Y_1 - Y_2}{L} = \frac{16 - 0}{20} = 0.8$$

Using the units of kips and inches, we evaluate the member's global stiffness matrix (Eq. (3.73)) as

$${}_1 = \frac{(29{,}000)(8)}{(20)(12)} \begin{bmatrix} 0.36 & 0.48 & -0.36 & -0.48 \\ 0.48 & 0.64 & -0.48 & -0.64 \\ -0.36 & -0.48 & 0.36 & 0.48 \\ -0.48 & -0.64 & 0.48 & 0.64 \end{bmatrix}$$

or

$$ {}_1 = \begin{array}{cc} & \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ & \begin{bmatrix} 348 & 464 & -348 & -464 \\ 464 & 618.67 & -464 & -618.67 \\ -348 & -464 & 348 & 464 \\ -464 & -618.67 & 464 & 618.67 \end{bmatrix} \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array} \end{array} \text{ k/in.} \qquad \textbf{(1)}$$

From Fig. 3.18(b), we observe that the code numbers for member 1 are 3, 4, 1, 2. These numbers are written on the right side and at the top of $_1$ (see Eq. (1)) to indicate the rows and columns, respectively, of the structure stiffness matrix , in which the elements of $_1$ must be stored. Note that the elements of $_1$, which correspond to the restrained coordinate numbers 3 and 4, are simply ignored. Thus, the element in row 3 and column 3 of $_1$ is stored in row 1 and column 1 of , as shown in Fig. 3.18(c); and the element in row 4 and column 3 of $_1$ is stored in row 2 and column 1 of . The remaining elements of $_1$ are stored in in a similar manner, as shown in Fig. 3.18(c).

e ber 2 From Fig. 3.18(b), we can see that joint 3 is the beginning joint, and joint 1 is the end joint, for member 2. Applying Eqs. (3.62), we write

$$L = \sqrt{(X_1 - X_3)^2 + (Y_1 - Y_3)^2} = \sqrt{(12-12)^2 + (16-0)^2} = 16 \text{ ft}$$

$$\cos\theta = \frac{X_1 - X_3}{L} = \frac{12-12}{16} = 0$$

$$\sin\theta = \frac{Y_1 - Y_3}{L} = \frac{16-0}{16} = 1$$

Thus, using Eq. (3.73),

$$\begin{array}{c} \begin{array}{cccc} 5 & 6 & 1 & 2 \end{array} \\ _2 = \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 906.25 & 0 & -906.25 \\ 0 & 0 & 0 & 0 \\ 0 & -906.25 & 0 & 906.25 \end{array}\right] \begin{array}{c} 5 \\ 6 \\ 1 \\ 2 \end{array} \text{ k/in.} \end{array}$$

From Fig. 3.18(b), we can see that the code numbers for this member are 5, 6, 1, 2. These numbers are used to store the pertinent elements of $_2$ in their proper positions in , as shown in Fig. 3.18(c).

e ber 3 It can be seen from Fig. 3.18(b) that joint 4 is the beginning joint, and joint 1 is the end joint, for member 3. Thus,

$$L = \sqrt{(X_1 - X_4)^2 + (Y_1 - Y_4)^2} = \sqrt{(12-24)^2 + (16-0)^2} = 20 \text{ ft}$$

$$\cos\theta = \frac{X_1 - X_4}{L} = \frac{12-24}{20} = -0.6$$

$$\sin\theta = \frac{Y_1 - Y_4}{L} = \frac{16-0}{20} = 0.8$$

Using Eq. (3.73),

$$\begin{array}{c} \begin{array}{cccc} 7 & 8 & 1 & 2 \end{array} \\ _3 = \left[\begin{array}{cccc} 348 & -464 & -348 & 464 \\ -464 & 618.67 & 464 & -618.67 \\ -348 & 464 & 348 & -464 \\ 464 & -618.67 & -464 & 618.67 \end{array}\right] \begin{array}{c} 7 \\ 8 \\ 1 \\ 2 \end{array} \text{ k/in.} \end{array}$$

The code numbers for this member are 7, 8, 1, 2. Using these numbers, the pertinent elements of $_3$ are stored in , as shown in Fig. 3.18(c).

The complete structure stiffness matrix , obtained by assembling the stiffness coefficients of the three members of the truss, is given in Fig. 3.18(c). Note that is symmetric.

oint Load Vector By comparing Figs. 3.18(a) and (b), we realize that

$$P_1 = 150 \text{ k} \qquad P_2 = -300 \text{ k}$$

Thus, the joint load vector is

$$\mathbf{P} = \begin{bmatrix} 150 \\ -300 \end{bmatrix} \text{ k} \tag{2}$$

oint Displacements By substituting **P** and into the structure stiffness relationship given by Eq. (3.89), we write

$$\mathbf{P} = \quad \mathbf{d}$$

$$\begin{bmatrix} 150 \\ -300 \end{bmatrix} = \begin{bmatrix} 696 & 0 \\ 0 & 2{,}143.6 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

Solving these equations, we determine the joint displacements:

$$d_1 = 0.21552 \text{ in.} \qquad d_2 = -0.13995 \text{ in.}$$

or

$$\mathbf{d} = \begin{bmatrix} 0.21552 \\ -0.13995 \end{bmatrix} \text{ in.}$$ **Ans**

To check that the solution of equations has been carried out correctly, we substitute the numerical values of joint displacements back into the structure stiffness relationship to obtain

$$\mathbf{P} = \quad \mathbf{d} = \begin{bmatrix} 696 & 0 \\ 0 & 2{,}143.6 \end{bmatrix} \begin{bmatrix} 0.21552 \\ -0.13995 \end{bmatrix} = \begin{bmatrix} 150 \\ -300 \end{bmatrix}$$ **Checks**

which is the same as the load vector **P** given in Eq. (2), thereby indicating that the calculated joint displacements do indeed satisfy the structure stiffness relations.

Member nd Displacements and nd Forces

e ber 1 The member end displacements in the global coordinate system can be obtained simply by comparing the member's global degree of freedom numbers with its code numbers, as follows:

$$_1 = \begin{array}{c} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \end{array}\!\!\begin{array}{l} 3 \\ 4 \\ 1 \\ 2 \end{array} = \begin{bmatrix} 0 \\ 0 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.21552 \\ -0.13995 \end{bmatrix} \text{ in.} \tag{3}$$

Note that the code numbers for the member (3, 4, 1, 2) are written on the right side of , as shown in Eq. (3). Because the code numbers corresponding to v_1 and v_2 are the restrained coordinate numbers 3 and 4, this indicates that $v_1 = v_2 = 0$. Similarly, the code numbers 1 and 2 corresponding to v_3 and v_4, respectively, indicate that $v_3 = d_1$ and $v_4 = d_2$. It should be clear that these compatibility equations could have been established alternatively by a simple visual inspection of the line diagram of the structure depicted in Fig. 3.18(b). However, as will be shown in Chapter 4, the use of the member code numbers enables us to conveniently program this procedure on a computer.

To determine the member end displacements in the local coordinate system, we first evaluate its transformation matrix as defined in Eq. (3.61):

$$_1 = \begin{bmatrix} 0.6 & 0.8 & 0 & 0 \\ -0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0.8 \\ 0 & 0 & -0.8 & 0.6 \end{bmatrix}$$

The member local end displacements can now be calculated, using the relationship $\mathbf{u} =$ (Eq. (3.63)), as

$$\mathbf{u}_1 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.8 & 0 & 0 \\ -0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0.8 \\ 0 & 0 & -0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.21552 \\ -0.13995 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.017352 \\ -0.25639 \end{bmatrix} \text{ in.}$$

Before we can calculate the member end forces in the local coordinate system, we need to determine its local stiffness matrix , using Eq. (3.27):

$$_1 = \begin{bmatrix} 966.67 & 0 & -966.67 & 0 \\ 0 & 0 & 0 & 0 \\ -966.67 & 0 & 966.67 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ k/in.}$$

Now, using Eq. (3.7), we compute the member local end forces as

$$= \mathbf{u}$$

$$_1 = \begin{bmatrix} _1 \\ _2 \\ _3 \\ _4 \end{bmatrix} = \begin{bmatrix} 966.67 & 0 & -966.67 & 0 \\ 0 & 0 & 0 & 0 \\ -966.67 & 0 & 966.67 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.017352 \\ -0.25639 \end{bmatrix} = \begin{bmatrix} -16.774 \\ 0 \\ 16.774 \\ 0 \end{bmatrix} \text{ k}$$

The member axial force is equal to the first element of the vector $_1$; that is,

$$_{a1} = -16.774 \text{ k}$$

in which the negative sign indicates that the axial force is tensile, or

$$_{a1} = 16.774 \text{ k (T)}$$ **Ans**

This member axial force can be verified by visually examining the free-body diagram of the member subjected to the local end forces, as shown in Fig. 3.18(d).

By applying Eq. (3.66), we determine the member end forces in the global coordinate system:

$$= {}^{T}$$

$$_1 = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.8 & 0 & 0 \\ 0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & -0.8 \\ 0 & 0 & 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} -16.774 \\ 0 \\ 16.774 \\ 0 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} -10.064 \\ -13.419 \\ \hdashline 10.064 \\ 13.419 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array} \text{ k} \tag{4}$$

Next, we write the member code numbers (3, 4, 1, 2) on the right side of $_1$ (see Eq. (4)), and store the pertinent elements of $_1$ in their proper positions in the reaction vector by matching the code numbers (on the side of $_1$) to the restrained coordinate numbers written on the right side of (see Fig. 3.18(e)). Thus, the element in row 1 of $_1$ (with code number 3) is stored in row 1 of (with restrained coordinate number 3); and the element in row 2 of $_1$ (with code number 4) is stored in row 2 of (with restrained coordinate number 4), as shown in Fig. 3.18(e). Note that the elements in rows 3 and 4 of $_1$, with code numbers corresponding to degrees of freedom 1 and 2 of the structure, are simply disregarded.

e ber 2 The member end displacements in the global coordinate system are given by

$$_2 = \begin{array}{c} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \end{array}\begin{array}{c} 5 \\ 6 \\ 1 \\ 2 \end{array} = \begin{bmatrix} 0 \\ 0 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.21552 \\ -0.13995 \end{bmatrix} \text{ in.}$$

The member end displacements in the local coordinate system can now be determined by using the relationship **u** = (Eq. (3.63)), with as defined in Eq. (3.61):

$$\mathbf{u}_2 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.21552 \\ -0.13995 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.13995 \\ -0.21552 \end{bmatrix} \text{ in.}$$

Using Eq. (3.7), we compute member end forces in the local coordinate system:

$$= \mathbf{u}$$

$$_2 = \begin{bmatrix} _1 \\ _2 \\ _3 \\ _4 \end{bmatrix} = 906.25 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.13995 \\ -0.21552 \end{bmatrix} = \begin{bmatrix} 126.83 \\ 0 \\ -126.83 \\ 0 \end{bmatrix} \text{ k}$$

from which we obtain the member axial force (see also Fig. 3.18(d)):

$$_{a2} = 126.83 \text{ k (C)}$$ **Ans**

Using the relationship $= \; ^T$ (Eq. (3.66)), we calculate the member global end forces to be

$$_2 = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 126.83 \\ 0 \\ -126.83 \\ 0 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 0 \\ 126.83 \\ \hdashline 0 \\ -126.83 \end{bmatrix} \end{array}\begin{array}{c} 5 \\ 6 \\ 1 \\ 2 \end{array} \text{ k}$$

The pertinent elements of $_2$ are now stored in their proper positions in the reaction vector , by using member code numbers (5, 6, 1, 2), as shown in Fig. 3.18(e).

e ber 3 The member global end displacements are

$$_3 = \begin{array}{c} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \end{array}\begin{array}{c} 7 \\ 8 \\ 1 \\ 2 \end{array} = \begin{bmatrix} 0 \\ 0 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.21552 \\ -0.13995 \end{bmatrix} \text{ in.}$$

As in the case of members 1 and 2, we can determine the end forces $_3$ and $_3$ for member 3 by using the relationships **u** = , = **u**, and $= \; ^T$, in sequence. However, such member forces can also be obtained by applying sequentially the global stiffness relationship = (Eq. (3.71)) and the transformation relation = (Eq. (3.60)). Let us apply this alternative approach to determine the end forces for member 3.

Applying Eq. (3.71), we compute the member end forces in the global coordinate system:

$$=$$

$$_3 = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} 348 & -464 & -348 & 464 \\ -464 & 618.67 & 464 & -618.67 \\ -348 & 464 & 348 & -464 \\ 464 & -618.67 & -464 & 618.67 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.21552 \\ -0.13995 \end{bmatrix}$$

$$= \begin{array}{c} \begin{bmatrix} -139.94 \\ 186.58 \\ 139.94 \\ -186.58 \end{bmatrix} \end{array} \begin{array}{c} 7 \\ 8 \\ 1 \\ 2 \end{array} \text{ k}$$

Using the member code numbers (7, 8, 1, 2), the pertinent elements of $_3$ are stored in the reaction vector , as shown in Fig. 3.18(e).

The member end forces in the local coordinate system can now be obtained by using the transformation relationship $=$ (Eq. (3.60)), with as defined in Eq. (3.61).

$$_3 = \begin{bmatrix} _1 \\ _2 \\ _3 \\ _4 \end{bmatrix} = \begin{bmatrix} -0.6 & 0.8 & 0 & 0 \\ -0.8 & -0.6 & 0 & 0 \\ 0 & 0 & -0.6 & 0.8 \\ 0 & 0 & -0.8 & -0.6 \end{bmatrix} \begin{bmatrix} -139.94 \\ 186.58 \\ 139.94 \\ -186.58 \end{bmatrix} = \begin{bmatrix} 233.23 \\ 0 \\ -233.23 \\ 0 \end{bmatrix} \text{k}$$

from which the member axial force is found to be (see also Fig. 3.18(d))

$$_{a3} = 233.23 \text{ k (C)}$$ **Ans**

Support Reactions The completed reaction vector is shown in Fig. 3.18(e), and the support reactions are depicted on a line diagram of the truss in Fig. 3.18(f). **Ans**

quilibrium Check Applying the equations of equilibrium to the free body of the entire truss (Fig. 3.18(f)), we obtain

$+ \rightarrow \sum F_X = 0 \qquad 150 - 10.064 - 139.94 = -0.004 \approx 0$ **Checks**

$+ \uparrow \sum F_Y = 0 \qquad -300 - 13.419 + 126.83 + 186.58 = -0.009 \approx 0$ **Checks**

$+\circlearrowleft \sum M_{①} = 0 \qquad -10.064(16) + 13.419(12) - 139.94(16)$

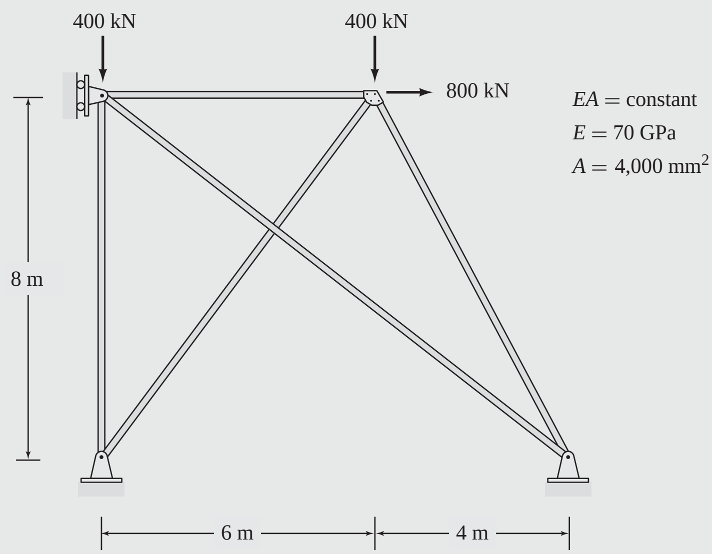
$+ 186.58(12) = -0.076 \text{ k-ft} \approx 0$ **Checks**

EXAMPLE 3.9

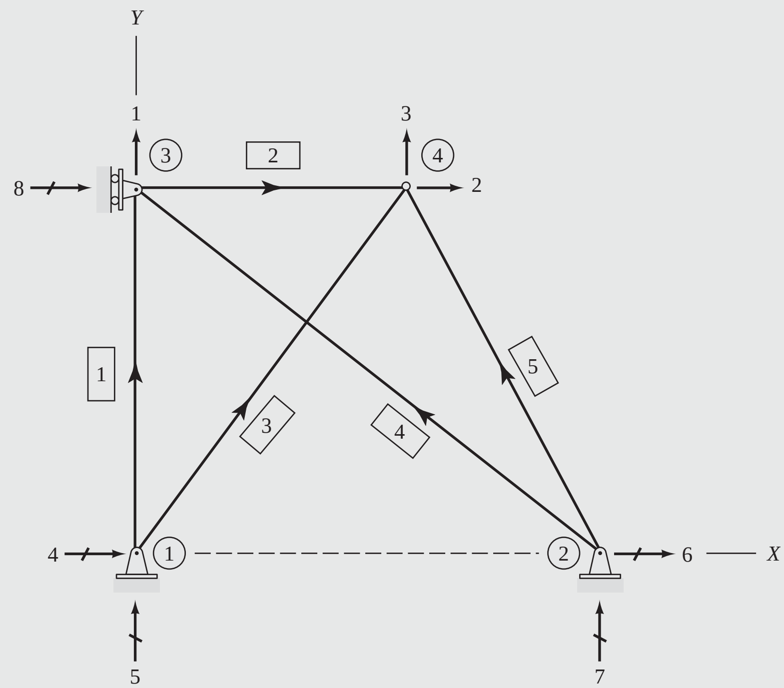
Determine the joint displacements, member axial forces, and support reactions for the truss shown in Fig. 3.19(a), using the matrix stiffness method.

SOLUTION

Analytical Model From the analytical model of the truss shown in Fig. 3.19(b), we observe that the structure has three degrees of freedom (numbered 1, 2, and 3), and five restrained coordinates (numbered 4 through 8). Note that for horizontal member 2, the left end joint 3 is chosen as the beginning joint, so that the positive directions of local axes are the same as the global axes. Thus, no coordinate transformations are necessary for this member; that is, the member stiffness relations in the local and global coordinate systems are the same.



(a) Truss



(b) Analytical Model

Fig. 3.19

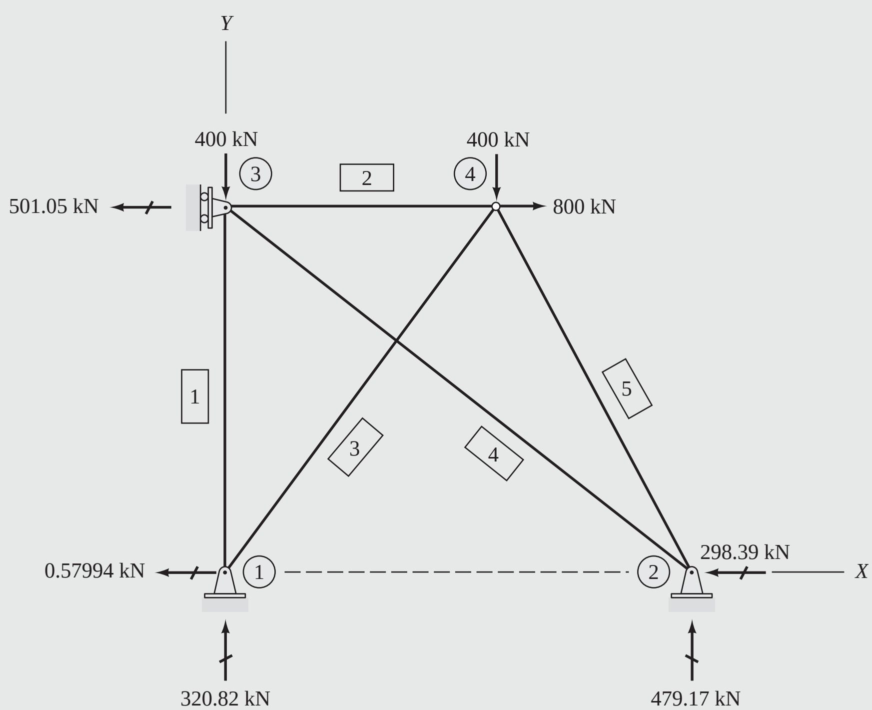
$$\mathbf{S} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 35{,}000 + 8{,}533 & 0 & 0 \\ 0 & 46{,}667 + 10{,}080 + 6{,}260.9 & 13{,}440 - 12{,}522 \\ 0 & 13{,}440 - 12{,}522 & 17{,}920 + 25{,}043 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

$$= \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 43{,}533 & 0 & 0 \\ 0 & 63{,}008 & 918 \\ 0 & 918 & 42{,}963 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \text{ kN/m}$$

(c) Structure Stiffness Matrix

$$\mathbf{R} = \left[\begin{array}{c} -0.57994 \\ 321.59 - 0.77325 \\ -98.008 - 200.38 \\ 78.407 + 400.76 \\ -599.06 + 98.008 \end{array}\right] \begin{array}{c} 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} = \left[\begin{array}{c} -0.57994 \\ 320.82 \\ -298.39 \\ 479.17 \\ -501.05 \end{array}\right] \begin{array}{c} 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \text{ kN}$$

(d) Support Reaction Vector



(e) Support Reactions

Fig. 3.19 (*continued*)

Structure Stiffness Matrix

e ber 1 Using Eqs. (3.62), we write

$$L = \sqrt{(X_3 - X_1)^2 + (Y_3 - Y_1)^2} = \sqrt{(0-0)^2 + (8-0)^2} = 8\,\text{m}$$

$$\cos\theta = \frac{X_3 - X_1}{L} = \frac{0-0}{8} = 0$$

$$\sin\theta = \frac{Y_3 - Y_1}{L} = \frac{8-0}{8} = 1$$

Using the units of kN and meters, we obtain the member global stiffness matrix (Eq. (3.73)):

$$_1 = \frac{70(10^6)(0.004)}{8}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \begin{array}{c c} \begin{array}{cccc} 4 & 5 & 8 & 1 \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 35{,}000 & 0 & -35{,}000 \\ 0 & 0 & 0 & 0 \\ 0 & -35{,}000 & 0 & 35{,}000 \end{bmatrix} & \begin{array}{c} 4 \\ 5 \\ 8 \\ 1 \end{array} \end{array} \text{kN/m}$$

From Fig. 3.19(b), we observe that the code numbers for member 1 are 4, 5, 8, 1. These numbers are written on the right side and at the top of $_1$, and the pertinent elements of $_1$ are stored in their proper positions in the structure stiffness matrix , as shown in Fig. 3.19(c).

e ber 2 As discussed, no coordinate transformations are needed for this horizontal member; that is, $_2 = \mathbf{I}$, and $_2 = {}_2$. Substituting $= 70(10^6)$ kN/m^2, $A = 0.004$ m^2, and $L = 6$ m into Eq. (3.27), we obtain

$$_2 = {}_2 = \begin{array}{c c} \begin{array}{cccc} 8 & 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 46{,}667 & 0 & -46{,}667 & 0 \\ 0 & 0 & 0 & 0 \\ -46{,}667 & 0 & 46{,}667 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} 8 \\ 1 \\ 2 \\ 3 \end{array} \end{array} \text{kN/m}$$

From Fig. 3.19(b), we can see that the code numbers for member 2 are 8, 1, 2, 3. These numbers are used to store the appropriate elements of $_2$ in , as shown in Fig. 3.19(c).

e ber 3

$$L = \sqrt{(X_4 - X_1)^2 + (Y_4 - Y_1)^2} = \sqrt{(6-0)^2 + (8-10)^2} = 10\text{ m}$$

$$\cos\theta = \frac{X_4 - X_1}{L} = \frac{6-0}{10} = 0.6$$

$$\sin\theta = \frac{Y_4 - Y_1}{L} = \frac{8-0}{10} = 0.8$$

$$_3 = \begin{array}{c c} \begin{array}{cccc} 4 & 5 & 2 & 3 \end{array} & \\ \begin{bmatrix} 10{,}080 & 13{,}440 & -10{,}080 & -13{,}440 \\ 13{,}440 & 17{,}920 & -13{,}440 & -17{,}920 \\ -10{,}080 & -13{,}440 & 10{,}080 & 13{,}440 \\ -13{,}440 & -17{,}920 & 13{,}440 & 17{,}920 \end{bmatrix} & \begin{array}{c} 4 \\ 5 \\ 2 \\ 3 \end{array} \end{array} \text{kN/m}$$

Using the code numbers (4, 5, 2, 3) of member 3, the relevant elements of $_3$ are stored in , as shown in Fig. 3.19(c).

e ber 4

$$L = \sqrt{(X_3 - X_2)^2 + (Y_3 - Y_2)^2} = \sqrt{(0 - 10)^2 + (8 - 0)^2} = 12.806 \text{ m}$$

$$\cos\theta = \frac{X_3 - X_2}{L} = \frac{0 - 10}{12.806} = -0.78088$$

$$\sin\theta = \frac{Y_3 - Y_2}{L} = \frac{8 - 0}{12.806} = 0.62471$$

$$\begin{array}{c} \\ {}_4 = \end{array} \begin{array}{c} \begin{array}{cccc} \quad 6 \quad & \quad 7 \quad & \quad 8 \quad & \quad 1 \quad \end{array} \\ \left[\begin{array}{cccc} 13{,}333 & -10{,}666 & -13{,}333 & 10{,}666 \\ -10{,}666 & 8{,}533 & 10{,}666 & -8{,}533 \\ -13{,}333 & 10{,}666 & 13{,}333 & -10{,}666 \\ 10{,}666 & -8{,}533 & -10{,}666 & 8{,}533 \end{array}\right] \end{array} \begin{array}{c} \\ 6 \\ 7 \\ 8 \\ 1 \end{array} \text{ kN/m}$$

The member code numbers are 6, 7, 8, 1. Thus, the element in row 4 and column 4 of $_4$ is stored in row 1 and column 1 of , as shown in Fig. 3.19(c).

e ber 5

$$L = \sqrt{(X_4 - X_2)^2 + (Y_4 - Y_2)^2} = \sqrt{(6 - 10)^2 + (8 - 0)^2} = 8.9443 \text{ m}$$

$$\cos\theta = \frac{X_4 - X_2}{L} = \frac{6 - 10}{8.9443} = -0.44721$$

$$\sin\theta = \frac{Y_4 - Y_2}{L} = \frac{8 - 0}{8.9443} = 0.89442$$

$$\begin{array}{c} \\ {}_5 = \end{array} \begin{array}{c} \begin{array}{cccc} \quad 6 \quad & \quad 7 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\ \left[\begin{array}{cccc} 6{,}260.9 & -12{,}522 & -6{,}260.9 & 12{,}522 \\ -12{,}522 & 25{,}043 & 12{,}522 & -25{,}043 \\ -6{,}260.9 & 12{,}522 & 6{,}260.9 & -12{,}522 \\ 12{,}522 & -25{,}043 & -12{,}522 & 25{,}043 \end{array}\right] \end{array} \begin{array}{c} \\ 6 \\ 7 \\ 2 \\ 3 \end{array} \text{ kN/m}$$

The code numbers for member 5 are 6, 7, 2, 3. These numbers are used to store the pertinent elements of $_5$ in .

The completed structure stiffness matrix is given in Fig. 3.19(c).

oint Load Vector By comparing Figs. 3.19(a) and (b), we obtain

$$\mathbf{P} = \begin{bmatrix} -400 \\ 800 \\ -400 \end{bmatrix} \text{ kN}$$

oint Displacements The structure stiffness relationship (Eq. (3.89)) can now be written as

$$\mathbf{P} = \ \mathbf{d}$$

$$\begin{bmatrix} -400 \\ 800 \\ -400 \end{bmatrix} = \begin{bmatrix} 43{,}533 & 0 & 0 \\ 0 & 63{,}008 & 918 \\ 0 & 918 & 42{,}963 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Solving these equations simultaneously, we determine the joint displacements.

$$\mathbf{d} = \begin{bmatrix} -0.0091884 \\ 0.012837 \\ -0.0095846 \end{bmatrix} \text{ m} = \begin{bmatrix} -9.1884 \\ 12.837 \\ -9.5846 \end{bmatrix} \text{ mm} \qquad \textbf{Ans}$$

To check our solution, the numerical values of **d** are back-substituted into the structure stiffness relation $\mathbf{P} = \ \mathbf{d}$ to obtain

$$\mathbf{P} = \ \mathbf{d} = \begin{bmatrix} 43{,}533 & 0 & 0 \\ 0 & 63{,}008 & 918 \\ 0 & 918 & 42{,}963 \end{bmatrix} \begin{bmatrix} -0.0091884 \\ 0.012837 \\ -0.0095846 \end{bmatrix} = \begin{bmatrix} -400 \\ 800.04 \approx 800 \\ -400 \end{bmatrix}$$

Checks

Member nd Displacements and nd Forces

e ber 1 The global end displacements of member 1 are obtained by comparing its global degree-of-freedom numbers with its code numbers. Thus,

$$_1 = \begin{array}{c} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \end{array} \begin{array}{c} 4 \\ 5 \\ 8 \\ 1 \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.0091884 \end{bmatrix} \text{ m}$$

To determine its local end displacements, we apply the relationship $\mathbf{u} =$ (Eq. (3.63)), with as given in Eq. (3.61):

$$\mathbf{u}_1 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.0091884 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.0091884 \\ 0 \end{bmatrix} \text{ m}$$

Next, we compute the end forces in the local coordinate system by using the relationship $=$ $\mathbf{u}$ (Eq. (3.7)), with as defined in Eq. (3.27). Thus,

$$_1 = \begin{bmatrix} _1 \\ _2 \\ _3 \\ _4 \end{bmatrix} = 35{,}000 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.0091884 \\ 0 \end{bmatrix} = \begin{bmatrix} 321.59 \\ 0 \\ -321.59 \\ 0 \end{bmatrix} \text{ kN}$$

Therefore, the member axial force, which equals the first element of the vector $_1$, is

$$_{a1} = 321.59 \text{ kN (C)}$$

Ans

The global end forces can now be obtained by using the relationship $= \ ^T$ (Eq. (3.66)):

$$_1 = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 321.59 \\ 0 \\ -321.59 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 321.59 \\ 0 \\ -321.59 \end{bmatrix} \begin{array}{c} 4 \\ 5 \\ 8 \\ 1 \end{array} \text{ kN}$$

Using the code numbers (4, 5, 8, 1), the elements of $_1$ corresponding to the restrained coordinates (4 through 8) are stored in their proper positions in , as shown in Fig. 3.19(d).

e ber 2

$$\mathbf{u}_2 = \ _2 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \begin{array}{c} 8 \\ 1 \\ 2 \\ 3 \end{array} = \begin{bmatrix} 0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.0091884 \\ 0.012837 \\ -0.0095846 \end{bmatrix} \text{ m}$$

Using the relationship $= \mathbf{u}$ (Eq. (3.7)), we determine the member end forces:

$$_2 = \ _2 = 46{,}667 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -0.0091884 \\ 0.012837 \\ -0.0095846 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} -599.06 \\ 0 \\ 599.06 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} 8 \\ 1 \\ 2 \\ 3 \end{array} \text{ kN}$$

from which the member axial force is found.

$$_{a2} = -599.06 \text{ kN} = 599.06 \text{ kN (T)}$$ **Ans**

The element in the first row of $_2$ (with code number 8) is stored in the fifth row of (with restrained coordinate number 8), as shown in Fig. 3.19(d).

e ber 3

$$_3 = \begin{bmatrix} 0 \\ 0 \\ 0.012837 \\ -0.0095846 \end{bmatrix} \begin{array}{c} 4 \\ 5 \\ 2 \\ 3 \end{array} \text{ m}$$

Using Eq. (3.63),

$$\mathbf{u} =$$

$$\mathbf{u}_3 = \begin{bmatrix} 0.6 & 0.8 & 0 & 0 \\ -0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0.8 \\ 0 & 0 & -0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.012837 \\ -0.0095846 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.00003452 \\ -0.01602 \end{bmatrix} \text{ m}$$

Applying Eq. (3.7),

$$= \mathbf{u}$$

$$_3 = 28{,}000 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.00003452 \\ -0.01602 \end{bmatrix} = \begin{bmatrix} -0.96656 \\ 0 \\ 0.96656 \\ 0 \end{bmatrix} \text{ kN}$$

from which,

$$_{a3} = -0.96656 \text{ kN} = 0.96656 \text{ kN (T)}$$ **Ans**

From Eq. (3.66), we obtain

$$= \ ^T$$

$$_3 = \begin{bmatrix} 0.6 & -0.8 & 0 & 0 \\ 0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & -0.8 \\ 0 & 0 & 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} -0.96656 \\ 0 \\ 0.96656 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.57994 \\ -0.77325 \\ 0.57994 \\ 0.77325 \end{bmatrix} \begin{array}{c} 4 \\ 5 \\ 2 \\ 3 \end{array} \text{ kN}$$

The pertinent elements of $_3$ are stored in , using the member code numbers (4, 5, 2, 3), as shown in Fig. 3.19(d).

e ber 4

$$_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.0091884 \end{bmatrix} \begin{matrix} 6 \\ 7 \\ 8 \\ 1 \end{matrix} \ \text{m}$$

$$\mathbf{u} =$$

$$\mathbf{u}_4 = \begin{bmatrix} -0.78088 & 0.62471 & 0 & 0 \\ -0.62471 & -0.78088 & 0 & 0 \\ 0 & 0 & -0.78088 & 0.62471 \\ 0 & 0 & -0.62471 & -0.78088 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.0091884 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -0.0057401 \\ 0.007175 \end{bmatrix} \text{m}$$

$$= \ \mathbf{u}$$

$$_4 = 21{,}865 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.0057401 \\ 0.007175 \end{bmatrix} = \begin{bmatrix} 125.51 \\ 0 \\ -125.51 \\ 0 \end{bmatrix} \text{kN}$$

from which,

$$_{a4} = 125.51 \text{ kN (C)}$$ **Ans**

$$= \ ^T$$

$$_4 = \begin{bmatrix} -0.78088 & -0.62471 & 0 & 0 \\ 0.62471 & -0.78088 & 0 & 0 \\ 0 & 0 & -0.78088 & -0.62471 \\ 0 & 0 & 0.62471 & -0.78088 \end{bmatrix} \begin{bmatrix} 125.51 \\ 0 \\ -125.51 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -98.008 \\ 78.407 \\ 98.008 \\ -78.407 \end{bmatrix} \begin{matrix} 6 \\ 7 \\ 8 \\ 1 \end{matrix} \ \text{kN}$$

The relevant elements of $_4$ are stored in , as shown in Fig. 3.19(d).

e ber 5

$$_5 = \begin{bmatrix} 0 \\ 0 \\ 0.012837 \\ -0.0095846 \end{bmatrix} \begin{matrix} 6 \\ 7 \\ 2 \\ 3 \end{matrix} \ \text{m}$$

$$\mathbf{u} =$$

$$\mathbf{u}_5 = \begin{bmatrix} -0.44721 & 0.89442 & 0 & 0 \\ -0.89442 & -0.44721 & 0 & 0 \\ 0 & 0 & -0.44721 & 0.89442 \\ 0 & 0 & -0.89442 & -0.44721 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.012837 \\ -0.0095846 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -0.014313 \\ -0.0071953 \end{bmatrix} \text{m}$$

$$= \mathbf{u}$$

$$_5 = 31{,}305 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.014313 \\ -0.0071953 \end{bmatrix} = \begin{bmatrix} 448.07 \\ 0 \\ -448.07 \\ 0 \end{bmatrix} \text{kN}$$

Thus,

$$_{a5} = 448.07 \text{ kN (C)}$$ **Ans**

$$= \ ^{T}$$

$$_5 = \begin{bmatrix} -0.44721 & -0.89442 & 0 & 0 \\ 0.89442 & -0.44721 & 0 & 0 \\ 0 & 0 & -0.44721 & -0.89442 \\ 0 & 0 & 0.89442 & -0.44721 \end{bmatrix} \begin{bmatrix} 448.07 \\ 0 \\ -448.07 \\ 0 \end{bmatrix}$$

$$= \begin{array}{c} \begin{bmatrix} -200.38 \\ 400.76 \\ 200.38 \\ -400.76 \end{bmatrix} \end{array} \begin{array}{c} 6 \\ 7 \\ 2 \\ 3 \end{array} \text{kN}$$

The pertinent elements of $_5$ are stored in , as shown in Fig. 3.19(d).

Support Reactions The completed reaction vector is given in Fig. 3.19(d), and the support reactions are shown on a line diagram of the structure in Fig. 3.19(e). **Ans**

quilibrium Check Considering the equilibrium of the entire truss, we write (Fig. 3.19(e)),

$$+ \rightarrow \sum F_X = 0 \quad -0.57994 - 298.39 - 501.05 + 800 = -0.02 \text{ kN} \approx 0$$ **Checks**

$$+ \uparrow \sum F_Y = 0 \quad 320.82 + 479.17 - 400 - 400 = -0.01 \text{ kN} \approx 0$$ **Checks**

$$+ \circlearrowleft \sum M_{①} = 0 \quad 479.17(10) + 501.05(8) - 800(8) - 400(6) = 0.1 \text{kN} \cdot \text{m} \approx 0$$ **Checks**

SUMMARY

In this chapter, we have studied the basic concepts of the analysis of plane trusses based on the matrix stiffness method. A block diagram that summarizes the various steps involved in this analysis is presented in Fig. 3.20.

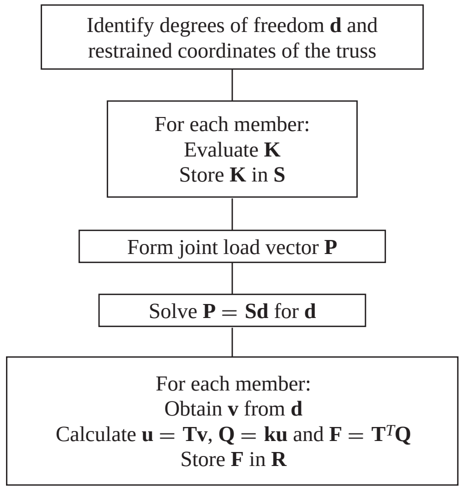


Fig. 3.20

PROBLEMS

Section 3.2

3.1 through 3.3 Identify by numbers the degrees of freedom and restrained coordinates of the trusses shown in Figs. P3.1–P3.3. Also, form the joint load vector **P** for the trusses.

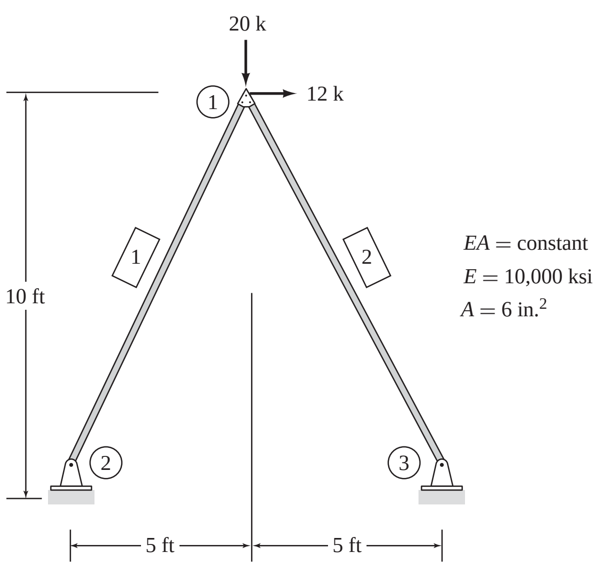


Fig. P3.1, P3.17

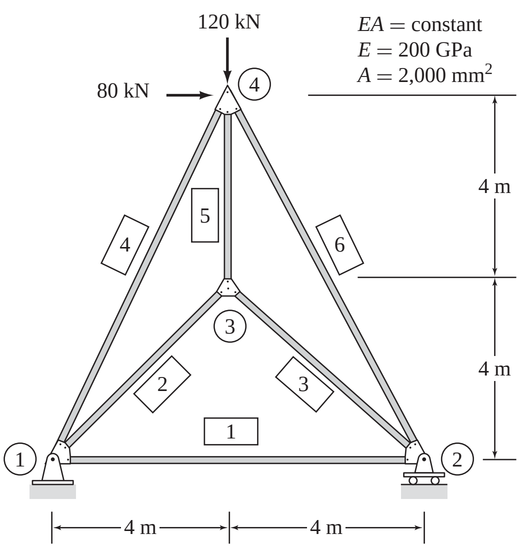


Fig. P3.2, P3.23

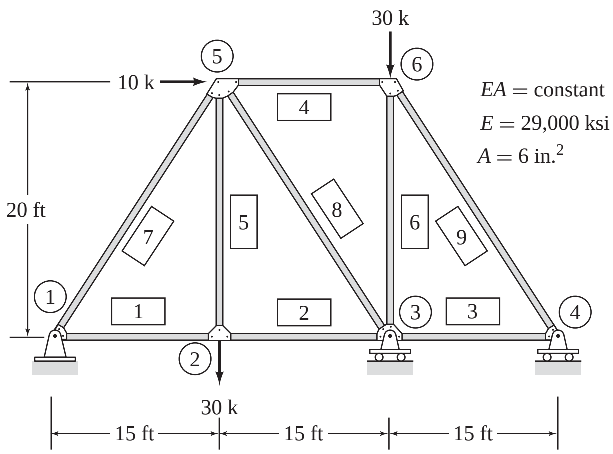


Fig. P3.3, P3.25

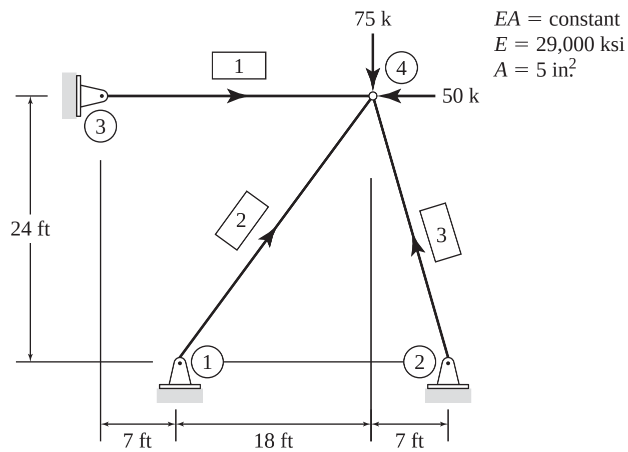


Fig. P3.5, P3.9, P3.15, P3.19

Section 3.3

3.4 a d 3.5 Determine the local stiffness matrix for each member of the trusses shown in Figs. P3.4 and P3.5.

3.6 If end displacements in the local coordinate system for member 5 of the truss shown in Fig. P3.6 are

$$\mathbf{u}_5 = \begin{bmatrix} -0.5 \\ 0.5 \\ 0.75 \\ 1.25 \end{bmatrix} \text{ in.}$$

calculate the axial force in the member.

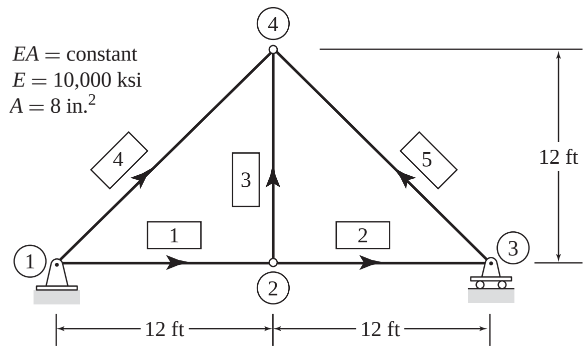


Fig. P3.6, P3.10, P3.12

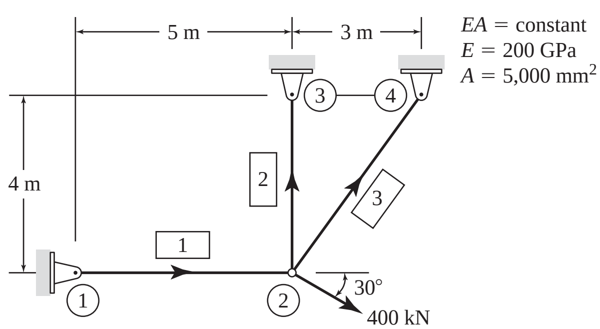


Fig. P3.4, P3.8, P3.14, P3.18

3.7 If end displacements in the local coordinate system for member 9 of the truss shown in Fig. P3.7 are

$$\mathbf{u}_9 = \begin{bmatrix} 17.6 \\ 3.2 \\ 33 \\ 6 \end{bmatrix} \text{ mm}$$

calculate the axial force in the member.

Section 3.5

3.8 a d 3.9 Determine the transformation matrix for each member of the trusses shown in Figs. P3.8 and P3.9.

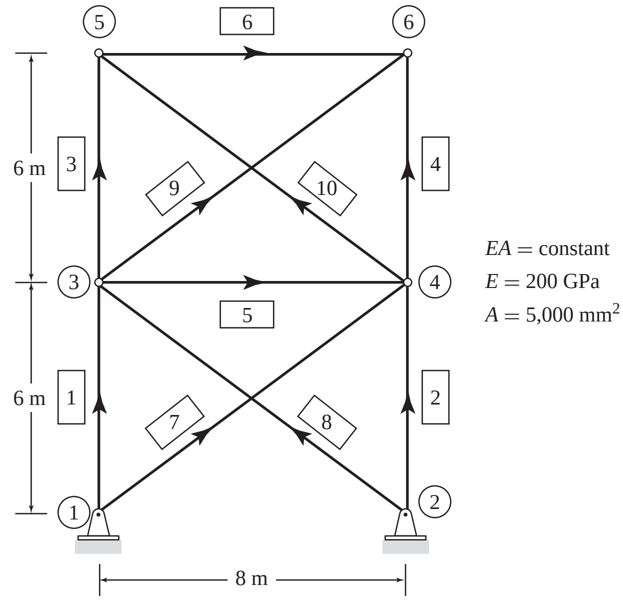


Fig. P3.7, P3.11, P3.13

3.10 If the end displacements in the global coordinate system for member 5 of the truss shown in Fig. P3.10 are

$$_{5} = \begin{bmatrix} 0.5 \\ 0 \\ 0.25 \\ -1 \end{bmatrix} \text{ in.}$$

calculate the end forces for the member in the global coordinate system. Is the member in equilibrium under these forces

3.11 If the end displacements in the global coordinate system for member 9 of the truss shown in Fig. P3.11 are

$$_{9} = \begin{bmatrix} 16 \\ -8 \\ 30 \\ -15 \end{bmatrix} \text{ mm}$$

calculate the end forces for the member in the global coordinate system. Is the member in equilibrium under these forces

Section 3.6

3.12 Solve Problem 3.10, using the member stiffness relationship in the global coordinate system, $=$.

3.13 Solve Problem 3.11, using the member stiffness relationship in the global coordinate system, $=$.

Section 3.7

3.14 a d 3.15 Determine the structure stiffness matrices for the trusses shown in Figs. P3.14 and P3.15.

Section 3.8

3.16 through 3.25 Determine the joint displacements, member axial forces, and support reactions for the trusses shown in Figs. P3.16 through P3.25, using the matrix stiffness method. Check the hand-calculated results by using the computer program on the publisher's website for this book (www.cengage.com/engineering), or by using any other general purpose structural analysis program available.

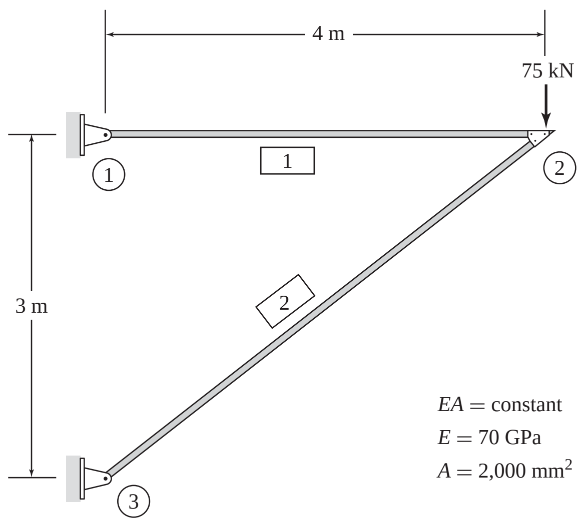


Fig. P3.16

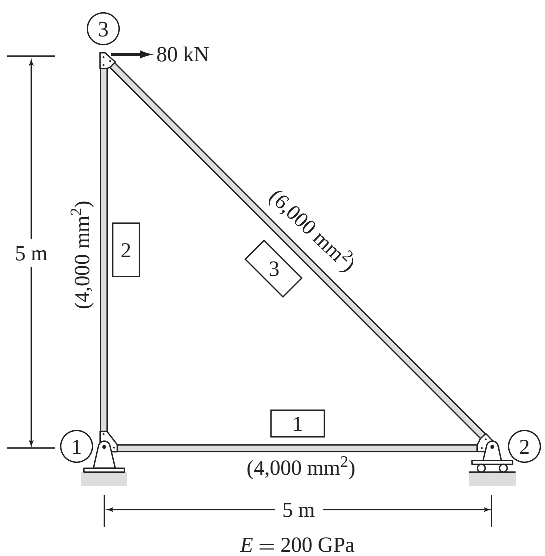


Fig. P3.20

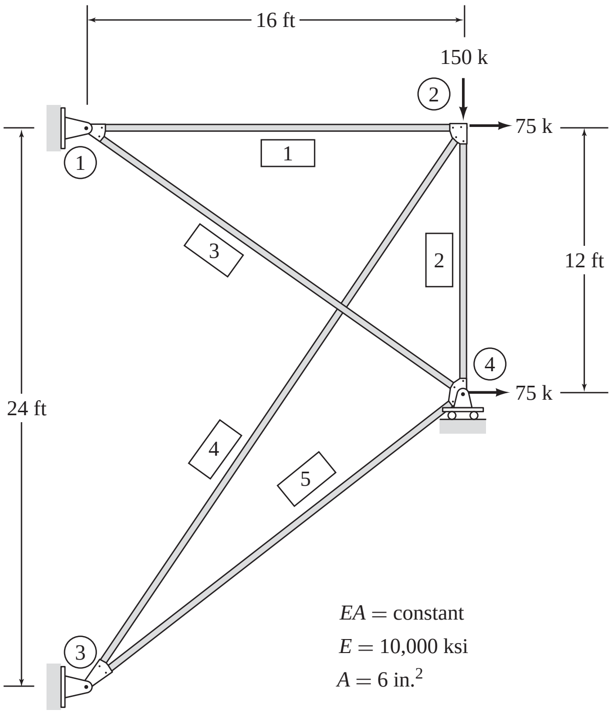


Fig. P3.21

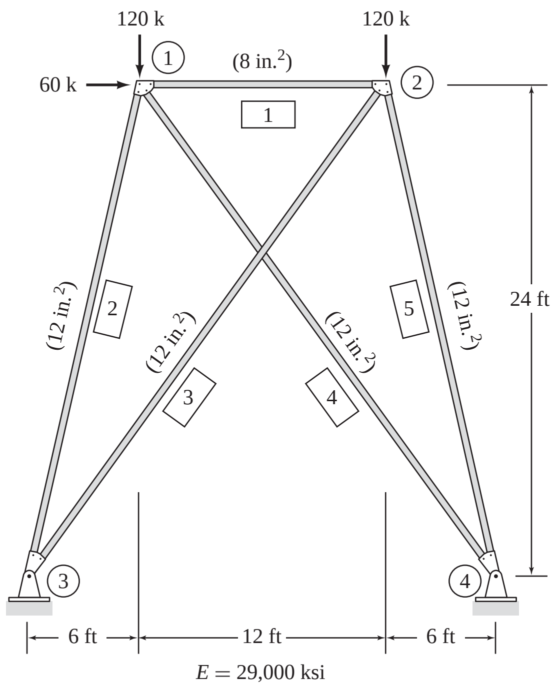


Fig. P3.22

3.26 a d 3.27 Using a structural analysis computer program, determine the joint displacements, member axial forces, and support reactions for the Fink roof truss and the Baltimore bridge truss shown in Figs. P3.26 and P3.27, respectively. er-ify the computer-generated results by manually checking the equilibrium equations for the entire truss, and for its joints numbered 5,10 and 15.

3.28 a d 3.29 Using a structural analysis computer program, determine the largest value of the load parameter P that can be applied to the trusses shown in Figs. P3.28 and P3.29 without causing yielding and buckling of any of the members.

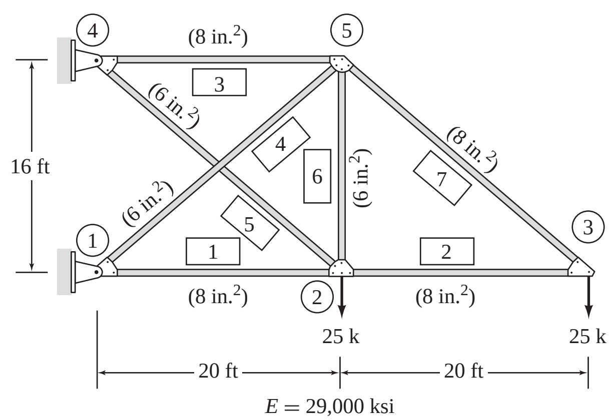


Fig. P3.24

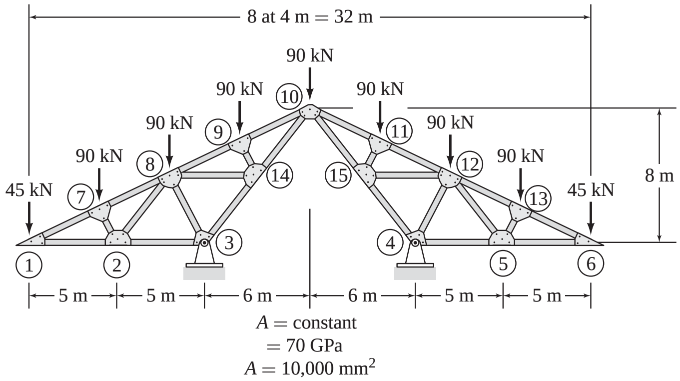


Fig. P3.26

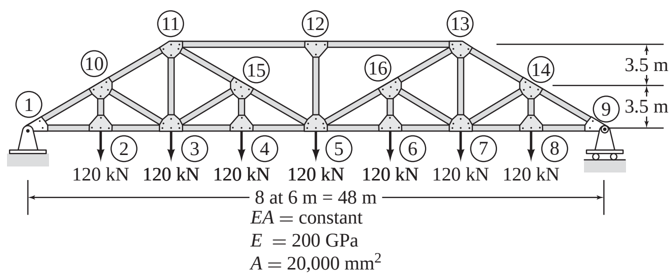
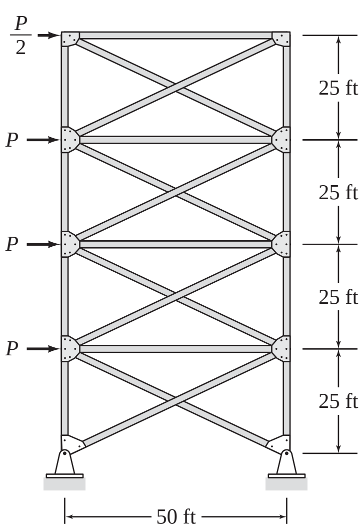
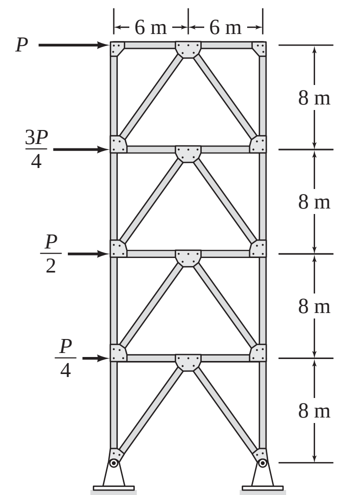


Fig. P3.27



E, A, I, σ_y = constant
E = 29,000 ksi A = 14.7 in.^{2}
Moment of Inertia: I = 400 in.^{4}
Yield Stress: σ_y = 36 ksi

Fig. P3.28



E, A, I, σ_y = constant
E = 200 GPa A = 14,600 mm^2
Moment of Inertia: I = 462(10^6) mm^4
Yield Stress: σ_y = 250 MPa

Fig. P3.29

4 COMPUTER PROGRAM FOR ANALYSIS OF PLANE TRUSSES

Truss Bridge
(Capricornis Photographic Inc. / Shutterstock)

In the previous chapter, we studied the basic principles of the analysis of plane trusses by the matrix stiffness method. In this chapter, we consider the computer implementation of the foregoing method of analysis. Our objective is to develop a general computer program that can be used to analyze any statically determinate or indeterminate plane truss, of any arbitrary configuration, subjected to any system of joint loads.

From a programming viewpoint, it is generally convenient to divide a structural analysis program into two parts or modules: (a) *input module,* and (b) *analysis module* (Fig. 4.1). The input module reads, and stores into the computer's memory, the structural and loading data necessary for the analysis; the analysis module uses the input data to perform the analysis, and communicates the results back to the user via an output device, such as a printer or a monitor. The development of a relatively simple input module is presented in Section 4.1; in the following five sections (4.2 through 4.6), we consider programming of the five analysis steps discussed in Chapter 3 (Fig. 3.20). The topics covered in these sections are as follows: assignment of the degree-of-freedom and restrained coordinate numbers for plane trusses (Section 4.2); generation of the structure stiffness matrix by assembling the elements of the member stiffness matrices (Section 4.3); formation of the joint load vector (Section 4.4); solution of the structure stiffness equations to obtain joint displacements (Section 4.5); and, finally, evaluation of the member axial forces and support reactions (Section 4.6).

The entire programming process is described by means of detailed flowcharts, so that readers can write this computer program in any programming language. It is important to realize that the programming process presented in this chapter represents only one of many ways in which the matrix stiffness method of analysis can be implemented on computers. Readers are strongly encouraged to conceive, and attempt, alternative strategies that can make the computer implementation (and/or application of the method) more efficient. One such strategy, which takes advantage of the banded form of the structure stiffness matrix, will be discussed in Chapter 9.

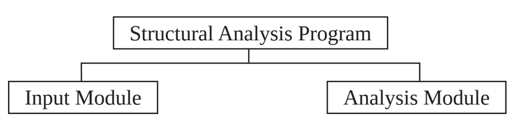


Fig. 4.1

4.1 DATA INPUT

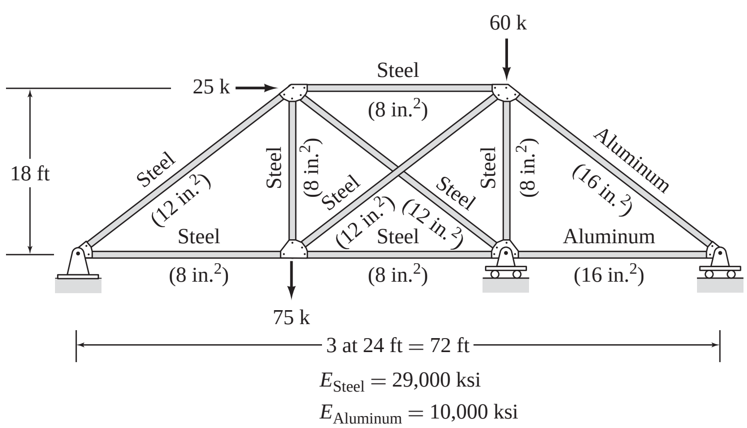
In this section, we focus our attention on the input module of our computer program. As stated previously, *the input module of a structural analysis program reads the structural and loading data necessary for analysis from a file or another type of input device, and stores it in the computer's memory so that it can be processed conveniently by the program for structural analysis.*

When structural analysis is carried out by hand calculations (e.g., as in Chapter 3), the information needed for the analysis is obtained by visually inspecting the analytical model of the structure (as represented by the line diagram). In computerized structural analysis, however, all of the data necessary for analysis must be specified in the form of numbers, and must be organized in the computer's memory in the form of matrices (arrays), in such a way that it can be used for analysis without any reference to a visual image (or line diagram) of the structure. This data in numerical form must completely and uniquely define the analytical model of the structure. In other words, a person with no knowledge of the actual structure or its analytical model should be able to reconstruct the visual analytical model of the structure, using only the numerical data and the knowledge of how this data is organized.

The input data necessary for the analysis of plane trusses can be divided into the following six categories:

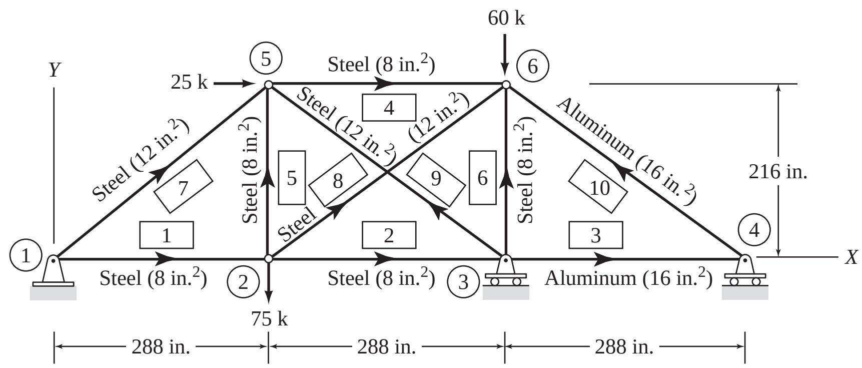
- joint data
- support data
- material property data
- cross-sectional property data
- member data
- load data

In the following, we discuss procedures for inputting data belonging to each of the foregoing categories, using the truss of Fig. 4.2(a) as an example. The analytical model of this truss is depicted in Fig. 4.2(b). Note that all the information in this figure is given in units of kips and inches. This is because we plan to design a computer program that can work with any *consistent* set of units. Thus, all the data must be converted into a consistent set of units before being input into the program.



(a) Actual Truss

Fig. 4.2



$E_{\text{Steel}} = 29{,}000$ ksi

$E_{\text{Aluminum}} = 10{,}000$ ksi

(b) Analytical Model

$$\begin{array}{cccc} & X \text{ coordinate} & Y \text{ coordinate} & \\ \mathbf{COORD} = & \left[\begin{array}{c} 0 \\ 288 \\ 576 \\ 864 \\ 288 \\ 576 \end{array}\right. & \left.\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 216 \\ 216 \end{array}\right] & \begin{array}{l} \leftarrow \text{Joint 1} \\ \leftarrow \text{Joint 2} \\ \leftarrow \text{Joint 3} \\ \leftarrow \text{Joint 4} \\ \leftarrow \text{Joint 5} \\ \leftarrow \text{Joint 6} \end{array} \end{array}$$

$NJ \times 2$

(c) Joint Coordinate Matrix

$$\mathbf{CP} = \begin{bmatrix} 8 \\ 12 \\ 16 \end{bmatrix} \begin{array}{l} \leftarrow \text{Cross-section type no. 1} \\ \leftarrow \text{Cross-section type no. 2} \\ \leftarrow \text{Cross-section type no. 3} \end{array}$$

$NCP \times 1$

(f) Cross-sectional Property Vector

$$\begin{array}{cl} \begin{array}{cccc} \text{Beginning joint} & \text{End joint} & \text{Material no.} & \text{Cross-section type no.} \end{array} & \\ \mathbf{MPRP} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 3 & 4 & 2 & 3 \\ 5 & 6 & 1 & 1 \\ 2 & 5 & 1 & 1 \\ 3 & 6 & 1 & 1 \\ 1 & 5 & 1 & 2 \\ 2 & 6 & 1 & 2 \\ 3 & 5 & 1 & 2 \\ 4 & 6 & 2 & 3 \end{bmatrix} & \begin{array}{l} \leftarrow \text{Member 1} \\ \leftarrow \text{Member 2} \\ \leftarrow \text{Member 3} \\ \leftarrow \text{Member 4} \\ \leftarrow \text{Member 5} \\ \leftarrow \text{Member 6} \\ \leftarrow \text{Member 7} \\ \leftarrow \text{Member 8} \\ \leftarrow \text{Member 9} \\ \leftarrow \text{Member 10} \end{array} \end{array}$$

$NM \times 4$

(g) Member Data Matrix

$$\begin{array}{c} \begin{array}{ccc} \text{Joint number} & \text{Restraint in } X \text{ direction } (0 = \text{free}, 1 = \text{restrained}) & \text{Restraint in } Y \text{ direction } (0 = \text{free}, 1 = \text{restrained}) \end{array} \\ \mathbf{MSUP} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & 1 \\ 4 & 0 & 1 \end{bmatrix} \end{array}$$

$NS \times (NCJT + 1)$

(d) Support Data Matrix

$$\mathbf{EM} = \begin{bmatrix} 29000 \\ 10000 \end{bmatrix} \begin{array}{l} \leftarrow \text{Material no. 1} \\ \leftarrow \text{Material no. 2} \end{array}$$

$NMP \times 1$

(e) Elastic Modulus Vector

$$\begin{array}{ccc} \text{Joint number} & & \text{Force in } X \text{ direction} \quad \text{Force in } Y \text{ direction} \\ \mathbf{JP} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} & & \mathbf{PJ} = \begin{bmatrix} 0 & -75 \\ 25 & 0 \\ 0 & -60 \end{bmatrix} \\ NJL \times 1 & & NJL \times NCJT \end{array}$$

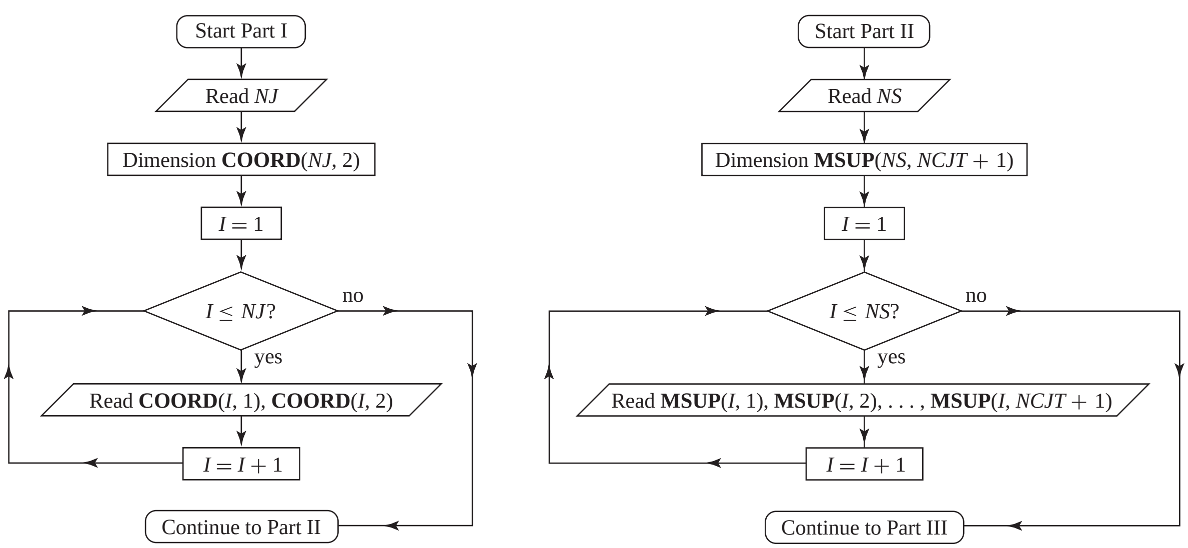
(h) Load Data Matrices

Fig. 4.2 (*continued*)

Joint Data

The joint data consists of: (a) the total number of joints (*N*) of the truss, and (b) the global (*X* and *Y*) coordinates of each joint. The relative positions of the joints of the truss are specified by means of the global (*X* and *Y*) coordinates of the joints. These joint coordinates are usually stored in the computer's memory in the form of a matrix, so that they can be accessed easily by the computer program for analysis. In our program, we store the joint coordinates in a matrix **COO D** of the order $N \times 2$ (Fig. 4.2(c)). The matrix, which is referred to as the *joint coordinate matrix,* has two columns, and its number of rows equals the total number of joints (*N*) of the structure. The *X* and *Y* coordinates of a joint *i* are stored in the first and second columns, respectively, of the *i*th row of the matrix **COO D**. Thus, for the truss of Fig. 4.2(b) (which has six joints), the joint coordinate matrix is a 6×2 matrix, as shown in Fig. 4.2(c). Note that the joint coordinates are stored in the sequential order of joint numbers. Thus, by comparing Figs. 4.2(b) and (c), we can see that the *X* and *Y* coordinates of joint 1 (i.e., 0 and 0) are stored in the first and second columns, respectively, of the first row of **COO D**. Similarly, the *X* and *Y* coordinates of joint 5 (288 and 216) are stored in the first and second columns, respectively, of the fifth row of **COO D**, and so on.

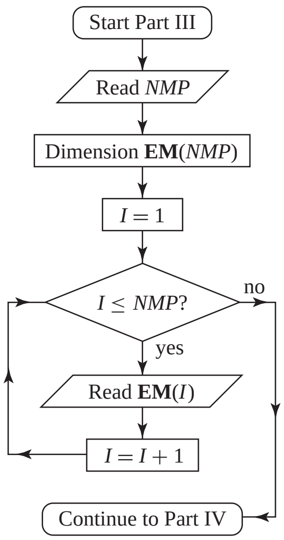
A flowchart for programming the reading and storing of the joint data for plane trusses is given in Fig. 4.3(a). As shown there, the program first reads the value of the integer variable *N* , which represents the total number of joints of the truss. Then, using a *Do Loop* command, the *X* and *Y* coordinates of each joint are read, and stored in the first and second columns, respectively, of the matrix **COO D**. The *Do Loop* starts with joint number 1 and ends with joint



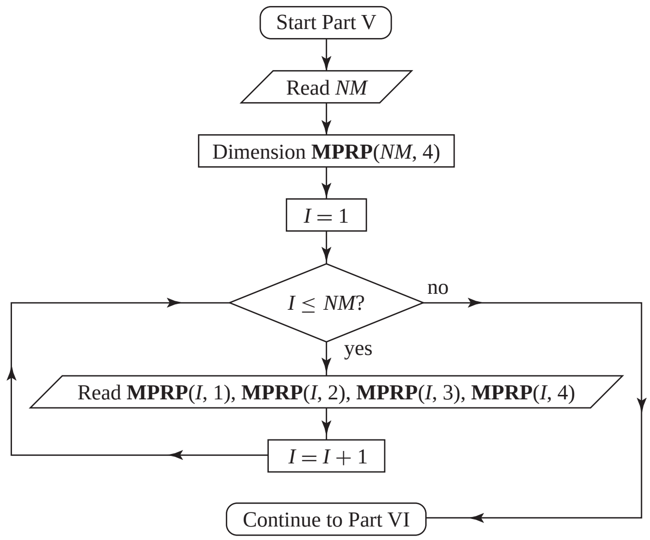
(a) Flowchart for Reading and Storing Joint Data

(b) Flowchart for Reading and Storing Support Data

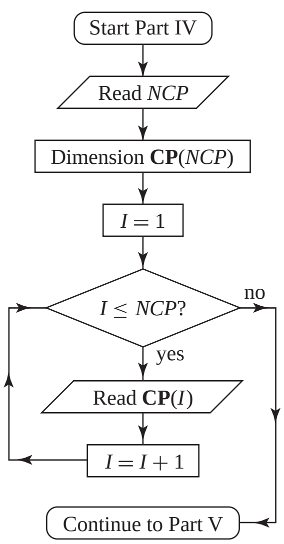
Fig. 4.3



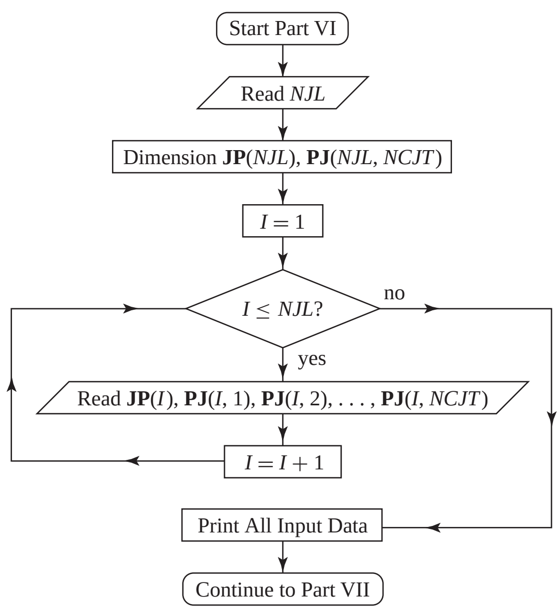
(c) Flowchart for Reading and Storing Material Property Data



(e) Flowchart for Reading and Storing Member Data



(d) Flowchart for Reading and Storing Cross-sectional Property Data



(f) Flowchart for Reading and Storing Load Data

Fig. 4.3 (*continued*)

number N. It should be noted that, depending upon the type of programming language and/or compiler being used, some additional statements (such as variable type declaration and formatted read/write statements) may be needed to implement the foregoing program. (It is assumed herein that the reader has a working knowledge of a programming language.)

The input data to be read by the computer program is either entered interactively by the user (responding to prompts on the screen), or is supplied in the form of a data file. The former approach is used in the computer software which can be downloaded from the publisher's website for this book. However, the latter approach is recommended for beginning programmers, because it is straightforward and requires significantly less programming. As an example, the input data file (in free-format) for the truss of Fig. 4.2(b) is given in Fig. 4.4. Note that the first line

6	
0, 0	
288, 0	
576, 0	Joint data
864, 0	
288, 216	
576, 216	
3	
1, 1, 1	
3, 0, 1	Support data
4, 0, 1	
2	
29000	Material property data
10000	
3	
8	Cross-sectional property data
12	
16	
10	
1, 2, 1, 1	
2, 3, 1, 1	
3, 4, 2, 3	
5, 6, 1, 1	
2, 5, 1, 1	Member data
3, 6, 1, 1	
1, 5, 1, 2	
2, 6, 1, 2	
3, 5, 1, 2	
4, 6, 2, 3	
3	
2, 0, −75	Joint load data
5, 25, 0	
6, 0, −60	

Fig. 4.4 *An xample of an Input Data File*

of this data file contains the total number of joints of the truss (i.e., 6); the next six lines contain the X and Y coordinates of joints 1 through 6, respectively.

Support Data

The support data consists of (a) the number of joints that are attached to supports (NS); and (b) the joint number, and the directions of restraints, for each support joint. Since there can be at most two restrained coordinates at a joint of a plane truss (i.e., $NCJT = 2$), the restraints at a support joint of such a structure can be conveniently specified by using a two-digit code in which each digit is either a 0 or a 1. The first digit of the code represents the restraint condition at the joint in the global X direction; it is 0 if the joint is free to translate in the X direction, or it is 1 if the joint is restrained in the X direction. Similarly, the second digit of the code represents the restraint condition at the joint in the global Y direction; a 0 indicates that the joint is unrestrained in the Y direction, and a 1 indicates that it is restrained. The restraint codes for the various types of supports for plane trusses are given in Fig. 4.5. (The special case of inclined roller supports will be considered in Chapter 9.)

Considering again the example truss of Fig. 4.2(b), we can see that joint 1 is attached to a hinged support that prevents it from translating in any direction.

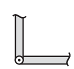
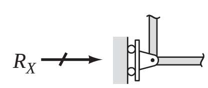
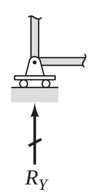
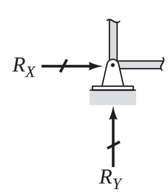
Type of Support		Restraint Code
Free joint (no support)		0, 0
Roller with horizontal reaction		1, 0
Roller with vertical reaction		0, 1
Hinge		1, 1

Fig. 4.5 *Restraint Codes for Plane Trusses*

Thus, the restraint code for joint 1 is 1,1 indicating that this joint is restrained from translating in both the X and Y directions. Similarly, the restraint codes for joints 3 and 4, which are attached to roller supports, are 0,1 because these joints are free to translate in the horizontal (X) direction, but are restrained by the rollers from translating in the vertical (Y) direction. The restraint codes of the remaining joints of the truss, which are free to translate in any direction, can be considered to be 0,0. However, it is not necessary to input codes for free joints, because the computer program considers every joint to be free, unless it is identified as a support joint.

The support data can be stored in the computer's memory in the form of an integer matrix **P** of order $NS \times (NC\ T + 1)$ (Fig. 4.2(d)). For plane trusses, because $NC\ T = 2$, the *support data matrix* **P** consists of three columns, with the number of rows equal to the number of support joints (NS). In each row of **P**, the support joint number is stored in the first column, and the first and second digits of the corresponding restraint code are stored in the second and third columns, respectively. Thus, for the truss of Fig. 4.2(b), which has three support joints, the support data matrix is a 3×3 matrix, as shown in Fig. 4.2(d). Note that in the first row of **P** the support joint number 1 is stored in the first column, and the first and second digits of the restraint code for this joint (i.e., 1 and 1) are stored in the second and third columns, respectively. Similarly, the second row of **P** consists of the support joint number 3 in the first column, and the two digits of the corresponding restraint code (i.e., 0 and 1) in the second and third columns, respectively, and so on.

A flowchart for programming the reading and storing of the support data is given in Fig. 4.3(b), in which, as noted previously, the integer variable $NC\ T$ denotes the number of structure coordinates per joint. Like this flowchart, many parts of the computer program presented in this chapter are given in a general form in terms of the variable $NC\ T$, so that they can be conveniently incorporated into computer programs for analyzing other types of framed structures (e.g., beams and plane frames), which are considered in subsequent chapters. For example, as discussed in this section, by setting $NC\ T = 2$, the flowchart of Fig. 4.3(b) can be used to input support data for plane trusses; whereas, as discussed subsequently in Chapter 6, the same flowchart can be used to input support data for plane frames, provided $NC\ T$ is set equal to three.

An example of how the support data for a plane truss may appear in an input data file is given in Fig. 4.4.

Material Property Data

The material property data involves (a) the number of materials used in the structure (NMP), and (b) the modulus of elasticity ( ) of each material. The elastic moduli are stored by the program in an *elastic modulus vector* . The number of rows of equals the number of materials (NMP), with the elastic modulus of material i stored in the ith row of the vector (Fig. 4.2(e)).

Consider, for example, the truss of Fig. 4.2(b). The truss is composed of two materials; namely, steel and aluminum. We arbitrarily select the steel (= 29,000 ksi) to be material number 1, and the aluminum (= 10,000 ksi) to be material number 2. Thus, the elastic modulus vector, , of the truss consists of two rows, as shown in Fig. 4.2(e); the elastic modulus of material number 1 (i.e., 29,000) is stored in the first row of , and the elastic modulus of material number 2 (i.e., 10,000) is stored in the second row.

Figure 4.3(c) shows a flowchart for programming the reading and storing of the material property data; Fig. 4.4 illustrates how this type of data may appear in an input data file.

Cross-Sectional Property Data

The cross-sectional property data consists of (a) the number of different cross-section types used for the truss members (*NCP*); and (b) the cross-sectional area (*A*) for each cross-section type. The cross-sectional areas are stored by the program in a *cross-sectional property vector* **CP**. The number of rows of **CP** equals the number of cross-section types (*NCP*), with the area of cross-section *i* stored in the *i*th row of the vector (Fig. 4.2(f)).

For example, three types of member cross-sections are used for the truss of Fig. 4.2(b). We arbitrarily assign the numbers 1, 2, and 3 to the cross-sections with areas of 8, 12, and 16 in.^{2}, respectively. Thus, the cross-sectional property vector, **CP**, consists of three rows; areas of cross-section types 1, 2, and 3 are stored in rows 1, 2, and 3, respectively, as shown in Fig. 4.2(f).

A flowchart for reading and storing the cross-sectional property data into computer memory is given in Fig. 4.3(d); Fig. 4.4 shows an example of an input data file containing this type of data.

Member Data

The member data consists of (a) the total number of members (*NM*) of the truss; and (b) for each member, the beginning joint number, the end joint number, the material number, and the cross-section type number.

The member data can be stored in computer memory in the form of an integer *member data matrix,* **P P**, of order $NM \times 4$ (Fig. 4.2(g)). The information corresponding to a member *i* is stored in the *i*th row of **P P**; its beginning and end joint numbers are stored in the first and second columns, respectively, and the material and cross-section numbers are stored in the third and fourth columns, respectively.

For example, since the truss of Fig. 4.2(b) has 10 members, its member data matrix is a 10×4 matrix, as shown in Fig. 4.2(g). From Fig. 4.2(b), we can see that the beginning and end joints for member 1 are 1 and 2, respectively; the material and cross-section numbers for this member are 1 and 1, respectively. Thus, the numbers 1, 2, 1, and 1 are stored in columns 1 through 4, respectively, of the first row of **P P**, as shown in Fig. 4.2(g). Similarly, we see from Fig. 4.2(b) that the beginning joint, end joint, material, and cross-section numbers for member 3 are 3, 4, 2, and 3, respectively, and they are stored, respectively, in columns 1 through 4 of row 3 of **P P**, and so on.

Figure 4.3(e) shows a flowchart for programming the reading and storing of the member data. An example of how member data may appear in an input data file is given in Fig. 4.4.

Load Data

The load data involves (a) the number of joints that are subjected to external loads ($N\ L$); and (b) the joint number, and the magnitudes of the force components in the global X and Y directions, for each loaded joint. The numbers of the loaded joints are stored in an integer vector **P** of order $N\ L \times 1$; the corresponding load components in the X and Y directions are stored in the first and second columns, respectively, of a real matrix **P** of order $N\ L \times NC\ T$, with $NC\ T = 2$ for plane trusses (see Fig. 4.2(h)). Thus, for the example truss of Fig. 4.2(a), which has three joints (2, 5, and 6) that are subjected to loads, the *load data matrices*, **P** and **P** , are of orders 3×1 and 3×2, respectively, as shown in Fig. 4.2(h). The first row of **P** contains joint number 2; the loads in the X and Y directions at this joint (i.e., 0 and -75 k) are stored in the first and second columns, respectively, of the same row of **P** . The information about joints 5 and 6 is then stored in a similar manner in the second and third rows, respectively, of **P** and **P** , as shown in the figure.

A flowchart for programming the reading and storing of the load data is given in Fig. 4.3(f), in which $NC\ T$ must be set equal to 2 for plane trusses. Figure 4.4 shows the load data for the example truss in an input file.

It is important to recognize that the numerical data stored in the various matrices in Figs. 4.2(c) through (h) *completely* and *uniquely* defines the analytical model of the example truss, without any need to refer to the line diagram of the structure (Fig. 4.2(b)).

After all the input data has been read and stored in computer memory, it is considered a good practice to print this data directly from the matrices in the computer memory (or view it on the screen), so that its validity can be verified (Fig. 4.3(f)). An example of such a printout, showing the input data for the example truss of Fig. 4.2, is given in Fig. 4.6.

```
                  p  e        a e

              M
                  ec  d  d

                   a   a    a

              ==========================
               e e a     c   a   a a
              ==========================
P   ec      e:   g  e 4-2
    c   e   pe : P a e
    e           : 6
    e     Me  e   : 10
    e     Ma e  a  P  pe     e       : 2
    e          - ec    a  P  pe      e   : 3
```

Fig. 4.6 *A Sample Printout of Input Data*

```
                     d  a e

     .           d  a e             d  a e
    1          0.0000  00         0.0000  00
    2          2.8800  02         0.0000  00
    3          5.7600  02         0.0000  00
    4          8.6400  02         0.0000  00
    5          2.8800  02         2.1600  02
    6          5.7600  02         2.1600  02
                   pp

         .       e   a             e   a
    1              e                  e
    3                                 e
    4                                 e

               Ma e  a  P  pe   e

 Ma e  a       M d                 -e  c e
      .        a   c            he  a   xpa
    1        2.9000  04          0.0000  00
    2        1.0000  04          0.0000  00

            - ec    a  P  pe   e

          P  pe      .        ea
               1            8.0000  00
               2            1.2000  01
               3            1.6000  01

                  Me  e  a a

Me  e     eg     g     d  Ma e  a        - ec    a
    .                          .      P  pe      .
 1          1       2      1             1
 2          2       3      1             1
 3          3       4      2             3
 4          5       6      1             1
 5          2       5      1             1
 6          3       6      1             1
 7          1       5      1             2
 8          2       6      1             2
 9          3       5      1             2
10          4       6      2             3

                        ad

          .              ce               ce
     2               0.0000  00      -7.5000  01
     5               2.5000  01       0.0000  00
     6               0.0000  00      -6.0000  01
                 d      p    a a
```

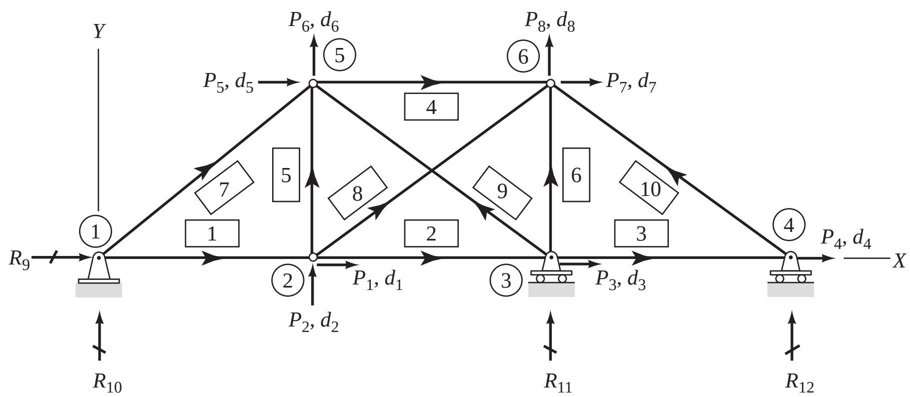
Fig. 4.6 (*continued*)

4.2 ASSIGNMENT OF STRUCTURE COORDINATE NUMBERS

Having completed the input module, we are now ready to develop the analysis module of our computer program. *The analysis module of a structural analysis program uses the input data stored in computer memory to calculate the desired response characteristics of the structure, and communicates these results to the user through an output device, such as a printer or a monitor*.

As discussed in Section 3.8, the first step of the analysis involves specification of the structure's degrees of freedom and restrained coordinates, which are collectively referred to as, simply, the *structure coordinates*. Recall that when the analysis was carried out by hand calculations (in Chapter 3), the structure coordinate numbers were written next to the arrows, in the global X and Y directions, drawn at the joints. In computerized analysis, however, these numbers must be organized in computer memory in the form of a matrix or a vector. In our program, the structure coordinate numbers are stored in an integer vector **C**, with the number of rows equal to the number of structure coordinates per joint ($NC\ T$) times the number of joints of the structure (N). For plane trusses, because $NC\ T = 2$, the number of rows of **C** equals twice the number of joints of the truss (i.e., $2N$). The structure coordinate numbers are arranged in **C** in the sequential order of joint numbers, with the number for the X coordinate at a joint followed by the number for its Y coordinate. In other words, the numbers for the X and Y structure coordinates at a joint i are stored in rows $(i-1)2+1$ and $(i-1)2+2$, respectively, of **C**. For example, the line diagram of the truss of Fig. 4.2(a) is depicted in Fig. 4.7(a) with its degrees of freedom and restrained coordinates indicated, and the corresponding 12×1 **C** vector is given in Fig. 4.7(b).

The procedure for assigning the structure coordinate (i.e., degrees of freedom and restrained coordinate) numbers was discussed in detail in Section 3.2.



(a) Line Diagram Showing Degrees of Freedom and Restrained Coordinates

Fig. 4.7

Structure coordinate number

$$\mathbf{NSC} = \begin{array}{c} \left[\begin{array}{c} 9 \\ 10 \\ \hline 1 \\ 2 \\ \hline 3 \\ 11 \\ \hline 4 \\ 12 \\ \hline 5 \\ 6 \\ \hline 7 \\ 8 \end{array}\right] \\ NCJT(NJ) \times 1 \end{array} \begin{array}{l} \leftarrow X \\ \leftarrow Y \\ \leftarrow X \\ \leftarrow Y \\ \leftarrow X \\ \leftarrow Y \\ \leftarrow X \\ \leftarrow Y \\ \leftarrow X \\ \leftarrow Y \\ \leftarrow X \\ \leftarrow Y \\ \end{array} \begin{array}{l} \left.\begin{array}{c} \\ \end{array}\right\} \text{Joint 1} \\ \left.\begin{array}{c} \\ \end{array}\right\} \text{Joint 2} \\ \left.\begin{array}{c} \\ \end{array}\right\} \text{Joint 3} \\ \left.\begin{array}{c} \\ \end{array}\right\} \text{Joint 4} \\ \left.\begin{array}{c} \\ \end{array}\right\} \text{Joint 5} \\ \left.\begin{array}{c} \\ \end{array}\right\} \text{Joint 6} \\ \end{array}$$

(b) Structure Coordinate Number Vector

Load magnitudes

Structure coordinate numbers

$$\mathbf{P} = \begin{array}{c} \left[\begin{array}{r} 0 \\ -75 \\ 0 \\ 0 \\ 25 \\ 0 \\ 0 \\ -60 \end{array}\right] \\ NDOF \times 1 \end{array} \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ \end{array}$$

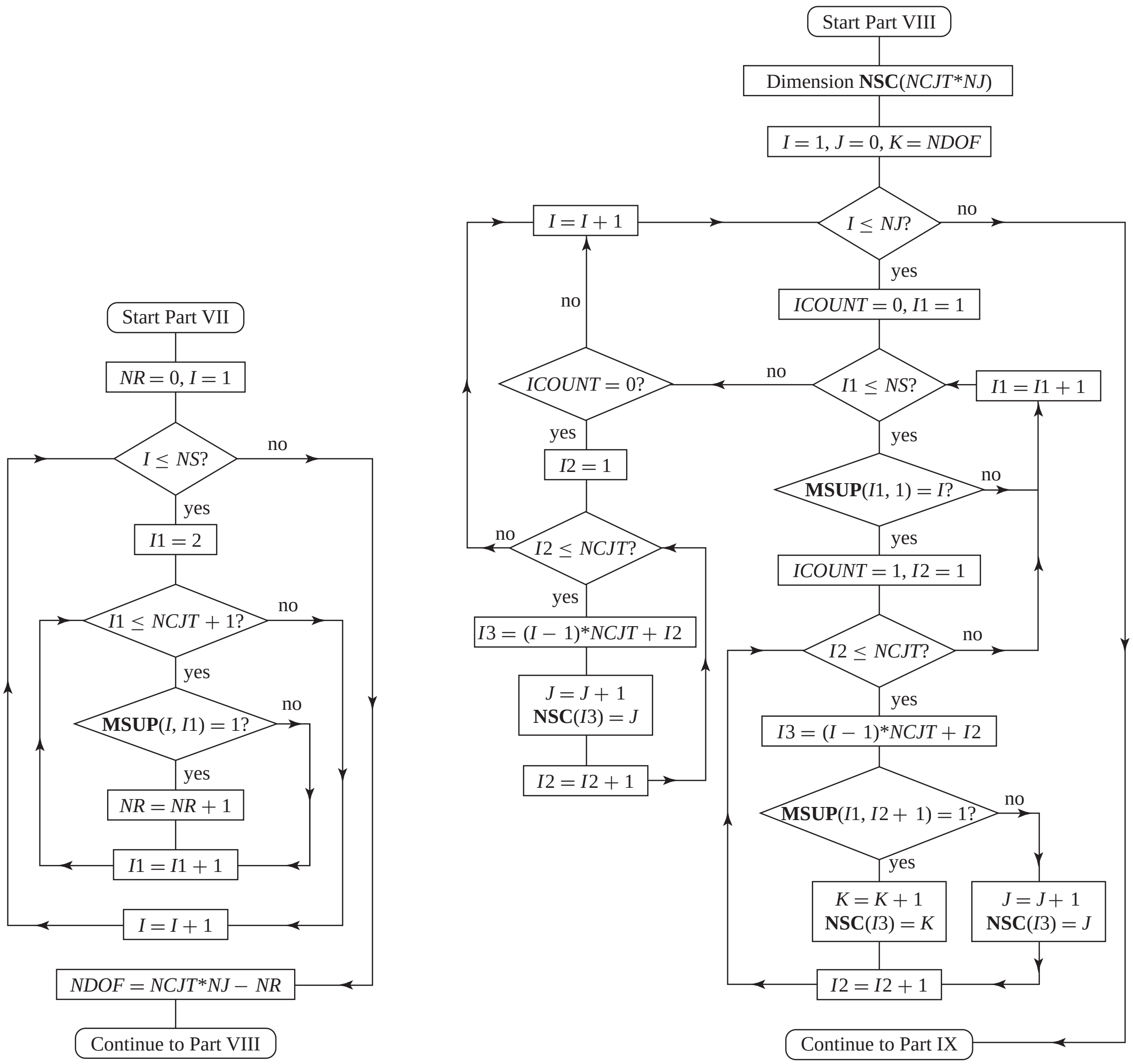
(c) Joint Load Vector

Fig. 4.7 (*continued*)

This procedure can be conveniently programmed using the flowcharts given in Fig. 4.8 on the next page. Figure 4.8(a) describes a program for determining the number of degrees of freedom and the number of restrained coordinates of the structure. (Note again that $NC\ \ T = 2$ for plane trusses.) The program first determines the number of restrained coordinates (NR) by simply counting the number of 1s in the second and third columns of the support data matrix **P**. Recall from our discussion of restraint codes in Section 4.1 that each 1 in the second or third column of **P** represents a restraint (in either the X or Y direction) at a joint of the structure. With NR known, the number of degrees of freedom ($NDOF$) is evaluated from the following relationship (Eq. (3.3)).

$$NDOF = 2(N\ \ ) - NR$$

For example, since the **P** matrix for the example truss, given in Fig. 4.2(d), contains four 1s in its second and third columns, the number of



(a) Flowchart for Determining Number of Degrees of Freedom

(b) Flowchart for Generating Structure Coordinate Numbers

Fig. 4.8

restrained coordinates of the truss is four. Furthermore, since the truss has six joints (Fig. 4.2(c)), its number of degrees of freedom equals

$$NDOF = 2(6) - 4 = 8$$

Once the number of degrees of freedom ($NDOF$) has been determined, the program generates the structure coordinate number vector **C**, as shown by the flowchart in Fig. 4.8(b). Again, *NC T* should be set equal to 2 for plane trusses. As this flowchart indicates, the program uses two integer counters, and , to keep track of the degrees-of-freedom and restrained coordinate numbers, respectively. The initial value of is set equal to 0, whereas the initial value of is set equal to *NDOF*.

The structure coordinates are numbered, one joint at a time, starting at joint 1 and proceeding sequentially to the joint number *N* . First, the number of the joint under consideration, *I*, is compared with the numbers in the first column of the support matrix **P** to determine whether or not *I* is a support joint. If a match is found between *I* and one of the numbers in the first column of **P**, then the counter *ICOUNT* is set equal to 1; otherwise, the value of *ICOUNT* remains 0 as initially assigned.

If joint *I* is not a support joint (i.e., $ICOUNT = 0$), then the number for its degree of freedom in the *X* direction is obtained by increasing the degrees-of-freedom counter by 1 (i.e., $= + 1$), and this value of is stored in row number $(I - 1)2 + 1$ of the **C** vector. Next, the value of is again increased by 1 (i.e., $= + 1$) to obtain the number for the degree of freedom of joint *I* in the *Y* direction, and the new value of is stored in row $(I - 1)2 + 2$ of the **C** vector.

If joint *I* is found to be a support joint (i.e., $ICOUNT = 1$), then the second column of the corresponding row of **P** is checked to determine whether joint *I* is restrained in the *X* direction. If the joint is restrained in the *X* direction, then the number for its *X*-restrained coordinate is obtained by increasing the restrained coordinate counter by 1 (i.e., $= + 1$), and this value of is stored in row $(I - 1)2 + 1$ of the **C** vector. However, if the joint is not restrained in the *X* direction, then the degrees-of-freedom counter is increased by 1, and its value (instead of that of) is stored in row $(I - 1)2 + 1$ of the **C** vector. Next, the restraint condition in the *Y* direction at joint *I* is determined by checking the third column of the corresponding row of **P**. If the joint is found to be restrained, then the counter is increased by 1; otherwise, the counter is increased by the same amount. The new value of either or is then stored in row $(I - 1)2 + 2$ of the **C** vector.

The computer program repeats the foregoing procedure for each joint of the structure to complete the structure coordinate number vector, **C**. As shown in Fig. 4.8(b), this part of the program (for generating structure coordinate numbers) can be conveniently coded using *Do Loop* or *For Next* types of programming statements.

4.3 GENERATION OF THE STRUCTURE STIFFNESS MATRIX

The structure coordinate number vector **C**, defined in the preceding section, can be used to conveniently determine the member code numbers needed to establish the structure stiffness matrix , without any reference to the visual image of the structure (e.g., the line diagram). The code numbers at the beginning of a member of a general framed structure are stored in the following rows of the **C** vector:

$$\left.\begin{array}{ll}\mathbf{C}\text{ row for the first code number} & = (B-1)NC\ T+1\\ \mathbf{C}\text{ row for the second code number} & = (B-1)NC\ T+2\\ \quad\vdots & \quad\vdots\\ \mathbf{C}\text{ row for the } NC\ T\text{th code number} & = (B-1)NC\ T+NC\ T\end{array}\right\}\quad\textbf{(4.1a)}$$

in which B is the beginning joint of the member. Similarly, the code numbers at the end of the member, connected to joint , can be obtained from the following rows of the **C**:

$$\left.\begin{array}{ll}\mathbf{C}\text{ row for the first code number} & = (\quad-1)NC\ T+1\\ \mathbf{C}\text{ row for the second code number} & = (\quad-1)NC\ T+2\\ \quad\vdots & \quad\vdots\\ \mathbf{C}\text{ row for the } NC\ T\text{th code number} & = (\quad-1)NC\ T+NC\ T\end{array}\right\}\quad\textbf{(4.1b)}$$

Suppose, for example, that we wish to determine the code numbers for member 9 of the truss of Fig. 4.2(b). First, from the member data matrix **P P** of this truss (Fig. 4.2(g)), we obtain the beginning and end joints for this member as 3 and 5, respectively. (This information is obtained from row 9, columns 1 and 2, respectively, of **P P**.) Next, we determine the row numbers of the **C** vector in which the structure coordinate numbers for joints 3 and 5 are stored. Thus, at the beginning of the member (Eq. (4.1a) with $NC\ T = 2$),

$$\begin{array}{ll}\mathbf{C}\text{ row for the first code number} & = (3-1)2+1 = 5\\ \mathbf{C}\text{ row for the second code number} & = (3-1)2+2 = 6\end{array}$$

Similarly, at the end of the member (Eq. (4.1b)),

$$\begin{array}{ll}\mathbf{C}\text{ row for the first code number} & = (5-1)2+1 = 9\\ \mathbf{C}\text{ row for the second code number} & = (5-1)2+2 = 10\end{array}$$

The foregoing calculations indicate that the code numbers for member 9 are stored in rows 5, 6, 9, and 10 of the **C** vector. Thus, from the appropriate rows of the **C** vector of the truss given in Fig. 4.7(b), we obtain the member's code numbers to be 3, 11, 5, 6. A visual check of the truss's line diagram in Fig. 4.7(a) indicates that these code numbers are indeed correct.

The procedure for forming the structure stiffness matrix by assembling the elements of the member global stiffness matrices was discussed in detail in Sections 3.7 and 3.8. A flowchart for programming this procedure is presented in Fig. 4.9, in which $NC\ T$ should be set equal to 2 for plane trusses. As indicated by the flowchart, this part of our computer program begins by initializing all the elements of the matrix to 0. The assembly of the structure stiffness matrix is then carried out using a *Do Loop*, in which the following operations are performed for each member of the structure.

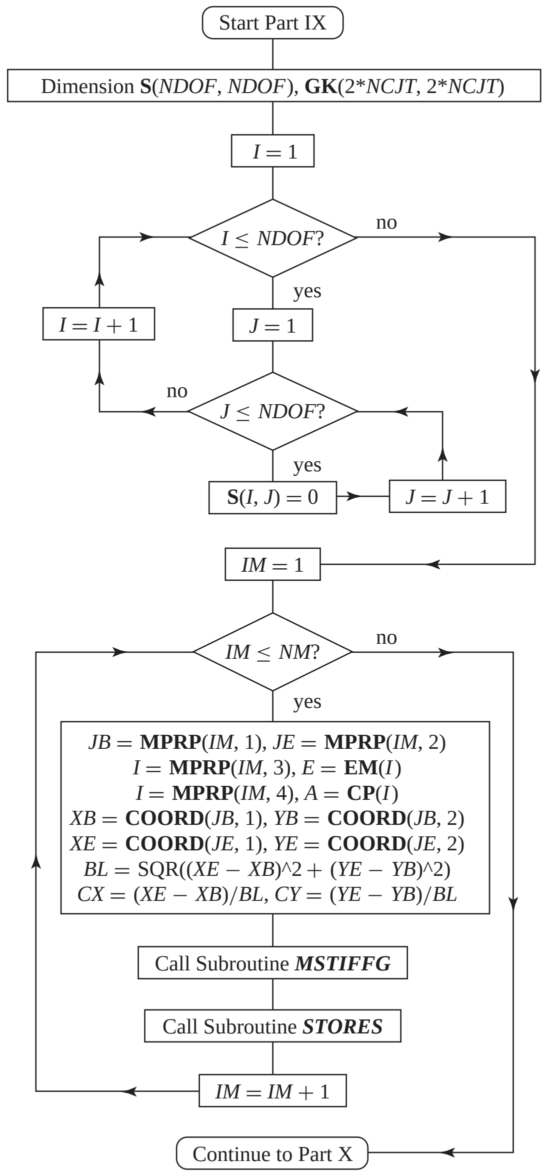


Fig. 4.9 *Flowchart for Generating Structure Stiffness Matrix for Plane Trusses*

1. *valuation of member properties.* For the member under consideration, *IM*, the program reads the beginning joint number, *B*, and the end joint number, , from the first and second columns, respectively, of the member data matrix **P P**. Next, the material property number is read from the third column of **P P**, and the corresponding value of the modulus of elasticity, , is obtained from the elastic modulus vector . The program then reads the number of the member cross-section type from the fourth column of **P P**,

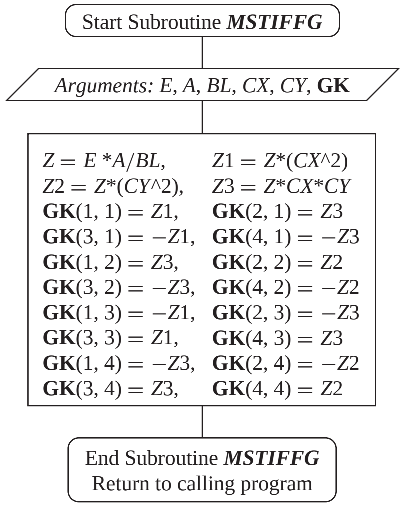


Fig. 4.10 *Flowchart of Subroutine **MSTIFFG** for Determining Member Global Stiffness Matrix for Plane Trusses*

and obtains the corresponding value of the cross-sectional area, A, from the cross-sectional property vector **CP**. Finally, the X and Y coordinates of the beginning joint B and the end joint are obtained from the joint coordinate matrix **COO D**, and the member's length, BL, and its direction cosines, CX ($= \cos\theta$) and CY ($= \sin\theta$), are calculated using Eqs. (3.62).

2. *Determination of member global stiffness matrix* **G** (=) *by subroutine **MSTIFFG**.* After the necessary properties of the member under consideration, IM, have been evaluated, the program calls on the subroutine ***MSTIFFG*** to form the member stiffness matrix in the global coordinate system. (A flowchart of this subroutine is shown in Fig. 4.10.) Note that in the computer program, the member global stiffness matrix is named **G** (instead of) to indicate that it is a real (not an integer) matrix. As the flowchart in Fig. 4.10 indicates, the subroutine simply calculates the values of the various stiffness coefficients, and stores them into appropriate elements of the **G** matrix, in accordance with Eq. (3.73).

3. *Storage of the elements of member global stiffness matrix* **G** *into structure stiffness matrix by subroutine **STORES**.* Once the matrix **G** has been determined for the member under consideration, IM, the program (Fig. 4.9) calls the subroutine ***STORES*** to store the pertinent elements of **G** in their proper positions in the structure stiffness matrix . A flowchart of this subroutine, which essentially consists of two nested *Do Loops*, is given in Fig. 4.11. As this flowchart indicates, the outer *Do Loop* performs the following operations sequentially for each row of the **G** matrix, starting with row 1 and ending with row 2($NC\ T$): (a) the member code number $N1$ corresponding to the row under consideration, I, is obtained from the **C** vector using the procedure discussed previously in this

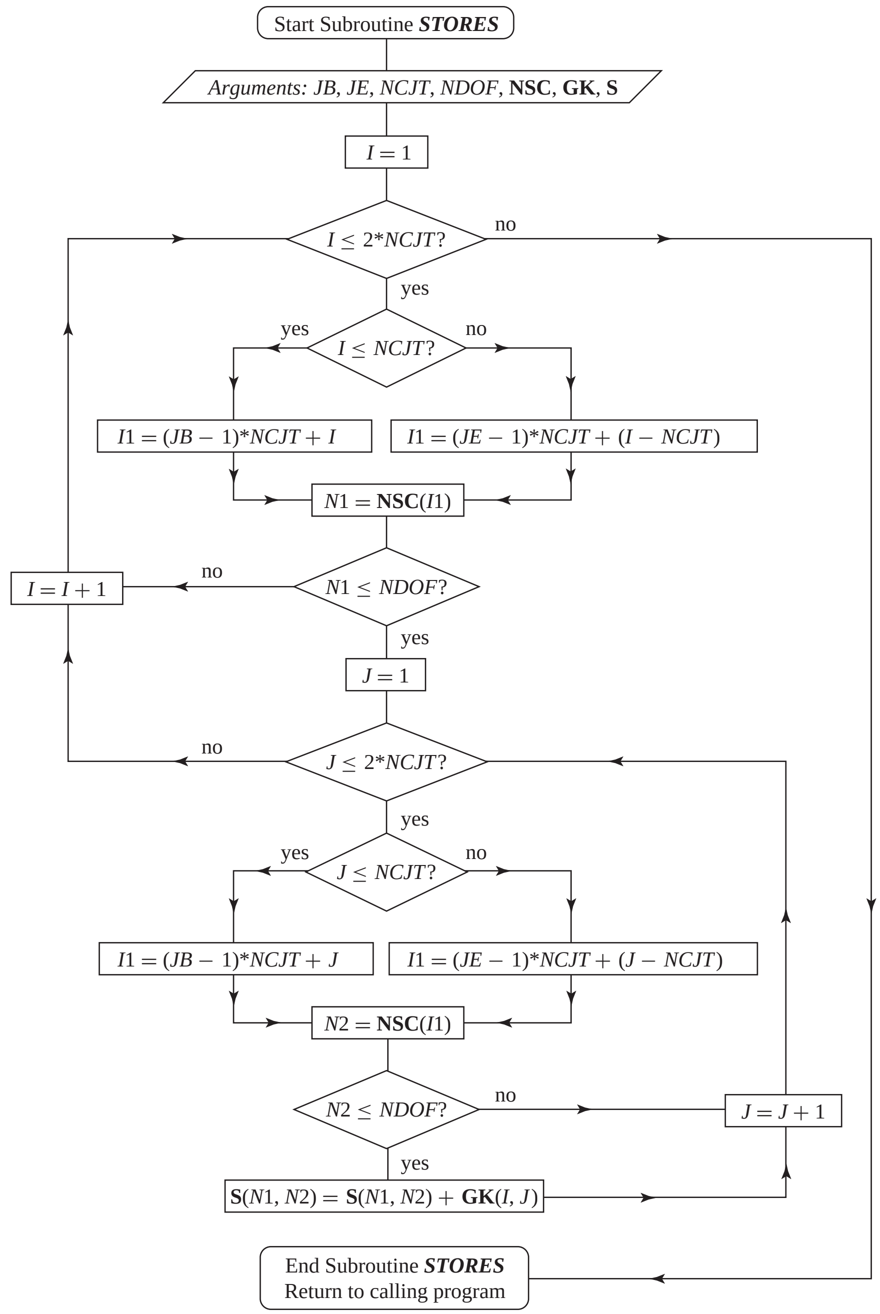


Fig. 4.11 *Flowchart of Subroutine **STORES** for Storing Member Global Stiffness Matrix in Structure Stiffness Matrix*

section; (b) if $N1$ is less than or equal to *NDOF*, then the inner *Do Loop* is activated; otherwise, the inner loop is skipped; and (c) the row number I is increased by 1, and steps (a) through (c) are repeated. The inner loop, activated from the outer loop, performs the following operations sequentially for each column of the **G** matrix, starting with column 1 and ending with column 2(*NC T*): (a) the member code number $N2$ corresponding to the column under consideration, , is obtained from the **C** vector; (b) if $N2$ is less than or equal to *NDOF*, then the value of the element in the Ith row and th column of **G** is added to the value of the element in the $N1$th row and $N2$th column of ; otherwise, no action is taken; and (c) the column number is increased by 1, and steps (a) through (c) are repeated. The inner loop ends when its steps (a) and (b) have been applied to all the columns of **G** ; the program control is then returned to step (c) of the outer loop. The subroutine ***STORES*** ends when steps (a) and (b) of the outer loop have been applied to all the rows of the **G** matrix, thereby storing all the pertinent elements of the global stiffness matrix of the member under consideration, *IM*, in their proper positions in the structure stiffness matrix .

Refocusing our attention on Fig. 4.9, we can see that formation of the structure stiffness matrix is complete when the three operations, described in the foregoing paragraphs, have been performed for each member of the structure.

4.4 FORMATION OF THE JOINT LOAD VECTOR

In this section, we consider the programming of the next analysis step, which involves formation of the joint load vector **P**. A flowchart for programming this process is shown in Fig. 4.12. Again, when analyzing plane trusses, the value of *NC T* should be set equal to 2 in the program. It is seen from the figure that this part of our computer program begins by initializing each element of **P** to 0. The program then generates the load vector **P** by performing the following operations for each row of the load data vector **P**, starting with row 1 and proceeding sequentially to row *N L*:

1. For the row under consideration, I, the number of the loaded joint $I1$ is read from the **P** vector.

2. The number of the X structure coordinate, N, at joint $I1$ is obtained from row $I2 = (I1 - 1)2 + 1$ of the **C** vector. If $N \leq NDOF$, then the value of the element in the Ith row and the first column of the load data matrix **P** (i.e., the X load component) is added to the Nth row of the load vector **P**; otherwise, no action is taken.

3. The **C** row number $I2$ is increased by 1 (i.e., $I2 = I2 + 1$), and the structure coordinate number, N, of the Y coordinate is read from the **C**. If $N \leq NDOF$, then the value of the element in the Ith row and the second column of **P** (i.e., the Y load component) is added to the Nth row of **P**; otherwise, no action is taken.

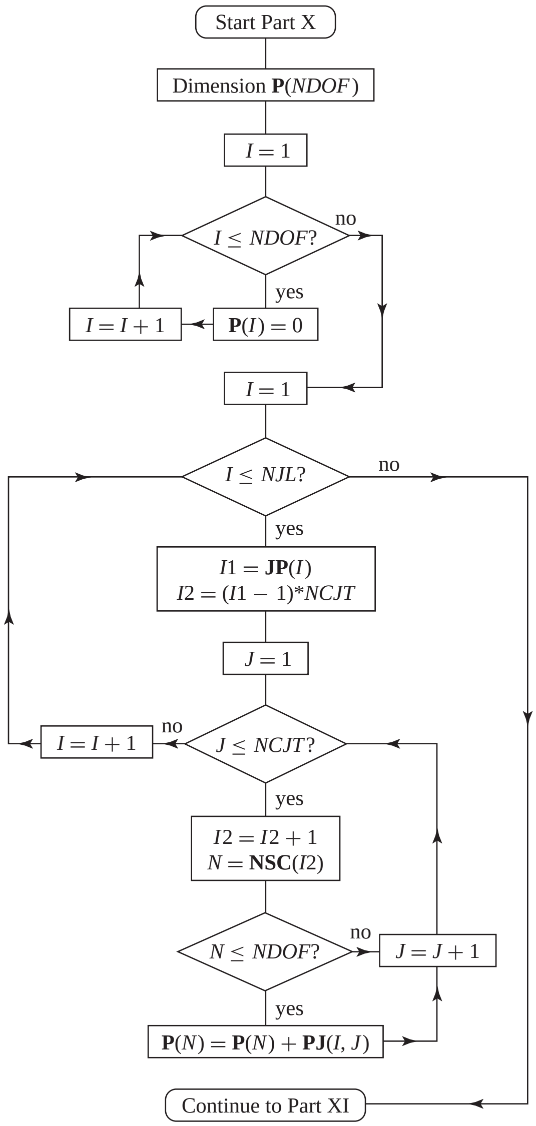


Fig. 4.12 *Flowchart for Forming oint Load Vector*

The foregoing operations are repeated for each loaded joint of the structure to complete the joint load vector **P**.

To illustrate this procedure, let us form the joint load vector **P** for the example truss of Fig. 4.2(a) without referring to its visual image or line diagram (i.e., using only the input data matrices and the **C** vector). Recall that in Section 4.2, using the **P** matrix, we determined that the number of degrees of freedom of this structure equals 8. Thus, the joint load vector **P** for the truss must be of order 8×1.

We begin generating **P** by focusing our attention on row 1 (i.e., $I = 1$) of the load data vector **P** (Fig. 4.2(h)), from which we determine the number of the first loaded joint, $I1$, to be 2. We then determine the row of the **C** in which the number of the X structure coordinate at joint 2 is stored, using the following relationship:

$$I2 = (2 - 1)2 + 1 = 3$$

From row 3 of the **C** vector given in Fig. 4.7(b), we read the number of the structure coordinate under consideration as 1 (i.e., $N = 1$). This indicates that the force component in the first row and first column of the load data matrix **P** (i.e., the X component of the load acting at joint 2) must be stored in the first row of **P**; that is, **P**$(1) = 0$. Next, we increase $I2$ by 1 (i.e., $I2 = 4$) and, from row 4 of the **C**, we find the number of the Y structure coordinate at the joint to be 2 (i.e., $N = 2$). This indicates that the load component in the first row and second column of **P** is to be stored in the second row of **P**; that is, **P**$(2) = -75$.

Having stored the loads acting at joint 2 in the load vector **P**, we now focus our attention on the second row of **P** (i.e., $I = 2$), and read the number of the next loaded joint, $I1$, as 5. We then determine the **C** row where the number of the X structure coordinate at joint 5 is stored as

$$I2 = (5 - 1)2 + 1 = 9$$

From row 9 of the **C** (Fig. 4.7(b)), we find the number of the structure coordinate under consideration to be 5 (i.e., $N = 5$). Thus, the force component in row 2 and column 1 of **P** must be stored in row 5 of **P**; or **P**$(5) = 25$. Next, we increase $I2$ by 1 to 10, and from row 10 of the **C** read the structure coordinate number, N, as 6. Thus, the load component in the second row and second column of **P** is stored in the sixth row of **P**; or **P**$(6) = 0$.

Finally, by repeating the foregoing procedure for row 3 of **P**, we store the X and Y force components at joint 6 in rows 7 and 8, respectively, of **P**. The completed joint load vector **P** thus obtained is shown in Fig. 4.7(c).

4.5 SOLUTION FOR JOINT DISPLACEMENTS

Having programmed the generation of the structure stiffness matrix and the joint load vector **P**, we now proceed to the next part of our computer program, which calculates the joint displacements, **d**, by solving the structure stiffness relationship, **d** = **P** (Eq. (3.89)). A flowchart for programming this analysis step is depicted in Fig. 4.13. The program solves the system of simultaneous equations, representing the stiffness relationship, **d** = **P**, using the Gauss–Jordan elimination method discussed in Section 2.4.

It should be recognized that the program for the calculation of joint displacements, as presented in Fig. 4.13, involves essentially the same operations as the program for the solution of simultaneous equations given in Fig. 2.2. However, in the previous program (Fig. 2.2), the elementary operations were

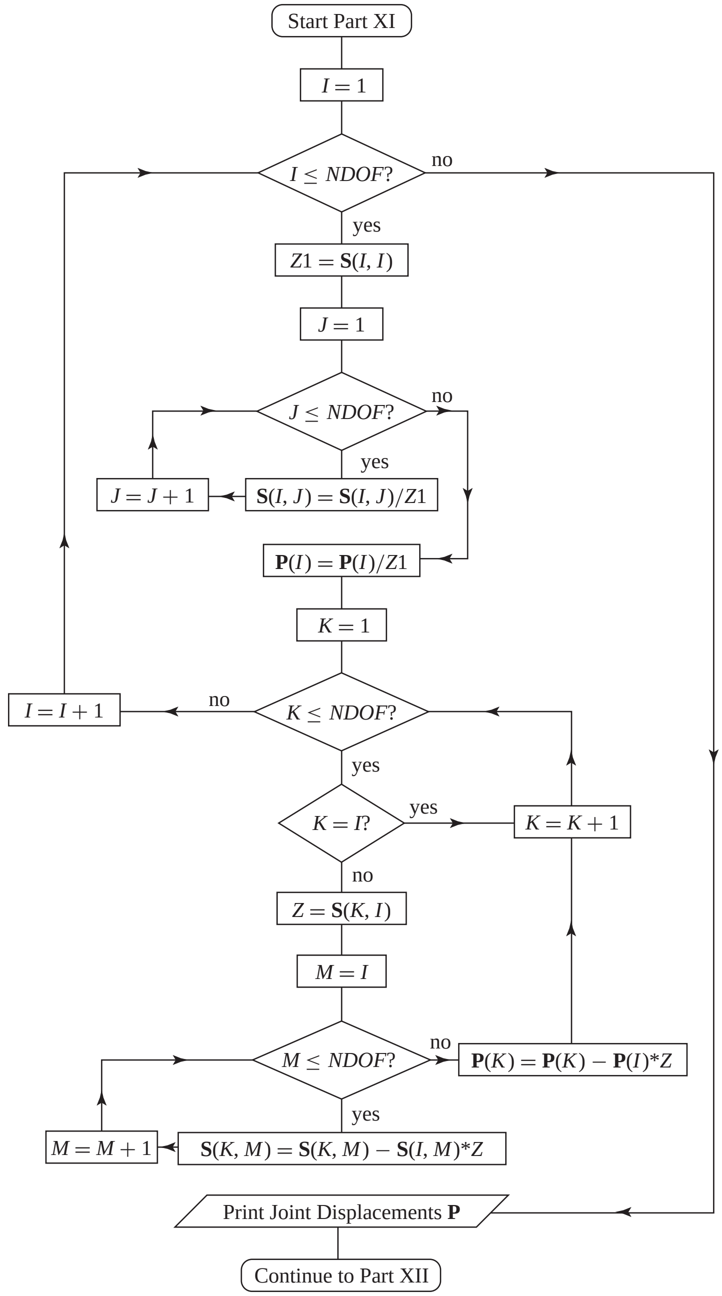


Fig. 4.13 *Flowchart for Calculation of oint Displacements by Gauss ordan Method*

applied to an augmented matrix; in the present program (Fig. 4.13), to save space in computer memory, no augmented matrix is formed, and the elementary operations are applied directly to the structure stiffness matrix and the joint load vector **P**. Thus, at the end of the Gauss–Jordan elimination process, the matrix is reduced to a unit matrix, and the **P** vector contains values of the joint displacements. In the rest of our computer program, therefore, **P** (instead of **d**) is considered to be the joint displacement vector. The joint displacements thus obtained can be communicated to the user through a printout or on the screen.

4.6 CALCULATION OF MEMBER FORCES AND SUPPORT REACTIONS

In this section, we consider programming of the final analysis step, which involves calculation of the member forces and support reactions. A flowchart for programming this analysis step is presented in Fig. 4.14, with $NC\ \ T = 2$ for plane trusses. As shown there, this part of our computer program begins by initializing each element of the reaction vector, , to 0. The member forces and support reactions are then determined by performing the following operations for each member of the structure, via a *Do Loop*.

1. *valuation of member properties.* For the member under consideration, *IM*, the program reads the beginning joint number *B*, the end joint number , the modulus of elasticity , the cross-sectional area *A*, and the *X* and *Y* coordinates of the beginning and end joints. It then calculates the member length, *BL*, and direction cosines, *CX* (= cos θ) and *CY* (= sin θ), using Eqs. (3.62).

2. *valuation of member global end displacements* **(=)** *by subroutine* ***MDISPG***. After the properties of the member under consideration, *IM*, have been calculated, the computer program calls subroutine ***MDISPG***, to obtain the member end displacements in the global coordinate system. A flowchart of this subroutine is given in Fig. 4.15. As this flowchart indicates, after initializing V to 0, the subroutine reads, in order, for each of the member end displacements, V_I, the number of the corresponding structure coordinate, *N*, at joint *B* or , from the NSC vector. If the structure coordinate number *N*, corresponding to an end displacement V_I, is found to be less than or equal to *NDOF*, then the value of the element in the *N*th row of the joint-displacement vector P (= d) is stored in the *I*th row of the member displacement vector V.

3. *Determination of member transformation matrix* T *by subroutine* ***MTRANS***. After the global end-displacement vector V for the member under consideration, *IM*, has been evaluated, the main program (Fig. 4.14) calls on the subroutine ***MTRANS*** to form the member transformation matrix T. A flowchart of this subroutine is shown in Fig. 4.16. As this figure indicates, the subroutine first initializes T to 0, and then simply stores the values of the direction

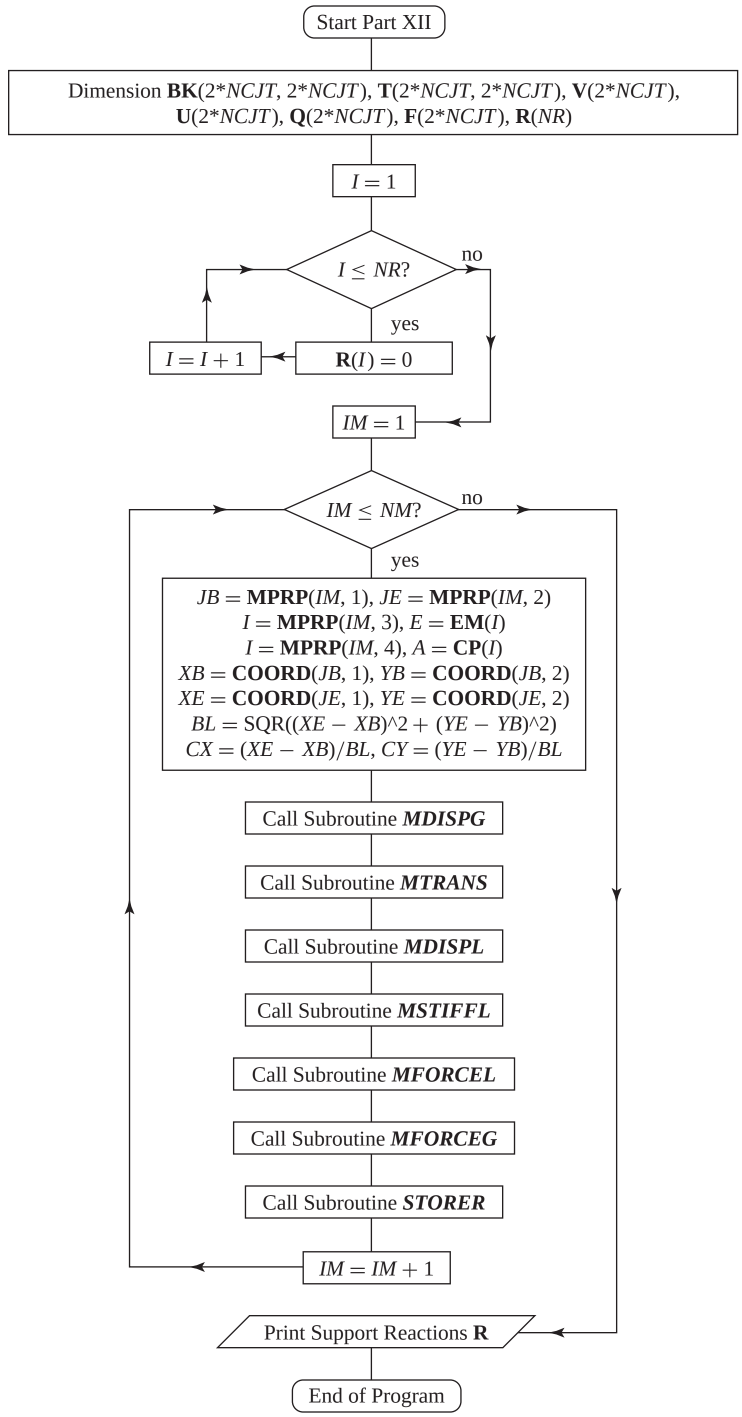


Fig. 4.14 *Flowchart for Determination of Member Forces and Support Reactions for Plane Trusses*

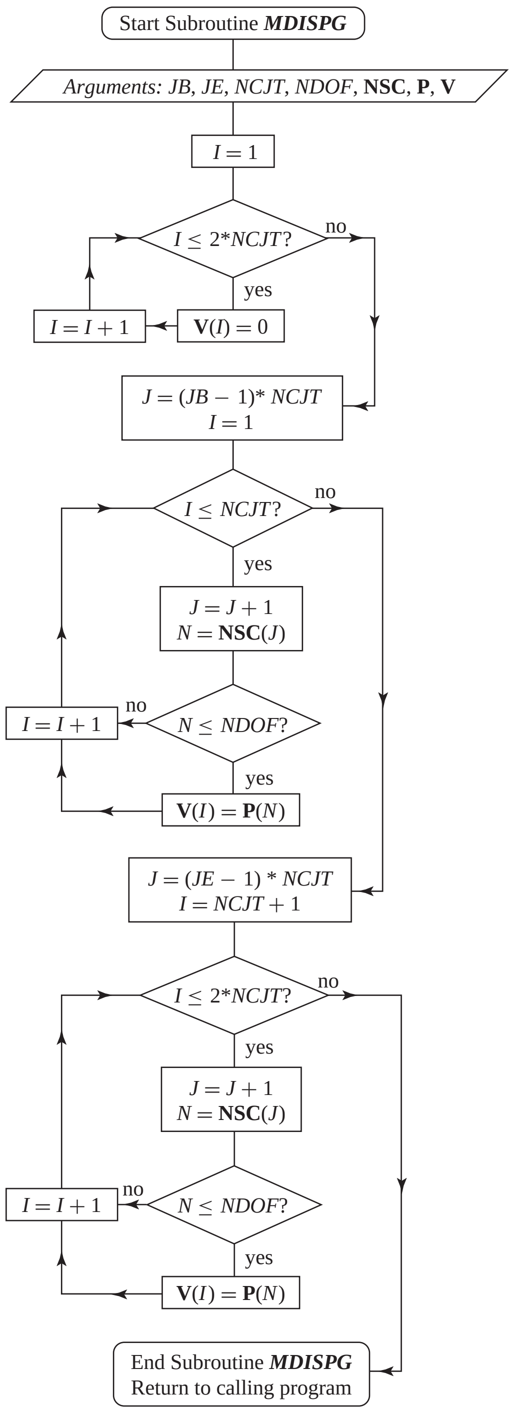


Fig. 4.15 *Flowchart of Subroutine* ***MDISPG*** *for Determining Member Global Displacement Vector*

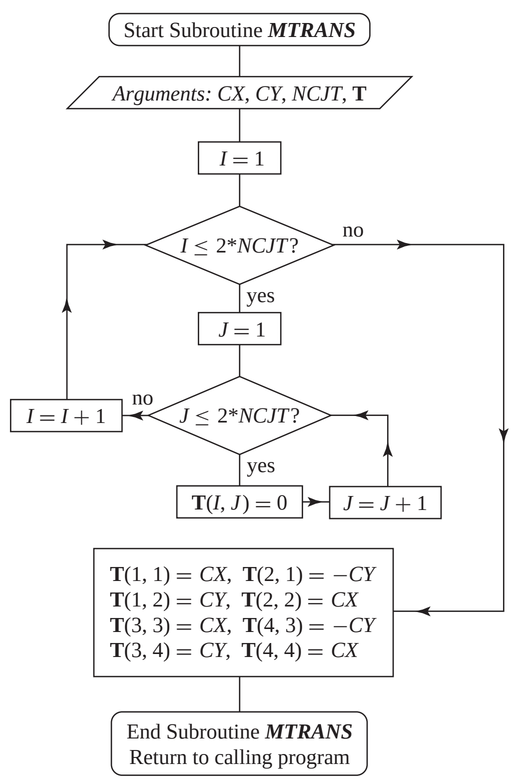


Fig. 4.16 *Flowchart of Subroutine* ***MTRANS*** *for Determining Member Transformation Matrix for Plane Trusses*

cosines *CX* and *CY*, with appropriate plus or minus signs, into various elements of in accordance with Eq. (3.61).

4. *Calculation of member local end displacements* **(= u)** *by subroutine **MDISPL**.* Next, as shown in Fig. 4.14, the program calls subroutine ***MDISPL*** to obtain the local end displacements of the member under consideration, *IM*. From the flowchart given in Fig. 4.17, we can see that after initializing to 0, this subroutine calculates the member local end-displacement vector by applying the relationship = (Eq. (3.63)). The procedure for multiplying matrices was discussed in Section 2.3, and subroutine ***MDISPL*** (Fig. 4.17) uses essentially the same operations as the program for matrix multiplication given in Fig. 2.1.

5. *Determination of member local stiffness matrix* **B** **(=** **)** *by subroutine **MSTIFFL**.* After the local end displacements of the member under

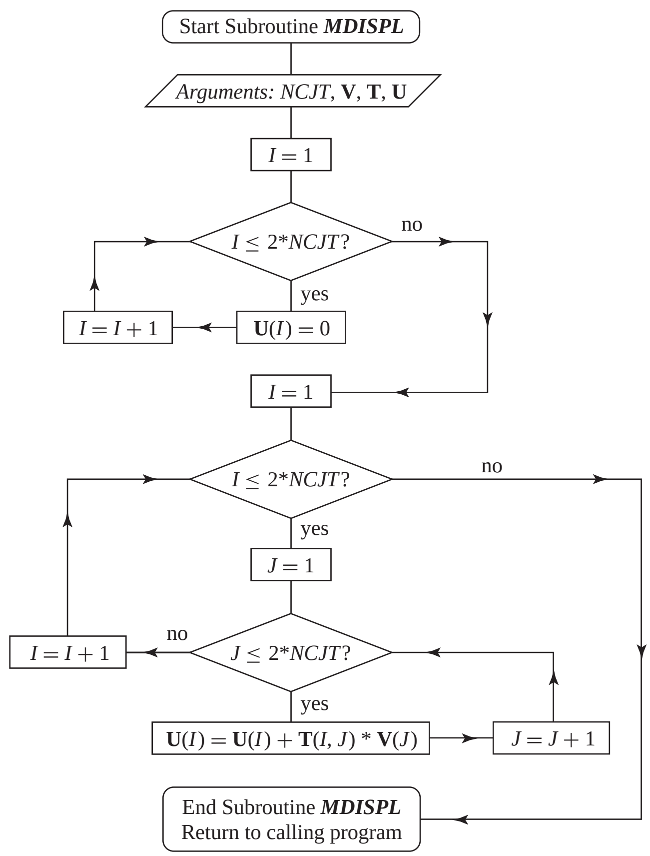


Fig. 4.17 *Flowchart of Subroutine **MDISPL** for Determining Member Local Displacement Vector*

consideration, *IM*, have been evaluated, the program calls subroutine ***MSTIFFL*** to form the member stiffness matrix in the local coordinate system. A flowchart of this subroutine is shown in Fig. 4.18, in which the member local stiffness matrix is identified by the name **B** (instead of) to indicate that it is a real matrix. As this figure indicates, the subroutine, after initializing **B** to 0, simply calculates the values of the various stiffness coefficients and stores them in appropriate elements of **B** , in accordance with Eq. (3.27).

6. *valuation of member local end forces by subroutine **MFORCEL**.* As shown in Fig. 4.14, the program then calls subroutine ***MFORCEL*** to obtain the local end forces of the member under consideration, *IM*. From the flowchart depicted in Fig. 4.19, we can see that, after initializing to 0, this

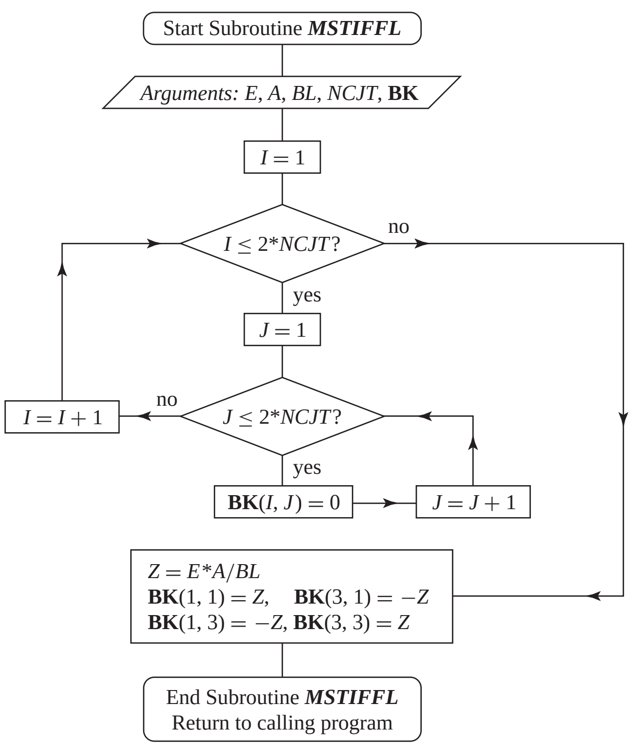


Fig. 4.18 *Flowchart of Subroutine **MSTIFFL** for Determining Member Local Stiffness Matrix for Plane Trusses*

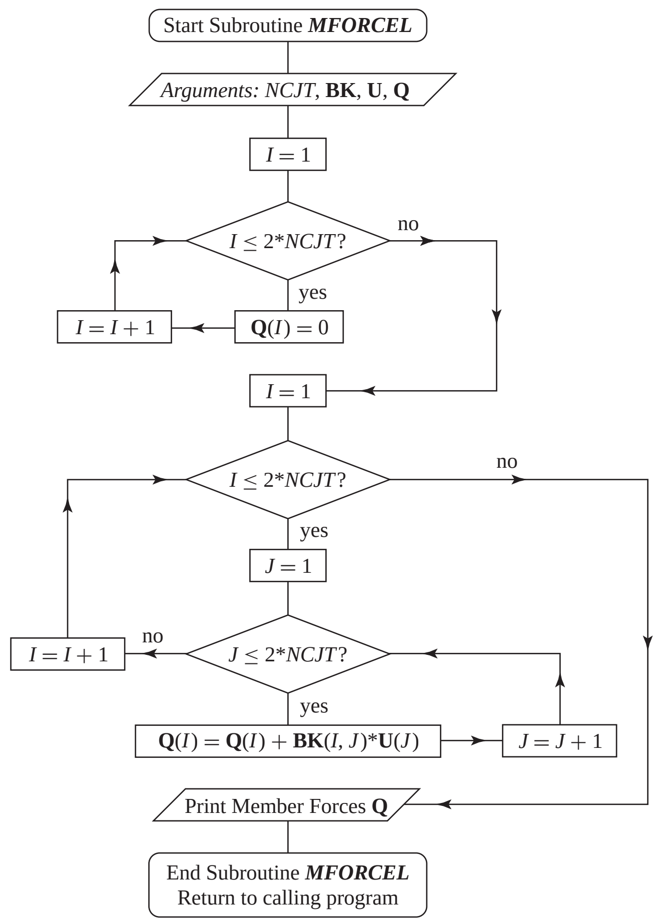


Fig. 4.19 *Flowchart of Subroutine **MFORCEL** for Determining Member Local Force Vector*

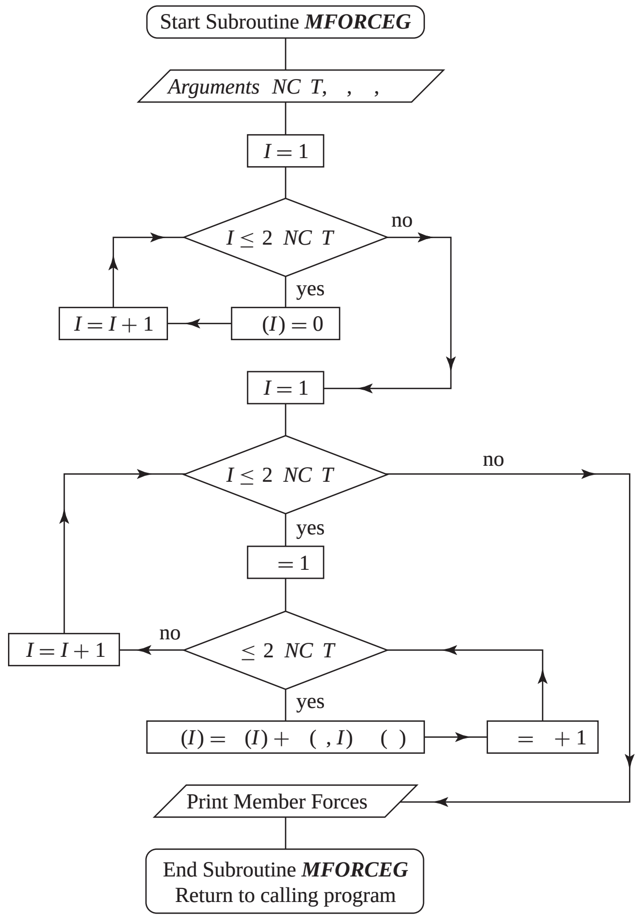


Fig. 4.20 *Flowchart of Subroutine **MFORCEG** for Determining Member Global Force Vector*

subroutine calculates the member local end forces using the relationship $= \mathbf{B}$ (Eq. (3.7)). The vector thus obtained is then printed or displayed on the screen.

7. *Calculation of member global end forces by subroutine **MFORCEG**.* After the local end forces of the member under consideration, *IM*, have been evaluated, the computer program calls subroutine ***MFORCEG*** to calculate the member end forces in the global coordinate system. A flowchart of this subroutine is given in Fig. 4.20. From the figure, we can see that after initializing to 0, the subroutine calculates the global end forces by applying the relationship $= \ ^{T}$ (Eq. (3.66)); these forces are then communicated to the user through a printer or on the screen.

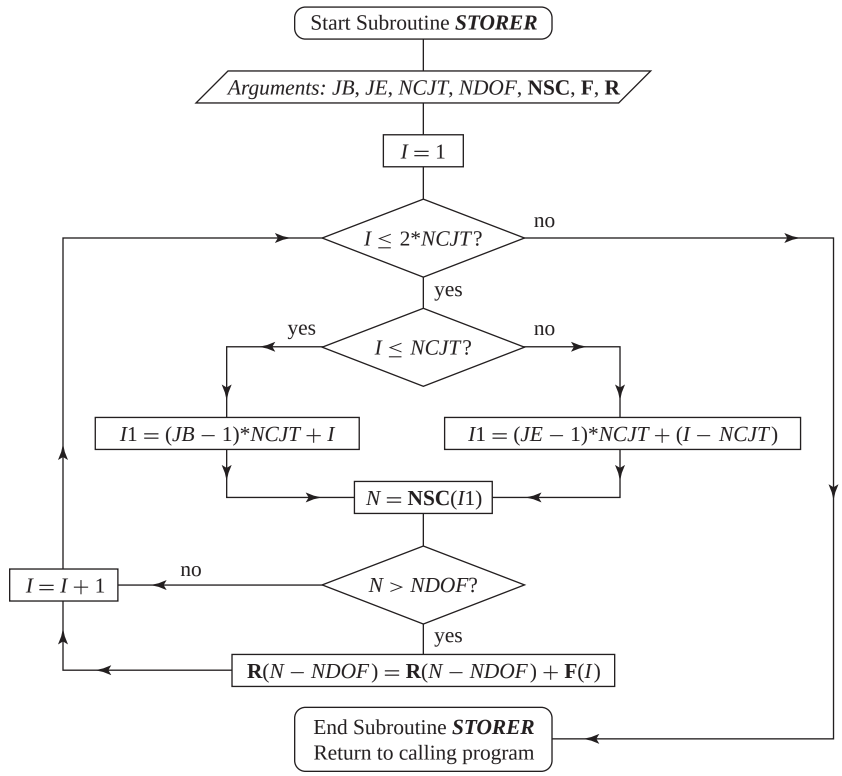


Fig. 4.21 *Flowchart of Subroutine* ***STORER*** *for Storing Member Global Forces in Support Reaction Vector*

8. *Storage of the elements of member global force vector in reaction vector by subroutine* ***STORER***. Once the global force vector has been determined for the member under consideration, *IM*, the program (Fig. 4.14) calls subroutine ***STORER*** to store the pertinent elements of in their proper positions in the support reaction vector . A flowchart of this subroutine, which essentially consists of a *Do Loop*, is given in Fig. 4.21. As shown in this flowchart, the subroutine reads, in order, for each of the member's forces, F_I, the number of the corresponding structure coordinate, *N*, from the **C** vector. If $N > NDOF$, then the value of F_I is added to the $(N - NDOF)$th row of the reaction vector .

Returning our attention to Fig. 4.14, we can see that the formation of the reaction vector is completed when the foregoing eight operations have been performed for each member of the structure. The support reactions thus obtained can then be communicated to the user via a printer or on the screen. A sample printout is given in Fig. 4.22, showing the results of the analysis for the example truss of Fig. 4.2.

```
                e           a

            ==========================
                    p ace e
            ==========================

      .             a   a                  a   a
  ----------    ---------------        --------------
      1          0.0000  00             0.0000  00
      2          7.4568 -02            -2.0253 -01
      3          1.1362 -01             0.0000  00
      4          1.0487 -01             0.0000  00
      5          5.7823 -02            -1.5268 -01
      6          2.8344 -02            -7.9235 -02

            ==========================
            Me   e   x a     ce
            ==========================

       Me   e          x a     ce   a
       -------      -------------------
          1             6.0069  01
          2             3.1459  01
          3             4.8629  00
          4             2.3747  01
          5             5.3543  01
          6             8.5105  01
          7             4.3836  01
          8             3.5762  01
          9             4.5402  01
         10             6.0787  00

            ==========================
               pp     eac
            ==========================

        .                 ce                  ce
    ----------     ------------       ------------
        1           -2.5000  01        2.6301  01
        3            0.0000  00        1.1235  02
        4            0.0000  00       -3.6472  00

                    d      a
```

Fig. 4.22 *A Sample Printout of Analysis Results*

SUMMARY

In this chapter, we have developed a general computer program for the analysis of plane trusses subjected to joint loads. The general program consists of a main program, which is subdivided into twelve parts, and nine subroutines. Brief descriptions of the various parts of the main program, and the subroutines, are provided in Table 4.1 for quick reference.

Table 4.1 *Computer Program for Analysis of Plane Trusses*

Main program part	Description
I	Reads and stores joint data (Fig. 4.3(a))
II	Reads and stores support data (Fig. 4.3(b))
III	Reads and stores material properties (Fig. 4.3(c))
I	Reads and stores cross-sectional properties (Fig. 4.3(d))
	Reads and stores member data (Fig. 4.3(e))
I	Reads and stores joint loads (Fig. 4.3(f))
II	Determines the number of degrees of freedom *NDOF* of the structure (Fig. 4.8(a))
III	Forms the structure coordinate number vector **C** (Fig. 4.8(b))
I	Generates the structure stiffness matrix (Fig. 4.9); subroutines called: ***MSTIFFG*** and ***STORES***
	Forms the joint load vector **P** (Fig. 4.12)
I	Calculates the structure's joint displacements by solving the stiffness relationship, **d** = **P**, using the Gauss–Jordan elimination method. The vector **P** now contains joint displacements (Fig. 4.13).
II	Determines the member end force vectors and , and the support reaction vector (Fig. 4.14); subroutines called: ***MDISPG, MTRANS, MDISPL, MSTIFFL, MFORCEL, MFORCEG,*** and ***STORER***

Subroutine	Description
MDISPG	Determines the member global displacement vector from the joint displacement vector **P** (Fig. 4.15)
MDISPL	Calculates the member local displacement vector = (Fig. 4.17)
MFORCEG	Determines the member global force vector = ^{T} (Fig. 4.20)
MFORCEL	Evaluates the member local force vector = **B** (Fig. 4.19)
MSTIFFG	Forms the member global stiffness matrix **G** (Fig. 4.10)
MSTIFFL	Forms the member local stiffness matrix **B** (Fig. 4.18)
MTRANS	Forms the member transformation matrix (Fig. 4.16)
STORER	Stores the pertinent elements of the member global force vector in the reaction vector (Fig. 4.21)
STORES	Stores the pertinent elements of the member global stiffness matrix **G** in the structure stiffness matrix (Fig. 4.11)

PROBLEMS

The objective of the following problems is to develop, incrementally, a computer program for the analysis of plane trusses; while testing each program increment for correctness, as it is being developed. The reader is strongly encouraged to manually solve as many of the problems (3.16 through 3.25) as possible, so that these hand-calculation results can be used to check the correctness of the various parts of the computer program.

Section 4.1

4.1 Develop an input module of a computer program for the analysis of plane trusses, which can perform the following operations:

a. read from a data file, or computer screen, all the necessary input data;
b. store the input data in computer memory in the form of scalars, vectors, and/or matrices, as appropriate; and
c. print the input data from computer memory.

Check the program for correctness by inputting data for the trusses of Problems 3.16 through 3.25, and by carefully examining the printouts of the input data to ensure that all data have been correctly read and stored.

Section 4.2

4.2 Extend the program developed in Problem 4.1, so that it can perform the following additional operations:

a. determining the number of degrees of freedom (*NDOF*) of the structure;
b. forming the structure coordinate number vector **C**; and
c. printing out the *NDOF* and **C**.

To check the program for correctness, use it to determine the *NDOF* and **C** for the trusses of Problems 3.16 through 3.25, and compare the computer-generated results to those obtained by hand calculations.

Section 4.3

4.3 Extend the program of Problem 4.2 to generate, and print, the structure stiffness matrix . Use the program to generate the structure stiffness matrices for the trusses of Problems 3.16 through 3.25, and compare the computer-generated matrices to those obtained by hand calculations.

Section 4.4

4.4 Extend the program developed in Problem 4.3 to form, and print, the joint load vector **P**. Apply the program to the trusses of Problems 3.16 through 3.25, and compare the computer-generated **P** vectors to those obtained by hand calculations.

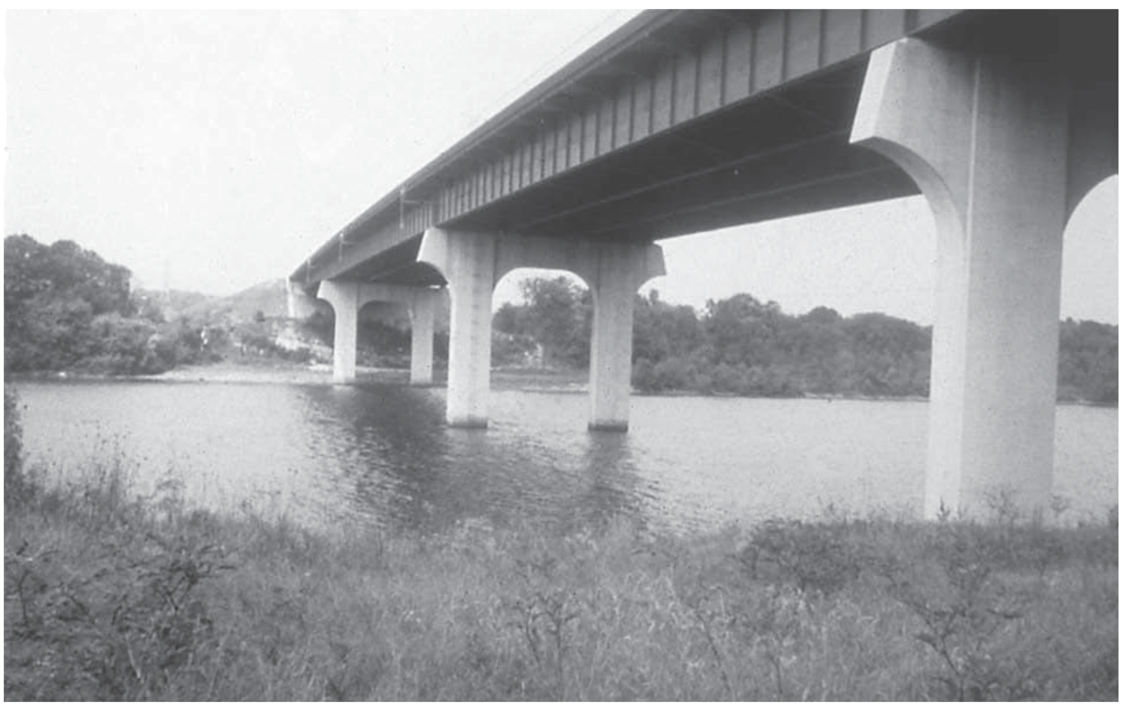
Section 4.5

4.5 Extend the program of Problem 4.4 so that it can: (a) calculate the structure's joint displacements by solving the stiffness relationship, $\mathbf{d} = \mathbf{P}$, using the Gauss–Jordan elimination method; and (b) print the joint displacements. Using the program, determine the joint displacements for the trusses of Problems 3.16 through 3.25, and compare the computer-generated results to those obtained by hand calculations.

Section 4.6

4.6 Extend the program developed in Problem 4.5 so that it can determine and print: (a) the local end forces, , for each member of the truss; and (b) the support reaction vector . Use the program to analyze the trusses of Problems 3.16 through 3.25, and compare the computer-generated results to those obtained by hand calculations.

5 BEAMS



A Continuous Beam Bridge
(Photo courtesy of Bethlehem Steel Corporation)

The term “beam” is used herein to refer to *a long straight structure, which is supported and loaded in such a way that all the external forces and couples (including reactions) acting on it lie in a plane of symmetry of its cross-section, with all the forces perpendicular to its centroidal axis.* Under the action of external loads, beams are subjected only to bending moments and shear forces (but no axial forces).

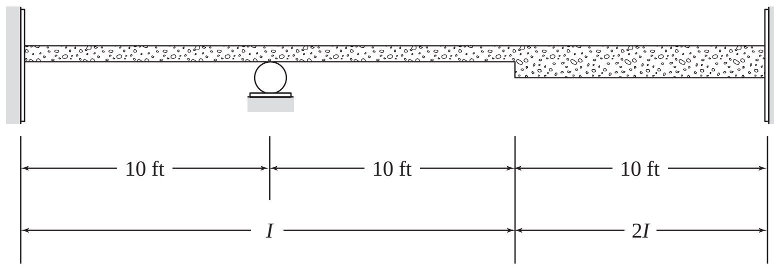
In this chapter, we study the basic concepts of the analysis of beams by the matrix stiffness method, and develop a computer program for the analysis of beams based on the matrix stiffness formulation. As we proceed through the chapter, the reader will notice that, although the member stiffness relations for beams differ from those for plane trusses, the overall format of the method of analysis remains essentially the same—and many of the analysis steps developed in Chapter 3 for the case of plane trusses can be directly applied to beams. Therefore, the computer program developed in Chapter 4 for the analysis of plane trusses can be modified with relative ease for the analysis of beams.

We begin by discussing the preparation of analytical models of beams in Section 5.1, where the global and local coordinate systems and the degrees of freedom of beams are defined. Next, we derive the member stiffness relations in the local coordinate system in Section 5.2; and present the finite-element formulation of the member stiffness matrix, via the principle of virtual work, in Section 5.3. The derivation of the member fixed-end forces, due to external loads applied to members, is considered in Section 5.4; and the formation of the stiffness relations for the entire beam, by combining the member stiffness relations, is discussed in Section 5.5. The procedure for forming the structure fixed-joint force vectors, and the concept of equivalent joint loads, are introduced in Section 5.6; and a step-by-step procedure for the analysis of beams is presented in Section 5.7. Finally, a computer program for the analysis of beams is developed in Section 5.8.

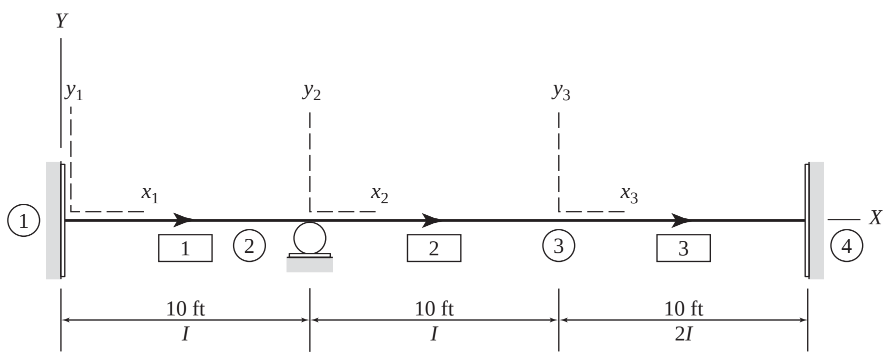
5.1 ANALYTICAL MODEL

For analysis by the matrix stiffness method, *the continuous beam is modeled as a series of straight prismatic members connected at their ends to joints, so that the unknown external reactions act only at the joints*. Consider, for example, the two-span continuous beam shown in Fig. 5.1(a). Although the structure actually consists of a single continuous beam between the two fixed supports at the ends, for the purpose of analysis it is considered to be composed of three members (1, 2, and 3), rigidly connected at four joints (1 through 4), as shown in Fig. 5.1(b). Note that joint 2 has been introduced in the analytical model so that the vertical reaction at the roller support acts on a joint (instead of on a member), and joint 3 is used to subdivide the right span of the beam into two members, each with constant flexural rigidity (I) along its length. This division of the beam into members and joints is necessary because the formulation of the stiffness method requires that the unknown external reactions act only at the joints (i.e., all the member loads be known in advance of analysis), and the member stiffness relationships used in the analysis (to be derived in the following sections) are valid for prismatic members only.

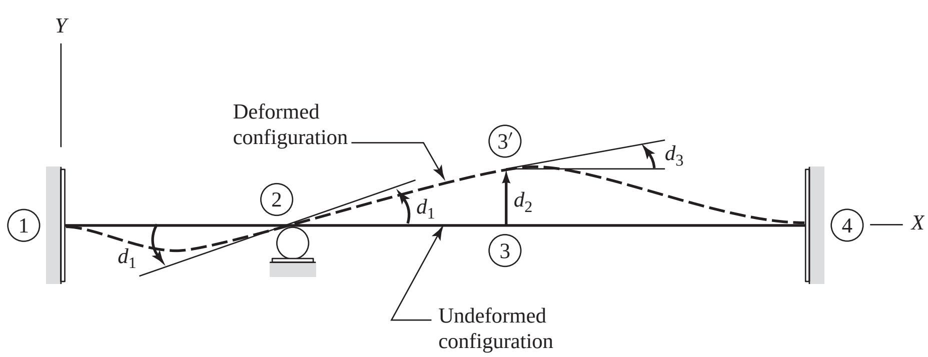
It is important to realize that because joints 1 through 4 (Fig. 5.1(b)) are modeled as rigid joints (i.e., the corresponding ends of the adjacent members are rigidly connected to the joints), they satisfy the continuity and restraint conditions of the actual structure (Fig. 5.1(a)). In other words, since the left end of member 1 and the right end of member 3 of the analytical model are rigidly



(a) Actual Beam

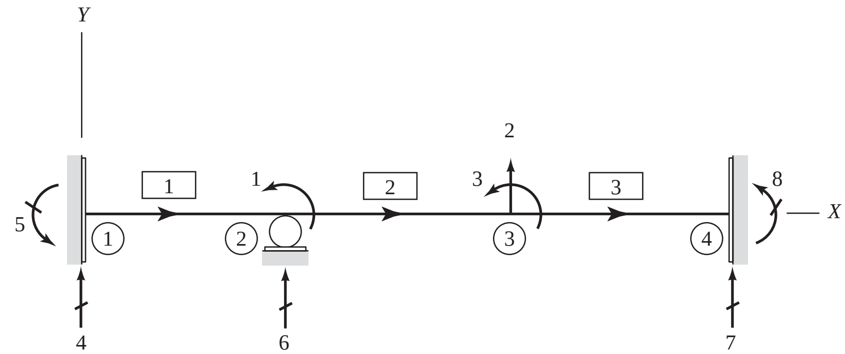


(b) Analytical Model Showing Global and Local Coordinate Systems

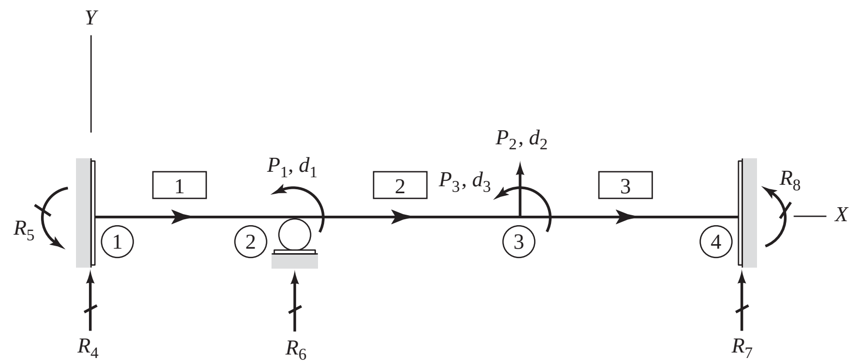


(c) Degrees of Freedom

Fig. 5.1



(d) Structure Coordinate Numbers



(e) Degrees of Freedom, Joint Loads, and Support Reactions

Fig. 5.1 (*continued*)

connected to joints 1 and 4, respectively, which are in turn attached to the fixed supports, the displacements and rotations at the exterior ends of the members are 0; thereby satisfying the restraint conditions of the actual beam at the two ends. Similarly, as the right end of member 1 and the left end of member 2 (Fig. 5.1(b)) are connected to the rigid joint 2, which is attached to a roller support, the displacements at the foregoing ends of members 1 and 2 are 0, and the rotations at the two ends are equal. This indicates that the analytical model satisfies the restraint and continuity conditions of the actual beam at the location of joint 2. Finally, the right end of member 2 and the left end of member 3 (Fig. 5.1(b)) are rigidly connected to joint 3, to ensure that the continuity of both the displacement and the rotation is maintained at the location of joint 3 in the analytical model.

Global and Local Coordinate Systems

As discussed in Chapter 3, the overall geometry, as well as the loads and displacements (including rotations) at the joints of a structure are described with reference to a Cartesian global (*XYZ*) coordinate system. The particular orientation of the global coordinate system, used in this chapter, is as follows.

> *The global coordinate system used for the analysis of beams is a right-handed XYZ coordinate system, with the X axis oriented in the horizontal (positive to the right) direction, and coinciding with the centroidal axis of the beam in the undeformed state. The Y axis is oriented in the vertical (positive upward) direction, with all the external loads and reactions of the beam lying in the XY plane.*

Although not necessary, it is usually convenient to locate the origin of the global XY coordinate system at the leftmost joint of the beam, as shown in Fig. 5.1(b), so that the X coordinates of all the joints are positive. As will become apparent in Section 5.8, this definition of the global coordinate system simplifies the computer programming of beam analysis, because only one (X) coordinate is needed to specify the location of each joint of the structure.

As in the case of plane trusses (Chapter 3), a local (right-handed, xyz) coordinate system is defined for each member of the beam, to establish the relationships between member end forces and end displacements, in terms of member loads. Note that *the terms forces (or loads) and displacements are used in this text in the general sense to include moments and rotations, respectively.* The local coordinate system is defined as follows.

> *The origin of the local xyz coordinate system for a member is located at the left end (beginning) of the member in its undeformed state, with the x axis directed along its centroidal axis in the undeformed state, and the y axis oriented in the vertical (positive upward) direction.*

The local coordinate systems for the three members of the example continuous beam are depicted in Fig. 5.1(b). As this figure indicates, the local coordinate system of each member is oriented so that the positive directions of the local x and y axes are the same as the positive directions of the global X and Y axes, respectively.

The selection of the global and local coordinate systems, as specified in this section, considerably simplifies the analysis of continuous beams by eliminating the need for transformation of member end forces, end displacements, and stiffnesses, from the local to the global coordinate system and vice-versa.

Degrees of Freedom

The degrees of freedom (or free coordinates) of a beam are simply its unknown joint displacements (translations and rotations). Since the axial deformations of the beam are neglected, the translations of its joints in the global X direction are 0. Therefore, a joint of a beam can have up to two degrees of freedom, namely, a translation in the global Y direction (i.e., in the direction perpendicular to the beam's centroidal axis) and a rotation (about the global Z axis). Thus, the number of structure coordinates (i.e., free and/or restrained coordinates) at a joint of a beam equals 2, or $NC\ T = 2$.

Let us consider the analytical model of the continuous beam as given in Fig. 5.1(b). The deformed shape of the beam, due to an arbitrary loading, is depicted in Fig. 5.1(c) using an exaggerated scale. From this figure, we can see that joint 1, which is attached to the fixed support, can neither translate nor rotate; therefore, it does not have any degrees of freedom. Since joint 2 of the beam is attached to the roller support, it can rotate, but not translate. Thus, joint 2 has only one degree of freedom, which is designated d_1 in the figure. As joint 3 is not attached to a support, two displacements—the translation d_2 in the Y direction, and the rotation d_3 about the Z axis—are needed to completely specify its deformed position 3′. Thus, joint 3 has two degrees of freedom. Finally, joint 4, which is attached to the fixed support, can neither translate nor rotate; therefore, it does not have any degrees of freedom. Thus, the entire beam has a total of three degrees of freedom.

As indicated in Fig. 5.1(c), joint translations are considered positive when vertically upward, and joint rotations are considered positive when counterclockwise. All the joint displacements in Fig. 5.1(c) are shown in the positive sense. The $NDOF \times 1$ joint displacement vector **d** for the beam is written as

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Since the number of structure coordinates per joint equals 2 (i.e., $NC\ T = 2$), the number of degrees of freedom, $NDOF$, of a beam can be obtained from Eq. (3.2) as

$$\boxed{\left.\begin{aligned} &NC\ T = 2 \\ &NDOF = 2(N\ \ ) - NR \end{aligned}\right\} \quad \text{for beams}} \qquad \textbf{(5.1)}$$

in which, as in the case of plane trusses, N represents the number of joints of the beam, and NR denotes the number of joint displacements restrained by supports (or the number of restrained coordinates). Let us apply Eq. (5.1) to the analytical model of the beam in Fig. 5.1(b). The beam has four joints (i.e., $N\ \ = 4$); two joints, 1 and 4, are attached to the fixed supports that together restrain four joint displacements (namely, the translations in the Y direction and the rotations of joints 1 and 4). Furthermore, the roller support at joint 2 restrains one joint displacement, which is the translation of joint 2 in the Y direction. Thus, the total number of joint displacements that are restrained by all supports of the beam is 5 (i.e., $NR = 5$). Substitution of the numerical values of N and NR into Eq. (5.1) yields

$$NDOF = 2(4) - 5 = 3$$

which is the same as the number of degrees of freedom of the beam obtained previously. As in the case of plane trusses, the free and restrained coordinates of a beam are collectively referred to simply as the structure coordinates.

When analyzing a beam, it is not necessary to draw its deformed shape, as shown in Fig. 5.1(c), to identify the degrees of freedom. Instead, all the structure coordinates (i.e., degrees of freedom and restrained coordinates) are usually directly specified on the beam's line diagram by assigning numbers to the arrows drawn at the joints in the directions of the joint displacements, as shown in Fig. 5.1(d). In this figure, a slash (/) has been added to the arrows corresponding to the restrained coordinates to distinguish them from those representing the degrees of freedom.

The procedure for assigning numbers to the structure coordinates of beams is similar to that for the case of plane trusses, discussed in detail in Section 3.2. The degrees of freedom are numbered first, starting at the lowest-numbered joint, that has a degree of freedom, and proceeding sequentially to the highest-numbered joint. If a joint has two degrees of freedom, then the translation in the Y direction is numbered first, followed by the rotation. The first degree of freedom is assigned the number 1, and the last degree of freedom is assigned a number equal to *NDOF*.

After all the degrees of freedom of the beam have been numbered, its restrained coordinates are numbered beginning with a number equal to $NDOF + 1$. Starting at the lowest-numbered joint that is attached to a support, and proceeding sequentially to the highest-numbered joint, all of the restrained coordinates of the beam are numbered. If a joint has two restrained coordinates, then the coordinate in the Y direction (corresponding to the reaction force) is numbered first, followed by the rotation coordinate (corresponding to the reaction couple). The number assigned to the last restrained coordinate of the beam is always 2(*N*). The structure coordinate numbers for the example beam, obtained by applying the foregoing procedure, are given in Fig. 5.1(d).

Joint Load and Reaction Vectors

Unlike plane trusses, which are subjected only to joint loads, the external loads on beams may be applied at the joints as well as on the members. The external loads (i.e., forces and couples or moments) applied at the joints of a structure are referred to as the *joint loads,* whereas the external loads acting between the ends of the members of the structure are termed the *member loads*. In this section, we focus our attention only on the joint loads, with the member loads considered in subsequent sections. As discussed in Section 3.2, an external joint load can, in general, be applied to the beam at the location and in the direction of each of its degrees of freedom. For example, the beam of Fig. 5.1(b), with three degrees of freedom, can be subjected to a maximum of three joint loads, P_1 through P_3, as shown in Fig. 5.1(e). As indicated there, a load corresponding to a degree of freedom d_i is denoted symbolically by P_i. The 3×1 joint load vector $\mathbf{P}$ for the beam is written in the form

$$\mathbf{P} = \underset{NDOF \times 1}{\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}}$$

As for the support reactions, when a beam is subjected to external joint and/or member loads, a reaction (force or moment) can develop at the location and in the direction of each of its restrained coordinates. For example, the beam of Fig. 5.1(b), which has five restrained coordinates, can develop up to five reactions, as shown in Fig. 5.1(e). As indicated in this figure, the reaction corresponding to the *i*th restrained coordinate is denoted symbolically by R_i. The 5×1 reaction vector for the beam is expressed as

$$= \underset{NR \times 1}{\begin{bmatrix} R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \end{bmatrix}}$$

EXAMPLE 5.1 Identify by numbers the degrees of freedom and restrained coordinates of the continuous beam with a cantilever overhang shown in Fig. 5.2(a). Also, form the beam's joint load vector **P**.

SOLUTION The beam has four degrees of freedom, which are identified by numbers 1 through 4 in Fig. 5.2(b). The four restrained coordinates of the beam are identified by numbers 5 through 8 in the same figure. **Ans**

By comparing Figs. 5.2(a) and (b), we can see that $P_1 = -50$ k-ft; $P_2 = 0$; $P_3 = -20$ k; and $P_4 = 0$. The negative signs assigned to the magnitudes of P_1 and P_3 indicate that these loads act in the clockwise and downward directions, respectively. Thus, the joint load vector can be expressed in the units of kips and feet, as

$$\mathbf{P} = \begin{bmatrix} -50 \\ 0 \\ -20 \\ 0 \end{bmatrix} \qquad \textbf{Ans}$$

Alternative Approach The analysis of beams with cantilever overhangs can be considerably expedited by realizing that the cantilever portions are statically determinate (in the sense that the shear and moment at a cantilever's end can be evaluated directly by applying the equilibrium equations to the free-body of the cantilever portion). Therefore, the cantilever portions can be removed from the beam, and only the remaining indeterminate part needs to be analyzed by the stiffness method. However, the end moments and the end forces exerted by the cantilevers on the remaining indeterminate part of the structure must be included in the stiffness analysis, as illustrated in the following paragraphs.

Since the beam of Fig. 5.2(a) has a cantilever member *CD*, we separate this statically determinate member from the rest of the beam, as shown in Fig. 5.2(c). The force S_{CD} and the moment M_{CD} at end *C* of the cantilever are then calculated by applying the equilibrium equations, as follows.

$$+\uparrow \sum F_Y = 0 \qquad S_{CD} - 20 = 0 \qquad S_{CD} = 20 \text{ k} \uparrow$$

$$+\circlearrowleft \sum M_C = 0 \qquad M_{CD} - 20(10) = 0 \qquad M_{CD} = 200 \text{ k-ft} \circlearrowright$$

Next, the moment M_{CD} is applied as a joint load, in the clockwise (opposite) direction, at joint C of the indeterminate part *AC* of the beam, as shown in Fig. 5.2(c). Note that

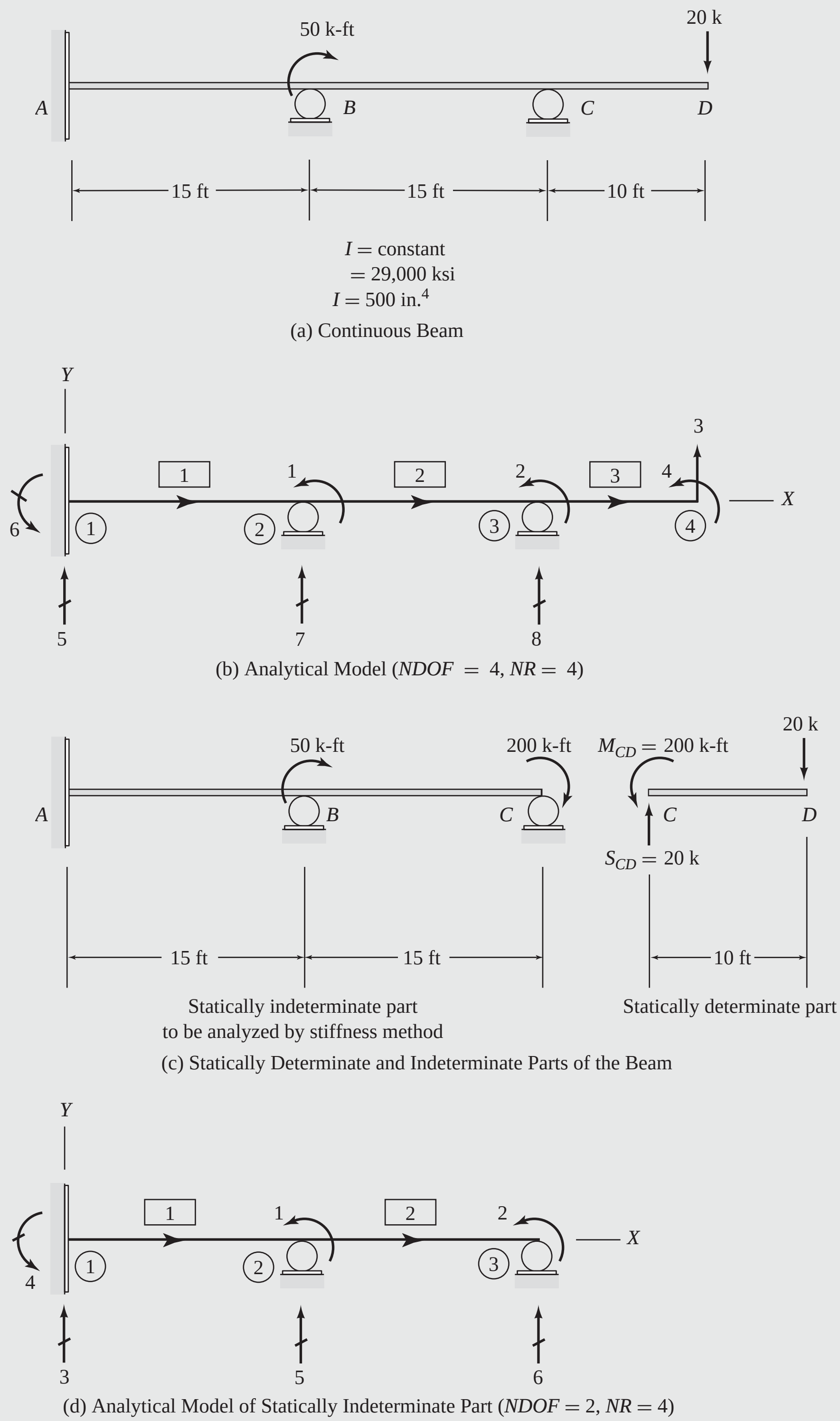

50 k-ft
20 k
A
B
C
D
15 ft
15 ft
10 ft
I = constant
= 29,000 ksi
I = 500 in.^{4}
(a) Continuous Beam
Y
X
1
2
3
4
6
5
7
8
(b) Analytical Model (NDOF = 4, NR = 4)
50 k-ft
200 k-ft
M_{CD} = 200 k-ft
20 k
S_{CD} = 20 k
15 ft
15 ft
10 ft
Statically indeterminate part
to be analyzed by stiffness method
Statically determinate part
(c) Statically Determinate and Indeterminate Parts of the Beam
Y
X
1
2
3
4
5
6
(d) Analytical Model of Statically Indeterminate Part (NDOF = 2, NR = 4)

Fig. 5.2

the end force S_{CD} ($= 20$ k) need not be considered in the analysis of the indeterminate part because its only effect is to increase the reaction at support C by 20 k.

The analytical model of the indeterminate part of the beam is drawn in Fig. 5.2(d). Note that the number of degrees of freedom has now been reduced to only two, identified by numbers 1 and 2 in the figure. The number of restrained coordinates remains at four, and these coordinates are identified by numbers 3 through 6 in Fig. 5.2(d). By comparing the indeterminate part of the beam in Fig. 5.2(c) to its analytical model in Fig. 5.2(d), we obtain the joint load vector as

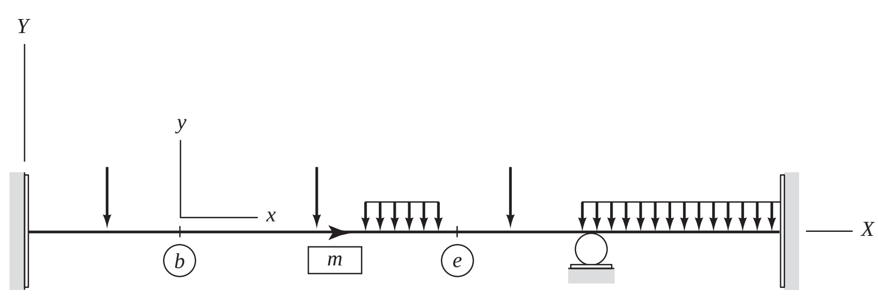
$$\mathbf{P} = \begin{bmatrix} -50 \\ -200 \end{bmatrix} \text{ k-ft}$$ **Ans**

Once the analytical model of Fig. 5.2(d) has been analyzed by the stiffness method, the reaction force R_6 must be adjusted (i.e., increased by 20 k) to account for the end force S_{CD} being exerted by the cantilever *CD* on support *C*.

5.2 MEMBER STIFFNESS RELATIONS

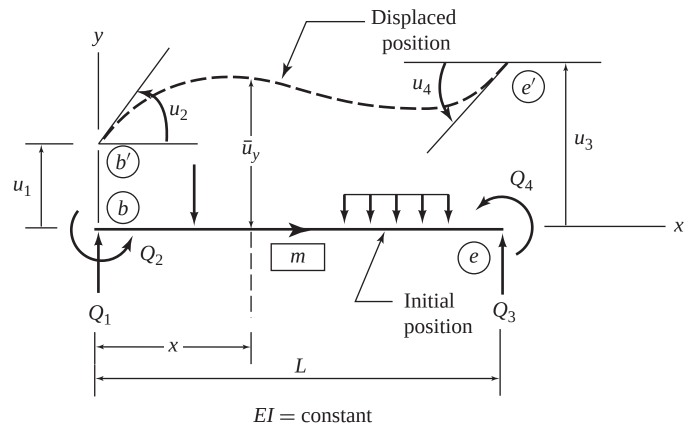
When a beam is subjected to external loads, internal moments and shears generally develop at the ends of its individual members. *The equations expressing the forces (including moments) at the end of a member as functions of the displacements (including rotations) of its ends, in terms of the external loads applied to the member, are referred to as the member stiffness relations.* Such member stiffness relations are necessary for establishing the stiffness relations for the entire beam, as discussed in Section 5.5. In this section, we derive the stiffness relations for the members of beams.

To develop the member stiffness relations, we focus our attention on an arbitrary prismatic member *m* of the continuous beam shown in Fig. 5.3(a). When the beam is subjected to external loads, member *m* deforms and internal shear forces and moments are induced at its ends. The initial and displaced positions of *m* are depicted in Fig. 5.3(b), in which L, , and I denote the length, Young's modulus of elasticity, and moment of inertia, respectively, of the member. It can be seen from this figure that two displacements—translation in the y direction and rotation about the z axis—are necessary to completely specify

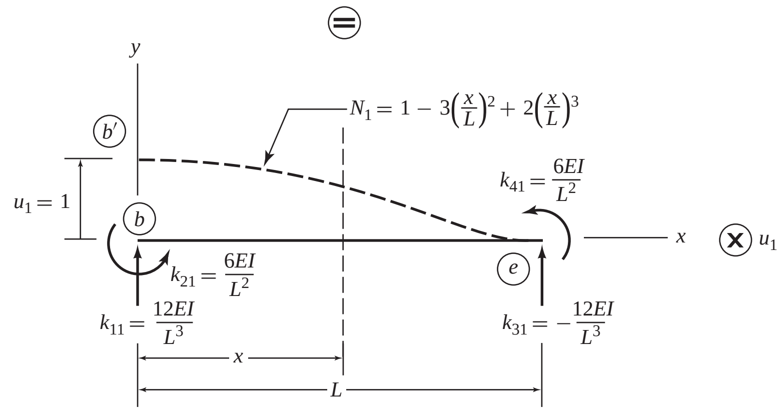


(a) Beam

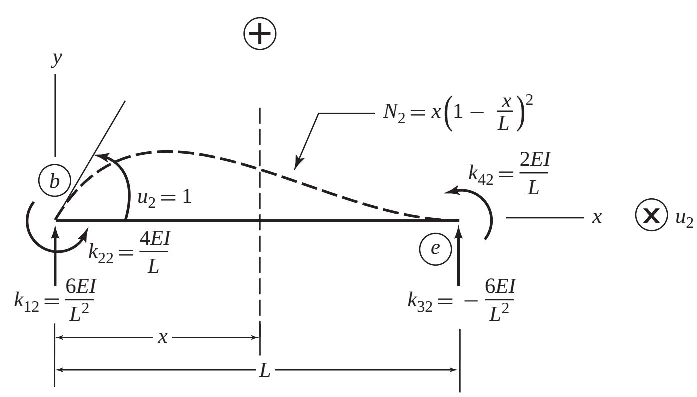
Fig. 5.3



(b) Member Forces and Displacements in the Local Coordinate System

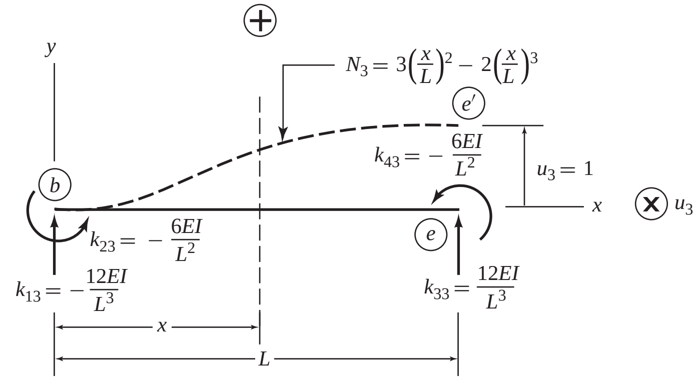


(c)

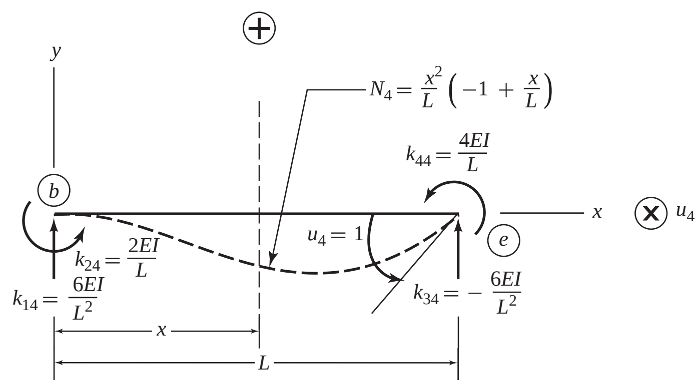


(d)

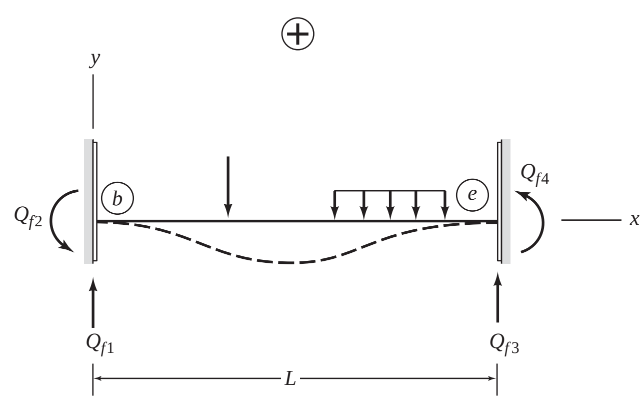
Fig. 5.3 (*continued*)



(e)



(f)



(g)

Fig. 5.3 (*continued*)

the displaced position of each end of the member. Thus, the member has a total of four end displacements or degrees of freedom. As Fig. 5.3(b) indicates, the member end displacements (including rotations) are denoted by u_1 through u_4, and the corresponding end forces (including moments) are denoted by ${}_1$ through ${}_4$. Note that the member end translations and forces are considered positive when vertically upward (i.e., in the positive direction of the local y axis), and the end rotations and moments are considered positive when counterclockwise. The numbering scheme used for identifying the member end displacements and forces is similar to that used previously for plane trusses in Chapter 3. As indicated in Fig. 5.3(b), *the member end displacements and forces are numbered by beginning at the left end b of the member, which is the origin of the local coordinate system, with the vertical translation and force numbered first, followed by the rotation and moment. The displacements and forces at the opposite end e of the member are then numbered in the same sequential order.*

The relationships between member end forces and end displacements can be conveniently established by subjecting the member, separately, to each of the four end displacements and external loads, as shown in Figs. 5.3(c) through (g); and by expressing the total member end forces as the algebraic sums of the end forces required to cause the individual end displacements and the forces caused by the external loads acting on the member with no end displacements. Thus, from Figs. 5.3(b) through (g), we can see that

$${}_1 = k_{11}u_1 + k_{12}u_2 + k_{13}u_3 + k_{14}u_4 + {}_{f1} \qquad \textbf{(5.2a)}$$

$${}_2 = k_{21}u_1 + k_{22}u_2 + k_{23}u_3 + k_{24}u_4 + {}_{f2} \qquad \textbf{(5.2b)}$$

$${}_3 = k_{31}u_1 + k_{32}u_2 + k_{33}u_3 + k_{34}u_4 + {}_{f3} \qquad \textbf{(5.2c)}$$

$${}_4 = k_{41}u_1 + k_{42}u_2 + k_{43}u_3 + k_{44}u_4 + {}_{f4} \qquad \textbf{(5.2d)}$$

in which, as defined in Chapter 3, *a stiffness coefficient k_{ij} represents the force at the location and in the direction of ${}_i$ required, along with other end forces, to cause a unit value of displacement u_j, while all other end displacements are* 0, *and the member is not subjected to any external loading between its ends.* The last terms, ${}_{fi}$ (with $i = 1$ to 4), on the right sides of Eqs. (5.2), represent *the forces that would develop at the member ends, due to external loads, if both ends of the member were fixed against translations and rotations* (see Fig. 5.3(g)). These forces are commonly referred to as the *member fixed-end forces* due to external loads. Equations (5.2) can be written in matrix form as

$$\begin{bmatrix} {}_1 \\ {}_2 \\ {}_3 \\ {}_4 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} {}_{f1} \\ {}_{f2} \\ {}_{f3} \\ {}_{f4} \end{bmatrix} \qquad \textbf{(5.3)}$$

or, symbolically, as

$$\boxed{\; = \; \mathbf{u} + {}_f} \qquad \textbf{(5.4)}$$

in which and **u** represent the member end force and member end displacement vectors, respectively, in the local coordinate system; is the member stiffness matrix in the local coordinate system; and $_f$ is called the *member fixed-end force vector in the local coordinate system*.

In the rest of this and the following section, we focus our attention on the derivation of the member stiffness matrix . The fixed-end force vector $_f$ is considered in detail in Section 5.4.

Derivation of Member Stiffness Matrix k

Various classical methods of structural analysis, such as the *method of consistent deformations* and the *slope-deflection equations*, can be used to determine the expressions for the stiffness coefficients k_{ij} in terms of member length and its flexural rigidity, I. In the following, however, we derive such stiffness expressions by directly integrating the differential equation for beam deflection. This direct integration approach is not only relatively simple and straightforward, but it also yields member shape functions as a part of the solution. The shape functions are often used to establish the member mass matrices for the dynamic analysis of beams [34]; they also provide insight into the finite-element formulation of beam analysis (considered in the next section).

It may be recalled from a previous course on *mechanics of materials* that the differential equation for small-deflection bending of a beam, composed of linearly elastic homogenous material and loaded in a plane of symmetry of its cross-section, can be expressed as

$$\boxed{\frac{d^2\bar{u}_y}{dx^2} = \frac{M}{I}} \tag{5.5}$$

in which $\bar{u}_y$ represents the deflection of the beam's centroidal axis (which coincides with the neutral axis) in the y direction, at a distance x from the origin of the xy coordinate system as shown in Fig. 5.3(b); and M denotes the bending moment at the beam section at the same location, x. It is important to realize that the bending moment M is considered positive in accordance with the *beam sign convention*, which can be stated as follows (see Fig. 5.4).

> *The bending moment at a section of a beam is considered positive when the external force or couple tends to bend the beam concave upward (in the positive y direction), causing compression in the fibers above (in the positive y direction), and tension in the fibers below (in the negative y direction), the neutral axis of the beam at the section.*

To obtain the expressions for the coefficients k_{i1} ($i = 1$ through 4) in the first column of the member stiffness matrix (Eq. (5.3)), we subject the

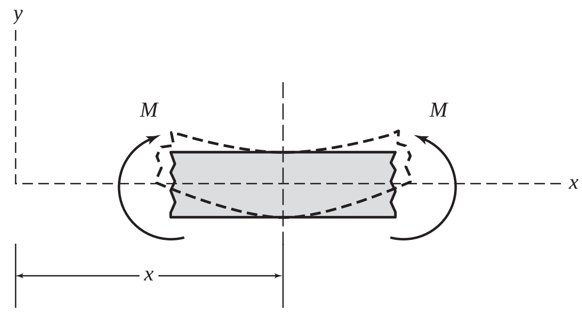


Fig. 5.4 *Beam Sign Convention*

member to a unit value of the end displacement u_1 at end b, as shown in Fig. 5.3(c). Note that all other end displacements of the member are 0 (i.e., $u_2 = u_3 = u_4 = 0$), and the member is in equilibrium under the action of two end moments k_{21} and k_{41}, and two end shears k_{11} and k_{31}. To determine the equation for bending moment for the member, we pass a section at a distance x from end b, as shown in Fig. 5.3(c). Considering the free body to the left of this section, we obtain the bending moment M at the section as

$$M = -k_{21} + k_{11}x \tag{5.6}$$

Note that the bending moment due to the couple k_{21} is negative, in accordance with the *beam sign convention*, because of its tendency to bend the member concave downward, causing tension in the fibers above and compression in the fibers below the neutral axis. The bending moment $k_{11}x$ due to the end shear k_{11} is positive, however, in accordance with the *beam sign convention*.

Substitution of Eq. (5.6) into Eq. (5.5) yields

$$\frac{d^2\bar{u}_y}{dx^2} = \frac{1}{I}(-k_{21} + k_{11}x) \tag{5.7}$$

in which the flexural rigidity I of the member is constant because the member is assumed to be prismatic. The equation for the slope θ of the member can be determined by integrating Eq. (5.7) as

$$\theta = \frac{d\bar{u}_y}{dx} = \frac{1}{I}\left(-k_{21}x + k_{11}\frac{x^2}{2}\right) + C_1 \tag{5.8}$$

in which C_1 denotes a constant of integration. By integrating Eq. (5.8), we obtain the equation for deflection as

$$\bar{u}_y = \frac{1}{I}\left(-k_{21}\frac{x^2}{2} + k_{11}\frac{x^3}{6}\right) + C_1x + C_2 \tag{5.9}$$

in which C_2 is another constant of integration. The four unknowns in Eqs. (5.8) and (5.9)—that is, two constants of integration C_1 and C_2, and two stiffness coefficients k_{11} and k_{21}—can now be evaluated by applying the following four boundary conditions.

$$\begin{array}{lll} \text{At end } b, & x = 0, & \theta = 0 \\ & x = 0, & \bar{u}_y = 1 \\ \text{At end } e, & x = L, & \theta = 0 \\ & x = L, & \bar{u}_y = 0 \end{array}$$

By applying the first boundary condition—that is, by setting $x = 0$ and $\theta = 0$ in Eq. (5.8)—we obtain $C_1 = 0$. Next, by using the second boundary condition—that is, by setting $x = 0$ and $\bar{u}_y = 1$ in Eq. (5.9)—we obtain $C_2 = 1$. Thus, the equations for the slope and deflection of the member become

$$\theta = \frac{1}{I}\left(-k_{21}x + k_{11}\frac{x^2}{2}\right) \tag{5.10}$$

$$\bar{u}_y = \frac{1}{I}\left(-k_{21}\frac{x^2}{2} + k_{11}\frac{x^3}{6}\right) + 1 \tag{5.11}$$

We now apply the third boundary condition—that is, we set $x = L$ and $\theta = 0$ in Eq. (5.10)—to obtain

$$0 = \frac{1}{I}\left(-k_{21}L + k_{11}\frac{L^2}{2}\right)$$

from which

$$k_{21} = k_{11}\frac{L}{2} \tag{5.12}$$

Next, we use the last boundary condition—that is, we set $x = L$ and $\bar{u}_y = 0$ in Eq. (5.11)—to obtain

$$0 = \frac{1}{I}\left(-k_{21}\frac{L^2}{2} + k_{11}\frac{L^3}{6}\right) + 1$$

from which

$$k_{21} = \frac{2\ \ I}{L^2} + k_{11}\frac{L}{3} \tag{5.13}$$

By substituting Eq. (5.12) into Eq. (5.13), we determine the expression for the stiffness coefficient k_{11}:

$$\boxed{k_{11} = \frac{12\ \ I}{L^3}} \tag{5.14}$$

and the substitution of Eq. (5.14) into Eq. (5.12) yields

$$\boxed{k_{21} = \frac{6\ \ I}{L^2}} \tag{5.15}$$

The remaining two stiffness coefficients, k_{31} and k_{41}, can now be determined by applying the equations of equilibrium to the free body of the member shown in Fig. 5.3(c). Thus,

$$+\uparrow \sum F_y = 0 \qquad \frac{12\quad I}{L^3} + k_{31} = 0$$

$$\boxed{k_{31} = -\frac{12\quad I}{L^3}} \tag{5.16}$$

$$+\circlearrowleft \sum M_e = 0 \qquad \frac{6\quad I}{L^2} - \frac{12\quad I}{L^3}(L) + k_{41} = 0$$

$$\boxed{k_{41} = \frac{6\quad I}{L^2}} \tag{5.17}$$

To determine the deflected shape of the member, we substitute the expressions for k_{11} (Eq. (5.14)) and k_{21} (Eq. (5.15)) into Eq. (5.11). This yields

$$\bar{u}_y = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 \tag{5.18}$$

Since the foregoing equation describes the variation of $\bar{u}_y$ (i.e., the y displacement) along the member's length due to a unit value of the end displacement u_1, while all other end displacements are zero, it represents the member shape function N_1; that is,

$$\boxed{N_1 = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3} \tag{5.19}$$

The expressions for coefficients k_{i2} ($i = 1$ through 4) in the second column of the member stiffness matrix (Eq. (5.3)) can be evaluated in a similar manner. We subject the member to a unit value of the end displacement u_2 at end b, as shown in Fig. 5.3(d). Note that all other member end displacements are 0 (i.e., $u_1 = u_3 = u_4 = 0$), and the member is in equilibrium under the action of two end moments k_{22} and k_{42}, and two end shears k_{12} and k_{32}. The equation for bending moment at a distance x from end b of the member can be written as

$$M = -k_{22} + k_{12}x \tag{5.20}$$

By substituting Eq. (5.20) into the differential equation for beam deflection (Eq. (5.5)), we obtain

$$\frac{d^2\bar{u}_y}{dx^2} = \frac{1}{I}(-k_{22} + k_{12}x) \tag{5.21}$$

By integrating Eq. (5.21) twice, we obtain the equations for the slope and deflection of the member as

$$\theta = \frac{d\bar{u}_y}{dx} = \frac{1}{I}\left(-k_{22}x + k_{12}\frac{x^2}{2}\right) + C_1 \tag{5.22}$$

$$\bar{u}_y = \frac{1}{I}\left(-k_{22}\frac{x^2}{2} + k_{12}\frac{x^3}{6}\right) + C_1 x + C_2 \tag{5.23}$$

The four unknowns, C_1, C_2, k_{12} and k_{22}, in Eqs. (5.22) and (5.23) can now be evaluated by applying the boundary conditions, as follows.

At end b,	$x = 0$,	$\theta = 1$
	$x = 0$,	$\bar{u}_y = 0$
At end e,	$x = L$,	$\theta = 0$
	$x = L$,	$\bar{u}_y = 0$

Application of the first boundary condition (i.e., $\theta = 1$ at $x = 0$) yields $C_1 = 1$; using the second boundary condition (i.e., $\bar{u}_y = 0$ at $x = 0$), we obtain $C_2 = 0$. By applying the third boundary condition (i.e., $\theta = 0$ at $x = L$), we obtain

$$0 = \frac{1}{I}\left(-k_{22}L + k_{12}\frac{L^2}{2}\right) + 1$$

from which

$$k_{22} = \frac{I}{L} + k_{12}\frac{L}{2} \tag{5.24}$$

and application of the last boundary condition (i.e., $\bar{u}_y = 0$ at $x = L$) yields

$$0 = \frac{1}{I}\left(-k_{22}\frac{L^2}{2} + k_{12}\frac{L^3}{6}\right) + L$$

from which

$$k_{22} = \frac{2\;\;I}{L} + k_{12}\frac{L}{3} \tag{5.25}$$

By substituting Eq. (5.24) into Eq. (5.25), we obtain the expression for the stiffness coefficient k_{12}:

$$\boxed{k_{12} = \frac{6\;\;I}{L^2}} \tag{5.26}$$

and by substituting Eq. (5.26) into either Eq. (5.24) or Eq. (5.25), we obtain

$$\boxed{k_{22} = \frac{4\;\;I}{L}} \tag{5.27}$$

To determine the two remaining stiffness coefficients, k_{32} and k_{42}, we apply the equilibrium equations to the free body of the member shown in Fig. 5.3(d):

$$+\uparrow \sum F_y = 0 \qquad \frac{6\ \ I}{L^2} + k_{32} = 0$$

$$\boxed{k_{32} = -\frac{6\ \ I}{L^2}} \tag{5.28}$$

$$+\circlearrowleft \sum M_e = 0 \qquad \frac{4\ \ I}{L} - \frac{6\ \ I}{L^2}(L) + k_{42} = 0$$

$$\boxed{k_{42} = \frac{2\ \ I}{L}} \tag{5.29}$$

The shape function (i.e., deflected shape) of the member, due to a unit end displacement u_2, can now be obtained by substituting the expressions for k_{12} (Eq. (5.26)) and k_{22} (Eq. (5.27)) into Eq. (5.23), with $C_1 = 1$ and $C_2 = 0$. Thus,

$$\boxed{N_2 = x\left(1 - \frac{x}{L}\right)^2} \tag{5.30}$$

Next, we subject the member to a unit value of the end displacement u_3 at end e, as shown in Fig. 5.3(e), to determine the coefficients k_{i3} ($i = 1$ through 4) in the third column of the member stiffness matrix . The bending moment at a distance x from end b of the member is given by

$$M = -k_{23} + k_{13}x \tag{5.31}$$

Substitution of Eq. (5.31) into the beam deflection differential equation (Eq. (5.5)) yields

$$\frac{d^2\bar{u}_y}{dx^2} = \frac{1}{\ I}(-k_{23} + k_{13}x) \tag{5.32}$$

By integrating Eq. (5.32) twice, we obtain

$$\theta = \frac{d\bar{u}_y}{dx} = \frac{1}{\ I}\left(-k_{23}x + k_{13}\frac{x^2}{2}\right) + C_1 \tag{5.33}$$

$$\bar{u}_y = \frac{1}{\ I}\left(-k_{23}\frac{x^2}{2} + k_{13}\frac{x^3}{6}\right) + C_1x + C_2 \tag{5.34}$$

The four unknowns, C_1, C_2, k_{13} and k_{23}, in Eqs. (5.33) and (5.34) are evaluated using the boundary conditions, as follows.

$$
\begin{array}{lll}
\text{At end } b, & x = 0, & \theta = 0 \\
 & x = 0, & \bar{u}_y = 0 \\
\text{At end } e, & x = L, & \theta = 0 \\
 & x = L, & \bar{u}_y = 1
\end{array}
$$

Using the first two boundary conditions, we obtain $C_1 = C_2 = 0$. Application of the third boundary condition yields

$$0 = \frac{1}{I}\left(-k_{23}L + k_{13}\frac{L^2}{2}\right)$$

from which

$$k_{23} = k_{13}\frac{L}{2} \tag{5.35}$$

and, using the last boundary condition, we obtain

$$1 = \frac{1}{I}\left(-k_{23}\frac{L^2}{2} + k_{13}\frac{L^3}{6}\right)$$

from which

$$k_{23} = -\frac{2 \quad I}{L^2} + k_{13}\frac{L}{3} \tag{5.36}$$

By substituting Eq. (5.35) into Eq. (5.36), we determine the stiffness coefficient k_{13} to be

$$\boxed{k_{13} = -\frac{12 \quad I}{L^3}} \tag{5.37}$$

and the substitution of Eq. (5.37) into Eq. (5.35) yields

$$\boxed{k_{23} = -\frac{6 \quad I}{L^2}} \tag{5.38}$$

The two remaining stiffness coefficients, k_{33} and k_{43}, are determined by considering the equilibrium of the free body of the member (Fig. 5.3(e)):

$$+\uparrow \sum F_y = 0 \qquad -\frac{12 \quad I}{L^3} + k_{33} = 0$$

$$\boxed{k_{33} = \frac{12 \quad I}{L^3}} \tag{5.39}$$

$$+\circlearrowleft \sum M_e = 0 \qquad -\frac{6\ \ I}{L^2} + \frac{12\ \ I}{L^3}(L) + k_{43} = 0$$

$$\boxed{k_{43} = -\frac{6\ \ I}{L^2}} \tag{5.40}$$

and the shape function N_3 for the member is obtained by substituting Eqs. (5.37) and (5.38) into Eq. (5.34) with $C_1 = C_2 = 0$. Thus,

$$\boxed{N_3 = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3} \tag{5.41}$$

To determine the stiffness coefficients k_{i4} ($i = 1$ through 4) in the last (fourth) column of , we subject the member to a unit value of the end displacement u_4 at end e, as shown in Fig. 5.3(f). The bending moment in the member is given by

$$M = -k_{24} + k_{14}x \tag{5.42}$$

Substitution of Eq. (5.42) into Eq. (5.5) yields

$$\frac{d^2\bar{u}_y}{dx^2} = \frac{1}{\ \ I}(-k_{24} + k_{14}x) \tag{5.43}$$

By integrating Eq. (5.43) twice, we obtain

$$\theta = \frac{d\bar{u}_y}{dx} = \frac{1}{\ \ I}\left(-k_{24}x + k_{14}\frac{x^2}{2}\right) + C_1 \tag{5.44}$$

$$\bar{u}_y = \frac{1}{\ \ I}\left(-k_{24}\frac{x^2}{2} + k_{14}\frac{x^3}{6}\right) + C_1x + C_2 \tag{5.45}$$

To evaluate the four unknowns, C_1, C_2, k_{14} and k_{24}, in Eqs. (5.44) and (5.45), we use the boundary conditions, as follows.

At end b, $\quad x = 0, \quad \theta = 0$
$\qquad\qquad\quad x = 0, \quad \bar{u}_y = 0$

At end e, $\quad x = L, \quad \theta = 1$
$\qquad\qquad\quad x = L, \quad \bar{u}_y = 0$

Application of the first two boundary conditions yields $C_1 = C_2 = 0$. Using the third boundary condition, we obtain

$$1 = \frac{1}{\ \ I}\left(-k_{24}L + k_{14}\frac{L^2}{2}\right)$$

or

$$k_{24} = -\frac{\ \ I}{L} + k_{14}\frac{L}{2} \tag{5.46}$$

and the use of the fourth boundary condition yields

$$0 = \frac{1}{I}\left(-k_{24}\frac{L^2}{2} + k_{14}\frac{L^3}{6}\right)$$

from which

$$k_{24} = k_{14}\frac{L}{3} \tag{5.47}$$

By substituting Eq. (5.47) into Eq. (5.46), we obtain the stiffness coefficient k_{14}:

$$\boxed{k_{14} = \frac{6\ \ I}{L^2}} \tag{5.48}$$

and by substituting Eq. (5.48) into Eq. (5.47), we obtain

$$\boxed{k_{24} = \frac{2\ \ I}{L}} \tag{5.49}$$

Next, we determine the remaining stiffness coefficients by considering the equilibrium of the free body of the member (Fig. 5.3(f)):

$$+\uparrow \sum F_y = 0 \qquad \frac{6\ \ I}{L^2} + k_{34} = 0$$

$$\boxed{k_{34} = -\frac{6\ \ I}{L^2}} \tag{5.50}$$

$$+\circlearrowleft \sum M_e = 0 \qquad \frac{2\ \ I}{L} - \frac{6\ \ I}{L^2}(L) + k_{44} = 0$$

$$\boxed{k_{44} = \frac{4\ \ I}{L}} \tag{5.51}$$

To obtain the shape function N of the beam, we substitute Eqs. (5.48) and (5.49) into Eq. (5.45), yielding

$$\boxed{N_4 = \frac{x^2}{L}\left(-1 + \frac{x}{L}\right)} \tag{5.52}$$

Finally, by substituting the expressions for the stiffness coefficients (Eqs. (5.14–5.17), (5.26–5.29), (5.37–5.40), and (5.48–5.51)), into the matrix form of given in Eq. (5.3), we obtain the following local stiffness matrix for the members of beams.

$$= \frac{I}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \qquad \textbf{(5.53)}$$

Note that the stiffness matrix is symmetric; that is, $k_{ij} = k_{ji}$.

EXAMPLE 5.2 Determine the stiffness matrices for the members of the beam shown in Fig. 5.5.

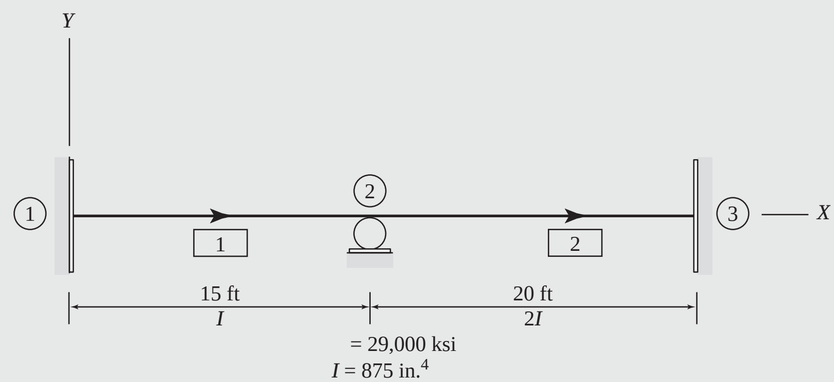


Fig. 5.5

SOLUTION **e ber 1** $= 29{,}000$ ksi, $I = 875$ in.^{4}, $L = 15$ ft $= 180$ in.

$$\frac{I}{L^3} = \frac{29{,}000\,(875)}{(180)^3} = 4.351 \text{ k/in.}$$

Substitution in Eq. (5.53) yields

$$_1 = \begin{bmatrix} 52.212 & 4{,}699.1 & -52.212 & 4{,}699.1 \\ 4{,}699.1 & 563{,}889 & -4{,}699.1 & 281{,}944 \\ -52.212 & -4{,}699.1 & 52.212 & -4{,}699.1 \\ 4{,}699.1 & 281{,}944 & -4{,}699.1 & 563{,}889 \end{bmatrix} \qquad \textbf{Ans}$$

e ber 2 $= 29{,}000$ ksi, $I = 1{,}750$ in.^{4}, $L = 20$ ft $= 240$ in.

$$\frac{I}{L^3} = \frac{29{,}000\,(1{,}750)}{(240)^3} = 3.6712 \text{ k/in.}$$

Thus, from Eq. (5.53)

$$_2 = \begin{bmatrix} 44.054 & 5{,}286.5 & -44.054 & 5{,}286.5 \\ 5{,}286.5 & 845{,}833 & -5{,}286.5 & 422{,}917 \\ -44.054 & -5{,}286.5 & 44.054 & -5{,}286.5 \\ 5{,}286.5 & 422{,}917 & -5{,}286.5 & 845{,}833 \end{bmatrix} \qquad \textbf{Ans}$$

5.3 FINITE-ELEMENT FORMULATION USING VIRTUAL WORK*

The member stiffness matrix , as given by Eq. (5.53), is usually derived in the finite-element method by applying the principle of virtual work. The formulation involves essentially the same general steps that were outlined in Section 3.4 for the case of the members of plane trusses.

Displacement Function

Consider a prismatic member of a beam, subjected to end displacements u_1 through u_4, as shown in Fig. 5.6. Since the member displaces only in the y direction, only one displacement function $\bar{u}_y$ needs to be defined. In Fig. 5.6, the displacement function $\bar{u}_y$ is depicted as the displacement of an arbitrary point G located on the member's centroidal axis (which coincides with the neutral axis) at a distance x from the end b.

As discussed in Section 3.4, in the finite-element method, a displacement function is usually *assumed* in the form of a complete polynomial of such a degree that all of its coefficients can be evaluated from the available boundary conditions of the member. From Fig. 5.6, we realize that the boundary conditions for the member under consideration are as follows.

$$\text{At end } b, \qquad x = 0, \qquad \bar{u}_y = u_1 \tag{5.54a}$$

$$x = 0, \qquad \theta = \frac{d\bar{u}_y}{dx} = u_2 \tag{5.54b}$$

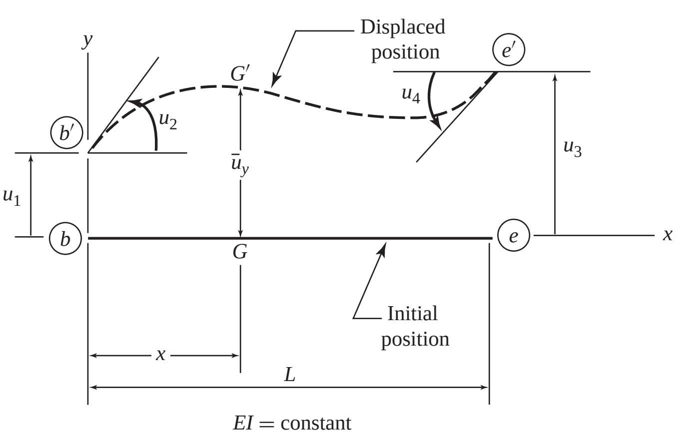


Fig. 5.6

*This section can be omitted without loss of continuity.

$$\text{At end } e, \qquad x = L, \qquad \bar{u}_y = u_3 \tag{5.54c}$$

$$x = L, \qquad \theta = \frac{d\bar{u}_y}{dx} = u_4 \tag{5.54d}$$

Since there are four boundary conditions, we can use a cubic polynomial (with four coefficients) for the displacement function $\bar{u}_y$, as

$$\bar{u}_y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{5.55}$$

in which a_0 through a_3 are the constants to be determined by applying the four boundary conditions specified in Eqs. (5.54). By differentiating Eq. (5.55) with respect to x, we obtain the equation for the slope of the member as

$$\theta = \frac{d\bar{u}_y}{dx} = a_1 + 2a_2 x + 3a_3 x^2 \tag{5.56}$$

Now, we apply the first boundary condition (Eq. (5.54a)) by setting $x = 0$ and $\bar{u}_y = u_1$ in Eq. (5.55). This yields

$$a_0 = u_1 \tag{5.57}$$

Similarly, using the second boundary condition—that is, by setting $x = 0$ and $\theta = u_2$ in Eq. (5.56)—we obtain

$$a_1 = u_2 \tag{5.58}$$

Next, we apply the third boundary condition, setting $x = L$ and $\bar{u}_y = u_3$ in Eq. (5.55). This yields

$$u_3 = a_0 + a_1 L + a_2 L^2 + a_3 L^3 \tag{5.59}$$

By substituting $a_0 = u_1$ (Eq. (5.57)) and $a_1 = u_2$ (Eq. (5.58)) into Eq. (5.59), we obtain

$$a_3 = \frac{1}{L^3}\left(-u_1 - u_2 L + u_3 - a_2 L^2\right) \tag{5.60}$$

To apply the fourth boundary condition (Eq. (5.54d)), we set $x = L$ and $\theta = u_4$ in Eq. (5.56). This yields

$$u_4 = a_1 + 2a_2 L + 3a_3 L^2 \tag{5.61}$$

By substituting Eqs. (5.57), (5.58), and (5.60) into Eq. (5.61), and solving the resulting equation for a_2, we obtain

$$a_2 = \frac{1}{L^2}\left(-3u_1 - 2u_2 L + 3u_3 - u_4 L\right) \tag{5.62}$$

and the backsubstitution of Eq. (5.62) into Eq. (5.60) yields

$$a_3 = \frac{1}{L^3}\left(2u_1 + u_2 L - 2u_3 + u_4 L\right) \tag{5.63}$$

Finally, by substituting Eqs. (5.57), (5.58), (5.62), and (5.63) into Eq. (5.55), we obtain the following expression for the displacement function $\bar{u}_y$, in terms of the end displacements u_1 through u_4.

$$\boxed{\begin{aligned}\bar{u}_y = &\left[1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3\right]u_1 + \left[x\left(1 - \frac{x}{L}\right)^2\right]u_2 \\ &+ \left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right]u_3 + \left[\frac{x^2}{L}\left(-1 + \frac{x}{L}\right)\right]u_4\end{aligned}} \tag{5.64}$$

Shape Functions

The displacement function $\bar{u}_y$, as given by Eq. (5.64), can alternatively be written as

$$\bar{u}_y = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4 \tag{5.65}$$

with

$$N_1 = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 \tag{5.66a}$$

$$N_2 = x\left(1 - \frac{x}{L}\right)^2 \tag{5.66b}$$

$$N_3 = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 \tag{5.66c}$$

$$N_4 = \frac{x^2}{L}\left(-1 + \frac{x}{L}\right) \tag{5.66d}$$

in which N_i ($i = 1$ through 4) are the member shape functions. A comparison of Eqs. (5.66a) through (5.66d) with Eqs. (5.19), (5.30), (5.41), and (5.52), respectively, indicates that the shape functions determined herein by assuming a cubic displacement function are identical to those obtained in Section 5.2 by exactly solving the differential equation for bending of beams. This is because a cubic polynomial represents the actual (or exact) solution of the governing differential equation (Eq. (5.5)), provided that the member is prismatic and it is not subjected to any external loading.

Equation (5.65) can be written in matrix form as

$$\bar{u}_y = [\, N_1 \quad N_2 \quad N_3 \quad N_4 \,] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \tag{5.67}$$

or, symbolically, as

$$\boxed{\bar{u}_y = \quad \mathbf{u}} \tag{5.68}$$

in which is the member shape-function matrix.

Strain–Displacement Relationship

We recall from *mechanics of materials* that the normal (longitudinal) strain ε in a fiber of a member, located at a distance y above the neutral axis, can be expressed in terms of the displacement $\bar{u}_y$ of the member's neutral axis, by the relationship

$$\varepsilon = -y \frac{d^2 \bar{u}_y}{dx^2} \tag{5.69}$$

in which the minus sign indicates that the tensile strain is considered positive. By substituting Eq. (5.68) into Eq. (5.69), we write

$$\varepsilon = -y \frac{d^2}{dx^2} (\quad \mathbf{u}) \tag{5.70}$$

Since the end-displacement vector **u** is not a function of x, it can be treated as a constant for the purpose of differentiation. Thus, Eq. (5.70) can be expressed as

$$\boxed{\varepsilon = \left(-y \frac{d^2}{dx^2}\right) \mathbf{u} = \mathbf{B}\mathbf{u}} \tag{5.71}$$

To determine the member strain-displacement matrix **B**, we write

$$\mathbf{B} = -y \frac{d^2}{dx^2} = -y \left[\frac{d^2 N_1}{dx^2} \quad \frac{d^2 N_2}{dx^2} \quad \frac{d^2 N_3}{dx^2} \quad \frac{d^2 N_4}{dx^2} \right] \tag{5.72}$$

By differentiating twice the equations for the shape functions as given by Eqs. (5.66), and substituting the resulting expressions into Eq. (5.72), we obtain

$$\mathbf{B} = -\frac{y}{L^2} \left[6\left(-1 + 2\frac{x}{L}\right) \quad 2L\left(-2 + 3\frac{x}{L}\right) \quad 6\left(1 - 2\frac{x}{L}\right) \quad 2L\left(-1 + 3\frac{x}{L}\right) \right] \tag{5.73}$$

Stress–Displacement Relationship

To establish the relationship between the member normal stress and the end displacements, we substitute Eq. (5.71) into the stress–strain relation $\sigma = \quad \varepsilon$. This yields

$$\boxed{\sigma = \quad \mathbf{B}\mathbf{u}} \tag{5.74}$$

Member Stiffness Matrix, k

With both the member strain and stress expressed in terms of end displacements, we can now establish the member stiffness matrix by applying the principle of virtual work for deformable bodies. Consider an arbitrary member of a beam in equilibrium under the action of end forces $_1$ through $_4$, as shown in Fig. 5.7. Note that the member is not subjected to any external loading between its ends; therefore, the fixed-end forces $_f$ are 0.

Now, assume that the member is given small virtual end displacements δu_1 through δu_4, as shown in Fig. 5.7. The virtual external work done by the real member end forces $_1$ through $_4$ as they move through the corresponding virtual end displacements δu_1 through δu_4 is

$$\delta W_e = {}_1\delta u_1 + {}_2\delta u_2 + {}_3\delta u_3 + {}_4\delta u_4$$

which can be written in matrix form as

$$\delta W_e = \delta \mathbf{u}^T \tag{5.75}$$

Substitution of Eq. (5.75) into the expression for the principle of virtual work for deformable bodies as given in Eq. (3.28) in Section 3.4, yields

$$\delta \mathbf{u}^T \quad = \int_V \delta \varepsilon^T \sigma \, dV \tag{5.76}$$

in which the right-hand side represents the virtual strain energy stored in the member. By substituting Eqs. (5.71) and (5.74) into Eq. (5.76), we obtain

$$\delta \mathbf{u}^T \quad = \int_V (\mathbf{B}\,\delta\mathbf{u})^T \quad \mathbf{B}\, dV\, \mathbf{u}$$

Since $(\mathbf{B}\,\delta\mathbf{u})^T = \delta\mathbf{u}^T\mathbf{B}^T$, the foregoing equation becomes

$$\delta \mathbf{u}^T \quad = \delta\mathbf{u}^T \int_V \mathbf{B}^T \quad \mathbf{B}\, dV\, \mathbf{u}$$

or

$$\delta \mathbf{u}^T \left(\quad - \int_V \mathbf{B}^T \quad \mathbf{B}\, dV\, \mathbf{u} \right) = 0$$

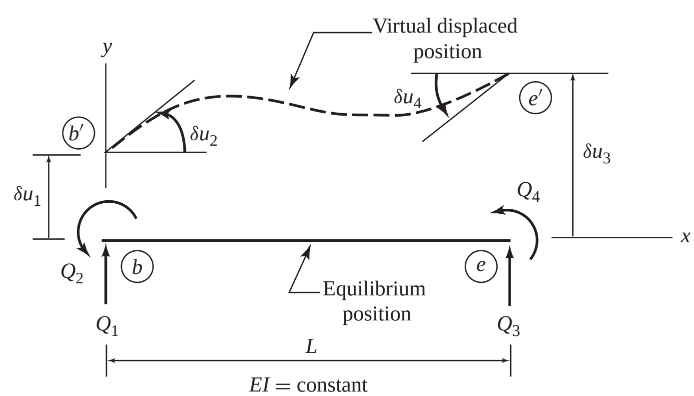


Fig. 5.7

As $\delta\mathbf{u}^T$ may be arbitrarily chosen and is not 0, the quantity in the parentheses must be 0; thus,

$$= \left(\int_V \mathbf{B}^T \quad \mathbf{B}\, dV \right) \mathbf{u} = \quad \mathbf{u} \tag{5.77}$$

with

$$\boxed{ = \int_V \mathbf{B}^T \quad \mathbf{B}\, dV } \tag{5.78}$$

Note that the foregoing general form of for beam members is the same as that obtained in Section 3.4 for the members of plane trusses (Eq. (3.55)). To explicitly determine the member stiffness matrix , we substitute Eq. (5.73) for **B** into Eq. (5.78). This yields

$$= \frac{\;}{L^4} \int_V y^2 \begin{bmatrix} 6\left(-1 + 2\dfrac{x}{L}\right) \\ 2L\left(-2 + 3\dfrac{x}{L}\right) \\ 6\left(1 - 2\dfrac{x}{L}\right) \\ 2L\left(-1 + 3\dfrac{x}{L}\right) \end{bmatrix} \left[6\left(-1 + 2\frac{x}{L}\right) \quad 2L\left(-2 + 3\frac{x}{L}\right) \quad 6\left(1 - 2\frac{x}{L}\right) \quad 2L\left(-1 + 3\frac{x}{L}\right) \right] dV \tag{5.79}$$

By substituting $dV = (dA)\, dx$ into Eq. (5.79), and realizing that $\int_A y^2\, dA = I$, we obtain

$$= \frac{I}{L^4} \int_0^L \\ \times \begin{bmatrix} 36\left(-1+2\frac{x}{L}\right)^2 & 12L\left(-2+3\frac{x}{L}\right)\left(-1+2\frac{x}{L}\right) & -36\left(-1+2\frac{x}{L}\right)^2 & 12L\left(-1+3\frac{x}{L}\right)\left(-1+2\frac{x}{L}\right) \\ 12L\left(-2+3\frac{x}{L}\right)\left(-1+2\frac{x}{L}\right) & 4L^2\left(-2+3\frac{x}{L}\right)^2 & 12L\left(-2+3\frac{x}{L}\right)\left(1-2\frac{x}{L}\right) & 4L^2\left(-2+3\frac{x}{L}\right)\left(-1+3\frac{x}{L}\right) \\ -36\left(-1+2\frac{x}{L}\right)^2 & 12L\left(-2+3\frac{x}{L}\right)\left(1-2\frac{x}{L}\right) & 36\left(-1+2\frac{x}{L}\right)^2 & 12L\left(-1+3\frac{x}{L}\right)\left(1-2\frac{x}{L}\right) \\ 12L\left(-1+3\frac{x}{L}\right)\left(-1+2\frac{x}{L}\right) & 4L^2\left(-2+3\frac{x}{L}\right)\left(-1+3\frac{x}{L}\right) & 12L\left(-1+3\frac{x}{L}\right)\left(1-2\frac{x}{L}\right) & 4L^2\left(-1+3\frac{x}{L}\right)^2 \end{bmatrix} dx \tag{5.80}$$

which, upon integration, becomes

$$= \frac{I}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Note that the foregoing expression for is identical to that derived in Section 5.2 (Eq. (5.53)) by directly integrating the differential equation for beam deflection and applying the equilibrium equations.

5.4 MEMBER FIXED-END FORCES DUE TO LOADS

It was shown in Section 5.2 that the stiffness relationships for a member of a beam can be written in matrix form (see Eq. (5.4)) as

$$= \mathbf{u} + {}_f$$

As the foregoing relationship indicates, the total forces that can develop at the ends of a member can be expressed as the sum of the forces **u** due to the end displacements **u**, and the fixed-end forces $_f$ that would develop at the member ends due to external loads if both member ends were fixed against translations and rotations.

In this section, we consider the derivation of the expressions for fixed-end forces due to external loads applied to the members of beams. To illustrate the procedure, consider a fixed member subjected to a concentrated load W, as shown in Fig. 5.8(a). As indicated in this figure, the fixed-end moments at the member ends b and e are denoted by FM_b and FM_e, respectively, whereas FS_b and FS_e denote the fixed-end shears at member ends b and e, respectively. Our objective is to determine expressions for the fixed-end moments and shears in terms of the magnitude and location of the load W; we will use the direct integration approach, along with the equations of equilibrium, for this purpose.

As the concentrated load W acts at point A of the member (Fig. 5.8(a)), the bending moment M cannot be expressed as a single continuous function of x over the entire length of the member. Therefore, we divide the member into two segments, bA and Ae; and we determine the following equations for bending moment in segments bA and Ae, respectively:

$$0 \le x \le l_1 \qquad M = -FM_b + FS_b x \tag{5.81}$$

$$l_1 \le x \le L \qquad M = -FM_b + FS_b x - W(x - l_1) \tag{5.82}$$

By substituting Eqs. (5.81) and (5.82) into the differential equation for beam deflection (Eq. (5.5)), we obtain, respectively,

$$0 \le x \le l_1 \qquad \frac{d^2\bar{u}_y}{dx^2} = \frac{1}{I}\left(-FM_b + FS_b x\right) \tag{5.83}$$

$$l_1 \le x \le L \qquad \frac{d^2\bar{u}_y}{dx^2} = \frac{1}{I}\left[-FM_b + FS_b x - W(x - l_1)\right] \tag{5.84}$$

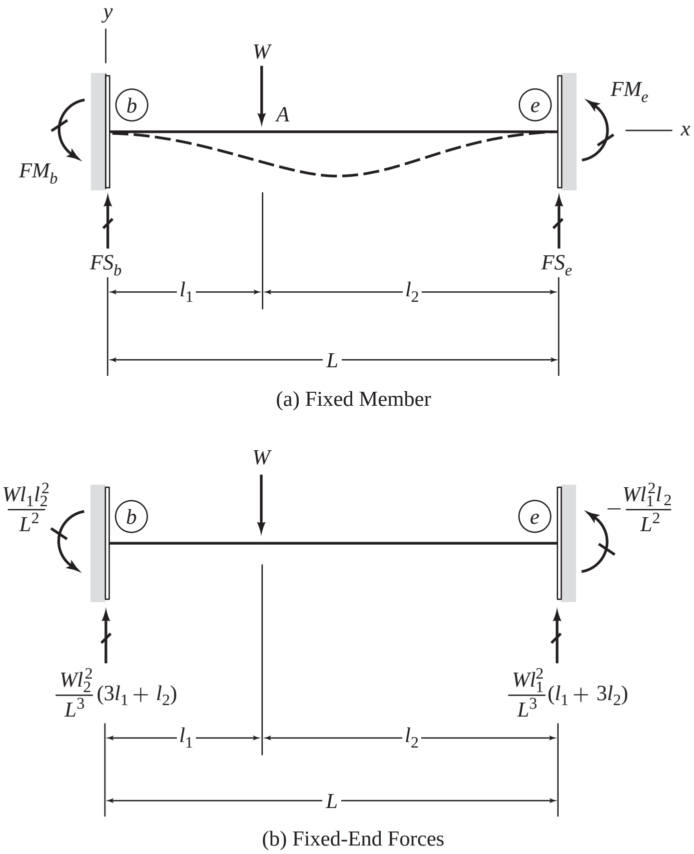


Fig. 5.8

By integrating Eq. (5.83) twice, we obtain the equations for the slope and deflection in segment *bA* of the member:

$$0 \le x \le l_1 \qquad \theta = \frac{d\bar{u}_y}{dx} = \frac{1}{I}\left(-FM_b x + FS_b \frac{x^2}{2}\right) + C_1 \tag{5.85}$$

$$0 \le x \le l_1 \qquad \bar{u}_y = \frac{1}{I}\left(-FM_b \frac{x^2}{2} + FS_b \frac{x^3}{6}\right) + C_1 x + C_2 \tag{5.86}$$

Similarly, by integrating Eq. (5.84) twice, we obtain the equations for the slope and deflection in the segment *Ae*:

$$l_1 \le x \le L \qquad \theta = \frac{1}{I}\left[-FM_b x + FS_b \frac{x^2}{2} - \frac{Wx}{2}(x - 2l_1)\right] + C_3 \tag{5.87}$$

$$l_1 \le x \le L \qquad \bar{u}_y = \frac{1}{I}\left[-FM_b \frac{x^2}{2} + FS_b \frac{x^3}{6} - \frac{Wx^2}{6}(x - 3l_1)\right] + C_3 x + C_4 \tag{5.88}$$

Equations (5.85) through (5.88) indicate that the equations for the slope and deflection of the member contain a total of six unknowns; that is, four constants of integration, C_1 through C_4, and two fixed-end forces, FM_b and FS_b. These six unknowns can be evaluated by applying four boundary conditions (i.e., the slopes and deflections at the two fixed ends, b and e, must be 0), and two continuity conditions requiring that the slope and the deflection of the member's elastic curve be continuous at point A. By applying the two boundary conditions for the fixed end b (i.e., at $x = 0, \theta = \bar{u}_y = 0$) to Eqs. (5.85) and (5.86), we obtain

$$C_1 = C_2 = 0 \tag{5.89}$$

Next, to evaluate the constant C_3, we use the condition that the slope must be continuous at point A. This condition requires that the two slope equations (Eqs. (5.85) and (5.87)) yield the same slope θ_A at $x = l_1$. By setting $x = l_1$ in Eqs. (5.85) and (5.87), and equating the resulting expressions, we obtain

$$\frac{1}{I}\left(-FM_b l_1 + FS_b \frac{l_1^2}{2}\right) = \frac{1}{I}\left(-FM_b l_1 + FS_b \frac{l_1^2}{2} + \frac{Wl_1^2}{2}\right) + C_3$$

By solving for C_3, we determine that

$$C_3 = -\frac{Wl_1^2}{2 \quad I} \tag{5.90}$$

In a similar manner, we evaluate the constant C_4 by applying the condition of continuity of deflection at point A. By setting $x = l_1$ in the two deflection equations (Eqs. (5.86) and (5.88)), and equating the resulting expressions, we obtain

$$\frac{1}{I}\left(-FM_b \frac{l_1^2}{2} + FS_b \frac{l_1^3}{6}\right)$$
$$= \frac{1}{I}\left(-FM_b \frac{l_1^2}{2} + FS_b \frac{l_1^3}{6} + \frac{Wl_1^3}{3}\right) - \frac{Wl_1^3}{2 \quad I} + C_4$$

from which

$$C_4 = \frac{Wl_1^3}{6 \quad I} \tag{5.91}$$

With the four constants of integration known, we can now evaluate the two remaining unknowns, FS_b and FM_b, by applying the boundary conditions that the slope and deflection at the fixed end e must be 0 (i.e., at $x = L, \theta = \bar{u}_y = 0$). By setting $x = L$ in Eqs. (5.87) and (5.88), with $\theta = 0$ in Eq. (5.87) and $\bar{u}_y = 0$ in Eq. (5.88), we obtain

$$\frac{1}{I}\left[-FM_b L + FS_b \frac{L^2}{2} - \frac{WL}{2}(L - 2l_1)\right] - \frac{Wl_1^2}{2 \quad I} = 0 \tag{5.92}$$

$$\frac{1}{I}\left[-FM_b \frac{L^2}{2} + FS_b \frac{L^3}{6} - \frac{WL^2}{6}(L - 3l_1)\right] - \frac{Wl_1^2 L}{2 \quad I} + \frac{Wl_1^3}{6 \quad I} = 0 \tag{5.93}$$

To solve Eqs. (5.92) and (5.93) for FS_b and FM_b, we rewrite Eq. (5.92) to express FM_b in terms of FS_b as

$$FM_b = FS_b \frac{L}{2} - \frac{W}{2}(L - 2l_1) - \frac{Wl_1^2}{2L} \tag{5.94}$$

By substituting Eq. (5.94) into Eq. (5.93), and solving the resulting equation for FS_b, we obtain

$$FS_b = \frac{W}{L^3}\left(L^3 - 3l_1^2 L + 2l_1^3\right)$$

Substitution of $L = l_1 + l_2$ into the numerator of the foregoing equation yields the expression for the fixed-end shear FS_b as

$$FS_b = \frac{Wl_2^2}{L^3}(3l_1 + l_2) \tag{5.95}$$

By back substituting Eq. (5.95) into Eq. (5.94), we obtain the expression for the fixed-end moment as

$$FM_b = \frac{Wl_1 l_2^2}{L^2} \tag{5.96}$$

Finally, the fixed-end forces, FS_e and FM_e, at the member end e, can be determined by applying the equations of equilibrium to the free body of the member (Fig. 5.8(a)). Thus,

$$+\uparrow \sum F_y = 0 \qquad \frac{Wl_2^2}{L^3}(3l_1 + l_2) - W + FS_e = 0$$

By substituting $L = l_1 + l_2$ into the numerator and solving for FS_e, we obtain

$$FS_e = \frac{Wl_1^2}{L^3}(l_1 + 3l_2) \tag{5.97}$$

and

$$+ \circlearrowleft \sum M_e = 0 \qquad \frac{Wl_1 l_2^2}{L^2} - \frac{Wl_2^2}{L^3}(3l_1 + l_2)L + Wl_2 + FM_e = 0$$

$$FM_e = -\frac{Wl_1^2 l_2}{L^2} \tag{5.98}$$

in which the negative answer for FM_e indicates that its actual sense is clockwise for the loading condition under consideration. Figure 5.8(b) depicts the four fixed-end forces that develop in a member of a beam subjected to a single concentrated load.

The expressions for fixed-end forces due to other types of loading conditions can be derived by using the direct integration approach as illustrated here, or by employing another classical method, such as the *method of consistent deformations*. The expressions for fixed-end forces due to some common types of member loads are given inside the front cover of this book for convenient reference.

Member Fixed-End Force Vector $\mathbf{Q}_f$

Once the fixed-end forces for a member have been evaluated, its fixed-end force vector $_f$ can be generated by storing the fixed-end forces in their proper positions in a 4×1 vector. In accordance with the scheme for numbering member end forces adopted in Section 5.2, the fixed-end shear FS_b and the fixed-end moment FM_b, at the left end b of the member, must be stored in the first and second rows, respectively, of the $_f$ vector; the fixed-end shear FS_e and the fixed-end moment FM_e, at the opposite member end e, are stored in the third and fourth rows, respectively, of the $_f$ vector. Thus, the fixed-end force vector for a member of a beam (Fig 5.8(a)) is expressed as

$$_f = \begin{bmatrix} _{f1} \\ _{f2} \\ _{f3} \\ _{f4} \end{bmatrix} = \begin{bmatrix} FS_b \\ FM_b \\ FS_e \\ FM_e \end{bmatrix} \tag{5.99}$$

When storing numerical values or fixed-end force expressions in $_f$, the appropriate sign convention for member end forces must be followed. In accordance with the sign convention adopted in Section 5.2, the fixed-end shears are considered positive when upward (i.e., in the positive direction of the local y axis); the fixed-end moments are considered positive when counterclockwise. For example, the fixed-end force vector for the beam member shown in Fig. 5.8(b) is given by

$$_f = \begin{bmatrix} \dfrac{Wl_2^2}{L^3}(3l_1 + l_2) \\ \dfrac{Wl_1l_2^2}{L^2} \\ \dfrac{Wl_1^2}{L^3}(l_1 + 3l_2) \\ -\dfrac{Wl_1^2l_2}{L^2} \end{bmatrix}$$

EXAMPLE 5.3 Determine the fixed-end force vectors for the members of the two-span continuous beam shown in Fig. 5.9. Use the fixed-end force equations given inside the front cover.

SOLUTION **e ber 1** By substituting $w = 2$ k/ft, $L = 30$ ft, and $l_1 = l_2 = 0$ into the fixed-end force expressions given for loading type 3, we obtain

$$FS_b = FS_e = \frac{2(30)}{2} = 30 \text{ k}$$

$$FM_b = \frac{2(30)^2}{12} = 150 \text{ k-ft}$$

$$FM_e = -\frac{2(30)^2}{12} = -150 \text{ k-ft}$$

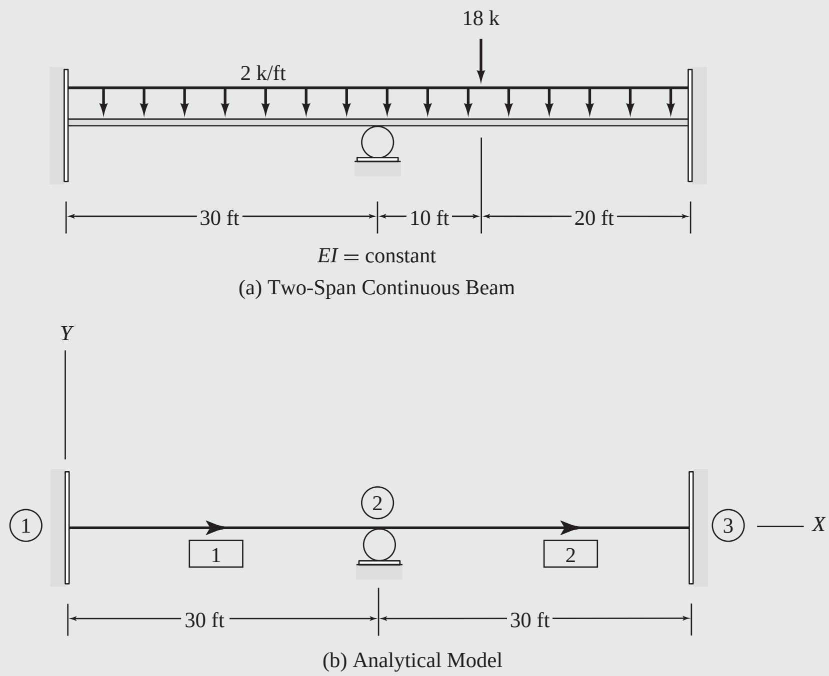


Fig. 5.9

By substituting these values of fixed-end forces into Eq. (5.99), we obtain the fixed-end force vector for member 1:

$$f_1 = \begin{bmatrix} 30 \\ 150 \\ 30 \\ -150 \end{bmatrix} \qquad \textbf{Ans}$$

e ber 2 From Fig. 5.9(a), we can see that this member is subjected to two different loadings—a concentrated load $W = 18$ k with $l_1 = 10$ ft, $l_2 = 20$ ft, and $L = 30$ ft (load type 1), and a uniformly distributed load $w = 2$ k/ft with $l_1 = l_2 = 0$ and $L = 30$ ft (load type 3). The fixed-end forces for such a member, due to the combined effect of several loads, can be conveniently determined by superimposing (algebraically adding) the fixed-end forces due to each of the loads acting individually on the member. By using superposition, we determine the fixed-end forces for member 2 to be

$$FS_b = \frac{18(20)^2}{(30)^3}[3(10) + (20)] + \frac{2\,(30)}{2} = 43.333 \text{ k}$$

$$FM_b = \frac{18\,(10)\,(20)^2}{(30)^2} + \frac{2\,(30)^2}{12} = 230 \text{ k-ft}$$

$$FS_e = \frac{18(10)^2}{(30)^3}[10 + 3(20)] + \frac{2\,(30)}{2} = 34.667 \text{ k}$$

$$FM_e = -\frac{18\,(10)^2\,(20)}{(30)^2} - \frac{2\,(30)^2}{12} = -190 \text{ k-ft}$$

Thus, the fixed-end force vector for member 2 is

$$f_2 = \begin{bmatrix} 43.333 \\ 230 \\ 34.667 \\ -190 \end{bmatrix}$$ **Ans**

5.5 STRUCTURE STIFFNESS RELATIONS

The procedure for establishing the structure stiffness relations for beams is essentially the same as that for plane trusses discussed in Section 3.7. The procedure, called the *direct stiffness method*, involves: (a) expressing the joint loads **P** in terms of the member end forces by applying the joint equilibrium equations; (b) relating the joint displacements **d** to the member end displacements **u** by using the compatibility conditions that the member end displacements and rotations must be the same as the corresponding joint displacements and rotations; and (c) linking the joint displacements **d** to the joint loads **P** by means of the member force-displacement relations $= \mathbf{u} + {}_{f}$.

Consider, for example, an arbitrary beam subjected to joint and member loads, as depicted in Fig. 5.10(a). The structure has three degrees of freedom, d_1 through d_3, as shown in Fig. 5.10(b). Our objective is to establish the structure stiffness relationships, which express the external loads as functions of the joint displacements **d**. The member end forces and end displacements **u** for the three members of the beam are given in Fig. 5.10(c), in which the superscript (i) denotes the member number.

By applying the equations of equilibrium $\sum F_Y = 0$ and $\sum M = 0$ to the free body of joint 2, and the equilibrium equation $\sum M = 0$ to the free body of joint 3, we obtain the following relationships between the external joint loads **P** and the internal member end forces .

$$P_1 = {}^{(1)}_{3} + {}^{(2)}_{1} \qquad \textbf{(5.100a)}$$

$$P_2 = {}^{(1)}_{4} + {}^{(2)}_{2} \qquad \textbf{(5.100b)}$$

$$P_3 = {}^{(2)}_{4} + {}^{(3)}_{2} \qquad \textbf{(5.100c)}$$

Next, we determine compatibility conditions for the three members of the beam. Since the left end 1 of member 1 is connected to fixed support 1 (Fig. 5.10(b)), which can neither translate nor rotate, the displacements $u_1^{(1)}$ and $u_2^{(1)}$ of end 1 of the member (Fig. 5.10(c)) must be 0. Similarly, since end 2 of this member is connected to joint 2, the displacements $u_3^{(1)}$ and $u_4^{(1)}$ of end 2 must be the same as the displacements d_1 and d_2, respectively, of joint 2. Thus, the compatibility equations for member 1 are:

$$u_1^{(1)} = u_2^{(1)} = 0 \qquad u_3^{(1)} = d_1 \qquad u_4^{(1)} = d_2 \qquad \textbf{(5.101)}$$

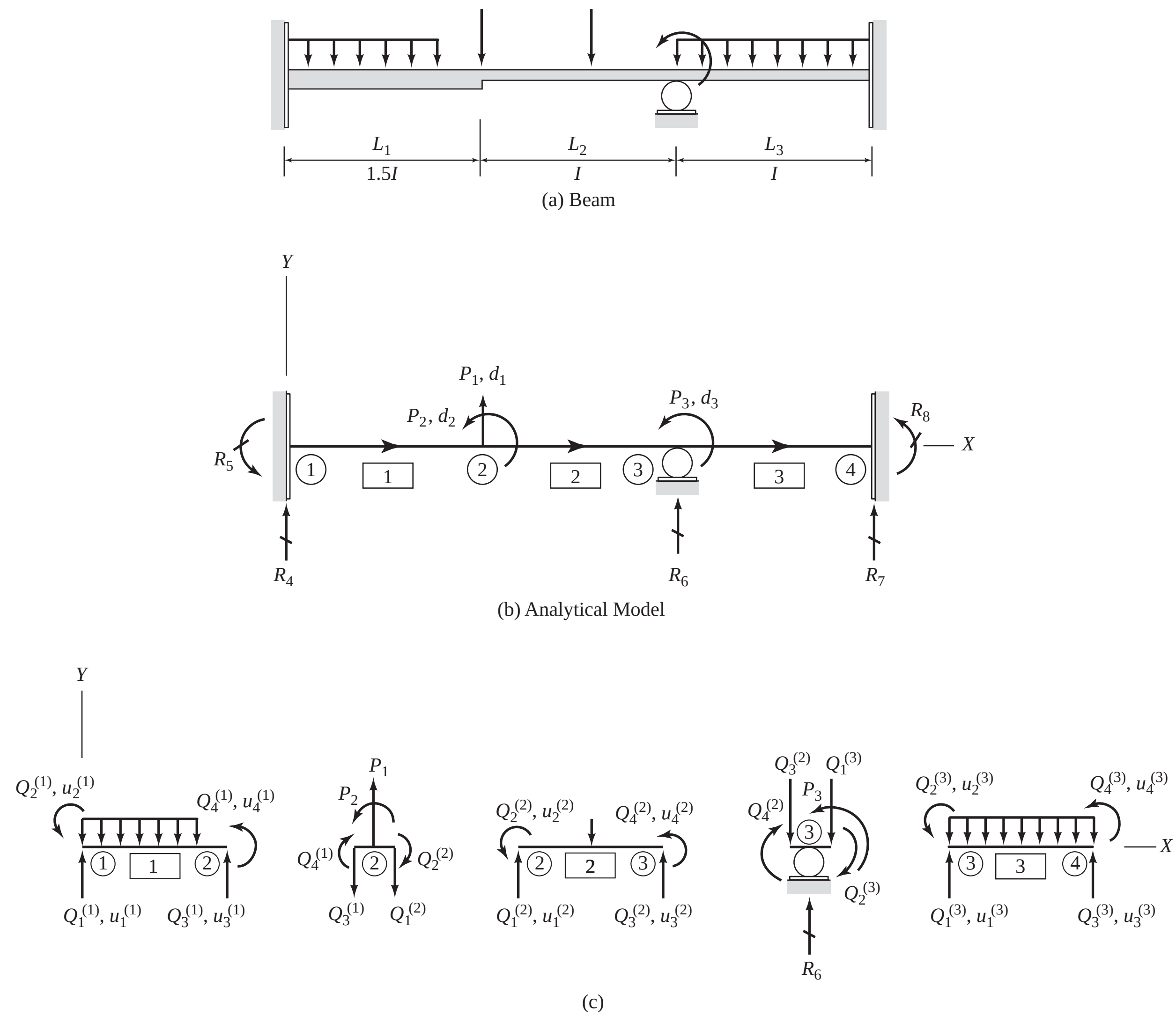


Fig. 5.10

In a similar manner, the compatibility equations for members 2 and 3, respectively, are given by

$$u_1^{(2)} = d_1 \qquad u_2^{(2)} = d_2 \qquad u_3^{(2)} = 0 \qquad u_4^{(2)} = d_3 \tag{5.102}$$

$$u_1^{(3)} = 0 \qquad u_2^{(3)} = d_3 \qquad u_3^{(3)} = u_4^{(3)} = 0 \tag{5.103}$$

The link between the joint equilibrium equations (Eqs. (5.100)) and the compatibility conditions (Eqs. (5.101) through (5.103)) is provided by the member stiffness relationship $\ = \mathbf{u} + \ _f$ (Eq. (5.4)). To express the six member end forces that appear in Eqs. (5.100) in terms of the member end displacements, we will use the expanded form of the member stiffness relationship given in Eqs. (5.2). Thus, the end forces ${}^{(1)}_{3}$ and ${}^{(1)}_{4}$, of member 1, can be expressed in terms of the member end displacements as

$$ {}^{(1)}_{3} = k^{(1)}_{31} u^{(1)}_{1} + k^{(1)}_{32} u^{(1)}_{2} + k^{(1)}_{33} u^{(1)}_{3} + k^{(1)}_{34} u^{(1)}_{4} + {}^{(1)}_{f3} \quad \textbf{(5.104a)} $$

$$ {}^{(1)}_{4} = k^{(1)}_{41} u^{(1)}_{1} + k^{(1)}_{42} u^{(1)}_{2} + k^{(1)}_{43} u^{(1)}_{3} + k^{(1)}_{44} u^{(1)}_{4} + {}^{(1)}_{f4} \quad \textbf{(5.104b)} $$

Similarly, the end forces ${}^{(2)}_{1}$, ${}^{(2)}_{2}$, and ${}^{(2)}_{4}$, of member 2, are written as

$$ {}^{(2)}_{1} = k^{(2)}_{11} u^{(2)}_{1} + k^{(2)}_{12} u^{(2)}_{2} + k^{(2)}_{13} u^{(2)}_{3} + k^{(2)}_{14} u^{(2)}_{4} + {}^{(2)}_{f1} \quad \textbf{(5.105a)} $$

$$ {}^{(2)}_{2} = k^{(2)}_{21} u^{(2)}_{1} + k^{(2)}_{22} u^{(2)}_{2} + k^{(2)}_{23} u^{(2)}_{3} + k^{(2)}_{24} u^{(2)}_{4} + {}^{(2)}_{f2} \quad \textbf{(5.105b)} $$

$$ {}^{(2)}_{4} = k^{(2)}_{41} u^{(2)}_{1} + k^{(2)}_{42} u^{(2)}_{2} + k^{(2)}_{43} u^{(2)}_{3} + k^{(2)}_{44} u^{(2)}_{4} + {}^{(2)}_{f4} \quad \textbf{(5.105c)} $$

and the end force ${}^{(3)}_{2}$, of member 3, is expressed as

$$ {}^{(3)}_{2} = k^{(3)}_{21} u^{(3)}_{1} + k^{(3)}_{22} u^{(3)}_{2} + k^{(3)}_{23} u^{(3)}_{3} + k^{(3)}_{24} u^{(3)}_{4} + {}^{(3)}_{f2} \quad \textbf{(5.106)} $$

Next, we relate the joint displacements **d** to the member end forces by substituting the compatibility equations, Eqs. (5.101), (5.102), and (5.103), into the member force-displacement relations given by Eqs. (5.104), (5.105), and (5.106), respectively. Thus,

$$ {}^{(1)}_{3} = k^{(1)}_{33} d_1 + k^{(1)}_{34} d_2 + {}^{(1)}_{f3} \quad \textbf{(5.107a)} $$

$$ {}^{(1)}_{4} = k^{(1)}_{43} d_1 + k^{(1)}_{44} d_2 + {}^{(1)}_{f4} \quad \textbf{(5.107b)} $$

$$ {}^{(2)}_{1} = k^{(2)}_{11} d_1 + k^{(2)}_{12} d_2 + k^{(2)}_{14} d_3 + {}^{(2)}_{f1} \quad \textbf{(5.107c)} $$

$$ {}^{(2)}_{2} = k^{(2)}_{21} d_1 + k^{(2)}_{22} d_2 + k^{(2)}_{24} d_3 + {}^{(2)}_{f2} \quad \textbf{(5.107d)} $$

$$ {}^{(2)}_{4} = k^{(2)}_{41} d_1 + k^{(2)}_{42} d_2 + k^{(2)}_{44} d_3 + {}^{(2)}_{f4} \quad \textbf{(5.107e)} $$

$$ {}^{(3)}_{2} = k^{(3)}_{22} d_3 + {}^{(3)}_{f2} \quad \textbf{(5.107f)} $$

Finally, by substituting Eqs. (5.107) into the joint equilibrium equations (Eqs. (5.100)), we establish the desired structure stiffness relationships as

$$ P_1 = \left(k^{(1)}_{33} + k^{(2)}_{11}\right) d_1 + \left(k^{(1)}_{34} + k^{(2)}_{12}\right) d_2 + k^{(2)}_{14} d_3 + \left({}^{(1)}_{f3} + {}^{(2)}_{f1} \right) \quad \textbf{(5.108a)} $$

$$ P_2 = \left(k^{(1)}_{43} + k^{(2)}_{21}\right) d_1 + \left(k^{(1)}_{44} + k^{(2)}_{22}\right) d_2 + k^{(2)}_{24} d_3 + \left({}^{(1)}_{f4} + {}^{(2)}_{f2} \right) \quad \textbf{(5.108b)} $$

$$ P_3 = k^{(2)}_{41} d_1 + k^{(2)}_{42} d_2 + \left(k^{(2)}_{44} + k^{(3)}_{22}\right) d_3 + \left({}^{(2)}_{f4} + {}^{(3)}_{f2} \right) \quad \textbf{(5.108c)} $$

Equations (5.108) can be conveniently expressed in matrix form as

$$ \mathbf{P} = \ \mathbf{d} + \mathbf{P}_f $$

or

$$ \boxed{\mathbf{P} - \mathbf{P}_f = \ \mathbf{d}} \quad \textbf{(5.109)} $$

in which

$$= \begin{bmatrix} k_{33}^{(1)} + k_{11}^{(2)} & k_{34}^{(1)} + k_{12}^{(2)} & k_{14}^{(2)} \\ k_{43}^{(1)} + k_{21}^{(2)} & k_{44}^{(1)} + k_{22}^{(2)} & k_{24}^{(2)} \\ k_{41}^{(2)} & k_{42}^{(2)} & k_{44}^{(2)} + k_{22}^{(3)} \end{bmatrix} \quad \textbf{(5.110)}$$

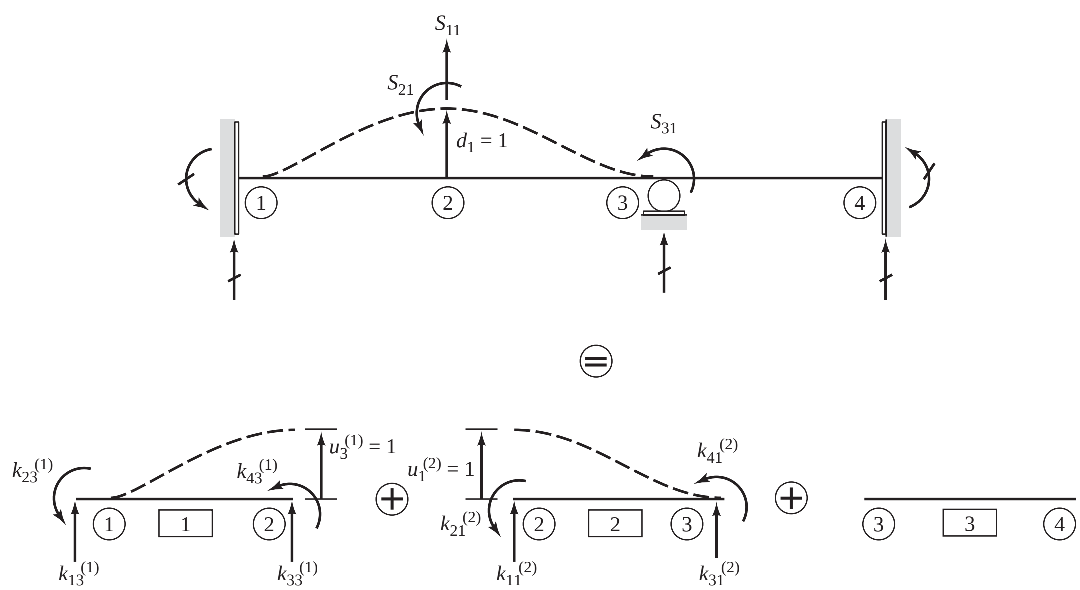
is the *NDOF* × *NDOF* structure stiffness matrix for the beam of Fig. 5.10(b), and

$$\mathbf{P}_f = \begin{bmatrix} {}^{(1)}_{f3} + {}^{(2)}_{f1} \\ {}^{(1)}_{f4} + {}^{(2)}_{f2} \\ {}^{(2)}_{f4} + {}^{(3)}_{f2} \end{bmatrix} \quad \textbf{(5.111)}$$

is the *NDOF* × 1 ***structure fixed-joint force vector***. The structure fixed-joint force vectors are further discussed in the following section. In the rest of this section, we focus our attention on the structure stiffness matrices.

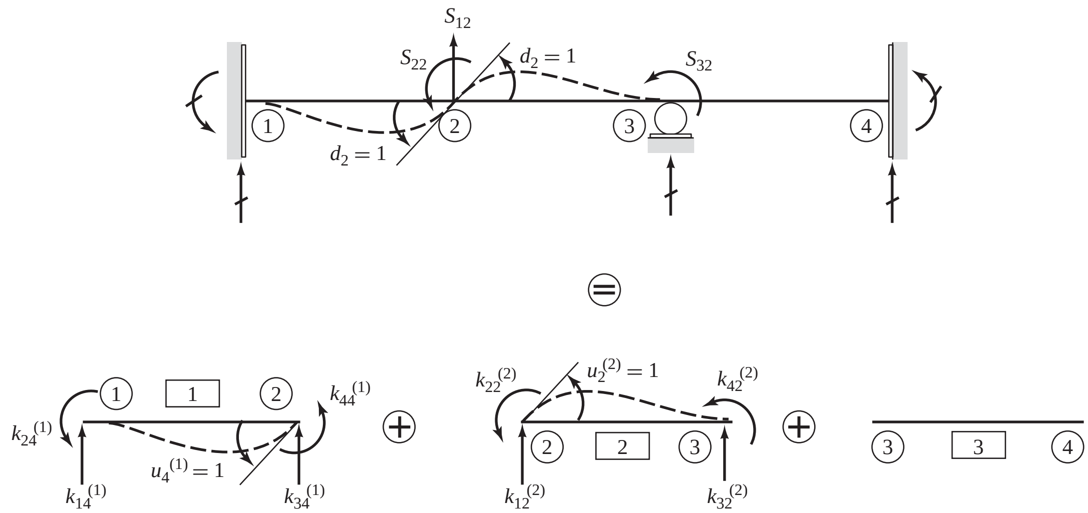
By examining Eq. (5.110), we realize that the structure stiffness matrix of the beam of Fig. 5.10(b) is symmetric, because of the symmetric nature of the member stiffness matrices (i.e., $k_{ij} = k_{ji}$). (The structure stiffness matrices of all linear elastic structures are always symmetric.) As discussed in Chapter 3, a structure stiffness coefficient S_{ij} represents the force at the location and in the direction of P_i required, along with other joint forces, to cause a unit value of the displacement d_j, while all other joint displacements are 0, and the structure is not subjected to any external loads. We can use this definition to verify the matrix (Eq. (5.110)) for the beam of Fig. 5.10.

In Figs. 5.11(a) through (c), the beam is subjected to the unit values of the three joint displacements d_1 through d_3, respectively. As depicted in Fig. 5.11(a),

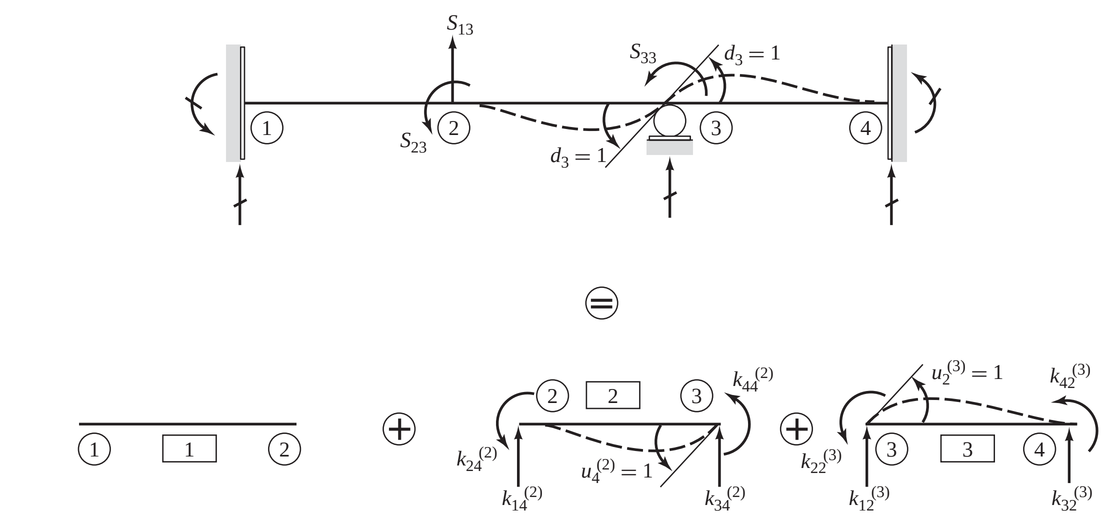


(a) First Column of **S** ($d_1 = 1$, $d_2 = d_3 = 0$)

Fig. 5.11



(b) Second Column of **S** ($d_2 = 1$, $d_1 = d_3 = 0$)



(c) Third Column of **S** ($d_3 = 1$, $d_1 = d_2 = 0$)

Fig. 5.11 (*continued*)

the joint displacement $d_1 = 1$ (with $d_2 = d_3 = 0$) induces unit displacements $u_3^{(1)} = 1$ at the right end of member 1 and $u_1^{(2)} = 1$ at the left end of member 2, while member 3 is not subjected to any displacements. The member stiffness coefficients (or end forces) necessary to cause the foregoing end displacements of the individual members are also shown in Fig. 5.11(a). (Recall that we derived the explicit expressions for member stiffness coefficients, in terms of , I, and L of the member, in Section 5.2.) From the figure, we can see that the total vertical joint force S_{11} at joint 2, required to cause the joint displacement $d_1 = 1$ (with $d_2 = d_3 = 0$), must be equal to the algebraic sum of the vertical forces at the two member ends connected to this joint; that is,

$$S_{11} = k_{33}^{(1)} + k_{11}^{(2)} \tag{5.112a}$$

Similarly, the total joint moment S_{21} at joint 2 must be equal to the algebraic sum of the moments at the ends of members 1 and 2 connected to joint 2. Thus, (Fig. 5.11(a)),

$$S_{21} = k_{43}^{(1)} + k_{21}^{(2)} \tag{5.112b}$$

and the total joint moment S_{31} at joint 3 must equal the algebraic sum of the moments at the two member ends connected to the joint; that is,

$$S_{31} = k_{41}^{(2)} \tag{5.112c}$$

Note that the foregoing expressions for s_{i1} ($i = 1$ through 3) are identical to those listed in the first column of in Eq. (5.110).

The second column of can be verified in a similar manner. From Fig. 5.11(b), we can see that the joint rotation $d_2 = 1$ (with $d_1 = d_3 = 0$) induces unit rotations $u_4^{(1)} = 1$ at the right end of member 1, and $u_2^{(2)} = 1$ at the left end of member 2. The member stiffness coefficients associated with these end displacements are also shown in the figure. By comparing the joint forces with the member end forces, we obtain the expressions for the structure stiffness coefficients as

$$S_{12} = k_{34}^{(1)} + k_{12}^{(2)} \tag{5.112d}$$

$$S_{22} = k_{44}^{(1)} + k_{22}^{(2)} \tag{5.112e}$$

$$S_{32} = k_{42}^{(2)} \tag{5.112f}$$

which are the same as those in the second column of in Eq. (5.110).

Similarly, by subjecting the beam to a unit rotation $d_3 = 1$ (with $d_1 = d_2 = 0$), as shown in Fig. 5.11(c), we obtain

$$S_{13} = k_{14}^{(2)} \tag{5.112g}$$

$$S_{23} = k_{24}^{(2)} \tag{5.112h}$$

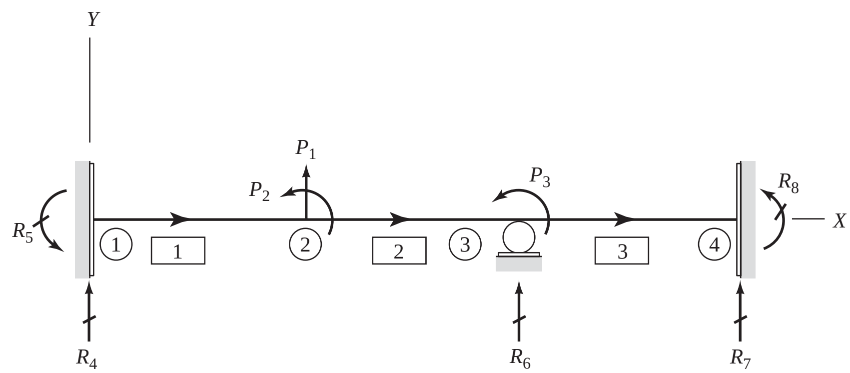
$$S_{33} = k_{44}^{(2)} + k_{22}^{(3)} \tag{5.112 }$$

The foregoing structure stiffness coefficients are identical to those listed in the third column of in Eq. (5.110).

Assembly of the Structure Stiffness Matrix Using Member Code Numbers

Although the procedures discussed thus far for formulating provide clearer insight into the basic concept of the structure stiffness matrix, it is more convenient from a computer programming viewpoint to directly form the structure stiffness matrix by assembling the elements of the member stiffness matrices . This technique, which is sometimes referred to as the *code number technique,* was described in detail in Section 3.7 for the case of plane trusses. The technique essentially involves storing the pertinent elements of the stiffness matrix for each member of the beam, in the structure stiffness matrix , by using the member code numbers. The code numbers for a member are simply the structure coordinate numbers at the location and in the direction of each of the member end displacements **u**, arranged in the order of the end displacements.

To illustrate this technique, consider again the three-member beam of Fig. 5.10. The analytical model of the beam is redrawn in Fig. 5.12(a), which shows its three degrees of freedom (numbered from 1 to 3) and five restrained coordinates (numbered from 4 to 8). In accordance with the notation for member end displacements adopted in Section 5.2, the first two end displacements of a member, u_1 and u_2, are always the vertical translation and rotation, respectively, at the left end (or beginning) of the member, whereas the last two end displacements, u_3 and u_4, are always the vertical translation and rotation, respectively, at the right end (or end) of the member. Thus, the first two code numbers for a member are always the structure coordinate numbers



Three degrees of freedom (1 through 3);
five restrained coordinates (4 through 8)

(a) Analytical Model

$$\mathbf{k}_1 = \begin{array}{c} \begin{array}{cccc} 4 & 5 & 1 & 2 \end{array} \\ \left[\begin{array}{cccc} k_{11}^{(1)} & k_{12}^{(1)} & k_{13}^{(1)} & k_{14}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} & k_{23}^{(1)} & k_{24}^{(1)} \\ k_{31}^{(1)} & k_{32}^{(1)} & k_{33}^{(1)} & k_{34}^{(1)} \\ k_{41}^{(1)} & k_{42}^{(1)} & k_{43}^{(1)} & k_{44}^{(1)} \end{array}\right] \end{array} \begin{array}{c} \\ 4 \\ 5 \\ 1 \\ 2 \end{array} \qquad \mathbf{k}_2 = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 6 & 3 \end{array} \\ \left[\begin{array}{cccc} k_{11}^{(2)} & k_{12}^{(2)} & k_{13}^{(2)} & k_{14}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} & k_{23}^{(2)} & k_{24}^{(2)} \\ k_{31}^{(2)} & k_{32}^{(2)} & k_{33}^{(2)} & k_{34}^{(2)} \\ k_{41}^{(2)} & k_{42}^{(2)} & k_{43}^{(2)} & k_{44}^{(2)} \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 6 \\ 3 \end{array} \qquad \mathbf{k}_3 = \begin{array}{c} \begin{array}{cccc} 6 & 3 & 7 & 8 \end{array} \\ \left[\begin{array}{cccc} k_{11}^{(3)} & k_{12}^{(3)} & k_{13}^{(3)} & k_{14}^{(3)} \\ k_{21}^{(3)} & k_{22}^{(3)} & k_{23}^{(3)} & k_{24}^{(3)} \\ k_{31}^{(3)} & k_{32}^{(3)} & k_{33}^{(3)} & k_{34}^{(3)} \\ k_{41}^{(3)} & k_{42}^{(3)} & k_{43}^{(3)} & k_{44}^{(3)} \end{array}\right] \end{array} \begin{array}{c} \\ 6 \\ 3 \\ 7 \\ 8 \end{array}$$

$$\mathbf{S} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} k_{33}^{(1)} + k_{11}^{(2)} & k_{34}^{(1)} + k_{12}^{(2)} & k_{14}^{(2)} \\ k_{43}^{(1)} + k_{21}^{(2)} & k_{44}^{(1)} + k_{22}^{(2)} & k_{24}^{(2)} \\ k_{41}^{(2)} & k_{42}^{(2)} & k_{44}^{(2)} + k_{22}^{(3)} \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

(b) Assembling of Structure Stiffness Matrix **S**

Fig. 5.12

corresponding to the vertical translation and rotation, respectively, of the beginning joint; and the third and fourth member code numbers are always the structure coordinate numbers corresponding to the vertical translation and rotation, respectively, of the end joint.

From Fig. 5.12(a) we can see that, for member 1 of the beam, the beginning joint is 1 with restrained coordinates 4 and 5, and the end joint is 2 with degrees of freedom 1 and 2. Thus, the code numbers for member 1 are 4, 5, 1, 2. Similarly, the code numbers for member 2, for which the beginning and end joints are 2 and 3, respectively, are 1, 2, 6, 3. In a similar manner, the code numbers for member 3 are found to be 6, 3, 7, 8.

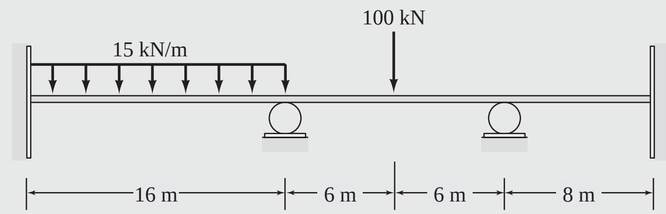
To establish the structure stiffness matrix , we write the code numbers of each member on the right side and at the top of its stiffness matrix $_i$ ($i = 1$, 2, or 3), as shown in Fig. 5.12(b). These code numbers now define the positions of the elements of the member stiffness matrices in the structure stiffness matrix . In other words, the code numbers on the right side of a matrix represent the row numbers of ; and the code numbers at the top represent the column numbers of . Furthermore, since the number of rows and columns of equal the number of degrees of freedom (*NDOF*) of the beam, only those elements of a matrix for which both the row and the column code numbers are less than or equal to *NDOF* belong in the structure stiffness matrix . The structure stiffness matrix is obtained by algebraically adding the pertinent elements of the matrices of all the members in their proper positions in the matrix.

To assemble the matrix for the beam of Fig. 5.12(a), we start by storing the pertinent elements of the stiffness matrix of member 1, $_1$, in the matrix (see Fig. 5.12(b)). Thus, the element $k_{33}^{(1)}$ is stored in row 1 and column 1 of , the element $k_{43}^{(1)}$ is stored in row 2 and column 1 of , the element $k_{34}^{(1)}$ is stored in row 1 and column 2 of , and the element $k_{44}^{(1)}$ is stored in row 2 and column 2 of . It should be noted that since the beam has three degrees of freedom, only those elements of $_1$ whose row and column code numbers both are less than or equal to 3 are stored in . The same procedure is then used to store the pertinent elements of $_2$ and $_3$, of members 2 and 3, respectively, in the matrix. Note that when two or more member stiffness coefficients are stored in the same element of , then the coefficients must be algebraically added. The completed structure stiffness matrix for the beam is shown in Fig. 5.12(b), and is identical to the one derived previously (Eq. (5.110)) by substituting the member compatibility and stiffness relations into the joint equilibrium equations.

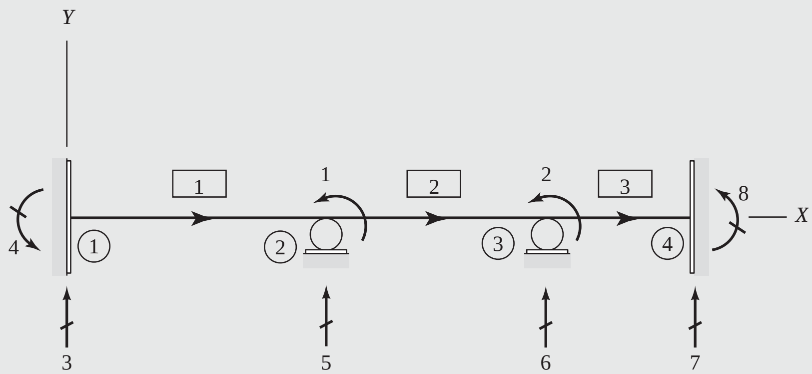
EXAMPLE 5.4 Determine the structure stiffness matrix for the three-span continuous beam shown in Fig. 5.13(a).

SOLUTION *Analytical Model* The analytical model of the structure is shown in Fig. 5.13(b). The beam has two degrees of freedom—the rotations of joints 2 and 3—which are identified by the structure coordinate numbers 1 and 2 in the figure.

Structure Stiffness Matrix To generate the 2×2 structure stiffness matrix , we will determine, for each member, the stiffness matrix and store its pertinent elements in their proper positions in by using the member code numbers.



(a) Three-Span Continuous Beam



(b) Analytical Model

Fig. 5.13

e ber 1 By substituting $L = 16$ m into Eq. (5.53), we obtain

$$
{}_1 = \;I \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 1 \end{array} \\ \left[\begin{array}{cccc} 0.0029297 & 0.023438 & -0.0029297 & 0.023438 \\ 0.023438 & 0.25 & -0.023438 & 0.125 \\ -0.0029297 & -0.023438 & 0.0029297 & -0.023438 \\ 0.023438 & 0.125 & -0.023438 & 0.25 \end{array}\right] \end{array} \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 1 \end{array} \qquad \textbf{(1)}
$$

From Fig. 5.13(b), we observe that the code numbers for this member are 3, 4, 5, 1. These numbers are written on the right side and at the top of $_1$ in Eq. (1), to indicate the rows and columns, respectively, of the structure stiffness matrix , where the elements of $_1$ are to be stored. Thus, the element in row 4 and column 4 of $_1$ is stored in row 1 and column 1 of , as

$$
= \;I \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left[\begin{array}{cc} 0.25 & 0 \\ 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array} \qquad \textbf{(2)}
$$

Note that the elements of $_1$ corresponding to the restrained coordinate numbers 3, 4, and 5 are disregarded.

e ber 2 $L = 12$ m. By using Eq. (5.53),

$$
{}_2 = \;I \begin{array}{c} \begin{array}{cccc} 5 & 1 & 6 & 2 \end{array} \\ \left[\begin{array}{cccc} 0.0069444 & 0.041667 & -0.0069444 & 0.041667 \\ 0.041667 & 0.33333 & -0.041667 & 0.16667 \\ -0.0069444 & -0.041667 & 0.0069444 & -0.041667 \\ 0.041667 & 0.16667 & -0.041667 & 0.33333 \end{array}\right] \end{array} \begin{array}{c} \\ 5 \\ 1 \\ 6 \\ 2 \end{array} \qquad \textbf{(3)}
$$

From Fig. 5.13(b), we can see that the code numbers for this member are 5, 1, 6, 2. These numbers are used to add the pertinent elements of $_2$ in their proper positions in the structure stiffness matrix given in Eq. (2), which now becomes

$$
= I \begin{array}{c} \begin{array}{cc} 1 & \qquad 2 \end{array} \\ \begin{bmatrix} 0.25 + 0.33333 & 0.16667 \\ 0.16667 & 0.33333 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array} \tag{4}
$$

e ber 3 $L = 8$ m. Thus,

$$
_3 = I \begin{array}{c} \begin{array}{cccc} 6 & 2 & 7 & 8 \end{array} \\ \begin{bmatrix} 0.023438 & 0.09375 & -0.023438 & 0.09375 \\ 0.09375 & \boxed{0.5} & -0.09375 & 0.25 \\ -0.023438 & -0.09375 & 0.023438 & -0.09375 \\ 0.09375 & 0.25 & -0.09375 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} \\ 6 \\ 2 \\ 7 \\ 8 \end{array} \tag{5}
$$

The code numbers for this member are 6, 2, 7, 8. Thus, the element in row 2 and column 2 of $_3$ is added in row 2 and column 2 of in Eq. (4), as

$$
= I \begin{array}{c} \begin{array}{cc} 1 & \qquad 2 \end{array} \\ \begin{bmatrix} 0.25 + 0.33333 & 0.16667 \\ 0.16667 & 0.33333 + 0.5 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array}
$$

Since the stiffnesses of all three members of the beam have now been stored in , the structure stiffness matrix for the given beam is

$$
= I \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0.58333 & 0.16667 \\ 0.16667 & 0.83333 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array} \qquad \textbf{Ans}
$$

Note that the structure stiffness matrix is symmetric.

5.6 STRUCTURE FIXED-JOINT FORCES AND EQUIVALENT JOINT LOADS

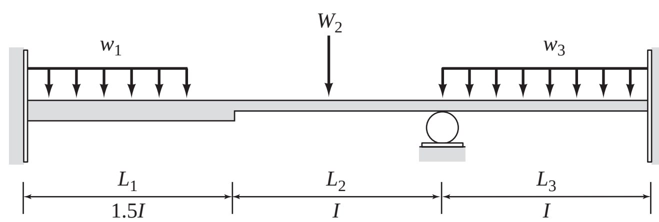
As discussed in the preceding section, the force–displacement relationships for an entire structure can be expressed in matrix form (see Eq. (5.109)) as

$$\mathbf{P} - \mathbf{P}_f = \ \mathbf{d}$$

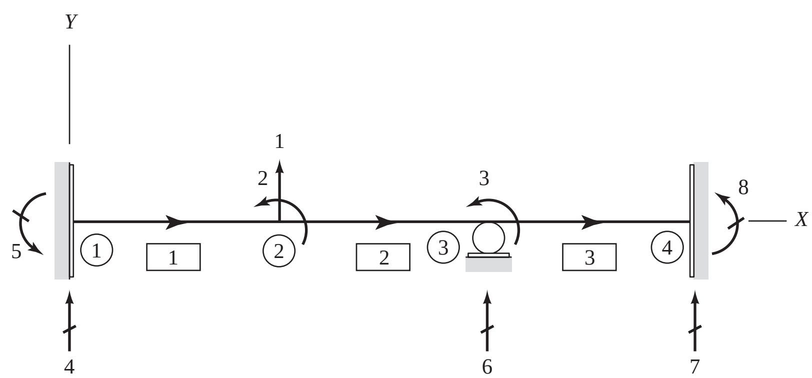
in which $\mathbf{P}_f$ represents the structure fixed-joint force vector. It was also shown in the preceding section that by using the basic equations of equilibrium, compatibility, and member stiffness, the structure fixed-joint forces $\mathbf{P}_f$ can be expressed in terms of the member fixed-end forces $_f$ (see Eq. (5.111)). In this section, we consider the physical interpretation of the structure fixed-joint forces; and discuss the formation of the $\mathbf{P}_f$ vector, by assembling the elements of the member $_f$ vectors, using the member code numbers.

The concept of the structure fixed-joint forces $\mathbf{P}_f$ is analogous to that of the member fixed-end forces $_f$. *The structure fixed-joint forces represent the reaction forces (and or moments) that would develop at the locations and in the directions of the structure's degrees of freedom, due to the external member* *loads, if all the joints of the structure were fixed against translations and rotations.*

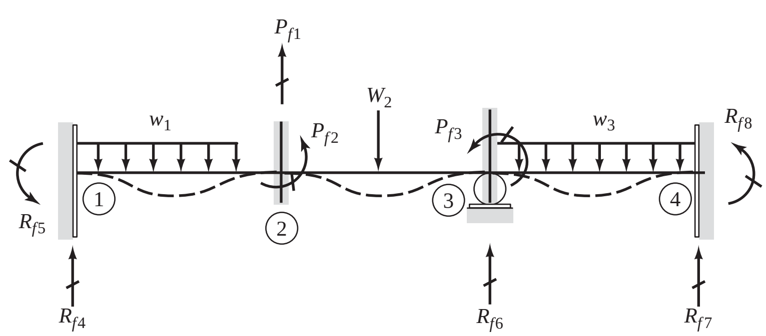
To develop some insight into the concept of structure fixed-joint forces, let us reconsider the beam of Fig. 5.10. The beam, subjected only to the member loads, is redrawn in Fig. 5.14(a), with its analytical model depicted in Fig. 5.14(b). Now, assume that joint 2, which is free to translate and rotate, is restrained against these displacements by an imaginary restraint (or clamp) applied to it, as shown in Fig. 5.14(c). Similarly, joint 3, which is free to rotate, is also restrained against rotation by means of an imaginary restraint (or clamp). When external loads are applied to the members of this hypothetical



(a) Beam

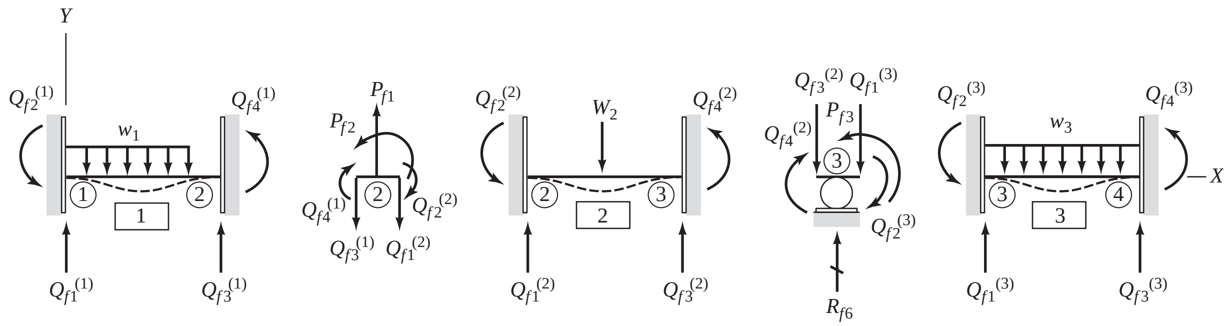


(b) Analytical Model

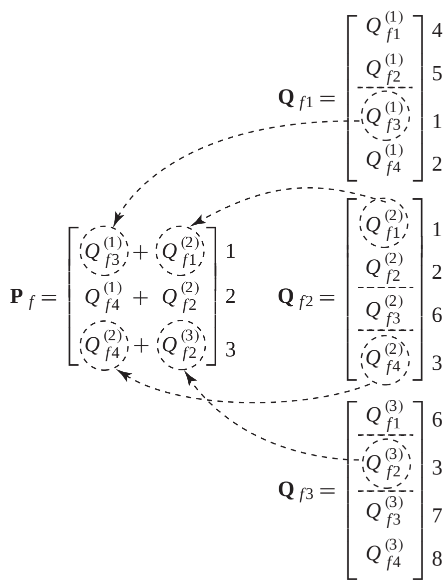


(c) Fixed Structure Subjected to Member Loads

Fig. 5.14



(d) Member Fixed-End Forces



(e) Assembly of Structure Fixed-Joint Force Vector $\mathbf{P}_f$

Fig. 5.14 (*continued*)

completely fixed structure, reaction forces and moments develop at each of its joints. Note that, in Fig. 5.14(c), the reactions due to the imaginary restraints are denoted symbolically by P_{fi} ($i = 1$ through 3), whereas the reactions at the actual supports are denoted by R_{fi} ($i = 4$ through 8). The imaginary reactions P_{f1}, P_{f2}, and P_{f3}, which are at the locations and in the directions of the structure's three degrees of freedom 1, 2, and 3, respectively, are considered the structure fixed-joint forces due to member loads. Thus, the structure fixed-joint force vector, $\mathbf{P}_f$, for the beam, can be written as

$$\mathbf{P}_f = \begin{bmatrix} P_{f1} \\ P_{f2} \\ P_{f3} \end{bmatrix} \tag{5.113}$$

To relate the structure fixed-joint forces $\mathbf{P}_f$ to the member fixed-end forces ${}_{f}$, we draw the free-body diagrams of the members and the interior joints of the hypothetical fixed beam, as shown in Fig. 5.14(d). In this figure, the superscript (i) denotes the member number. Note that, because all the joints of the beam are completely restrained, the member ends, which are rigidly connected to the joints, are also fixed against any displacements. Therefore, only the fixed-end forces due to member loads, ${}_{f}$, can develop at the ends of the three members of the beam. By applying the equations of equilibrium $\sum F_Y = 0$ and $\sum M = 0$ to the free body of joint 2, and the equilibrium equation $\sum M = 0$ to the free body of joint 3, we obtain the following relationships between the structure fixed-joint forces and the member fixed-end forces.

$$P_{f1} = {}^{(1)}_{f3} + {}^{(2)}_{f1}$$

$$P_{f2} = {}^{(1)}_{f4} + {}^{(2)}_{f2}$$

$$P_{f3} = {}^{(2)}_{f4} + {}^{(3)}_{f2}$$

which can be expressed in vector form as

$$\mathbf{P}_f = \begin{bmatrix} P_{f1} \\ P_{f2} \\ P_{f3} \end{bmatrix} = \begin{bmatrix} {}^{(1)}_{f3} + {}^{(2)}_{f1} \\ {}^{(1)}_{f4} + {}^{(2)}_{f2} \\ {}^{(2)}_{f4} + {}^{(3)}_{f2} \end{bmatrix}$$

Note that the foregoing $\mathbf{P}_f$ vector is identical to that determined for the example beam in the preceding section (Eq. (5.111)).

Assembly of Structure Fixed-Joint Force Vector Using Member Code Numbers

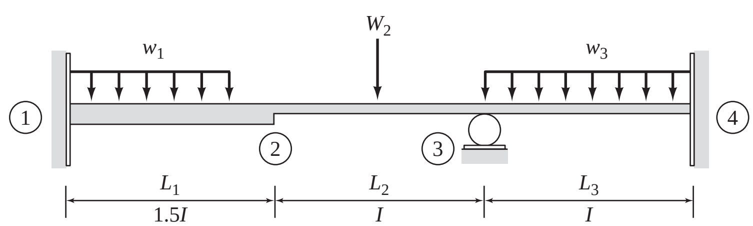
The structure fixed-joint force vector $\mathbf{P}_f$ can be conveniently assembled from the member fixed-end force vectors ${}_{f}$, using the member code number technique. The technique is similar to that for forming the structure stiffness matrix , described in the preceding section. Essentially, the procedure involves storing the pertinent elements of the fixed-end force vector ${}_{f}$ for each member of the beam in their proper positions in the structure fixed-joint force vector $\mathbf{P}_f$, using the member code numbers.

The foregoing procedure is illustrated for the example beam in Fig. 5.14(e). As shown there, the code numbers of each member are written on the right side of its fixed-end force vector ${}_{f}$. Any member code number that is less than or equal to the number of degrees of freedom of the structure (*NDOF*), now identifies the row of $\mathbf{P}_f$ in which the corresponding member force is to be stored. Thus, as shown in Fig. 5.14(e), the third and fourth elements of ${}_{f1}$, with code numbers 1 and 2, respectively, are stored in the first and second rows of $\mathbf{P}_f$. The same procedure is then repeated for members 2 and 3. Note that the completed $\mathbf{P}_f$ vector for the beam is identical to that obtained previously (Eq. (5.111)).

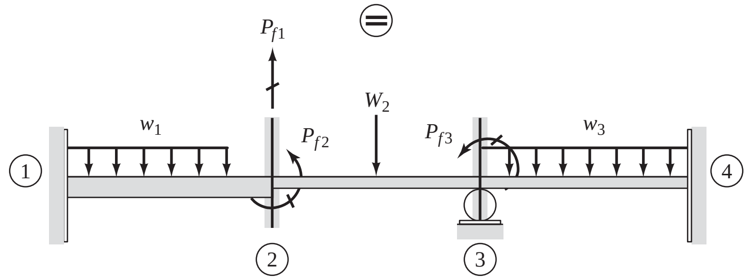
Equivalent Joint Loads

The negatives of the structure fixed-joint forces (i.e., $-\mathbf{P}_f$) are commonly known as the *equivalent joint loads*. This is because the structure fixed-joint forces, when applied to a structure with their directions reversed, cause the same joint displacements as the actual member loads.

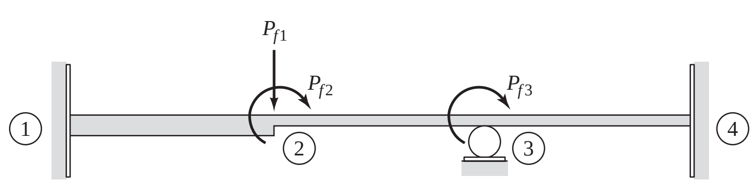
The validity of the foregoing interpretation can be shown easily using the principle of superposition (Section 1.7), as illustrated in Fig. 5.15. Figure 5.15(a) shows a continuous beam subjected to arbitrary member loads. (This beam was considered previously, and its analytical model is given in Fig. 5.14(b).) In Fig. 5.15(b), joints 2 and 3 of the beam are fixed by imaginary restraints so that the translation and rotation of joint 2, and the rotation of joint 3, are 0. This hypothetical completely fixed beam is then subjected to member loads, causing the structure fixed-joint forces P_{f1}, P_{f2}, and P_{f3} to develop at the imaginary restraints, as shown in Fig. 5.15(b). Lastly, as shown in Fig. 5.15(c),



(a) Actual Beam Subjected to Member Loads



(b) Fixed Beam Subjected to Member Loads



(c) Actual Beam Subjected to Equivalent Joint Loads

Fig. 5.15

the actual beam is subjected to external loads at its joints, which are equal in magnitude to P_{f1}, P_{f2}, and P_{f3}, but are reversed in direction.

By comparing Figs. 5.15(a), (b), and (c), we realize that the actual loading applied to the beam in Fig. 5.15(a) equals the algebraic sum of the loadings given in Figs. 5.15(b) and (c), because the reactive forces P_{f1}, P_{f2}, and P_{f3} in Fig. 5.15(b) cancel the corresponding applied loads in Fig. 5.15(c). Thus, according to the principle of superposition, any joint displacement of the actual beam due to the member loads (Fig. 5.15(a)) must equal the algebraic sum of the corresponding joint displacement of the fixed beam due to the member loads (Fig. 5.15(b)), and the corresponding joint displacement of the actual beam subjected to no member loads, but to the fixed-joint forces with their directions reversed. However, the joint displacements of the fixed beam (Fig. 5.15(b)) are 0. Therefore, the joint displacements of the beam due to the member loads (Fig. 5.15(a)) must be equal to the corresponding joint displacements of the beam due to the negatives of the fixed-joint forces (Fig. 5.15(c)). In other words, the negatives of the structure fixed-joint forces do indeed cause the same joint displacements of the beam as the actual member loads; and in that sense, such forces can be considered to be the equivalent joint loads. It is important to realize that this equivalency between the negative fixed-joint forces and the member loads is valid only for joint displacements, and cannot be generalized to member end forces and reactions, because such forces are generally not 0 in fixed structures subjected to member loads.

Based on the foregoing discussion of equivalent joint loads, we can define the *equivalent joint load vector* $\mathbf{P}_e$ for a structure as simply the negative of its fixed-joint force vector $\mathbf{P}_f$; that is,

$$\boxed{\mathbf{P}_e = -\mathbf{P}_f} \tag{5.114}$$

An alternative form of the structure stiffness relations, in terms of the equivalent joint loads, can now be obtained by substituting Eq. (5.114) into Eq. (5.109). This yields

$$\boxed{\mathbf{P} + \mathbf{P}_e = \quad \mathbf{d}} \tag{5.115}$$

Once , $\mathbf{P}_f$ (or $\mathbf{P}_e$), and $\mathbf{P}$ have been evaluated, the structure stiffness relations (Eq. (5.109) or Eq. (5.115)), which now represent a system of simultaneous linear equations, can be solved for the unknown joint displacements $\mathbf{d}$. With $\mathbf{d}$ known, the end displacements $\mathbf{u}$ for each member can be determined by applying the compatibility equations defined by its code numbers; then the corresponding end forces can be computed by applying the member stiffness relations. Finally, the support reactions are determined from the member end forces , by considering the equilibrium of the support joints in the directions of the restrained coordinates via member code numbers, as discussed in Chapter 3 for the case of plane trusses.

EXAMPLE 5.5 Determine the fixed-joint force vector and the equivalent joint load vector for the propped-cantilever beam shown in Fig. 5.16(a).

SOLUTION *Analytical Model* See Fig. 5.16(b).

Structure Fixed- oint Force Vector To generate the 3×1 structure fixed-joint force vector $\mathbf{P}_f$, we will, for each member: (a) determine the fixed-end force vector $_f$, using the fixed-end force equations given inside the front cover; and (b) store the pertinent elements of $_f$ in their proper positions in $\mathbf{P}_f$, using the member code numbers.

e ber 1 By substituting $w = 30$ kN/m, $L = 9$ m, and $l_1 = l_2 = 0$ into the fixed-end force expressions for loading type 3, we obtain

$$FS_b = FS_e = \frac{30(9)}{2} = 135 \text{ kN}$$

$$FM_b = \frac{30(9)^2}{12} = 202.5 \text{ kN} \cdot \text{m}$$

$$FM_e = -\frac{30(9)^2}{12} = -202.5 \text{ kN} \cdot \text{m}$$

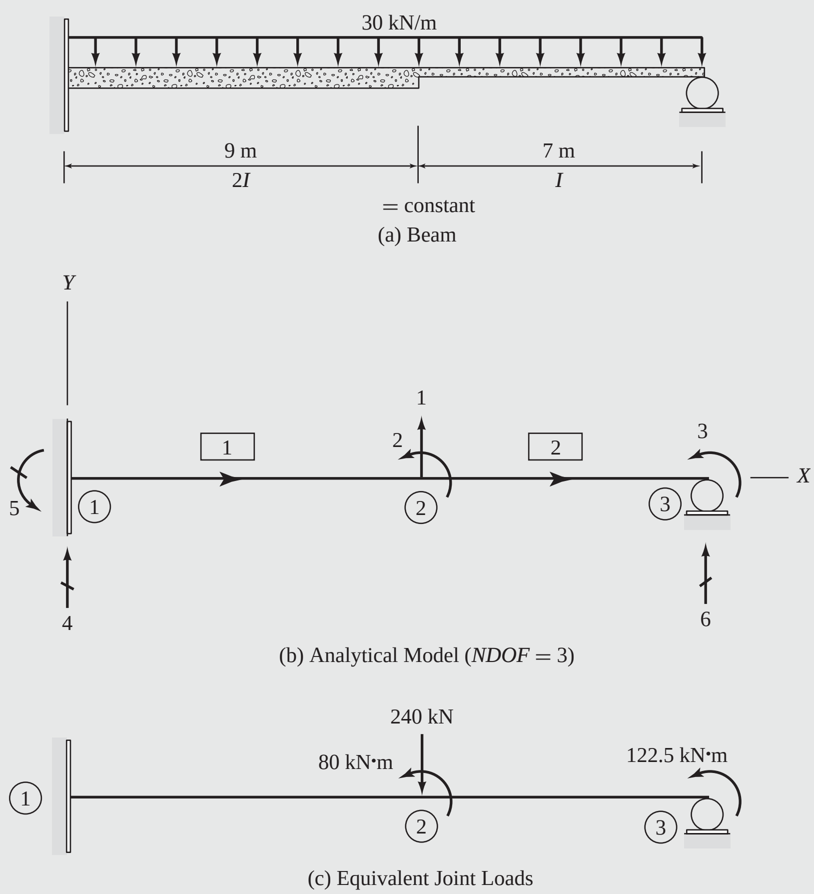


Fig. 5.16

Thus, the fixed-end force vector for member 1 is given by

$$_{f1} = \begin{array}{c} \left[\begin{array}{c} 135 \\ 202.5 \\ \hdashline 135 \\ -202.5 \end{array}\right] \end{array} \begin{array}{c} 4 \\ 5 \\ 1 \\ 2 \end{array} \tag{1}$$

From Fig. 5.16(b), we can see that the code numbers for member 1 are 4, 5, 1, 2. These numbers are written on the right side of $_{f1}$ in Eq. (1) to indicate the rows of the structure fixed-joint vector $\mathbf{P}_f$, where the elements of $_{f1}$ are to be stored. Thus, the elements in the third and fourth rows of $_{f1}$ are stored in rows 1 and 2, respectively, of $\mathbf{P}_f$, as

$$\mathbf{P}_f = \left[\begin{array}{c} 135 \\ -202.5 \\ 0 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \tag{2}$$

Note that the elements of $_{f1}$ corresponding to the restrained coordinate numbers 4 and 5 are disregarded.

e ber 2 By substituting $w = 30$ kN/m, $L = 7$ m, and $l_1 = l_2 = 0$ into the fixed-end force expressions for loading type 3, we obtain

$$FS_b = FS_e = \frac{30(7)}{2} = 105 \text{ kN}$$

$$FM_b = \frac{30(7)^2}{12} = 122.5 \text{ kN} \cdot \text{m}$$

$$FM_e = -\frac{30(7)^2}{12} = -122.5 \text{ kN} \cdot \text{m}$$

Thus,

$$_{f2} = \left[\begin{array}{c} 105 \\ 122.5 \\ \hdashline 105 \\ \hdashline -122.5 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 6 \\ 3 \end{array} \tag{3}$$

From Fig. 5.16(b), we observe that the code numbers for this member are 1, 2, 6, 3. These numbers are used to add the pertinent elements of $_{f2}$ in their proper positions in $\mathbf{P}_f$ given in Eq. (2), which now becomes

$$\mathbf{P}_f = \left[\begin{array}{c} 135 + 105 \\ -202.5 + 122.5 \\ -122.5 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

Since the fixed-end forces for both members of the beam have now been stored in $\mathbf{P}_f$, the structure fixed-joint force vector for the given beam is

$$\mathbf{P}_f = \left[\begin{array}{c} 240 \\ -80 \\ -122.5 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \qquad \textbf{Ans}$$

quivalent oint Load Vector By using Eq. (5.114), we obtain

$$\mathbf{P}_e = -\mathbf{P}_f = \left[\begin{array}{c} -240 \\ 80 \\ 122.5 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \qquad \textbf{Ans}$$

These equivalent joint loads, when applied to the beam as shown in Fig. 5.16(c), cause the same joint displacements as the actual 30 kN/m uniformly distributed load given in Fig. 5.16(a).

5.7 PROCEDURE FOR ANALYSIS

Based on the concepts presented in the previous sections, we can develop the following step-by-step procedure for the analysis of beams by the matrix stiffness method. The reader should note that the overall format of this procedure is essentially the same as the procedure for analysis of plane trusses presented in Chapter 3.

1. Prepare an analytical model of the beam, as follows.
   - **a.** Draw a line diagram of the beam, and identify each joint and member by a number. The origin of the global *XY* coordinate system is usually located at the farthest left joint, with the *X* and *Y* axes oriented in the horizontal (positive to the right) and vertical (positive upward) directions, respectively. For each member, establish a local *xy* coordinate system, with the origin at the left end (beginning) of the member, and the *x* and *y* axes oriented in the horizontal (positive to the right) and vertical (positive upward) directions, respectively.
   - **b.** Number the degrees of freedom and restrained coordinates of the beam, as discussed in Section 5.1.
2. Evaluate the structure stiffness matrix and fixed-joint force vector $\mathbf{P}_f$. The number of rows and columns of must be equal to the number of degrees of freedom (*NDOF*) of the beam; the number of rows of $\mathbf{P}_f$ must equal *NDOF*. For each member of the structure, perform the following operations.
   - **a.** Compute the member stiffness matrix (Eq. (5.53)).
   - **b.** If the member is subjected to external loads, then evaluate its fixed-end force vector $_f$, using the expressions for fixed-end forces given inside the front cover.
   - **c.** Identify the member code numbers and store the pertinent elements of and $_f$ in their proper positions in the structure stiffness matrix , and the fixed-joint force vector $\mathbf{P}_f$, respectively. The complete structure stiffness matrix , obtained by assembling the stiffness coefficients of all the members of the beam, must be symmetric.
3. If the beam is subjected to joint loads, then form the $NDOF \times 1$ joint load vector $\mathbf{P}$.
4. Determine the joint displacements $\mathbf{d}$. Substitute $\mathbf{P}$, $\mathbf{P}_f$, and into the structure stiffness relations, $\mathbf{P} - \mathbf{P}_f = \ \mathbf{d}$ (Eq. (5.109)), and solve the resulting system of simultaneous equations for the unknown joint displacements $\mathbf{d}$. To check that the simultaneous equations have been solved correctly, substitute the numerical values of the joint displacements $\mathbf{d}$ back into the structure stiffness relations, $\mathbf{P} - \mathbf{P}_f = \ \mathbf{d}$.

If the solution is correct, then the stiffness relations should be satisfied. It should be noted that joint translations are considered positive when in the positive direction of the Y axis, and joint rotations are considered positive when counterclockwise.

5. Compute member end displacements and end forces, and support reactions. For each member of the beam, do the following.

a. Obtain member end displacements **u** from the joint displacements **d**, using the member code numbers.

b. Compute member end forces, using the relationship $= \mathbf{u} + \quad _f$ (Eq. (5.4)).

c. Using the member code numbers, store the pertinent elements of in their proper positions in the support reaction vector (as discussed in Chapter 3).

6. Check the calculation of member end forces and support reactions by applying the equations of equilibrium, $\sum F_Y = 0$ and $\sum M = 0$, to the free body of the entire beam. If the calculations have been carried out correctly, then the equilibrium equations should be satisfied.

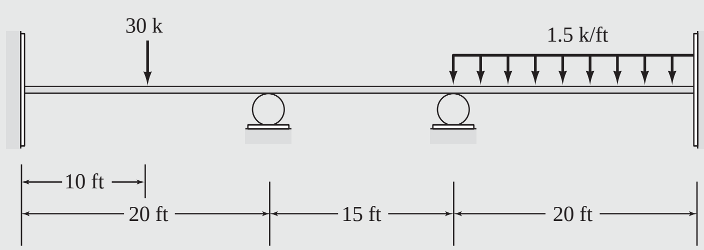
EXAMPLE 5.6 Determine the joint displacements, member end forces, and support reactions for the three-span continuous beam shown in Fig. 5.17(a), using the matrix stiffness method.

SOLUTION *Analytical Model* See Fig. 5.17(b). The beam has two degrees of freedom—the rotations of joints 2 and 3—which are numbered 1 and 2, respectively. The six restrained coordinates of the beam are numbered 3 through 8.

Structure Stiffness Matrix and Fixed- oint Force Vector

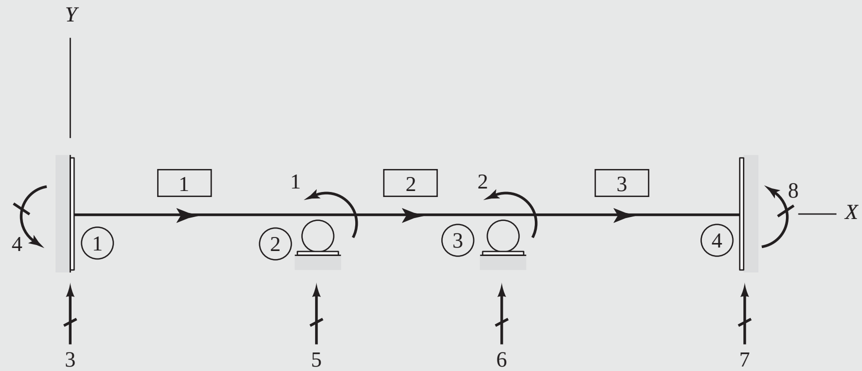
e ber 1 By substituting $= 29{,}000$ ksi, $I = 510$ in.^{4}, and $L = 240$ in. into Eq. (5.53), we obtain

$$
_1 = \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 1 \end{array} \\ \left[\begin{array}{cccc} 12.839 & 1{,}540.6 & -12.839 & 1{,}540.6 \\ 1{,}540.6 & 246{,}500 & -1{,}540.6 & 123{,}250 \\ -12.839 & -1{,}540.6 & 12.839 & -1{,}540.6 \\ 1{,}540.6 & 123{,}250 & -1{,}540.6 & 246{,}500 \end{array}\right] \end{array} \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 1 \end{array} \qquad \textbf{(1)}
$$



(a) Three-Span Continuous Beam

Fig. 5.17

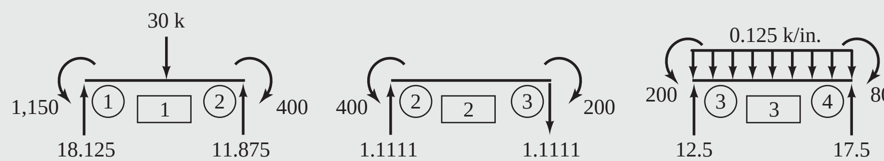


(b) Analytical Model

$$\mathbf{S} = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} \begin{array}{cc} \quad 1 \quad\quad\quad\quad & \quad\quad 2 \end{array} \\ \begin{bmatrix} 246{,}500 + 328{,}667 & 164{,}333 \\ 164{,}333 & 328{,}667 + 246{,}500 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \end{array} = \begin{array}{c} \begin{array}{cc} 1 \quad\quad & \quad 2 \end{array} \\ \begin{bmatrix} 575{,}167 & 164{,}333 \\ 164{,}333 & 575{,}167 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \end{array}$$

$$\mathbf{P}_f = \begin{bmatrix} -900 \\ 600 \end{bmatrix}\begin{array}{c} 1 \\ 2 \end{array}$$

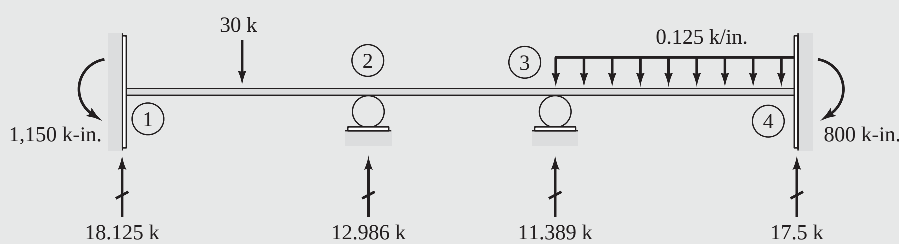
(c) Structure Stiffness Matrix and Fixed-Joint Force Vector



(d) Member End Forces

$$= \begin{bmatrix} 18.125 \\ 1{,}150 \\ 11.875 + 1.1111 \\ -1.1111 + 12.5 \\ 17.5 \\ -800 \end{bmatrix}\begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} = \begin{bmatrix} 18.125\text{ k} \\ 1{,}150\text{ k-in.} \\ 12.986\text{ k} \\ 11.389\text{ k} \\ 17.5\text{ k} \\ -800\text{ k-in.} \end{bmatrix}$$

(e) Support Reaction Vector



(f) Support Reactions

Fig. 5.17 (*continued*)

Using the fixed-end force equations given inside the front cover, we evaluate the fixed-end forces due to the 30 k concentrated load as

$$FS_b = \frac{30(120)^2}{(240)^3}[3(120)+120] = 15 \text{ k}$$

$$FM_b = \frac{30(120)(120)^2}{(240)^2} = 900 \text{ k-in.}$$

$$FS_e = \frac{30(120)^2}{(240)^3}[120+3(120)] = 15 \text{ k}$$

$$FM_e = -\frac{30(120)^2(120)}{(240)^2} = -900 \text{ k-in.}$$

Thus, the fixed-end force vector for member 1 is

$$\mathbf{Q}_{f1} = \begin{array}{c} \left[\begin{array}{r} 15 \\ 900 \\ 15 \\ \hdashline -900 \end{array}\right] \end{array} \begin{array}{c} 3 \\ 4 \\ 5 \\ 1 \end{array} \qquad (2)$$

From Fig. 5.17(b), we observe that the code numbers for member 1 are 3, 4, 5, 1. Using these code numbers, the pertinent elements of $\mathbf{k}_1$ and $\mathbf{Q}_{f1}$ are stored in their proper positions in the 2 × 2 structure stiffness matrix $\mathbf{S}$ and the 2 × 1 structure fixed-joint force vector $\mathbf{P}_f$, respectively, as shown in Fig. 5.17(c).

M b 2 = 29,000 ksi, $I = 510$ in.^{4}, and $L = 180$ in.

$$\mathbf{k}_2 = \begin{array}{c} \begin{array}{cccc} 5 & 1 & 6 & 2 \end{array} \\ \left[\begin{array}{rrrr} 30.432 & 2{,}738.9 & -30.432 & 2{,}738.9 \\ 2{,}738.9 & 328{,}667 & -2{,}738.9 & 164{,}333 \\ -30.432 & -2{,}738.9 & 30.432 & -2{,}738.9 \\ 2{,}738.9 & 164{,}333 & -2{,}738.9 & 328{,}667 \end{array}\right] \end{array} \begin{array}{c} \\ 5 \\ 1 \\ 6 \\ 2 \end{array} \qquad (3)$$

Since this member is not subjected to any external loads, its fixed-end force vector is 0; that is,

$$\mathbf{Q}_{f2} = \qquad (4)$$

Using the code numbers 5, 1, 6, 2 for this member (see Fig. 5.17(b)), the relevant elements of $\mathbf{k}_2$ are stored into $\mathbf{S}$, as shown in Fig. 5.17(c).

M b 3 = 29,000 ksi, $I = 510$ in.^{4}, and $L = 240$ in.

$$\mathbf{k}_3 = \begin{array}{c} \begin{array}{cccc} 6 & 2 & 7 & 8 \end{array} \\ \left[\begin{array}{rrrr} 12.839 & 1{,}540.6 & -12.839 & 1{,}540.6 \\ 1{,}540.6 & 246{,}500 & -1{,}540.6 & 123{,}250 \\ -12.839 & -1{,}540.6 & 12.839 & -1{,}540.6 \\ 1{,}540.6 & 123{,}250 & -1{,}540.6 & 246{,}500 \end{array}\right] \end{array} \begin{array}{c} \\ 6 \\ 2 \\ 7 \\ 8 \end{array} \qquad (5)$$

The fixed-end forces due to the 0.125 k/in. (=1.5 k/ft) uniformly distributed load are

$$FS_b = \frac{0.125(240)}{2} = 15 \text{ k}$$

$$FM_b = \frac{0.125(240)^2}{12} = 600 \text{ k-in.}$$

$$FS_e = \frac{0.125(240)}{2} = 15 \text{ k}$$

$$FM_e = -\frac{0.125(240)^2}{12} = -600 \text{ k-in.}$$

Thus,

$$Q_{f3} = \begin{array}{c} \left[\begin{array}{r} 15 \\ \hdashline 600 \\ \hdashline 15 \\ -600 \end{array}\right] \end{array} \begin{array}{l} 6 \\ 2 \\ 7 \\ 8 \end{array} \tag{6}$$

The relevant elements of k_3 and Q_{f3} are stored in S and P_f, respectively, using the member code numbers 6, 2, 7, 8.

The completed structure stiffness matrix S and structure fixed-joint force vector P_f are given in Fig. 5.17(c). Note that the S matrix is symmetric.

oint Load Vector Since no external loads (i.e., moments) are applied to the beam at joints 2 and 3, the joint load vector is 0; that is,

$$P =$$

oint Displacements By substituting the numerical values of P, P_f, and S into Eq. (5.109), we write the stiffness relations for the entire continuous beam as

$$P - P_f = Sd$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -900 \\ 600 \end{bmatrix} = \begin{bmatrix} 575{,}167 & 164{,}333 \\ 164{,}333 & 575{,}167 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

or

$$\begin{bmatrix} 900 \\ -600 \end{bmatrix} = \begin{bmatrix} 575{,}167 & 164{,}333 \\ 164{,}333 & 575{,}167 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

By solving these equations simultaneously, we determine the joint displacements to be

$$d = \begin{bmatrix} 2.0284 \\ -1.6227 \end{bmatrix} \times 10^{-3} \text{ rad}$$

Ans

To check the foregoing solution, we substitute the numerical values of d back into the structure stiffness relationship to obtain

$$P - P_f = Sd = \begin{bmatrix} 575{,}167 & 164{,}333 \\ 164{,}333 & 575{,}167 \end{bmatrix} \begin{bmatrix} 2.0284 \\ -1.6227 \end{bmatrix} \times 10^{-3} = \begin{bmatrix} 900.01 \\ -599.99 \end{bmatrix}$$

Checks

Member nd Displacements and nd Forces

M b 1 The member end displacements u can be obtained simply by comparing the member's degree of freedom numbers with its code numbers, as follows:

$$u_1 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \begin{array}{l} 3 \\ 4 \\ 5 \\ 1 \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2.0284 \end{bmatrix} \times 10^{-3} \tag{7}$$

Note that the member code numbers (3, 4, 5, 1), when written on a side of u as shown in Eq. (7), define the compatibility equations for the member. Since the code numbers corresponding to u_1, u_2, and u_3 are the restrained coordinate numbers 3, 4, and 5, respectively, this indicates that $u_1 = u_2 = u_3 = 0$. Similarly, the code number 1 corresponding to u_4 indicates that $u_4 = d_1$. The foregoing compatibility equations can be easily verified by a visual inspection of the beam's line diagram, given in Fig. 5.17(b).

The member end forces can now be calculated, using the member stiffness relationship $Q = ku + Q_f$ (Eq. (5.4)). Using k_1 and Q_{f1} from Eqs. (1) and (2), respectively, we write

$$\mathbf{Q}_1 = \begin{bmatrix} 12.839 & 1{,}540.6 & -12.839 & 1{,}540.6 \\ 1{,}540.6 & 246{,}500 & -1{,}540.6 & 123{,}250 \\ -12.839 & -1{,}540.6 & 12.839 & -1{,}540.6 \\ 1{,}540.6 & 123{,}250 & -1{,}540.6 & 246{,}500 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2.0284 \end{bmatrix} \times 10^{-3}$$

$$+ \begin{bmatrix} 15 \\ 900 \\ 15 \\ -900 \end{bmatrix} = \begin{array}{c} \left[\begin{array}{c} 18.125 \text{ k} \\ 1{,}150 \text{ k-in.} \\ 11.875 \text{ k} \\ \hdashline -400 \text{ k-in.} \end{array}\right] \end{array} \begin{array}{c} 3 \\ 4 \\ 5 \\ 1 \end{array} \qquad (8)$$

Ans

The end forces for member 1 are shown in Fig. 5.17(d). We can check our calculation of end forces by applying the equilibrium equations, $\sum F_y = 0$ and $\sum M = 0$, to the free body of member 1 to ensure that it is in equilibrium. Thus,

$$+\uparrow \sum F_y = 0 \qquad 18.125 - 30 + 11.875 = 0$$ **Checks**

$$+ \circlearrowleft \sum M_{①} = 0 \qquad 1{,}150 - 30(120) - 400 + 11.875(240) = 0$$ **Checks**

Next, to generate the support reaction vector R, we write the member code numbers (3, 4, 5, 1) on the right side of $\mathbf{Q}_1$, as shown in Eq. (8), and store the pertinent elements of $\mathbf{Q}_1$ in their proper positions in R by matching the code numbers on the side of $\mathbf{Q}_1$ to the restrained coordinate numbers on the right side of R (see Fig. 5.17(e)).

M b 2 The member end displacements are given by

$$\mathbf{u}_2 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \begin{array}{c} 5 \\ 1 \\ 6 \\ 2 \end{array} = \begin{bmatrix} 0 \\ d_1 \\ 0 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2.0284 \\ 0 \\ -1.6227 \end{bmatrix} \times 10^{-3}$$

By using $\mathbf{k}_2$ from Eq. (3) and $\mathbf{Q}_{f2} =$, we compute member end forces as

$$\mathbf{Q} = \mathbf{ku} + \mathbf{Q}_f$$

$$\mathbf{Q}_2 = \begin{bmatrix} 30.432 & 2{,}738.9 & -30.432 & 2{,}738.9 \\ 2{,}738.9 & 328{,}667 & -2{,}738.9 & 164{,}333 \\ -30.432 & -2{,}738.9 & 30.432 & -2{,}738.9 \\ 2{,}738.9 & 164{,}333 & -2{,}738.9 & 328{,}667 \end{bmatrix} \begin{bmatrix} 0 \\ 2.0284 \\ 0 \\ -1.6227 \end{bmatrix} \times 10^{-3}$$

$$= \left[\begin{array}{c} 1.1111 \text{ k} \\ \hdashline 400 \text{ k-in.} \\ \hdashline -1.1111 \text{ k} \\ \hdashline -200 \text{ k-in.} \end{array}\right] \begin{array}{c} 5 \\ 1 \\ 6 \\ 2 \end{array}$$

Ans

The foregoing member end forces are shown in Fig. 5.17(d). To check our calculations, we apply the equations of equilibrium to the free body of member 2 as

$$+\uparrow \sum F_y = 0 \qquad 1.1111 - 1.1111 = 0$$ **Checks**

$$+ \circlearrowleft \sum M_{②} = 0 \qquad 400 - 200 - 1.1111(180) = 0.002 \approx 0$$ **Checks**

Next, we store the pertinent elements of $\mathbf{Q}_2$ in their proper positions in the reaction vector R, using the member code numbers (5, 1, 6, 2), as shown in Fig. 5.17(e).

M b 3

$$\mathbf{u}_3 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \begin{matrix} 6 \\ 2 \\ 7 \\ 8 \end{matrix} = \begin{bmatrix} 0 \\ d_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1.6227 \\ 0 \\ 0 \end{bmatrix} \times 10^{-3}$$

By substituting $\mathbf{k}_3$ and $\mathbf{Q}_{f3}$ from Eqs. (5) and (6), respectively, into the member stiffness relationship $\mathbf{Q} = \mathbf{k}\mathbf{u} + \mathbf{Q}_f$, we determine the end forces for member 3 to be

$$\mathbf{Q}_3 = \begin{bmatrix} 12.839 & 1{,}540.6 & -12.839 & 1{,}540.6 \\ 1{,}540.6 & 246{,}500 & -1{,}540.6 & 123{,}250 \\ -12.839 & -1{,}540.6 & 12.839 & -1{,}540.6 \\ 1{,}540.6 & 123{,}250 & -1{,}540.6 & 246{,}500 \end{bmatrix} \begin{bmatrix} 0 \\ -1.6227 \\ 0 \\ 0 \end{bmatrix} \times 10^{-3}$$

$$+ \begin{bmatrix} 15 \\ 600 \\ 15 \\ -600 \end{bmatrix} = \begin{bmatrix} 12.5 \text{ k} \\ 200 \text{ k-in.} \\ 17.5 \text{ k} \\ -800 \text{ k-in.} \end{bmatrix} \begin{matrix} 6 \\ 2 \\ 7 \\ 8 \end{matrix}$$ **Ans**

These member end forces are shown in Fig. 5.17(d). To check our calculations, we apply the equilibrium equations:

$+\uparrow \sum F_y = 0 \qquad 12.5 - 0.125(240) + 17.5 = 0$ **Checks**

$+\circlearrowleft \sum M_{③} = 0 \qquad 200 - 0.125(240)(120) - 800 + 17.5(240) = 0$ **Checks**

Next, by using the code numbers (6, 2, 7, 8) for member 3, we store the relevant elements of $\mathbf{Q}_3$ in their proper positions in $\mathbf{R}$.

Support Reactions The completed reaction vector $\mathbf{R}$ is shown in Fig. 5.17(e), and the support reactions are depicted on a line diagram of the structure in Fig. 5.17(f). **Ans**

quilibrium Check Finally, applying the equilibrium equations to the free body of the entire beam (Fig. 5.17(f)), we write

$+\uparrow \sum F_y = 0$

$18.125 - 30 + 12.986 + 11.389 - 0.125(240) + 17.5 = 0$ **Checks**

$+\circlearrowleft \sum M_{①} = 0$

$1{,}150 - 30(120) + 12.986(240) + 11.389(420) - 0.125(240)(540)$

$+ 17.5(660) - 800 = 0.02 \approx 0$ **Checks**

EXAMPLE 5.7 Determine the joint displacements, member end forces, and support reactions for the beam shown in Fig. 5.18(a), using the matrix stiffness method.

SOLUTION *Analytical Model* See Fig. 5.18(b). The beam has four degrees of freedom (numbered 1 through 4) and four restrained coordinates (numbered 5 through 8).