STATISTICAL PROPERTIES OF DYNAMICS INTRODUCTION TO THE FUNCTIONAL ANALYTIC APPROACH.

STEFANO GALATOLO

ABSTRACT. These are lecture notes for a simple minicourse approaching the satistical properties of a dynamical system by the study of the associated transfer operator (considered on a suitable function space).

The following questions will be addressed:
- existence of a regular invariant measure;
- Lasota Yorke inequalities and spectral gap;
- decay of correlations and some limit theorem;
- stability under perturbations of the system.

The point of view taken is to present the general construction and ideas needed to obtain these results in the simplest way. For this, some theorem is proved in a form which is weaker than usually known, but whit an elementary and simple proof. The application of the tools and the general construction introduced is explained in details for circle expanding maps.

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http://www.math.sci.hokudai.ac.jp/~Hokkaido-Pisa/index.html

CONTENTS

1. Physical measures

Let X be a metric space, $T : X \mapsto X$ a Borel measurable map and μ a T-invariant Borel probability measure (a measure such that for each measurable set A it holds $\mu(A) = \mu(T^{-1}(A))$). Invariant measures represent *equilibrium* states, in the sense that probabilities of events do not change in time. We will see that under suitable assumptions some invariant measure is also an attractor of many other regular measures, and the speed of convergence to this equilibrium has important consequences for the statistical properties of the dynamics.

A set A is called T-invariant if $T^{-1}(A) = A$ (*mod* 0). The system (X, T, μ) is said to be ergodic if each T-invariant set has total or null measure.

Birkhoff ergodic theorem says that in this case, time averages computed along μ-typical orbits coincides with space average with respect to μ. More precisely, in ergodic systems, for any $f \in L^1(X, \mu)$ it holds

$$(1) \qquad \lim_{n \to \infty} \frac{S_n^f(x)}{n} = \int f \, d\mu,$$

for μ almost each x, where $S_n^f = f + f \circ T + \ldots + f \circ T^{n-1}$.

Note that the equality in (1) is up to small sets according to μ. A given map $T : X \mapsto X$ may have many invariant measures corresponding to many possible statistical limit behaviors.

It is important to select the physically relevant ones; the ones which come from the time averages of a large set of points. Large according to the natural measure we can consider on our phase space; when X is a manifold this could be the Lebesgue measure.

Definition 1. *We say that a point x belongs to the basin of an invariant measure μ if (1) holds at x for each bounded continuous f.*

In case X is a manifold (possibly with boundary), a physical measure is an invariant measure whose basin has positive Lebesgue measure.

Often these physical measures also have other interesting features such as:

- they are as regular as possible among the invariant ones;
- they have a certain stability under perturbations of the system;
- they are in some sense limits of iterates of the Lebesgue measure (by the transfer operator we define in the next section).

These measures hence encodes important information about the statistical behavior of the system (see [19] for a general survey). In the following we will see

some method to select those measures, prove their existence and some of its main statistical properties.

2. Transfer operator

Let us consider the space $SM(X)$ of Borel measures with sign on X (equivalently complex valued measures can be considered). A function T between metric spaces naturally induces a function $L : SM(X) \to SM(X)$ *which is linear* and is called transfer operator (associated to T). Let us define L: if $\nu \in SM(X)$ then $L[\nu] \in SM(X)$ is such that

$$L[\nu](A) = \nu(T^{-1}(A)).$$

Remark that if the measure is absolutely continuous: $d\nu = f \, dm$ (here we are considering the Lebesgue measure m as a reference measure, note that other measures can be considered) and if T is nonsingular[1], the operator induces another operator $\tilde{L} : L^1(m) \to L^1(m)$ acting on the measure densities ($\tilde{L}f = \frac{L(f \, dm)}{dm}$). By a small abuse of notation we will still indicate by L this operator.

It is straightforward to see that In this case $L : L^1 \to L^1$ is a *positive operator*[2] and preserves the integral

$$\int Lf \, dm = \int f \, dm,$$

let us see some other important basic properties

Proposition 2. $L : L^1 \to L^1$ *is a* weak contraction *for the L^1 norm. If f is a L^1 density, then*

$$||Lf||_1 \leq ||f||_1.$$

Proof.

$$||Lf||_1 = \int |Lf| \, dm \leq \int L|f| \, dm = \int_{T^{-1}X} |f| \, dm = ||f||_1.$$

$\square$

Proposition 3. *Consider $f \in L^1(m)$, and $g \in L^\infty(m)$, then:*

$$\int g \, L(f) \, dm = \int g \circ T \, f \, dm.$$

Proof. Let us first prove it for simple functions if $g = 1_B$ then

$$\int g \circ T \, f \, dm \quad = \quad \int 1_B \circ T \, f \, dm =$$

$$\int 1_{T^{-1}B} \, f \, dm \quad = \quad \int_{T^{-1}B} f \, dm.$$

If $g \in L^\infty$ we can approximated by a combination of simple functions $\hat{g} = \sum_i a_i 1_{A_i}$ in a way that $||g - \hat{g}||_\infty \leq \epsilon$ and

$$\int \hat{g} \circ T \, f \, dm = \int \hat{g} \, L(f) \, dm.$$

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[1]A map is nonsingular (with respect to the Lebesgue measure) when $m(T^{-1}(A)) = 0 \iff m(A) = 0$

[2]$f \geq 0 \implies Lf \geq 0$

Then

$$\int g \circ T \ f \ dm \quad = \quad \int [g - \hat{g} + \hat{g}] \circ T \ f \ dm$$

$$= \quad \int [g - \hat{g}] \circ T \ f \ dm + \int \hat{g} \circ T \ f \ dm.$$

Moreover

$$\int g \ L(f) \ dm \quad = \quad \int [g - \hat{g} + \hat{g}] \ L(f) \ dm$$

$$= \quad \int [g - \hat{g}] \ L(f) \ dm + \int \hat{g} \ L(f) \ dm$$

and since T is nonsingular

$$|\int [g - \hat{g}] \circ T \ f \ dm| \quad \leq \quad ||[g - \hat{g}] \circ T||_\infty ||f||_1$$

$$\leq \quad ||[g - \hat{g}]||_\infty ||f||_1 \underset{\epsilon \to 0}{\to} 0,$$

moreover

$$|\int [g - \hat{g}] \ L(f) \ dm| \leq ||[g - \hat{g}]||_\infty ||Lf||_1 \underset{\epsilon \to 0}{\to} 0.$$

$\square$

Measures which are invariant for T are fixed points of L. Since physical measures usually have some "as good as possible" regularity property we will find such invariant measures in some space of "regular" measures. A first example which will be explained in more details below is the one of expanding maps, where we are going to find physical measures in the space of invariant measures having an absolutely continuous density.

3. Expanding maps: regularizing action of the transfer operator and existence of a regular invariant measure

In this section we illustrate one approach which allows to prove the existence of regular invariant measures. The approach is quite general, but we will show it on a class of one dimensional maps, where the construction is relatively simple. An important step is to find a suitable function space on which the transfer operator has good properties.

Let us consider a map T which is expanding on the circle. i.e.

- $T : S^1 \to S^1$,
- $T \in C^2$,
- $|T'(x)| > 1 \ \forall x$.

Let us consider the Banach space $W^{1,1}$ of absolutely continuous functions[3] with the norm

$$||f|| = ||f||_1 + ||f'||_1.$$

We will show that the transfer operator is regularizing for the $|| \ ||$ norm. This implies that iterates of a starting measure have bounded $|| \ ||$ norm, allowing to find a suitable invariant measure (and much more information on the statistical behavior of the system, as it will be described in the following sections).

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[3]For which $f' \in L^1$, $f(x) = f(0) + \int_0^x f'(t) \ dt$.

3.1. **Lasota-Yorke inequalities.** A main tool to implement this idea is the so called Lasota Yorke inequality ([16]) [4], let us see what it is about: we consider the operator L restricted to some Banach space $(B_s, || \ ||_s)$, and we consider another function space $B_w \supset B_s$ equipped with a weaker norm $|| \ ||_w$ such that $||L^n||_{B_w \to B_w} \leq M$ is uniformly bounded (as it is for the L^1 norm, see Remark 2). In this context, if the two spaces are well chosen it is possible in many interesting cases to prove that there are $A \geq 0, 0 \leq \lambda < 1$ such that for each n

$$||L^n g||_s \leq A\lambda^n ||g||_s + B||g||_w.$$

This means that the iterates $L^n g$ have bounded strong norm $|| \ ||_s$ and then by suitable compactness arguments[5] this inequality may show the existence of an invariant measure in B_s. Similar inequalities can be proved in many systems, and they are a main tool for the study of statistical properties of dynamical systems.

Now let us see how the above inequality can be obtained in our case. Let us consider a nonsingular transformation and see its action on densities. In the case of expanding maps we are considering we have an explicit formula for the transfer operator[6]:

$$(2) \qquad [Lf](x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{|T'(y)|}.$$

Taking the derivative of (2) (remember that $T'(y) = T'(T^{(-1)}(x))$)

$$(Lf)' = \sum_{y \in T^{-1}(x)} \frac{1}{(T'(y))^2} f'(y) - \frac{T''(y)}{(T'(y))^3} f(y).$$

Note that

$$(3) \qquad (Lf)' = L(\frac{1}{T'} f') - L(\frac{T''}{(T')^2} f)$$

$$||(Lf)'||_1 \quad \leq \quad ||\frac{1}{T'} f'||_1 + ||\frac{T''}{(T')^2} f||_1$$

$$\leq \quad \alpha ||f'||_1 + ||\frac{T''}{(T')^2}||_\infty ||f||_1$$

Where $\alpha = \max(\frac{1}{T'})$.
Hence

$$||(Lf)'||_1 + ||Lf||_1 \leq \alpha ||f'||_1 + \alpha ||f||_1 + (||\frac{T''}{(T')^2}||_\infty + 1)||f||_1$$

and

$$||(Lf)|| \leq \alpha ||f|| + (||\frac{T''}{(T')^2}||_\infty + 1)||f||_1.$$

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[4]In the probabilistic context, this kind of estimations are often called as Doeblin Fortet inequalities.

[5]We will see this in the next paragraphs.

[6]To prove the formula consider a small neighborhood $B(x, \epsilon)$ of x and check the amount of measure which is sent there by L. The value of the density $Lf(x)$ will a.e. be the value of the limit $\lim_{\epsilon \to 0} \frac{\mu(T^{-1}(B(x,\epsilon)))}{2\epsilon}$ (or see [10] pag. 85).

Iterating the inequality[7]

$$
(4) \qquad ||(L^n f)|| \leq \alpha^n ||f|| + \frac{(||\frac{T''}{(T')^2}||_\infty + 1)}{1 - \alpha} ||f||_1.
$$

Hence if we start with $f \in W^{1,1}$ all the elements of the sequence $L^n f'$ of iterates of f are in $W^{1,1}$, and their strong norms are uniformly bounded.

3.1.1. *A Lipschitz Lasota Yorke inequality.* By Equation 4, $||(L^n f)||$ is uniformly bounded, then also $||(L^n f)||_\infty$ is. Let us remark that since the transfer operator is positive

$$
M := \sup_{n, ||f||_\infty = 1} ||(L^n f)||_\infty = \sup_n ||(L^n 1)||_\infty.
$$

Then there is n_1 such that $\alpha^{n_1} M < 1$.

Now consider a new map, $T_2 = T^{n_1}$. This map still has the same regularity properties as before and is uniformly expanding on the circle. Let L_2 be its transfer operator. From (3) we have

$$
(5) \qquad ||(L_2 f)'||_\infty \quad \leq \quad ||L_2(\frac{1}{T_2'} f')||_\infty + ||L_2(\frac{T_2''}{(T_2')^2} f)||_\infty
$$

$$
(6) \qquad\qquad\qquad \leq \quad \alpha^{n_1} M ||f'||_\infty + M ||\frac{T_2''}{(T_2')^2}||_\infty ||f||_\infty.
$$

Then L_2 satisfies a Lasota Yorke inequality, with the norms $|| \, ||_l$ defined as $||f||_l = ||f'||_\infty + ||f||_\infty$ and $|| \, ||_\infty$, that is

$$
||L^{nn_1} f||_l \leq \lambda^n ||f||_l + B ||f||_\infty
$$

with $\lambda = \alpha^{n_1} M < 1$. By this

$$
(7) \qquad ||L^{nn_1 + q} f||_l \quad \leq \quad \lambda^n ||L^q f||_l + B ||L^q f||_\infty
$$

$$
\qquad\qquad\qquad\qquad \leq \quad \lambda^n M ||f||_l + BM ||f||_\infty
$$

and then also L satisfies a Lasota Yorke inequality with these norms.

Remark 4. *In this section we have shown two examples of regularization estimations on different function spaces. This kind of estimations are possible over many kind of systems having some uniform contracting/expanding behavior. In these cases the choice of the good Banach spaces involved is crucial. When the system is expanding, even on higher dimension and with low regularity, spaces of bounded variation functions and absolutely continuous measures are usually considered. If the system has contracting directions, the physical measure usually has fractal support and it is often included in some suitable distributions space on which a Lasota Yorke estimation can be proved (see. e.g. [15],[2],[10],[3], [14]).*

3.2. **Existence of a regular invariant measure.** The following theorem provides a compactness argument to prove the existence of an invariant measure in L^1. (see [11] for more details and generalizations)

Proposition 5 (Rellich-Kondrachov). $W^{1,1}$ *is compactly immersed in* L^1. *If* $B \subset W^{1,1}$ *is a strongly bounded set:* $B \subseteq B(0, K)$ *then for each* ϵ, B *has a finite* ϵ-net *for the* L^1 *topology.*

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[7] $||L^2 f|| \leq \alpha ||Lf|| + B ||Lf||_1$
$\leq \alpha^2 ||f|| + \alpha B ||f||_1 + B ||f||_1, ...$

In particular any bounded subsequence $f_n \in W^{1,1}$ has a weaky converging subsequence. There is f_{n_k} and $f \in L^1$ such that

$$f_{n_k} \to f$$

in L^1.

Proof. (sketch) Let us consider a subdivision $x_1, ..., x_m$ of I with step ϵ.

Let us consider πf_n to be the piecewise linear approximation of f such that

$$\pi f_n(x_i) = f_n(x_i).$$

Now, if $x_i \leq \overline{x} \leq x_{i+1}$ then

$$|f_n(\overline{x}) - \pi f_n(\overline{x})| \leq \int_{x_i}^{x_{i+1}} |f'(t)| \, dt := h_i.$$

Remark that $\sum h_i \leq K$. Hence

$$||f_n - \pi f_n||_1 \leq \epsilon \sum h_i \leq \epsilon K.$$

Since π has finite rank it is then standard to construct a $2\epsilon K$-net and we have that there is f_{n_k}, $f \in L^1$ s.t. $f_{n_k} \to f$ in L^1. $\square$

Now let us consider the sequence $g_n = \frac{1}{n}\sum_{i=0}^{n-1} L^n 1$ where 1 is the density of the normalized Lebesgue measure. By the Lasota Yorke inequality the sequence has uniformly bounded $W^{1,1}$norm and by the last Lemma has a subsequence g_{n_k} converging in L^1 to a limit h.

Now recall that L is continuous in the L^1 norm. By this

$$Lh = L(\lim_{k \to \infty} g_{n_k}) = \lim_{k \to \infty} Lg_{n_k} = h.$$

Then h is an invariant density.

Proposition 6. *The density h found above has the following properties:*

- $h \in W^{1,1}$ *and*
- $||h|| \leq \frac{(||\frac{T''}{(T')^2}||_\infty + 1)}{1-\alpha}$.

Proof. Consider $g_{n,m} = L^m(g_n)$. Remark that, $g_{n,m} \in W^{1,1}$ and the norms are uniformly bounded. By the Lasota Yorke inequality

$$(8) \quad ||g_{n_1,a+m} - g_{n_2,b+m}||_{W^{1,1}} \leq \alpha^m ||g_{n_1,a} - g_{n_2,b}||_{W^{1,1}} + B||g_{n_1,a} - g_{n_2,b}||_1$$

also remark that if $||g_{n_k,0} - h||_1 \leq \epsilon$ then $||g_{n_k,j} - h||_1 \leq \epsilon$ for all $j \geq 0$. Then the sequence $g_{n_k,k} \to h$ in L^1, and by 8 is a Cauchy sequence in the $W^{1,1}$norm, suppose $k_1 \leq k_2$

$$(9)$$
$$||g_{n_{k_1},k_1} - g_{n_{k_2},k_2}||_{W^{1,1}} \leq \alpha^{k_1} ||g_{n_{k_1},0} - g_{n_{k_2},k_2-k_1}||_{W^{1,1}} + B||g_{n_{K_1},0} - g_{n_{K_2},k_2-k_1}||_1.$$

This implies that it converges to some limit which is forced to be h. Hence $h \in W^{1,1}$.

By the Lasota Yorke inequality, since h is invariant then $||h|| = ||Lh|| \leq \frac{(||\frac{T''}{(T')^2}||_\infty + 1)}{1-\alpha}$.
$\square$

4. Convergence to equilibrium and mixing

Let us consider again an expanding map of the circle T as defined before and its transfer operator L. Let us consider the strong and weak space of zero average densities

$$V_s = \{g \in W^{1,1} \ s.t. \ \int g \, dm = 0\}$$

and

$$V_w = \{g \in L^1 \ s.t. \ \int g \, dm = 0\}.$$

Proposition 7. *For each $g \in V_s$, it holds*

$$\lim_{n \to \infty} ||L^n g||_1 = 0.$$

Proof. First let us suppose that $||g||_l < \infty$. By 7 we know that all the iterates of g have uniformly bounded l norm

$$||L^n g||_l \leq \overline{M}.$$

Let us denote by g^+, g^- the positive and negative parts of g. Remark that $||g||_1 = 2 \int g^+ \, dm$. There is a point $\overline{x}$ such that $g^+(\overline{x}) \geq \frac{1}{2}||g||_1$. Around this point consider a neighborhood $N = B(\overline{x}, \frac{1}{4}||g||_1 \overline{M}^{-1})$. For each point $x \in N$, $g^+(x) \geq \frac{1}{4}||g||_1$.

Now let $d = \min |T'|$, $D = \max |T'|$. If n_1 is the smallest integer such that $d^{n_1} \frac{1}{2}||g||_1 \overline{M}^{-1} > 1$ (i.e. $n_1 > \frac{\log(2||g||_1^{-1}\overline{M})}{\log d}$) then $T^{n_1}(N) = S^1$ and $L^{n_1}g^+$ has then density at least

$$\frac{||g||_1}{4D^{n_1}} \geq \frac{||g||_1}{4D e^{\log(2||g||_1^{-1}\overline{M})\frac{\log D}{\log d}}} = \frac{||g||_1}{4D}(2||g||_1\overline{M}^{-1})^{\frac{\log D}{\log d}}$$

on S^1. Then, setting $C = \frac{(2\overline{M}^{-1})^{\frac{\log D}{\log d}}}{4D}$

$$||L^{n_1}g||_1 \leq ||g||_1 - C||g||_1^{\frac{\log D}{\log d}+1}.$$

Let us denote $g_1 = L^{n_1}g$. We can repeat the above construction and obtain n_2 such that

$$||g_2||_1 = ||L^{n_2}g_1||_1 \leq ||g_1||_1 - C||g_1||_1^{\frac{\log D}{\log d}+1}$$

and so on. Continuing, we have a sequence g_n such that

$$||g_{n+1}||_1 \leq ||g_n||_1 - C||g_n||_1^{\frac{\log D}{\log d}+1}$$

and then $||g_n||_1 \to 0$.

If now more generally, $g \in W^{1,1}$ we can approximate g with a $\tilde{g}$ such that $||\tilde{g}||_l < \infty$ in a way that $||g - \tilde{g}||_1 \leq \epsilon$. Since $||L||_{L^1 \to L^1} \leq 1$, $\lim_{n \to \infty} ||L^n(g - \tilde{g})||_1 \leq \epsilon$. And then the statement follows. $\square$

It easily follows (see Proposition 29)

Corollary 8. *Expanding maps of the circle, considered with its $W^{1,1}$ invariant measure are mixing.*

Corollary 9. *For expanding maps of the circle, there is only one invariant measure in $W^{1,1}$.*

Proof. If there are two invariant probability densities h_1, h_2 then $h_1 - h_2 \in V_s$ and invariant, impossible. $\qquad\square$

Corollary 10. *The whole sequence $g_n = \frac{1}{n} \sum_{i=0}^{n-1} L^n 1$ converges to h.*

Remark 11. *By Proposition 7 it also follows that, if $g \in W^{1,1}$ is a probability density (and then $g - h \in V_s$) then $L^n g \to h$ in the L^1 norm. By the Lasota Yorke inequality we also get the convergence in $W^{1,1}$ (see the proof of Proposition 6).*

4.1. **Speed of convergence to equilibrium.** For several applications, it is important to quantify the speed of mixing or convergence to equilibrium.

Let us see how to quantify: consider two vector subspaces of the space of signed (complex) measures on X

$$B_s \subseteq B_w,$$

endowed with two norms, the strong norm $|| \ ||_s$ on B_s and the weak norm $|| \ ||_w$ on B_w, such that $|| \ ||_s \geq || \ ||_w$ on B_w (the $W^{1,1}$ norm and the L^1 norm e.g.).

Definition 12. *We say that the transformation (X, T) has convergence to equilibrium[8] with speed Φ with respect to these norms if for any $f \in V_s$.*

$$(10) \qquad\qquad ||L^n f||_w \leq \Phi(n) \ ||f||_s.$$

We remark that in this case if ν is a starting probability measure in B_s and μ is the invariant measure, still in B_s, then $\nu - \mu \in V_s$ and then

$$||L^n \nu - \mu||_w \leq \Phi(n) \ ||\nu - \mu||_s.$$

and then $L^n \nu$ converges to μ at a speed $\Phi(n)$. Depending on the strong norm, one may prove that $||\nu - \mu||_s \leq C||\nu||_s$ where C does not depend on ν, obtaining

$$||L^n \nu - \mu||_w \leq C\Phi(n) \ ||\nu||_s$$

(see Section 7 for one example).

In the next section we will see that the Lasota Yorke inequality and the properties of the spaces we have chosen allows to prove exponential speed of convergence for our circle expanding maps.

5. Spectral gap and consequences

Now we see a general result that easily implies that the convergence rate of iterates of the transfer operator is exponentially fast.

We recall some basic concepts on the spectrum of operators. Let L be an operator acting on a complex Banach space $(B, || \ ||)$:

- the spectrum of an operator is defined as

$$spec(L) = \{\lambda : (\lambda I - L) \text{ has no bounded inverse}\}$$

- the spectral radius of L is defined as

$$\rho(L) = \sup\{|z| : z \in spec(L)\}.$$

An important connection between the spectral properties of the operator, and the asymptotic behavior of its iterates is given by the following formula

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[8]This speed is also related to the decay of correlation integrals, like

$$| \int f \circ T^n \ g \ d\mu - \int f \ d\mu \int g \ d\mu |$$

for observables f, g in suitable function spaces, see Section 7

Proposition 13 (Spectral radius formula)**.**

$$\rho(L) = \lim_{n\to\infty} \sqrt[n]{||L^n||} = \inf_n \sqrt[n]{||L^n||}.$$

Definition 14 (Spectral gap)**.** *The operator L has spectral gap if*

$$L = \lambda \, \mathrm{P} + \mathrm{N}$$

where

- P *is a projection (i.e. $\mathrm{P}^2 = \mathrm{P}$) and $\dim(Im(\mathrm{P})) = 1$;*
- *the spectral radius of* N *satisfies $\rho(\mathrm{N}) < |\lambda|$;*
- $\mathrm{P}\,\mathrm{N} = \mathrm{N}\,\mathrm{P} = 0$.

The following is an elementary tool to verify spectral gap of L on B_s.

Let us consider a transfer operator L acting on two normed vector spaces of complex or signed measures $(B_s, || \; ||_s)$, $(B_w, || \; ||_w)$, $B_s \subseteq B_w \subseteq CM(X)$ with $|| \; ||_s \geq || \; ||_w$.

Theorem 15. *Suppose:*

(1) *(Lasota Yorke inequality). For each $g \in B_s$*

$$||L^n g||_s \leq A\lambda_1^n ||g||_s + B||g||_w;$$

(2) *(Mixing) for each $g \in V_s$, it holds*

$$\lim_{n\to\infty} ||L^n g||_w = 0;$$

(3) *(Compact inclusion) the strong zero average space V_s is compactly immersed in the weak one V_w (the strong unit ball in the weak topology has a finite ϵ net for each epsilon);*

(4) *(Weak boundedness) the weak norm of the operator restricted to V_s satisfies*

$$\sup_n ||L^n|_{V_s}||_w < \infty.$$

Under these assumptions there are $C_2 > 0, \rho_2 < 1$ such that for all $g \in V_s$

(11) $$||L^n g||_s \leq C_2 \rho_2^n ||g||_s.$$

Proof. We first show that assumption 2 and 3 and 4 imply that L is uniformly contracting from V_s to V_w: there is $n_1 > 0$ such that $\forall g \in V_s$

(12) $$||L^{n_1} g||_w \leq \lambda_2 ||g||_s$$

where $\lambda_2 B < 1$.

Indeed, by item (3), for any ϵ there is a finite set $\{g_i\}_{i\in(1,\dots,k)}$ in the strong unit ball $\mathcal{B}$ of V_s such that for each g in $\mathcal{B}$ there is a $g_i \in V_s$ such that $||g - g_i||_w \leq \epsilon$.

Hence

$$\sup_{g\in V_s, ||g||_s \leq 1} ||L^n g||_w \leq \sup_{1\leq i\leq k, v\in\{v\in V_s \; s.t. \; ||v||_w\leq\epsilon\}} ||L^n(g_i + v)||_w.$$

Now, by item (4) suppose that $\forall n \;\; ||L^n|_{V_s}||_w \leq M$, then

$$\sup_i ||L^n(g_i + v)||_w \leq \sup_i ||L^n(g_i)||_w + M\epsilon.$$

Since by item (2) for each i, $\lim_{n\to\infty} ||L^n(g_i)||_w = 0$ and ϵ can be chosen as small as wanted we have Eq. (12).

Let us apply the Lasota Yorke inequality. For each $f \in V_s$

$$||L^{n_1+m} f||_s \leq A\lambda_1^m ||L^{n_1} f||_s + B||L^{n_1} f||_w$$

then

$$\begin{aligned}
||L^{n_1+m}f||_s &\leq A\lambda_1^m||L^{n_1}f||_s + B\lambda_2||f||_s \\
&\leq A\lambda_1^m[A\lambda_1^{n_1}||f||_s + B||f||_w] + B\lambda_2||f||_s.
\end{aligned}$$

If m is big enough

$$||L^{n_1+m}f||_s \leq \lambda_3||f||_s$$

with $\lambda_3 < 1$.

This easily implies the statement. Indeed set $n_2 = n_1 + m$, for each $k, q \in \mathbb{N}$, $q \leq n_2$, $g \in V_s$,

$$\begin{aligned}
||L^{kn_2+q}g||_s &\leq \lambda_3^k||L^q g||_s \\
&\leq \lambda_3^k(\lambda_1^q||g||_s + B||g||_w).
\end{aligned}$$

Implying that for each $g \in V_s$ there are $C_2 > 0, \rho_2 < 1$ such that

(13) $$||L^n g||_s \leq C_2\rho_2^n||g||_s.$$

$\square$

By this theorem and the spectral radius formula, the spectral radius of L restricted to V_s is strictly smaller than 1, and the spectral gap (as defined in Definition 14) follows: first remark that by the Lasota Yorke inequality and the spectral radius formula, the spectral radius of L on B_s is not greater than than 1. Since there is an invariant measure in B_s then this radius is 1. By item 2) there can be only one fixed point of L in B_s and only one invariant probability measure which we denote by h (if there are two, consider the difference which is in V_s...).

Now let us remark that every $g \in B_s$ can be written as follows:

$$g = [g - h\ g(X)] + [h\ g(X)].$$

the function $P : B_s \to B_s$ defined as

$$P(g) = h\ g(X)$$

is a projection. The function $N : B_s \to B_s$ defined as

$$N(g) = L[g - h\ g(X)]$$

is such that $N(B_s) \subseteq V_s$, $N|V_s = L|V_s$, and by (11) satisfies $\rho(N) < |\lambda|$. It holds

$$L = P + N$$

and $PN = PN = 0$. Thus, under the assumption of Proposition 15, L has spectral gap according to the Definition 14.

We remark that in several texts the role of Theorem 15 is played by a more general result referred to Hennion, Hervé or Ionescu-Tulcea and Marinescu (see e.g. [15], [17]). Theorem 15 is less general, but has a simple and somewhat constructive proof.

Remark 16. *Equation 13 obviously implies exponential convergence to equilibtium.*

Remark 17 (spectral gap for expanding maps of the circle)**.** *By Proposition 5[9] , Proposition 7 and the Lasota Yorke inequality, the assumptions of Theorem 15 are verified on our expanding maps of the circle for the $W^{1,1}$ norm (whit L^1 norm as a weak norm). Then their transfer operator have spectral gap.*

---

[9]Which can be easily adapted to V_s, by considering an integral preserving projection $\pi_2 f = \pi f - \int \pi f$.

5.1. **Central limit.** We see an application of Theorem 15 to the estimation of the fluctuations of an observable, obtaining a sort of central limit theorem. A proof of the result can be found in [17].

Theorem 18. *Let (X, T, μ) be a mixing probability preserving transformation having spectral gap on some Banch space $(\mathcal{B}, || \; ||)$ containing the constants and satisfying*

$$
\begin{array}{rcl}
(14) \qquad\qquad ||fg|| & \leq & ||f|| \; ||g|| \\
|| \; || & \geq & || \; ||_1.
\end{array}
$$

Let $f \in \mathcal{B}$ be bounded and $\int f \; d\mu = 0$. If there is no $\nu \in \mathcal{B}$ such that $f = \nu - \nu \circ T$ a.e., then $\exists \sigma > 0$ s.t. for all intervals $[a, b]$,

$$
\mu \left\{ x : \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f \circ T^k \in [a, b] \right\} \rightarrow \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^b e^{-t^2/2\sigma^2} \, dt.
$$

Remark 19. *We remark that the assumption in Equation (14) can be relaxed. Indeed if $||fg|| \leq K||f|| \; ||g||$, then considering the new rescaled norm $|| \; ||_K = K|| \; ||$ it holds*

$$
||fg||_K = K||fg|| \leq K^2||f|| \; ||g|| \leq ||f||_K ||g||_K.
$$

We remark, that the $W^{1,1}$ norm satisfies (14) after rescaling. Since there is spectral gap, then Theorem 18 applies to our class of expanding maps on S^1.

6. STABILITY AND RESPONSE TO PERTURBATION

In this section we consider small perturbations of a given system and try to study the dependence of the invariant measure on the perturbation. If the measure varies continuously, we know that many of the statistica properties of the system are stable under perturbation (see [1] and [8] for examples of results in this direction, in several classes of systems).

It is known that even in families of piecewise expanding maps, the physical invariant measure may change discontinuously (see e.g. [12]).

We will see that under certain general assumptions related to the convergence to equilibrium of the system and to a uniform Lasota Yorke estimation, the physical measure changes continuously, and we have estimations on the modulus of continuity. If stronger, spectral gap assumptions can be made on the system, the dependence is Lipschitz, or even differentiable.

We remark that with more work, other stability results can be proved for the whole spectral picture of the system (see [15]).

Consider again two vector subspaces of measures with sign on X

$$
B_s \subseteq B_w \subseteq SM(X),
$$

endowed with two norms, the strong norm $|| \; ||_s$ on B_s and the weak norm $|| \; ||_w$ on B_w, such that $|| \; ||_s \geq || \; ||_w$ and let V_s be compactly immersed in V_w as before. Denote as before by V_s, V_w the "zero average" spaces.

A uniform family of operators. Let us consider a one parameter family of operators L_δ, $\delta \in [0, 1)$. Suppose that:

UF1 $\forall \delta$, $L_\delta(B_s) \subseteq B_s$ and each operator satisfies a Lasota Yorke inequality. There exists constants $A, B, \lambda_1 \in \mathbb{R}$ and $\lambda_1 < 1$ such that $\forall f \in B_s, \forall n \geq 1$

$$
(15) \qquad\qquad ||L_\delta^n f||_s \leq A\lambda_1^n ||f||_s + B||f||_w.
$$

UF2 Suppose that L_δ approximates L_0 when δ is small in the following sense: there is $C \in \mathbb{R}$ such that $\forall g \in B_s$:

$$\tag{16} ||(L_\delta - L_0)g||_w \leq \delta C ||g||_s.$$

UF3 Suppose that L_0 has exponential convergence to equilibrium, with respect to the norms $|| \; ||_w$ and $|| \; ||_s$.

UF4 $\forall \delta, n, g \in B_s \;\; ||L_\delta^n g||_w \leq M ||g||_w.$

We will see that under these assumptions we can ensure that the invariant measure of the system varies continuously (in the weak norm) when L_0 is perturbed to L_δ for small values of δ.

6.1. **Stability of fixed points, a general statement.** We state a general result on the stability of fixed points satisfaying certain assumptions. This will be a flexible tool to obtain the stability of the invariant measure under small perturbations.

Let us consider two operators L_0 and L_δ preserving a normed space of measures $\mathcal{B} \subseteq SM(X)$ with norm $|| \; ||_\mathcal{B}$. Let us suppose that $f_0, f_\delta \in \mathcal{B}$ are fixed points, respectively of L_0 and L_δ.

Lemma 20. *Suppose that:*

    a): $||L_\delta f_\delta - L_0 f_\delta||_\mathcal{B} < \infty$
    b): L_0^i *is continuous on* $\mathcal{B}$; $\exists C_i$ *s.t.* $\forall g \in \mathcal{B}, \; ||L_0^i g||_\mathcal{B} \leq C_i ||g||_\mathcal{B}.$

    Then for each N

$$\tag{17} ||f_\delta - f_0||_\mathcal{B} \leq ||L_0^N (f_\delta - f_0)||_\mathcal{B} + ||L_\delta f_\delta - L_0 f_\delta||_\mathcal{B} \sum_{i \in [0, N-1]} C_i.$$

Proof. The proof is a direct computation

$$
\begin{aligned}
||f_\delta - f_0||_\mathcal{B} &\leq ||L_\delta^N f_\delta - L_0^N f_0||_\mathcal{B} \\
&\leq ||L_0^N f_0 - L_0^N f_\delta||_\mathcal{B} + ||L_0^N f_\delta - L_\delta^N f_\delta||_\mathcal{B}
\end{aligned}
$$

Hence

$$||f_0 - f_\delta||_\mathcal{B} \leq ||L_0^N (f_0 - f_\delta)||_\mathcal{B} + ||L_0^N f_\delta - L_\delta^N f_\delta||_\mathcal{B}$$

but

$$L_0^N - L_\delta^N = \sum_{k=1}^N L_0^{N-k} (L_0 - L_\delta) L_\delta^{k-1}$$

and

$$
\begin{aligned}
(L_0^N - L_\delta^N) f &= \sum_{k=1}^N L_0^{N-k} (L_0 - L_\delta) L_\delta^{k-1} f_\delta \\
&= \sum_{k=1}^N L_0^{N-k} (L_0 - L_\delta) f_\delta
\end{aligned}
$$

by item b)

$$
\begin{aligned}
||(L_0^N - L_\delta^N) f_\delta||_\mathcal{B} &\leq \sum_{k=1}^N C_{N-k} ||(L_0 - L_\delta) f_\delta||_\mathcal{B} \\
&\leq ||(L_0 - L_\delta) f_\delta||_\mathcal{B} \sum_{i \in [0, N-1]} C_i
\end{aligned}
$$

then
$$||f_\delta - f_0||_\mathcal{B} \leq ||L_0^N(f_0 - f_\delta)||_\mathcal{B} + ||(L_0 - L_\delta)f_\delta||_\mathcal{B} \sum_{i \in [0, N-1]} C_i.$$

$\square$

Now, let us apply the statement to our family of operators satisfying assumptions UF 1,...,4, supposing $B_w = \mathcal{B}$. We have the following

Proposition 21. *Suppose L_δ is a uniform family of operators satisfying UF1,...,4. f_0 is the unique invariant measure of L_0, f_δ is an invariant measure of L_δ. Then*
$$||f_\delta - f_0||_w = O(\delta \log \delta).$$

Proof. We remark that by the uniform Lasota Yorke inequality $||f_\delta||_s \leq M$ are uniformly bounded.

Hence
$$||L_\delta f_\delta - L_0 f_\delta||_w \leq \delta C M$$
(see item a) of Lemma 20). Moreover by UF4, $C_i \leq M_2$.

Hence
$$||f_\delta - f_0||_w \leq \delta C M M_2 N + ||L_0^N(f_0 - f_\delta)||_w.$$

Now by the exponential convergence to equilibrium of L_0
$$\begin{aligned}
||L_0^N(f_\delta - f_0)||_w &\leq C_2 \rho_2^N ||(f_\delta - f_0)||_s \\
&\leq C_2 \rho_2^N M
\end{aligned}$$

hence
$$||f_\delta - f_0||_\mathcal{B} \leq \delta C M M_2 N + C_2 \rho_2^N M$$
choosing $N = \left\lfloor \frac{\log \delta}{\log \rho_2} \right\rfloor$

$$\begin{aligned}
(18) \qquad ||f_\delta - f_0||_w &\leq \delta C M M_2 \left\lfloor \frac{\log \delta}{\log \rho_2} \right\rfloor + C_2 \rho_2^{\left\lfloor \frac{\log \delta}{\log \rho_2} \right\rfloor} M \\
&\leq \delta \log \delta C M_2 M \frac{1}{\log \rho_2} + C_2 \delta M.
\end{aligned}$$

$\square$

6.2. **Application to expanding maps.** In the previous section we considered the stability of the invariant measure under small perturbations of the transfer operator.

There are many kinds of interesting perturbations to be considered. Two main classes are deterministic or stochatsic ones.

In the deterministic ones the transfer operator is perturbed by small changes on the underlying dynamics (the map).

The stochastic ones can be of several kinds. The simplest one is the adding of some noise perturbing the result of the deterministic dynamics at each iteration (see [15] for some example and related estimations).

We now consider small deterministic perturbations of our expanding maps on S^1. Let us consider a one parameter family T_δ, $\delta \in [0, 1]$ of expanding maps of the circle satisfying the properties stated at beginning of Section 3 and

UFM: $||T_\delta - T_0||_{C^2} \leq K\delta$ for some $K \in \mathbb{R}$.

To each of these maps it is associated a transfer operator L_δ acting on $W^{1,1}$

Proposition 22. *If L_0 and L_δ are trsnfer operators of expanding maps T_0 and T_δ satisfying UFM, then there is a $C \in \mathbb{R}$ such that $\forall g \in W^{1,1}$:*

$$(19) \qquad\qquad ||(L_\delta - L_0)f||_1 \leq \delta C ||f||_{W^{1,1}}$$

and assumption **UF2** *is satisfied.*

Proof.

$$(20) \qquad\qquad [L_\delta f](x) = \sum_{y \in T_\delta^{-1}(x)} \frac{f(y)}{|T'_\delta(y)|}.$$

$$
\begin{aligned}
|[L_\delta f](x) - [L_0 f](x)| \quad = \quad & |\sum_{y \in T_\delta^{-1}(x)} \frac{f(y)}{|T'_\delta(y)|} - \sum_{y \in T_0^{-1}(x)} \frac{f(y)}{|T'_0(y)|}| \\
\leq \quad & |\sum_{y \in T_\delta^{-1}(x)} \frac{f(y)}{|T'_\delta(y)|} - \sum_{y \in T_\delta^{-1}(x)} \frac{f(y)}{|T'_0(y)|}| + \\
& +|\sum_{y \in T_\delta^{-1}(x)} \frac{f(y)}{|T'_0(y)|} - \sum_{y \in T_0^{-1}(x)} \frac{f(y)}{|T'_0(y)|}|.
\end{aligned}
$$

The first summand can be estimated as follows

$$
\begin{aligned}
|\sum_{y \in T_\delta^{-1}(x)} \frac{f(y)}{|T'_\delta(y)|} - \sum_{y \in T_\delta^{-1}(x)} \frac{f(y)}{|T'_0(y)|}| \quad \leq \quad & |\sum_{y \in T_\delta^{-1}(x)} \frac{f(y)}{|T'_\delta(y)|} [1 - \frac{|T'_\delta(y)|}{|T'_0(y)|}]| \\
\leq \quad & D_1 |\sum_{y \in T_\delta^{-1}(x)} \frac{f(y)}{|T'_\delta(y)|}| \\
\leq \quad & D_1 |L_\delta f(x)|
\end{aligned}
$$

where $D_1 = \sup |1 - \frac{|T'_\delta(y)|}{|T'_0(y)|}|$ and remark that $D_1 = O(\delta)$. For second summand let us denote $T_\delta^{-1}(x) = \{y_1, ..., y_n\}$, $T_0^{-1}(x) = \{y_1^0, ..., y_n^0\}$. Let $\Delta_y = \sup_{x,i}(|y_i - y_i^0|)$ note that $\Delta_y = O(\delta)$

$$
\begin{aligned}
|\sum_{y \in T_\delta^{-1}(x)} \frac{f(y)}{|T'_0(y)|} - \sum_{y \in T_0^{-1}(x)} \frac{f(y)}{|T'_0(y)|}| \quad \leq \quad & |\sum_{i=1}^{n} \frac{f(y_i) - f(y_i^0)}{|T'_0(y_i)|}| + |\sum_{i=1}^{n} f(y_i)(\frac{1}{|T'_0(y_i)|} - \frac{1}{|T'_0(y_i^0)|})| \\
\leq \quad & |\sum_{i=1}^{n} \frac{f(y_i) - f(y_i^0)}{|T'_0(y_i)|}| + |\sum_{i=1}^{n} \frac{f(y_i)}{|T'_0(y_i)|}(1 - \frac{|T'_0(y_i)|}{|T'_0(y_i^0)|})| \\
\leq \quad & |\sum_{i=1}^{n} \frac{f(y_i) - f(y_i^0)}{|T'_0(y_i)|}| + D_2 |\sum_{i=1}^{n} \frac{f(y_i)}{|T'_0(y_i)|}| \\
\leq \quad & |\sum_{i=1}^{n} \frac{\int_{y_i^0}^{y_i} f'(t) dt}{|T'_0(y_i)|}| + D_2 D_3 |L_\delta f(x)|
\end{aligned}
$$

where $D_2 := \sup |1 - \frac{|T_0'(y_i)|}{|T_0'(y_i^0)|}| = O(\delta)$ and $D_3 = \sup |\frac{T_\delta'}{T_0'}| \underset{\delta \to 0}{\to} 1$. Hence

$$
\begin{aligned}
||L_\delta f - L_0 f||_1 &\leq D_1 ||L_\delta f(x)||_1 + ||\sum_{i=1}^n \frac{\int_{y_i^0}^{y_i} f'(t)dt}{|T_0'(y_i)|}||_1 + D_2 D_3 ||L_\delta f(x)||_1 \\
&\leq (D_1 + D_2 D_3)||f(x)||_1 + ||\sum_{i=1}^n \frac{\int_{y_i^0}^{y_i} f'(t)dt}{|T_0'(y_i)|}||_1 \\
&\leq O(\delta)||f(x)||_1 + ||\sum_{i=1}^n \frac{\int_{y_i - \Delta_y}^{y_i} |f'(t)|dt}{|T_0'(y_i)|}||_1 \\
&\leq O(\delta)||f(x)||_1 + ||\sum_{i=1}^n \frac{[1_{[-\Delta_y,0]} * |f'|](y_i)}{|T_0'(y_i)|}||_1 \\
&\leq O(\delta)||f(x)||_1 + D_3||L_\delta[1_{[-\Delta_y,0]} * |f'|]||_1 \\
&\leq O(\delta)||f(x)||_1 + D_3||1_{[-\Delta_y,0]} * |f'|||_1 \\
&\leq O(\delta)||f(x)||_1 + D_3||1_{[-\Delta_y,0]}||_1||f'||_1 \\
&\leq O(\delta)||f||_{W^{1,1}}.
\end{aligned}
$$

where $[1_{[-\Delta_y,0]} * |f'|]$ stands for the convolution function between the characteristic of the interval $_{[-\Delta_y,0]}$ (mod 1) and $|f'|$. And the statement is proved. $\square$

It is easy to verify that a family of expanding maps uniformly satisfying the assumptions at beginning of section 3 satysfy UF1,3,4. UFM implies UF2, this allow to apply Proposition 21 and prove the stability of the invariant measure for this family of mappings and have an explicit estimation for the modulus of continuity:

Corollary 23. *Let h_δ be the family of invariant measures in L^1 for the maps T_δ described above. Then*

$$||h_0 - h_\delta||_1 = O(\delta \log \delta).$$

6.3. **Uniform family of operators and uniform V_s contraction.** Now we show how a suitable uniform family of operators, not only has a certain stability on the invariant measure as seen above, but also a uniform rate of contraction of the space V_s and hence a uniform convergence to equilibrium and spectral gap (we remark that stability results on the whole spectral picture are known, see [15], [2] e.g.).

Consider again two vector subspaces of the space of signed measures on X

$$B_s \subseteq B_w \subseteq SM(X),$$

endowed with two norms, the strong norm $|| \ ||_s$ on B_s and the weak norm $|| \ ||_w$ on B_w, such that $|| \ ||_s \geq || \ ||_w$. Denote as before by V_s, V_w the "zero average" strong and weak spaces.

Uniform family of operators. Let us consider a one parameter family of operators L_δ, $\delta \in [0,1)$. Suppose that they satisfy UF1,...UF4:

Proposition 24 (Uniform V_s contraction for the uniform family of operators.)**.** *Under the above assumptions there are $\lambda_4 < 1$, $A_2, \delta_0 \in \mathbb{R}$ such that for each $\delta \leq \delta_0$, $f \in V_s$*

$$(21) \qquad\qquad ||L_\delta^k f||_s \leq A_2 \lambda_4^k ||f||_s.$$

Before the proof we need the following

Lemma 25. *Suppose that L_0 satisfies a Lasota Yorke inequality and*

- $\forall g \in B_s \quad ||(L_\delta - L_0)g||_w \le C\delta||g||_s$;
- $\forall \delta, n, g \in B_s \quad ||L_\delta^n g||_w \le M||g||_w$

then L_δ^n approximates L_0^n in the following sense: there are constants C, D such that $\forall g \in B_s, \forall n \ge 0$

$$\text{(22)} \qquad ||(L_\delta^n - L_0^n)g||_w \le \delta(C||g||_s + nD||g||_w).$$

Proof.

$$
\begin{aligned}
||(L_\delta^n - L_0^n)g||_w &\le \sum_{k=1}^n ||L_\delta^{n-k}(L_\delta - L_0)L_0^{k-1}g||_w \le M\sum_{k=1}^n ||(L_\delta - L_0)L_0^{k-1}g||_w \\
&\le M\sum_{k=1}^n \delta C||L_0^{k-1}g||_s \\
&\le \delta MC\sum_{k=1}^n (A\lambda_1^{k-1}||g||_s + B||g||_w) \\
&\le \delta MC(\frac{A}{1-\lambda_1}||g||_s + Bn||g||_w).
\end{aligned}
$$

$\square$

of Proposition 24. Let us apply the Lasota Yorke inequality

$$||L_\delta^{n+m}f||_s \le A\lambda_1^n||L_\delta^m f||_s + B||L_\delta^m f||_w$$

by the above assumptions and Lemma 25

$$
\begin{aligned}
||L_\delta^{n+m}f||_s &\le A\lambda_1^n||L_\delta^m f||_s + B[E\lambda_2^m||f||_s + \delta(C||f||_s + mD||f||_w)] \\
&\le A\lambda_1^n[A\lambda_1^m||f||_s + B||f||_w] + B[E\lambda_2^m||f||_s + \delta(C||f||_s + mD||f||_w)].
\end{aligned}
$$

If n, m are big enough and δ small enough then we have that

$$||L_\delta^{n+m}f||_s \le \lambda_3||f||_s$$

and thus there are $\lambda_4 < 1$, $A_2 \in \mathbb{R}$ such that for each $f \in V_s$

$$\text{(23)} \qquad ||L_\delta^k f||_s \le A_2\lambda_4^k||f||_s.$$

$\square$

6.4. **Lipschitz continuity.** In this paragraph (essentially taken from [15]) we see that exploiting the uniform contraction rate of V_s and some further assumptions we can prove Lipschitz dependence of the relevant invariant measure under system perturbations. Further work also lead to differentiable dependence (see next section).

Proposition 26. *Let us consider a uniform family L_δ, $\delta \in [0, 1)$ of operators satisfying UF1,...,UF4. Suppose that each operator L_δ has a unique invariant measure h_δ in B_s and*

$$||(L_\delta - L_0)h_0||_s \le C_{h_0}\delta;$$

then the dependence is Lipschitz (with respect to the strong norm)

$$||h_0 - h_\delta||_s \le O(\delta).$$

Proof. Denote $\Delta h = h_\delta - h_0$:

$$
\begin{aligned}
(I - L_\delta)\Delta h &= (I - L_\delta)(h_\delta - h_0) \\
&= h_\delta - L_\delta h_\delta - h_0 + L_\delta h_0 \\
&= (L_\delta - L_0)h_0.
\end{aligned}
$$

By the uniform contraction (21) we have that $(I - L_\delta)$ is invertible on V_s, and $(I - L_\delta)^{-1} = \sum_0^\infty L_\delta^i$ is uniformly bounded $||(I - L_\delta)^{-1}||_{V_s \to V_s} \leq M_2$.

Since $(L_\delta - L_0)h_0 \in V_s$, then

$$
\Delta h = (I - L_\delta)^{-1}(L_\delta - L_0)h_0.
$$

and

(24)
$$
||\Delta h||_s \leq \delta M_2 C_{h_0}.
$$

hence we have the statement. $\square$

6.5. **Linear Response.** Under some additional assumptions, there is a kind of differentiable dependence of the invariant measure under small perturbations.

Consider the action of a family of operators on the spaces $B_w, B_s, B_{ss} \subseteq SM(X)$ of Borel measures on X equipped with norms $|| \ ||_w, || \ ||_s, || \ ||_{ss}$ respectively, such that $|| \ ||_w \leq || \ ||_s \leq || \ ||_{ss}$. We suppose that L_δ has a unique fixed point $h_\delta \in B_{ss}$. The following proposition is essentially proved in [15] (see [5] for the proof of the proposition in this form).

Proposition 27. *Suppose that the following assumptions hold:*

(1) *The norms $||L_0^k||_{B_w \to B_w}$ and $||L_\delta^k||_{B_w \to B_w}$ are uniformly bounded with respect to k and $\delta > 0$.*

(2)

(25)
$$
||L_\delta - L_0||_{B_s \to B_w} \leq C\delta.
$$

(3) *The operators L_δ, $\delta \geq 0$, have uniform rate of contraction on V_s: there are $C_1 > 0$, $0 < \rho < 1$, such that*

(26)
$$
||L_\delta^n||_{V_s \to B_s} \leq C_1 \rho^n.
$$

(4) *$\exists$ an operator $\hat{L} : B_{ss} \to B_s$ such that*

(27)
$$
\lim_{\delta \to 0} ||\delta^{-1}(L_0 - L_\delta)f - \hat{L}f||_s = 0 \ \forall f \in B_{ss}.
$$

Let

$$
\hat{h} = (Id - L_0)^{-1}\hat{L}h.
$$

Then

$$
\lim_{\delta \to 0} ||\delta^{-1}(h_0 - h_\delta) - \hat{h}||_w = 0;
$$

i.e. $\hat{h}$ represents the derivative of h_δ for small increments of δ.

6.6. **Rigorous numerical methods for the computation of invariant measures.** In this section we briefly mention a possible application of the general methods here exposed. It is possible to use stability results to design efficient numerical methods for the approximation of the invariant measure and other important statistical features of a system.

The approximation can also be rigorous, in the sense that an explicit bound on the approximation error can be provided (for example it is possible to approximate the absolutely continuous invariant measure of a system up to a small given error in the L^1 distance). Thus the result of the computation has a mathematical meaning .

This can be done by approximating the transfer operator L_0 of the system by a suitable finite rank one L_δ which is essentially a matrix, of which we can compute fixed points and other properties.

There are many ways to construct a suitable L_δ depending on the system which is considered. The most used one (for L^1 approximation) is the so called Ulam discretization. In this approximation, the system is approximated by a Markov chain.

In the *Ulam Discretization* method the phase space X is discretized by a partition $I_\delta = \{I_i\}$ and the system is approximated by a (finite state) Markov Chain with transition probabilities

$$(28) \qquad P_{ij} = m(T^{-1}(I_j) \cap I_i)/m(I_i)$$

(where m is the normalized Lebesgue measure on the phase space). The approximated operator L_δ can be seen in the following way: let F_δ be the $\sigma-$algebra associated to the partition I_δ, then:

$$(29) \qquad L_\delta(f) = \mathbf{E}(L(\mathbf{E}(f|F_\delta))|F_\delta)$$

($\mathbf{E}$ is the conditional expectation).

In a series of works it was proved that in several cases the fixed point f_δ of L_δ converges to the fixed point of L. Explicit bounds on the error have been given, rigorous methods implemented and experimented in several classes of cases. (see e.g. [4],[6],[7],[9], and [13] where several nontrivial experiments are also shown).

7. Appendix: Convergence speed and correlation integral

Convergence to equilibrium is often estimated or applied in the form of correlation integrals. In this section we show how to relate these with the notion we used in Section 4.1.

We consider here the spaces $W^{1,1}$ and L^1 as strong and weak spaces. Similar arguments applies to many other spaces.

7.1. **Estimating $\int \psi \circ T^n\ g\ dm - \int g\ dm\ \int \psi\ d\mu$ (another kind of convergence to equilibrium estimation).** From the convergence to equilibrium we get that for each g such that $\int g\ dm = 0$

$$||L^n g||_1 \le \Phi(n)||g||_{W^{1,1}}$$

$\forall \psi \in L^\infty$

$$\int \psi \circ T^n\ g\ dm \le \Phi(n)||\psi||_\infty ||g||_{W^{1,1}}.$$

Now let us consider g such that $\int g \ dm \neq 0$

$$\int \psi \circ T^n \ g \ dm - \int g \ dm \ \int \psi \ d\mu \ \leq$$

$$\leq \ (\int g \ dm)[\int \psi \circ T^n \ \frac{g}{\int g \ dm} \ dm - \int \psi \circ T^n \ h \ dm]$$

$$\leq \ (\int g \ dm) \int \psi \circ T^n \ [\frac{g}{\int g \ dm} - h] \ dm$$

and since $\int \frac{g}{\int g \ dm} - h \ dm = 0$

$$(30) \quad \int \psi \circ T^n \ g \ dm - \int g \ dm \ \int \psi \ d\mu \ \leq \ \Phi(n)||\psi||_\infty ||g - h \int g \ dm||_{W^{1,1}}$$

$$(31) \qquad\qquad\qquad\qquad\qquad\qquad \leq \ C_h \Phi(n)||\psi||_\infty ||g||_{W^{1,1}}$$

Hence modulo a constant which depends on h and not on g, ψ and n we have that the decay of the integrals is the same as the convergence to equilibrium defined in 4.1.

7.2. **Estimating $\int \psi \circ T^n \ g \ d\mu - \int g \ d\mu \ \int \psi \ d\mu$ (a decay of correlation estimation).** For many applications it is useful to estimate the speed of decreasing of the following integral

$$(32) \qquad\qquad |\int g \cdot (\psi \circ F^n) \ d\mu - \int \psi \ d\mu \int g \ d\mu|.$$

where μ is invariant. Again, supposing $\int g \ d\mu = 0$, $\psi \in L^\infty$

$$|\int g \cdot (\psi \circ F^n) \ d\mu| \ \leq \ |\int g \cdot (\psi \circ F^n) \ h \ dm| \leq \Phi(n)||\psi||_\infty ||gh||_{W^{1,1}}$$

$$\leq \ C_h \Phi(n)||\psi||_\infty ||g||_{W^{1,1}}.$$

Now let us consider g such that $\int g \ d\mu \neq 0$

$$\int \psi \circ T^n \ g \ d\mu - \int g \ d\mu \ \int \psi \ d\mu \ \leq \ (\int g \ d\mu)[\int \psi \circ T^n \ \frac{g}{\int g \ d\mu} \ h \ dm - \int \psi \circ T^n \ h \ dm]$$

$$\leq \ (\int g \ d\mu) \int \psi \circ T^n \ [\frac{g}{\int g \ d\mu} - 1] \ h \ dm$$

and since $\int \frac{g}{\int g \ d\mu} - 1 \ d\mu = 0$

$$(33) \qquad \int \psi \circ T^n \ g \ d\mu - \int g \ d\mu \ \int \psi \ d\mu \leq C_h \Phi(n)||\psi||_\infty ||g||_{W^{1,1}}.$$

8. Appendix: absolutely continuous invariant measures, convergence and mixing

In this Section we present some general tool which are useful when working with absolutely continuous invariant measures.

Let us suppose that X is a manifold, let us denote by m the normalized Lebesgue measure, consider non singular transformations, and iterate absolutely continuous measures.

Proposition 28. *If for **some** $f \geq 0$ with $\int f \, dm = 1$,*

$$L^n f \to h$$

weakly in L^1. Then T has an absolutely continuous invariant probability measure with density h.

Proof. Let us suppose without loss of generality $\int h \, dm = 1$. $L^n f \xrightarrow{w} h \int f \, dm$ if and only if $\forall \phi \in L^\infty$

$$
\begin{aligned}
\int \phi \, h \, dm &= \lim_{n \to \infty} \int \phi \, L^{n+1} f \, dm = \lim_{n \to \infty} \int (\phi \circ T) \, L^n f \, dm \\
&= \int (\phi \circ T) \, h \, dm = \int \phi \, Lh \, dm.
\end{aligned}
$$

$\square$

Proposition 29. *If the system has an absolutely continuous invariant measure with density $h \in L^1$ and for **all** $f \in L^1$ such that $\int f \, dm = 0$*

$$L^n f \xrightarrow{weaky} 0$$

then T is mixing.

Proof. By the assumptions, given any f the sequence $L^n f - h \int f \, dm \xrightarrow{w} 0$. And then $L^n f \xrightarrow{w} h \int f \, dm$, $\forall f \in L^1$.

Now let us consider two measurable sets E and F. Let us denote by hm, the measure with density h with respect to the Lebesgue measure m.

$$
[hm](E \cap T^{-n}F) = \int 1_E \, (1_F \circ T^n) \, hdm = \int 1_F \, L^n(h1_E) \, dm
$$

$$
\xrightarrow[n \to \infty]{} \int 1_F \, [h \int (h1_E) \, dm] \, dm = \int (h1_F) \, dm \, \cdot \, \int (h1_E) \, dm = [hm](E) \, [hm](F).
$$

$\square$

9. Bibliographic remarks and acknowledgements

The books [2], [10], [18] are detailed introductions to the subject considered in these notes from different points of view. Our point of view is closer to the one given in the notes [15] and [17], from which some definition and statement is taken. We recommend to consult all these texts for many related topics, generalizations, applications and complements we cannot include in these short introductory notes.

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E-mail address: galatolo@dm.unipi.it

Dipartimento di Matematica, Universita di Pisa, Via Buonarroti 1,Pisa - Italy