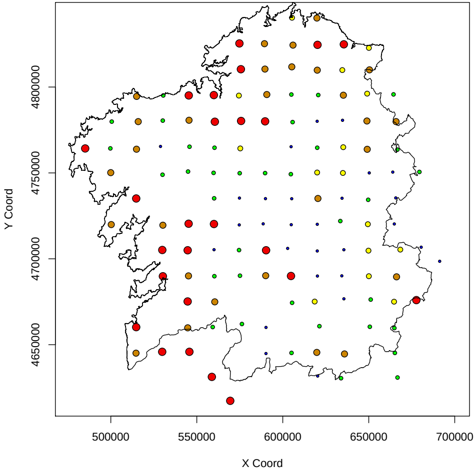
# Lecture 2

# The linear geostatistical model

1. Formulation of linear geostatistical models and assumptions.
2. Brief introduction to Gaussian processes.
3. Understanding the nugget effect.
4. Parameter estimation via the maximum likelihood method.

# Where we observe matters



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- Example: $\rho(u) = \exp\{-u/\phi\}$

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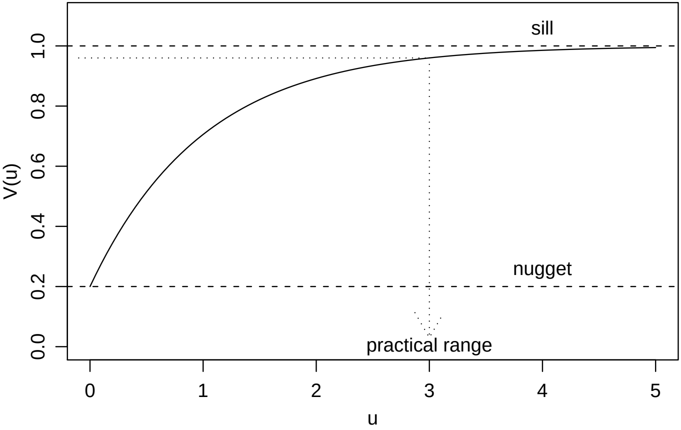

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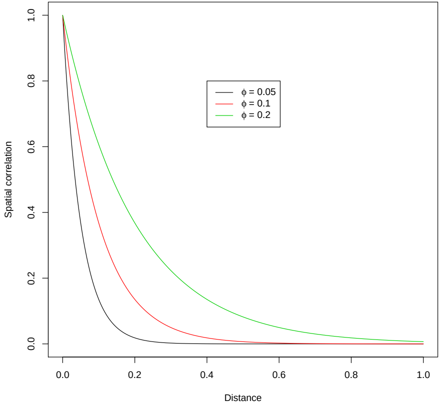

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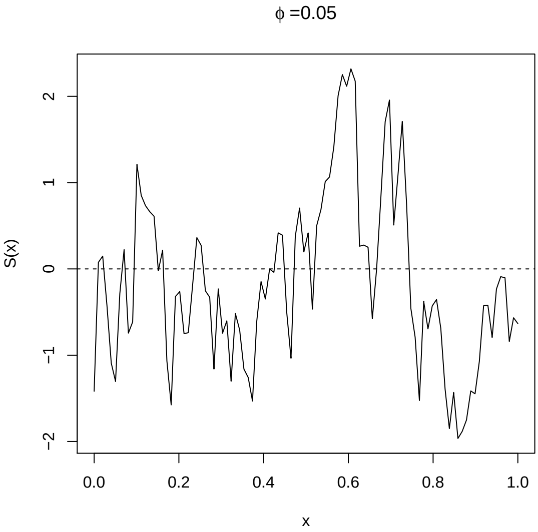

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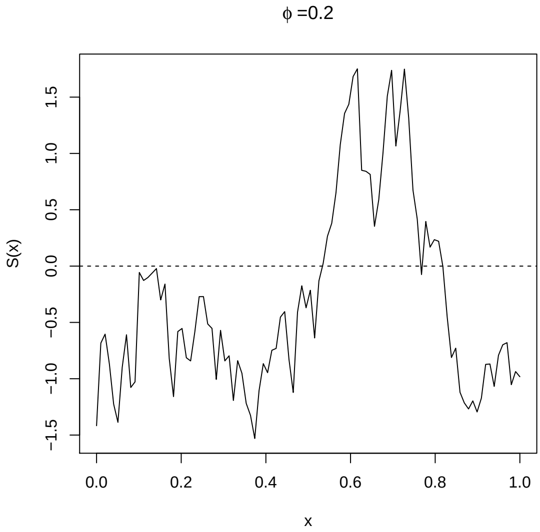

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  - $\kappa = 0.5$ gives $\rho(u) = \exp\{u/\phi\}$
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- Often sufficient to choose amongst $\kappa = 0.5, 1.5, 2.5$

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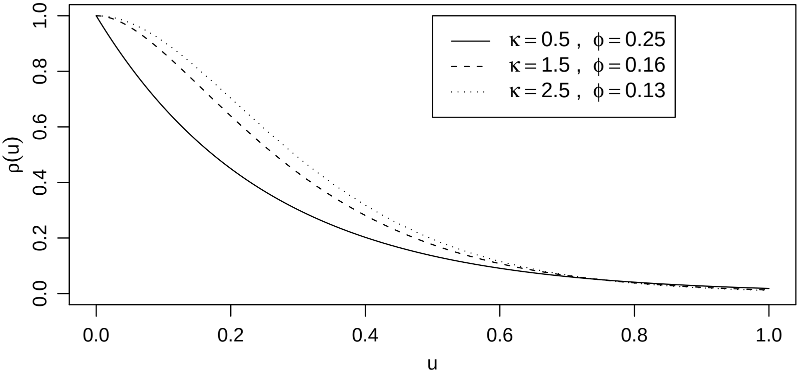

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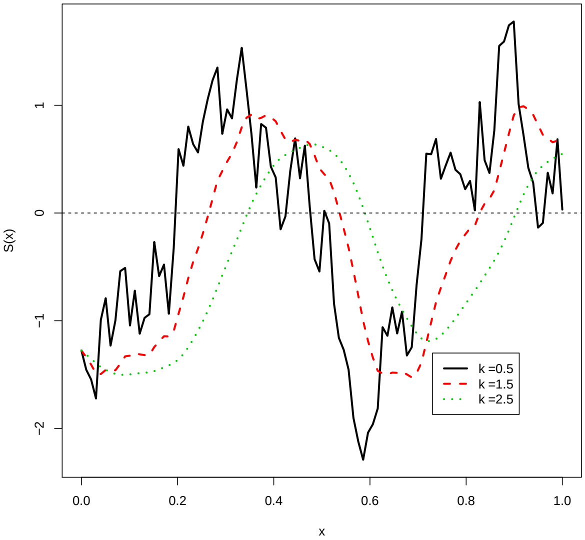

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If $u^* < u_{min}$ then $Z(x_i)$ is pure noise.

# Getting initial parameter estimates

- Widely used, but not recommended except for initial analysis.
- $\theta = (\sigma^2, \phi, \tau^2)$
- Weighted least squares criterion:

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- Standard errors not available.

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- Multivariate Gaussian distribution: $Y \sim MVN(D\beta, \sigma^2 R + \tau^2 I)$.
  - $D$ matrix of covariates: $[D]_{ik} = d_k(x_i)$
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- Fitting process
  1. Initialise $\beta$, e.g. using ordinary least squares.
  2. Initialise $\theta$, e.g. using the empirical variogram
  3. Maximize

     $$l(\theta) = \log\{f(y; \beta, \theta)\}$$

     where $f(\cdot; \beta; \theta)$ denotes the density of the multivariate Gaussian distribution.