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MODERN MISSION DESIGN: THE FUNDAMENTAL THEORETICAL FRAMEWORK

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Abstract: The main aim of this contribution is to present a brief review of the theoretical framework applied in modern space mission design, based on the planar restricted three-body problem, illustrating its applicability with the two-patched three-body problem approximation for a Earth-Moon transfer mission design. Predicted by Lyapunov-Moser Theorem, the dynamical solutions of the linear analogue, valid in a neighborhood of a collinear Lagrangian equilibrium, can be grouped in four categories of orbits, namely, transit, nontransit, asymptotic and periodic solutions. The former two categories are separated by the two-dimensional invariant manifolds, which contain the asymptotic solutions, and are associated to the periodic solution of a given energy surface. Space mission projects in the solar system, involving three or more bodies, can strategically take advantage of these dynamical structures leading to alternative low energy solutions.

Keywords: space mission design, three-body problem, low energy transfer orbits.

1. INTRODUCTION

The traditional techniques applied in transfer orbit design are based on the two-body problem and its solutions. The most important of them are the Hohmann [1] and the tri-impulsive bi-elliptic transfers [2,3]. Although these orbital maneuvers are associated with small typical transfer times, they require a large amount of fuel. In modern missions in the solar system, which may involve three or more bodies, and the flyby through distinct planets or satellites, these traditional transfers may become very expensive or even prohibitive. So, new techniques, relied on properties of systems of three or more bodies, are imperative. In this context, the dynamical invariant sets and the dynamical properties of the circular planar restricted three body problem, which is the main mathematical model applied in this alternative approach, play a crucial role in the low energy transfer orbit design. The purpose of this paper is to present a self-contained review of the fundamental theoretical framework which provides the essential dynamical ingredients for modern mission analysis. This task is accomplished with the examination of the Lyapunov-Moser theorem [4], and of the seminal contributions of Conley [5,6]. Conley verified the applicability of this theorem in the planar circular restricted three-body problem, and investigated the linear analysis dynamical consequences due this validity. After Conley, McGehee [7] - proving the existence of homoclinic orbits in both the interior and exterior regions - the Barcelona group formed by Llibre, Martinez and Simó [8] - with the investigation of the transversality of the invariant manifolds associated to Lyapunov family of periodic orbits near the Lagrangian point L_1 - and the Caltech team, with Koon, Lo, Marsden, and Ross [9] - proving the existence of heteroclinic connections - among others, gave outstanding contributions in the area.

Besides, two or more patched 3B problem are usually considered in a complete mission design, such as in an Earth-Moon transfer, or in the case known as the *Petit Grand Tour of the Jovian Moons.* Some dynamical conditions, satisfied in the Earth-Moon transfer, are required to the real system in order to guarantee the suitability of this patched theoretical approach.

This contribution is organized as follows: in the next section the mathematical model and its main dynamical features are introduced. In section 3, a summary of the relevant theoretical results is presented and in the section 4, some illustrative applications are given through an actual mission design scheme. Conclusions are found in the last section.

2. THE MATHEMATICAL MODEL

The planar circular restricted three body problem (PR3BP) [10] describes the motion of a third body P_3 moving in the gravitational field of two main bodies, P_1 and P_2, called the primaries. Once that, this third body mass is much smaller than the masses of the primaries, it is assumed that the motion of the primaries is not perturbed by P_3, being a solution of a two-body (2B) problem, constituted by P_1 and P_2. In the circular version of the model, the orbits of P_1 and P_2 are coplanar circles centered in the barycenter of this 2B system.

Another possibility is to consider elliptic solutions for the primaries, as it is the case of the elliptical model. In the planar model, the third body trajectories are restricted to the same plane of P_1 and P_2, while in the spatial version, P_3 moves in the three-dimensional space. In the solar system, the primaries are usually the Sun and Earth or the Earth and Moon or Jupiter and a Jupiter's Moon, while the third body usually represents a spacecraft or a comet. The actual eccentricity of Moon's orbit is 0.055 and the Earth's one is 0.017, which are almost circular. The model is expressed in non-dimensional variables, in such a way that the sum of the masses of the primaries, the distance of P_1 and P_2, and the angular velocity of the P_1 and P_2 motion are normalized to one. The masses of P_1 and P_2 are given, respectively, by *1-μ* and μ, where the only parameter of the model is $\mu = m_2/(m_1 + m_2)$, with $m_1 > m_2$.

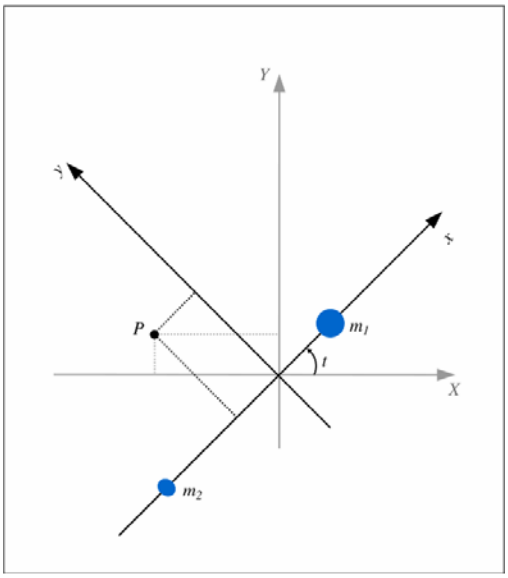

Fig. 1. Inertial and rotating coordinate frame. The *x-y* rotating system moves counterclockwise with unitary angular velocity in relation to the *X-Y* inertial one.

Choosing a rotating coordinate system, with the origin in the center of mass, and the static primaries localized in the *x*-axis in $(\mu, 0)$ and $(-1+\mu, 0)$, as depicted by Fig. 1, the position of the spacecraft in the plane can be represented by *(x,y)*. The spacecraft's equations of motion are given by

$$\ddot{x} - 2\dot{y} = \Omega_x, \quad \ddot{y} - 2\dot{x} = \Omega_y, \tag{1}$$

where

$$\Omega(x, y) = \frac{x^2 + y^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{\mu(1-\mu)}{2}, \tag{2}$$

$$r_1^2 = (x-\mu)^2 + y^2, \quad r_2^2 = (x+1-\mu)^2 + y^2. \tag{3}$$

The integral of motion J defines a three-dimensional invariant manifold by

$$\begin{gathered} \mathcal{M}(\mu, C) = \left\{ (x, y, \dot{x}, \dot{y}) \in \mathbf{R}^4 \mid J(x, y, \dot{x}, \dot{y}) = \text{constant} \right\}, \\ J(x, y, \dot{x}, \dot{y}) = 2\Omega(x, y) - (\dot{x}^2 + \dot{y}^2) = C. \end{gathered} \tag{4}$$

The conservation associated to J restricts the motion in the four-dimensional phase space to a three-dimensional invariant manifold $\mathcal{M}$. The Jacobi constant C is related with the third body's energy E by $C = -2E$, given that, as usual,

$$E(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \Omega(x, y). \tag{5}$$

2.1. Lagrangian Points

This dynamical model has five equilibria defined by

$$\begin{aligned} &\frac{\partial J}{\partial x} = \Omega_x = 0, && \frac{\partial J}{\partial y} = \Omega_y = 0, \\ &\frac{\partial J}{\partial \dot{x}} = 0 \Rightarrow \dot{x} = 0, && \frac{\partial J}{\partial \dot{y}} = 0, \Rightarrow \dot{y} = 0. \end{aligned} \tag{6}$$

These equilibria are called Lagrangian points, being:

- three collinear points, namely, L_1, L_2, and L_3, localized in the *x*-axis *(saddle-center points);*
- two triangular points, called L_4 and L_5, localized in the vertices of equilateral triangles *(stable points, if* $m_1/m_2 > 24.96$*).*

Figure 2 presents the Earth-Moon system potential energy *Ω(x,y)* with the corresponding five equilibria L_k, k=1,…,5.

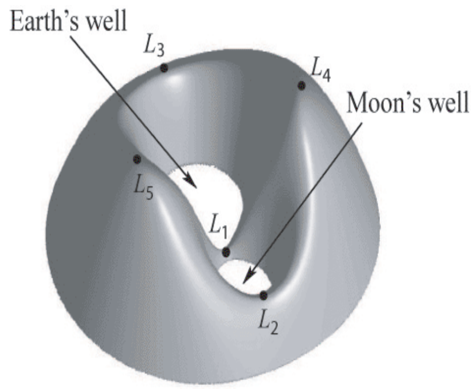

Fig. 2. The potential energy surface with the five Lagrangian points for the Earth-Moon system.

The Jacobi constant values evaluated in these static solutions are denoted by C_k, for k=1,2,3,4,5. For the Earth-Moon system, these values are $C_1 \approx 3.20034$, $C_2 \approx 3.18416$, $C_3 \approx 3.02415$, and $C_4 = C_5 = 3$. In Fig. 3, the energy constant values of the five equilibria are given as a function of the mass parameter μ.

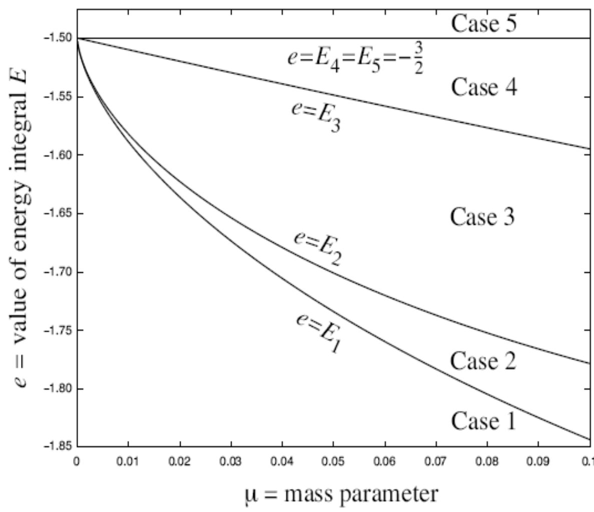

Fig. 3. The energy constant values for the five equilibria as a function of the μ. These constants define the five cases of the possible Hill´s regions.

The regions obtained by the projection of the $\mathcal{M}$ surface onto the position space *x-y* are called *the Hill´s Regions*. Since

$$\dot{x}^2 + \dot{y}^2 = 2\Omega(x,y) - C \geq 0, \qquad (7)$$

the Hill regions M are defined by

$$M(\mu, C) = \{(x,y) \mid \Omega(x,y) \geq C/2\}, \qquad (8)$$

and constitute the accessible regions to the trajectories for a given C value. The boundaries of these regions are the called *zero-velocity curves*, since they are the locus in the *x-y* space where kinetic energy vanishes. It is easy to verify that there are five possible cases for the Hill´s regions, defined by the C_k values, as is shown in the Fig. 4.

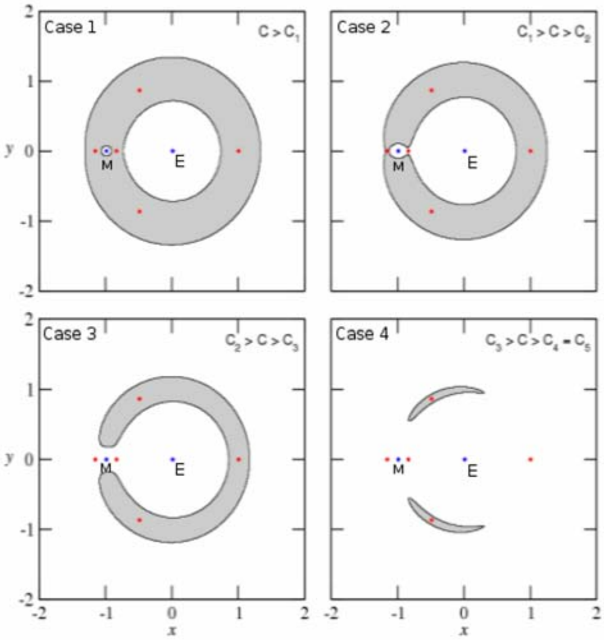

Fig. 4. The five cases of the possible Hill´s regions. The white regions represent the Hill´s regions, while the dark regions correspond to the inaccessible ones. In Case 1, $C>C_1$, no transfer orbit between the primaries is possible. In Case 2, $C_1>C>C_2$ there are only internal transfers, while in Case 3 ($C_2>C>C_3$) and Case 4 ($C_3>C>C_4=C_5$), internal and external connections are possible. The fifth case is not shown, and it corresponds to $C<C_4=C_5$. In Case 5, all the plane is accessible.

Figures 5, 6, 7 and 8 show the projections in the *x-y* plane of a solution, respectively, for the Cases 1, 2, 3, and 4. It is interesting to note the nonlinear behavior of these solutions, contrasting with the conic solutions of the two-body system.

Around the collinear Lagrangian points, there are periodic orbits, called Lyapunov orbits in the planar case and Halo orbits in the spatial case. The stability of these periodic solutions is given by the monodromy matrix.

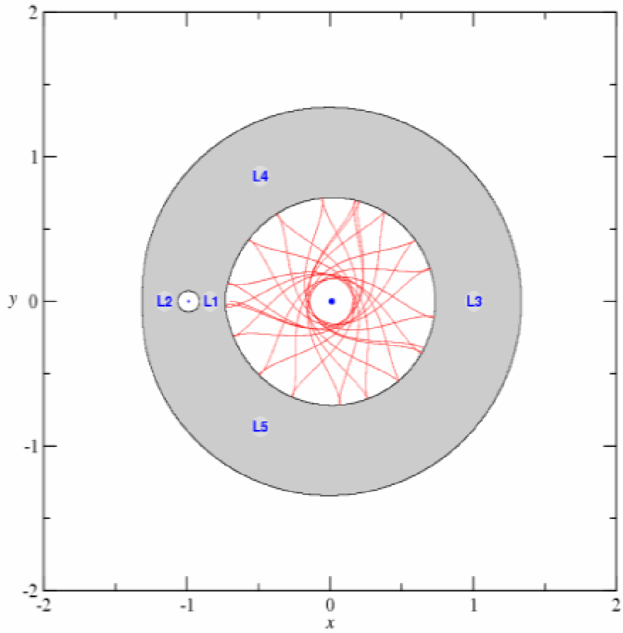

Fig. 5. The projection of a trajectory in the *x-y* plane (red line) (Case 1) for the Earth-Moon system, corresponding to a particle orbiting around the main primary. The prohibited region (dark grey) is shown.

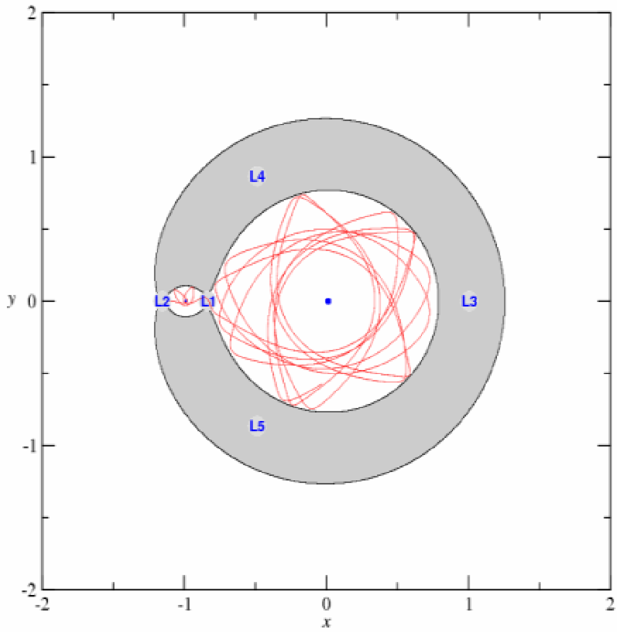

Fig. 6. The projection of a trajectory in the *x-y* plane (red line) (Case 2) for the Earth-Moon system, corresponding to a particle orbiting around the two primaries. The prohibited region (dark grey) is shown.

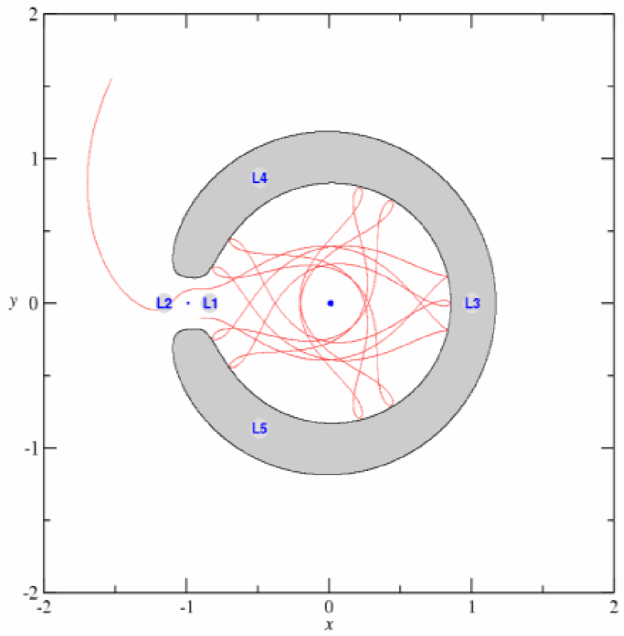

Fig. 7. The projection of a trajectory in the *x*-*y* plane (red line) (Case 3) for the Earth-Moon system, corresponding to a particle orbiting around the main primary and escaping to the exterior region. The prohibited region (dark grey) is shown.

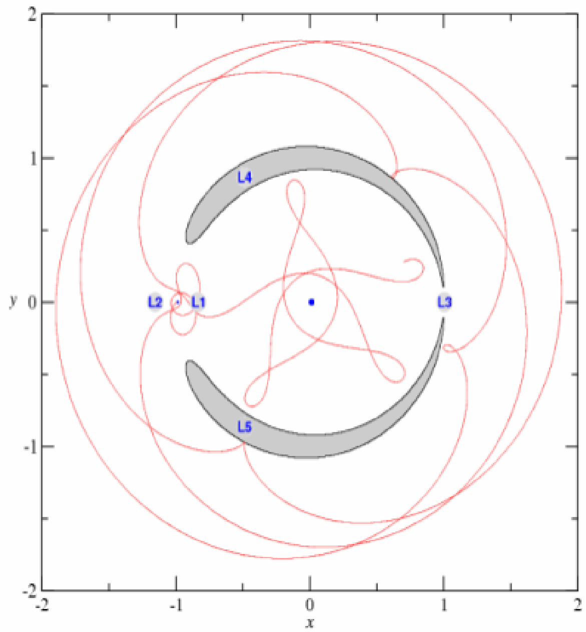

Fig. 8. The projection of a trajectory in the *x*-*y* plane (red line) (Case 4) for the Earth-Moon system. The reduced prohibited region (dark grey) is shown.

3. SUMMARY OF THE MOST RELEVANT THEORETICAL FRAMEWORK

In this section a summary of the most fundamental theoretical results concerning the orbit transfer design, based on the PR3BP approach, is presented. We start stating a generalization of a Lyapunov's theorem, due to Moser, and then, following Conley, the relevant dynamical structures are extracted from the linear analysis of the Hamiltonian description.

Lyapunov-Moser Theorem [4]: *Let $H(x,y)$ be a real analytic function in a neighborhood of the origin, and let*

$$\begin{aligned} x_k &= H_{y_k}(x,y), \\ y_k &= -H_{x_k}(x,y), \qquad k = 1,\dots,n, \end{aligned} \tag{9}$$

be the Hamiltonian system with n degrees of freedom and with an equilibrium $x_k=y_k=0$. Let $\alpha_1,\dots,\alpha_n,-\alpha_1,\dots,-\alpha_n$ be the eigenvalues of the associated Jacobian matrix. Assume that these eigenvalues are 2n different complex numbers; α_1, α_2 independent over the reals and $\alpha_k \neq n_1\alpha_1+n_2\alpha_2$ for all integers n_1,n_2 and $k\geq 3$. Then there exists a four-parameter family of solutions of (9), given by

$$\begin{aligned} x_k &= \varphi_k(\xi,\eta,\gamma,\bar{\gamma}), \\ y_k &=, \psi_k(\xi,\eta,\gamma,\bar{\gamma}), \end{aligned} \tag{10}$$

where

$$\begin{aligned} \xi &= \xi^\circ e^{ta_1(\nu_1^\circ,\nu_2^\circ)}, \\ \eta &= \eta^\circ e^{ta_2(\nu_1^\circ,\nu_2^\circ)}, \\ \gamma &= \gamma^\circ e^{-ta_1(\nu_1^\circ,\nu_2^\circ)}, \\ \varsigma &= \varsigma^\circ e^{-ta_2(\nu_1^\circ,\nu_2^\circ)}, \end{aligned} \tag{11}$$

and

$$\begin{aligned} a_1(\nu_1^\circ,\nu_2^\circ) &= \alpha_1 + \cdots, \\ a_2(\nu_1^\circ,\nu_2^\circ) &= \alpha_2 + \cdots \end{aligned} \tag{12}$$

are convergent power series in ν_1°,ν_2°, where $\nu_1^\circ = \xi^\circ\gamma^\circ$ and $\nu_2^\circ = \eta^\circ\varsigma^\circ$. The series φ_k, ψ_k converge in a neighborhood of the origin and the rank of the matrices

$$\begin{bmatrix} \varphi_{k_\xi} & \varphi_{k_\gamma} \\ \psi_{k_\xi} & \psi_{k_\gamma} \end{bmatrix} \text{ and } \begin{bmatrix} \phi_{k_\eta} & \phi_{k_\varsigma} \\ \psi_{k_\eta} & \psi_{k_\varsigma} \end{bmatrix} \tag{13}$$

is four. The solutions (11) depend on four complex parameters, $\xi^\circ,\eta^\circ,\gamma^\circ,\varsigma^\circ$. If in addition α_1, α_2, $-\alpha_1$, $-\alpha_2$ contain their complex conjugates, the solutions can be chosen to be real, depending on four real parameters.

It is worth of note that not just a family of solutions but all solutions of the system are described by this result when $n=2$.

Conley [5,6] verified that the Hamiltonian description of the PR3BP satisfies the conditions required by this theorem, in any case of the three collinear Lagrangian points, as can be seen as follows.

The system of equations of Section 1 can be rewritten in Hamiltonian form where

$$\begin{aligned} H = &\frac{(p_x+y)^2+(p_y+x)^2}{2} - \frac{x^2+y^2}{2} \\ &-\frac{1-\mu}{r_1} - \frac{\mu}{r_2} - \frac{\mu(1-\mu)}{2} \end{aligned} \tag{14}$$

is the Hamiltonian function. Then, the equations of motion can be expressed as

$$
\begin{aligned}
\dot{x} &= \frac{\partial H}{\partial p_x} = p_x + y, \\
\dot{y} &= \frac{\partial H}{\partial p_y} = p_y - x, \\
\dot{p}_x &= -\frac{\partial H}{\partial x} = p_y - x + \Omega_x, \\
\dot{p}_x &= -\frac{\partial H}{\partial x} = -p_x - y + \Omega_y.
\end{aligned}
\tag{15}
$$

From these equations we proceed to the linearization of the dynamics near the equilibrium solution. Given one collinear Lagrangian point with coordinates (q,0) the Hamiltonian (14) may be expanded up to second order around such equilibrium. After making a coordinate change to translate (q,0) to the origin, the Hamiltonian for the linearized equations, denoted by H_l, formed by quadratic terms, is given by

$$H = \frac{1}{2}\left[\left(p_x + y\right)^2 + \left(p_y - x\right)^2 - ax^2 + bx^2\right], \tag{16}$$

where a and b are defined by $a = 2\rho + 1$ and $b = \rho - 1$, and

$$\rho = \mu \; \left|q + 1 - \mu\right|^{-3} + \left(1 - \mu\right) \; \left|q - \mu\right|^{-3}. \tag{17}$$

Yet another transformation is introduced in order to obtain variables with a simpler geometric meaning. Let $\upsilon_x = p_x + y$ and $\upsilon_y = p_y - x$, where υ_x and υ_y correspond to the velocity of the particle in the rotating coordinate system. The new equations of motion now appear

$$
\begin{aligned}
\dot{x} &= \upsilon_x, \qquad \dot{\upsilon}_x = 2\upsilon_y + ax, \\
\dot{y} &= \upsilon_y, \qquad \dot{\upsilon}_y = -2\upsilon_x - by,
\end{aligned}
\tag{18}
$$

which is the linearization of the system of equations (1) around the equilibrium point. The energy integral H_l of (16) is now

$$E_l = \tfrac{1}{2}\left(\upsilon_x^2 + \upsilon_x^2 - ax^2 + by^2\right). \tag{19}$$

The zero-surface of the energy integral E_l corresponds to the Jacobi integral surface which passes through the libration point.

The eingenvalues of the linear system (18) have the form $\pm\lambda$ and $\pm\, i\tau$, where λ and τ are positive real constants. So, these three fixed points are classified as saddle-center equilibria. The corresponding eingenvectors are

$$
\begin{aligned}
u_1 &= (1, -\sigma_1, \lambda, -\sigma_1\lambda), \\
u_2 &= (1, \sigma_1, -\lambda, -\sigma_1\lambda), \\
w_1 &= (1, -i\sigma_2, i\tau, \sigma_2\tau), \\
w_2 &= (1, i\sigma_2, -i\tau, \sigma_2\tau),
\end{aligned}
\tag{20}
$$

with σ_1 and σ_2 being real constants. Now a linear change of coordinates is performed with the eingenvectors, u_1, u_2, w_1, w_2, as the axes of the new system. The equations of motion assume the simpler form

$$
\begin{aligned}
\dot{\xi} &= \lambda\xi, \qquad \dot{\gamma}_1 = \tau\gamma_2, \\
\dot{\eta} &= -\lambda\eta, \qquad \dot{\gamma}_2 = -\tau\gamma_1,
\end{aligned}
\tag{21}
$$

where $\gamma = \gamma_1 + i\gamma_2$, and the energy function (19) reads

$$E_l = \lambda\xi\eta + \frac{\tau}{2}\left(\gamma_1^2 + \gamma_2^2\right). \tag{22}$$

Solutions of the equations (21) can be written as

$$
\begin{aligned}
&\xi(t) = \xi^\circ\, e^{\lambda t}, \qquad \eta(t) = \eta^\circ\, e^{-\lambda t}, \\
&\gamma(t) = \gamma_1(t) + i\gamma_2(t) = \gamma^\circ\, e^{-i\tau t},
\end{aligned}
\tag{23}
$$

where the constants, $\xi^\circ, \eta^\circ, \gamma^\circ = \gamma_1^\circ + i\gamma_2^\circ$ are the initial conditions. Besides the energy function (22), two other integrals of motion, $\xi\eta$ and $|\gamma|^2$, are associated to the solutions of the linear dynamical system (eq.(21)). It is interesting to note that the Hamiltonian function (22) depends only on these variables and that the sign of *ξ(t)* and *η(t)* are also conserved by the linear dynamics. These general conservations in the linear system are related with local conservations in the original nonlinear system.

3.1. The transit orbit existence

There exists a region R defined by a fixed energy surface $E_l = \varepsilon$ and $\left|\eta - \xi\right| \le c$, which is homeomorphic to the product of a two-sphere and an interval. Given a fixed positive value of ε, with an additional coordinate change of the form

$$\xi = r + s, \eta = r - s, \text{ leading to } \xi - \eta = 2s, \xi\eta = r^2 - s^2 \tag{24}$$

one can see that a bi-ellipsoid is defined by each value of s in [-c,c], as prescribed by

$$\varepsilon + \lambda s^2 = \lambda r^2 + \frac{\tau}{2}\left(\gamma_1^2 + \gamma_2^2\right). \tag{25}$$

In this way, the two bi-spheres corresponding to $\xi - \eta = -c$ and $\xi - \eta = +c$ are the boundary of R. Conley proved that, for small enough (positive) values of c and ε, the region R is in the interior of a closed ball B, the region in which all the power series of Lyapunov-Moser theorem converge and in which these series as well as their first partial derivatives are dominated by their lowest order terms. Conley called the connecting region which appears for the Jacobi constant C near and bellow the respective C_k, *for* k=1,2,3 of a neck, and based on the Moser results, proved the existence of transit orbits.

3.2. The phase space trajectories in the neck region

Given the linearly independent solutions (23), the general solution of the linear systems can be expressed as

$$\boldsymbol{u}(t) = \alpha_u u_1 \,\mathbf{exp}(\lambda t) + \alpha_s u_2 \,\mathbf{exp}(-\lambda t) + 2\,\mathbf{Re}(\beta w_1 \,\mathbf{exp}(i\tau t)), \tag{26}$$

where the arbitrary constants α_u and α_s are real and β is complex.

Due to the sign conservation of *ξ(t)* and *η(t)*, the three possibilities for each of these signs $(-, 0, +)$, defined respectively by α_u and α_s, result in nine different classes of solutions on the surface of energy. The analysis of the asymptotic behavior of x and y can lead to the solution classification as a function of the real constants α_u and α_s. As $t \to \infty$, it can occur that $x \to -\infty$, or that x be bounded, or that $x \to +\infty$, according to α_u<0, α_u=0, α_u>0. If α_s

replaces α_u, the same is valid if $t \to -\infty$. If x keeps bounded for both $t \to \pm\infty$, the trajectory corresponds to the periodic solution defined by $\alpha_u=\alpha_s=0$. Given that, Conley grouped the nine classes of orbits into four categories, or types of motion, namely, (a) asymptotic orbits for $\alpha_u=0$ or $\alpha_s=0$, but not both null, (this category contain four classes) (b) transit orbits for $\alpha_u\alpha_s<0$ (with two classes), (c) nontransit orbits for $\alpha_u\alpha_s>0$ (with two classes), and (d) periodic orbits for $\alpha_u=\alpha_s=0$ (with one class). These four categories in a neck region near the L_2 equilibrium point are shown in Fig. 9. Let $\Delta=(n_1,n_2)$ be the ordered pair of signs of α_u and α_s, respectively. The four classes of the asymptotic orbits are related with two pairs of hyperbolic invariant tubes which depart from each periodic orbit, one of them associated with the asymptotic stable set, W_s, for $\Delta=(+,0)$ or $(-,0)$, and the other pair with the unstable one, W_u, for $\Delta=(0,+)$ or $(0,-)$. These tubes are topologically equivalent to two-dimensional cylinders, while the orbits which rest on them are spirals. With respect to the centered equilibrium, the two classes of transit orbits match the left to right sense and the right to left sense, for respectively, $\Delta=(-,+)$ and $(+,-)$, while the left restricted orbits $(\Delta=(-,-))$ and the right restricted orbits $(\Delta=(+,+))$ are the classes of the nontransit category. A single class is associated to the periodic orbit, for $\Delta=(0,0)$. Actually, the set of periodic solutions associated to each collinear Lagrangian point constitutes the family of Lyapunov periodic orbits, with two parameters C and μ.

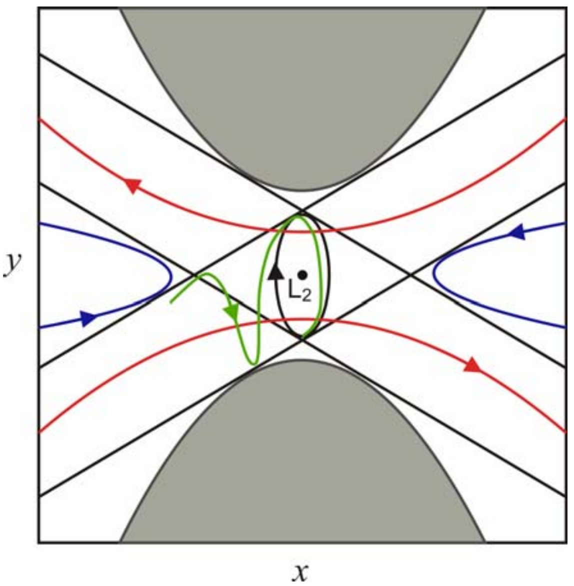

Fig. 9. The four categories of trajectories in the neck near the collinear L_2 Lagrangian point are shown, namely, the periodic orbit (black closed curve around L_2), the transit orbits (red lines), the asymptotic orbits (green line), and the nontransit orbits (blue lines).

Furthermore, the unstable and stable invariant manifolds associated to the Lyapunov periodic orbits separate the two kinds of motion in the three-dimensional energy surface, immersed in the four dimensional phase space: the transit and the nontransit orbits, in such a way that, the transit solutions are internal to the two-dimensional tubes, while the nontransit solutions are external to them. So, the stable and unstable manifold tubes control the spacecraft transport to and from the capture region. Figure 10 presents the projection of the stable and unstable invariant manifolds of a Lyapunov orbit in the plane *x-y*.

Conley [2] originally conjectured that periodic low energy Earth-Moon orbit design should follow the criteria: (i) cost per cycle must be as small as possible; (ii) control and stability should as easy as possible; and (iii) as much flexibility should be built into the scheme as possible. The author argued that the first requirement could be accomplished if orbits whose Jacobi constant is just above that of the critical point between the Earth and the Moon were considered. Among the possible objections to this choice are the long flight time and the strong sensitivity to initial conditions, in the sense that, small change in initial condition set could imply in orbits belonging to different classes, i.e., nontransit instead of nontransit solutions. By the other side, this sensibility could be used to fulfill the second and the third requirements.

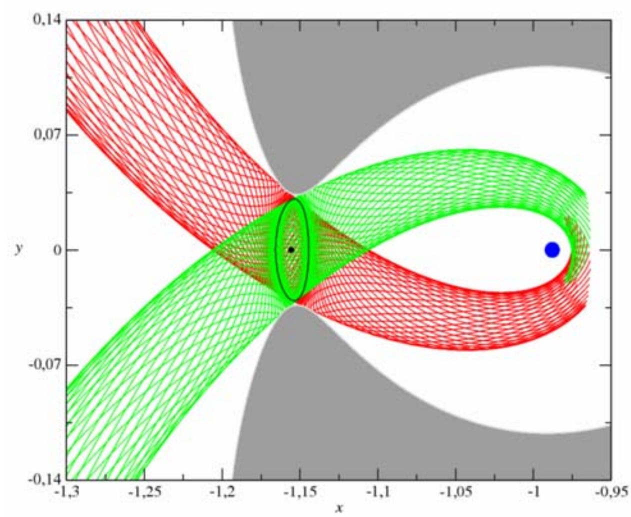

Fig. 10. The projection of the unstable (red curves) and stable (green curves) invariant manifolds associated to a Lyapunov orbit (closed curve) in the *x-y* plane for the Earth-Moon system. The blue circle and the black dot represent, respectively, the Moon and the L_2 Lagrangian point. The white region corresponds to the Hill region associated to this Jacobi constant value, while the dark gray to the prohibited zone.

Nowadays, the complete trajectory of the Earth-Moon mission is constructed including the solar gravitational assist in a four-body problem [9]. In this case, in a preliminary phase of the project, a two-patched three body problem approximation is considered, in such a way that the transfer trajectory can be separated in two pieces, each of them being built based on the invariant tubes of a particular R3BP, as explained in the next section.

4. REAL MISSION DESIGN APPLICATIONS

The low energy transfer from Earth to Moon is the main mission example explored here to illustrate the application of the theoretical results obtained above. Due to the fact that the invariant tubes restrict the transfer trajectory and that these tubes do not get close enough to the larger primary, a transfer based on a single three-body system is not suitable for standard purposes. With the solar gravitational assistance, a four body problem is constituted. However, since the structure of the phase space solutions of the four body problem is poorly understood, a valuable preliminary sketch can be built based on the three-body problem dynamical structures, considering two-patched restricted three-body problem approximation. In this case, the Sun-Earth and the Earth-Moon systems are the two pieces of the complete problem.

The success of this approach depends strongly on the specific configuration of the involved four bodies. Among the restrictions, the most significant is that the invariant manifold tubes of the two three-body systems must intersect within a reasonable time. Specifically, the unstable invariant structures of the L_2 Earth-Moon Lyapunov orbits grow in a typical period of one month to the circular region around the Earth with a mean radius of 1.000.000 km. Similarly, the L_1 and L_2 Sun-Earth manifolds are risen in the same time order to the same circular region. This typical time intervals become this approach useful for practical applications.

Let the Poincaré section γ in the Sun-Earth rotating frame be defined as the plane of constant x-position passing through the Earth position, x_E. The overlap of the relevant associated tubes, as illustrated by Fig. 11, defines possible candidates to orbit solutions.

Considering initial conditions in the Poincaré section γ, for a fixed Jacobi integral value of each R3BP, the two-piece strategy for the low energy transfer project must satisfy the requirements:

(i) Through the direct integration, the spacecraft transit trajectory must be guided by the L_2 stable manifold of the Earth-Moon System, being a transit orbit of the L_2 Earth-Moon equilibrium region - and, therefore, a $W_s(L_2)$ tube interior orbit - in such a way that the spacecraft must be temporarily captured by the Moon.

(ii) Conversely, departing from the same point in the Poincaré section γ, defining an initial condition for the other 3BP, the retrograde evolution must lead the spacecraft trajectory into the neighborhood of the Earth. Being a nontransit orbit associated to the stable and unstable invariant manifold of L_1 or L_2 of the Sun-Earth system, the correspondent direct trajectory must leave a stationary orbit around the Earth, twist around the correspondent Lyapunov orbit and be scattered back to the circular region around the Earth, externally to the unstable invariant tube.

Hence, in practice, the candidates to a transfer orbit can be defined in the Poincaré section γ, that is, in the plane $(y, \dot{y})$, for $x=x_E$, as the subset $A \cap B$, where A is the subset constituted by the interior points of the intersection of the $W_s(L_2)$ of Earth-Moon system with γ, and B is constituted by the exterior points of the intersection of the $W_s(L_1$ or $L_2)$ of the Sun-Earth system with γ. It is remarkable that the orbit branch associated to the Sun-Earth system presents a strong sensitivity to initial conditions, and must be exterior, but not distant of the respective tubes. If the subset $A \cap B$ is empty or too small, the Moon phase can be varied in order to obtain a reasonable intersection. A low thrust maneuver can be considered to provide an adequate patch of the initial conditions on the Poincaré section γ, producing a $\dot{x}$ variation. Finally, the initial condition of solution of the two patched three-body problem can be integrated with a bi-circular (four body) problem and then differentially corrected to a fully integrated trajectory with JPL ephemeris.

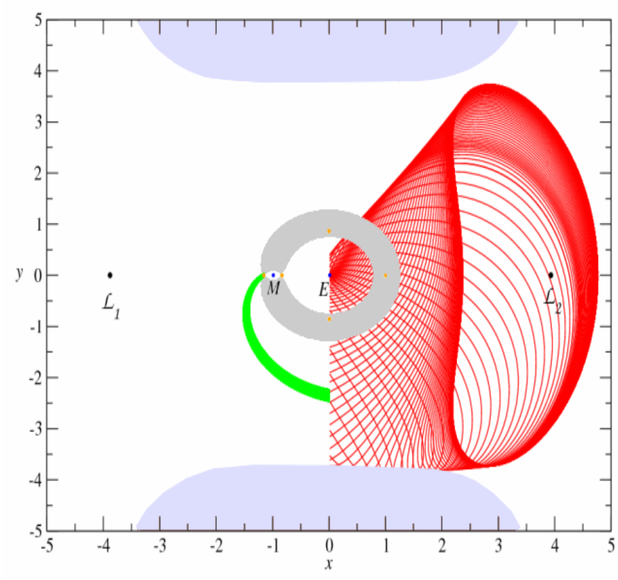

Fig. 11. The two patched three body system approximation is illustrated with the projection of the unstable manifold (red curves) of the L_2 Sun-Earth Lyapunov orbit, and the stable manifold (green curves) of the L_2 Earth-Moon Lyapunov orbit, in the plane x-y. The $\mathcal{L}_1$ and $\mathcal{L}_2$ Lagrangian points of the Sun-Earth system are marked with black dots, while the five equilibria of the Earth-Moon system are marked with orange dots.

Besides the low energy Earth-Moon transfer, successfully applied in the failed Japanese Muses-A recovery in 1991 and renamed as the Híten mission, many other actual space missions can be cited to corroborate the application of the restricted three body problem theoretical framework reviewed in this paper. Among them, the Genesis Discovery mission (2001-2004) and the *Petit Grand Tour of the Jovian Moons* are some of the most famous cases.

5. CONCLUSION

The most relevant theoretical skeleton used in modern mission design, involving the circular planar restricted three body problem, is reviewed in this contribution. Verifying the applicability of the Moser-Lyapunov Theorem to the R3BP, Conley was able to extract all the relevant dynamical structures from the linearized analogue. From this, four categories of orbits were identified in an equilibrium region, explicitly, the transit, nontransit, asymptotic and periodic orbits. In this way, the two-dimensional invariant manifolds were identified as separatrices in the four-dimensional phase space for a given energy surface. Furthermore, the two patched 3B approximation is illustrated as a preliminary phase of a low energy Earth-Moon transfer orbit design.

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