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Foreword

The subject of this treatise, the restricted problem of three bodies, occupies a central place in analytical dynamics, celestial mechanics, and space dynamics. Entry into celestial mechanics and space dynamics can be gained by the study of the problem of two bodies. To penetrate the fundamental problems, the number of participating bodies must be increased from two to three. This step is critical. Not only is the two-body problem solved—and the meaning of "solution" may be different for astronomers, engineers, and mathematicians—but a general understanding exists regarding this dynamical system. The problem of three bodies on the other hand is neither solved nor is the behavior of the dynamical system completely understood.

The solar system provides few applications of the general problem of three bodies. This results in an unusual situation where a more general problem having considerable complexity is less useful than a comparatively simple formulation. Also it is important to realize that more is known about the restricted problem than about the general problem.

This volume is strongly influenced by the creators of modern dynamics, H. Poincaré and G. D. Birkhoff. Poincaré, in his *Méthodes Nouvelles de la Mécanique Céleste* and also in his famous *Mémoire Couronné*, "Sur le problème des trois corps et les équations de la dynamique," uses the problem of three bodies as his favorite example when presenting his work in dynamics. The same is true for G. D. Birkhoff's *Dynamical Systems*, and for C. L. Siegel's *Vorlesungen über Himmelsmechanik*. A. Wintner's *Analytical Foundations of Celestial Mechanics* was originally planned to treat the problem of three bodies, especially the restricted problem, but it actually presented more of the mathematical foundations than of the celestial mechanics. It is interesting to note that H. Happel's book is entitled *Das Dreikörperproblem*, and the subtitle reads "Vorlesungen über Himmelsmechanik," while the second volume of K. Stumpff's *Himmelsmechanik* displays the subtitle "Das Dreikörperproblem." How intimately the problem of three bodies is connected with celestial mechanics and with dynamics in general when titles, subtitles, contents, applications, and examples become interchangeable!

The applications of the restricted problem to celestial mechanics form the basis of some lunar and planetary theories. The modern applications to space mechanics are probably even more cogent if not more numerous than the classical applications. The implications of the restricted problem for cosmogony and stellar dynamics are also numerous. Finally, it can be shown that a great variety of dynamical systems can be presented by equations of motion which are formally identical with the equations of the restricted problem. One measure of the importance of a scientific endeavor is its effect on peripheral fields. While authors from Euler to Siegel recognized astronomy and dynamics as the only peripheral fields, today we know that space mechanics and stellar dynamics are fields which benefit equally.

The interest in space sciences rejuvenated celestial mechanics, and the well-established tools of the latter were immediately applied. Some of the problems were not really new and the proven methods of classical celestial mechanics—in the hands of the masters—produced immediate results. I think of several solutions of the drag-free earth-satellite problem, for instance, which today may be considered settled. It is a perturbation of the two-body problem, and the success in solving it is partly explained by the popularity of satellite problems in classical celestial mechanics. Other problems in space dynamics, closely associated with the restricted problem, are of considerable importance and interest today. Many of these problems are new, and in what follows one of them will be contrasted to a classical problem. Consider the famous classical three-body problem, the sun-earth-moon combination and the determination of the motion of the moon. We might think about two large bodies, the sun and the earth, which move around each other in approximate circles, and in their field a third body, the moon, which moves on an approximate ellipse. This configuration is stationary in a sense, since no collisions take place. This is also true for the motion of a Trojan asteroid under the continued influence of the sun and Jupiter. On the other hand, one of the central problems in space science is to create artificial bodies which may be required to move on orbits connecting the close neighborhood of two natural celestial bodies. Sometimes collision orbits are desired. Problems with close approaches and collisions were hardly ever treated in classical celestial mechanics and these problems became important in the new science of space dynamics.

The use of three essentially different approaches to dynamics, the qualitative, the quantitative, and the formalistic, is dictated by the special advantages of each and is described in the Introduction, where a number of references to the history of the restricted problem are also given.

The first chapter introduces the problem of three bodies and formulates the equations of motion in inertial and in rotating coordinate systems. The relation of the restricted problem to the general problem of three bodies is described and illustrated with examples. Several applications to cosmogony and stellar dynamics are also outlined. Chapter 2 discusses reductions of the problem and offers a comprehensive treatment of streamline analogies.

Chapter 3 is concerned with regularization and shows how the equations of motion can be written in a system free of singularities. This subject is the feature which distinguishes a work on classical celestial mechanics from one on modern applications. This chapter is probably the most important one for the reader who is working in the field of space mechanics. Chapter 4 is devoted to the principal qualitative aspect of the restricted problem—the curves of zero velocity, several uses of which are discussed. The regions of permissible motion and the location and properties of the libration points are established. Motion and nonlinear stability in the neighborhood of these equilibrium points are treated in detail in Chapter 5.

Chapter 6 contains a short introductory treatment of Hamiltonian dynamics in the extended phase space. Chapter 7 applies the principles and methods of the previous chapter to the restricted problem and to its regularization. The generating functions that are used are derived with emphasis on justification and motivation. A natural way to introduce the concept of perturbation theory is presented.

Chapter 8 discusses the problem of two bodies in a rotating coordinate system and treats periodic orbits in the restricted problem, following H. Poincaré and G. D. Birkhoff. Chapter 9 presents the quantitative aspects of the restricted problem. The results of G. Darwin, E. Strömgren, and F. R. Moulton are discussed and several of the recently established lunar and interplanetary orbits in the Soviet and American literature are compared. Chapter 10 is devoted to modifications of the restricted problem, such as the elliptic problem, the three-dimensional problem, and Hill's problem.

V. Szebehely

Preface

This volume has been developed from my lectures and seminars on various aspects of celestial mechanics, dynamics, the restricted problem of three bodies, periodic orbits, regularization, and space dynamics. While directed primarily to the graduate student, it is intended to be sufficiently comprehensive to serve as a reference and advanced text on many applications of celestial mechanics. One purpose is to familiarize those readers who are concerned with the space applications of celestial mechanics with the next step after the problem of two bodies. The student of celestial mechanics will find both classical studies and recent developments in the restricted problem of three bodies with a survey of the pertinent literature.

This is the first book devoted to the theory of orbits in the restricted problem. My aim is to build a bridge between books written for the astronomer, mathematician, space engineer, and student of dynamics. Instead of developing the subject separately for each of these professions, it is hoped that the single subject of this volume will be useful for all its readers. Astronomers will find more references to analytical dynamics than is usual in textbooks on celestial mechanics; workers in the field of dynamics will read about astronomical applications; the needs of mathematicians and engineers will be met by the problem of establishing the totality of possible motions of our dynamical system.

Teaching experience shows that students are interested in historical reviews and remarks in the field of celestial mechanics, which is so rich in traditions and in cultural background material. Such comments are collected at the end of each chapter with the discussions of the pertinent references. Most chapters contain a generous amount of basic mathematical information. I make it a point to extend the foundations more than necessary for the building, in order to establish a more solid edifice and offer to the reader the opportunity of proceeding with his own applications.

My guiding principle has been to inform the reader of the motivation and purpose of the developments, hoping to inspire his enthusiastic interest in the subject. I try to avoid unnecessary *epsilontics* in the mathematical parts and highly specialized and undefined terms in the applications. Mathematics is a tool in dynamics, not a goal. The Wintnerian turnaround from the problem of three bodies to mathematics is avoided, and an attempt is made to emphasize the dynamics. I subject the brilliance of Poincaré and of G. D. Birkhoff to scrutiny and explanation rather than to competition. My aims are to summarize G. Darwin's eloquence, to expand Siegel's terseness, to generalize Charlier, and to particularize Moulton and E. Strömgren. Special attention is paid to the Soviet literature of the past two or three decades; it contains many significant contributions to celestial mechanics and space dynamics. Recent numerical results on earth-moon trajectories are compared with previous results, and classical orbit computations are brought up to date.

April, 1967 V. SZEBEHELY

Acknowledgments

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I extend sincere thanks to my graduate students. The computations involved in preparing the major tables were performed by Miss C. Williams. Several of the complicated figures were drawn by an automatic plotter, imputs to which were prepared by Dr. D. Pierce and Mr. M. Standish. Several of the orbits were computed by Messrs. S. Knowles and J. Lieske. In these undertakings the facilities of the Yale Computer Center were used. The burden of reading the proofs was shared mostly by Messrs. D. Bettis, J. Breedlove, P. Esposito, R. Laubscher, and P. Nacozy, graduate students. The tables were put in their final form by Dr. and Mrs. R. Duncombe of the U.S. Naval Observatory.

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Contents

Introduction

The purpose of dynamics is to characterize the totality of possible motions of a given dynamical system. Such a characterization does not necessarily mean an explicit, closed-form, general solution of the problem since this is seldom possible, and when it is possible, it is most of the time neither meaningful nor helpful in understanding the behavior of the system. An example is the problem of two bodies, which is considered solved since the properties of the totality of possible motions are known. Although the coordinates describing the motion of the bodies participating in the problem cannot be represented as explicit functions of the time in closed form, the problem is nevertheless considered solved.

Qualitative, quantitative, and formalistic dynamics are the three major approaches to the understanding of the behavior of dynamical systems. The qualitative approach is probably the most elegant and sometimes the most powerful. The formalistic method is the basis of classical celestial mechanics. The quantitative approach is often the most popular among astronomers and engineers who may want to find one particular solution of a problem rather than to study the behavior of the dynamical system. Examples are the ephemerides of the planets, representing particular solutions of the astronomers' n-body problem and Apollo trajectories, being particular solutions of the engineers' problem.

Qualitative methods in dynamics are well suited to the treatment of such questions as stability, existence problems, integrability, and reducibility. The names of H. Poincaré and G. D. Birkhoff are associated with qualitative dynamics; Hill's name is seldom thought of in this connection, in spite of his use of the zero velocity curves to establish limiting regions. His method is probably one of the most powerful and successful qualitative ideas in the restricted problem.

It is sometimes said of qualitative dynamics that its results are not helpful to "practical" men (to the "users" as opposed to the "creators"). This misconception is partly because some of the qualitative results in dynamics have not yet been interpreted and some of these results are of theoretical interest only.

Knowledge of certain qualitative properties of a dynamical system may be much more valuable than numerical solutions. An example is the existence question of periodic orbits. Solutions of nonintegrable dynamical systems are never known along the whole time axis unless they are of periodic or asymptotic nature. This is seen when we consider an attempt to establish a particular solution of the differential equations of a dynamical system with an electronic computer. Not attempting for the moment to evaluate such an undertaking, let us visualize the computer output as the time increases without limit and as various error sources contribute to the printouts. Unless some systematic behavior of the result is discovered, sooner or later the computer output becomes meaningless and no valuable information about the dynamical system will be obtained along the whole time axis. The orbit or the behavior of the system will remain unknown in spite of the numerical work.

Another example is furnished by one of the fundamental questions of dynamics: the description of the totality of possible motions of a dynamical system. For nonintegrable systems this is a major problem as no closed-form general solution is available. The practical importance of knowing all possible orbits between the earth and the moon does not need emphasis, since selection of an orbit "best" suited for a certain mission requires information regarding the possible choices. A formalistic approach to this problem is not fruitful, for even if it should furnish convergent series which give the general solution, the nature, the properties, and the totality of the solution could not in general be determined from such series. The quantitative approach to this problem is to select a region of the initial conditions which is of practical interest and to compute as many orbits as possible in this region. This set of orbits is called the "totality of orbits of interest." The deficiency in this approach is the possible omission of useful orbits or of whole families of useful orbits. When the possible range of initial conditions is considerable, the establishment of families of orbits according to six varying initial conditions is almost a hopeless task numerically. The description of the totality of possible motions should come from a combined approach (numerical and formalistic) with qualitative dynamics leading and organizing the steps. One of the most practical and most important problems in applied celestial mechanics, the selection of a suitable orbit, is therefore equivalent to one of the most advanced problems of qualitative dynamics.

Turning now to the formalistic approach we enter the stronghold of classical celestial mechanics. The formalistic methods are also called general perturbation methods, and the principal mathematical tools are series expansions. In order to have a general perturbation method the initial conditions are kept arbitrary in the solution. Justification of the method from a mathematical point of view requires scrutiny of the convergence of the series with respect to the variables. It is ironic that one of the qualitative results of dynamics, attributed to Poincaré, states that the series used in celestial mechanics are in general divergent. Nevertheless, finite parts of such series are often extremely useful in celestial mechanics since they do give results in agreement with observations. Questions connected with the behavior of the system as the time increases to infinity cannot, of course, be answered by such series solutions. The classical series of celestial mechanics become of little use when bodies approach each other closely and when they collide. Since such orbits are of central importance in modern dynamics, new formalistic approaches have had to be devised.

The hopefully expected ultimate answer of representing the totality of solutions as "simple" functions of the initial conditions and of time may come from formalistic dynamics. Such a result can probably be expected from a combined effort of the three major approaches with the formalistic approach taking the lead. Newton's approach to dynamics was to find just such explicit expressions representing the motion of dynamical systems. Advances in celestial mechanics and in other branches of science with mathematical orientation show more or less the same steps. First comes the attempt to describe the field of interest with simple analytic expressions. This leads of necessity to successive approximations and series solutions if the first attempt for simple closed-form solutions fails. Those fields, such as the "solvable" dynamical systems, in which the first step furnishes results, are considered solved and are soon abandoned. The quantitative approach to dynamics is not unlike the first step because it gives a particular solution in a simple form: a set of numbers representing the coordinates as functions of time. Those fields in which sufficient interest exists for establishing general solutions, but which at the same time are not "integrable" and therefore are not amenable to simple general solutions, are graduated to the second phase of mathematical physics: to series solutions. Some problems are solved at this stage if the series solutions furnish the properties of the general solution. This is seldom the case for systems of appreciable complexity where neither the convergence of the successive approximations nor the physical meaning of the series solution is completely clear. Those physical problems that fall in this last category are elevated to the domain of qualitative methods.

The quantitative approach to dynamics corresponds to experimentation. Its significance cannot be overestimated, especially in nonintegrable dynamical systems since, after all, this is the only method which furnishes an orbit when the convergence of the formalistic approach is in doubt. The power of properly designed experiments and the importance of proper interpretation of results are well known in physics. Dynamics' experimental tool, the computer, only recently became efficient enough to handle complex problems; therefore, experimentation in dynamics has not advanced as far as it has for example in physics. Famous classical computational results in the restricted problem were obtained without the use of digital computers by E. Strömgren and G. Darwin. Recent high-speed computational results, together with the older results, reveal several significant properties of the system that can be verified theoretically. Such a combined theoretical-experimental approach has shown great potentialities in dynamics.

The history of the restricted problem begins with Euler and Lagrange in 1772, continues with Jacobi (1836) and Hill (1878), and is followed by Poincaré (1899), Levi-Civita (1905), and Birkhoff (1915). The span of almost 200 years, from Euler until now, includes other great names and important contributions; this short historical review nevertheless will concentrate on the accomplishments of Euler, Jacobi, and Poincaré.

The first contribution was made by *Euler* in 1772 in connection with his lunar theories. His work was the first important contribution to the restricted problem and its influence on further developments of lunar theories and even on some very recent work in space dynamics is clearly evident. His principal accomplishment was the introduction of a synodic (rotating) coordinate system, the use of which led to an integral of the equations of motion, known today as the Jacobian integral. Euler himself did not discover the Jacobian integral which was first given by Jacobi (1836) who, as Wintner remarks, "rediscovered" the synodic system. The actual situation is somewhat complex since Jacobi published his integral in a sidereal (fixed) system in which its significance is definitely less than in the synodic system. The tongue-in-cheek remark of Wintner which is mentioned before, is not completely accurate, nor is his immediately following recommendation, citing Newcomb's report on lunar theory as a useful reference for the history of the restricted problem.

Prior to his lunar theory Euler (1760) gave the solution of the problem of two fixed centers of force, in which two fixed masses act on a third

body according to the Newtonian law of gravitation. This dynamical
system is a special and highly simplified case of the restricted problem
since centrifugal and Coriolis forces do not enter. Its direct significance
is limited as fixed force centers do not occur either in celestial mechanics
or in its applications. In view of the fact, however, that Euler's problem
of two fixed force centers can be solved in closed form its indirect
applications are numerous. In the literature of space mechanics attention
was called to this problem (and to its simplification by use of Bonnet's
theorem) in 1959 in connection with the reliability and accuracy of
digital-computer solutions. Euler's solution of the problem of two
fixed centers of force can also be used in connection with the artificial
satellite problem and as a reference orbit for general-perturbation
calculations in the restricted problem. In fact Vinti's solution of the
artificial satellite problem, established independently of Euler's result,
turns out to be essentially analytically identical with it. The idea to use
Euler's solution as a reference orbit in the restricted problem is not new.
Unfortunately, this solution involving elliptic functions is less useful
than the far simpler Encke method using conic sections as reference
orbits. On the other hand, Euler's problem does include the effect of
both masses while Encke's method considers only one. The third
application of Euler's two fixed force centers is related to the problem
of regularization. The coordinate transformation employed by Euler to
treat the problem of two fixed centers of force when used for the
restricted problem eliminates the singularities or in other words
"regularizes" the problem. Not only did Thiele (1892) and Burrau (1906)
make use of this transformation for the restricted problem, performing
the regularizing process, but also the large amount of numerical work
performed by the Copenhagen school under the direction of Strömgren
(1935) was based on this transformation.

The *Jacobian* integral of the restricted problem is attached to the
name of the second major contributor.

Implications of this integral are numerous. Since it connects the
magnitude of the velocity vector (the speed) of the third body to its
location, it allows us to make certain general, *qualitative* statements
regarding the motion without actually solving the equations of motion.
This fact gives great importance to an integral applicable to an
"unsolvable" dynamical problem. It permits the establishment of a certain
forbidden region from which the third body is excluded. The application
of this principle to celestial mechanics was first made by Hill (1878) to
show that the earth-moon distance must remain bounded from above
for all time, which is to say that *if* Hill's model for the sun-earth-moon
system is accepted, then the moon cannot depart from the earth's
neighborhood arbitrarily far.

Poincaré's famous three volumes of *Méthodes Nouvelles* were completed in 1899. This work was so new and original that many of its implications are still not entirely clear. Probably the most significant contribution made by Poincaré was his emphasis on the qualitative aspects of celestial mechanics as opposed to the quantitative approach. Just as Euler proposed the lunar theories, which may be considered the highest computational accomplishments of mankind, Poincaré initiated analytical methods which seem to be the highest theoretical accomplishments. Just as Euler's work on the restricted problem was followed by Hill and Brown (1896), who gave the most precise lunar theory, so was Poincaré followed by Birkhoff (1915), who elevated the methods of qualitative dynamics to heights still unconquered by those who wish to apply his results.

The problem of regularization, which is so prominent in certain applications of space dynamics, is associated with the names of Thiele (1892), Painlevé (1897), Levi-Civita (1903), Burrau (1906), Sundman (1912), and Birkhoff (1915).

Interpretation and continuation of the undertaking begun by Euler and culminating in Birkhoff's work was by no means finished by the latter. In the 1920's Moulton's school published its results, while in the thirties Moiseev and again Birkhoff made their qualitative contributions, and the final quantitative results of the Strömgren school were published. In 1941 Wintner's book was published, and in the fifties came Kolmogorov's important work, and also Siegel's book. In the sixties Russian writers following Kolmogorov's work took significant steps in qualitative analysis. Also during the sixties a considerable amount of numerical experimental dynamics was performed on digital computers.

The literature of the restricted problem is closely associated with publications in celestial mechanics and with the appearance of books on dynamics. Chapters in books on celestial mechanics by Plummer (1918), by Charlier (1907), by Moulton (1914), by Brouwer–Clemence (1961), by Danby (1962), and by McCuskey (1963) offer informative descriptions of the restricted problem. Whittaker's *Analytical Dynamics* (1904) may be considered the outstanding reference text in dynamics on the general and on the restricted problem of three bodies, while chapters by Pars (1965) and Pollard (1966) give concise treatments from the points of view of dynamics and mathematics.

The justification given by Birkhoff in 1927 for the appearance of a book like this may be quoted here to close the introduction. "At a time when no physical theory can properly be termed fundamental—the known theories appear to be merely more or less fundamental in certain directions—it may be asserted with confidence that ordinary differential equations in the real domain, and particularly equations of dynamical origin, will continue to hold a position of the highest importance."

Chapter **1**

Description of the Restricted Problem

1.1 Introduction

It is often the case in physical sciences that the major difficulty in attacking a problem is the lack of clear definitions, and once the problem is stated the solution is on its way. In this spirit we shall put emphasis on describing the restricted problem in the clearest and simplest terms possible.

The offering of a clear definition is of course a necessary but not a sufficient condition for making progress. The problem of three bodies is a good example where with a little care the most precise statement of the problem can be given. This statement describes a rather simple sounding problem, the solution of which is not available.

In this chapter the simplest and most frequently occurring version of the restricted problem is described. The basic formulation seems to appear first in Euler's memoir on his second lunar theory; therefore, it is almost 200 years old.

After defining the problem, the equations of motion are derived in inertial (sidereal) and rotating (synodic) coordinate systems using physical (dimensional) and dimensionless variables. The four equations are compared and it is shown how the introduction of synodic coordinates results in the existence of the Jacobian constant and of the Jacobian integral. The derivations of these four sets of equations of motion are performed starting with basic and simple principles and using rather elementary methods in order to facilitate the understanding of the physical picture. In a later chapter (Chapter 7) the Lagrangian and Hamiltonian formulations will be given and a more sophisticated picture will be revealed.

The appearance of the restricted problem as the degenerate case of the general three-body problem is shown next to serve as the basis for two important items. As the first outcome the review of various modifications of the basic restricted problem is presented. In the last section the applicability of the restricted problem is analyzed.

1.2 Statement of the problem and equations of motion in a sidereal system

We define our problem as follows: Two bodies revolve around their center of mass in circular orbits under the influence of their mutual gravitational attraction and a third body (attracted by the previous two but not influencing their motion) moves in the plane defined by the two revolving bodies. *The restricted problem of three bodies is to describe the motion of this third body.*

The two revolving bodies are called the primaries (or the primary and the secondary, a nomenclature popular in stellar dynamics but which we will *not* follow). The masses m_1 and m_2 of these bodies are arbitrary but the bodies have such internal mass distributions that they may be considered point masses. The mass of the third body m_3 is an intricate subject which will be discussed in some detail later in this chapter. At this point the approximate statement is accepted, that m_3 is much smaller than either m_1 or m_2. This is intuitively correct since m_3 does not influence the motion of m_1 and m_2.

The circular motion of m_1 and m_2 around their mass center 0 is shown in Fig. 1.1.

Balance between the gravitational and centrifugal forces requires that

$$k^2 \frac{m_1 m_2}{l^2} = m_2 a n^2 = m_1 b n^2, \tag{1}$$

where k is the Gaussian constant of gravitation, n is the (common) angular velocity of m_1 and m_2, l is their mutual distance, and a and b are as shown in Fig. 1.1. The quantity n in celestial mechanics is called the mean motion and the angle nt^* the longitude of m_1. The symbol t^* is used for time, this way preserving t for the dimensionless time.

From this

$$k^2 m_1 = a n^2 l^2, \qquad k^2 m_2 = b n^2 l^2, \qquad k^2(m_1 + m_2) = n^2 l^3, \tag{2}$$

the last equation being Kepler's third law.

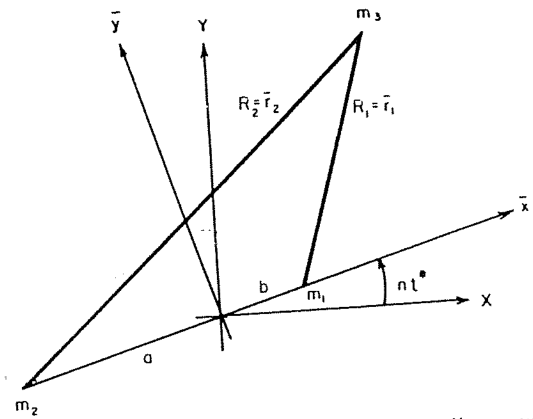

Fig. 1.1. The fixed (sidereal) and the rotating (synodic) coordinate systems ($m_1 > m_2$).

Also

$$a = \frac{m_1 l}{M} \quad \text{and} \quad b = \frac{m_2 l}{M}, \tag{3}$$

where $M = m_1 + m_2$.

The equations of motion of m_3 in an inertial (fixed) rectangular coordinate system, X being the abscissa and Y the ordinate of m_3, are

$$d^2X/dt^{*2} = \partial F/\partial X \quad \text{and} \quad d^2Y/dt^{*2} = \partial F/\partial Y. \tag{4}$$

We note that the inertial coordinate system X, Y shown in Fig. 1.1 is called the sidereal system. F is the force function or the negative potential and is given by

$$F = k^2(m_1/R_1 + m_2/R_2). \tag{5}$$

The distances R_1 and R_2 are given by

$$R_1 = [(X - X_1)^2 + (Y - Y_1)^2]^{1/2}, \tag{6}$$
$$R_2 = [(X - X_2)^2 + (Y - Y_2)^2]^{1/2},$$

where (X_1, Y_1) and (X_2, Y_2) are the time-dependent coordinates of m_1 and m_2, respectively, which are obtainable by inspecting Fig. 1.1:

$$X_1 = b \cos nt^*, \qquad X_2 = -a \cos nt^*,$$
$$Y_1 = b \sin nt^*, \qquad Y_2 = -a \sin nt^*.$$

The time dependence of the coordinates of m_1 and m_2 introduces the time explicitly in the equations of motion. This is intuitively expected since m_1 and m_2 *move* in the fixed system of coordinates. The formal proof of the explicit occurrence of the time consists of substituting the preceding time dependencies into (6), (6) into (5), and (5) into (4). Equations (4) become

$$\frac{d^2X}{dt^{*2}} = -k^2\left[\frac{m_1(X - b\cos nt^*)}{R_1^3} + \frac{m_2(X + a\cos nt^*)}{R_2^3}\right],$$
$$\frac{d^2Y}{dt^{*2}} = -k^2\left[\frac{m_1(Y - b\sin nt^*)}{R_1^3} + \frac{m_2(Y + a\sin nt^*)}{R_2^3}\right], \tag{7}$$

or simply

$$\frac{d^2X}{dt^{*2}} = \frac{\partial F(X, Y, t^*)}{\partial X} \quad\text{and}\quad \frac{d^2Y}{dt^{*2}} = \frac{\partial F(X, Y, t^*)}{\partial Y}. \tag{8}$$

1.3 An invariant relation and the total energy of the system

This section discusses three questions of considerable theoretical importance. In Part (A) an invariant relation for the restricted problem in the inertial system is derived, Part (B) gives the definition of an integral of a dynamical system in general, and Part (C) discusses the conservation of energy.

(A) Inasmuch as the gravitational force field possesses a potential, an attempt might be made to derive an invariant relation corresponding to the energy conservation of the dynamical system. Multiplying Eqs. (8) by dX/dt^* and dY/dt^*, respectively, adding and integrating with respect to the time gives

$$\frac{1}{2}\left[\left(\frac{dX}{dt^*}\right)^2 + \left(\frac{dY}{dt^*}\right)^2\right] = \int_{t_0^*}^{t^*}\left(\frac{\partial F}{\partial X}\frac{dX}{dt^*} + \frac{\partial F}{\partial Y}\frac{dY}{dt^*}\right)dt^*, \tag{9}$$

where, for the time being, the constant of integration is disregarded. Since

$$dF = F_X\,dX + F_Y\,dY + F_{t^*}\,dt^*,$$

where subscripts denote partial derivatives, the quadrature on the right side of (9) becomes

$$\int_{t_0^*}^{t^*} dF - F_{t^*}\,dt^* = F - \int_{t_0^*}^{t^*} F_{t^*}\,dt^*. \tag{10}$$

Equations (9) and (10) give

$$\tfrac{1}{2}V^2 = F - \int_{t_0^*}^{t^*} F_{t^*}\,dt^*, \tag{11}$$

where V is the velocity of m_3. The energy is therefore not conserved in the system since $\frac{1}{2}V^2 - F$ is a function of the time and not constant. The interpretation of the quadrature

$$\int_{t_0^*}^{t^*} \frac{\partial F(X, Y, t^*)}{\partial t^*}\,dt^*$$

occurring in (11) is as follows. Consider the solution given in the form $X = X(\alpha_i, t^*)$, $Y = Y(\alpha_i, t^*)$, where the constants α_i $(i = 1, 2,..., 4)$ represent the initial conditions. Substituting this solution into the quadrature gives a function depending on the initial conditions and on the time. Therefore (11) becomes

$$\tfrac{1}{2}V^2 - F = C(\alpha_i, t^*).$$

The essential point is that a relation like (11) is of limited use unless additional approximations are made, since in general its interpretation requires the solution of the problem. On the other hand, if F should not depend explicitly on the time, then $F_{t^*} = 0$ and $\frac{1}{2}V^2 - F = C(\alpha_i)$; i.e., the constant of integration depends only on the initial conditions.

It is to be noticed that Eq. (11) is an invariant relation of the dynamical system in question. While it is not immediately helpful in establishing the "solution" it does serve as a useful check in numerical and formal computations. In the following we give the definition of an integral of a dynamical system to avoid any misunderstanding.

(B) Consider a dynamical system of n degrees of freedom with coordinates $q_1, q_2, \ldots, q_n$ and write the equations of motion in the form

$$d^2q_i/dt^2 = Q_i(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t), \tag{12}$$

where $i = 1,..., n$ and t is used for the time. This set of n second-order equations form a system of the $2n$th order, which can be written as $2n$ equations of the first order by introducing

$$x_i = q_i \quad\text{and}\quad x_{n+i} = \dot{q}_i$$

or

$$x_1 = q_1, x_2 = q_2, \ldots, x_n = q_n, x_{n+1} = \dot{q}_1, x_{n+2} = \dot{q}_2, \ldots, x_{2n} = \dot{q}_n. \tag{13}$$

Equations (12) become

$$\frac{dx_i}{dt} = x_{n+i},$$
$$dx_{n+i}/dt = Q_i(x_1, x_2, \ldots, x_{2n}, t). \tag{14}$$

Explicit and detailed forms of Eqs. (14) are

$$\begin{aligned} \dot{x}_1 &= x_{n+1}, \dot{x}_2 = x_{n+2}, \ldots, \dot{x}_n = x_{2n}, \\ \dot{x}_{n+1} &= Q_1(x_1, \ldots, x_{2n}, t), \\ \dot{x}_{n+2} &= Q_2(x_1, \ldots, x_{2n}, t), \\ &\vdots \\ \dot{x}_{2n} &= Q_n(x_1, \ldots, x_{2n}, t), \end{aligned} \tag{15}$$

or simply

$$\dot{x}_k = P_k(x_1, \ldots, x_m, t), \tag{16}$$

where $k = 1, 2, \ldots, m$, $m = 2n$, and the functions P_k and their partial derivatives are defined and continuous in some domain.

Considering now the $2n$ first-order differential equations given by Eq. (16) we define an integral of this system as follows. If a function $G(x_1, x_2, \ldots, x_m, t)$ with the same properties as P_k satisfies the condition

$$dG/dt = 0 \tag{17}$$

when any set of solution $x_1(t), x_2(t), \ldots, x_m(t)$ is substituted, we call $G(x_1, x_2, \ldots, x_m, t) = \text{const}$ an integral of the system (16). Equation (17) can be expanded:

$$\sum_{k=1}^{m} \frac{\partial G}{\partial x_k} \frac{dx_k}{dt} + \frac{\partial G}{\partial t} = 0,$$

or using (16)

$$\sum_{k=1}^{m} \frac{\partial G}{\partial x_k} P_k + \frac{\partial G}{\partial t} = 0. \tag{18}$$

Equation (18) is the condition to be satisfied in order that G be an integral.

(C) We will now show that the total energy in the restricted problem is not constant. The total energy of m_3 per unit mass is

$$h_3 = \tfrac{1}{2}(dX/dt^*)^2 + \tfrac{1}{2}(dY/dt^*)^2 - F, \tag{19}$$

which, as shown in Part (A), is not constant. The question arises whether the total energy of the dynamical system formed by the three bodies is constant. The total energy of the individual particles is, of course, the sum of their separate energies, so we have

$$H = m_3 h_3 + H_{12}, \tag{20}$$

where H_{12} is the energy of the m_1, m_2 system, and

$$H_{12} = \tfrac{1}{2}n^2(m_1 b^2 + m_2 a^2) - k^2 \frac{m_1 m_2}{l}. \tag{21}$$

The first term is the kinetic energy and the second the potential energy. Using the elementary relations given by Eqs. (1) and (2) one finds that

$$\tfrac{1}{2}n^2(m_1 b^2 + m_2 a^2) = \tfrac{1}{2}k^2 \frac{m_1 m_2}{l}; \tag{22}$$

therefore, $H_{12} = -\frac{1}{2}k^2 m_1 m_2/l = \text{const}$, and the total energy of the system formed by the three bodies becomes

$$H = m_3 h_3 - \tfrac{1}{2}k^2 \frac{m_1 m_2}{l} \neq \text{const}. \tag{23}$$

The formal reason for the result that the total energy of the three bodies participating in the restricted problem is not constant is simply that the total energy of m_3 is not constant while that of the m_1, m_2 system is constant; therefore, the sum of the energies cannot be constant. The deeper reason for this "violation of the energy conservation" will become clear in the next chapter when the general problem of three bodies will be discussed. In the restricted problem we neglected the effect of m_3 on the motion of m_1 and m_2, creating in a way a dynamical situation which, strictly speaking, exists only when $m_3 = 0$. If this condition is satisfied then, of course, Eq. (23) gives for the total energy of the three bodies $H = -\frac{1}{2}k^2 m_1 m_2/l = \text{const}$.

If the mass of m_3 is different from zero it must have an effect on the motion of m_1 and m_2, which cannot move any more on their assumed circular orbits. In this case Eq. (21) is not valid and H_{12} is not constant. The total energy of the dynamical system formed by the three bodies without restrictions on the motion of m_1 and m_2 possesses a potential which does not depend explicitly on the time; therefore, the total energy of this system is conserved.

1.4 Equations of motion in a synodic coordinate system and the Jacobian integral

It was discussed in some detail that the force function F contains the time explicitly because of the motion of the primaries. Consequently the Hamiltonian depends on the time explicitly, it is not an integral, and it is not constant along an orbit as will be shown in Chapter 7.

Based on the principle that one purpose of mathematics is indeed to verify intuitive results, the question is proposed: what coordinate system would result in a force function which would show no explicit dependence on the time? The intuitive answer is that since the time dependence is a consequence of the motion of the primaries in a fixed (sidereal) system one should expect that a coordinate system in which m_1 and m_2 are fixed will show superior qualities. The following derivation shows the correctness of this intuitive guess.

The coordinate transformation is the well-known rotation which, with the notation of Fig. 1.1, becomes

$$\begin{aligned} X &= \bar{x} \cos nt^* - \bar{y} \sin nt^*, \\ Y &= \bar{x} \sin nt^* + \bar{y} \cos nt^*, \end{aligned} \tag{24}$$

or, in the notation of matrices,

$$\mathbf{R} = \mathbf{A}\bar{\mathbf{r}},$$

where the vector $\mathbf{R}$ has the components X, Y; the components of the vector $\bar{\mathbf{r}}$ are $\bar{x}$, $\bar{y}$ and the matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{pmatrix} \cos nt^* & -\sin nt^* \\ \sin nt^* & \cos nt^* \end{pmatrix}. \tag{25}$$

The transformation of Eqs. (7) is probably simplest when complex variables are introduced. Let

$$Z = ze^{int^*}, \tag{26}$$

where

$$z = \bar{x} + i\bar{y}, \qquad Z = X + iY, \qquad i = (-1)^{1/2}.$$

The distances R_1 and R_2, for instance, are given by Eqs. (6) as

$$R_1 = |Z - Z_1| \quad \text{and} \quad R_2 = |Z - Z_2|, \tag{27}$$

where according to Eqs. (7)

$$Z_1 = be^{int^*} \quad \text{and} \quad Z_2 = -ae^{int^*}. \tag{28}$$

Substituting for Z from Eq. (26) and for Z_1, Z_2 from Eqs. (28), the distances given by Eqs. (27) become

$$\begin{aligned} R_1 &= |z - b| = [(\bar{x} - b)^2 + \bar{y}^2]^{1/2}, \\ R_2 &= |z + a| = [(\bar{x} + a)^2 + \bar{y}^2]^{1/2}. \end{aligned} \tag{29}$$

The left-hand sides of Eqs. (7) in complex notation become

$$\frac{d^2Z}{dt^{*2}} = \left(\frac{d^2z}{dt^{*2}} + 2in\frac{dz}{dt^*} - n^2z\right)e^{int^*}, \tag{30}$$

and the collection and arrangement of the remaining terms of Eqs. (7) is left to the reader as a simple example. The complex form of the equations of motion in the rotating system is

$$\frac{d^2z}{dt^{*2}} + 2in\frac{dz}{dt^*} - n^2z = -k^2\left[m_1\frac{(z-b)}{|z-b|^3} + m_2\frac{(z+a)}{|z+a|^3}\right]. \tag{31}$$

The real and imaginary parts give

$$\begin{aligned} \frac{d^2\bar{x}}{dt^{*2}} - 2n\frac{d\bar{y}}{dt^*} - n^2\bar{x} &= -k^2\left[m_1\frac{(\bar{x}-b)}{\bar{r}_1^3} + m_2\frac{(\bar{x}+a)}{\bar{r}_2^3}\right], \\ \frac{d^2\bar{y}}{dt^{*2}} + 2n\frac{d\bar{x}}{dt^*} - n^2\bar{y} &= -k^2\left[\frac{m_1\bar{y}}{\bar{r}_1^3} + \frac{m_2\bar{y}}{\bar{r}_2^3}\right], \end{aligned} \tag{32}$$

where the $\bar{r}_1$, $\bar{r}_2$ notation was introduced for R_1, R_2, indicating that in the rotating coordinate system the distances show no explicit dependence on the time [cf. Eq. (29)]. Equations (32) verify the intuition that in the rotating coordinate system the force function is not expected to show explicit dependence on the time. The right-hand sides of Eqs. (7) have been simplified since Eqs. (32) do not contain t^*; the left-hand sides have become more complicated by the appearance of the first derivatives and linear terms. The terms $n^2\bar{x}$ and $n^2\bar{y}$ are of small concern since they can be combined with terms in the right-hand members, but the presence of the first derivatives raises the question whether the transformation serves any interest or whether it only complicates matters. The answer to this question is connected with the fact that the new Eqs. (32) possess a "useful" integral. In fact the only known integral of the restricted problem can be obtained directly from (32) in the same way as Eq. (9) was obtained from (8). Prior to this step, it is convenient to establish the force function belonging to Eqs. (32). For this purpose we write them as

$$\begin{aligned} \frac{d^2\bar{x}}{dt^{*2}} - 2n\frac{d\bar{y}}{dt^*} &= \frac{\partial F^*}{\partial \bar{x}}, \\ \frac{d^2\bar{y}}{dt^{*2}} + 2n\frac{d\bar{x}}{dt^*} &= \frac{\partial F^*}{\partial \bar{y}}, \end{aligned} \tag{33}$$

and find the function F^* so that

$$\begin{aligned} \frac{\partial F^*}{\partial \bar{x}} &= n^2\bar{x} - k^2\left[m_1\frac{(\bar{x}-b)}{\bar{r}_1^3} + m_2\frac{(\bar{x}+a)}{\bar{r}_2^3}\right], \\ \frac{\partial F^*}{\partial \bar{y}} &= n^2\bar{y} - k^2\left[\frac{m_1\bar{y}}{\bar{r}_1^3} + \frac{m_2\bar{y}}{\bar{r}_2^3}\right]. \end{aligned} \tag{34}$$

This problem is well known in potential theory and, since in establishing the generating functions of canonical transformations similar problems and methods will be used later, we offer a short discussion of this question in Section 1.7. At this point we give the answer to the problem proposed by Eqs. (34):

$$F^* = \frac{n^2}{2}(\bar{x}^2 + \bar{y}^2) + k^2\left(\frac{m_1}{\bar{r}_1} + \frac{m_2}{\bar{r}_2}\right). \tag{35}$$

Note that F^* may also be obtained directly from the transformation.

Equations (33) possess an integral, as may be shown by multiplying the first by $d\bar{x}/dt^*$, the second by $d\bar{y}/dt^*$, and adding and integrating with respect to the time t^*. This gives

$$\frac{1}{2}\left[\left(\frac{d\bar{x}}{dt^*}\right)^2 + \left(\frac{d\bar{y}}{dt^*}\right)^2\right] = \int_{t_0^*}^{t^*}\left(\frac{\partial F^*}{\partial \bar{x}}\,d\bar{x} + \frac{\partial F^*}{\partial \bar{y}}\,d\bar{y}\right) = F^* - \frac{C^*}{2} \tag{36}$$

since now

$$dF^* = \frac{\partial F^*}{\partial \bar{x}}\,d\bar{x} + \frac{\partial F^*}{\partial \bar{y}}\,d\bar{y}.$$

This integral and C^* are known as the Jacobian integral and the Jacobian constant after Jacobi.

Another form of Eq. (36) is obtained if we write $\bar{v}$ for the magnitude of the velocity relative to the rotating coordinate system and obtain

$$\bar{v}^2 = 2F^* - C^*. \tag{37}$$

Substituting F^* from (35) and writing $\bar{r}^2$ for $\bar{x}^2 + \bar{y}^2$ we obtain

$$\bar{v}^2 = n^2\bar{r}^2 + 2k^2\left(\frac{m_1}{\bar{r}_1} + \frac{m_2}{\bar{r}_2}\right) - C^*. \tag{38}$$

1.5 Equations of motion in dimensionless coordinates

(A) The equations of motion in the inertial system [Eqs. (7)] contain k^2, a, b, m_1, m_2, and n as physical parameters which are not all independent. It will now be shown by means of dimensionless variables that the restricted problem depends on only one parameter. For this purpose let Greek letters (excepting for the time) represent dimensionless quantities as follows:

$$\begin{gathered}\xi = X/l, \quad \eta = Y/l, \quad t = nt^*, \quad \mu_1 = m_1/M = a/l, \\ \mu_2 = m_2/M = b/l, \quad \rho_1 = R_1/l, \quad \rho_2 = R_2/l.\end{gathered} \tag{39}$$

The equations of motion (7) become

$$d^2\xi/dt^2 = \partial\phi/\partial\xi, \qquad d^2\eta/dt^2 = \partial\phi/\partial\eta, \tag{40}$$

where

$$\phi = F/l^2n^2 = \mu_1/\rho_1 + \mu_2/\rho_2 \tag{41}$$

and

$$\begin{aligned}\rho_1^2 &= (\xi - \mu_2\cos t)^2 + (\eta - \mu_2\sin t)^2, \\ \rho_2^2 &= (\xi + \mu_1\cos t)^2 + (\eta + \mu_1\sin t)^2.\end{aligned} \tag{42}$$

The variables occurring in the equations of motion are the dimensionless coordinates (ξ, η) and the dimensionless time t. The only remaining parameters (constants) are μ_1 and μ_2, but since $m_1/M + m_2/M = 1$ we have $\mu_1 + \mu_2 = 1$, i.e., given one of the dimensionless masses, the other is determined. We are therefore left with only one parameter (either μ_1 or μ_2), the selection of which determines the problem.

The dimensionless equations corresponding to Eqs. (7) are

$$\begin{aligned}\frac{d^2\xi}{dt^2} &= -\left[\mu_1\frac{(\xi - \mu_2\cos t)}{\rho_1^3} + \mu_2\frac{(\xi + \mu_1\cos t)}{\rho_2^3}\right], \\ \frac{d^2\eta}{dt^2} &= -\left[\mu_1\frac{(\eta - \mu_2\sin t)}{\rho_1^3} + \mu_2\frac{(\eta + \mu_1\sin t)}{\rho_2^3}\right].\end{aligned} \tag{43}$$

(B) Now we establish the equations of motion in the rotating coordinate system using dimensionless variables. We will observe that in this way we obtain the simplest form of the differential equation of motion.

Introducing

$$\begin{gathered}x = \bar{x}/l, \quad y = \bar{y}/l, \quad t = nt^*, \\ r_1 = \bar{r}_1/l, \quad r_2 = \bar{r}_2/l, \quad \mu_{1,2} = m_{1,2}/M,\end{gathered} \tag{44}$$

Eqs. (32) of Section 1.4 become

$$\begin{aligned}\ddot{x} - 2\dot{y} &= \bar{\Omega}_x, \\ \ddot{y} + 2\dot{x} &= \bar{\Omega}_y,\end{aligned} \tag{45}$$

where dots denote derivatives with respect to the dimensionless time (t) and subscripts signify partial derivatives. The function $\bar{\Omega}$ corresponds to the previously introduced function ϕ for the fixed coordinate system by Eq. (41), and to the dimensionless form of F given by Eqs. (35); i.e.,

$$\bar{\Omega} = \frac{F^*}{l^2n^2} \tag{46}$$

or

$$\bar{\Omega} = \tfrac{1}{2}(x^2 + y^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}, \tag{47}$$

where

$$\begin{aligned} r_1^2 &= (x - \mu_2)^2 + y^2, \\ r_2^2 &= (x + \mu_1)^2 + y^2. \end{aligned} \tag{48}$$

Equations (45) and (47), which define the problem in a synodic coordinate system, are widely used. A modification of $\bar{\Omega}$ by the addition of a constant will not affect the equations of motion and will offer a more symmetric form. Let

$$\Omega = \bar{\Omega} + \tfrac{1}{2}\mu_1\mu_2, \tag{49}$$

which results in

$$\Omega = \tfrac{1}{2}[\mu_1 r_1^2 + \mu_2 r_2^2] + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} \tag{50}$$

or

$$\Omega = \mu_1\left(\frac{r_1^2}{2} + \frac{1}{r_1}\right) + \mu_2\left(\frac{r_2^2}{2} + \frac{1}{r_2}\right). \tag{51}$$

The equations of motion are

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x, \\ \ddot{y} + 2\dot{x} &= \Omega_y. \end{aligned} \tag{52}$$

The Jacobian integral of Eqs. (45) is

$$\dot{x}^2 + \dot{y}^2 = 2\bar{\Omega} - \bar{C} \tag{53}$$

and, using Ω instead of $\bar{\Omega}$, the integral becomes

$$\dot{x}^2 + \dot{y}^2 = 2\Omega - C, \tag{54}$$

giving

$$C = \bar{C} + \mu_1\mu_2. \tag{55}$$

The Jacobian integral (54) in the dimensionless synodic system connects the dimensionless relative velocity with the position coordinates through the Jacobian constant.

The return to the fixed system (dimensionless sidereal) is effected by the transformations

$$\begin{aligned} \xi &= x\cos t - y\sin t, \\ \eta &= x\sin t + y\cos t, \end{aligned}$$

similar to Eqs. (29), or by

$$\zeta = ze^{it},$$

where now $\zeta = x + iy$.

The left side of the Jacobian integral (54) transforms into $(\dot{\xi} + \eta)^2 + (\dot{\eta} - \xi)^2$ and the right side becomes

$$\xi^2 + \eta^2 + \frac{2\mu_1}{\rho_1} + \frac{2\mu_2}{\rho_2} - \bar{C},$$

where ρ_1 and ρ_2 are given by Eqs. (42).

So the Jacobian integral in the sidereal system is

$$\dot{\xi}^2 + \dot{\eta}^2 = 2(\xi\dot{\eta} - \eta\dot{\xi}) + \frac{2\mu_1}{\rho_1} + \frac{2\mu_2}{\rho_2} - \bar{C}. \tag{56}$$

Example. Derive Eq. (56) directly from the equations of motion in the sidereal system.

(C) The parameters μ_1, μ_2 are connected by the relation $\mu_1 + \mu_2 = 1$. In the dimensionless system of variables, μ_1 and μ_2 represent the masses of the primaries whose total mass therefore is unity and whose distance is also unity since linear dimensions occurring in the equations were made dimensionless by dividing by the distance of the primaries l. The dimensionless time t is also a measure of the angle by which the rotating system rotated during the actual time t^*. Another physical interpretation of t is that it is the longitude of m_1 or of μ_1.

The dependent variables x and y refer to the rotating system; for instance, if they are constant the associated physical picture is that the third body is fixed relative to the rotating coordinate system, i.e., it is moving on a circle with constant velocity in the fixed system. The mean motion of the dimensionless rotating system itself is unity (its angular velocity is one) and the gravitational constant is $k^2 = 1$. It is often stated in the literature that Eqs. (52) *assume* unit distance between the primaries, unit total mass, unit mean motion, etc. This is a misleading statement, seeming to imply that the equations are not applicable when the distance between the primaries is say 240,000 miles. The simple fact is that Eqs. (52) contain dimensionless variables and parameters, which are related to physical quantities by Eq. (44). Figure 1.2 shows the dimensionless synodic system.

It has been mentioned before that Eqs. (51) and (52) contain one parameter only. In order to eliminate either μ_1 or μ_2, the relation $\mu_1 = 1 - \mu_2$ is used. Further, if $\mu_2 = \mu$, then $\mu_1 = 1 - \mu$. The value of the dimensionless mass μ_2 is non-negative and less than 1, therefore $0 \leqslant \mu \leqslant 1$. The values $\mu = 0$ and $\mu = 1$ correspond to a unit mass at the origin. The value $\mu = \frac{1}{2}$ corresponds to two equal masses located at

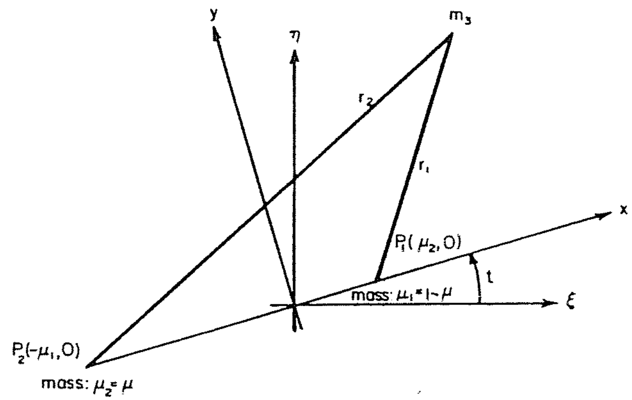

FIG. 1.2. Notation for the dimensionless synodic system (x, y) to be used throughout the text $(\mu_1 > \mu_2)$.

$x = \pm\frac{1}{2}$, both having dimensionless mass $\frac{1}{2}$. When $0 \leqslant \mu < \frac{1}{2}$ the larger mass is located at $0 \leqslant x < \frac{1}{2}$ and the smaller mass at $-1 \leqslant x < -\frac{1}{2}$, and when $\frac{1}{2} < \mu \leqslant 1$ the larger mass is at $-\frac{1}{2} < x \leqslant 0$ and the smaller at $\frac{1}{2} < x \leqslant 1$. The case $\mu = \frac{1}{2}$ corresponds to symmetry regarding the primaries. If μ appears as a small parameter ($\mu \sim 0$, in fact $\mu < \frac{1}{2}$), the larger mass will be to the right of the origin. Selecting $\mu_1 = \mu$ results in the opposite situation; i.e., for $\mu < \frac{1}{2}$ the larger mass will be located to the left of the origin. The literature is divided regarding the location of the larger mass, and the choice of convention to be followed is not important especially since Eqs. (51) and (52) are given in their generality. The case $\mu_2 = \mu$ as shown in Fig. 1.2 puts the larger mass ($\mu_1 = 1 - \mu$) for $\mu < \frac{1}{2}$ to the right of the center of mass and the selection $\mu_1 = \mu$ locates the smaller mass (again for $\mu < \frac{1}{2}$) to the right. Further notational variation in the literature offers two possibilities in naming the distances r_1 and r_2, and the reader's attention is called to these, altogether four combinations, i.e., the location of the larger mass and the naming of the distances (cf. Section 9.3 for details).

The equations of motion will not be affected by these choices in the form they are given by Eqs. (51) and (52).

Placing the larger mass to the right results in

$$\Omega = \tfrac{1}{2}[(1-\mu)r_1^2 + \mu r_2^2] + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}, \tag{57}$$

and if the larger mass is placed on the left side of the origin,

$$\Omega = \tfrac{1}{2}[\mu r_1^2 + (1-\mu)r_2^2] + \frac{\mu}{r_1} + \frac{1-\mu}{r_2}. \tag{58}$$

The transition from Eq. (57) to (58) is obtained by writing μ for $1-\mu$ and $1-\mu$ for μ.

Note that only the range $0 \leqslant \mu \leqslant \frac{1}{2}$ is of interest since the range $\frac{1}{2} \leqslant \mu \leqslant 1$ is the reflection of the previous one with respect to the y axis.

1.6 Summary of the equations of motion

The equations of motion in the inertial or fixed (or sidereal) coordinate system X, Y, using dimensional quantities X, Y, R_1, R_2, t^*, k^2, m_1, m_2, a, and b, are

$$\begin{aligned} \frac{d^2X}{dt^{*2}} &= -k^2\left[m_1\frac{(X - b\cos nt^*)}{R_1^3} + m_2\frac{(X + a\cos nt^*)}{R_2^3}\right], \\ \frac{d^2Y}{dt^{*2}} &= -k^2\left[m_1\frac{(Y - b\sin nt^*)}{R_1^3} + m_2\frac{(Y + a\sin nt^*)}{R_2^3}\right], \end{aligned} \tag{7}$$

where the meaning of the symbols is shown in Fig. 1.1.

The dimensionless form of these equations is

$$\begin{aligned} \frac{d^2\xi}{dt^2} &= -\left[\mu_1\frac{(\xi - \mu_2\cos t)}{\rho_1^3} + \mu_2\frac{(\xi + \mu_1\cos t)}{\rho_2^3}\right], \\ \frac{d^2\eta}{dt^2} &= -\left[\mu_1\frac{(\eta - \mu_2\sin t)}{\rho_1^3} + \mu_2\frac{(\eta + \mu_1\sin t)}{\rho_2^3}\right], \end{aligned} \tag{43}$$

where ξ, η correspond to X, Y; μ_1, μ_2 to m_1, m_2; ρ_1, ρ_2 to R_1, R_2; and t to t^*.

The equations of motion in the rotating (synodic) coordinate system, using dimensional quantities, are

$$\begin{aligned} \frac{d^2\bar{x}}{dt^{*2}} - 2n\frac{d\bar{y}}{dt^*} - n^2\bar{x} &= -k^2\left[m_1\frac{(\bar{x} - b)}{\bar{r}_1^3} + m_2\frac{(\bar{x} + a)}{\bar{r}_2^3}\right], \\ \frac{d^2\bar{y}}{dt^{*2}} + 2n\frac{d\bar{x}}{dt^*} - n^2\bar{y} &= -k^2\left[\frac{m_1\bar{y}}{\bar{r}_1^3} + \frac{m_2\bar{y}}{\bar{r}_2^3}\right], \end{aligned} \tag{32}$$

where Fig. 1.2 shows the notation.

The nondimensional form of these equations is

$$\begin{aligned} \frac{d^2x}{dt^2} - 2\frac{dy}{dt} &= -\left[\frac{\mu_1(x - \mu_2)}{r_1^3} + \frac{\mu_2(x + \mu_1)}{r_2^3}\right] + x, \\ \frac{d^2y}{dt^2} + 2\frac{dx}{dt} &= -\left[\frac{\mu_1 y}{r_1^3} + \frac{\mu_2 y}{r_2^3}\right] + y, \end{aligned} \tag{59}$$

where x, y corresponds to $\bar{x}$, $\bar{y}$ and r_1, r_2 to $\bar{r}_1$, $\bar{r}_2$.

The Jacobian integral in the dimensional and dimensionless systems is given by

$$\left(\frac{d\bar{x}}{dt^*}\right)^2 + \left(\frac{d\bar{y}}{dt^*}\right)^2 = n^2(\bar{x}^2 + \bar{y}^2) + 2k^2\left(\frac{m_1}{\bar{r}_1} + \frac{m_2}{\bar{r}_2}\right) - C^* \tag{38}$$

and

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \mu_1 r_1^2 + \mu_2 r_2^2 + \frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2} - C. \tag{54}$$

Selecting $\mu_2 = \mu$, the equations of motion in the synodic system, using dimensionless variables, become

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x, \\ \ddot{y} + 2\dot{x} &= \Omega_y, \end{aligned} \tag{52}$$

where

$$\Omega = \tfrac{1}{2}[(1-\mu)r_1^2 + \mu r_2^2] + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}, \tag{57}$$

$$r_1^2 = (x-\mu)^2 + y^2,$$

$$r_2^2 = (x+1-\mu)^2 + y^2.$$

The Jacobian integral becomes

$$\dot{x}^2 + \dot{y}^2 = 2\Omega - C. \tag{54}$$

1.7 A method of finding the potential

The reader will find the following short deviation from the main stream of development useful in several chapters.

Consider the problem of finding a function $\phi(x_1, x_2, x_3)$ which should satisfy the equations

$$\begin{aligned} \frac{\partial\phi}{\partial x_1} &= F_1(x_1, x_2, x_3), \\ \frac{\partial\phi}{\partial x_2} &= F_2(x_1, x_2, x_3), \\ \frac{\partial\phi}{\partial x_3} &= F_3(x_1, x_2, x_3). \end{aligned} \tag{60}$$

First we establish a condition for the existence of ϕ by the simple consideration that

$$\frac{\partial^2\phi}{\partial x_i\,\partial x_j} = \frac{\partial^2\phi}{\partial x_j\,\partial x_i}, \tag{61}$$

or

$$\frac{\partial F_1}{\partial x_2} = \frac{\partial F_2}{\partial x_1}, \qquad \frac{\partial F_2}{\partial x_3} = \frac{\partial F_3}{\partial x_2}, \qquad \frac{\partial F_3}{\partial x_1} = \frac{\partial F_1}{\partial x_3}. \tag{62}$$

This condition may also be stated by saying that the field with the components F_1, F_2, F_3 is irrotational, or curl $\mathbf{F} = 0$, where the vector $\mathbf{F}$ has the components F_1, F_2, F_3.

If the existence of ϕ is established, it may be determined by several methods, one of which will be described shortly since it suits the problem at hand. From the first of Eqs. (60),

$$\phi = \int F_1(x_1, x_2, x_3)\,dx_1 + \psi(x_2, x_3), \tag{63}$$

where the integration is to be performed treating x_2 and x_3 as constants. Equation (63) will now give a function ϕ which will satisfy the first of Eqs. (60) with an arbitrary function ψ. Substituting Eq. (63) into the second and third of Eqs. (60) gives

$$\begin{aligned} \frac{\partial}{\partial x_2}\int F_1\,dx_1 + \frac{\partial}{\partial x_2}\psi &= F_2, \\ \frac{\partial}{\partial x_3}\int F_1\,dx_1 + \frac{\partial}{\partial x_3}\psi &= F_3, \end{aligned} \tag{64}$$

which can also be written as

$$\begin{aligned} \frac{\partial\psi}{\partial x_2} &= F_2 - \frac{\partial}{\partial x_2}\int F_1\,dx_1 = G_2(x_2, x_3), \\ \frac{\partial\psi}{\partial x_3} &= F_3 - \frac{\partial}{\partial x_3}\int F_1\,dx_1 = G_3(x_2, x_3). \end{aligned} \tag{65}$$

Note that

$$\partial G_2/\partial x_1 = \partial G_3/\partial x_1 = 0.$$

This follows from the fact that the function $\psi(x_2, x_3)$ is independent of x_1 and also directly from Eqs. (65). Taking partial derivatives of the last parts of Eqs. (65) with respect to x_1 we have

$$\frac{\partial G_2}{\partial x_1} = \frac{\partial F_2}{\partial x_1} - \frac{\partial^2}{\partial x_1\,\partial x_2}\int F_1\,dx_1 = \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} = 0$$

and

$$\frac{\partial G_3}{\partial x_1} = \frac{\partial F_3}{\partial x_1} - \frac{\partial^2}{\partial x_1\,\partial x_3}\int F_1\,dx_1 = \frac{\partial F_3}{\partial x_1} - \frac{\partial F_1}{\partial x_3} = 0.$$

The problem of finding the function ψ is the original problem reduced by one degree. From the first of Eqs. (65)

$$\psi = \int G_2\, dx_2 + \chi(x_3), \tag{66}$$

where the integration is performed by treating x_3 as a constant. The arbitrary function $\chi(x_3)$ is obtained by substituting (66) into the second of Eqs. (65):

$$\frac{\partial}{\partial x_3} \int G_2\, dx_2 + \frac{d\chi}{dx_3} = G_3 .$$

Solution of (66) for $\chi(x_3)$ is possible since

$$G_3 - \frac{\partial}{\partial x_3} \int G_2\, dx_2$$

depends on x_3 only.

This is shown by evaluating partial derivatives of the preceding expression with respect to x_1 and x_2. First,

$$\frac{\partial G_3}{\partial x_1} - \frac{\partial^2}{\partial x_1\, \partial x_3} \int G_2\, dx_2 = 0,$$

since G_2 and G_3 depend on x_2 and x_3 only as was shown above. Second,

$$\frac{\partial G_3}{\partial x_2} - \frac{\partial^2}{\partial x_2\, \partial x_3} \int G_2\, dx_2 = \frac{\partial G_3}{\partial x_2} - \frac{\partial G_2}{\partial x_3} = 0,$$

since according to Eqs. (65)

$$\frac{\partial G_3}{\partial x_2} - \frac{\partial G_2}{\partial x_3} = \frac{\partial^2 \psi}{\partial x_2\, \partial x_3} - \frac{\partial^2 \psi}{\partial x_3\, \partial x_2} = 0.$$

The function $\psi(x_2, x_3)$ is given by Eq. (66) and ϕ by (63). In fact the reader can perform the substitutions and arrive at a rather impressive formula for the direct computation of the desired function $\phi(x_1, x_2, x_3)$.

Often the function ϕ can be written by inspection, especially in problems in which Newtonian gravitational fields are acting. Remembering that the gradient of the potential of the gravitational field is

$$\operatorname{grad} \frac{1}{|\mathbf{r}|} = - \frac{\mathbf{r}}{|\mathbf{r}|^3}$$

we have

$$F_1 = - \frac{x}{|\mathbf{r}|^3}, \qquad F_2 = - \frac{y}{|\mathbf{r}|^3}, \qquad F_3 = - \frac{z}{|\mathbf{r}|^3},$$

with

$$|\mathbf{r}|^2 = x^2 + y^2 + z^2, \qquad \phi = \frac{1}{|\mathbf{r}|}.$$

With any of the preceding methods, the solution of the problem mentioned in Section 1.4 and defined by Eqs. (34) is

$$F^* = \frac{n^2}{2}(\bar{x}^2 + \bar{y}^2) + k^2 \left[\frac{m_1}{\bar{r}_1} + \frac{m_2}{\bar{r}_2}\right]. \tag{35}$$

1.8 Relation to the general problem of three bodies

The approximately circular motion of the planets around the sun and the small masses of the asteroids and the satellites of the planets compared to the planets' masses originally suggested the formulation of the restricted problem.

Section 1.3, Part (C), dealt with the questions regarding the total energy of the three bodies involved in the restricted problem, and it was shown that the energy is not conserved. On the other hand it is well known that the general problem of three bodies possesses an energy integral and that the total energy is conserved. It is, therefore, in the transition from the general to the restricted problem that the principle of energy conservation is lost, and a further examination of this transition is made in what follows. As a rather important side-product, the conditions to be satisfied in order that the methods of the restricted problem be applicable will be given. The general problem of three bodies presents one of the very few cases in the physical sciences when the applicability of a higher-order approach or a more complex physical model is more limited than a simpler proposition. The standard procedure is just the opposite and it is usually expected that a complicated model gives a better approximation to the actual physical situation than a simpler one. The general problem of three bodies has only a few applications in celestial mechanics (one is the dynamics of multiple-star systems) and only a very few in space dynamics and in solar-system dynamics. In addition the existing knowledge about the general problem is considerably less than about the restricted problem. In those problems where the restrictions of the "proper" restricted problem are found to be so severe as to prevent applications, modifications and generalizations of the restricted problem in most cases suffice, and transition to the general problem is not required.

The following short treatment of the general problem is offered for the purpose of showing its relation to the restricted problem, and not to attempt to give a comprehensive treatment of this problem, which seems to be more complex and less fruitful than the restricted problem.

The recognition of this fact came probably first with Euler (1772). Prior to him, publications about the general problem were more numerous than after him, though even today new and significant papers appear on the general problem.

The problem of three bodies is defined as follows: three particles with arbitrary masses attract each other according to the Newtonian law of gravitation; they are free to move in space and are initially moving in any given manner; find their subsequent motion.

The difference between this and the restricted problem is first of all that in the latter the masses of only two particles are arbitrary; the third mass is much smaller than the other two. The general problem allows any sets of initial conditions for the three particles involved; the restricted problem requires circular orbits for the primaries.

Figure 1.3 shows the conventional notation. The masses of the three bodies are m_1, m_2, m_3 and their position vectors are

$$\mathbf{r}_1(q_1, q_2, q_3), \qquad \mathbf{r}_2(q_4, q_5, q_6), \qquad \mathbf{r}_3(q_7, q_8, q_9).$$

The vectors connecting the mass points are

$$\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2, \qquad \mathbf{r}_{23} = \mathbf{r}_2 - \mathbf{r}_3, \qquad \mathbf{r}_{31} = \mathbf{r}_3 - \mathbf{r}_1, \tag{67}$$

and the distances between m_1, m_2, and m_3 are

$$\begin{aligned} |\mathbf{r}_{12}| &= [(q_1 - q_4)^2 + (q_2 - q_5)^2 + (q_3 - q_6)^2]^{1/2}, \\ |\mathbf{r}_{23}| &= [(q_4 - q_7)^2 + (q_5 - q_8)^2 + (q_6 - q_9)^2]^{1/2}, \\ |\mathbf{r}_{31}| &= [(q_7 - q_1)^2 + (q_8 - q_2)^2 + (q_9 - q_3)^2]^{1/2}. \end{aligned} \tag{68}$$

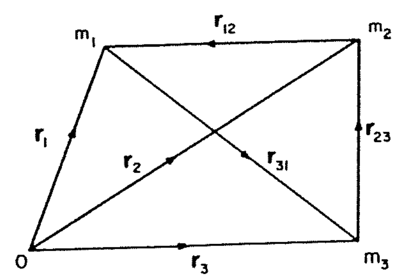

FIG. 1.3. The general problem of three bodies: the masses are m_1, m_2, and m_3; the corresponding position vectors are $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$.

The force function is

$$F = k^2\left(\frac{m_1 m_2}{|\mathbf{r}_{12}|} + \frac{m_2 m_3}{|\mathbf{r}_{23}|} + \frac{m_3 m_1}{|\mathbf{r}_{31}|}\right) \tag{69}$$

and the equations of motion are

$$m_i \ddot{\mathbf{r}}_i = \partial F/\partial \mathbf{r}_i, \qquad i = 1, 2, 3. \tag{70}$$

The system consists of three second-order differential equations for the vectors $\mathbf{r}_i$ or of nine second-order equations for the coordinates q_i. The resulting eighteenth-order system as compared to the fourth-order system describing the restricted problem explains the inherently more complex nature of the general problem in three dimensions.

Equations (70) in explicit form become

$$\begin{aligned} \ddot{\mathbf{r}}_1 &= -k^2 m_2 \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} + k^2 m_3 \frac{\mathbf{r}_3 - \mathbf{r}_1}{|\mathbf{r}_3 - \mathbf{r}_1|^3}, \\ \ddot{\mathbf{r}}_2 &= -k^2 m_3 \frac{\mathbf{r}_2 - \mathbf{r}_3}{|\mathbf{r}_2 - \mathbf{r}_3|^3} + k^2 m_1 \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}, \\ \ddot{\mathbf{r}}_3 &= -k^2 m_1 \frac{\mathbf{r}_3 - \mathbf{r}_1}{|\mathbf{r}_3 - \mathbf{r}_1|^3} + k^2 m_2 \frac{\mathbf{r}_2 - \mathbf{r}_3}{|\mathbf{r}_2 - \mathbf{r}_3|^3}. \end{aligned} \tag{71}$$

Example. Derive the equations of motion of the general problem of three bodies in vectorial form by computing the gradients indicated in Eq. (70). Note that $\partial|\mathbf{r}|/\partial\mathbf{r} = \mathbf{r}/|\mathbf{r}|$.

The eighteenth-order system is reducible to a sixth-order system and the not-at-all trivial execution of such reductions is one of the major subjects of the classical literature on the general problem, the best example of which was given by Lagrange in 1772.

(1) Since no external forces are acting on the system the center of mass will move on a straight line with constant velocity, i.e.,

$$\sum_{i=1}^{3} m_i \dot{\mathbf{r}}_i = \mathbf{a} \qquad \text{and} \qquad \sum_{i=1}^{3} m_i \mathbf{r}_i = \mathbf{a}t + \mathbf{b}. \tag{72}$$

These two vector equations, corresponding to six scalar equations with six scalar constants of integration (i.e., the coordinates of $\mathbf{a}$ and $\mathbf{b}$), represent six integrals with which the system may be reduced from the eighteenth to the twelfth order.

(2) The conservation of angular momentum may be written as

$$\sum_{i=1}^{3} \mathbf{r}_i \times m_i \dot{\mathbf{r}}_i = \mathbf{c}. \tag{73}$$

When the corresponding three scalar integrals (with the components of $\mathbf{c}$ as constants of integration) are used in conjunction with the introduction of an ignorable coordinate, the system is reduced from the twelfth to the eighth order.

(3) Further reduction is possible by two by using the integral of energy and eliminating the time.

The energy integral is obtained from Eqs. (70) by the usual method, i.e., multiplication by $\dot{\mathbf{r}}_i$, summation, and integration:

$$\sum_{i=1}^{3} m_i \int \dot{\mathbf{r}}_i \ddot{\mathbf{r}}_i \, dt = \sum_{i=1}^{3} \int \frac{\partial F}{\partial \mathbf{r}_i} \dot{\mathbf{r}}_i \, dt \tag{74}$$

or

$$m_1 \dot{\mathbf{r}}_1^2 + m_2 \dot{\mathbf{r}}_2^2 + m_3 \dot{\mathbf{r}}_3^2 = 2F - C, \tag{75}$$

showing that the principle of conservation of the energy applies here.

At this point it may be shown how the general problem is reduced to the restricted problem.

Equations (71) describe the general case of the problem of three bodies with Newtonian gravitational forces. The structure of these equations is of some interest inasmuch as the masses m_1, m_2, and m_3 are missing from the first, second, and third equations, respectively. This fact does not "uncouple" the equations since all three position vectors occur in all three equations.

The terms appearing on the right-hand side have specific physical significance. The first term on the right side of the first equation, for instance, represents the force per unit mass acting on the first body due to the presence of the second mass. The second term of the right side of the same equation is the effect of the third mass m_3 on m_1.

Decreasing the mass of the third body will reduce its influence on the motion of m_1 and of m_2; i.e., as $m_3 \to 0$ the first two equations of (71) approach (with $k^2 = 1$)

$$\ddot{\mathbf{r}}_1 = -m_2 \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}, \qquad \ddot{\mathbf{r}}_2 = -m_1 \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3}, \tag{76}$$

while the third equation does not change. This step does uncouple the equations since the motion of m_1 and of m_2 can now be determined, without considering the effect of the third mass, by solving the twelfth-order system of differential equations (76).

The third equation of (71) requires comment since if in the first two equations $m_3 = 0$ [which gives Eqs. (76)], then the third equation reduces to $0 = 0$ because in this case, $m_3 = 0$, no division by m_3 is allowed. Herein lies the approximation which creates the restricted problem of three bodies. The assumption is made that $m_3 \neq 0$ but that m_3 is sufficiently small so that it does not affect the motion of m_1 and of m_2. That is, Eqs. (76) are approximate while the third equation of (71) is exact. The system of equations consisting of Eqs. (76) and of the third equation of (71) represents our dynamical system only approximately; the degree of approximation is given by the "smallness" of the terms:

$$m_3 \frac{\mathbf{r}_3 - \mathbf{r}_1}{|\mathbf{r}_3 - \mathbf{r}_1|^3} \quad \text{and} \quad m_3 \frac{\mathbf{r}_2 - \mathbf{r}_3}{|\mathbf{r}_2 - \mathbf{r}_3|^3}$$

as compared to the terms

$$m_2 \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \quad \text{and} \quad m_1 \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3},$$

respectively.

The effect of m_1 and m_2 on the motion of m_3 is given by the third equation of (71). Accepting the above-mentioned approximation one might solve Eqs. (76), substitute the solution into the third equation of (71), and obtain the sixth-order differential equation for the motion of the third body:

$$\ddot{\mathbf{r}}_3 = -m_1 \frac{\mathbf{r}_3 - \mathbf{r}_1}{|\mathbf{r}_3 - \mathbf{r}_1|^3} + m_2 \frac{\mathbf{r}_2 - \mathbf{r}_3}{|\mathbf{r}_2 - \mathbf{r}_3|^3}, \tag{77}$$

where now $\mathbf{r}_1$ and $\mathbf{r}_2$ are given functions of the time and of the initial conditions.

Equation (77) describes the restricted problem of three bodies. The "restriction" is equivalent to the assumption that the motion of m_1 and m_2 is not influenced by m_3, while the motion of m_3 is determined by the masses and by the motion of m_1 and m_2.

1.9 Classification and modifications of the restricted problem

The last result in Section 1.8 allows an orderly presentation of the various generalizations of the restricted problem. Equation (77) may be written as

$$\ddot{\mathbf{r}}_3 = \mathbf{f}(m_1, m_2, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3), \tag{78}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are given functions of time and of their initial conditions since the original three equations have been uncoupled. The function $\mathbf{f}$ represents the force field which in Eq. (77) is the Newtonian gravitational field. From this point of view the most general form of the restricted problem is simply

$$\ddot{\mathbf{r}} = \mathbf{g}(\mathbf{r}, t), \tag{79}$$

i.e., the problem is the determination of the motion of a body with position vector $\mathbf{r}$ in a field $\mathbf{g}$ depending on the position of the body and on the time. Equation (79) describes a system of two or three degrees of freedom depending on two- or three-dimensional motion of the body.

The problem introduced as the restricted problem of three bodies in Section 1.2 specifies circular motion for m_1 and m_2 and also requires that the motion of m_3 take place in the plane defined by the motion of m_1 and m_2.

The following modifications of this basic problem are of interest:

(i) The motion of m_1 and m_2 is not circular. If Newtonian gravitational forces are involved the two-body problem of m_1 and m_2, depending on their initial conditions, may have any conic section as its solution. The most important case is when m_1 and m_2 move in elliptic orbits. This "elliptic restricted problem"—also called the "pseudo restricted problem"—has many applications and will be discussed in considerable detail in Section 10.3.

(ii) The three-dimensional restricted problem is concerned with a motion of m_3 that does not take place in the plane of motion of m_1 and m_2. This problem appears when the initial conditions of the third body are such that the body initially is not in the plane of motion of m_1 and m_2, or when its initial velocity vector has a component not in this plane. Within the framework of Newtonian gravitational forces, if the initial position and velocity vectors of the third body are in the plane of motion of m_1 and m_2 then the motion of the third body will be confined to this plane since there are no forces taking it out of this plane. The applications of an essentially three-dimensional motion of the third body in celestial mechanics occur in the study of the orbits of some minor planets with large inclination to the ecliptic. Applications to cosmogony and to the space sciences are also of importance and are discussed in Section 10.2.

(iii) The value of the mass ratio m_1/m_2 and the previously introduced parameter μ connected with it have important effects on the motion of the third body as well as on the approach to the problem. The value $\mu = 0$ changes the restricted problem to the two-body problem, so problems with small values of μ appear as perturbation problems of the two-body problem. On the other hand the problem with $\mu = \frac{1}{2}$ in general is essentially a three-body problem. Specification of μ is therefore important, so much so that, for instance, the case $\mu = \frac{1}{2}$ is distinguished by a special name and it is generally known as the Copenhagen problem. Poincaré's restricted problem assumes a small value of μ so that perturbation theories become applicable.

(iv) If masses of the participating primaries vary with time, the basic equations of motion are to be significantly modified but the basic idea of the restricted problem is still useful in stellar dynamics (close binaries) and cosmogony.

(v) If forces that are not central are involved in the problem, the important question is whether a circular or conic-section motion of the primaries is still meaningful or not. If so, then the modified field becomes significant only when the motion of the third body is studied. This is rather important in the early parts of certain earth-to-moon trajectories, which may be significantly influenced by the higher-order gravitational harmonics of the earth.

The expression "restricted problem" will be used for the circular, planar, Newtonian case, provided $\mu \neq 0$ or $\neq 1$, and all modifications will be described by their prominent features; for instance, elliptic restricted problem, three-dimensional restricted problem, etc. Chapter 10 is devoted to the modifications of the restricted problem.

1.10 Applications

Three simple applications are presented.

The basic assumption made in the derivation of the equations of motion of the restricted problem will first be examined in connection with two important applications. This assumption is that the mass of the third particle is so small in comparison to the masses of the primaries that it does not affect their motion. The first application will be to the motion of the moon [Part (A)], the second to the motion of a space probe in the earth-moon environment [Part (B)]. The third application deals with the Jacobian integral [Part (C)].

(A) The three masses in the sun-earth-moon system are related by the mass ratios $m_\odot : m_\oplus : m_☾ = 2 \times 10^{33}\,\mathrm{g} : 6 \times 10^{27}\,\mathrm{g} : 7.4 \times 10^{25}\,\mathrm{g}$ or approximately $m_\odot : m_\oplus : m_☾ = 300{,}000 : 1 : 0.01$. The relative distances are $d_{\odot\oplus} : d_{\oplus☾} = 149.6 \times 10^{11}\,\mathrm{cm} : 3.844 \times 10^{10}\,\mathrm{cm}$ or approximately $390 : 1$, and of course $d_{\odot\oplus} \cong d_{\odot☾}$. The equations of motion for the general problem of three bodies are given by (71) and are reproduced here with $m_1 = m_\odot$, $m_2 = m_\oplus$, $m_3 = m_{\leftmoon}$, $|\mathbf{r}_{23}| = d_{\oplus\leftmoon}$, $|\mathbf{r}_{31}| = d_{\odot\leftmoon}$ and $|\mathbf{r}_{12}| = d_{\odot\oplus}$:

$$
\begin{aligned}
m_\odot \ddot{\mathbf{r}}_\odot &= -k^2 \left[m_\odot m_\oplus \frac{\mathbf{r}_\odot - \mathbf{r}_\oplus}{d^3_{\odot\oplus}} + m_\odot m_{\leftmoon} \frac{\mathbf{r}_{\leftmoon} - \mathbf{r}_\odot}{d^3_{\odot\leftmoon}} \right], \\
m_\oplus \ddot{\mathbf{r}}_\oplus &= -k^2 \left[m_\oplus m_{\leftmoon} \frac{\mathbf{r}_\oplus - \mathbf{r}_{\leftmoon}}{d^3_{\oplus\leftmoon}} + m_\oplus m_\odot \frac{\mathbf{r}_\odot - \mathbf{r}_\oplus}{d^3_{\odot\oplus}} \right], \qquad (80) \\
m_{\leftmoon} \ddot{\mathbf{r}}_{\leftmoon} &= -k^2 \left[m_{\leftmoon} m_\odot \frac{\mathbf{r}_{\leftmoon} - \mathbf{r}_\odot}{d^3_{\odot\leftmoon}} + m_{\leftmoon} m_\oplus \frac{\mathbf{r}_\oplus - \mathbf{r}_{\leftmoon}}{d^3_{\oplus\leftmoon}} \right].
\end{aligned}
$$

First, an order-of-magnitude evaluation is made to show the effect of the earth on the motion of the sun as compared to the effect of the moon on the sun. This requires the comparison of the terms

$$m_\odot m_\oplus \frac{\mathbf{r}_\odot - \mathbf{r}_\oplus}{d^3_{\odot\oplus}} \quad \text{and} \quad m_\odot m_{\leftmoon} \frac{\mathbf{r}_{\leftmoon} - \mathbf{r}_\odot}{d^3_{\odot\leftmoon}}$$

or of

$$m_\oplus / d^2_{\odot\oplus} \quad \text{and} \quad m_{\leftmoon} / d^2_{\odot\oplus} .$$

The effect of the earth (first term) is 100 times the effect of the moon (second term) since $m_\oplus : m_{\leftmoon} \cong 100$. So omitting the term $m_\odot m_{\leftmoon} (\mathbf{r}_{\leftmoon} - \mathbf{r}_\odot)/d^3_{\odot\leftmoon}$ from the first equation of Eqs. (80) is equivalent to a 1% error in formulating the equations of motion. The second equation of (80) is modified by omitting the first term on the right. Compared with the second term this gives

$$\frac{m_{\leftmoon}}{m_\odot} \left(\frac{d_{\odot\oplus}}{d_{\oplus\leftmoon}} \right)^2 \cong 0.005,$$

i.e., the effect of the moon on the motion of the earth is 5×10^{-3} times the effect of the sun on the motion of the earth.

So the results of the order-of-magnitude evaluations of the terms on the right sides in the first two equations may be written as

$$
\begin{aligned}
&O(1) + O(0.01), \\
&O(0.01) + O(1), \qquad (81)
\end{aligned}
$$

where 0.005 became $O(0.01)$.

The terms on the right side of the last equation of Eqs. (80) correspond to the effect of the sun and of the earth on the motion of the moon. The ratio of these terms gives

$$\frac{m_\odot}{m_\oplus} \left(\frac{d_{\oplus\leftmoon}}{d_{\odot\leftmoon}} \right)^2 \cong 2,$$

showing that the motion of the moon is governed by the sun, rather than by the earth—a favorite question of examiners, but not pertinent to the present argument. What is important is that the effect of the third body (the moon) on the sun-earth system is two orders of magnitude smaller than the two-body force governing the motion of the sun-earth system. The requirement of circular motion of the earth-sun system is satisfied also approximately since the eccentricity of the earth's orbit is only 0.017. The orbit of the third body is not exactly in the plane of the motion of m_1 and m_2; in fact, the inclination of the moon's orbit to the plane of the ecliptic is $\cong 5°$. So application of the model of the restricted problem to the motion of the moon is valid with the following approximations:

(i) the motion of the third body is two dimensional, though in fact there is a $\pm 5°$ deviation;

(ii) the motion of the earth is circular, though in fact there is an eccentricity of less than 2 %;

(iii) the effect of the moon on the sun-earth system is zero, though in fact there is an effect of 1 %.

Application of the restricted problem to the motion of the moon forms the basis of several lunar theories. Those effects which the restricted problem does not include are handled as perturbations of the solution of the restricted problem proper.

(B) The second application is pertinent to the motion of a space probe in the earth-moon region. The three bodies considered are $m_1 = m_\oplus$, $m_2 = m_{\leftmoon}$, $m_3 = m_p$, and assuming a 6000-kg space probe, $m_\oplus : m_{\leftmoon} : m_p \cong 100 : 1 : 10^{-19}$. The pertinent distances are $d_{\oplus\leftmoon}$, $d_{p\oplus} \geqslant R_\oplus$, $d_{p\leftmoon} \geqslant R_{\leftmoon}$, where $d_{\oplus\leftmoon} : R_\oplus : R_{\leftmoon} \cong 400 : 6.38 : 1.74$, $R_\oplus$ and $R_{\leftmoon}$ being the radii of the earth and of the moon. The inequalities restrict the motion to the region exterior to the surface of the earth and moon.

The three equations of motion may again be examined. The first equation, giving now the acceleration of the earth, contains on the right side the effect of the moon,

$$m_\oplus m_{\leftmoon} / d^2_{\oplus\leftmoon} ,$$

and the effect of the space probe,

$$m_\oplus m_p / d^2_{\oplus p} .$$

The ratio of these forces is

$$2.5 \times 10^{15} \leqslant \frac{m_{\leftmoon}}{m_p} \left(\frac{d_{\oplus p}}{d_{\oplus\leftmoon}} \right)^2 < \infty,$$

where the lower bound was found by using for $d_{\oplus p}$ its minimum value, $R_{\oplus}$. The upper bound results from the situation when the probe departs indefinitely from the earth-moon system; its distance from the earth, $d_{\oplus p}$, approaches infinity. It is concluded that the effect of the probe as compared to the effect of the moon on the earth is between 4×10^{-16} and zero and it does not become larger than 4×10^{-16} times the lunar effect.

The second equation in this application gives the acceleration of the moon. The right side contains the effect of the probe,

$$m_{\leftmoon} m_p/(d_{\leftmoon p})^2,$$

and the effect of the earth,

$$m_{\leftmoon} m_{\oplus}/(d_{\oplus \leftmoon})^2.$$

The ratio of the forces is

$$2 \times 10^{16} \leqslant \frac{m_{\oplus}}{m_p}\left(\frac{d_{\leftmoon p}}{d_{\oplus \leftmoon}}\right)^2 < \infty,$$

where the lower bound results from $(d_{\leftmoon p})_{\min} = R_{\leftmoon}$. The force exerted by the probe on the moon therefore varies between zero and 0.5×10^{-16} times the effect of the earth on the moon.

The right sides of the first two equations of motion give the order-of-magnitude evaluation:

$$\begin{aligned} &O(1) + O(10^{-16}), \\ &O(10^{-16}) + O(1). \end{aligned} \tag{82}$$

The third equation of motion refers to the space probe:

$$m_p \ddot{\mathbf{r}}_p = -k^2 \left[m_p m_{\oplus} \frac{\mathbf{r}_p - \mathbf{r}_{\oplus}}{d^3_{p\oplus}} + m_p m_{\leftmoon} \frac{\mathbf{r}_{\leftmoon} - \mathbf{r}_p}{d^3_{p\leftmoon}} \right].$$

The effect of the earth on the probe is proportional to the first term on the right side, i.e., to $m_{\oplus}/d^2_{p\oplus}$. The effect of the moon is given by the second term, $m_{\leftmoon}/d^2_{p\leftmoon}$. The ratio of these forces is $(m_{\oplus}/m_{\leftmoon})(d_{p\leftmoon}/d_{p\oplus})^2$.

The maximum of this ratio corresponds to the case when the probe is in the vicinity of the earth, i.e., when $d_{p\leftmoon} = (d_{p\leftmoon})_{\max} \cong d_{\oplus\leftmoon}$ and at the same time $d_{p\oplus} = (d_{p\oplus})_{\min} = R_{\oplus}$. So

$$\max \frac{m_{\oplus}}{m_{\leftmoon}}\left(\frac{d_{p\leftmoon}}{d_{p\oplus}}\right)^2 = \frac{m_{\oplus}}{m_{\leftmoon}}\left(\frac{d_{\oplus\leftmoon}}{R_{\oplus}}\right)^2 = 4 \times 10^5.$$

The minimum of the force ratio occurs when the probe is on the surface of the moon, having $d_{p\leftmoon} = (d_{p\leftmoon})_{\min} = R_{\leftmoon}$ and $d_{p\oplus} = (d_{p\oplus})_{\max} \cong d_{\oplus\leftmoon}$, or

$$\min \frac{m_{\oplus}}{m_{\leftmoon}}\left(\frac{d_{p\leftmoon}}{d_{p\oplus}}\right)^2 = \frac{m_{\oplus}}{m_{\leftmoon}}\left(\frac{R_{\leftmoon}}{d_{\oplus\leftmoon}}\right)^2 = 2 \times 10^{-3}.$$

Therefore, the variation of the ratio is approximately from 10^5 to 10^{-3}, while the probe travels from the earth to the moon. In other words, the effect of the earth on the probe is 4×10^5 times that of the moon when the probe is in the vicinity of the earth and the effect of the moon is $\frac{1}{2} \times 10^3$ times that of the earth when the probe is close to the moon. While this is an interesting result, at this point it is more significant to observe Eqs. (82), which show that neglecting the probe's effect amounts to an error of 10^{-16}. Note that the corresponding number for the moon in the sun-earth-moon system is 10^{-2}, as given in Eqs. (81).

The calculation shows that the basic assumptions of the restricted problem are very nearly satisfied in problems associated with space dynamics. In fact, questions pertinent to space dynamics always belong to the category of restricted problems in the sense that probes do not influence the motion of natural celestial bodies—with the possible exception of comets—and the determination of the orbits of space probes may always be reduced to the problem of the motion in a given force field according to Eq. (79). This is also true for collision trajectories because of the nonzero radius of the celestial bodies as shown in the previous application. The above earth-moon space probe problem satisfies also the circular motion requirement of the primaries approximately since the eccentricity of the moon's orbit is 0.055. The plane of the orbits of the moon and earth moves a few minutes of arc during one year (this variation is relative to the 5°.15 angle to the ecliptic) and space probe orbits are essentially in the earth-moon plane since any deviations call for large additional energies.

The effect of the sun on the motion of the lunar probe is omitted by the formulation of the restricted problem. This omission is justified for short duration missions (say a few days) only.

We see that the assumptions of the restricted problem are not exactly satisfied for space probes, but for qualitative studies the approximations are acceptable. For precise calculations of trajectories the results from the restricted problem may be used only as reference orbits or as first approximations.

Precise statements regarding the approximations made when the model of the restricted problem is used to study earth-moon trajectories are as follows:

(i) the motion of the primaries (earth and moon) takes place approximately in a plane (with a few minutes deviation per year);

(ii) the motion of the third body takes place essentially in the same plane (typical trajectories show small out-of-plane deviations);

(iii) the effect of a typical probe on the motion of the primaries is of the order of 10^{-16} as compared to the forces acting between the primaries;

(iv) the motion of the primaries is essentially circular (with an eccentricity of 0.055).

Example 1. Investigate the applicability of the restricted problem to the sun-Jupiter-asteroid problem.

Example 2. Investigate the applicability of the restricted problem to the sun-asteroid-space probe problem using a density of 2 g/cm³ and a radius of 1 km for the asteroid.

(C) One application of the Jacobian integral in the sidereal system, Eq. (56), is known as Tisserand's criterion for the identification of comets. This is our third application in this section.

In the sidereal system the energy integral with $\mu = 0$ is

$$\dot{\xi}^2 + \dot{\eta}^2 = 2/\rho_1 - 1/a, \tag{83}$$

where $\rho = \rho_1$ and a is the semimajor axis of the comet. The angular momentum is also conserved when $\mu = 0$ and it may be written as

$$\xi\dot{\eta} - \eta\dot{\xi} = [a(1 - e^2)]^{1/2}, \tag{84}$$

where e is the eccentricity of the comet's orbit around the sun. Substituting these relations into Eq. (56) gives

$$\frac{2}{\rho} - \frac{1}{a} = 2[a(1 - e^2)]^{1/2} + \frac{2\mu}{\rho_2} + \frac{2(1-\mu)}{\rho_1} - \bar{C}. \tag{85}$$

Equating now ρ with ρ_1 and

$$\mu(1/\rho_2 - 1/\rho_1)$$

with zero, we have

$$\frac{1}{a} + 2[a(1 - e^2)]^{1/2} = \bar{C}, \tag{86}$$

a constant. Note first that $\rho \cong \rho_1$, since for the sun-Jupiter system $\mu < 10^{-3}$. Furthermore ρ_1 and ρ_2 may also be approximately equal if the epochs are carefully selected so that $\rho_1 \cong \rho_2 > 1$; consequently,

$$\mu\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) = \mu\,\frac{\rho_1 - \rho_2}{\rho_1\rho_2} \cong 0.$$

Equation (86) is the result of the derivation. The number of assumptions made in obtaining this equation reflects its approximate nature. The fact that the Jacobian integral (even when applied in its exact form) is not a good measure of orbit accuracy is discussed in Chapter 9. Consequently the significance of Eq. (86) is limited, nevertheless its engineering use to evaluate the critical parameters of so-called "swing-by" trajectories should be kept in mind (cf. Section 10.26).

1.11 Notes

The excellent reference book by Marcolongo [1], and also Whittaker's famous report [2], both give credit to Jacobi [3] for first formulating the restricted problem in 1836. I must, however, share Wintner's [4, p. 436] opinion and consider the publication date of Euler's [5] second lunar theory, 1772, as the birth date of the restricted problem. The term "restricted" seems to have been originated by Poincaré who in his *Méthodes* [6] first calls the restricted problem "a particular case of the problem of three bodies" and later introduces the word "restreint."

The statement of the problem as given in Section 1.2 is universally accepted; its mathematical formulation shows considerable variety and, curiously, several errors. Poincaré [6, Vol. 1, p. 22] gives the equations defining the distances between the third particle and the primaries with a typographical error: the printed r_2^1 should read r_1^2.

G. D. Birkhoff might disturb the reader more in his famous Palermo paper [7, p. 5]. The description given in the text, his Fig. 1, and his Eq. (2) all contradict. In the text the mass of S (sun, presumably) is μ and that of J is $1 - \mu$. (This in itself is somewhat strange since throughout the paper $\mu \leqslant \frac{1}{2}$.) In Fig. 1 the mass of S is clearly larger than the mass of J; presumably they are $1 - \mu$ and μ. Finally, according to the notation introduced in the figure,

$$r_1^2 = (x - 1 + \mu)^2 + y^2, \qquad r_2^2 = (x + \mu)^2 + y^2,$$

while the equations in the text are

$$r_1^2 = (x - \mu)^2 + y^2, \qquad r_2^2 = (x - \mu + 1)^2 + y^2.$$

A. Wintner's definition [4, p. 347] is given by making references to *five* previous sections (207, 214, 241, 276, and 343). The resulting description of the restricted problem is certainly correct if not necessarily helpful.

Figure 1.1 places the larger mass (m_1) at $t^* = 0$ to the right of the origin. This convention will be maintained throughout.

The quantity n occurring in Eq. (1) is the mean motion, so called because it is the average (mean) angular velocity with respect to the time (also when n is not constant) and velocity and "motio" were synonyms in the seventeenth and eighteenth centuries. See more on this subject by Wintner [4, p. 377].

Equations (36)–(38), (53), (54), and (56) of Sections 1.4 and 1.5 are various forms of the Jacobian integral, though Eq. (54) is referred to most frequently as the Jacobian integral. It is of interest to note, however, that Jacobi announced his integral [3] in the sidereal coordinate system [cf. Eq. (56)]. Jacobi's original form differs, though, from Eq. (56) since the origin of his coordinate system coincides with the primary having the larger mass. In [3] Jacobi shows how the conservation of energy and angular momentum may be combined for certain dynamical systems so as to obtain an integral. Derivation of Eq. (56), as given in the text by transforming the Jacobian integral from the synodic system to the sidereal, seems to be simpler than either the original derivation by Jacobi [3] or Tisserand's method of direct integration described in [8, Vol. 4, p. 203]. The derivations of Eq. (56) and consequently of Tisserand's criterion of identity of a comet by Plummer [9, p. 236] and by Moulton [10, p. 297] differ in that Plummer places the origin of the coordinate system at the sun (following Tisserand and Jacobi) while Moulton uses essentially the same system as in Eq. (56). Regarding the practical applicability of the criterion see Brouwer and Clemence [11, p. 256] and Chapter 10. For readers with vectorial preference, the book of Danby [12, p. 188] is recommended.

Note that the Jacobian integral, as given by Eq. (38), connects the square of the velocity with the centrifugal potential ($n^2\bar{r}^2$) and with the gravitational effect. This is *not* an expression of the conservation of energy, in spite of several authors' insistence on calling it either the energy integral or the relative energy integral. Whittaker [13, p. 354] carefully avoids this mistake in spite of using his physically undefined variables. Instead of contributing either erroneous or mysterious meanings to Eq. (38) it should be simply regarded as an integral of the differential equations of motion of the restricted problem, using rotating Cartesian coordinates.

The many possible choices regarding the arrangement of the masses, uses of notations, selections of the coordinate systems, etc., are shown in Section 9.3 since for the presentation of the numerical results information regarding these conventions is essential. The *smaller* mass is on the right in the writings of Brouwer and Clemence [11], Danby [12], Darwin [14], Moulton [10], Wintner [4], etc. The *larger* mass is on the right in Charlier's book [15], in Hill's papers [16], in Egorov's memoir [17], and in Birkhoff's work [7].

The contents of Section 1.7 will also be utilized in Chapters 2, 6, and 7. The method of finding the potential as given in the text is standard. Another method, based on guided intuition, is simply to integrate F_1, F_2, F_3 with respect to x_1, x_2, and x_3, keeping the other variables always constant. In this way three functions of x_1, x_2, x_3 are obtained which will furnish the desired function $\phi(x_1, x_2, x_3)$ by selecting all the *different* terms appearing as the results of the integration.

The reduction of the general problem of three bodies (Section 1.8) by Lagrange has not utilized the Hamiltonian approach. For this see Whittaker's book [13, p. 339] and also his report [2], which is one of the most complete and thorough treatments available on the general problem. The introduction of a cyclic or ignorable coordinate mentioned in connection with Eq. (73) and related to the second step in the reduction process is identical with Jacobi's [18] elimination of nodes. Another explanation of step number two is that the integrals (73) give two functions which are in involution with each other and with the Hamiltonian, and consequently the order of the canonical equations may be reduced by two units per integral. A large number of reductions exist, though according to Sylvester [19] there are in principle only two approaches. Besides reducing the order of the system of equations by using integrals or by various transformations (in other words by *elimination* of variables) one might introduce such variables *ab initio* that the system appears in its lower order (process of *ablimination*). Along these lines see articles by Radau [20] and Egerváry [21], and for the planar general problem of three bodies by Szebehely [22].

The classification and terminology used in Section 1.9 is not uniform in the literature. In fact even Poincaré's generally accepted key word "restreint" is modified to "asteroidische" by Charlier [15, Vol. 2, p. 104], notwithstanding the use of this latter term in the more recent literature for the elliptic (Szebehely and Giacaglia [23]) or pseudo (Colombo *et al.* [24]) restricted problem.

Additional applications (cf. Section 1.10) are proposed by Klose [25]. Regarding the major area of application, lunar theories, see Newcomb's excellent report [26].

The precise values of the astronomical constants are of no importance in Section 1.10, where only orders of magnitude enter the computations. The value given for the astronomical unit, $d_{\odot\oplus} = 149.6 \times 10^{11}$ cm, corresponds to 8″.7941 solar parallax and follows the recommendation of the International Astronomical Union [27].

1.12 References

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Chapter 2

Reduction

2.1 Introduction

Reducibility is probably the most challenging problem in dynamics. A general interpretation of reducibility includes various transformations and changes of the original problem not only along mathematical lines but also in a physical sense. The analogies in dynamics can be considered as reductions, and this chapter also deals with analogies of the restricted problem.

The reduction of the order of the differential equations representing a problem is of primary interest and it is closely associated with the question of integrability and with the solvable or unsolvable nature of the problem. If a dynamical system is considered integrable when all the coordinates can be expressed as "known" functions of the time, then integrability depends on Cauchy's existence theorem on one hand and a quite arbitrary definition of what are considered "known" functions on the other hand. Allowing convergent infinite series as "known" functions, the restricted problem is certainly integrable in the whole finite (x, y) plane if the singularities are eliminated by the process of regularization. Since such "global" regularizations are available for the restricted problem, the general solution certainly can be expressed by convergent infinite series. Such "solutions" seldom contribute significantly to information regarding the behavior of a dynamical system. Questions of stability, regions of possible motion, and other fundamentally important behaviorial problems can not very well be answered from complicated infinite series "solutions."

Another definition of integrability of a dynamical system is associated with the possibility of reducing the differential equations to quadratures. Examples show that this often does not mean that qualitative information about the dynamical system has been obtained.

The general problem of three bodies is described by an eighteenth-order system of differential equations which is reducible to a sixth-order system, as outlined in Section 1.8. The equations of the restricted problem form a fourth-order system. Its reduction to the third order by use of the Jacobian integral and further reduction to the second order by elimination of the time are shown in detail in this chapter. Such reductions are not unique, and a number of third-order and second-order systems of interest can be derived.

The original fourth-order system of equations can be interpreted as a representation of a four-dimensional flow without much physical applicability. The reduced system on the other hand is analogous to a three-dimensional or two-dimensional flow.

Inasmuch as this chapter is based on the use of integrals to reduce the equations of motion, it might be justified to offer a short survey of the existence of integrals in the n-body, three-body, and restricted problems.

The ten integrals of the general problem of three bodies mentioned in Section 1.8 are the only algebraic integrals by *Bruns' theorem*. More precisely (and more generally) the theorem may be stated as follows. In the problem of n bodies, the only integrals which involve the coordinates and velocities algebraically, and which do not involve the time explicitly, are composed of the integrals of the center of mass, of angular momentum, and of energy. In other words, with the exception of the known integrals, no integrals exist which are algebraic functions of the time, the coordinates, and the velocities, and so the classical integrals are the only independent algebraic integrals.

The definition of an integral previously given in Section 1.3 is recalled as well as the fact that the ten integrals discussed in Section 1.8 are algebraic functions of the positions and of the velocities.

Painlevé's generalization of Bruns' theorem allows the inclusion of integrals which are algebraic functions of the velocities and any analytic functions of the coordinates. Consequently Painlevé's theorem says that every integral of the n-body problem which contains the velocities algebraically and the coordinates arbitrarily is a combination of the classical integrals.

Bruns' theorem concerns algebraic integrals and therefore it contains a serious restriction. *Poincaré* removed Bruns' restriction but introduced others and so his theorem on the nonexistence of integrals must also shortly be mentioned. We state it for the restricted problem, referring the reader to the references for its more general versions. Let, for this purpose, H be the Hamiltonian of our dynamical system which can be expanded with respect to a parameter μ in a convergent power series, for a sufficiently small value of μ, as follows:

$$H = H_0 + \mu H_1 + \mu^2 H_2 + \cdots .$$

Here $H = H(q, p)$, $H_0 = H_0(p)$, $H_1 = H_1(q, p)$,.... The Hamiltonian is periodic in the variables q_1 and q_2, with period 2π. Such expansions the reader will find in Chapters 7 and 8, where a simple transformation is offered which gives a nonvanishing second-rank Hessian for H_0. Note that q and p stand for the components of the coordinates q_1, q_2 and of the momenta p_1, p_2.

Let now $\varphi = \varphi(q, p, \mu)$ be a periodic function of the variables q_1 and q_2 with period 2π. Furthermore let φ be a single-valued analytic function for any real values of q_1 and q_2, for sufficiently small values of μ, and for values of p_1 and p_2 which form a domain arbitrarily small. Consequently φ may be expanded in a convergent power series of μ, using single-valued analytic functions

$$\varphi_i = \varphi_i(q, p)$$

which are periodic in q_1, q_2. This series is

$$\varphi = \varphi_0 + \mu\varphi_1 + \mu^2\varphi_2 + \cdots .$$

Poincaré's theorem now states that the restricted problem has no integral except the Jacobian integral, which is of the form

$$\varphi = \text{const.}$$

It is important to mention that, as Cherry has shown, the theorem does not hold if we relax the requirement that the integral is tb be developable in powers of μ. Poincaré has generalized this theorem to the general problem of three bodies and Painlevé showed a similar generalization as already mentioned in connection with Bruns' result. This relaxes the requirement regarding the way the coordinates enter the integral.

The theorem corresponding to Bruns' theorem and related to the nonexistence of algebraic integrals of the restricted problem was given by Siegel. It states that the restricted problem has no other algebraic integral besides the Jacobian integral. In other words, if $\phi(z)$ is an algebraic function of z then Siegel's theorem states that every algebraic integral of the restricted problem is of the form

$$\phi(\dot{x}^2 + \dot{y}^2 - 2\Omega) = 0.$$

Note that Siegel's theorem is independent of Bruns' since the reduction of the latter to the restricted problem is not feasible.

2.2 Reduction to the third order

The fourth-order system described by the equations

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x , \\ \ddot{y} + 2\dot{x} &= \Omega_y , \end{aligned} \tag{1}$$

can be reduced to a third-order system by means of the Jacobian integral

$$\dot{x}^2 + \dot{y}^2 = 2\Omega - C. \tag{2}$$

This reduction can be effected by a number of ways and the following general method will be considered first. Let the desired third-order system be written as

$$\begin{aligned} \dot{x} &= f(x, y, z), \\ \dot{y} &= g(x, y, z), \\ \dot{z} &= h(x, y, z), \end{aligned} \tag{3}$$

where the functions f, g, h as well as the variable z are unknown at this point. Determination of these follows from the requirement that Eqs. (3) must satisfy one of Eqs. (1) and Eq. (2). From Eq. (2) the quantity $\dot{y}$, viz., the function g, can be obtained if $f(x, y, z)$ is known:

$$g = \pm (\Lambda^2 - f^2)^{1/2}, \tag{4}$$

where

$$\Lambda(x, y) = [2\Omega(x, y) - C]^{1/2}. \tag{5}$$

The function $h(x, y, z)$ can be derived from the first of Eqs. (1) since, from this,

$$\ddot{x} = \Omega_x + 2\dot{y} = f_x\dot{x} + f_y\dot{y} + f_z\dot{z}$$

or

$$\Omega_x + 2g = f_x f + f_y g + f_z h. \tag{6}$$

The function h therefore becomes

$$h = \frac{\Omega_x - ff_x - g(f_y - 2)}{f_z} . \tag{7}$$

At this point Eq. (4) gives $g(x, y, z)$, and Eq. (7) determines $h(x, y, z)$ provided $f(x, y, z)$ is known. The second equation of motion (1) has not been utilized yet but it contains no new information as is seen by evaluating h again from the second equation of (1):

$$h = \frac{\Omega_y - gg_y - f(2 + g_x)}{g_z}. \tag{8}$$

This result is identical with Eq. (7), and one way to show this is to eliminate Ω_x, Ω_y and Ω_z by means of

$$\begin{aligned} ff_x + gg_x &= \Omega_x, \\ ff_y + gg_y &= \Omega_y, \\ ff_z + gg_z &= 0, \end{aligned} \tag{9}$$

which equations are obtained by taking partial derivatives of the Jacobian integral:

$$f^2 + g^2 = 2\Omega - C. \tag{10}$$

So once the function $f(x, y, z)$ is selected, Eqs. (4) and (7) contain the solution to the reduction problem.

Example 1. Show that the selection of the special separable function gives

$$\begin{aligned} f(x, y, z) &= \Lambda(x, y)\,\varphi(z), \\ g &= \Lambda[1 - \varphi^2(z)]^{1/2}, \\ h &= \frac{(1-\varphi^2)^{1/2}}{\varphi'}\,[\Lambda_x(1-\varphi^2)^{1/2} - \Lambda_y\varphi + 2], \end{aligned} \tag{11}$$

where

$$\varphi' = d\varphi/dz \neq 0.$$

Example 2. Consider the general form of Eq. (1):

$$\begin{aligned} \ddot{x} - 2\,\lambda(x, y)\,\dot{y} &= \Omega_x, \\ \ddot{y} + 2\,\lambda(x, y)\,\dot{x} &= \Omega_y, \end{aligned} \tag{12}$$

and show the results corresponding to Eqs. (4) and (7).

Example 3. Show that if $\varphi(z) = \cos z$ in Eqs. (11) then

$$z = \arctan dy/dx. \tag{13}$$

This example treats the special case introduced by Birkhoff. In this case z is the angle between the tangent to the orbit and the positive x axis, and the equations of motion (3) become

$$\begin{aligned} \dot{x} &= \Lambda(x, y)\cos z, \\ \dot{y} &= \Lambda(x, y)\sin z, \\ \dot{z} &= -2 - \Lambda_x \sin z + \Lambda_y \cos z. \end{aligned} \tag{14}$$

The direct proof is as follows. Let s denote the arc length along an orbit. Then

$$dx/ds = \cos z, \qquad dy/ds = \sin z,$$

and

$$\dot{x} = \frac{dx}{ds}\dot{s} = \Lambda \cos z, \qquad \dot{y} = \Lambda \sin z,$$

since the absolute value of the velocity vector is obtained from the Jacobian integral as

$$(ds/dt)^2 = \dot{x}^2 + \dot{y}^2 = 2\Omega - C = \Lambda^2.$$

The last equation of (14) follows from (13) when $\dot{z}$ is evaluated:

$$\dot{z} = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{s}^2}.$$

Eliminating $\ddot{x}$ and $\ddot{y}$ by means of Eqs. (1) and substituting for $\dot{x}$ and $\dot{y}$ from (14), the desired result [the third equation of Eqs. (14)] is obtained.

Inasmuch as no unique reduction exists from the fourth to the third order, the above results contain one arbitrary function, $f(x, y, z)$, and a further condition is required to obtain a definite reduction.

2.3 Reduction to the second order

In addition to the use of the Jacobian integral, the time can be eliminated from the equations of motion and a second-order differential equation can be found representing the dynamical system. In general, proceeding from the third-order system given by Eqs. (3), the following is obtained:

$$dx/f = dy/g = dz/h \tag{15}$$

or

$$dz/dx = h/f = \alpha(x, y, z) \tag{16}$$

and

$$y' = dy/dx = g/f = \beta(x, y, z). \tag{17}$$

The last equation gives $z = B(x, y, y')$ from which

$$\frac{dz}{dx} = \frac{\partial B}{\partial x} + \frac{\partial B}{\partial y}y' + \frac{\partial B}{\partial y'}y''. \tag{18}$$

Substituting Eq. (18) into (16) gives

$$\frac{\partial B}{\partial x} + \frac{\partial B}{\partial y} y' + \frac{\partial B}{\partial y'} y'' = \alpha[x, y, B(x, y, y')] \tag{19}$$

or

$$y'' = \gamma(x, y, y'), \tag{20}$$

which is the desired second-order differential equation.

Note that the reduction to the second order does not have to be performed in two separate steps. In fact, writing $y = y(x)$, $dy/dx = \dot{y}/\dot{x}$, and $\dot{y} = y'\dot{x}$, from the Jacobian integral it follows that

$$\dot{x}^2 = \frac{\Lambda^2}{1 + (y')^2} = \Gamma^2(x, y, y'). \tag{21}$$

Time derivatives on both sides become

$$2\dot{x}\ddot{x} = \frac{\partial \Gamma^2}{\partial x}\dot{x} + \frac{\partial \Gamma^2}{\partial y}\dot{y} + \frac{\partial \Gamma^2}{\partial y'}\frac{dy'}{dt}. \tag{22}$$

Here

$$\ddot{x} = 2\dot{y} + \Omega_x = 2y'\dot{x} + \Lambda\Lambda_x \tag{23}$$

and

$$dy'/dt = y''\dot{x}. \tag{24}$$

Substitution of Eqs. (23) and (24) into (22) gives, when $\dot{x} \neq 0$,

$$2(\Lambda\Lambda_x \pm 2y'\Gamma) = \frac{\partial \Gamma^2}{\partial x} + \frac{\partial \Gamma^2}{\partial y} y' + \frac{\partial \Gamma^2}{\partial y'} y''.$$

By means of Eq. (21) we evaluate the partial derivatives and obtain

$$y'' = \frac{1 + (y')^2}{\Lambda}\{\Lambda_y - \Lambda_x y' \pm 2[1 + (y')^2]^{1/2}\}. \tag{25}$$

Example 1. Show that, using a function $f(x, y, z)$ and Eqs. (4) and (7) for the functions g and h, Eq. (25) can be obtained by the method indicated at the beginning of this section [Eqs. (15–20)].

Example 2. Show the second-order reduction of Eqs. (12).

The general solution of Eq. (25) will contain three constants: the Jacobian constant C, which is contained in the function Λ, and two constants of integration. This solution gives information only about the

geometry of the problem, i.e., the dependence of y on x. To complete the solution of the fourth-order system, the time dependence of the original variables is needed. The Jacobian integral gives, from Eq. (21),

$$\dot{x}^2[1 + (y')^2] = \Lambda^2$$

and, from this,

$$t = \pm \int \frac{[1 + (y')^2]^{1/2}}{\Lambda} dx + C_4, \tag{26}$$

where C_4 is the fourth constant of the solution.

Here y' and Λ are functions of x since the solution $y(x)$ has been obtained from Eq. (25).

The original fourth-order system requires four initial conditions: at $t = t_0$, $x = x_0$, $y = y_0$, $\dot{x} = \dot{x}_0$, and $\dot{y} = \dot{y}_0$. The corresponding initial conditions for the second-order system are y_0 and $y_0' = \dot{y}_0/\dot{x}_0$. The relation between these initial conditions and the Jacobian constant is $C = 2\Omega(x_0, y_0) - (\dot{x}_0^2 + \dot{y}_0^2)$.

There are many ways in which Eq. (25) can be written as a system of two first-order differential equations.

For instance, by writing $y = y_1$ and $y' = y_2$ we have

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= \frac{1 + y_2^2}{\Lambda(x, y_1)}\left[\frac{\partial \Lambda}{\partial y_1} - y_2\frac{\partial \Lambda}{\partial x} \pm 2(1 + y_2^2)^{1/2}\right], \end{aligned} \tag{27}$$

with the same initial conditions as before: at $t = t_0$, $x = x_0$, $y = y_1 = y_0 = y_{10}$, and $y' = y_2 = y_0' = y_{20} = \dot{y}_0/\dot{x}_0$.

The second-order presentation has the unquestionable disadvantage that, since $y' = \dot{y}/\dot{x}$, at those points on the orbit where $\dot{x} = 0$, $\dot{y} \neq 0$, y' has a singularity, a problem which does not arise in the third-order and fourth-order formulations.

Note the important difference between the third-order and second-order presentations. The former can be written as

$$\begin{aligned} \dot{x} &= f(x, y, z), \\ \dot{y} &= g(x, y, z), \\ \dot{z} &= h(x, y, z), \end{aligned} \tag{28}$$

where the right-hand sides do not contain the independent (time) variable. The second-order presentation (27) can be written as

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= G(y_1, y_2, x), \end{aligned} \tag{29}$$

or, using x for y_1, y for y_2, and t for x, we have

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= G(x, y, t). \end{aligned} \tag{30}$$

A comparison of the third-order (28) and second-order (30) systems now reveals the essential difference, viz., the occurrence of the independent variable on the right side of Eqs. (30).

The original fourth-order system is reducible to a third-order system by means of the Jacobian integral. The third-order system can be further reduced because the independent variable does not occur in the equations. The second-order system cannot be further reduced by elimination of the independent variable.

The multiplicity of the second-order presentation is to be kept in mind. A systematic approach to the various transformations is given in the next section and its strong relation to flow analogies is pointed out.

2.4 Four-dimensional streamline analogy

Consider the original fourth-order system (1) and write it as a system of first-order equations. This transformation is once again not unique but one possibility is:

$$\begin{aligned} \dot{x}_1 &= x_3, \\ \dot{x}_2 &= x_4, \\ \dot{x}_3 &= 2x_4 + \frac{\partial \Omega(x_1, x_2)}{\partial x_1}, \\ \dot{x}_4 &= -2x_3 + \frac{\partial \Omega(x_1, x_2)}{\partial x_2}, \end{aligned} \tag{31}$$

where $x_1 = x$, $x_2 = y$, $x_3 = \dot{x}$, and $x_4 = \dot{y}$. If the original physical meaning of the variables is forgotten for a moment, Eqs. (31), or their generalization

$$\dot{x}_i = f_i(x_1, x_2, x_3, x_4) \qquad (i = 1,..., 4), \tag{32}$$

can be looked at as the description of a four-dimensional velocity field with velocity components $\dot{x}_1, ..., \dot{x}_4$. This velocity field depends on the position only and not on the time, therefore it is termed a steady flow. Introducing $\mathbf{v}$ for the velocity vector, with components $\dot{x}_1, ..., \dot{x}_4$, and $\mathbf{r}$ for the position vector, with components $x_1, ..., x_4$, Eq. (32) becomes

$$\mathbf{v} = \mathbf{f}(\mathbf{r}) = d\mathbf{r}/dt.$$

The solution of the dynamical problem is $\mathbf{r} = \mathbf{r}(t, \mathbf{r}_0)$. This means that the position vector as the function of the time and of the initial conditions is known. (The fact that the "position vector" $\mathbf{r}$ contains components which physically, in the dynamical problem, correspond to velocities is of no concern at present.) The solution of the hydrodynamical problem cannot be found until the proper problem is formulated. This is very important in hydrodynamics since the knowledge of the velocity field as a function of the time often constitutes the "solution." This is true in what is called the *Eulerian* formulation of hydrodynamics. Therefore, in this sense the hydrodynamical problem is solved since the right-hand-side functions in Eqs. (31) are given. In what is known as the *Lagrangian* formulation of fluid mechanics, the paths of the individual fluid particles are of interest. The path depends on the particle in question, i.e., on the initial conditions given to the particle. Those initial conditions are simply the four position coordinates $\mathbf{r}_0(x_{10}, x_{20}, x_{30}, x_{40})$ at $t = t_0$. In the dynamical problem the initial conditions are of course the same, since $x_{10} = x_0$, $x_{20} = y_0$, $x_{30} = \dot{x}_0$, and $x_{40} = \dot{y}_0$. So the solution of the Lagrangian hydrodynamical problem and of the original dynamical problem are equivalent.

The paths of the various fluid particles are different since their starting places are different and, excepting singularities, the starting places will completely determine the paths, just as in the dynamical problem the initial conditions determine the orbit. The path of a fluid particle takes place in a four-dimensional space with coordinates x_1, x_2, x_3, x_4. The orbit of the third body in the dynamical problem is in a two-dimensional space, x_1, x_2. The projection of the path of the fluid particles on the x_1, x_2 plane will give the orbit of the third body in the restricted problem. The projection of the path of the third particle on the x_3, x_4 plane will give the velocity of the third body in the restricted problem. Another terminology is to speak about the motion of the third body in the phase space with coordinates $(x, y, \dot{x}, \dot{y}) = (x_1, x_2, x_3, x_4)$. The path of the fluid particle in its four-dimensional space is then identical with the orbit of the third body in the phase space. The totality of possible motions in the dynamical problem corresponds to the motion of all fluid particles, or to the motion of the whole fluid.

In a hydrodynamical situation when the flow velocity is independent of the time, i.e., when $\mathbf{v} = \mathbf{f}(\mathbf{r})$, as in the present case, we speak of a steady flow and define the streamlines of the flow as the paths of the particles. At any given point $P(x_1, x_2, x_3, x_4)$, in the four-dimensional flow there is at any moment only one fluid particle (with its well-defined velocity)—again excepting singularities. Therefore streamlines, i.e., the paths of fluid particles, do not cross. The corresponding dynamical situation is that orbits do not intersect *in the phase space*. An intersection in the four-dimensional flow space would mean that Eqs. (31) do not determine a unique velocity. Intersection of orbits in the dynamical problem (in the plane $x = x_1$, $y = x_2$) simply means that the *projection* of the flow paths intersect.

The projection of closed streamlines will give closed orbits and, since there are no intersections in the flow space, the four coordinates and four velocity components must be matched at the beginning and at re-entry. (By re-entering orbits in dynamics we mean periodic orbits which return to their initial conditions.) Such periodic paths in the four-dimensional flow space correspond to periodic orbits in the dynamical problem.

A stagnation point in the flow is defined by $\dot{x}_1 = \cdots = \dot{x}_4 = 0$. This, in the dynamical problem, corresponds to $x_3 = x_4 = \partial\Omega/\partial x_1 = \partial\Omega/\partial x_2 = 0$, i.e., to the so-called equilibrium points (see Chapter 4) since with $\dot{x} = \dot{y} = 0$ the equations of motion give $\ddot{x} = \partial\Omega/\partial x_1 = \partial\Omega/\partial x$ and $\ddot{y} = \partial\Omega/\partial y$; consequently, the partial derivatives are zero so we have $\dot{x} = \ddot{x} = \dot{y} = \ddot{y} = 0$.

The pressure anywhere in the flow is given by Euler's equation

$$\boldsymbol{\varphi} - \operatorname{grad} p = \frac{d\mathbf{v}}{dt}\rho = \left(\frac{\partial \mathbf{v}}{\partial \mathbf{r}}\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}\right)\rho, \tag{33}$$

where $\boldsymbol{\varphi}$ is the external force acting on the fluid, p is the pressure, and ρ the density. The gradient is defined by $\partial p/\partial x_i$ and

$$\frac{\partial \mathbf{v}}{\partial \mathbf{r}}\mathbf{v} \quad \text{by} \quad \sum_{j=1}^{4} \frac{\partial v_i}{\partial x_j} v_j .$$

Example. Show that, for zero external force and for a flow with unit density, the negative pressure gradient has the components $\ddot{x}, \ddot{y}, \dddot{x}, \dddot{y}$.

The fourth-order system (31) possesses the Jacobian integral which can either be written down directly or be derived by multiplying the last two equations of Eqs. (31) by x_3 and x_4, adding and integrating. The result is

$$x_3^2 + x_4^2 = 2\Omega(x_1, x_2) - C. \tag{34}$$

Inasmuch as the solutions of (31), i.e., the equations of the streamlines, will have to satisfy (34), the points describing a streamline must lie on the above-mentioned three-dimensional hypersurface. Once the initial conditions of a fluid particle are given and $\mathbf{r}_0(x_{10}, x_{20}, x_{30}, x_{40})$ is known, the constant C may be evaluated from Eq. (34). Or, once the value of C for a particle and three of its initial conditions are given, its

motion is determined. Thus C may always take the place of one of the initial conditions.

Two questions of interest regarding the properties of a flow are: (i) is it a potential flow, and (ii) what is its divergence? The velocity field given by Eqs. (31) represents the flow of an incompressible fluid since

$$\operatorname{div} \mathbf{v} = \sum_{i=1}^{4} \frac{\partial v_i}{\partial x_i} = 0, \tag{35}$$

but it is not a potential flow since in general

$$\partial v_i/\partial x_j \neq \partial v_j/\partial x_i \qquad (i \neq j). \tag{36}$$

2.5 Three-dimensional streamline analogy

By means of the Jacobian integral our fourth-order system can be reduced to a third-order system as shown before. This means that the four-dimensional hydrodynamic analogy becomes three dimensional and Eqs. (3) may now be interpreted as defining the three velocity components of a flow field. Since the right-hand members do not contain the time explicitly the flow is steady. If the three-dimensional flow is to represent the motion of an incompressible fluid then

$$\partial f/\partial x + \partial g/\partial y + \partial h/\partial z = 0. \tag{37}$$

In Section 2.2 it was shown that selecting $f(x, y, z)$ the other two velocity components become

$$g = (\Lambda^2 - f^2)^{1/2} \tag{4}$$

and

$$h = \frac{\Omega_x - ff_x + g(2 - f_y)}{f_z}. \tag{7}$$

Evaluating g_y and h_z and substituting in Eq. (37) we obtain

$$F(\Lambda, \Lambda_x, \Lambda_y, f, f_x, f_y, f_z, f_{xz}, f_{yz}, f_{zz}) = 0. \tag{38}$$

The solution of this nonlinear second-order partial differential equation for $f(x, y, z)$ will furnish the totality of functions f which, when used in (4) and (7), gives the velocity components of all incompressible flows analogous to the restricted problem.

Example. Find the partial differential equation, Eq. (38), and prove that it can be satisfied by $f = \Lambda(x, y)\,\varphi(z)$ as suggested by Eq. (11). Solve the resulting ordinary second-order differential equation for $\varphi(z)$ which is obtained after substituting $\Lambda\varphi$, and show that the result is a slight generalization of Birkhoff's flow field [see Eq. (14)].

It is essential to recall that the reduction to the third order utilizes the Jacobian integral; i.e., the equations for the velocity field contain the Jacobian constant C:

$$\begin{aligned} \dot{x} &= f(x, y, z, C), \\ \dot{y} &= g(x, y, z, C), \\ \dot{z} &= h(x, y, z, C). \end{aligned} \tag{39}$$

This is clearly shown in Eqs. (14), where $\Lambda = (2\Omega - C)^{1/2}$. The physical meaning is that the three-dimensional flow now has a parameter C. If the value of C is fixed the totality of the streamlines in the three-dimensional flow corresponds to orbits with the same Jacobian constant. The coordinates x and y in the dynamical problem represent the coordinates of the third particle. The orbit in the dynamical problem therefore corresponds to the projection of the steamlines on the xy plane in the hydrodynamical problem. The third coordinate of a fluid particle (z) has no immediate physical significance in the dynamical problem. In Birkhoff's analogy,

$$z = \arctan y', \tag{40}$$

so the third position-coordinate z of the fluid particle measures the angle between the tangent to the orbit and the x axis. The third velocity component of the flow $\dot{z}$ measures the rate of change of this angle with time.

Example. Show that, for the two-body problem in fixed coordinates

$$\ddot{x} = \Omega_x, \qquad \ddot{y} = \Omega_y, \qquad \Omega = 1/r, \tag{41}$$

circular orbits correspond to spiral flows using Birkhoff's analogy, since $z = (\text{const})\, t$.

2.6 General remarks on the relation between Hamiltonian systems and flow in the phase space

Consider a dynamical system with n degrees of freedom and with the Hamiltonian

$$H = H(q_1, \dots, q_n, p_1, \dots, p_n, t), \tag{42}$$

where the q_i's are the generalized coordinates and the p_i's are the generalized momenta.

The equations of motion are

$$\dot{q}_i = \partial H/\partial p_i, \qquad \dot{p}_i = -\partial H/\partial q_i, \tag{43}$$

where $i = 1, 2, \dots, n$.

Introduce $x_1 = q_1, \dots, x_n = q_n, x_{n+1} = p_1, \dots, x_{2n} = p_n$. The equations of motion become

$$\begin{aligned} \dot{x}_1 &= \frac{\partial H}{\partial x_{n+1}}, & \dot{x}_2 &= \frac{\partial H}{\partial x_{n+2}}, & \dots \quad \dot{x}_n &= \frac{\partial H}{\partial x_{2n}}, \\ \dot{x}_{n+1} &= -\frac{\partial H}{\partial x_1}, & \dot{x}_{n+2} &= -\frac{\partial H}{\partial x_2}, & \dots \quad \dot{x}_{2n} &= -\frac{\partial H}{\partial x_n}, \end{aligned}$$

and, considering $\dot{x}_1, \dots, \dot{x}_{2n}$ as the $2n$ components of a velocity vector and $x_1, \dots, x_{2n}$ as the $2n$ components of a position vector, we have

$$\mathbf{v} = \mathbf{f}(\mathbf{r}, t). \tag{44}$$

The divergence of this velocity field is zero. Proof:

$$\begin{aligned} \operatorname{div} \mathbf{v} &= \frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial v_n}{\partial x_n} + \frac{\partial v_{n+1}}{\partial x_{n+1}} + \cdots + \frac{\partial v_{2n}}{\partial x_{2n}} \\ &= \frac{\partial^2 H}{\partial x_{n+1}\, \partial x_1} + \cdots + \frac{\partial^2 H}{\partial x_{2n}\, \partial x_n} - \frac{\partial^2 H}{\partial x_1\, \partial x_{n+1}} - \cdots - \frac{\partial^2 H}{\partial x_n\, \partial x_{2n}} = 0. \end{aligned} \tag{45}$$

Therefore, a system of canonical equations can always be represented by an incompressible flow. The converse of this theorem is not true in general since the incompressibility condition may be satisfied by other than canonical systems. See, for instance, the fourth-order system given by Eq. (31) which satisfies the incompressibility condition but is *not* a canonical set.

Since the Hamiltonian given by Eq. (42) depends explicitly on the time, the corresponding flow field is unsteady,

$$\partial \mathbf{v}/\partial t \neq 0. \tag{46}$$

Steady flows require time-independent Hamiltonians. A time-independent Hamiltonian represents an integral of the dynamical system, and a steady flow possesses streamlines which are identical with the paths of the fluid particles. If the flow is unsteady the streamline pattern changes from time to time and the path lines and streamlines will differ. The concept of streamlines can be extended to unsteady flows but its usefulness is questionable. This situation is analogous to the role of the integral discussed in Section 1.4, and we see now that unsteady flows and time-dependent Hamiltonians are related the same way as the nonexistence of streamlines in an unsteady flow is related to the nonexistence of an integral in the usual sense. Allowing the concept of changing streamlines in hydrodynamics and time-dependent integration constants in dynamics we can once again establish the analogy.

Example 1. Show that the components of the negative pressure gradient $-\partial p/\partial x_k$ $(k = 1, 2, \ldots, 2n)$ are $\ddot{q}_1, \ddot{q}_2, \ldots, \ddot{q}_n, \ddot{p}_1, \ddot{p}_2, \ldots, \ddot{p}_n$.

Example 2. Show that the flow in general is not a potential flow, in other words, that there is no function ϕ of which the velocity vector of the flow is the gradient.

Realizing that the canonical form of the equations of motion leads to incompressible flow analogies, it follows that canonical transformations of a dynamical system result in new flows which, however, will retain their incompressible character. Regarding the time dependence of the Hamiltonian and the associated unsteadiness of the flow field, the concept of canonical transformations simply states that, if the original Hamiltonian is time independent and the generating function is also time independent, then the new flow (as well as the original) will be steady. By means of time-dependent generating functions a steady flow might be turned into an unsteady one or vice versa. Therefore the unsteadiness or steadiness of the flow is not an invariant property of canonical transformations, while the incompressibility is.

The unsteadiness of a flow can be characterized by the dimensionless ratio of the absolute values of the local and convective acceleration vectors:

$$S = \frac{\left|\dfrac{\partial \mathbf{v}}{\partial t}\right|}{\left|\dfrac{\partial \mathbf{v}}{\partial \mathbf{r}}\,\mathbf{v}\right|} \tag{47}$$

or, using the summation convention,

$$S^2 = \frac{\dfrac{\partial v_i}{\partial t}\cdot\dfrac{\partial v_i}{\partial t}}{\dfrac{\partial v_j}{\partial x_k}\cdot\dfrac{\partial v_j}{\partial x_l}\,v_k v_l}. \tag{48}$$

This allows one to establish a measure of unsteadiness of a flow. For instance, when the flow is steady and its velocity vector depends only on the location $(\mathbf{r})$ in the flow, $\partial\mathbf{v}/\partial t = \mathbf{0}$ and so $S = 0$. When the time variation of the velocity vector is much larger than its change due to displacement, S becomes large.

In this way we may also speak about the effect of time on a Hamiltonian and on adiabatic invariants with well-defined measure of "slow" variations. More will be said about this in Section 10.33 in connection with the elliptic restricted problem.

Example. Evaluate the measure of unsteadiness (47) for a dynamical system of n degrees of freedom when the Hamiltonian depends explicitly on the time. Show that

$$S = G(q_1, \ldots, q_n, p_1, \ldots, p_n)\,\frac{1/f(t)}{dt} \tag{49}$$

when

$$H = H(q_1, \ldots, q_n, p_1, \ldots, p_n)f(t).$$

2.7 Two-dimensional streamline analogy

The dynamical system with two degrees of freedom under consideration possesses an integral. Therefore it is reducible to a second-order system by use of this integral and by elimination of the time. This second-order system—if it is canonical—will correspond to an incompressible two-dimensional flow. In order to obtain a two-dimensional incompressible flow analogy, therefore, the reductions are to be executed in a canonical system. How this can be done will be shown in this section.

Consider the time-independent Hamiltonian of a system of two degrees of freedom:

$$H = H(q_1, q_2, p_1, p_2). \tag{50}$$

The equations of motion are

$$\dot{q}_1 = \frac{\partial H}{\partial p_1}, \qquad \dot{q}_2 = \frac{\partial H}{\partial p_2}; \qquad \dot{p}_1 = -\frac{\partial H}{\partial q_1}, \qquad \dot{p}_2 = -\frac{\partial H}{\partial q_2}, \tag{51}$$

and the system has the integral

$$H(q_1, q_2, p_1, p_2) = H_0 = \text{const.} \tag{52}$$

If q_1, say, is considered as the independent variable, the equations of motion may be written as

$$dq_2/dq_1 = H_{p_2}/H_{p_1}, \qquad dp_2/dq_1 = -H_{q_2}/H_{p_1}, \tag{53}$$

since, for instance,

$$\frac{dq_2}{dt} = \frac{dq_2}{dq_1}\dot{q}_1 . \tag{54}$$

Here q_2 and p_2 are the only remaining dependent variables since q_1 becomes the new independent variable and the corresponding momentum p_1 may be obtained by solving Eq. (52) for p_1:

$$p_1 = p_1(q_2, p_2; q_1; H_0), \tag{54a}$$

where the semicolons indicate the separation between the dependent q_2, p_2 and independent q_1 variables and also the distinction between the latter and an integration constant H_0. When the partial derivatives of H are computed as indicated in Eqs. (53), using H as given by (50), the derivatives H_{p_1}, H_{p_2}, and H_{q_2} are first established as functions of q_1, q_2, p_1, and p_2. Then in these results Eq. (54a) is used to eliminate p_1. Therefore Eqs. (53) will read

$$\begin{aligned} dq_2/dq_1 &= \alpha(q_2, p_2; q_1; H_0), \\ dp_2/dq_1 &= \beta(q_2, p_2; q_1; H_0). \end{aligned} \tag{55}$$

These equations constitute the second-order system that was obtained by use of the integral (52) and by elimination of the time (54).

To solve the dynamical problem we may proceed as follows. First, Eqs. (55) are solved for q_2 and p_2:

$$\begin{aligned} q_2 &= q_2(q_1, H_0, a, b), \\ p_2 &= p_2(q_1, H_0, a, b), \end{aligned} \tag{56}$$

where a, b are constants of integration introduced when Eqs. (55) were solved. Then from Eq. (54a) we have

$$p_1 = p_1(q_1, H_0, a, b), \tag{57}$$

and, from the first of Eqs. (51),

$$t - t_0 = \int_{q_{10}}^{q_1} \frac{dq_1}{H_{p_1}} . \tag{58}$$

Here $H_{p_1} = \partial H/\partial p_1$ is computed from (50) which gives

$$H_{p_1} = \gamma(q_1, q_2, p_1, p_2). \tag{59}$$

Prior to integration, q_2, p_1, and p_2 as functions of q_1, H_0, a, and b are to be substituted so that with $\bar{\gamma}(q_1, H_0, a, b) = \gamma(q_1, q_2, p_1, p_2)$,

$$t = \int_c^{q_1} \frac{dq_1}{\bar{\gamma}(q_1, H_0, a, b)} , \tag{60}$$

the inverse of which gives the time dependence of q_1 together with four constants of integration: H_0, a, b, and c. Using (56) and (57) again the time dependence of q_2, p_1, and p_2 is obtained.

Returning now to Eqs. (55) we investigate whether these equations represent a two-dimensional incompressible flow, considering

$$dq_2/dq_1 \quad \text{and} \quad dp_2/dq_1$$

as flow-velocity components, q_2, p_2 as coordinates in the hydrodynamical problem, and q_1 as the independent variable. To show that the answer to the above question is in the affirmative we prove that (55) is a canonical system, i.e., that there is a Hamiltonian $\bar{H} = \bar{H}(q_2, p_2; q_1; H_0)$ that gives the equations of motion as

$$dq_2/dq_1 = \partial\bar{H}/\partial p_2 \quad \text{and} \quad dp_2/dq_1 = -\partial\bar{H}/\partial q_2 ; \tag{61}$$

in fact,

$$\bar{H} = -p_1(q_2, p_2; q_1; H_0). \tag{62}$$

Since q_1 became the new independent variable, its conjugate, p_1, becomes the new Hamiltonian with a minus sign. This is a well-known theorem in connection with canonical systems.

It is to be noted that, by selecting q_2 as the independent variable, the role of the variables will be altered and, unless H is symmetric in the subscripts 1 and 2, the final outcome (55) will be different.

At this point we return to the restricted problem before further theoretical developments are offered. The Hamiltonian corresponding to Eqs. (50) for the restricted problem is

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + p_1 q_2 - p_2 q_1 - f(q_1, q_2), \tag{63}$$

where

$$p_1 = \dot{q}_1 - q_2, \qquad p_2 = \dot{q}_2 + q_1, \qquad q_1 = x, \qquad q_2 = y, \tag{64}$$

and

$$f = \frac{1-\mu}{r_1} + \frac{\mu}{r_2}, \tag{65}$$

$$r_1^2 = (q_1 - \mu)^2 + q_2^2, \qquad r_2^2 = (q_1 + 1 - \mu)^2 + q_2^2. \tag{66}$$

Equations (63) to (66) are discussed, derived, and analyzed in considerable detail in Section 7.2. At this point they only serve to demonstrate the reduction of a fourth-order system.

Since the Hamiltonian does not contain the time explicitly, $H = H_0 =$ const and the order may be reduced from 4 to 2. Solving the equation $H = H_0$ for p_1 we obtain

$$p_1 = -q_2 \pm (q_2^2 - p_2^2 + 2p_2q_1 + 2f + 2H_0)^{1/2}, \tag{67}$$

corresponding to Eq. (55). To preserve the notation for the previously introduced Jacobian constant $\bar{C}$, we substitute $-\bar{C}/2$ for H_0. The new Hamiltonian, $\tilde{H} = -p_1$, as given by Eq. (62), now becomes

$$\tilde{H} = q_2 \mp [2\Omega - C - (p_2 - q_1)^2]^{1/2}, \tag{68}$$

where under the square root the formula given by Eq. (67) was changed, using the definition of Ω:

$$\Omega = \tfrac{1}{2}(q_1^2 + q_2^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \tfrac{1}{2}\mu(1-\mu). \tag{69}$$

Hence

$$2\Omega - C - (p_2 - q_1)^2 = q_2^2 - p_2^2 + 2p_2q_1 + 2f + \mu(1-\mu) - C$$

and, since

$$H_0 = -\bar{C}/2, \qquad \bar{C} = C - \mu(1-\mu),$$

we have

$$\mu(1-\mu) - C = 2H_0.$$

In what follows Eq. (68) will be used instead of its equivalent (67). The equations of motion corresponding to (61) are

$$\frac{dq_2}{dq_1} = \frac{\partial\tilde{H}}{\partial p_2} = \pm\frac{p_2 - q_1}{[2\Omega - C - (p_2 - q_1)^2]^{1/2}},$$
$$\frac{dp_2}{dq_1} = -\frac{\partial\tilde{H}}{\partial q_2} = -1 \pm \frac{\Omega_{q_2}}{[2\Omega - C - (p_2 - q_1)^2]^{1/2}}. \tag{70}$$

This second-order system may now be interpreted as a two-dimensional flow of an incompressible fluid. The flow velocities are on the left side, the coordinates of a fluid particle are q_2 and p_2, and the independent variable is q_1. Corresponding to this physical picture the variables may be renamed. The independent variable will be called t, the q_2, p_2 variables y and x, and the Hamiltonian ψ, since this is an

accepted symbol for the stream function. So the hydrodynamical picture is represented by

$$\frac{dx}{dt} = u = -\frac{\partial\psi}{\partial y} = -1 \pm \frac{\Omega_y}{[A^2 - (x-t)^2]^{1/2}},$$
$$\frac{dy}{dt} = v = \frac{\partial\psi}{\partial x} = \pm\frac{x - t}{[A^2 - (x-t)^2]^{1/2}}, \tag{71}$$

where $A^2 = 2\Omega - C$ depends on q_1, q_2, i. e., on t, y.

Subjecting Eqs. (70) to canonical transformations, different forms of the hydrodynamic Eqs. (71) may be obtained, all representing incompressible flows. Indeed, from Eqs. (71)

$$\operatorname{div} \mathbf{v} = \partial u/\partial x + \partial v/\partial y = -\psi_{yx} + \psi_{xy} = 0, \tag{72}$$

as expected since the dynamical problem (70) was formulated with canonical variables.

The flow is not a potential flow,

$$|\operatorname{curl} \mathbf{v}| = \partial v/\partial x - \partial u/\partial y = \Delta\psi \neq 0. \tag{73}$$

The flow described by Eqs. (71) is unsteady since t occurs on the right side of the equations. This is of course the consequence of the occurrence of q_1 in the right side of Eqs. (70) which in turn is caused by q_1 appearing explicitly in $\tilde{H}$. Since $\tilde{H}$ depends explicitly on the independent variable, $\tilde{H}$ is not a constant, and there is no useful integral.

In general, if in a dynamical problem H contains the time explicitly, the invariant relation $H =$ const is not available for reduction along these lines. The corresponding hydrodynamical analog will be an unsteady flow. Using the previously defined [Eqs. (47) and (48)] measure of flow unsteadiness (S) the preceding result states that reducible dynamical systems and steady flows are equivalent.

Example. Show that the two-body problem may be represented as a two-dimensional unidirectional unsteady incompressible flow (one of the flow-velocity components is zero). The nonzero flow-velocity component depends only on the "time" and on the coordinate which is normal to the flow direction. Show that $S = \infty$, curl $\mathbf{v} \neq \mathbf{0}$, and div $\mathbf{v} = 0$.

2.8 Further general remarks

The continuity equation of hydrodynamics expresses the conservation of matter in the following form:

$$\partial\rho/\partial t + \operatorname{div} \rho\mathbf{v} = 0, \tag{74}$$

where ρ is the density and $\mathbf{v}$ the flow velocity. For incompressible fluids $\rho = \text{const}$ and Eq. (74) becomes

$$\operatorname{div} \mathbf{v} = 0. \tag{75}$$

In a steady flow none of the variables depends explicitly on the time, therefore $\partial\rho/\partial t = 0$ and so the continuity equation for the steady flow of a compressible fluid is

$$\operatorname{div} \rho\mathbf{v} = \rho \operatorname{div} \mathbf{v} + \mathbf{v} \cdot \operatorname{grad} \rho = 0. \tag{76}$$

Consider a region of the three-dimensional space enclosed by a surface S. The rate at which mass flows into S is

$$\int \rho\mathbf{v} \cdot \mathbf{dS}, \tag{77}$$

where the vector $\mathbf{dS}$ is normal to S and is directed inward. By the divergence (Gauss') theorem the above integral becomes

$$-\int_V \operatorname{div} \rho\mathbf{v}\, dV, \tag{78}$$

where V is the volume enclosed by S.

The mass of fluid in S is $\int_V \rho\, dV$ and its time rate of change is

$$\frac{\partial}{\partial t} \int \rho\, dV.$$

The conservation of matter, therefore, can be expressed by equating the above two volume-integrals; i.e., the total mass enclosed in the volume does not change:

$$\int_V (\partial\rho/\partial t + \operatorname{div} \rho\mathbf{v})\, dV = 0. \tag{79}$$

For an incompressible fluid the mass of fluid in S is

$$\rho \int_V dx_1\, dx_2\, dx_3 = \text{const}, \tag{80}$$

expressing the constancy of the volume.

Now consider with Liouville the equations

$$dx_i/dt = X_i(x_1, \ldots, x_m\,;\, t) \tag{81}$$

and the velocities (dx_i/dt) such that

$$\partial X_i/\partial x_i = \operatorname{div} \mathbf{v} = 0. \tag{82}$$

The increase in volume is measured by div $\mathbf{v}$ and, since this is zero, the volume

$$\int_V dx_1\, dx_2 \cdots dx_m$$

is conserved.

Now considering Eqs. (43) in place of (81). The integral invariant, expressing the volume conservation of the $m = 2n$ variables, becomes

$$\int_v dq_1 \cdots dq_n\, dp_1 \cdots dp_n\,.$$

This is certainly an invariant since the canonical equations satisfy the zero divergence condition, but we see that Liouville's theorem is more general and is not restricted to Hamiltonian systems. In fact, the three-dimensional flow analogy discussed in Section 2.5, because of its odd order may not be considered a Hamiltonian system. Nevertheless if the zero divergence condition is satisfied, the integral invariant exists in the form

$$\int dx\, dy\, dz.$$

A rather elegant consideration and proof of Liouville's theorem for canonical systems consists simply of realizing that the Jacobian determinant of a canonical transformation is unity, i.e.,

$$\int_v dq_1 \cdots dq_n\, dp_1 \cdots dp_n = \int_V dQ_1 \cdots dQ_n\, dP_1 \cdots dP_n\,,$$

where the lower-case variables represent the system before and the capital variables after transformation. Furthermore, motion is equivalent to infinitesimal canonical transformations, therefore motion corresponds to preservation of the integral

$$\int dq_1 \cdots dq_n\, dp_1 \cdots dp_n\,.$$

Example. Show that $\int dx\, dy\, d\dot{x}\, d\dot{y}$ is invariant and finite within a circle of radius $\ll 1$ centered at one of the primaries.

2.9 Applications

The two purposes of the developments in this chapter were (i) to present the equations of motion in basically different forms and (ii) to acquaint the reader with the hydrodynamic analogies of dynamics.

By *basically different forms* of the equations we mean that the order of the system of the differential equations was changed from four to three and then to two. The numerical solution of the pertinent differential equations is not the subject of this chapter (it will be discussed in Chapter 9), neither are general perturbation methods our subjects here. Unquestionably, however, the order of the equations to be solved is of the utmost importance whatever approach we choose. The fourth-order form of the equations is generally used in special or general perturbation methods. The third-order form has not been used as yet; the second-order form has been studied only recently. Of course, the complete solution of the problem requires the determination of the dependent variables (the coordinates) as functions of the independent variable (time), which demands a further integration, but the advantage of the second-order presentation is that this process is isolated. The solution of the second-order system [as given, for instance, by Eqs. (27)] will furnish the dependence of y on x; i.e., the geometry or shape of the orbit will be known. The motion of the third particle on this curve may be determined separately; see Eq. (26).

The disadvantage of the second-order presentation is that it introduces a new singularity. In the fourth-order presentation the conditions $\dot{x} = 0$ or $\dot{y} = 0$ do not present problems, while in the second-order form these lead to an infinite value of dy/dx or dx/dy. This difficulty of course can be avoided by switching from y' to $1/y'$ in the numerical process whenever $\dot{x} = \pm\epsilon$, but nevertheless it is present and requires attention especially in the study of those periodic orbits which cross the x axis perpendicularly, i.e., with $\dot{x} = 0$. Closely connected with this question are, of course, the second-order systems discussed in connection with the hydrodynamic analogies. Inasmuch as these systems are in canonical forms, they can be further transformed with ease, in this way offering a great variety of second-order presentations. The advantages and disadvantages of the second-order forms, from the point of view of special or general perturbations, cannot be really evaluated until the actual transformed forms have been thoroughly studied.

Hydrodynamic analogies have two uses: conceptual and physical. We mean by the first the search for qualitative information and by the second the actual construction of an analog device.

The applications of Liouville's theorem to statistical mechanics are well known, and one of Poincaré's applications to the restricted problem

is essentially covered by the example at the end of the previous section. The invariant integral associated with a ring transformation and the latter's fixed points led Poincaré to prove the existence of an infinite number of periodic orbits for small mass ratio and Birkhoff to show the existence of an infinite number of symmetric periodic orbits in the restricted problem.

The construction of a physical model of the hydrodynamic analogy must be restricted to flows in two and three dimensions. The complexities of the three-dimensional steady flow and the two-dimensional unsteady flow may be comparable, especially since in the first case the projection of the three-dimensional streamlines on the x, y plane will furnish the orbits, while in the second the actual path lines will have to be observed.

Naturally the question of further equations controlling the motion of the fluid arises since only the continuity equation was used up to this point. The Eulerian equations of hydrodynamics are given by Eq. (33). With some well-known transformations in vector analysis this can be written as

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \operatorname{grad}\frac{\mathbf{v}^2}{2} - \mathbf{v} \times \operatorname{curl}\mathbf{v}\right) = \boldsymbol{\varphi} - \operatorname{grad} p, \tag{83}$$

which, when ρ is constant and the flow is steady, becomes

$$\boldsymbol{\varphi}/\rho + \mathbf{v} \times \operatorname{curl}\mathbf{v} = \operatorname{grad}(\mathbf{v}^2/2 - p/\rho). \tag{84}$$

Taking the curl of this equation results in

$$\operatorname{curl}(\mathbf{w} \times \mathbf{v}) = \operatorname{curl}\boldsymbol{\varphi}/\rho, \tag{85}$$

where $\mathbf{w} = \operatorname{curl}\mathbf{v}$ is the vorticity vector.

The Navier–Stokes equation for a viscous fluid is

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \operatorname{grad}\frac{\mathbf{v}^2}{2} - \mathbf{v} \times \mathbf{w}\right) = \boldsymbol{\varphi} - \operatorname{grad} p + \rho\nu \operatorname{curl}\mathbf{w}, \tag{86}$$

where ν is the kinematic viscosity.

Again, assuming incompressibility and steadiness, we have

$$\operatorname{curl}(\mathbf{w} \times \mathbf{v}) = \operatorname{curl}\boldsymbol{\varphi}/\rho - \nu \operatorname{curl}\operatorname{curl}\mathbf{w}. \tag{87}$$

So in discussing an analogy between the restricted problem and the steady flow of an incompressible fluid, the flow-velocity vector must satisfy either Eq. (85) or (87). This, of course, is possible since $\boldsymbol{\varphi}$ in these equations was left arbitrary.

Note that in general $\operatorname{curl}\boldsymbol{\varphi} \neq \mathbf{0}$, therefore, the external field acting on the fluid is not conservative. This fact prevents the use of a gravitational field for generating φ.

Example. Evaluate curl φ for Birkhoff's analogy given by Eqs. (14).

2.10 Notes

The problem of integrability mentioned in Section 2.1 will be taken up also in Chapter 8. Instead of speaking of integrable or nonintegrable systems, the Soviet literature of the 1960's suggests the terms integrated and nonintegrated systems; see Arnol'd [1], for instance.

The Bruns theorem given in the introduction, Section 2.1, appeared originally in 1887 by Bruns [2]. Whittaker's treatments [3, p. 358; 4, p. 523; 5, p. 157] are the chief references. Poincaré's [6] correction of Bruns' paper was published in 1896. This was followed by Pailevé's [7] generalization one year later and MacMillan's [8] rediscussion. Poincaré's paper [9] and book [10, Vol. 1, p. 233] and Whittaker's interpretations [3, p. 380; 4, p. 525] are clear on the restricted validity of Poincaré's theorem. Cherry's [11] remarks are of considerable interest as is Painlevé's [12] generalization. Siegel's [13] theorem followed Bruns' by 49 years. The theorem, corresponding to Bruns' work but related to Hill's problem, will be mentioned in Chapter 10. All these theorems do not exclude the existence of additional integrals of other types as, for example, Levi-Civita [14] shows in terms of his regularizing coordinates (see Chapter 3).

The subject of the "third integral" belongs to the study of stellar dynamics where potential functions different from the one applicable to the restricted problem are used. Nevertheless the concept that in addition to the classical integrals others (it is the *second* in the restricted problem, the *eleventh* in the general problem of three bodies, and the *third* in stellar dynamics) might exist has been extensively studied. Whittaker's [15] fundamental article on the adelphic integral and the corresponding passages in his book [3, p. 432] are the basic references, together with Cherry's [11] and Siegel's [16] papers. Applications to stellar dynamics are comprehensively treated by Contopoulos [17]. Goudas and Barbanis [18] describe some new properties of the third integral and Hénon and Heiles [19] give computational results. Preliminary studies by Hénon [20] and, in some detail, by Bozis [21] deal with a second integral for the restricted problem. These results, especially in connection with the restricted problem, are still not completely developed; consequently they and the last reference of [17] are not discussed here. Kotsakis' [22] work, however must be mentioned

since it is an interesting application of some of these results to the estimation of the dispersion of fragments in the restricted problem with obvious cosmogonical implications.

The general form of reduction to the third order given in Section 2.2 and the result of the reduction to the second order, Eq. (25) of Section 2.3, are not widely known. An interesting example regarding Eq. (25) is given by Whittaker [3, p. 421]. In fact, this example is also applicable to Eqs. (146) and (147) of Section 3.9 and to Eqs. (142) of Section 10.5.2, Part (D). A general formulation of the second-order equation (25) and its form in polar coordinates is given by Szebehely [23]. Birkhoff's transformation given by Eq. (13) of Section 2.2 appeared first in his 1915 memoir [23a].

Regarding the flow analogies discussed in this chapter, the reader will find the most readable and enjoyable references in Lanczos' book [24, pp. 172–183 and 219–228]. The many passages dedicated to this subject by Poincaré [10, Vol. 3, pp. 142–174] are not so clear as Lanczos' presentation; nevertheless, I consider Lanczos an excellent introduction to Poincaré.

Liouville's [25] theorem is presented, without the mysticism found in other papers, by Lanczos [24, pp. 177–180] and in an article by Synge [26, pp. 172–175], and also, without a physical picture, by Wintner [27, p. 88]. The statistical aspects and those of measure theory are emphasized by Khinchin [28, pp. 15–19].

The measure of unsteadiness in hydrodynamics [see Eqs. (47)–(48)] is introduced by Szebehely [29] and is discussed along with other dimensionless numbers by Truesdell [30, pp. 379, 432].

Sections 2.6 and 2.7 require familiarity, in fact, in a few instances intimate familiarity, with canonical systems. Chapter 6 may be helpful, but any other treatment of Hamiltonian dynamics will suffice and will furnish the reader with the necessary information; see, for instance, Brouwer and Clemence [31, p. 530] or Goldstein [32, p. 215].

A discussion of the reduction to the second-order system, similar to Section 2.7, is offered by Szebehely [33]. The second-order system in polar coordinates, already mentioned in connection with Section 2.3, is treated by Meffroy [34] and from the general point of view by Tisserand [35, Vol. 1, p. 87].

The example at the end of Section 2.8 is from Poincaré [10, Vol. 3, p. 157]. The physical feasibility of analogy devices is discussed in [33]. I acknowledge gratefully J. D. Mulholland's penetrating comments on this subject.

2.11 References

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Chapter 3

Regularization

3.1 Introduction

A significant difference between the motion of natural (celestial) bodies and that of artificial bodies is that close approaches are common occurrences in the latter case while in the former they happen but seldom.

The consequences of this fact can be understood when the property of the Newtonian gravitational force field is recalled, according to which the forces acting between particles approach infinity when the distance between the bodies approaches zero. Therefore at collision (when r_1 or r_2 is zero) the equations show singularities.

Space probes often require orbits which connect two celestial bodies. The actual orbits, of course, do not go through the points of singularity since before this happens the trajectory ends at the point of impact between the probe and the surface of the celestial body. In the framework of the problem of three bodies this impact can take place only at the singularity since the participating bodies are point masses. Therefore, in the physical sense singularities are never reached by means of collision in celestial mechanics. In numerical (computational) and conceptual respects the singularity aspects of the problem of three bodies are of utmost importance.

Both the force acting on the third body and its velocity increase as the body approaches the vicinity of one of the primaries. The step size of a numerical integration must be decreased significantly in this region in order to obtain reliable results. The physical aspects of space

trajectories also require an increased accuracy in these critical regions near the singularities.

The conceptual aspects of the singularities in the field are connected with the existence of the solution of differential equations.

Since the singularities occurring at collisions are not of essential character, they can be eliminated by the proper choice of the independent variable. Once this has been done the following have been effected:

(i) existence of solutions has been established for an arbitrary selection of the initial conditions;
(ii) solutions going through singularities can be traced analytically;
(iii) solutions may be established numerically before, at, and after collision;
(iv) close approaches may be treated with analytical and numerical precision.

In this chapter we proceed from discussing the simplest straightline collision-orbit without regularization, to the complete regularization of the restricted problem. Accordingly, the first subject is the problem of two bodies. The corresponding equations of motion are regularized in two steps. First we treat collision orbits (Section 3.2), then the general problem is regularized (Section 3.3). This is followed by the regularization of the equations of motion of the restricted problem. Here first we solve the problem "locally," by which in this context we mean that we regularize the equations of motion only at one of the two singularities (Section 3.4). Following Birkhoff's terminology, we then affect "global" regularizations; we eliminate both singularities simultaneously. Both the "global" and the "local" regularizations may be, according to mathematical usage, considered local operations; nevertheless, we accept Birkhoff's terminology at this point and defer additional remarks to Section 3.10. The global regularization may be performed with several transformations, and we describe three methods in Sections 3.5–3.7. Generalizations and comparisons of global regularization techniques are offered then in Sections 3.8 and 3.9. Section 3.10 contains the theorem of existence of solutions of the restricted problem for finite intervals of time. The chapter is concluded with two major areas of application: space dynamics and stellar dynamics,

The following two additional remarks are in order at this point. The *regularization of the differential equations* of motion is the principal problem and the main subject of this chapter. The *regularization of the solution* at collision can always be accomplished by introducing the eccentric anomaly since the collision of two bodies in any problem can be regularized in this way.

The second remark is that the inclusion of a chapter on regularization in a book on celestial mechanics might serve two purposes. It helps to establish the existence of solutions from the point of view of analysis and it extends the applicability of celestial mechanics to collision orbits. Books on classical celestial mechanics often omit entirely the subject of regularization since the analyticity of solutions is not of central interest to astronomers and because collisions do not occur in classical celestial mechanics.

For our purposes inclusion of the chapter on regularization is a necessity in order to satisfy our interest in applications to space dynamics as well as to establish the existence of the solutions of the restricted problem. To these solutions, after all, our whole volume is dedicated.

The questions of existence are treated at the end of the chapter, only after a relatively leisurely journey through the methods of regularization. The first three sections after the introduction (Sections 3.2, 3.3, and 3.4) the reader will find comfortable, mostly easy going, and at some places even repetitious. Teaching experience suggests such a treatment to lay a sound foundation for the more complex aspects. Since regularization is essential in space dynamics, the presentation of Sections 3.2, 3.3, and 3.4 is directed to workers in this field, leaving some of the analytical aspects to Section 3.10.

3.2 Regularization of collision orbits in the problem of two bodies

Part (A) of this section describes the dynamics of a simple collision orbit without the use of regularizing variables. In Part (B) regularizing variables are introduced in a general form, and in Part (C) a special transformation concerning only the independent variable is treated. Part (D) offers an explanation of the mathematical result according to which the particle is reflected after collision, shows the regularizing role of the eccentric anomaly, and proves that the distance to collision varies as the 2/3 power of the time. In Part (E) regularization is affected by transforming the independent as well as the dependent variables.

(A) Consider the equations of motion [Section 1.5, Eq. (52)] of the restricted problem and let $\mu_1 = 1$, $\mu_2 = 0$. The corresponding function Ω is

$$\Omega = \frac{1}{r} + \frac{1}{2} r^2 \tag{1}$$

since $r^2 = r_1{}^2 = x^2 + y^2$ [see Fig. 3.1(a)]. The equations of motion are

$$\ddot{x} - 2\dot{y} = x(1 - 1/r^3), \qquad \ddot{y} + 2\dot{x} = y(1 - 1/r^3), \tag{2}$$

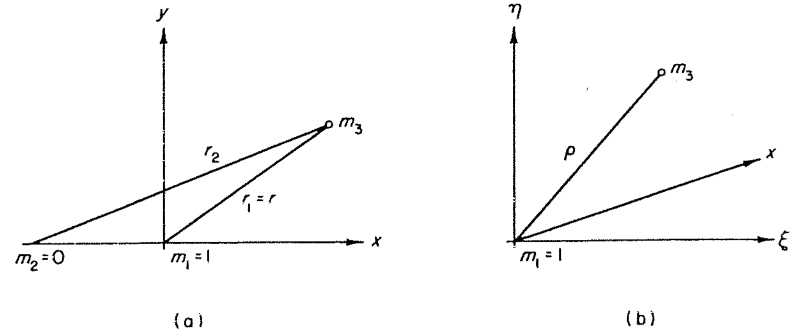

FIG. 3.1. Problem of two bodies.

and the Jacobian integral is

$$\dot{x}^2 + \dot{y}^2 = r^2 + 2/r - C. \tag{3}$$

Equations (2) describe the problem of two bodies; in fact they refer to a simplified restricted problem of three bodies in which the mass of one of the two primaries is zero. This description, however, is in a synodic coordinate system which renders the equations rather complicated. The equations of motion in a corresponding fixed system are

$$\ddot{\xi} = -\xi/\rho^3 \quad \text{and} \quad \ddot{\eta} = -\eta/\rho^3, \tag{4}$$

where $\rho^2 = r^2 = x^2 + y^2 = \xi^2 + \eta^2$ [see Fig. 3.1(b)]. The energy integral of Eqs. (4) is

$$\dot{\xi}^2 + \dot{\eta}^2 = 2/\rho - C. \tag{5}$$

In the first instance the simplest possible close approach, a collision, will be considered. Since a two-body collision orbit in a fixed system of coordinates is a straight line, we might specify the following initial conditions: at $t = 0$, $\xi = \xi_0$, $\dot{\xi} = \dot{\xi}_0$, and $\eta \equiv 0$, $\dot{\eta} \equiv 0$, for any t. Noting that $\rho = |\xi|$ we have

$$\ddot{\xi} = \mp 1/\xi^2 \tag{6}$$

for $\xi \gtrless 0$. The energy integral gives

$$\dot{\xi}^2 = 2/|\xi| - C = \pm 2/\xi - C, \tag{7}$$

again for $\xi \gtrless 0$, and in order to evaluate C we can substitute the initial conditions, obtaining

$$C = \pm 2/\xi_0 - \dot{\xi}_0{}^2. \tag{8}$$

In order to discuss a specific problem, let $\dot{\xi}_0 = 0$ and $\xi_0 > 0$. Thus $C = 2/\xi_0 > 0$, and Eq. (7) gives

$$\pm \int_{\xi_0}^{\xi} \left(\frac{\xi}{2 - C\xi}\right)^{1/2} d\xi = t. \tag{9}$$

Note that, since $2/|\xi| - 2/\xi_0 \geqslant 0$, $|\xi| \leqslant \xi_0$; the particle released at $t = 0$ from $\xi = \xi_0 > 0$ will never depart farther from the origin than its initial position. The velocity of the particle is directed toward the origin and is negative during the time interval $0 < t < t_c$, where t_c is the time of collision; that is,

$$\dot{\xi} = -\left(\frac{2}{|\xi|} - C\right)^{1/2}, \tag{10}$$

for $0 < \xi < \xi_0$. Equation (6) shows that this negative velocity increases in absolute value as the particle approaches the origin (point of collision), since $\ddot{\xi} < 0$ for $\xi > 0$. As $\xi \to 0$, $|\dot{\xi}| \to \infty$, and therefore Eq. (10) can be integrated from $t = 0$ to $t_c - \delta$, where the time $t_c - \delta$ corresponds to $\xi = \epsilon > 0$ and the time $t = 0$ to $\xi = \xi_0$. The limit is to be evaluated as $\epsilon \to 0$ and $\delta \to 0$. Now Eq. (9) may be evaluated with the negative sign, giving

$$t = \frac{1}{C}[\xi(2 - C\xi)]^{1/2} + \frac{2}{C^{3/2}} \arctan\left(\frac{2 - C\xi}{C\xi}\right)^{1/2}. \tag{11}$$

As $\xi \to 0$, from Eq. (11)

$$t \to t_c = \pi/C^{3/2} \tag{12}$$

which connects the initial conditions with the instant of collision since $C = 2/\xi_0$ and so $\xi_0 = 2(t_c/\pi)^{2/3}$.

The monotonically decreasing function $\xi = \xi(t)$ as obtained from Eq. (11) is shown in Fig. 3.2. From Eq. (10), $|\dot{\xi}| \to \infty$ as $\xi \to 0$ (i.e., as $t \to t_c$), so the curve intersects the t axis perpendicularly. The initial conditions show that the curve starts perpendicularly to the ξ axis; $t = 0$, $\xi = \xi_0$, $\dot{\xi} = \dot{\xi}_0 = 0$. Note that the second derivative $\ddot{\xi}$ is negative in the region $0 \leqslant t < t_c$ so the curve is concave.

What happens to the particle following the instant t_c for $t \geqslant t_c$ is, of course, not clear from the preceding results, which cease to be meaningful at $t = t_c$. The continuation of the orbit after collision is not feasible since the solution encounters the singularity present in the problem.

(B) To eliminate the singularity, new dependent and independent variables are introduced:

$$\xi = f(u) \tag{13}$$

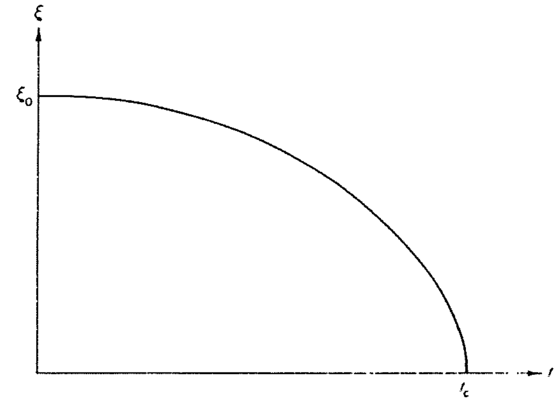

FIG. 3.2. One-dimensional collision orbit. Displacement ξ as a function of time t.

and

$$\tau = \int_{t_0}^{t} \frac{dt}{g(u)}. \tag{14}$$

The motion in the new system is described by the function $u = u(\tau)$; the new variable u depends on its own independent variable τ. When the functions $f(u)$ and $g(u)$ are known, Eq. (13) also gives u as a function of t since $\xi = \xi(t)$. Consequently Eq. (14) gives the relation between the old t and new τ time variables.

Another form of (14), which is frequently used in the literature, is

$$dt/d\tau = g(u). \tag{15}$$

This gives the ratio of the differentials of the old and new times as a function of the new, or, by Eq. (13), of the old dependent variable.

To understand what happens at and close to collision, the phenomenon must be slowed down by stretching the time scale so that the approach of the actual velocity to infinity can be handled.

The considerations leading to the selection of the functions f and g are presented in the following paragraphs.

The new velocity $u' = du/d\tau$ is related to the actual (physical) velocity $\dot{\xi} = d\xi/dt$ by

$$\frac{d\xi(t)}{dt} = \frac{df(u)}{du}\frac{du(\tau)}{d\tau}\frac{d\tau}{dt} \tag{16}$$

or

$$\dot{\xi} = u'f'/g, \tag{17}$$

if the notation $f' = df/du$ is introduced. Equation (16) follows from (13) using (15) and gives the new velocity:

$$u' = g\dot{\xi}/f'. \tag{18}$$

In order to have a finite value of this new velocity at collision, it is necessary that the ratio g/f' approach zero as $\dot{\xi} \to \infty$.

The energy integral (7) may be written as

$$\dot{\xi}^2 = 2/\xi - C = 2U, \tag{19}$$

where the $\pm$ sign is omitted; it should be remembered that $\xi > 0$. The energy integral in terms of the new variables becomes

$$(u')^2 = \frac{g^2}{(f')^2}\left(\frac{2}{f} - C\right) = \frac{g^2}{(f')^2}2U. \tag{20}$$

As $\xi \to 0$, $f(u) \to 0$ and $2U \to \infty$. In order to have a finite velocity u' at collision, we must have $[g^2/(f')^2]U$ finite as $\xi \to 0$.

Since $2U = (2/\xi) - C$, $U \to \infty$ as $\xi \to 0$, and close to collision $U = 1/\xi = 1/f$. So the requirement for finite velocity in the (u, τ) system is that

$$\frac{g^2}{(f')^2 f}$$

remain finite as $\xi \to 0$ or that

$$\frac{g}{f'}\frac{1}{f^{1/2}}$$

be finite as $f \to 0$.

If g/f' is represented by a power series in $f^{1/2}$, the lowest term in this series must be (const)$f^{1/2}$. This follows from writing $g/f' = (f^{1/2})^n$ which gives

$$\frac{g}{f'}\frac{1}{f^{1/2}} = (f^{1/2})^{n-1}.$$

Therefore the above limit requirement is satisfied if $n - 1 \geqslant 0$, and the lowest allowable power of $f^{1/2}$ in the series must be 1. The series is

$$\frac{g}{f'} = A_1 f^{1/2} + A_2 f + A_3 f^{3/2} + A_4 f^2 + \cdots. \tag{21}$$

Consequently

$$\frac{g}{f' f^{1/2}} = A_1 + A_2 f^{1/2} + \cdots, \tag{22}$$

and, as $f \to 0$, $(g/f' f^{1/2}) \to A_1$. Only if $A_1 = 0$ does the limit process lead to $g/(f' f^{1/2}) = 0$. Therefore, the velocity in the system u, τ is finite at the singularity, $u' = 2^{1/2}A_1$, provided g and f are selected satisfying Eq. (22).

For instance, if $\xi = f(u) = u^n$, we have

$$g = A_1 f' f^{1/2} = nA_1 u^{(3/2)n-1}. \tag{23}$$

The next step is to investigate the equation of motion regarding singularities. Similarly to Eq. (16), we have

$$\ddot{\xi} = f'u' \frac{d^2\tau}{dt^2} + (f'u'' + f''u'^2)\left(\frac{d\tau}{dt}\right)^2$$

or, since

$$\frac{d^2\tau}{dt^2} = \frac{d}{dt}\frac{1}{g(u)} = -\frac{g'u'}{g^3},$$

we have

$$\ddot{\xi} = -u'^2 \frac{f'g'}{g^3} + \frac{f'u'' + f''u'^2}{g^2}. \tag{24}$$

The equation of motion in the system u, τ therefore becomes, from Eq. (6) using only the region $\xi > 0$,

$$u''\frac{f'}{g^2} + u'^2\left(\frac{f''}{g^2} - \frac{f'g'}{g^3}\right) = \frac{1}{f'}\frac{dU}{du} \tag{25}$$

or

$$u'' + u'^2 \frac{gf'' - f'g'}{f'g} = \frac{g^2}{f'^2}\frac{dU}{du}. \tag{26}$$

In connection with the regularization of the velocity in the (u, τ) system, Eq. (20) leads to the requirement that $(g^2/f'^2)2U$ must be finite. In order to utilize this requirement we compute

$$\frac{d}{du} U \frac{g^2}{f'^2} = \frac{g^2}{f'^2}\frac{dU}{du} + \frac{u'^2}{f'g}(g'f' - gf''). \tag{27}$$

Solving this equation for the required term on the right side and substituting it into (26) we have

$$u'' = \frac{d}{du}\frac{g^2}{f'^2} U, \tag{28}$$

which also follows, of course, from (20) by differentiation since $du'/du = u''/u'$.

(C) Regarding the selection of the functions f and g we first recall Sundman's and Levi-Civita's idea according to which the essential part of the regularization is the time transformation, the selection of the function g. The basic idea follows from the previously mentioned process of slowing down the phenomenon, and the equations

$$dt/d\tau = g(\xi) \tag{29}$$

and

$$\xi = u \tag{30}$$

take the place of Eqs. (13) and (14). The new velocity $u' = \xi' = d\xi/d\tau$ becomes, from Eq. (20),

$$\xi'^2 = g^2(2/\xi - C), \tag{31}$$

and in order to maintain a finite ξ' as $\xi \to 0$ we must have

$$g^2 = A_1\xi + B_1\xi^2 + C_1\xi^3 + \cdots. \tag{32}$$

This gives

$$\xi'^2 = 2A_1 + \xi(2B_1 - CA_1) + \xi^2(2C_1 - CB_1) + \cdots, \tag{33}$$

and so the new velocity is $(2A_1)^{1/2}$ at collision. If $A_1 = C_1 = \cdots = 0$, but $B_1 = 1$, we have $g = \xi$, $\xi'^2 = 2\xi - C\xi^2$, and from Eq. (29)

$$dt/d\tau = \xi, \tag{34}$$

or

$$d\tau = dt/\xi = \Omega(\xi)\, dt. \tag{35}$$

The new time variable τ is therefore directly related to the potential $\Omega(\xi) = 1/\xi$ of the dynamical problem. Since ξ in the present problem is the distance of the moving particle from the singularity, we have $\Omega(\xi) = 1/r$ or

$$\tau = \int_{t_0}^{t} \frac{dt}{r}, \tag{36}$$

which is a form popular in the literature.

The actual solution of the problem in the (ξ, τ) system follows from integrating Eq. (31). Instead of using the general form, Eq. (32), for $g(\xi)$, the special case $B_1 = 1$, $A_1 = C_1 = \cdots = 0$ is continued:

$$\xi'^2 = \xi(2 - C\xi). \tag{37}$$

The initial conditions are $t = 0$, $\tau = 0$, $\xi = 2/C$ and the solution is obtained from Eq. (37) in the form

$$\xi = \frac{1}{C}(1 + \cos C^{1/2}\tau) \tag{38}$$

with

$$t = \frac{1}{C}[\tau + (1/C^{1/2}) \sin C^{1/2}\tau]. \tag{39}$$

The functions $\xi(\tau)$ and $t(\tau)$ are shown in Figs. 3.3 and 3.4. Equations (38) may be obtained either from Eq. (37) or from

$$\xi'' + C\xi - 1 = 0, \tag{40}$$

which is a consequence of (26) or (28). Equation (39) follows from

$$= \int_0^{\tau} \xi\, d\tau. \tag{41}$$

At collision, $t = t_c = \pi/C^{3/2}$, $\tau = \tau_c = \pi/C^{1/2}$, and at the beginning of the motion $t = \tau = 0$.

The motion of the particle, according to Eqs. (38) and (39) and as shown in the corresponding Figs. 3.3 and 3.4, is oscillatory. At the beginning of the motion $t = \tau = 0$, $\xi = 2/C > 0$, $\dot{\xi} = 0$, and $\xi' = 0$. The particle moves toward the origin ($\xi = 0$) with $\dot{\xi} < 0$ and $\xi' < 0$ after the beginning of the motion. At $\tau = \tau_c/2$, $t = [(2 + \pi)/2\pi]\, t_c$, $\xi = 1/C$, $\dot{\xi} < 0$, $\xi' < 0$. As the particle approaches the origin, $|\dot{\xi}|$

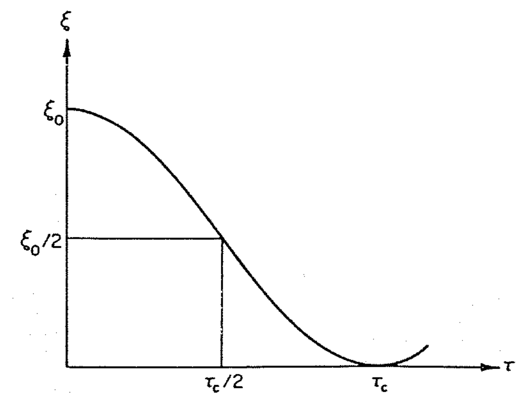

FIG. 3.3. Displacement ξ as a function of the regularized time τ for a one-dimensional collision orbit.

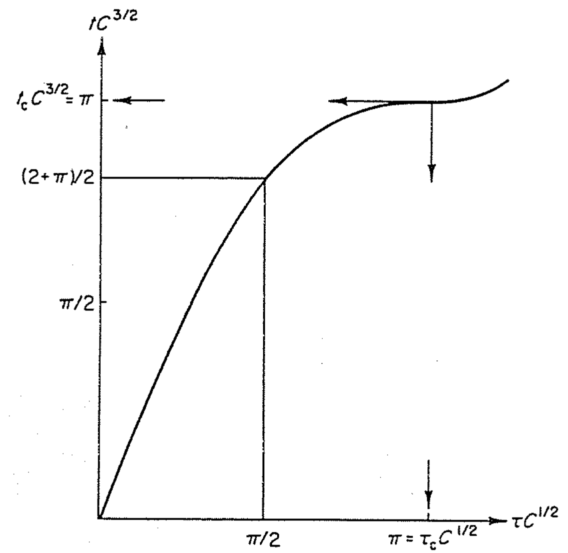

FIG. 3.4. Relation between physical t and pseudo time τ.

increases as $1/\xi^{1/2}$ and ξ' approaches zero. At collision $\xi = 0$, $t = t_c$, $\tau = \tau_c$, $\xi' = 0$, and $|\dot{\xi}| = +\infty$. Shortly after collision $t > t_c$, $\tau > \tau_c$, $\dot{\xi} > 0$, and $\xi' > 0$. At $t = 2t_c$, $\tau = 2\tau_c$, the particle is back at $\xi = 2/C$ with $\dot{\xi} = \xi' = 0$ and the cycle repeats itself. The function $\xi(\tau)$ is regular everywhere and so is the new velocity,

$$\xi'(\tau) = -\frac{1}{C^{1/2}} \sin C^{1/2}\tau.$$

The oscillation takes place along the positive ξ axis between $\xi_0 = 2/C$ and $\xi = 0$ with period $2t_c = 2\pi/C^{3/2}$. Therefore, for unit mean motion $n = 2\pi/2t_c = 1$, we have $t_c = \pi$, $C = 1$, and $\xi_0 = 2$.

(D) The regularized velocity ξ' changes sign at collision; $\xi' < 0$ before collision and $\xi' > 0$ after collision. Since the actual velocity $|\dot{\xi}| = \infty$ at collision, the incoming particle in this system arrives with a velocity $\dot{\xi} \to -\infty$ and leaves with a velocity $\dot{\xi} \to \infty$. Visualization of this is helped if the limiting degeneration of an elliptical two-dimensional orbit is considered.

As in Fig. 3.5, let $\dot{\eta}_0 > 0$ be a small vertical velocity at point $\xi = \xi_0$, $\eta = 0$ so that

$$2/\xi_0 > \dot{\eta}_0^2.$$

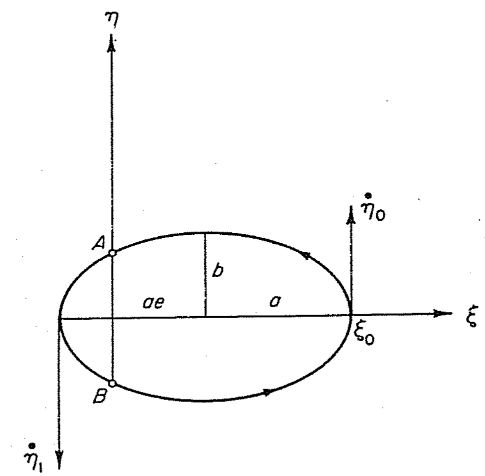

FIG. 3.5. Close approach as $\dot{\eta}_0 \to 0$.

The particle describes an ellipse with focus at the origin of the ξ, η coordinate system. The semimajor axis of this ellipse is related to the initial conditions by the energy integral $(2/r) - (1/a) = v^2$ or

$$\frac{1}{a} = \frac{2}{\xi_0} - \dot{\eta}_0^2, \tag{42}$$

and the eccentricity of the elliptic orbit is obtained from $\xi_0 = a(1+e)$ or

$$e = \frac{\xi_0 - a}{a}. \tag{43}$$

When $\xi_0 = 2a$, then $e = 1$, $\dot{\eta}_0 = 0$, and with $v^2 = \dot{\xi}_0^2$ we have

$$\frac{2}{\xi_0} - \frac{1}{a} = \dot{\xi}_0^2 \tag{44}$$

from which $C = 1/a$.

The velocity at the pericenter is

$$\dot{\eta}_1 = \frac{1}{a^{1/2}}\left(\frac{1+e}{1-e}\right)^{1/2},$$

which goes to infinity as $e \to 1$. The velocity at the apocenter is

$$\dot{\eta}_0 = \frac{1}{a^{1/2}}\left(\frac{1-e}{1+e}\right)^{1/2}, \tag{45}$$

and when $e \to 1$, $\dot{\eta}_0 \to 0$.

The speed at point A with coordinates $\xi = 0$, $\eta = \eta_0$ is

$$v_A = \frac{1}{a^{1/2}}\left(\frac{1+e^2}{1-e^2}\right)^{1/2} \tag{46}$$

and the velocity components at points A and B become

$$\begin{aligned} \dot{\xi}_A &= -\frac{1}{[a(1-e^2)]^{1/2}} = -\dot{\xi}_B, \\ \dot{\eta}_A &= -\frac{e}{[a(1-e^2)]^{1/2}} = \dot{\eta}_B. \end{aligned} \tag{47}$$

As $e \to 1$, the ellipse flattens out ($b = a(1-e^2)^{1/2} \to 0$) and the above components tend toward ∞. The ξ component of the velocity changes sign as the particle goes around the singularity, $\dot{\xi}_A = -\dot{\xi}_B$.

The period of the motion on the ellipse is $T = 2\pi/n$, with $n = a^{-3/2}$. When $\eta_0 \to 0$, $a^{-1} \to 2/\xi_0$ and so $T \to 2\pi(2/\xi_0)^{-3/2}$. Since $C = 2/\xi_0$ and $t_c = T/2$, the limiting condition of the elliptic motion furnishes the previously obtained time for collision, t_c.

It is not expected that the motion can be followed completely in the case $e = 1$; nevertheless, the preceding discussion suggests that the results obtained with the regularization process are reasonable not only from a mathematical but also from a physical point of view.

Prior to leaving this simplest case of regularization, attention is called to Eq. (38) connecting the distance $\xi = r$ with the new time variable τ. If the substitutions $a = 1/C$ and $na = aa^{-3/2} = C^{1/2}$ are made, we obtain

$$r = a(1 + \cos na\tau),$$

where $na\tau = u$ is the eccentric anomaly, and consequently Kepler's equation

$$nt = u + e \sin u$$

becomes Eq. (39) with $e = 1$. Note that in the last two equations the plus signs become minus signs if $u = \tau = 0$ corresponds to perihelion instead of to aphelion as in Fig. 3.5.

The conclusion is that the eccentric anomaly is a regularizing variable for the problem of two bodies. In fact, a comparison of Eq. (36),

$$d\tau = dt/r,$$

with

$$du = \frac{na}{r}\,dt$$

shows that our "new time" τ is essentially the eccentric anomaly.

If the time is measured from collision by $t^* = t - t_c$ or by $\tau^* = \tau - \tau_c$ with $Ct_c = \tau_c = \pi/C^{1/2}$ then Eqs. (38) and (39) become

$$\xi = \frac{1}{C}(1 - \cos C^{1/2}\tau^*)$$

and

$$t^* = \frac{1}{C}\left(\tau^* - \frac{1}{C^{1/2}}\sin C^{1/2}\tau^*\right).$$

Power series expansions for these functions may be written as

$$\xi = \frac{\tau^{*2}}{2!} - \frac{C\tau^{*4}}{4!} \pm \cdots = \tau^{*2}\,\Xi(\tau^*)$$

and

$$t^* = \frac{\tau^{*3}}{3!} - \frac{C\tau^{*5}}{5!} \pm \cdots = \tau^{*3}\,T(\tau^*),$$

where the series $\Xi(\tau^*)$ and $T(\tau^*)$ are convergent for any τ^* since they are essentially Taylor series for the functions sine and cosine. The constant terms in the series Ξ and T are $(2!)^{-1}$ and $(3!)^{-1}$, respectively. Consequently for sufficiently small t^* we have

$$\xi = (t^*)^{2/3}X(t^{*1/3}),$$

where once again the function $X(t^{*1/3})$ is a power series with the constant term $(9/2)^{1/3}$. The original solution therefore has at t^* a branch point of order 2 and the synodic path possesses a cusp at collision.

(E) Equations (13) and (14) propose the performance of the regularization by introducing two transformations,

$$\xi = f(u) \quad \text{and} \quad dt/d\tau = g(u).$$

The selection for $f(u)$ in the previous example was simply $f(u) = \xi = u$, and so the dependent variable was *not* transformed. The transformation of the independent variable followed the formula $dt/d\tau = g(u) = g(\xi) = \xi$. Regularization was performed, therefore, by transforming only the independent variable, as was expected since this was the essential step in the regularization process.

If the dependent variable is also transformed the situation is different because Eq. (23) furnishes a possible function $g(u)$, once $f(u)$ is selected.

The selection of

$$\xi = f(u) = u^2 \tag{48}$$

is the next natural step. Note that, when n is even, Eq. (23) gives rational functions for g. In the present case

$$g(u) = Bu^2, \tag{49}$$

where $B = \text{const}$. The new velocity, from Eq. (20), becomes

$$(u')^2 = \left(\frac{1}{2} - \frac{Cu^2}{4}\right) B^2$$

and the selection of $B = 4$ gives

$$u' = \pm\, 2(2 - Cu^2)^{1/2}. \tag{50}$$

The selection of the constant B is quite arbitrary. If $B = 4$ we have $g = 4u^2 = (f')^2$, which will be of interest later.

Regarding the sign in Eq. (50), note that, since $u = \pm\xi^{1/2}$, from Eq. (48), and

$$\dot{\xi} = \frac{u'f'}{g} = \frac{u'}{2u}, \tag{51}$$

from Eq. (17), we have the result that, when $\dot{\xi} > 0$, either both $u > 0$, $u' > 0$ or both $u < 0$, $u' < 0$. When $\dot{\xi} < 0$, the signs of u and u' must be opposite. It is to be realized that there are two distinct values of u corresponding to any value of ξ except to $\xi = 0$.

The equation of motion is Eq. (50). The second-order equation can be established either by differentiating Eq. (50) or by using Eq. (28). The result is

$$u'' + 4Cu = 0. \tag{52}$$

The solution of Eq. (50) or (52) is

$$u = (2/C)^{1/2} \cos 2C^{1/2}\tau, \tag{53}$$

where the initial conditions $\tau = 0$, $u = u_0 = \xi_0^{1/2} = (2/C)^{1/2}$, and $u_0' = 0$ are used. The last condition follows from Eq. (51), since

$$u_0' = 2u_0\dot{\xi}_0 = 0.$$

Collision corresponds to $\xi = u = 0$, i.e., to $\tau_c = \pi/4C^{1/2}$. The new velocity, from Eq. (53), becomes

$$u' = -\,2(2)^{1/2} \sin 2C^{1/2}\tau \tag{54}$$

and its value at collision is

$$u_c' = -2(2)^{1/2};$$

the same results can also be obtained from Eq. (50) with $u = u_c = 0$.

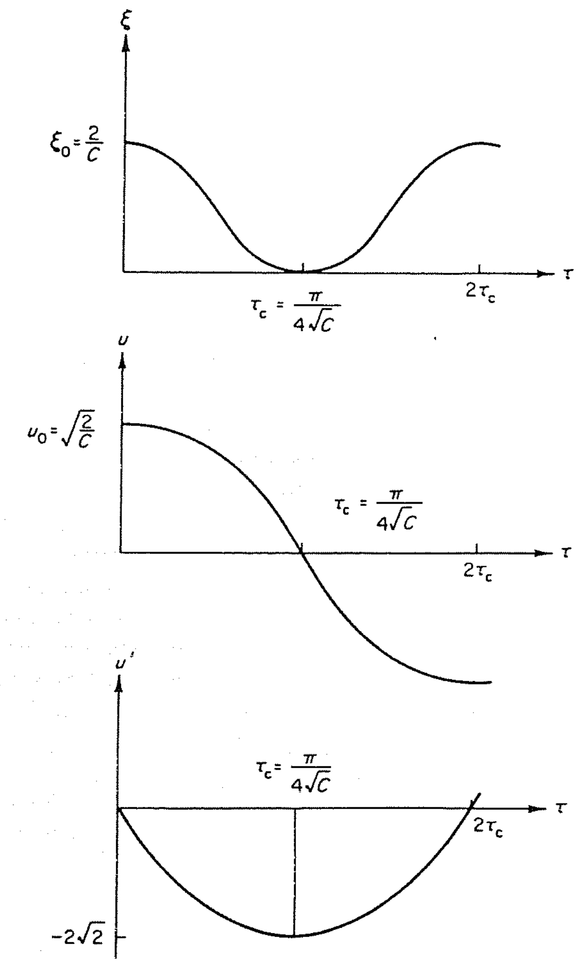

Fig. 3.6. Collision of two bodies with transformed time and coordinate.

In Fig. 3.6 are shown the relations $\xi(\tau)$, $u(\tau)$, and $u'(\tau)$. Only the positive value of the relation $u = \xi^{1/2}$ is shown. Between the initial position and collision, $0 \leqslant \tau \leqslant \tau_c$, $u_0 \geqslant u \geqslant 0$, $\xi_0 \geqslant \xi \geqslant 0$, and $0 \geqslant u' \geqslant -2(2)^{1/2}$. Observe that in this region $0 \geqslant \dot{\xi} \geqslant -\infty$, i.e., $\dot{\xi} < 0$ and $\operatorname{sign} u = -\operatorname{sign} u'$. After the collision in the time domain $\tau_c \leqslant \tau \leqslant 2\tau_c$ the particle returns from the origin to the position $\xi = \xi_0$, which it occupied at $\tau = 0$. During this time $0 \geqslant u \geqslant -u_0$, $0 \leqslant \xi \leqslant \xi_0$, and $-2(2)^{1/2} \leqslant u' \leqslant 0$. We have $\operatorname{sign} u = \operatorname{sign} u'$ in this region and $\dot{\xi} > 0$.

To find the relation between the new and old times τ and t we have $dt/d\tau = 4u^2$ from Eq. (15). Using (53) we obtain

$$t = \int_0^\tau \frac{8}{C} \cos^2 2C^{1/2}\tau \, d\tau,$$

or

$$t = \frac{4}{C}\left(\tau + \frac{\sin 4C^{1/2}\tau}{4C^{1/2}}\right), \tag{55}$$

the general shape of which is also shown in Fig. 3.4, since in this case $\tau_c = \pi/4C^{1/2}$.

There is no essential difference between the double transformation when both the time and the dependent variable are transformed, and the previously shown single time transformation. Both methods regularize the equation of motion and represent the solution. The first regularization results in zero velocity at collision, the second gives $2(2)^{1/2}$.

The transformation of the independent variable regularized the equation of motion which became

$$\xi'' + C\xi - 1 = 0$$

and the double transformation gave

$$u'' + 4Cu = 0.$$

Both these results are of the same mathematical simplicity.

The two-dimensional case to be discussed in the next section differs from the straight-line motion (from the point of view of regularization) in that the integral of energy does not furnish the equations of motion in the two-dimensional case. The introduction of the energy relation (20) into the equation of motion (26) eliminates the quadratic term in the velocity in Eq. (26), resulting in Eq. (28). It will be shown that in the two-dimensional case the introduction of a geometrical (coordinate) transformation, in addition to the time transformation, is necessary to eliminate the quadratic velocity term from the equations of motion.

Inasmuch as the regularization of the restricted problem is performed by double transformations, the above considerations [Eqs. (48) and (55)] might be looked upon as preparatory exercises.

3.3 Regularization of the general problem of two bodies

We now return to Eqs. (4) and study their general form. Introducing $\zeta = \xi + i\eta$ we have $\rho = |\zeta|$ and

$$\ddot{\zeta} = -\zeta/|\zeta|^3. \tag{56}$$

It is recognized that the singularity is at $\zeta = 0$, i.e., at collision; therefore, regularization from a mathematical point of view is of interest only when the conic-section orbit degenerates into a straight-line solution. From a practical point of view so-called close approach orbits also require regularization since, when an actual orbit is to be computed under these circumstances, problems of accuracy arise. For this reason we will regularize Eq. (56) by introducing a coordinate transformation,

$$\zeta = f(w), \tag{57}$$

and a time transformation,

$$dt/d\tau = g(w), \tag{58}$$

quite similar to Eqs. (13) and (14). Here $w = u + iv$, so while in the case of the straight-line motion the transformation established a relation between the points of the physical line ξ and the transformed line u, now the relation is between the physical plane ζ and the transformed plane w. The function $g(w)$ is a real function of the complex variable w, introducing in this way the new time τ as a real quantity.

The first transformation represents a conformal mapping; it contains the geometric information and it controls the accuracy of the shape of the orbit. The second transformation is the essential one, as was shown before, since it controls the kinematic aspects and it performs the regularization. The introduction of two transformations gives greater freedom and it will be shown in this section that significant simplifications of the transformed equations of motion are obtained by properly (and not independently) selecting the functions f and g.

The energy integral in the physical plane is

$$|\dot{\zeta}|^2 = 2/|\zeta| - C, \tag{59}$$

which is identical with the previously given Eq. (5).

Computation of $\dot{\zeta}$ follows the pattern established for the one-dimensional case:

$$\dot{\zeta} = \frac{d\zeta}{dw}\frac{dw}{d\tau}\frac{d\tau}{dt}, \tag{60}$$

or

$$|\dot{\zeta}|^2 = \frac{|f'(w)|^2 |w'|^2}{g^2}. \tag{61}$$

Combining this with Eq. (59) gives the energy integral in the new system of variables:

$$|w'|^2 = \left(\frac{2}{|f|} - C\right)\frac{g^2}{|f'|^2}, \tag{62}$$

similar to Eq. (20). On the left side of Eq. (62) is the square of the new velocity and

$$|w'|^2 = (du/d\tau)^2 + (dv/d\tau)^2. \tag{63}$$

Prior to the treatment of the general situation and in complete analogy to the previously discussed case let $\zeta = w^2$, corresponding to $\xi = u^2$, as given by Eq. (48). Equation (36) suggests for $d\tau/dt$ the reciprocal of the distance; i.e., in the present case $d\tau/dt = r^{-1} = |\zeta|^{-1} = |w|^{-2}$. [The factor 4 which will be applied here as well as in Eq. (49) is of no consequence.] Therefore the transformation equations are

$$\zeta = f(w) = w^2 \tag{64}$$

and

$$\frac{dt}{d\tau} = g(w) = 4\,|w|^2. \tag{65}$$

The energy integral becomes

$$|w'|^2 = 8 - 4\,|w|^2 C \tag{66}$$

and the relation between the physical and the transformed velocities is

$$|\dot{\zeta}| = \frac{|w'|}{2\,|w|}, \tag{67}$$

which generalizes Eq. (51) in a straightforward manner.

The equation of motion using the transformed variables is

$$w'' + 4Cw = 0, \tag{68}$$

which equation can be obtained by transforming the original equation of motion, Eq. (56), and using Eq. (66).

Considering now the equation of motion (56), we write it as

$$\ddot{\zeta} = \mathrm{grad}_\zeta \frac{1}{|\zeta|}, \tag{69}$$

where for the real function F of a complex variable ζ we define

$$\mathrm{grad}_\zeta F(\zeta) = \frac{\partial F}{\partial \xi} + i\,\frac{\partial F}{\partial \eta}. \tag{70}$$

Using the notation

$$U = \frac{1}{|\zeta|} - \frac{C}{2} \tag{71}$$

as before, we have

$$\ddot{\zeta} = \mathrm{grad}_\zeta U. \tag{72}$$

The energy integral, Eq. (59), is

$$|\dot{\zeta}|^2 = 2U. \tag{73}$$

In order to transform Eq. (72) the expression given by Eq. (60) for $\dot{\zeta}$ is used:

$$\dot{\zeta} = \frac{df}{dw}\frac{dw}{d\tau}\frac{d\tau}{dt},$$

which, with

$$\frac{d\tau}{dt} = \dot{\tau}, \qquad \frac{df}{dw} = f'(w), \qquad \frac{dw}{d\tau} = w'(\tau),$$

becomes

$$\dot{\zeta} = f'w'\dot{\tau}. \tag{74}$$

The second derivative is

$$\ddot{\zeta} = f'w'\ddot{\tau} + (f''w'^2 + f'w'')\dot{\tau}^2. \tag{75}$$

The gradient operator will have to be transformed also and it will be shown that

$$\bar{f}'\,\mathrm{grad}_\zeta U = \mathrm{grad}_w U, \tag{76}$$

where the bar denotes the conjugate, i.e., $\bar{f}' = \overline{(df/dw)}$.

Therefore Eq. (72) becomes

$$w'' + w'\frac{\ddot{\tau}}{\dot{\tau}^2} + w'^2\frac{f''}{f'} = \frac{\mathrm{grad}_w U}{\dot{\tau}^2\,|f'|^2}. \tag{77}$$

Computation of $\ddot{\tau}$ requires some care. Since $\dot{\tau} = 1/g$, we have

$$\ddot{\tau}/\dot{\tau}^2 = -\dot{g}. \tag{78}$$

Also, $g = g(w)$ is a real function of a complex variable, and so we may write it as $g = h(w)\,\bar{h}(w) = |h|^2$, where we require differentiability of $h(w)$. The derivative of g becomes

$$\frac{dg}{dt} = \left(h\,\frac{\overline{dh}}{dw}\frac{\overline{dw}}{d\tau} + \bar{h}\,\frac{dh}{dw}\frac{dw}{d\tau}\right)\dot{\tau}. \tag{79}$$

In computing the derivative of $\bar{h}$ we use the fact that

$$(\bar{h})' = \frac{d\bar{h}}{dw} = \overline{\frac{dh}{dw}} = (\overline{h'}).$$

Consequently,

$$\frac{d\bar{h}}{dt} = \overline{\frac{dh}{dw}\,\frac{dw}{d\tau}}\,\frac{d\tau}{dt} = \overline{\frac{dh}{dw}}\;\overline{\frac{dw}{d\tau}}\,\dot{\tau} = \frac{\bar{h}'\bar{w}'}{hh}$$

and

$$\frac{\ddot{\tau}}{\dot{\tau}^2} = -\dot{g} = -\left(\frac{\bar{h}'\bar{w}'}{\bar{h}} + \frac{h'w'}{h}\right).$$

Substitution in Eq. (77) gives

$$w'' - \frac{|\,w'\,|^2}{\bar{h}}\,\overline{\frac{dh}{dw}} + w'^2\left(\frac{f''}{f'} - \frac{h'}{h}\right) = \frac{|\,h\,|^4}{|\,f'\,|^2}\operatorname{grad}_w U. \tag{80}$$

The energy integral can be used to change the second term on the left side since

$$|\,w'\,|^2 = \frac{|\,h\,|^4}{|\,f'\,|^2}\,2U. \tag{81}$$

The equation of motion becomes

$$w'' + (w')^2\frac{d}{dw}\left(\ln\frac{f'}{h}\right) = \frac{|\,h\,|^4}{|\,f'\,|^2}\left(2U\,\overline{\frac{d\ln h}{dw}} + \operatorname{grad}_w U\right), \tag{82}$$

which may also be written as

$$w'' = \operatorname{grad}_w\left|\frac{h^2}{f'}\right|^2 U - 2iw'\operatorname{Im}\left(w'\frac{d}{dw}\ln\frac{f'}{h}\right), \tag{83}$$

where the symbol Im stands for the imaginary part of the complex expression following it.

Before filling in a few steps in the derivation between Eqs. (82) and (83) we note that, if $f' = h$, we have

$$w'' = \operatorname{grad}_w |\,f'\,|^2\,U, \tag{84}$$

and with $\zeta = f(w) = w^2$ as proposed by Eq. (64) we have

$$w'' = \operatorname{grad}_w 4\,|\,w\,|^2\left(\frac{1}{|\,w\,|^2} - \frac{C}{2}\right) = -4Cw, \tag{85}$$

in complete agreement with Eq. (68).

The reader can readily verify Eq. (83) by considering that

(i) $\operatorname{grad}_w g_1(w)\,g_2(w) = g_1 \operatorname{grad}_w g_2 + g_2 \operatorname{grad}_w g_1$,

where g_1 and g_2 are real functions of the complex variable w, and that

(ii) $\operatorname{grad}_w |\,G(w)\,|^2 = 2G\,\overline{\dfrac{dG}{dw}}$,

where G is a complex function of the complex variable w.

The proof of Statement (i) is left to the reader, while that of (ii) is shown as follows.

Since $G(w)$ is analytic, we have

$$G_u = \frac{\partial G}{\partial u} = \frac{dG}{dw} = -iG_v \qquad\text{and}\qquad \bar{G}_u = \overline{\frac{dG}{dw}} = i\bar{G}_v\,.$$

Furthermore,

$$\operatorname{grad}_w |\,G\,|^2 = G\operatorname{grad}_w \bar{G} + \bar{G}\operatorname{grad}_w G,$$

where

$$\operatorname{grad}_w G = G_u + iG_v = 0$$

and

$$\operatorname{grad}_w \bar{G} = \bar{G}_u + i\bar{G}_v = 2\,\overline{\frac{dG}{dw}}\,.$$

Q.E.D.

At this point Eq. (76) is also proved as follows. Consider the expanded form of $\operatorname{grad}_w U(w) = U_u + iU_v$,

$$\operatorname{grad}_w U = U_\xi \xi_u + U_\eta \eta_u + i(U_\xi \xi_v + U_\eta \eta_v).$$

The Cauchy–Riemann relations transform the imaginary part into

$$-U_\xi \eta_u + U_\eta \xi_u\,,$$

and with this expression $\operatorname{grad}_w U$ becomes

$$(\xi_u - i\eta_u)(U_\xi + iU_\eta) = \bar{f}'\operatorname{grad}_\zeta U.$$

Q.E.D.

Note that, with the selection of $h = f'$, the nonlinear part of Eq. (83) (regarding w') disappears as anticipated. The consequence of this selection is that $g = |\,f'\,|^2$, and the time transformation is related to the coordinate transformation by

$$dt/d\tau = |\,f'\,|^2. \tag{86}$$

The elegance and simplicity of Eq. (68) cannot but fill the reader with joy. The original equation of motion [Eq. (56)],

$$d^2\zeta/dt^2 + \zeta/|\zeta|^3 = 0,$$

is replaced by Eq. (68),

$$d^2w/d\tau^2 + 4Cw = 0,$$

the solution of which can be immediately written as

$$w = A\cos 2C^{1/2}\tau + B\sin 2C^{1/2}\tau \qquad \text{for} \quad C > 0 \tag{87}$$

or as

$$w = A\cosh 2(-C)^{1/2}\tau + B\sinh 2(-C)^{1/2}\tau \qquad \text{for} \quad C < 0, \tag{88}$$

and finally, for $C = 0$, we have

$$w = A + B\tau. \tag{89}$$

Such a uniform presentation of the problem of two bodies is, of course, not new and the uniformization of the introduction of universal variables was not the primary purpose but only a side product of this discussion. Nevertheless, the process of regularization also accomplished the introduction of such variables.

In summary we can make the following statements regarding the regularization of the problem of two bodies:

The transformations $\zeta = w^2$ and

$$t = 4\int_0^\tau |w|^2\, d\tau$$

give the equations of motion as

$$d^2w/d\tau^2 + 4Cw = 0,$$

where the constant

$$C = \frac{2}{|\zeta|} - |\dot\zeta|^2$$

is determined by the initial conditions. The initial conditions are transformed from the values ζ_0, $\dot\zeta_0$ to the values w_0, $w_0' = (dw/d\tau)_0$, according to $w_0 = \zeta^{1/2}$ and $w_0' = 2\dot\zeta_0\bar{w}_0$.

The following generalization of the preceding statements is also available and is based on the result given as Eq. (84).

The second-order differential equation

$$\ddot\zeta = \operatorname{grad}_\zeta U$$

can be transformed into

$$w'' = \operatorname{grad}_w |f'(w)|^2 U,$$

where the geometric transformation, $\zeta = f(w)$, and the time transformation, $dt = |f'(w)|^2\, d\tau$, are governed by the same function $f(w)$. If the time transformation is related to the function $f(w)$ in any other way, then the quadratic term in the first derivative will be present in the transformed equation.

In the above equation

$$\operatorname{grad}_w |f'(w)|^2 U = \frac{\partial}{\partial u}|f'|^2 U + i\frac{\partial}{\partial v}|f'|^2 U,$$

and the new (transformed) and original (physical) velocities are connected by

$$\dot\zeta = f'w'/|f'|^2 \qquad \text{or} \qquad w' = \bar{f}'\dot\zeta.$$

3.4 Local regularization of the restricted problem

The general formulation is not different from the problem of two bodies. We introduce the two transformations again as

$$z = f(w) \tag{90}$$

and

$$dt/d\tau = g(w) = |h(w)|^2,$$

similar to Eqs. (57) and (58). Note that Eq. (90) establishes a relation between z and w instead of between ζ and w since we deal with transformations of the nondimensional rotating coordinate system (z).

The equation of motion of the restricted problem in a complex form is

$$\ddot z + 2i\dot z = \operatorname{grad}_z U, \tag{91}$$

which, after performing the transformations, becomes

$$w'' + 2i\,g(w)\,w' = \operatorname{grad}_w \left|\frac{h^2}{f'}\right|^2 U - 2iw' \operatorname{Im}\left(w'\frac{d}{dw}\ln\frac{f'}{h}\right), \tag{92}$$

with

$$U \equiv \Omega - C/2.$$

The derivation of Eq. (92) is similar to that of Eq. (83) except that the term linear in w' enters into (92). It is once again observed that, if $f' = h$ or $g(w) = |f'|^2$, the equation of motion becomes

$$w'' + 2i|f'|^2 w' = \operatorname{grad}_w |f'|^2 U. \tag{92'}$$

The Jacobian integral in the physical system is

$$|\dot{z}|^2 = 2U, \tag{93}$$

which, in the transformed system, becomes

$$|dw/d\tau|^2 = 2\,|f'|^2 U. \tag{94}$$

The physical velocity $|\dot{z}| \to \infty$ at the singularities of the function U, where $r_1 \to 0$ or $r_2 \to 0$.

Example. Derive the transformed equations of motion (92) from Eq. (91) using steps similar to the derivation of the regularized equations of the two-body problem.

The selection of the time transformation associated with the coordinate transformation is governed in Sections 3.3 and 3.4 by Eqs. (83) and (92), respectively. According to these equations if

$$dt/d\tau = g(w) = |h(w)|^2 = |f'(w)|^2,$$

then the regularized equation is linear in the first and second derivatives of the new dependent variable w. With another choice for $g(w)$, once $f(w)$ is selected, $(w')^2$ or its equivalent appears in the transformed equations. There are, of course, other possible choices for $g(w)$. One principle for the selection of the function governing the time transformation proposed by Arenstorf is particularly pleasing because of its dynamic significance. In this case we require that any two "equivalent" solutions of the regularized equation (92) show geometric and dynamic correspondence with respect to their time variable. Equivalent solutions in the w plane, say $w_1(\tau)$ and $w_2(\tau)$, are those which map into the same solution curve in the z plane. Such equivalence possesses only geometric significance but if we require that, for a given value of τ_0, points on the regularized solutions $w_1(\tau_0)$ and $w_2(\tau_0)$ must map into the same $z(t_0)$ point, then we achieve dynamic equivalence as well. Note that the relation between the two times t and τ depends on the solution in the w plane since

$$\tau = \int_0^t \frac{dt}{g(w)},$$

and consequently the requirement of dynamic equivalence is not necessarily satisfied. We return to this question at the end of Section 3.7.

The function $z = f(w) = w^2$ regularized the singularity at the origin of the two-body problem. The transformation function $Z = z - \mu$ moves the point $z = \mu$ of the z plane to the origin of the Z plane; therefore, the transformation $Z = w^2$ will again regularize the singularity at the origin of the Z plane, and consequently also at the point $z = \mu$ of the z plane. Since in the restricted problem the primaries are located at $z_1 = \mu$ and at $z_2 = \mu - 1$, the transformation $z = \mu + w^2$ will regularize the singularity at m_1 and the transformation $z = \mu - 1 + w^2$ will handle the singularity at m_2. Such transformations are called *local* regularizations (after Birkhoff) since, by selecting either one of the above transformations, only one of the singularities is eliminated.

Consider the function

$$z = f(w) = w^2 + \mu, \tag{95}$$

which transforms the point $P_1(\mu, 0)$ of the physical plane to the origin of the w plane and at the same time the point $P_2(\mu - 1, 0)$ is transformed to $w_{1,2} = \pm i$. The origin of the physical plane moves to $w_{1,2} = \pm i\mu^{1/2}$ and the upper half of the z plane ($y > 0$) transforms into the first quadrant of the w plane ($u > 0$, $v > 0$). Since $w = \pm(z - \mu)^{1/2}$ the $y > 0$ part of the z plane also transforms into the third quadrant of the w plane ($u < 0$, $v < 0$). The relation between points of the z and w planes is a one-to-two correspondence; to every point (except $z = \mu$) in the z plane corresponds two distinct points in the w plane. This is the well-known property of the function given by Eq. (95) which takes the one leaf of the w plane into the two leaves of the z plane. Figures 3.7(a) and (b) show the pertinent information regarding the transformation $z = w^2 + \mu$. Figure 3.7(a) illustrates the transformation of the important points with their coordinates. Figure 3.7(b) shows the mapping of the z plane into the w plane. The 90° sector marked 1 of the z plane becomes the two 45° sectors of the w plane. The line segment to the left of P_2, along the negative x axis marked a, transforms into segments along the v axis. Note that the sectors in the z plane are separated by the vertical line crossing the x axis at P_1 and *not* by the y axis.

Transformation of $U = \Omega - (C/2)$ into the new variable w requires the expression of r_1 and r_2 in terms of w. Since $r_1 = |z - \mu|$ and $r_2 = |z - \mu + 1|$, we have $r_1 = |w|^2$ and $r_2 = |1 + w^2|$. The function U is given by

$$U = \tfrac{1}{2}[(1-\mu)r_1^2 + \mu r_2^2] + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} - \frac{C}{2}, \tag{96}$$

or by

$$U = \tfrac{1}{2}[(1-\mu)|w|^4 + \mu|1 + w^2|^2] + \frac{1-\mu}{|w|^2} + \frac{\mu}{|1+w^2|} - \frac{C}{2}. \tag{97}$$

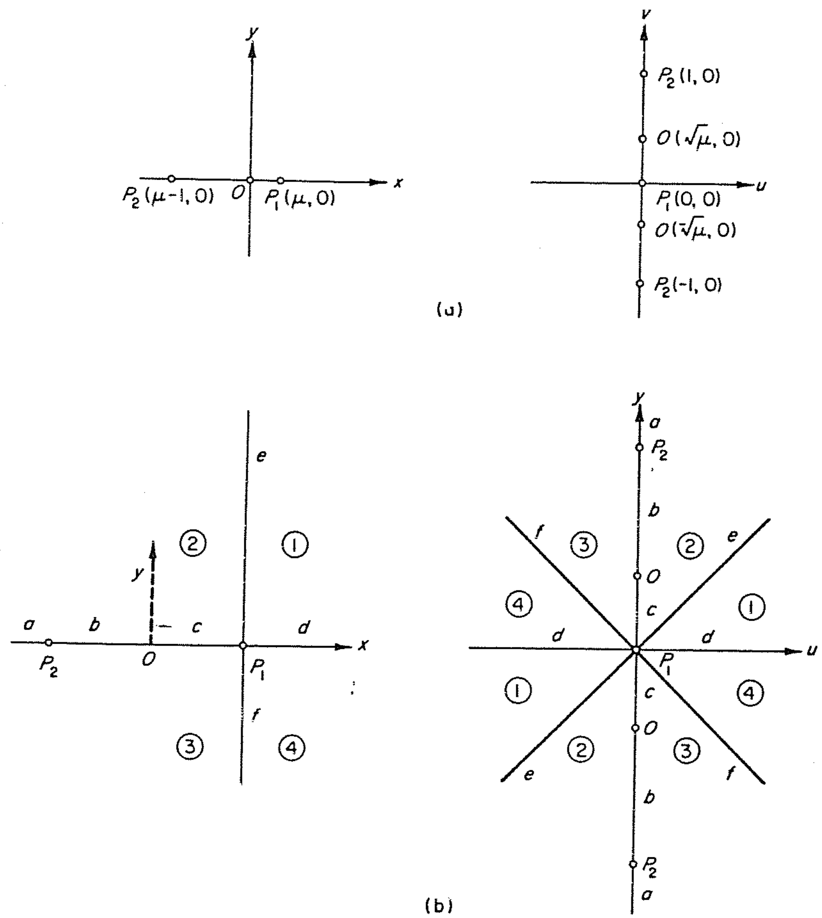

FIG. 3.7. The $z = w^2 + \mu$ transformation of Levi-Civita.

Since $|f'|^2 = 4|w|^2$, the new velocity from Eq. (94) becomes

$$\left|\frac{dw}{d\tau}\right|^2 = 8(1-\mu) + |w|^2\left[\frac{8\mu}{|1+w^2|} + 4(1-\mu)|w|^4 + \mu|w^2+1|^2 - 4C\right]. \tag{98}$$

At m_1, $r_1 = 0$, $z = \mu$, $w = 0$, and Eq. (98) gives the finite velocity in the w plane as

$$|dw/d\tau| = 2[2(1-\mu)]^{1/2}. \tag{99}$$

At m_2, $r_2 = 0$, $z = \mu - 1$, $w = \pm i$, and Eq. (98) gives infinite velocity.

The differential equations of motion in the w plane, according to Eq. (92′), contain the partial derivatives of $|f'|^2 U$. These can be obtained from

$$U|f'|^2 = 2\Big[(u^2+v^2)^3 + 2\mu(u^4 - v^4) + (\mu - C)(u^2 + v^2) + 2(1-\mu) + \frac{2\mu(u^2+v^2)}{[(u^2+v^2)^2 + 1 + 2(u^2 - v^2)]^{1/2}}\Big]. \tag{100}$$

The equations of motion from Eq. (92′) become

$$\begin{aligned} u'' - 8(u^2+v^2)v' &= (U|f'|^2)_u, \\ v'' + 8(u^2+v^2)u' &= (U|f'|^2)_v, \end{aligned} \tag{101}$$

since $|f'|^2 = 4(u^2 + v^2)$.

At the point $P_1(\mu, 0)$ in the physical plane we have $w = 0$, grad $U|f'|^2 = 0$, and $|w'|$ is finite; therefore, at collision with m_1 we have $w'' = 0$. Equation (100) shows that, at $P_2(\mu - 1, 0)$ in the physical plane, $w = \pm i$, $U|f'|^2 \to \infty$, $|w'| \to \infty$, and $\text{grad}_w\, U|f'|^2$ is not defined.

The functions $z = \mu + w^2$ and $z = \mu - 1 + w^2$ are also called Levi-Civita's transformations since they are equivalent to the transformations introduced by Levi-Civita for the two-body problem. The fact that a two-body regularization eliminates the local singularities of the restricted problem is not surprising. Finite perturbing effects near and at the singularity become zero when compared to the effect of the singularity.

Nevertheless, the question arises regarding the possibility of regularizing both singularities simultaneously, i.e., of finding a transformation which allows collision with both primaries. We do not speak of simultaneous collision of the three bodies participating in the restricted problem, of course, since two of the bodies are moving in concentric circular orbits. By simultaneous regularization we mean a transformation which with *one* single function $z = f(w)$ establishes new equations of motion in a form in which points of collision with both m_1 and m_2 are regular points. Such transformations accomplish *global* regularization as opposed to *local* regularization, and these transformations are discussed in the next section.

3.5 Birkhoff's global regularization of the restricted problem

In order to present various regularizing transformations in a concise manner, the origin of the coordinate system is changed. Instead of using the mass center of the primaries as the origin of the coordinate system, we select the midpoint between the primaries and let $q = z + \frac{1}{2} - \mu$ be the new complex variable. The primaries are now located at $q = \pm\frac{1}{2}$ and the equations of motion become

$$\ddot{q} + 2i\dot{q} = \mathrm{grad}_q\, U(q), \tag{102}$$

where

$$U(q) = \Omega(q) - C/2 \tag{103}$$

and

$$\Omega(q) = \tfrac{1}{2}[(1-\mu)r_1^2 + \mu r_2^2] + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \tag{104}$$

as before, but now

$$r_1 = |q - \tfrac{1}{2}| \quad \text{and} \quad r_2 = |q + \tfrac{1}{2}|, \tag{105}$$

or

$$r_1 = [(q_1 - \tfrac{1}{2})^2 + q_2^2]^{1/2} \quad \text{and} \quad r_2 = [(q_1 + \tfrac{1}{2})^2 + q_2^2]^{1/2}. \tag{106}$$

The Jacobian integral is

$$|\dot{q}|^2 = 2\Omega(q) - C = 2U(q). \tag{107}$$

The complex vector q and the associated coordinate system are shown in Fig. 3.8.

The regularizing function is $q = f(w)$, and we experiment with a generalization of the transformation first suggested by Birkhoff,

$$q = \alpha w + \beta/w = f(w). \tag{108}$$

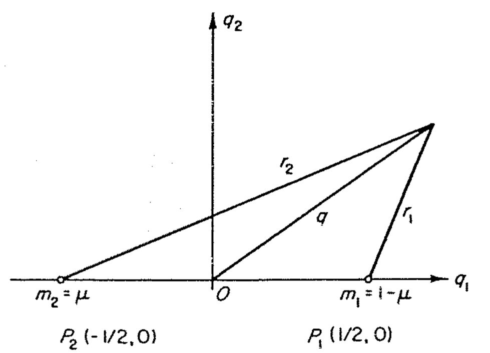

Fig. 3.8. Rotating coordinate system with midpoint as origin.

From this

$$|f'(w)|^2 = \frac{|\alpha w^2 - \beta|^2}{|w|^4}. \tag{109}$$

The constants α and β are determined by two conditions. It is required that the function $f(w)$ should eliminate both singularities, and the points P_1 and P_2 be fixed points of the transformation. Regarding the first requirement we must consider the product of the critical part of $\Omega(q)$,

$$\frac{1-\mu}{r_1} + \frac{\mu}{r_2} = \frac{1-\mu}{|\alpha w + \beta/w - 1/2|} + \frac{\mu}{|\alpha w + \beta/w + 1/2|}$$

with $|f'(w)|^2$. This product can be written as

$$\frac{1}{|w|^3}\left[\frac{(1-\mu)\,|\alpha w^2 - \beta|^2}{|\alpha w^2 + \beta - w/2|} + \frac{\mu\,|\alpha w^2 - \beta|^2}{|\alpha w^2 + \beta + w/2|}\right]. \tag{110}$$

One singularity is located at $q = \frac{1}{2}$ corresponding to

$$w_{1,2} = \frac{1}{4\alpha}[1 \pm (1 - 16\alpha\beta)^{1/2}], \tag{111}$$

which result follows from Eq. (108) by writing $q = \frac{1}{2}$ and solving the equation for w. Equation (111) also gives, of course, the roots of the denominator of the first term in the product given by (110). In order to eliminate these singularities we must have the same roots present in the numerator, i.e.,

$$\frac{1}{4\alpha}[1 \pm (1 - 16\alpha\beta)^{1/2}] = \pm\left(\frac{\beta}{\alpha}\right)^{1/2}$$

or

$$16\alpha\beta = 1. \tag{112}$$

This result shows immediately, from Eq. (111), that $w_1 = w_2 = \frac{1}{4}\alpha$, i.e., that m_1 transforms into one point. If P_1 is to be a fixed point of the transformation, we must have $\frac{1}{4}\alpha = \frac{1}{2}$ or $\alpha = \frac{1}{2}$. Equation (112) then gives $\beta = \frac{1}{8}$ and the transformation function becomes

$$q = \tfrac{1}{4}(2w + 1/2w). \tag{113}$$

The second term in the critical part of $|f'|^2\,\Omega(q)$ has a singularity at P_2 and its elimination is identical with the previous procedure.

It is important to note the factor $1/|w|^3$ appearing in the new potential function as given by (110). This brings in a new singularity as $w = 0$, corresponding to $q \to \infty$. While singularities at P_1 and P_2 are eliminated in the transformed plane, its origin becomes a new singular point. This, however, corresponds to infinity in the physical plane and therefore we may say that all points of the finite physical plane have been regularized by the transformation.

The various equations describing the transformed system can be written in an elegant form by introducing the distances ρ_1, ρ_2, and ρ in the w plane corresponding to the quantities r_1, r_2, and r in the physical q plane. Let

$$\rho_1 = |w - \tfrac{1}{2}|, \qquad \rho_2 = |w + \tfrac{1}{2}|, \qquad \rho = |w|. \tag{114}$$

Then

$$|f'(w)|^2 = \frac{|4w^2 - 1|^2}{64|w|^4},$$

or

$$|f'(w)|^2 = \frac{\rho_1^2\rho_2^2}{4\rho^4}. \tag{115}$$

The function $\Omega(q)$ also becomes simpler using these variables since

$$r_1 = \rho_1^2/2\rho, \qquad r_2 = \rho_2^2/2\rho, \qquad |f'|^2 = r_1 r_2/\rho^2. \tag{116}$$

The equations of motion require the evaluation of the function $U|f'|^2 = (\Omega - C/2)|f'|^2$, which in our case becomes

$$U|f'|^2 = \frac{\rho_1^2\rho_2^2}{32\rho^6}[(1-\mu)\rho_1^4 + \mu\rho_2^4] + \frac{1}{2\rho^3}[(1-\mu)\rho_2^2 + \mu\rho_1^2] - \frac{C}{2}\left(\frac{\rho_1\rho_2}{2\rho^2}\right)^2. \tag{117}$$

Note that the only singularity is at $\rho = 0$, corresponding to $r_1 \to \infty$, $r_2 \to \infty$, $|w| = 0$, and $|q| \to \infty$ as mentioned before. The origin of the w plane and the corresponding infinity of the physical plane q become singularities.

Figures 3.9–3.11 show the geometry of the transformation

$$q = \tfrac{1}{4}(2w + 1/2w).$$

The points $P_1(\frac{1}{2}, 0)$, $P_2(-\frac{1}{2}, 0)$ are fixed points of the transformation as the reader can verify by substitution. The origin of the q plane becomes the points $w = \pm i/2$ marked by $\mathcal{O}_{1,2}(0, \pm\frac{1}{2})$. The circular arc in the first quadrant of the w plane with radius $\frac{1}{2}$ and connecting the points $P_1(\frac{1}{2}, 0)$ and $\mathcal{O}_1(0, \frac{1}{2})$ is the mapping of the $\overline{P_1\mathcal{O}}$ segment of the q_1 axis since with

$$w = \tfrac{1}{2}e^{is}$$

we have

$$q = \tfrac{1}{2}\cos s.$$

As $0 \to s \to \pi/2$ this path is described. Also, as $0 \to s \to -\pi/2$, the path in the q plane is the same, from P_1 to 0 along the positive q_1 axis, while the corresponding path in the w plane is the quarter of a circle with radius $\frac{1}{2}$, connecting $P_1(\frac{1}{2}, 0)$ with $\mathcal{O}_2(0, -\frac{1}{2})$. The correspondence between the points of the q plane and of the w plane is one-to-two, in fact

$$4w^2 - 8wq + 1 = 0, \tag{118}$$

and therefore $w_1w_2 = \frac{1}{4}$. This means that two values of w corresponding to a single value of q are separated by the real axis of the w plane and, if $|w_1| \leqslant \frac{1}{2}$, then $|w_2| \geqslant \frac{1}{2}$ and vice versa. While the geometric mean of the two values of w corresponding to the same q is $\frac{1}{2}$, $(w_1w_2)^{1/2} = \frac{1}{2}$, their arithmetic mean is q since $(w_1 + w_2)/2 = q$.

The mapping of the q plane into the w plane is shown in Figs. 3.9 and 3.10. Figure 3.11 shows the construction of the vectors q, w_1, and w_2. In this figure $\overline{\mathcal{O}A} = w_1$, $\overline{\mathcal{O}C} = w_2$, $\overline{\mathcal{O}B} = q$, $\overline{\mathcal{O}D} = |4w_1|^{-1} = |w_2| = \overline{\mathcal{O}C}$, $\overline{CB} = \overline{AB}$, and $\overline{AF} = \overline{AE}$. The construction starts with w_1 and with the drawing of the tangents to the circle of radius $\frac{1}{2}$ from A. In this way points E, F, and D are obtained. The endpoint of w_2 is C which is the mirror image of point D with respect to the line P_1P_2. The endpoint of q is at the midpoint between A and C.

The velocity expressed in the regularized system of variables is

$$(du/d\tau)^2 + (dv/d\tau)^2 = 2U|f'|^2. \tag{119}$$

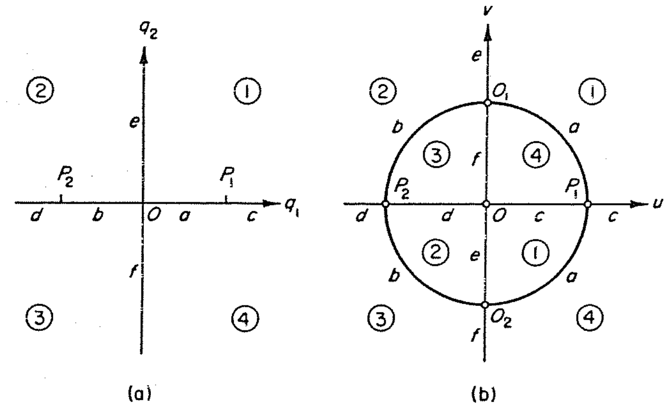

FIG. 3.9. Corresponding sectors and segments in Birkhoff's transformation: (a) the physical plane q; (b) the regularized plane w.

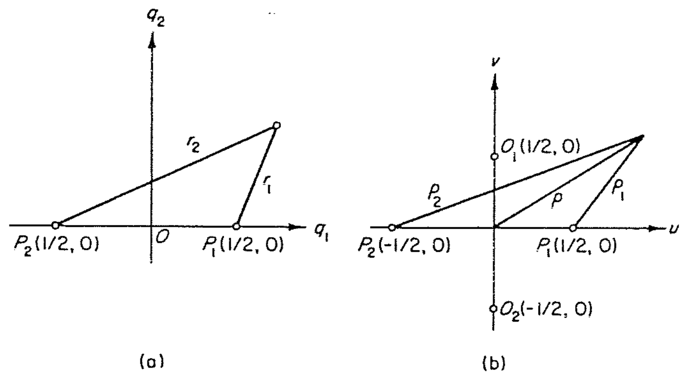

FIG. 3.10. Corresponding points in Birkhoff's transformation and definition of the distances ρ_1, ρ_2, and ρ: (a) the physical plane q; (b) the regularized plane w.

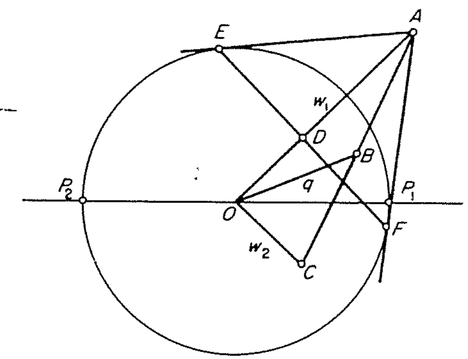

FIG. 3.11. Construction of the complex vectors q, w_1, and w_2.

At the first primary P_1, $\rho_1 = 0$, $\rho_2 = 1$, $\rho = \frac{1}{2}$ and Eq. (117) gives $2U|f'|^2 = 8(1-\mu)$, so the velocity from Eq. (119) becomes $2[2(1-\mu)]^{1/2}$. Similarly, at P_2 the velocity is $2(2\mu)^{1/2}$. These finite values of the velocities are expected since P_1 and P_2 are regular points.

It is of more interest to note that the absolute value of the transformed velocity is fixed at the primaries and therefore it is not subject to arbitrary selection once the mass ratio is determined. In other words, the initial conditions for an orbit at, say, m_1 are determined by the location of m_1: $u = \frac{1}{2}$, $v = 0$, and, by the direction of the velocity vector at this point, say $\tan\alpha = v'/u'$. The fourth-order system of differential equations for u and v contains the parameter C. The solution of the fourth-order system requires knowledge of the value of C, which,

together with $u = \frac{1}{2}$, $v = 0$, and $\alpha = \alpha_0$, furnishes the four initial conditions. It is to be noted that initial conditions cannot be given at the singular points in the physical plane since, at $q = q_1 = \pm\frac{1}{2}$, $(\dot{q})^2 \to \infty$ (and C is not defined).

3.6 The Thiele-Burrau transformation

This global regularization was first proposed by T. N. Thiele for $\mu = \frac{1}{2}$, and his basic idea was later generalized by C. Burrau for arbitrary values of μ. The transformation is the same as used by Euler to integrate the problem of two fixed gravitational centers.

We again start with the synodic coordinate system, the origin of which is located at the midpoint between the primaries. The equations of motion, the Jacobian integral, and various other equations of interest in this system are given by Eqs. (102)–(107). The Thiele–Burrau transformation is described by

$$q = \tfrac{1}{2}\cos w \tag{120}$$

or

$$q = \tfrac{1}{4}(e^{iw} + e^{-iw}). \tag{121}$$

Equation (121) exhibits a close relation between Eqs. (113) and (121). In fact if $w = w_B$ in Eq. (113), where the subscript B refers to Birkhoff's transformation, and if $w = w_T$ in (121), then we have

$$w_B = \tfrac{1}{2}e^{iw_T}. \tag{122}$$

From (120)

$$|f'|^2 = \tfrac{1}{4}(\cosh^2 v - \cos^2 u) = \tfrac{1}{8}(\cosh 2v - \cos 2u), \tag{123}$$

since $f' = -\frac{1}{2}\sin w = -\frac{1}{2}(\sin u \cosh v + i\cos u \sinh v)$.

The r, r_1, and r_2 distances become

$$r = \tfrac{1}{2}|\cos w| = \tfrac{1}{2}(\sinh^2 v + \cos^2 u)^{1/2} = \frac{1}{2(2)^{1/2}}(\cosh 2v + \cos 2u)^{1/2}, \tag{124}$$

$$\begin{aligned} r_1 &= \tfrac{1}{2}|\cos w - 1| = \tfrac{1}{2}(\cosh v - \cos u), \\ r_2 &= \tfrac{1}{2}|\cos w + 1| = \tfrac{1}{2}(\cosh v + \cos u). \end{aligned} \tag{125}$$

From Eqs. (120) and (125)

$$|f'(w)|^2 = r_1 r_2,$$

and therefore the elimination of both singularities can be expected when $\Omega(q)$ is multiplied by $|f'|^2$. In fact

$$U|f'|^2 = \frac{r_1 r_2}{2}[(1-\mu)r_1^2 + \mu r_2^2 - C] + (1-\mu)r_2 + \mu r_1, \tag{126}$$

and, after substituting from Eqs. (125), we have

$$\begin{aligned} U|f'|^2 &= \frac{1}{16}(\cosh 2v - \cos 2u) \\ &\quad \times [\tfrac{1}{4} - C + \tfrac{1}{8}(\cosh 2v + \cos 2u) + (\mu - \tfrac{1}{2})\cosh v \cos u] \\ &\quad + (\tfrac{1}{2} - \mu)\cos u + \tfrac{1}{2}\cosh v = \Omega^*(u, v). \end{aligned} \tag{127}$$

The geometry of the transformation follows from Eq. (120) which may be written as

$$q_1 = \tfrac{1}{2}\cos u \cosh v, \qquad q_2 = -\tfrac{1}{2}\sin u \sinh v.$$

The lines $u = \text{const}$ are hyperbolas and the lines $v = \text{const}$ are ellipses in the physical plane q_1, q_2, since

$$\frac{q_1^2}{\cos^2 u} - \frac{q_2^2}{\sin^2 u} = \frac{1}{4} \quad \text{and} \quad \frac{q_1^2}{\cosh^2 v} + \frac{q_2^2}{\sinh^2 v} = \frac{1}{4}. \tag{128}$$

The center of the confocal hyperbolas and ellipses is at the origin $q = 0$, and the focuses are at the primaries.

The origin of the physical plane $q = 0$ transforms into the points $v = 0$, $u = \pm\pi/2, \ldots, \pm(2n+1)\pi/2$. The origin of the w plane

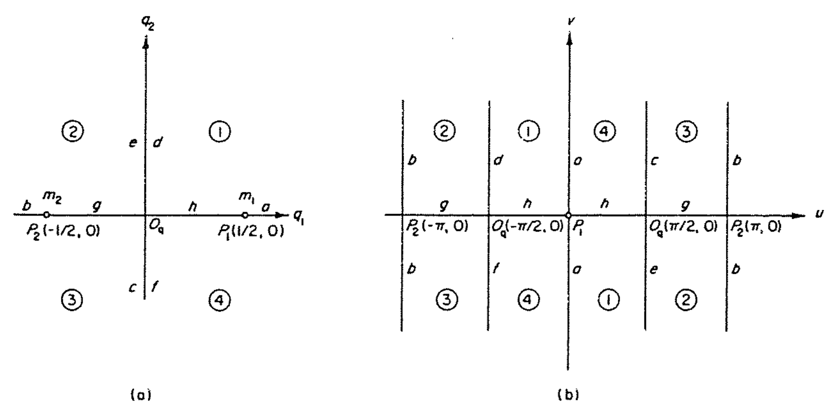

FIG. 3.12. The Thiele–Burrau transformation: (a) the physical plane q; (b) the regularized plane w.

corresponds to m_1 in the physical plane and points of the w plane corresponding to m_1 have the coordinates $v = 0$, $u = 0, \ldots, \pm 2k\pi$. The second primary m_2, located at $P_2(-\frac{1}{2}, 0)$ in the physical plane, transforms into $v = 0$, $u = \pm\pi, \ldots, \pm(2n+1)\pi$. The pertinent features of the transformation are shown in Fig. 3.12.

An expression for the velocity can again be obtained from Eq. (119):

$$(du/d\tau)^2 + (dv/d\tau)^2 = 2\Omega^*(u, v).$$

At m_1, $u = v = 0$, $2\Omega^*(u, v) = 2(1-\mu)$, and, at m_2, $u = \pi$, $v = 0$, $2\Omega^*(u, v) = 2\mu$. Therefore the absolute value of the regularized velocity at m_1 is $[2(1-\mu)]^{1/2}$ and at m_2 the regularized velocity becomes $(2\mu)^{1/2}$.

The differential equations of motion give the same results at collisions $u'' = v'' = 0$, as found using the Levi-Civita transformation, Eq. (101).

3.7 Lemaître's regularization

This transformation may be related to Birkhoff's transformation when in Eq. (113) we write $w_B = w$,

$$q = \tfrac{1}{4}(2w_B + 1/2w_B),$$

and then substitute $2w_B = w_L^2$. In this way we obtain

$$q = \tfrac{1}{4}(w_L^2 + 1/w_L^2),$$

and therefore the transformation to be investigated is

$$q = \tfrac{1}{4}(w^2 + 1/w^2), \tag{129}$$

with

$$|f'(w)|^2 = \frac{|w^4 - 1|^2}{4|w|^6} = \frac{dt}{d\tau}. \tag{130}$$

The distances of the third particle from m_1 and m_2 become

$$r_1 = \frac{|w^2 - 1|^2}{4|w|^2} \quad \text{and} \quad r_2 = \frac{|w^2 + 1|^2}{4|w|^2}. \tag{131}$$

The formula for $|f'(w)|^2$ therefore becomes

$$|f'(w)|^2 = 4\frac{r_1 r_2}{|w|^2}. \tag{132}$$

Elimination of both singularities is accomplished. In fact

$$U|f'|^2 = \frac{|1-w^4|^2}{128|w|^{10}}[-16C|w|^4 + |1-w^2|^4 + \mu(|1-w^2|^4 - |1+w^2|^4)]$$

$$+ \frac{1}{|w|^4}[(1-\mu)|1+w^2|^2 + \mu|1-w^2|^2] \quad \Omega^*(u, v). \qquad (133)$$

The geometry of the transformation is somewhat involved. The primaries appear in the w plane at the points $w = \pm 1$ and $w = \pm i$, since from Eq. (129) we have $q = \frac{1}{2}$ for $w = \pm 1$ and $q = -\frac{1}{2}$ when $w = \pm i$. The origin of the physical plane ($q = 0$) becomes $w = (-1)^{1/4}$, and the origin of the regularized plane ($w = 0$) corresponds to the infinity point of the physical plane. Note that $\Omega^*(u, v)$ is regular everywhere in the w plane except at $w = 0$. Lemaître's transformation with the corresponding regions of the q and w planes is shown in Fig. 3.13.

It is to be noted that the transformation orders four points of the w plane to every point of the q plane with the exception of the primaries; this follows from Eq. (129):

$$w^4 - 4qw^2 + 1 = 0, \qquad (134)$$

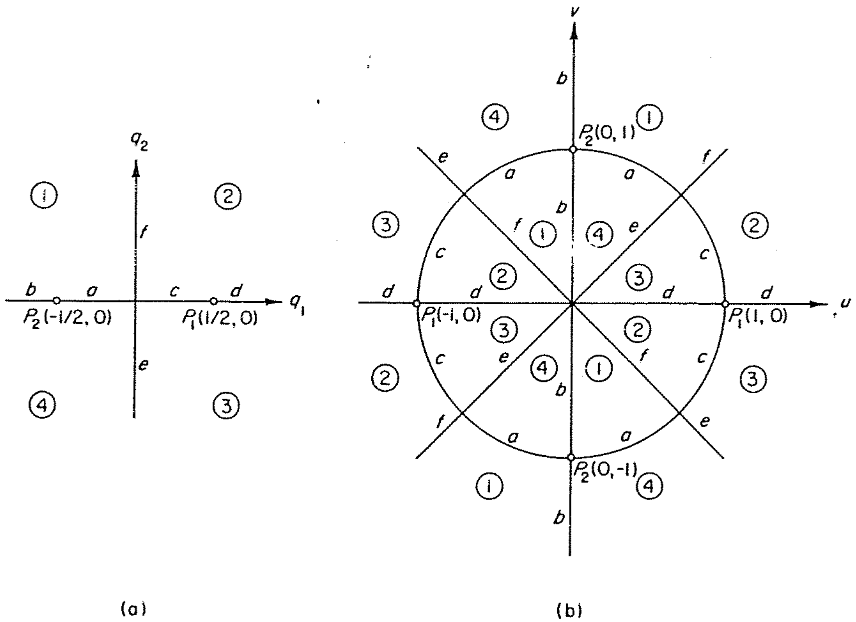

FIG. 3.13. Mapping of Lemaître's transformation: (a) physical plane q; (b) regularized plane w.

the solution of which is

$$w = \pm [2q \pm (4q^2 - 1)^{1/2}]^{1/2}. \qquad (135)$$

Here the two $\pm$ signs are to be taken independently so as to obtain all four roots of Eq. (134). If $4q^2 = 1$, and $q = \pm\frac{1}{2}$, we have $w = \pm(\pm 1)^{1/2}$, $w_{1,2} = \pm 1$, and $w_{3,4} = \pm i$, corresponding to the location of the primaries.

The fixed points of the transformation may be obtained from the solution of Eq. (134) when q is substituted for w:

$$w^4 - 4w^3 + 1 = 0. \qquad (136)$$

Fixed points along the real axes are solutions of the equation

$$h(u) = u^4 - 4u^3 + 1 = 0.$$

According to Descartes' sign rule $h(u) = 0$ has no negative roots and it may have two positive roots. Since $h(+\frac{1}{2}) > 0$ and $h(+1) < 0$, one of the positive roots is located in the region d on the q plane, on the positive q_1 axis between the first primary m_1 and the point $q_1 = +1$. Inasmuch as the minimum of $h(u)$ is at $u = +3$, $h(+3) < 0$, and $h(+4) = +1$, the other positive root is between $+3 < w = q = u = q_1 < +4$, so it is located along the positive u axis outside the unit circle in the region d. The other two roots of the equation $h(u) = 0$ are complex and therefore are of no interest since u is real. There are no other fixed points of the transformations.

In order to obtain the transformed velocities at the points of collision, $\Omega^*(u, v)$ from Eq. (133) at $w = \pm 1$ (corresponding to m_1) and at $w = \pm i$ (corresponding to m_2) is evaluated. The first term of Eq. (133) does not contribute to the value of Ω^* at these points, and we get

$$\Omega^*(P_1) = 4(1 - \mu) \qquad \text{and} \qquad \Omega^*(P_2) = 4\mu.$$

The Jacobian integral in the transformed system is

$$u'^2 + v'^2 = 2\Omega^*,$$

and therefore the magnitude of the regularized velocity at m_1 is $[2(1-\mu)]^{1/2}$ and at m_2 it becomes $(2\mu)^{1/2}$.

The regularizing transformation introduced by Arenstorf is an interesting modification of the method described in this section. In fact Eq. (129) giving Lemaître's transformation is equivalent to the function proposed by Arenstorf. The essential difference is that the time transformation $|f'|^2 = dt/d\tau$ used by Lemaître's method becomes $|wf'|^2$ in Arenstorf's regularization. This time transformation satisfies the requirement of dynamical equivalence mentioned in Section 3.4.

3.8 Generalization of global regularization methods

The three methods previously mentioned, which offer the global regularization of the restricted problem, are based on the general transformation function

$$q = A\left[k(w) + \frac{1}{k(w)}\right] = f(w), \tag{137}$$

where $A = \frac{1}{4}$ in all three cases and

$$k(w) = 2w,\ e^{iw},\ \text{and } w^2$$

for the Birkhoff, the Thiele–Burrau, and the Lamaître transformations, respectively. It is not necessary to select $A = \frac{1}{4}$; nevertheless, a reason for selecting this value for A is as follows. From Eq. (137),

$$|f'(w)|^2 = |A^2|\frac{|k'|^2|k^2-1|^2}{|k|^4}. \tag{138}$$

On the other hand, from the relations $r_1 = |q - \frac{1}{2}|$ and $r_2 = |q + \frac{1}{2}|$ we have

$$r_1 = \frac{|2Ak^2 - k + 2A|}{2|k|}, \qquad r_2 = \frac{|2Ak^2 + k + 2A|}{2|k|}. \tag{139}$$

Since $|f'(w)|^2$ is the multiplier of U — which has poles at $r_1 = 0$ and $r_2 = 0$ — in order to eliminate these singularities we must have $(r_1 r_2)^n \gamma(w) = |f'|^2$ so that when the product $U|f'|^2$ is formed the terms $1/r_1$ and $1/r_2$ become regular. The function $\gamma(w)$ is to be regular at the singularities and the exponent n is, of course, to be larger than or equal to one. As a matter of fact it is necessary that $n = 1$ since $n > 1$ gives $|f'|^2 U = 0$ at the singularities where the velocities will also be zero consequently. In fact, if grad $|f'|^2 U = 0$ at the singularities, the acceleration will also be zero and a steady-state solution is obtained whereby the particle rests at the singularities.

The function $\gamma(w)$ is $|w|^{-2}$, 1, and $4|w|^{-2}$ for Birkhoff's, Thiele's, and Lemaître's transformations, respectively. We note that the point $w = 0$ does not coincide with the transformed location of either of the primaries.

Computing $r_1 r_2$ from (139) and comparing it with (138) gives

$$\frac{|(k^2+1)^2 - k^2/4A^2|}{|k|^2}\gamma(w) = \frac{|k^2-1|^2|k'|^2}{|k|^4}.$$

If $A = \frac{1}{4}$,

$$\gamma(w) = \left|\frac{k'}{k}\right|^2 \quad \text{and} \quad |f'|^2 = r_1 r_2\left|\frac{k'}{k}\right|^2. \tag{140}$$

Once $A = \frac{1}{4}$ is selected, the locations of the singular points in the transformed plane are to be established. If w_1 represents the transformed position of the first primary and w_2 that of the second, then from Eq. (137) we have

$$[k(w_1) + k^{-1}(w_1)] = 2 \quad \text{and} \quad [k(w_2) + k^{-1}(w_2)] = -2 \tag{141}$$

or $k(w_1) = 1$ and $k(w_2) = -1$. These equations, of course, might have multiple roots as we have seen before.

The singularities will be eliminated since

$$\Omega^*(u, v) = U|f'|^2 = \left|\frac{k'}{k}\right|^2 \frac{r_1 r_2}{2}[r_1^2 - C + \mu(r_2^2 - r_1^2)]$$
$$+ \left|\frac{k'}{k}\right|^2 [(1-\mu)r_2 + \mu r_1]. \tag{142}$$

The only critical term in this expression regarding the singularities is $k'(w)$ since we have seen that $k(w) \neq 0$ at the primaries. Therefore, as long as $k'(w_1)$ and $k'(w_2)$ are finite at the singularities, the above expression $\Omega^*(u, v)$ is regular. This condition is, of course, satisfied by the functions $k(w) = 2w$, e^{iw}, and w^2, since the corresponding derivatives are $k' = 2$, ie^{iw}, and $2w$. In the case of Birkhoff's transformation, $k' = 2 =$ const and therefore it is finite at w_1 and w_2. Using Thiele's transformation, $k' = ie^{iw} = ik$ and therefore $k'(w_1) = ik(w_1) = i$ and $k'(w_2) = ik(w_2) = -i$. Both $k'(w_1)$ and $k'(w_2)$ are therefore finite. For Lemaître's transformation, $k' = 2w = 2k^{1/2}$ and consequently $k'(w_1) = \pm 2$ and $k'(w_2) = \pm 2i$, both being finite.

The absolute value of the transformed velocity at m_1 is obtained from Eq. (142) by writing $r_1 = 0$:

$$|w'|_{m_1} = [2\Omega^*(r_1 = 0, r_2 = 1)]^{1/2} = \left|\frac{k'}{k}\right|[2(1-\mu)]^{1/2} \tag{143}$$

and the velocity at m_2 becomes

$$|w'|_{m_2} = [2\Omega^*(r_1 = 1, r_2 = 0)]^{1/2} = \left|\frac{k'}{k}\right|(2\mu)^{1/2}.$$

Since at both primaries

$$|k(w_1)| = |k(w_2)| = 1,$$

we have

$$
\begin{aligned}
|w'|_{m_1} &= |h'(w_1)|\,[2(1-\mu)]^{1/2} \\
|w'|_{m_2} &= |h'(w_2)|\,(2\mu)^{1/2}.
\end{aligned}
\tag{144}
$$

Example. Find the pertinent relations for

$$q = f(w) = \frac{\alpha^2 w^{2n} + 1}{4\alpha w^n}.$$

3.9 Comparison of global regularizations

The three previously treated methods of global regularization will now be compared regarding their actual use. This is not a simple task since in general no one of the three methods is superior to the other two in every respect.

A comparison regarding multivaluedness favors Birkhoff's transformation since the correspondence between points of the physical and regularized planes is one to two. The correspondence is one to four in the case of Lemaître's transformation and one to infinity using the Thiele–Burrau transformation. In general, with $h(w) = \alpha w^n$, we have

$$\alpha^2 w^{2n} - 4\alpha q w^n + 1 = 0,$$

and so the correspondence is twice that of the exponent in the function $h(w)$.

A comparison from the point of view of the simplicity of the regularized differential equation throws favorable light on the Thiele–Burrau transformation. The pertinent expressions to be considered are $|f'(w)|^2$ and $\Omega^*(u, v)$, since these enter the regularized equations of motion. The expression $|f'(w)|^2$ in Thiele's transformation is

$$|f'(w)|_T^2 = \tfrac{1}{4}|\sin w|^2,$$

in Birkhoff's transformation it is

$$|f'(w)|_B^2 = \frac{|4w^2 - 1|^2}{64|w|^4},$$

and in Lemaître's transformation it becomes

$$|f'(w)|_L^2 = \frac{|w^4 - 1|^2}{4|w|^6}.$$

If $w = u + iv$ is substituted, we obtain

$$|f'(w)|_T^2 = \tfrac{1}{8}(\cosh 2v - \cos 2u),$$

while the corresponding expressions for $|f'|_B^2$ and $|f'|_L^2$ become fractions of polynomials of high degree.

Similar remarks apply regarding the function Ω^*, which is relatively simple for Thiele's transformation [see Eq. (127)], while it is much more involved for the other two mappings. The form of the function Ω^* for Birkhoff's transformation is given by Eq. (117) and for Lemaître's transformation by Eq. (133). The significant difference between these two equations is that the latter contains only rational fractions of polynomials while the former contains irrational expressions because of the presence of the term $\rho^{-3} = |w|^{-3} = (u^2 + v^2)^{-3/2}$. Lemaître's transformation therefore rationalizes the Ω^* function in addition to regularizing it. The disadvantage is, of course, the appearance of the sixteenth power of the dependent variable in the expression for Ω^* using Lemaître's transformation while only the eighth power occurs when Birkhoff's transformation is applied.

The relatively simple trigonometric and hyperbolic functions occurring in the expressions for $|f'(w)|^2$ and for $\Omega^*(u, v)$ in the case of Thiele's transformation suit this method ideally for hand calculation and the method was originally introduced for this purpose. The use of digital computers might give preference to Birkhoff's or to Lemaître's transformation.

Analytical developments are equally simple for Birkhoff's and Lemaître's methods but published results exist only for the former transformation.

Example. Reduce the regularized equations of motions to a second-order form by referring to Section 2.3, Eq. (25).

The basic similarity between the equations of motion obtained by the various regularization methods is to be emphasized once again. Either in the "center-of-mass" system $[z = z(t)]$ or in the "midpoint" system $[q = q(t)]$ the original equations of motion, before regularization, can be written as

$$\ddot{z} + 2i\dot{z} = \operatorname{grad}_z \Omega, \qquad \text{or} \qquad \ddot{z} + \lambda(z)\,\dot{z} = \operatorname{grad}_z \Omega, \tag{145}$$

with $\lambda(z) = 2i$.

Transformation of this equation by $z = f(w)$ and by $dt = |f'(w)|^2\, d\tau$ gives

$$w'' + 2i\,|f'(w)|^2\, w' = \operatorname{grad}_w \Omega^*$$

or

$$w'' + \lambda^*(w)\, w' = \operatorname{grad}_w \Omega^*, \tag{146}$$

with

$$\lambda^*(w) = 2i\,|f'(w)|^2 \qquad \text{and} \qquad \Omega^* = (\Omega - C/2)\,|f'(w)|^2.$$

Note the formal similarity between Eqs. (145) and (146) and the invariance of the basic form of the equations with respect to regularizing transformations. The starred functions are obtained essentially by multiplication by $|f'(w)|^2$. The same invariance appears also in the corresponding Jacobian integrals:

$$|\dot{z}|^2 = 2\Omega - C \qquad \text{and} \qquad |w'|^2 = 2\Omega^*.$$

Example. Show that $|f'(w)|^2$ and the Jacobian determinant of the transformation $z = f(w)$ are identical when $f(w)$ is analytic.

Based on this invariance one may generalize the restricted problem and express it by the equation

$$\frac{d^2y}{dx^2} + \lambda(y)\frac{dy}{dx} = \operatorname{grad}_y F(y), \tag{147}$$

where y is the (complex) dependent variable, x the (real) independent variable, and the functions $F(y)$ and $\lambda(y)$ are regular in the region of interest. This equation has an integral of the form

$$|dy/dx|^2 = 2F - C, \tag{148}$$

for any function λ. By means of Eq. (148), Eq. (147) can be reduced to a second-order equation, further reduction of which depends on the actual forms of the functions λ and F. If y is the position vector and x the time, then Eq. (147) represents a two-degree-of-freedom irreversible dynamical system. Equation (147) is always reducible to a second-order system of differential equations.

3.10 The existence of solutions

(A) Any real finite singularity of the solution of the equations of motion $x(t)$, $y(t)$ corresponds to a collision with one of the primaries. This statement is considered in what follows in some detail.

The equations of motion and the Jacobian integral in the original (physical) system are given by

$$\ddot{x} - 2\dot{y} = \Omega_x, \qquad \ddot{y} + 2\dot{x} = \Omega_y, \qquad \dot{x}^2 + \dot{y}^2 = 2\Omega - C,$$

where

$$\Omega = \tfrac{1}{2}[(1-\mu)r_1^2 + \mu r_2^2] + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}.$$

We first note that Ω remains bounded if and only if r_1, r_2, r_1^{-1}, and r_2^{-1} remain bounded. This means that neither collisions nor departures to infinity can be allowed if Ω is to remain bounded. For fixed values of the mass parameter and of the Jacobian constant, Ω stays bounded if and only if both $\dot{x}$ and $\dot{y}$ remain bounded. The requirement of the boundedness of the previously mentioned four quantities r_1, r_2, r_1^{-1}, and r_2^{-1} suggests the introduction of a function

$$\rho(t) = \min(r_1, r_2, r_1^{-1}, r_2^{-1}),$$

which, along a solution, is either positive or zero. When its value is zero, Ω is unbounded. The following theorem by Wintner is offered here without proof:

For fixed values of μ and C there exist for every $\rho^* > 0$ two numbers $\alpha^* > 0$ and $\beta^* > 0$ such that any solution $x(t)$, $y(t)$, for which the inequality $\rho > \rho^*$ is satisfied at some time $t = t_0$, has the following two properties:

(i) the solution is regular analytic in the interval of time given by $|t - t_0| < \alpha^*$;

(ii) the solution is bounded according to

$$[x(t) - x(t_0)]^2 + [y(t) - y(t_0)]^2 < \beta^*$$

for every time in the same interval.

Accordingly if a solution is not regular analytic at t_c, then we have

$$\lim \rho(t) = 0 \qquad \text{as} \quad t \to t_c.$$

But this occurs if and only if one of the following three conditions is satisfied:

(i) $\lim r_1(t) = 0$,
(ii) $\lim r_2(t) = 0$,
(iii) $\lim r_1^{-1}(t) = 0$,

as $t \to t_c$.

In Cases (i) and (ii) we have collisions; in Case (iii) we have departure to infinity, Case (iii) being equivalent to $r_2^{-1}(t) \to 0$. The problems of collision can be treated and the solution can be regularized as shown in this chapter. The only remaining singularity, that of rejection to infinity, cannot occur in a finite time in the restricted problem as may be seen from the following considerations.

As $r_1(t) \to \infty$ the centrifugal forces dominate over the gravitational forces and the equations of motion and the Jacobian integral become

$$\ddot{x} - 2\dot{y} = x, \qquad \ddot{y} + 2\dot{x} = y, \qquad \dot{x}^2 + \dot{y}^2 = x^2 + y^2 - \bar{C}.$$

The solutions of this system of linear differential equations are regular analytic for all finite t and so $x^2 + y^2$ must approach a finite limit as $t \to t_c$. This is a contradiction since $\dot{x}^2 + \dot{y}^2$ and consequently $x^2 + y^2$ approaches infinity as $t \to t_c$. We conclude therefore that the coordinates $x(t)$, $y(t)$ must remain bounded for every solution of the restricted problem as t varies over a finite range.

But if departures to infinity are impossible then the only singularities correspond to collisions. Now with regularization we have shown that the solution can be continued analytically through the collision. *Consequently the solutions of the restricted problem exist for all finite times.*

(B) The following two questions must still be examined.

First, we must inquire regarding the analyticity of t as a function of τ, especially as $\tau \to \tau^*$. In fact for the binary collision discussed in Section 3.2 we have, according to Eq. (36),

$$\tau(t) = \int_{t_c}^{t} \frac{dt}{r(t)} + \tau_c$$

and, since

$$r(t) = A(t - t_c)^{2/3} + \cdots,$$

the integrand becomes infinite as $t \to t_c$.

In fact $d\tau/dt$ is not bounded at collision in any of the regularizing transformations discussed, so the existence of $\lim \tau = \tau_c$ as $t \to t_c$ must be carefully investigated.

Returning to the regularization with r^{-1} we have, after performing the integration,

$$\tau(t) \to B(t - t_c)^{1/3} + \tau_c,$$

so the integral and $\tau(t)$ exist for $t > t_c$ and $\tau \to \tau_c$ as $t \to t_c$.

Another problem is connected with the dates of the collisions, which, if they would cluster at a finite $t = t^*$, the solution would not be defined for $-\infty < t < +\infty$. The fact that if there are an infinite number of collisions they cannot have a finite accumulation point has been proven by Wintner and by Arenstorf; consequently, this question is also settled.

We note that τ increases monotonically with t for all transformations. Nevertheless, as t varies from $-\infty$ to $+\infty$ the regularizing time variable τ does not necessarily do so. We have seen, for instance, that the Birkhoff regularization introduces a new singularity corresponding to infinity in

the physical plane. This point in turn corresponds to $t \to \infty$. Consequently we have the case when a solution in the w plane approaches the singularity (the midpoint between the primaries) at a finite value of τ which must correspond to $t \to \infty$.

3.11 Applications

Primary applications of regularization techniques are encountered in space dynamics, in cosmogony, and in stellar dynamics.

Using the earth-moon system as the primaries in a restricted problem, families of space probe trajectories connecting the two force centers can be established. Such an orbit is shown in Fig. 3.14.

The solid curve connecting the points E and M represents the orbit in a synodic coordinate system. The numbers along the curves represent the time in days. The other solid curve is the orbit in a fixed coordinate system. The dotted lines are the regularized representations of the orbit. To a single orbit in the x, y system there correspond two orbits in the u, v system since Birkhoff's transformation, which was used in this case, establishes a one-to-two relation between the physical x, y plane and the regularized u, v planes. The pertinent data are shown in the figure. The

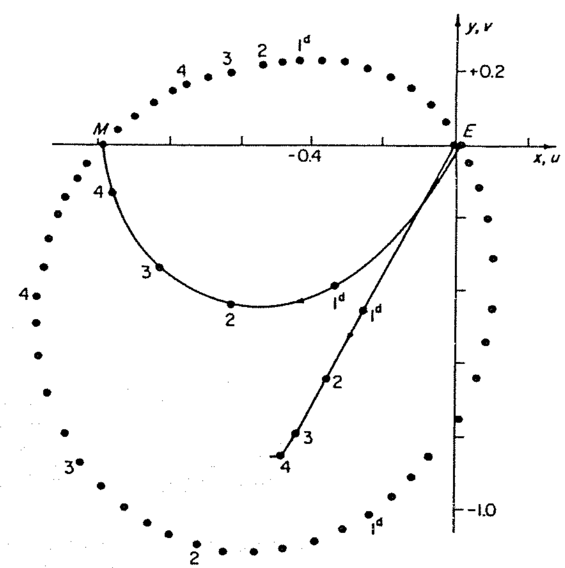

FIG. 3.14. Consecutive collision orbit in the earth-moon system. $\theta = 60^\circ.855$; $\psi_1 = 120^\circ.428$, $\psi_2 = 300^\circ.428$; $C = 2.000009$; $P = 4^d.41$.

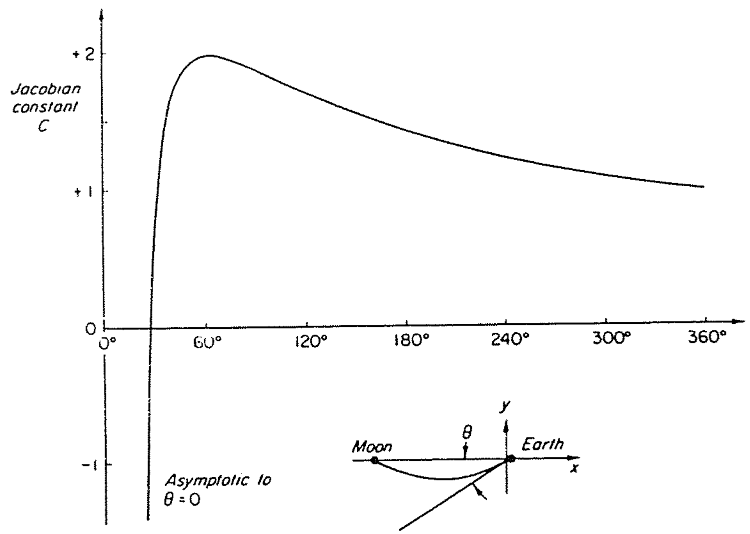

FIG. 3.15. Family of consecutive collision orbits.

angle θ is the "firing angle" between the tangent to the orbit at the earth and the negative x axis. This angle is also shown in Fig. 3.15, which represents a family of consecutive collision orbits. The angles ψ_1 and ψ_2 are the corresponding angles in the regularized plane.

The reader will find a detailed discussion of this family and of its extension in the references listed at the end of this chapter and also in Chapter 9.

The practical importance of such trajectories is that the sensitivity to errors in guidance is small. Considering a trajectory which goes, in extension, through the center of the moon, say, it can be seen that, if the initial conditions are slightly changed, then this orbit will still hit the moon—even if it misses its center.

The cosmogonical importance of regularization is twofold. First, when predictions for a long time are essential, no singularities should remain in the equations of motion. Second, many cosmogonical theories are based on close approaches, collisions, and captures which cannot be treated without regularization.

In cosmogony and in stellar dynamics the restricted problem competes with the general problem of three bodies in importance. Nevertheless questions of collision are equally prominent and therefore a regularization of the gravitational n-body problem might be in order now.

The equations of motion for n bodies may be written as

$$m_i \ddot{\mathbf{r}}_i = k^2 \sum_{i \neq j, j=1}^{n} \frac{m_i m_j}{|\mathbf{r}_{ji}|^3} \mathbf{r}_{ji}, \tag{149}$$

which, by introducing the force function

$$F = \frac{k^2}{2} \sum_{i=1}^{n} \sum_{i \neq j, j=1}^{n} \frac{m_i m_j}{|\mathbf{r}_{ij}|}, \tag{150}$$

becomes

$$m_i \ddot{\mathbf{r}}_i = \frac{\partial F}{\partial \mathbf{r}_i} \qquad (i = 1, 2, \ldots, n). \tag{151}$$

Here m_i and $\mathbf{r}_i$ are the mass and the position vector of the ith body, $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ is the position vector of the ith body relative to the jth body, and k^2 is the Gaussian constant of gravitation.

The new independent variable is introduced by

$$d\tau = F(\mathbf{r}_1, \ldots, \mathbf{r}_n)\, dt, \tag{152}$$

giving

$$\dot{\mathbf{r}} = F\mathbf{r}',$$

where the prime denotes the derivative with respect to τ.

The integral of energy of the original system is

$$\sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i^2 = 2F - C$$

which now becomes

$$\sum m_i \mathbf{r}_i'^2 = 2/F - C/F^2.$$

The equations of motion are

$$\mathbf{r}_i'' + \mathbf{r}_i' \frac{F'}{F} = \frac{1}{m_i F^2} \frac{\partial F}{\partial \mathbf{r}_i}, \tag{153}$$

where F' can be obtained by a dot product,

$$F' = \frac{\partial F}{\partial \mathbf{r}_i} \mathbf{r}_i',$$

and consequently

$$\mathbf{r}_i'' + \mathbf{r}_i' \left(\mathbf{r}_i' \frac{\partial \ln F}{\partial \mathbf{r}_i} \right) = -\frac{1}{m_i} \frac{\partial F^{-1}}{\partial \mathbf{r}_i}. \tag{154}$$

Note that the square of the first derivative $(\mathbf{r}_i')$ appears as expected; nevertheless, Eq. (154) is regularized for binary collisions. The usefulness of such transformations must be decided by their applicability to computers.

An implication in the stellar dynamics of this regularized equation is that there appears to be a tendency of stars in a cluster to form binaries. This, of course, necessitates handling close approaches within the framework of the n-body problem.

3.12 Notes

The fundamental ideas regarding the regularization of the general problem of three bodies were contributed by Sundman [1] who clearly shows Poincaré's influence [2].

Section 3.2 treats the possible simplest case in order to free the reader from algebraic complications. Already Euler [2a] regularized the problem of collision between two bodies moving on the same straight line, as was called to my attention by Miss C. Williams. Note that Euler's method corresponds to the selection of $n = 1$ in Eq. (23) of Section 3.2.

The development of the basic ideas, with many references, can be found in the books by Whittaker [3] and by Leimanis and Minorsky [4]. Wintner's book [5] offers a somewhat decentralized treatment of the problem of regularization. Happel [6] dedicates a fourteen-page chapter to the subject.

Regularization may be achieved in the case of the general problem of two bodies (Section 3.3) without transformation of the coordinates. I am grateful to Dr. R. F. Arenstorf for his communication in which he offers a delightfully short proof that the time transformation

$$dt/d\tau = r$$

may be used to obtain from the conservation of angular momentum and from Eqs. (56) and (59) the following result:

$$\zeta'' + C\zeta = \text{const},$$

where prime stands for derivative with respect to τ. This is analogous to the regularization applied for the collision problem in Section 3.2 where the above time transformation and the eccentric anomaly as the regularizing variable are introduced in Eq. (40). See also [4] and [5, p. 193].

Levi-Civita's contributions to the subject are numerous [7]. The third and the last references quoted are preferred. Closely related is the transformation proposed by Bohlin [8]. With slight modification this

becomes what in the text (Section 3.4) is called Levi-Civita's transformation.

The history of the Thiele–Burrau transformation (Section 3.6) begins with Euler [9] who established the integrability of the problem of two centers of gravitation. This problem is discussed in the books by Whittaker [3, p. 97] and Charlier [10] as well as in Plummer's book [11], where the connection between the regularization of the restricted problem and Euler's problem is made clear. Euler's transformation replaces rectangular coordinates by elliptic coordinates and in this way the problem becomes of the Liouville type and can be integrated by elliptic functions. It is to be realized that, with either one of the force centers, conic-section orbits exist. When the two centers operate simultaneously, conic-section solutions still exist according to Bonnet's theorem [12] due to Legendre [13] and generalized by Egorov [14]. This theorem and its application to the problem of two fixed centers of gravitation and the application of this latter problem to the restricted problem are described by Szebehely [15]; for details see Section 10.5.1. The regularization of the restricted problem for $\mu = \frac{1}{2}$, using essentially Euler's transformation, was performed by Thiele [16] (without reference to Euler) and for arbitrary μ by Burrau [17]. The original problem was "oversolved" since local regularization would have allowed the continuation of Strömgren's class "a" orbits to their natural termination (see Chapter 9). In 1892, Thiele proposed the study of the motion around the third libration point (see Chapter 4), L_3, as an astronomy prize problem in the Danish Royal Academy. The small elliptic orbits around L_3 resulting from the solution of the linearized equations of motion will be treated in detail in Chapter 5. As the amplitude increases, a member of this family becomes a so-called ejection or collision orbit colliding with m_1. Thiele's problem was to establish this member of the family. The regularization of the restricted problem therefore was prompted by the necessity to complete a family of orbits.

The Thiele–Burrau regularization was adopted by the Strömgren school for their extensive numerical integrations, and the various transformation formulas were tabulated and presented in graphs and in nomograms. Inasmuch as this transformation was used mostly, and is recommended primarily, for hand calculations, only Lindow's [18] nomogram and Strömgren's (Burrau and Strömgren [19]) tables of the transformations are mentioned here.

Historically it is significant that Burrau had settled Thiele's prize problem before Thiele's application of Euler's transformation appeared. Reference should be made in this respect to Burrau's papers [20] in which he recognizes the above-mentioned fact that only local regularization is necessary for the solution of the problem and he discovers consequently the $\tau = at^{1/3}$ transformation discussed at the end of Section 3.2. The derivation given there is made more precise by Wintner [5], which is followed in Section 3.2, Part D.

The introduction of Birkhoff's transformation (Section 3.5) was much less eventful. An extensive discussion of his transformation is given in [21]. No numerical applications are offered and the transformation is used in connection with the topology of the dynamical problem. Note that on p. 13, Eq. 21 is in error and ρ_1 and ρ_2 should be interchanged. See also a later work by Birkhoff [22].

Prior to Birkhoff's result, Armellini's strange paper [23], evidently primed by Sundman, appeared and discussed regularization of the restricted problem by introducing $d\tau = (r_1 + r_2 - 1)^{-1}\,dt$.

It is interesting to review the series of publications associated with increasing the exponent of $h(w) = w$ from one to two: $h(w) = w^2$. This is the process from Birkhoff's to Lemaître's transformation (Section 3.7). The first to propose the transformation $z = (c/2)(w^2 + w^{-2})$ without specific applications to regularization was Plummer [24]. Several papers by Lemaître of the University of Louvain treat the regularization of the general problem of three bodies [25]. Application of this work to the global regularization of the restricted problem is contributed by Deprit in a series of papers [26, 26a]. Lemaître's 1955 paper and Deprit's excellent 1963 summary in Icarus are recommended primarily. The underlying principle of a symmetric treatment of the general plane problem of three bodies is given by Murnaghan [27], by Szebehely [28], by Deprit and Delie [29], by Deprit and Roels [30], and finally by Deprit and Delie [31]. Essentially the same method was proposed by Arenstorf in a remarkable and excellent paper with strong analytical orientation [32]. The mathematically inclined reader will find thorough enjoyment in this paper where the concept of dynamical equivalence (Section 3.4) is introduced.

The transformation in question was named after Lemaître by me in 1964 (see [35]) since at that time the earliest references (1954) I could locate were written by him (see [25]). I propose to preserve the terminology, in spite of Plummer's paper [24] of a much earlier date (1914), for two reasons: First, Lemaître's name is now associated with this transformation in the literature through several contributions by others in addition to myself. Second, Plummer's aim in introducing the transformation was not directed to regularization; in fact, he does not mention regularization in connection with this transformation, though he discusses other transformations in the same paper with regularization in mind, regarding applications. On the other hand, Lemaître's main purpose was to regularize the general problem of three bodies.

Broucke [34] pointed out that, with the function $h(w) = w^n$ mentioned

in Sections 3.8 and 3.9, not only is regularization affected but also certain multiple periodic orbits (with repeated loops around the origin) can be transformed into simple periodic orbits without loops. This also implies that series solutions obtained in the w plane for a given, say, satellite problem will be valid for more revolutions in the physical (q or z) plane than in the w plane.

The form in which the conformal mappings are presented in Figs. 3.7(b), 3.9(a), (b), and 3.13(a), (b) is conventional; see, for instance, Deprit's and Broucke's [26a] paper.

The importance of regularization in space mechanics is discussed by Szebehely [33], and a comparison of the various regularization methods mentioned in Section 3.9 is offered by the same author [35]. This latter paper discusses a general perturbation scheme for the solution of the equations of motion of the restricted problem after Birkhoff's regularizing transformation was performed. An alternate derivation of the regularized equations is given by Szebehely and Giacaglia [35a], in a paper in which the regularization of the elliptic restricted problem is also treated (see Section 10.3). Examples for general perturbation analyses are given by Pierce [36] using Levi-Civita's regularization. A derivation of Eq. (92) in Section 3.4 is offered by Szebehely [36a].

A discussion of universal variables mentioned in Section 3.3 in connection with Eqs. (68) and (87) to (89) is presented by Herrick [37] who — not from the point of view of regularization — introduced such variables first in 1945 [38].

A comprehensive treatise on regularization with a number of examples is available as an article by Szebehely (39).

The process of regularization within the framework of Hamiltonian dynamics will be presented in Chapter 7.

Section 3.10 discusses the question of the existence of solutions of the restricted problem. The outlines given for the theorems and for some of the proofs show that the analytical background is well established in the literature. The three major references regarding this section are by Leimanis (Leimanis and Minorsky [4, p. 59]), Wintner [5, pp. 269, 329, 356–359], and Arenstorf [32]. Wintner's fundamental theorem regarding the existence of solutions of the restricted problem is an application of one of Painlevé's results. The solution of the problem of establishing $\lim \tau \to t^*$ for the various regularization transformations is offered by Arenstorf [32]. The simple discussion given in Section 3.10 for the transformation introducing the eccentric anomaly is not entirely rigorous, in spite of the fact that every collision in the restricted problem is essentially a (straight-line) two-body collision. The last remark of Section 3.10 is at variance with Wintner's conclusion [5, p. 359] and has been first observed by Arenstorf [32].

Numerical results outlined in Section 3.11 and obtained for the restricted problem by use of regularized equations were first given by the Copenhagen school using the Thiele Burrau transformation. This work, for which a general paper by Strömgren [40] is given as a representative reference, is discussed in considerable detail in Chapter 9. The Copenhagen school established several single-collision orbits and a few consecutive-collision orbits, but no families consisting solely of collision orbits were found.

Consecutive-collision orbits without periodicity but forming a well-defined family were established by V. Szebehely and co-workers. The first paper by Szebehely *et al.* [41] shows a family of trajectories connecting the earth's center with the moon's center. One member of this family is close to such an actual earth-to-moon trajectory which could be utilized for lunar landing missions. The two-body approximation of some of the computed consecutive collision orbits is established. The continuation of the family of orbits is described by Pierce and Standish [42]. Consecutive collision orbits of high complexity are systematically developed from the basic family in [42] and are discussed in Chapter 9.

Computation of collision trajectories in the restricted problem using Levi-Civita's regularization have been performed by Benedikt [43] and and Broucke [44].

Benedikt in the same paper discusses the power series solution for the problem of finding initial conditions which lead to collision. The same problem is treated by Levi-Civita [45], by Bisconcini [46], and by Whittaker [3, p. 424].

The general problem of three bodies mentioned in Section 3.11 is not the subject of this treatise; nevertheless, the fact that practitioners of celestial mechanics were reluctant to accept Sundman's developments cannot be ignored. Considering double collisions only, the general problem is solved in principle since Sundman [1] regularized this problem. (Simultaneous (i.e., triple) collisions are to be excluded by specifying nonzero angular momentum vector. This condition is only necessary but not sufficient, since triple collisions can be excluded by proper choice of the initial conditions even when the angular momentum is zero [5, p. 265]).

The same applies to the restricted problem which, after a global regularization was affected, can be solved, at least in principle, by power series. Regarding the question of local and global regularization, see Wintner [47].

In an interesting article, Cesco [48] points out that astronomers take only limited interest in regularization because of its ill-chosen applications. Motion around the equilateral libration points with small amplitude, when not even close approaches occur, does not seem to be

a good test case for judging the merits of regularization. Cesco proposes an example with double collision which shows the power, in fact the necessity, of regularization. I believe the argument to be correct but the basic issue somewhat different. Close approaches do not usually occur in celestial mechanics, and this is the reason for not applying regularizing transformations. When collisions and close approaches enter the problem, such as in space mechanics and in stellar dynamics, regularization does become essential.

The regularization of the n-body problem discussed in Section 3.11 is treated in considerable detail by Leimanis (Leimanis and Minorsky [4, p. 94]). The qualitative aspects are discussed by Khilmi [49]. See also a note by Ebert [50] on the use of regularization to find a new invariant relation for the problem of n bodies.

I am grateful to Dr. M. Lecar for calling to my attention the importance of regularization in stellar dynamics.

3.13 References

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Chapter 4

Totality of Solutions

4.1 Introduction

The motion of a dynamical system with two degrees of freedom is described by a system of differential equations of the fourth order. A solution of this system is determined—excepting singularities—when four initial conditions are given. A point in the four-dimensional space formed by these four initial conditions will, therefore, represent a set of initial conditions as well as a particular solution. The totality of solutions is represented by the points of the previously mentioned four-dimensional space which is often referred to as the *phase space*.

The concept of phase space may be introduced through systems possessing a single degree of freedom. The initial conditions in this case are simply x and $\dot{x}$ say; therefore, the phase space becomes a two-dimensional x, $\dot{x}$ phase plane in which the various solutions may be shown. A single solution with the initial conditions x, $\dot{x}$ may be just a point or may be represented in the form of a curve since as the time varies from $t = t_0$ to $t = t$ the corresponding values of x and $\dot{x}$ will, in general, also vary. For instance, if the phase plane shows a straight line parallel to the x axis representing $\dot{x} = \dot{x}_0 = $ const, the corresponding motion takes place according to $x = \dot{x}_0 t + x_0$. A line having an angle of 45° with the x axis and passing through the origin corresponds to $x = x_0 e^t$ and the circle $\dot{x}^2 + x^2 = a^2\omega^2$ represents harmonic oscillation. The actual motion takes place along the x axis in all three examples.

If an integral of a dynamical system with a single degree of freedom

establishes a relation between x and $\dot{x}$, $C = F(x, \dot{x})$, then this might be represented by a curve in the x, $\dot{x}$ plane.

The points of this curve serve two purposes. They represent first all initial conditions which give the same value of C; that is, selecting any point $(x_0, \dot{x}_0)$, a solution can be generated which will be associated with the given value of C. But if C remains constant during the motion (the function F is an integral) then the solution as shown on the phase plane must coincide with the curve for which $C = $ const. Therefore the second purpose of the plot $C = F(x, \dot{x})$ is actually to display the motion of the system since the particle will have to perform such a motion as will keep its trace on this curve.

In this way the points of this curve represent solutions corresponding to a certain given value of C. As this value of C changes, the integral curves are generated. The harmonic oscillator serves as an example. Here $\ddot{x} + x = 0$ is the equation of motion—assuming unit mass and unit value for the spring constant. The points of the x, $\dot{x}$ plane represent the initial conditions and also the totality of motions. The integral expresses the principle of conservation of energy: $\dot{x}^2 + x^2 = C$. These integral curves are circles in the phase plane. The "isoenergetic" solutions (solutions with such initial conditions that the energy is the same) are located on the same circle. An orbit also appears as a circle. The two-dimensional manifold of the totality of solutions is reduced to the points of a line by means of the integral.

The phase space of four dimensions associated with dynamical systems of two degrees of freedom is reducible when an integral of the system exists. Such an integral has the form

$$F(x, y, \dot{x}, \dot{y}) = C \tag{1}$$

and represents a three-dimensional subspace or hypersurface for a fixed value of C. Another integral, if it exists, and if it is independent of the first one, will further reduce the representation to the points of a surface. This is seen when the two integrals

$$\begin{aligned} F_1(x, y, \dot{x}, \dot{y}) &= C_1 \\ F_2(x, y, \dot{x}, \dot{y}) &= C_2 \end{aligned} \tag{2}$$

are combined giving $0 = f(C_1, C_2, x, y, \dot{y})$, which is the equation of a surface in the space $\dot{y}$, x, y. All solutions, therefore, which have initial conditions as in Eqs. (2) will be located on this surface. Not only the initial conditions and the points representing the solution, but also the orbits themselves are on the surface $f = 0$. The cut $x = 0$ of this surface with the y, $\dot{y}$ plane will, in general, be a curve on which the points of intersection of the orbit with the plane $x = 0$ (say, for $\dot{x} > 0$)

will be located. The existence of this curve implies the existence of the two integrals, Eqs. (2). If only one of the two integrals exists, then, of course, no curve is obtained in the $y, \dot{y}$ plane but an area will be filled with points corresponding to the intersection of the plane $x = 0$ with the volume $F_1(x, y, \dot{x}, \dot{y}) = C_1$.

Another avenue of reduction of the four-dimensional phase space is to investigate certain classes of solutions. Reduction to a three-dimensional phase space by means of an integral is identical with considering only those solutions for which the constant of integration is the same. In addition to this requirement we might also restrict our study to solutions with certain special initial conditions, say $x_0 \neq 0$, $y_0 = 0$, $\dot{x}_0 = 0$, $\dot{y}_0 \neq 0$. If the constant of integration is fixed, the selection of all four initial conditions is contradictory since, for instance, C, x_0, y_0, and $\dot{x}_0$ determine $\dot{y}_0$ from Eq. (1). The geometry of the above initial conditions is that orbits starting on the x axis normally are investigated. The three-dimensional phase space associated with a given value of C, therefore, can be reduced to a one-parameter family if only orbits starting perpendicularly to the x axis are investigated. In other words the one-dimensional family appears extracted from the three-dimensional phase space.

Another reduction of the multidimensional manifold results when isoenergetic orbits are studied and the magnitude of the velocity is restricted to zero. Equation (1) gives

$$C = F(x, y, 0, 0)$$

which is a curve in the plane x, y for some value of C. These curves are called curves of zero velocity and often Hill curves.

A dynamical system with three degrees of freedom, as, for instance, the three-dimensional restricted problem, leads to a six-dimensional phase-space representation, where various reductions are again feasible. For instance, with a fixed value of the constant of integration C, the manifold of zero velocity appears as surfaces, since now

$$C = F(x, y, z, \dot{x}, \dot{y}, \dot{z}),$$

from which

$$z = f(C, x, y),$$

when $\dot{x} = \dot{y} = \dot{z} = 0$. These manifolds are called surfaces of zero velocity.

In the study of the manifolds, analytical properties are of importance and it will be shown that the singularities of these manifolds are needed for their complete characterization. After this, it may be shown that certain properties of possible orbits are associated with these manifolds.

This chapter is concluded with applications which are of special importance in every-day calculations of engineering systems, since the manifolds contain information about the energy and therefore requirements for engine performance may be established for space vehicles.

4.2 The manifold of the states of motion

The points $P(x, y, \dot{x}, \dot{y})$ of the four-dimensional phase space represent states of motion since every point corresponds to an initial condition which in turn represents an orbit. The Jacobian integral gives a relation between these quantities; therefore, for a fixed value of the Jacobian constant, C, only three initial conditions can be selected freely. Points of the three-dimensional manifold corresponding to a fixed value of C also can be considered as states of motion, but only in an isoenergetic sense.

Since the dynamical problem may be described by the variables x, y as well as by other sets, say, by the regularizing coordinates u, v, the states of motion may be described by $x, y, \dot{x}, \dot{y}$ or by u, v, u', v'. Since the system x, y contains singularities (at the primaries), studies of the state of motion often are performed in the regularized system, u, v. Care is to be exercised since, for any given state of motion in the x, y system, there may correspond several distinct sets of values of u, v, u' and v'.

For instance, the Birkhoff transformation establishes a two-to-one correspondence between the transformed w and the physical z planes. The two values of w which are ordered to one given value of z are related by

$$2 + \frac{1}{w_1 - \mu} + \frac{1}{w_2 - \mu} = 0. \tag{3}$$

Equation (3) follows from the basic formula of the Birkhoff transformation,

$$z = f(w) = \frac{w^2 + \mu(1 - \mu)}{2w + 1 - 2\mu}, \tag{4}$$

which can be written as

$$1 + \frac{1}{z - \mu} = \left(1 + \frac{1}{w - \mu}\right)^2. \tag{5}$$

If w_1 corresponds to the positive value of the square root of the left-hand side and w_2 to the negative square root, then Eq. (3) is immediately obtained.

The two regularized velocities for any two-to-one transformation are connected by

$$w_1' \frac{f'(w_1)}{|f'(w_1)|^2} = w_2' \frac{f'(w_2)}{|f'(w_2)|^2}, \tag{6}$$

since

$$\frac{dz}{dt} = \frac{dz}{dw}\frac{dw}{d\tau}\frac{d\tau}{dt}$$

or

$$\dot{z} = \frac{f'(w)\, w'}{|f'(w)|^2}.$$

This result is equivalent to Eq. (6) because the same $\dot{z}$ is to be obtained whether w_1 or w_2 is used to evaluate it.

Equation (6) may also be written as

$$\frac{w_1'}{w_2'} = \frac{\overline{f'(w_1)}}{\overline{f'(w_2)}}. \tag{7}$$

For instance, the Levi-Civita transformation,

$$z = \mu + w^2,$$

gives

$$\frac{w_1'}{w_2'} = \frac{\bar{w}_1}{\bar{w}_2} = \frac{+(\bar{z} - \mu)^{1/2}}{-(\bar{z} - \mu)^{1/2}} = -1.$$

Example. Show that Birkhoff's transformation results in $w_1' = -w_2'$ at the singularities.

In the regularized Birkhoff system there are always two distinct sets of state variables which correspond to a single state of motion in the physical space. We note that the two transformed position vectors w_1 and w_2 have the property that they either are both on the real axis u or are separated by it. This follows from Eq. (3), which may also be written as

$$1 + \frac{1}{w_1 - \mu} = -1 - \frac{1}{w_2 - \mu}$$

or

$$1 + \frac{\bar{w}_1 - \mu}{|w_1 - \mu|^2} = -1 - \frac{\bar{w}_2 - \mu}{|w_2 - \mu|^2}.$$

Taking the imaginary parts of both sides gives

$$\frac{-v_1}{|w_1 - \mu|^2} = \frac{v_2}{|w_2 - \mu|^2},$$

and therefore v_1 and v_2 are of opposite signs. If one of the w values is real, the other is also real ($v_1 = v_2 = 0$) and the w's are located on the real axis, u.

Representation of the manifold of the state of motion is therefore quite simple whether the original physical variables x, y or their corresponding regularized set is utilized, this latter set having analytic advantages.

4.3 Singularities of the manifold of the states of motion

Any set of numbers x, y, $\dot{x}$, $\dot{y}$ satisfying the Jacobian equation $\dot{x}^2 + \dot{y}^2 = 2\Omega(x, y) - C$ will represent a possible motion for a given C. Alternatively, any set of numbers u, v, u', v' satisfying the Jacobian equation $u'^2 + v'^2 = 2\Omega^*(u, v)$ will represent a possible motion for the given value of C, since Ω^* contains C.

The manifold

$$F(u, v, u', v') = u'^2 + v'^2 - 2\Omega^*(u, v) = 0 \tag{8}$$

possesses singularities at the points u, v, u', v', where

$$\frac{\partial F}{\partial u} = 0, \qquad \frac{\partial F}{\partial v} = 0, \qquad \frac{\partial F}{\partial u'} = 0, \qquad \frac{\partial F}{\partial v'} = 0. \tag{9}$$

These singular points of the regularized manifold (of the states of motion) are distinct from the singularities occurring in the original differential equations of motion. In Chapter 3 the singularities of the equations of motion were discussed and it was shown that they were associated with the primaries. Equations (9) are equivalent to

$$\Omega_u^* = 0, \qquad \Omega_v^* = 0, \qquad u' = 0, \qquad v' = 0. \tag{10}$$

The last two equations represent zero velocity in the regularized plane and because of Eq. (8) we also have $\Omega^*(u, v) = 0$. The relation to the physical plane is given by $\Omega^*(u, v) = (\Omega - C/2)|f'(w)|^2 = 0$, and so $\Omega = C/2$, or $\dot{x}^2 + \dot{y}^2 = 0$, or $\dot{x} = 0$, $\dot{y} = 0$. The condition of zero velocity obtained in the regularized system of variables is also valid, therefore, in the physical system. This result also follows from a study of the manifold

$$G(x, y, \dot{x}, \dot{y}) = \dot{x}^2 + \dot{y}^2 - 2\Omega(x, y) + C = 0,$$

which may be complicated because of the singularities of the function $\Omega(x, y)$.

Note that the equation

$$(\Omega - C/2)\,|f'(w)|^2 = 0$$

is not satisfied in general at the locations of the primaries where the quantity $|f'(w)|^2$ is zero. This can be seen by realizing that the velocity is not zero at the primaries in the regularized system in general, since $\Omega^* \neq 0$ at P_1 and P_2.

We conclude, therefore, that the last two conditions [Eqs. (10)] regarding the singularities of the manifold of the states of motion ($u' = 0$, $v' = 0$) lead to all those points of the planes u, v or x, y where both components of the velocity are zero, that is, to the curves of zero velocity. Inasmuch as the function Ω (and also Ω^*) is analogous to a potential function, the equations $\Omega^* = 0$ or $\Omega = C/2$ describe equipotential lines (cf. Section 4.7.4).

The two additional conditions [Eqs. (10)] regarding the singularities of the manifold of the states of motion are $\Omega_u^* = 0$ and $\Omega_v^* = 0$, which also turn out to be equivalent to the conditions $\Omega_x = 0$ and $\Omega_y = 0$. To show this, we write the condition as $\operatorname{grad}_w \Omega^* = 0$ or as

$$\operatorname{grad}_w\{(\Omega - c/2)\,|f'(w)|^2\} = 0.$$

After expansion we have

$$2f'\bar{f}''(\Omega - C/2) + |f'|^2\bar{f}' \operatorname{grad}_z \Omega = 0$$

which gives $\operatorname{grad}_z \Omega = 0$ or $\Omega_x = 0$, $\Omega_y = 0$, since the first term must be zero as shown previously.

We conclude that the *singularities of the manifold* of the states of motion are located at those points of the curves of zero velocity where $\Omega_u^* = \Omega_v^* = 0$ or $\Omega_x = \Omega_y = 0$. In other words the singularities are defined by either of the following two sets of equations:

$$\Omega^* = 0, \qquad \Omega_u^* = 0, \qquad \Omega_v^* = 0, \tag{11}$$

or

$$\Omega = C/2, \qquad \Omega_x = 0, \qquad \Omega_y = 0. \tag{12}$$

In what follows the physical plane will be discussed and the interpretation of Eqs. (12) will be given.

The equations of motion

$$\ddot{x} - 2\dot{y} = \Omega_x$$

and

$$\ddot{y} + 2\dot{x} = \Omega_y$$

at the points of *singularity of the manifold* of the states of motion ($\dot{x} = \dot{y} = \Omega_x = \Omega_y = 0$) give $\ddot{x} = \ddot{y} = 0$, for which reason the points described by Eqs. (12) are also called *equilibrium points*. In fact all derivatives of the coordinates with respect to the time are zero at these points. For instance, from the first equation of motion

$$\dddot{x} = 2\ddot{y} + \Omega_{xx}\dot{x} + \Omega_{xy}\dot{y} = 0.$$

Therefore, if the third particle is placed at an equilibrium point with zero velocity, it will stay there. For this reason the singularities are also called *stationary points*. The terms "*libration points*" and "*Lagrangian points*" are also used with the notation $L_1, \dots L_5$, see Chapter 5.

The equations $\Omega_x = 0$, $\Omega_y = 0$ in expanded form become

$$x - \frac{(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu(x+1-\mu)}{r_2^3} = 0 \tag{13}$$

and

$$y\left(1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3}\right) = 0. \tag{14}$$

The solution of Eqs. (13) and (14) results in five points, three on the x axis, called the collinear solutions, and two forming equilateral triangles with the primaries, called the triangular or equilateral solutions. These latter ones are given by $r_1 = r_2 = 1$, which values, if substituted in Eqs. (13) and (14), will furnish the solution $y \neq 0$. Furthermore, to find the collinear points, we write $y = 0$ in Eqs. (13) and (14) and solve Eq. (13) for x.

The triangular points have the coordinates $x = \mu - \frac{1}{2}$ and $y = \pm 3^{1/2}/2$, the plus sign corresponding to L_4 and the minus to L_5. Computation of the values of the abscissas of the collinear points requires the solution of quintic equations since for $y = 0$ Eq. (13) may be written as

$$x + \frac{A}{(x-\mu)^2} + \frac{B}{(x+1-\mu)^2} = 0, \tag{15}$$

which leads to a fifth-order algebraic equation.

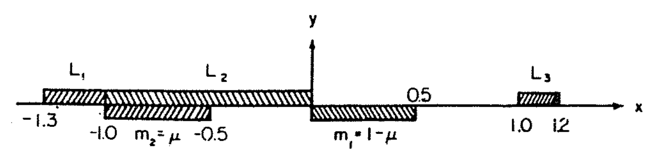

FIG. 4.1. Location of the collinear equilibrium points.

Figure 4.1 shows the locations of the equilibrium points. The first collinear point is located left of the second primary, the second is between the primaries, and the third collinear equilibrium point is to the right of the first primary. The regions in which the collinear points are located are shown in Fig. 4.1 and are marked L_1, L_2, and L_3.

Narrower limits can be given for the locations as follows:

$$\begin{aligned} \mu - 2 &\leqslant x_1 \leqslant \mu - 1, \\ \mu - 1 &\leqslant x_2 \leqslant \mu, \\ \mu &\leqslant x_3 \leqslant \mu + 1. \end{aligned} \tag{16a}$$

The limits can be made independent of the value of μ with the following result:

$$\begin{aligned} -1.3 &\leqslant x_1 \leqslant -1, \\ -1 &\leqslant x_2 \leqslant 0, \\ 1 &\leqslant x_3 \leqslant 1.2. \end{aligned} \tag{16b}$$

The actual limits are given approximately by

$$\begin{aligned} -1.271630 &\leqslant x_1 \leqslant -1, \\ -1 &\leqslant x_2 \leqslant 0, \\ +1 &\leqslant x_3 \leqslant 1.198406. \end{aligned} \tag{16c}$$

4.4 Computation of the location of the collinear points

The determination of the abscissas of the collinear points requires the solution of Eq. (13) or (15). Depending on the range in which the point is located, the coefficients A and B of Eq. (15) may be found and the specific quintic equation may be solved.

(A) Taking Range 1 first (see Fig. 4.2) we have $r_1 = \mu - x$ and $r_2 = \mu - x - 1$, which, when substituted in Eq. (13), results in

$$x + \frac{1-\mu}{(x-\mu)^2} + \frac{\mu}{(x+1-\mu)^2} = 0, \tag{17}$$

and so the coefficients in Eq. (15) become $A = 1 - \mu$ and $B = \mu$.

Putting $r_2 = \xi^{(1)}$, $r_1 = 1 + \xi^{(1)}$, $x = \mu - 1 - \xi^{(1)}$, we obtain

$$\xi^{(1)} + 1 - \mu + \frac{\mu - 1}{(1+\xi^{(1)})^2} - \frac{\mu}{(\xi^{(1)})^2} = 0. \tag{18}$$

A variety of methods are available for the solution of this fifth-order algebraic equation for $\xi^{(1)}$ with μ as a parameter. The actual quintic is

$$\xi^5 + (3-\mu)\xi^4 + (3-2\mu)\xi^3 - \mu\xi^2 - 2\mu\xi - \mu = 0, \tag{19}$$

where

$$\xi = \xi^{(1)}.$$

Descartes' sign rule indicates one and only one positive root for $0 < \mu \leqslant \frac{1}{2}$, and there is one real root ($\xi^{(1)} = 0$) for $\mu = 0$.

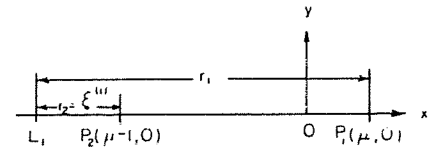

Fig. 4.2. Notation for the computation of $x_1 = x_1(L_1)$.

One method of solution, especially convenient for high-speed computers, is as follows. Factoring ξ^3 from the first three terms and solving for ξ^3 we have

$$\xi^3 = \frac{\mu(1+\xi)^2}{3 - 2\mu + \xi(3 - \mu + \xi)}. \tag{20}$$

An iteration process can start at this point using $\xi = 0$ as the starting value on the right side and obtaining $\xi = [\mu/(3-2\mu)]^{1/3}$ for the first iteration, etc. An alternate process starts with $\xi = (\mu/3)^{1/3}$, but the reader will find a third choice preferable. This starting value is $[\mu/3(1-\mu)]^{1/3}$ and it is suggested by the following considerations. Equation (18) may be written as

$$\xi + (1-\mu)[1 - (1+\xi)^{-2}] - \mu\xi^{-2} = 0$$

or as

$$(1-\mu)[1 + \xi - (1+\xi)^{-2}] + \mu(\xi - \xi^{-2}) = 0,$$

from which

$$\frac{\mu}{3(1-\mu)} = \frac{\xi^3(1 + \xi + \xi^2/3)}{(1+\xi)^2(1-\xi^3)}. \tag{21}$$

A series solution in powers of the quantity $\nu = [\mu/3(1-\mu)]^{1/3}$ is

$$\xi^{(1)} = \nu\left(1 + \frac{1}{3}\nu - \frac{1}{9}\nu^2 - \frac{31}{81}\nu^3 - \frac{119}{243}\nu^4 - \frac{1}{9}\nu^5\right) + \mathcal{O}(\nu^7). \tag{22}$$

An expansion with respect to $(\mu/3)^{1/3}$ is

$$\xi^{(1)} = \left(\frac{\mu}{3}\right)^{1/3}\left[1 + \frac{1}{3}\left(\frac{\mu}{3}\right)^{1/3} - \frac{1}{9}\left(\frac{\mu}{3}\right)^{2/3} + \cdots\right] \tag{23}$$

which follows from Eq. (22) with the substitution

$$\frac{\mu}{3(1-\mu)} = \frac{\mu}{3}(1 + \mu + \mu^2 + \cdots) \cong \frac{\mu}{3}.$$

The reader will realize that the rates of convergence of these series [(22) and (23)], which are given mainly for historical interest, are different (see Section 4.9).

Actual experience shows that the solution of Eq. (19) is most simply obtained by an iterative process using Eq. (20) and $[\mu/3(1-\mu)]^{1/3}$ as starting value.

(B) Range 2 is between the two primaries. According to Fig. 4.3

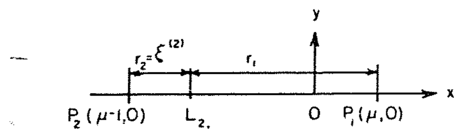

FIG. 4.3. Notation for the computation of $x_2 = x_2(L_2)$.

now $r_1 = \mu - x$, $r_2 = 1 - \mu + x$, which, when substituted in Eq. (13), gives

$$x + \frac{1-\mu}{(x-\mu)^2} - \frac{\mu}{(x+1-\mu)^2} = 0 \tag{24}$$

and so $A = 1 - \mu$ and $B = -\mu$. Writing again $r_2 = \xi^{(2)}$, $r_1 = 1 - \xi^{(2)}$, $x = \mu - 1 + \xi^{(2)}$ we have

$$\xi^{(2)} - 1 + \mu + \frac{1-\mu}{(\xi^{(2)}-1)^2} - \frac{\mu}{(\xi^{(2)})^2} = 0 \tag{25}$$

or

$$\xi^3 = \frac{\mu(1-\xi)^2}{3 - 2\mu - \xi(3 - \mu - \xi)}, \tag{26}$$

where $\xi = \xi^{(2)}$. The quintic becomes

$$\xi^5 - (3-\mu)\xi^4 + (3-2\mu)\xi^3 - \mu\xi^2 + 2\mu\xi - \mu = 0,$$

which shows the existence of at least one positive root according to Descartes' rule of signs.

Expansions similar to Eqs. (22) and (23) are available. An iterative process, starting again with $\xi = [\mu/3(1-\mu)]^{1/3}$, is the favored method of solution. The expansion with respect to

$$\nu = \left[\frac{\mu}{3(1-\mu)}\right]^{1/3}$$

is

$$\xi^{(2)} = \nu\left(1 - \frac{1}{3}\nu - \frac{1}{9}\nu^2 - \frac{23}{81}\nu^3 + \frac{151}{243}\nu^4 - \frac{1}{9}\nu^5\right) + \mathcal{O}(\nu^7), \tag{27}$$

which, with respect to $(\mu/3)^{1/3}$, becomes

$$\xi^{(2)} = \left(\frac{\mu}{3}\right)^{1/3}\left[1 - \frac{1}{3}\left(\frac{\mu}{3}\right)^{1/3} - \frac{1}{9}\left(\frac{\mu}{3}\right)^{2/3} + \cdots\right]. \tag{28}$$

(C) Finally, in the third region we have, according to Fig. 4.4, $r_1 = x - \mu$, $r_2 = 1 + x - \mu$, which, when substituted in Eq. (13), gives

$$x - \frac{1-\mu}{(x-\mu)^2} - \frac{\mu}{(1+x-\mu)^2} = 0 \tag{29}$$

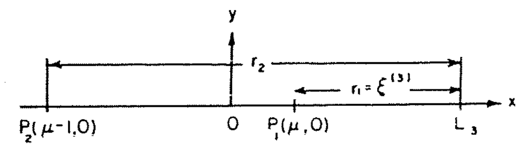

FIG. 4.4. Notation for the computation of $x_3 = x_3(L_3)$.

and so $A = \mu - 1$ and $B = -\mu$. Writing $r_1 = \xi^{(3)} = x - \mu$, $r_2 = 1 + \xi^{(3)}$, $x = \mu + \xi^{(3)}$, we have

$$\xi^{(3)} + \mu - \frac{1-\mu}{[\xi^{(3)}]^2} - \frac{\mu}{[1+\xi^{(3)}]^2} = 0, \tag{30}$$

or

$$\xi^3 = \frac{(1-\mu)(1+\xi)^2}{1 + 2\mu + \xi(2 + \mu + \xi)}, \tag{31}$$

where $\xi^{(3)} = \xi$.

The quintic becomes

$$\xi^5 + (2+\mu)\xi^4 + (1+2\mu)\xi^3 - (1-\mu)\xi^2 - 2(1-\mu)\xi - (1-\mu) = 0, \tag{32}$$

which, because of the single sign-change present, indicates one and only one positive root. Since this root is near $+1$ it is advantageous to introduce $\eta = \xi - 1$ which gives

$$\eta^5 + (7 + \mu)\eta^4 + (19 + 6\mu)\eta^3 + (24 + 13\mu)\eta^2 + 2(6 + 7\mu)\eta + 7\mu = 0. \tag{33}$$

The starting value

$$\eta = -\frac{7}{12}\mu$$

follows immediately from the last two terms. In fact, the existence of at least one negative root follows from the five changes of sign in Eq. (33) when the transition $\eta \to -\eta$ is introduced. A series solution in powers of $\nu = 7\mu/12$ is

$$\eta = -\nu\left(1 + \frac{23}{84}\nu^2 + \frac{23}{84}\nu^3 + \frac{761}{2352}\nu^4 + \frac{3163}{7056}\nu^5 + \frac{30703}{49392}\nu^6\right) + \mathcal{O}(\nu^8). \tag{34}$$

(D) Note that the function $x_1 = x_1(\mu)$ possesses a minimum at $\mu_0 = 0.178944$. This can be found by first noting that the minimum must satisfy Eq. (18) which connects $\xi^{(1)}$ or, through $\xi^{(1)} = \mu - 1 - x_1$, the quantity x_1 with μ. Furthermore, at the extremum $dx_1/d\mu = 0$ or $d\xi^{(1)}/d\mu = 1$, which, when substituted in the differentiated form of Eq. (18), results in

$$\xi^3(3 - 2\mu + \xi) + (\xi + 1)^3(2\mu - \xi) = 0, \tag{35}$$

where ξ is written, once again, for $\xi^{(1)}$.

Solution of (18) and (35) for μ gives

$$\mu = \frac{\xi^5 + 3\xi^4 + 3\xi^3}{\xi^4 + 2\xi^3 + \xi^2 + 2\xi + 1} = \frac{3\xi^2 + \xi}{6\xi^2 + 6\xi + 2}, \tag{36}$$

from which, when $\xi \neq 0$, we obtain

$$6\xi^6 + 21\xi^5 + 31\xi^4 + 19\xi^3 - \xi^2 - 5\xi - 1 = 0. \tag{37}$$

The only positive value satisfying this equation is $\xi = 0.450574$, which gives $(x_1)_{\min} = -1.271630$ with the previously given value of μ_0.

4.5 Description of Appendixes I, II, and III for the characteristics of the collinear equilibrium points

Inasmuch as Appendixes I, II, and III (see pp. 214–225) furnish in identical form the values for the first, second, and third collinear equilibrium points, respectively, only Appendix I is discussed in detail.

It consists of four parts: (1)A and B, (2)C, (3)D, (4)E. The distinction between these parts is the values of μ included. Sections A and B of Appendix I contain information about the following ranges of μ: $10^{-6} \leqslant \mu \leqslant 9 \times 10^{-6}$; $10^{-5} \leqslant \mu \leqslant 9 \times 10^{-5}$; $10^{-4} \leqslant \mu \leqslant 9 \times 10^{-4}$; $10^{-3} \leqslant \mu \leqslant 9 \times 10^{-3}$; $10^{-2} \leqslant \mu \leqslant 9 \times 10^{-2}$; and $0.1 \leqslant \mu \leqslant 0.5$. Each of the first five ranges contains 9 equidistant values, while the last range gives 40 entries. Appendix IC covers those values of μ which cluster around Jupiter's mass parameter,

$$\mu = \frac{m_{♃}}{m_{♃} + m_{\odot}} \cong 0.00095,$$

and furnishes 21 equidistant values in the range $0.00085 \leqslant \mu \leqslant 0.00105$. Appendix ID gives similar results for the earth-moon system, clustering around

$$\mu = \frac{m_{☾}}{m_{\oplus} + m_{☾}} \cong 0.012,$$

and covers the range $0.011 \leqslant \mu \leqslant 0.013$ with 21 equidistant entries. Appendix IE contains 13 entries of different values of μ. The first nine correspond to the planets. The tenth and eleventh give the formerly accepted and the most recently (1965) available values for the earth-moon system. The twelfth entry marked "critical" corresponds to the value of the mass parameter which separates the linearly stable and unstable

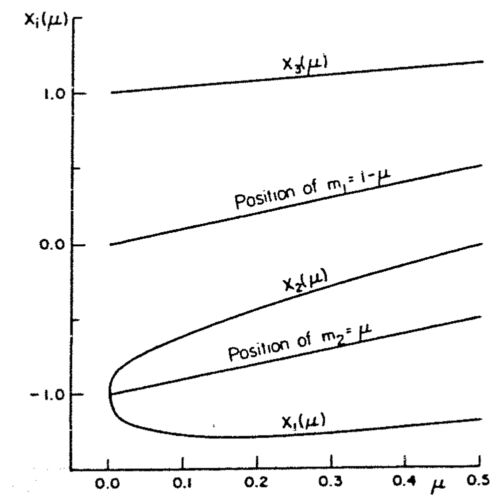

FIG. 4.5. Location of the collinear equilibrium points versus μ.

motions around the triangular libration points; see Chapter 5. The last entry corresponds to the value of μ used by G. Darwin; see Chapter 9.

The appendixes give the value of the mass parameter μ, the location of the collinear libration point x_i, the value of the Jacobian constant C_i, and the values of Ω_{xx} and Ω_{yy}, all at the collinear libration points. The error of the entries is expected to be less than $\pm 0.5 \times 10^{-11}$.

Figure 4.5 shows the abscissas of the collinear points as functions of the mass parameter.

The singularities of the manifold of the states of motion are associated with the solutions of the equations $\Omega_x = 0$, $\Omega_y = 0$, $2\Omega = C$. The complete solution of the problem therefore requires the evaluation of the value of the Jacobian constant, C, at the equilibrium points. It can be shown that at the triangular points $\Omega(L_4) = \Omega(L_5) = \frac{3}{2}$, independently of the value of the mass parameter, so $C(L_4) = C(L_5) = 3$.

The evaluation of $C(L_1) = C_1$, $C(L_2) = C_2$, and $C(L_3) = C_3$ leads to Fig. 4.6, which shows the monotonically increasing functions $C_2(\mu)$ and $C_3(\mu)$, and a maximum ($C_1 = 3.769683$) for the function $C_1(\mu)$ at $\mu = 0.334364$.

The tabulated values of C_1, C_2, and C_3 as functions of μ are given in Appendixes I, II, and III as mentioned before.

It is important to view these tables in the proper perspective. There are the always-present two approaches: that of the astronomer and

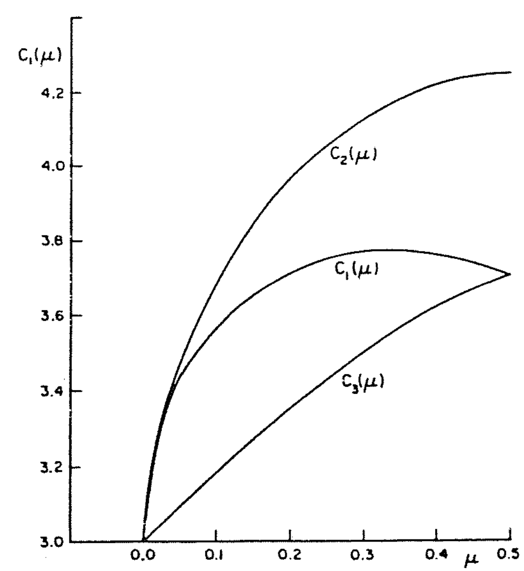

FIG. 4.6. Values of the Jacobian constants (C_i) at the collinear equilibrium points versus μ.

engineer who wish to apply the formulation and the results of the restricted problem to actual physical situations, and that of the mathematically inclined person who studies the problem per se. The values of the mass parameters of the planets, satellites, binary stars, and of any and all natural and artificial configurations are known with varying accuracy, but certainly with larger errors than shown in, say, Appendixes IE, IIE, and IIIE. The locations of the libration points are meaningful only when the assumptions of the restricted problem are acceptable. In actual physical situations various perturbations relative to and in addition to the restricted problem are present; consequently, one might question the significance of giving the values of $x(L_i)$ to ten decimals. On the other hand anyone computing periodic orbits around L_i with double precision might need more than ten decimals for the location of the libration points.

For the approximate construction of curves of zero velocity the second derivatives of the function Ω are not needed to ten figures, but any detailed computation around the collinear libration points requires the values of the slopes with at least that much accuracy.

To find the value of any entry for a value of μ not given in the tables presents the problem of solving a fifth-order algebraic equation in the case when the root is given approximately from interpolation. The accuracy required from such an operation depends on the problem at hand. Considering the physical situation, the associated limited accuracy with which the value of μ is known, the series solutions given by Eqs. (22), (27), and (34), and finally the auxiliary tables around the values $\mu = 0.012$ and $\mu = 0.0095$, it is felt that the tables as given in Appendixes I, II, and III might furnish the answers to the majority of pertinent questions.

4.6 Analysis of the function $\Omega(x, y)$

The Jacobian integral

$$\dot{x}^2 + \dot{y}^2 = 2\Omega(x, y) - C \tag{38}$$

allows the establishment of well-defined regions in the (x, y) plane, where motion with given initial conditions may take place. Such investigations are based on the simple observation that the quantity on the left-hand side of Eq. (38) is always positive. Therefore, for given initial conditions, the value of the Jacobian constant C may be evaluated and then the curve $2\Omega(x, y) = C$ will establish the desired region. This strikingly simple principle seems to cause more difficulties in the applications than many of the much more sophisticated operations;

therefore, it will be discussed in more detail in Section 4.7 as well as in the section devoted to applications (Section 4.8).

The regularized plane is also available, of course, for establishing the regions of motion since there the condition for possible motion is simply $\Omega^* \geq 0$.

Since the function $\Omega(x, y)$ plays the principal role in studying the regions of possible motions, its properties will be summarized and several propositions will be proved in the following sections (4.6.1–4.6.10). First we list these properties and then offer explanations and proofs.

List of properties of the function $\Omega(x, y)$:

(1) $\Omega(x, y) \geq \frac{3}{2}$,
$\lim_{r\to\infty} \Omega(x, y) = \lim_{r_1 \to 0} \Omega(x, y) = \lim_{r_2\to 0} \Omega(x, y) = \infty$,
$\Omega(x, y) = \Omega(x, -y)$.

(2) The absolute minima of the function $\Omega(x, y)$ occur at L_4 and L_5; the collinear equilibrium points are saddle points.

(3) The slope of the curves of zero velocity is given by $S = -\Omega_x/\Omega_y$, except at the collinear points, where $S = \pm(-\Omega_{xx}/\Omega_{yy})^{1/2}$.

(4) The minima of the function $\Omega(x, 0)$ occur at the collinear points.

(5) $1.5 = \Omega(L_{4,5}) \leq \Omega(L_3) \leq \Omega(L_1) \leq \Omega(L_2) \leq 2.125$.

(6) If $\mu = 0$, $x_1 = x_2 = -x_3 = -1$ and $C_i = 3$, $i = 1, 2, \ldots, 5$.

(7) $C_1 = C_3 = 3.706796$, $C_2 = (C_2)_{\max} = 4.25$, and $x_1 = -x_3 = -1.198406$, $x_2 = 0$ when $\mu = \frac{1}{2}$.

(8) The functions $C_i(\mu)$ monotonically increase when $i = 2$ or 3 and possess a maximum for $i = 1$; $(C_1)_{\max} = 3.769683$ at $\mu = 0.334364$.

(9) The distances between the masses and the collinear equilibrium points $r_2(L_1) = \mu - 1 - x_1(\mu)$, $r_2(L_2) = x_2(\mu) + 1 - \mu$, and $r_1(L_3) = x_3(\mu) - \mu$ increase monotonically with μ in the first two cases and decrease in the last. Also $r_2(L_1) \leq r_1(L_3)$, $r_2(L_2) \leq r_1(L_2)$, and $r_1(L_2) < r_1(L_3)$; the signs of equality corresponding to $\mu = \frac{1}{2}$. Here $r_i(L_j)$ is the distance between the ith primary and the jth collinear point.

(10) The distance between the center of mass and the second collinear equilibrium point, $[-x_2(\mu)]$, monotonically decreases; between the center of mass and the third collinear equilibrium point, $x_3(\mu)$, monotonically increases; and the distance between the first collinear equilibrium point and the center of mass, $[-x_1(\mu)]$, possesses a maximum: $(-x_1)_{\max} = 1.271630$ at $\mu = 0.178944$.

4.6.1 Bounds and a symmetry property of the function $\Omega(x, y)$

The first part of the first proposition may be shown by writing the function $\Omega(x, y)$ in the form

$$\Omega = \frac{3}{2} + (1-\mu)(r_1 - 1)^2\left(\frac{1}{2} + \frac{1}{r_1}\right) + \mu(r_2 - 1)^2\left(\frac{1}{2} + \frac{1}{r_2}\right). \tag{39}$$

The second and third terms are positive for $0 < \mu \leq \frac{1}{2}$, except when $r_1 = r_2 = 1$, in which case they are zero and $(\Omega)_{\min} = \Omega(r_1 = 1, r_2 = 1) = \frac{3}{2}$.

Regarding the upper bound of the function $\Omega(x, y)$, we recall the discussion in Chapter 3 according to which at the primaries $\Omega \to \infty$.

Writing $\Omega(x, y)$ in the form

$$\Omega = \tfrac{1}{2}r^2 + \tfrac{1}{2}\mu(1-\mu) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2},$$

we see that $\Omega(x, y) \to \infty$ as $r \to \infty$.

The last one of the first set of properties, the symmetry of the function $\Omega = \Omega(r_1, r_2)$, follows from its definition [Eq. (50) of Section 1.5] when the meanings of r_1 and r_2 are recalled:

$$r_1 = [(x-\mu)^2 + y^2]^{1/2}, \qquad r_2 = [(x - \mu + 1)^2 + y^2]^{1/2}.$$

So $r_i(x, y) = r_i(x, -y)$ for $i = 1, 2$, which proves the property that the curves $\Omega = \text{const}$ are symmetric to the x axis.

Note that using Birkhoff's transformation we have

$$\Omega^*(u, v) = \Omega^*(u, -v),$$

which can be verified by inspecting Eqs. (114) and (117) of Section 3.5.

4.6.2 Properties of the function Ω at the equilibrium points

The first part of the second proposition is connected with Proposition (1), but here we proceed in a more formal way by computing the Hessian at the triangular and collinear points.

The previously computed functions Ω_x and Ω_y may be written as

$$\Omega_x = x - \frac{(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu(x+1-\mu)}{r_2^3} = g(x, y) \tag{40}$$

and

$$\Omega_y = y\left[1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3}\right] = y f(x, y). \tag{41}$$

Note that, at the equilateral points,

$$r_1 = r_2 = 1, \qquad y \neq 0, \qquad f(L_{4,5}) = g(L_{4,5}) = 0.$$

The second derivatives of Ω become $\Omega_{xx} = g_x$, $\Omega_{xy} = g_y = yf_x$, $\Omega_{yy} = yf_y + f$, and at the equilateral points these derivatives have the values of $\Omega_{xx} = g_x(L_{4,5})$, $\Omega_{xy} = \pm(3^{1/2}/2) f_x(L_{4,5})$, $\Omega_{yy} = \pm(3^{1/2}/2) f_y(L_{4,5})$, where the positive sign belongs to L_4 and the negative to L_5.

After computing partial derivatives of g and f and substituting we obtain

$$g_x(L_{4,5}) = \tfrac{3}{4}, \qquad f_x(L_{4,5}) = 3(\mu - \tfrac{1}{2}), \qquad f_y(L_{4,5}) = \pm 3\frac{3^{1/2}}{2}$$

and consequently

$$\Omega_{xx}(L_{4,5}) = \tfrac{3}{4}, \qquad \Omega_{xy}(L_{4,5}) = \pm 3\frac{3^{1/2}}{2}(\mu - \tfrac{1}{2})$$

and

$$\Omega_{yy}(L_{4,5}) = 9/4.$$

The Hessian determinant of the second derivatives is

$$H(L_{4,5}) = \begin{vmatrix} \Omega_{xx} & \Omega_{xy} \\ \Omega_{xy} & \Omega_{yy} \end{vmatrix} = \begin{vmatrix} \frac{3}{4} & \pm 3\frac{3^{1/2}}{2}(\mu - \frac{1}{2}) \\ \pm 3\frac{3^{1/2}}{2}(\mu - \frac{1}{2}) & \frac{9}{4} \end{vmatrix} = \frac{27}{4}\mu(1-\mu). \tag{42}$$

Since $\Omega_{xx} > 0$ and $H(L_{4,5}) > 0$, excepting the uninteresting cases $\mu = 0$ or 1, we conclude that at the points L_4 and L_5 the function $\Omega(x, y)$ has minima. In fact $\Omega(L_{4,5}) = \frac{3}{2}$ for any value of μ.

The saddle points of the surface $z = \Omega(x, y)$ may be found by computing the Hessian determinant at the collinear points. Examination of the second derivatives from Eqs. (40) and (41) on the x axis requires additional attention:

$$\Omega_{xx} = g_x(x, 0) = 1 + \frac{2(1-\mu)(x-\mu)^2}{r_1^5} + \frac{2\mu(x+1-\mu)^2}{r_2^5}, \tag{43}$$

$$\Omega_{yy} = f(x, 0) = 1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3}, \tag{44}$$

and

$$\Omega_{xy} = y f_x(x, 0) = 0. \tag{45}$$

At the first collinear point $x - \mu = -r_1$ and $x + 1 - \mu = -r_2$, and so

$$g_x(x_1, 0) = 1 + \frac{2(1-\mu)}{r_1^3} + \frac{2\mu}{r_2^3} > 0. \tag{46}$$

Also, of course, at all collinear points $g(x_i, 0) = 0$, so

$$\mu - r_1 + \frac{1-\mu}{r_1^2} + \frac{\mu}{r_2^2} = 0. \tag{47}$$

Equation (44) can be written in the form

$$f(x, 0) = 1 - \frac{1-\mu}{r_1^2}\frac{1}{r_1} - \frac{\mu}{r_2^3}$$

and using Eq. (47) to express $(1-\mu)r_1^{-2}$ we obtain

$$f(x_1, 0) = 1 - \frac{1}{r_1}\left(r_1 - \mu - \frac{\mu}{r_2^2}\right) - \frac{\mu}{r_2^3}$$

or

$$f(x_1, 0) = \mu\left(\frac{1}{r_1} + \frac{1}{r_1 r_2^2} - \frac{1}{r_2^3}\right) = \mu\left(\frac{1}{r_2} - \frac{1}{r_1}\right)\left(r_2 - \frac{1}{r_2^2}\right)$$

and finally

$$f(x_1, 0) = \frac{\mu}{r_1}\left(1 - \frac{1}{r_2^3}\right). \tag{48}$$

Since $r_2 < 1$, $f(x_1, 0) < 0$.

The Hessian determinant is $H(L_1) = g_x(x_1, 0) f(x_1, 0) < 0$, which means that L_1 is a saddle point.

For L_2 and L_3 the analysis is quite similar and it presents no new problems.

The same results may be expressed by means of the indicatrix. The Dupin indicatrix of the surface $z = \Omega(x, y)$ at a given point is an ellipse, hyperbola, or parabola, according to the sign of the Gaussian curvature K of the surface at that point. If the principal curvatures of the surface at a given point are $C_1 = 1/R_1$ and $C_2 = 1/R_2$, where R_1 and R_2 are the principal radii of curvature, then

$$K = C_1 C_2 = \frac{\Omega_{xx}\Omega_{yy} - \Omega_{xy}^2}{(\Omega_x^2 + \Omega_y^2 + 1)^2}.$$

The average or mean curvature is defined by

$$M = \tfrac{1}{2}(C_1 + C_2) = \frac{(1 + \Omega_y^2)\Omega_{xx} - 2\Omega_x\Omega_y\Omega_{xy} + (1 + \Omega_x^2)\Omega_{yy}}{2(\Omega_x^2 + \Omega_y^2 + 1)^{3/2}}.$$

The curvatures C_1 and C_2, consequently, may be obtained from the quadratic equation

$$C^2 - 2MC + K = 0.$$

We speak of an elliptic, hyperbolic, or parabolic point when K is positive, negative, or zero.

At the five libration points, $\Omega_x = \Omega_y = 0$ and so the Gaussian curvature becomes the Hessian

$$K = \Omega_{xx}\Omega_{yy} - (\Omega_{xy})^2 = H.$$

The mean curvature at the equilibrium points is

$$M = \tfrac{1}{2}(\Omega_{xx} + \Omega_{yy}).$$

The Gaussian curvature at the triangular libration points is positive, while at the collinear points it is always negative. The mean curvature is $\frac{3}{2}$ at the triangular points and it is always positive at the collinear points.

The collinear points are hyperbolic and Dupin's indicatrices are hyperbolas at these points. The triangular points are elliptic points, the Dupin indicatrices are ellipses, and the surface $z = \Omega(x, y)$ stays on one side of its tangent plane at these points.

Example 1. Show that

$$\begin{aligned} &\Omega_{xx}(L_1) \to 9, && \Omega_{xx}(L_2) \to 9, && \Omega_{xx}(L_3) \to 3, \\ &\Omega_{yy}(L_1) \to -3, && \Omega_{yy}(L_2) \to -3, && \Omega_{yy}(L_3) \to 0, \end{aligned}$$

as $\mu \to 0$.

Example 2. Show that

$$\Omega_{xx}(L_2) = 17 \qquad \text{and} \qquad \Omega_{yy}(L_2) = -7$$

when $\mu = \frac{1}{2}$.

Example 3. Show that when the three collinear equilibrium points are considered $\Omega_{xy} = 0$, $17 \geqslant \Omega_{xx} \geqslant 3$, and $-7 \leqslant \Omega_{yy} \leqslant 0$ in the range $0 \leqslant \mu \leqslant \frac{1}{2}$.

4.6.3 The slopes of the curves of zero velocity

The curves $\Omega = \text{const}$ do not intersect except at the collinear (singular) points of the manifold.

The slopes of the curves $\Omega = \text{const}$ at any point, except at the singularities, are computed from the formula

$$S = \tan\alpha = -\frac{\Omega_x(x, y)}{\Omega_y(x, y)} \tag{49}$$

or

$$S = -\frac{g(x, y)}{y\, f(x, y)}. \tag{50}$$

The slopes at the collinear singular points are computed by a series expansion around the libration points $(x_L, 0)$, since $g = 0$ and $y = 0$ in this case.

The Taylor series of the function $\Omega(x, y)$ around $x = x_L$ and $y = 0$ is

$$\begin{aligned} \Omega(x, y) = {} & \Omega(x_L, 0) + \Omega_x(x_L, 0)(x - x_L) \\ & + \Omega_y(x_L, 0)\, y + \tfrac{1}{2}\Omega_{xx}(x_L, 0)(x - x_L)^2 \\ & + \Omega_{xy}(x_L, 0)(x - x_L)y + \tfrac{1}{2}\Omega_{yy}(x_L, 0)\, y^2 + \cdots. \end{aligned} \tag{51}$$

In this expansion $\Omega_x(x_L, 0) = \Omega_y(x_L, 0) = \Omega_{xy}(x_L, 0) = 0$ since the point $(x_L, 0)$ is a collinear equilibrium point. Writing now $x = x_L + \Delta x$ and $y = \Delta y$ we have

$$\Omega(x_L + \Delta x, \Delta y) = \Omega(x_L, 0) + \tfrac{1}{2}\Omega_{xx}(x_L, 0)(\Delta_x)^2 + \tfrac{1}{2}\Omega_{yy}(x_L, 0)(\Delta y)^2 + \cdots,$$

which in the limit gives

$$S = \frac{dy}{dx} = \pm\left(\frac{-\Omega_{xx}(x_L, 0)}{\Omega_{yy}(x_L, 0)}\right)^{1/2} = \pm\left(\frac{-g_x(x_L, 0)}{f(x_L, 0)}\right)^{1/2}. \tag{52}$$

An alternate way to obtain this result is to compute $\Omega_x(x, y)$ and $\Omega_y(x, y)$ by taking partial derivatives of $\Omega(x, y)$ as given by Eq. (51) and then substituting these results into Eq. (49):

$$\begin{aligned} \Omega_x(x, y) &= \Omega_x(x_L, 0) + \Omega_{xx}(x_L, 0)(x - x_L) + \Omega_{xy}(x_L, 0)\, y + \cdots, \\ \Omega_y(x, y) &= \Omega_y(x_L, 0) + \Omega_{yy}(x_L, 0)\, y + \Omega_{xy}(x_L, 0)(x - x_L) + \cdots, \end{aligned} \tag{53}$$

or

$$\begin{aligned} \Omega_x(x, y) &= \Omega_{xx}(x_L, 0)(x - x_L) + \cdots, \\ \Omega_y(x, y) &= \Omega_{yy}(x_L, 0)\, y + \cdots. \end{aligned}$$

Writing now Δx for $x - x_L$ and Δy for y, once again Eq. (52) is obtained.

Since $f(x_i, 0) < 0$ and $g_x(x_i, 0) > 0$, for $i = 1, 2, 3$, we see that S is real. Appendixes I, II, and III, in addition to giving the location of the collinear libration points and the values of the Jacobian constants, also list the second partial derivatives needed to evaluate the slopes.

4.6.4 The function $\Omega(x, 0)$

First it is shown that, in the ranges where the respective collinear points are located, the minima of the function $\Omega(x, 0)$ occur at the collinear libration points. For this the expression

$$\Omega(x, 0) = \tfrac{1}{2}x^2 + \tfrac{1}{2}\mu(1 - \mu) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} \tag{54}$$

is needed, which follows from the definition of $\Omega(x, y)$. Here $r_1(x)$ and $r_2(x)$ are defined differently in the three ranges of interest, as discussed before. Note that, at the singularities $r_1 = 0$ and $r_2 = 0$, $\Omega(x, 0) \to \infty$. Also from Eq. (54) it follows that $\Omega(x, 0) > 0$. Finally $\Omega(x, 0) \to \frac{1}{2}x^2 \to \infty$ as $x \to \pm\infty$.

From Eq. (54) we have

$$\frac{d\Omega(x, 0)}{dx} = x - \frac{1 - \mu}{r_1^2}\frac{dr_1}{dx} - \frac{\mu}{r_2^2}\frac{dr_2}{dx} \tag{55}$$

and

$$\frac{d^2\Omega(x, 0)}{dx^2} = 1 + \frac{2(1 - \mu)}{r_1^3}\left(\frac{dr_1}{dx}\right)^2 + \frac{2\mu}{r_2^3}\left(\frac{dr_2}{dx}\right)^2 > 1. \tag{56}$$

Since the second derivative is everywhere positive, the condition

$$\frac{d\Omega(x, 0)}{dx} = 0$$

furnishes the minima. It is to be noted that when $d^2\Omega/dx^2$ is computed the second derivatives of r_1 and r_2 are zero.

Now, in Range 1, on the left side of m_2, we have $r_1 = \mu - x$, $r_2 = \mu - x - 1$; consequently, $dr_1/dx = dr_2/dx = -1$. Equation (55) gives

$$\frac{d\Omega(x, 0)}{dx} = x + \frac{1 - \mu}{r_1^2} + \frac{\mu}{r_2^2}, \tag{57}$$

and therefore the location of the minimum is computed from the previously given equation for x_1 [cf. Eq. (17)].

In Range 2, between m_1 and m_2, we have $r_1 = \mu - x$, $r_2 = 1 - \mu + x$, and consequently $dr_1/dx = -dr_2/dx = -1$. Equation (55) gives

$$\frac{d\Omega(x, 0)}{dx} = x + \frac{1 - \mu}{r_1^2} - \frac{\mu}{r_2^2}, \tag{58}$$

and once again reference is made to Eq. (24) furnishing x_2 as the location of the minimum of Ω in this range.

In Range 3, to the right of m_1, we have $r_1 = x - \mu$, $r_2 = 1 + x - \mu$, and so $dr_1/dx = dr_2/dx = 1$. Equation (55) gives

$$\frac{d\Omega(x, 0)}{dx} = x - \frac{1 - \mu}{r_1^2} - \frac{\mu}{r_2^2}. \tag{59}$$

This is to be compared with Eq. (29), and the proof that the function $\Omega(x, 0)$ is minimum at the x_1, x_2, x_3 locations has been completed.

Figure 4.7 shows the function $C = 2\Omega(x, 0)$ for several values of μ, and Appendix IV gives the corresponding numerical values.

The value $\mu = 0.5$ represents symmetry with respect to the C axis as shown in Fig. 4.7(a). The asymmetry increases as μ decreases from

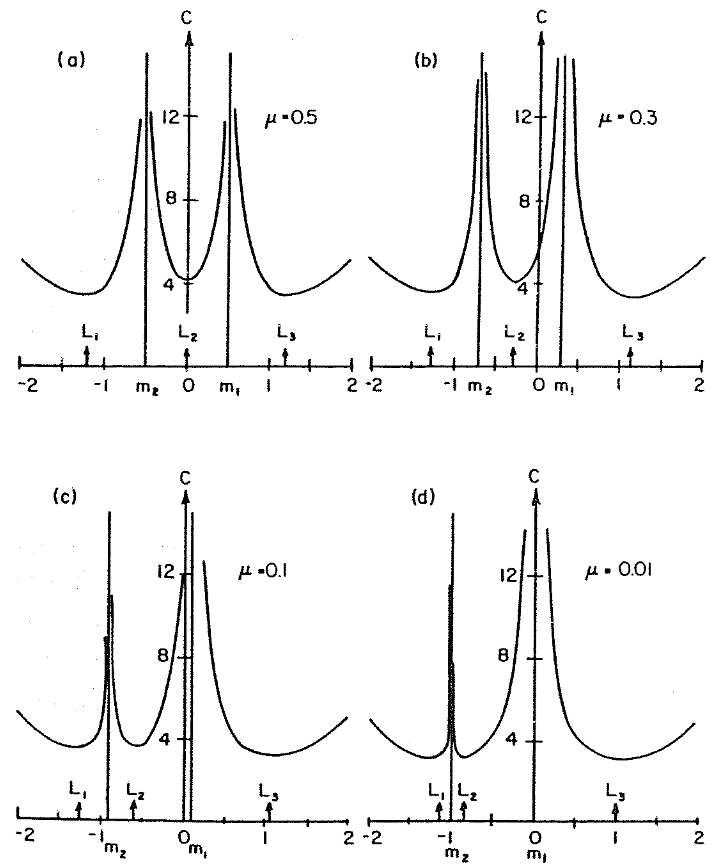

FIG. 4.7. The Jacobian constant along the x axis.

0.5 to 0. In Fig. 4.7(d) the minima corresponding to L_1 and L_2 are close to each other and also to the asymptote located at $x = \mu - 1 = -0.99$.

Appendix IV is constructed for $\mu = 10^{-6}$, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 0.1, 0.2, 0.3, 0.4, and 0.5. The error of the values of C given in Appendix IV is less than $\pm 5 \times 10^{-11}$ with the following remarks. At $x = \mu$ and at $x = \mu - 1$, where the primaries are located, $C = \infty$ as shown in the table. For small values of μ computation of the Jacobian constant at $x = 0$ and at $x = -1$ may be performed by series. From Eq. (54) we have with $r_1 = 1 + \mu$ and $r_2 = \mu$

$$C(-1, 0) = 3 + \mu - \mu^2 + 2\frac{1-\mu}{1+\mu}$$

or

$$C(-1, 0) = 5 - 3\mu(1 - \mu) + 4\sum_{n=3}^{\infty} (-\mu)^n.$$

Computation at $x = 0$ is quite different since now Eq. (54) gives

$$C(0, 0) = \mu - \mu^2 + \frac{2(1-\mu)}{\mu} + \frac{2\mu}{1-\mu}$$

from which

$$C(0, 0) = \frac{2}{\mu} - 2 + 3\mu + \mu^2 + 2\sum_{n=3}^{\infty} \mu^n.$$

For small values of μ we have

$$C(0, 0) \cong 2/\mu$$

and

$$C(-1, 0) \cong 5 - 3\mu.$$

The values of C given in Appendix IV corresponding to $C(0, 0)$ are simply $2/\mu$, since the construction of the appendix does not allow the display of the exact values. These approximate values of $C(0, 0)$ are indicated in Appendix IV by the sign $\sim$. The error of the values of C in the neighborhood of $x = -1$, $y = 0$ is the same as for the other entries, $\pm 5 \times 10^{-11}$. The range $-3 \leqslant x \leqslant 3$ is covered in 80 steps for every value of μ listed with the exception of $\mu = \frac{1}{2}$, for which, because of symmetry, only the range $0 \leqslant x \leqslant 3$ is given.

4.6.5 Values of the function $\Omega(x, y)$ at the equilibrium points

The four different values of the function $\Omega(x, y)$ at the five equilibrium points are ordered as follows:

$$1.5 = \Omega(L_4) = \Omega(L_5) \leqslant \Omega(L_3) \leqslant \Omega(L_1) \leqslant \Omega(L_2) \leqslant 2.125. \tag{60}$$

Consequently the corresponding values of the Jacobian constant satisfy the following inequalities:

$$3 = C_4 = C_5 \leqslant C_3 \leqslant C_1 \leqslant C_2 \leqslant 4.25. \tag{61}$$

Since it was shown previously that the function Ω has its absolute minima at the points L_4 and L_5, the first part of the inequality is considered proved. The second part refers to the collinear equilibrium points which are located on the x axis and therefore Eq. (60) may be written as

$$1.5 \leqslant \Omega(x_3, 0) \leqslant \Omega(x_1, 0) \leqslant \Omega(x_2, 0) \leqslant 2.125,$$

where x_1, x_2, and x_3 are the abscissas of the collinear equilibrium points.

First we show that $\Omega(\mu + \alpha, 0) < \Omega(\mu - \alpha, 0)$, where $0 < \alpha < 1$. Since

$$\Omega(\mu + \alpha, 0) = \tfrac{1}{2}(\mu + \alpha)^2 + \frac{\mu}{2}(1 - \mu) + \frac{1-\mu}{\alpha} + \frac{\mu}{1+\alpha}$$

and

$$\Omega(\mu - \alpha, 0) = \tfrac{1}{2}(\mu - \alpha)^2 + \frac{\mu}{2}(1 - \mu) + \frac{1-\mu}{\alpha} + \frac{\mu}{1-\alpha},$$

we have

$$\Omega(\mu - \alpha, 0) - \Omega(\mu + \alpha, 0) = \frac{2\mu}{1-\alpha^2}(\alpha^2 - \alpha + 1) > 0,$$

because the discriminant of the quadratic form is negative. Furthermore, $\Omega(x_3, 0) < \Omega(\mu + \alpha, 0)$ provided $0 < \alpha < x_3 - \mu$, for the minimum of the function $\Omega(x, 0)$ is at x_3 in the $\mu < x < \infty$ region. Also $x_3 - \mu > \mu - x_2$, or $r_1(L_3) > r_1(L_2)$. Consequently if $\mu - \alpha = x_2$, we have $\Omega(x_3, 0) < \Omega(x_2, 0)$, which is part of our proposition. The inequality $\Omega(x_1, 0) < \Omega(x_2, 0)$ may be verified similarly. The inequality used in this derivation, $r_1(L_3) > r_1(L_2)$, will be proved in Section 4.6.9.

4.6.6 Limiting values when μ is zero

Equations (57), (58), and (59) with the right side equated to zero may be evaluated when $\mu = 0$. The critical term occurring in the limit process when Eqs. (57) and (58) are used is

$$\varphi = \lim_{\mu \to 0} \frac{\mu}{[\mu - x_i(\mu) - 1]^2},$$

with $i = 1, 2$. The value of φ, however, may be seen to be zero and only l'Hospital's rule is needed to evaluate the limit since both $|dx_i/d\mu|$ and $|dx_i/d\mu|\,|\mu - x_i - 1| \to \infty$, as $\mu \to 0$, for $i = 1, 2$.

The result $x_3 \to 1$ as $\mu \to 0$ may be established directly from Eq. (59) without the appearance of any critical terms. In this way the first part of Proposition (6), regarding the locations of the collinear equilibrium points for $\mu \to 0$, is established with the exception of the verification of the limits regarding the derivative $dx_i/d\mu$.

These limits may be obtained from Eqs. (22) and (27) as follows. For $i = 1$ and 2 we have $x_i = \mu - 1 + \xi^{(i)}(-1)^i$, which notation is introduced in Section 4.4, Parts (A) and (B). Consequently

$$\frac{dx_i}{d\mu} = 1 + \frac{d\xi^{(i)}}{d\nu}\frac{d\nu}{d\mu}(-1)^i,$$

where the symbol

$$\nu = [\mu/3(1-\mu)]^{1/3}$$

was introduced also in Section 4.4. Since

$$\lim_{\mu\to 0}\frac{d\nu}{d\mu} = \lim_{\mu\to 0}[3\mu^{1/2}(1-\mu)]^{-4/3} = \infty$$

and since

$$\lim_{\substack{\mu\to 0\\(\nu\to 0)}}\frac{d\xi^{(i)}}{d\nu} = 1,$$

we have

$$\lim_{\mu\to 0}\frac{dx_i}{d\mu} = (-1)^i\,\infty, \qquad \text{for} \quad i = 1, 2.$$

Note that the result $dx_3/d\mu = 5/12$ may be obtained without difficulty since, using Section 4.4 Part (C),

$$dx_3/d\mu = 1 + d\eta/d\mu = 1 - 7/12 + \mathcal{O}(\mu^2),$$

and so, at $\mu = 0$, $dx_3/d\mu = 5/12$.

The second part of Proposition (6) refers to the value of the Jacobian constant at the equilibrium points when $\mu \to 0$. A limit process gives $\Omega(L_i) = \Omega(x_i, 0) = 3/2$ for $i = 1, 2, 3$, using Eq. (54), and $\Omega(L_i) = 3/2$ for $i = 4, 5$, using Eq. (39).

The only critical term occurring is

$$\bar{\varphi} = \lim_{\mu\to 0}\frac{(-1)^i\mu}{1-\mu+x_i(\mu)},$$

which may be treated similarly to the previous limit denoted by φ.

4.6.7 Limiting values when $\mu = \frac{1}{2}$

First we show that $x_1 = -x_3$ when μ is $\frac{1}{2}$ by writing $\mu = \frac{1}{2}$ in Eqs. (17) and (29). In this way we obtain

$$x_1 + \frac{\frac{1}{2}}{(x_1 - \frac{1}{2})^2} + \frac{\frac{1}{2}}{(x_1 + \frac{1}{2})^2} = 0 \tag{62}$$

and

$$x_3 - \frac{\frac{1}{2}}{(x_3 - \frac{1}{2})^2} - \frac{\frac{1}{2}}{(x_3 + \frac{1}{2})^2} = 0, \tag{63}$$

or $x_1 = -x_3$.

Regarding the value of $x_2(\frac{1}{2})$ consider the quintic equation given in Section 4.4, Part (B), for ξ. This equation becomes, with $\mu = \frac{1}{2}$ for $\xi = \xi^{(2)} = \frac{1}{2} + x_2$,

$$2\xi^5 - 5\xi^4 + 4\xi^3 - \xi^2 + 2\xi - 1 = 0$$

or

$$(\xi - \tfrac{1}{2})[\xi^2(\xi-1)^2 + 1] = 0.$$

Consequently the only real root is $\xi = \frac{1}{2}$, and therefore $x_2 = 0$.

The location of the first equilibrium point for $\mu = \frac{1}{2}$ may be computed from Eq. (62) or from $x_1 = -1.2 + \eta$, where an approximate value of η (accurate to eight decimals) is the smaller positive root of the equation

$$14.48\eta^2 - 5.8705\eta + 0.00932 = 0.$$

For reasons of symmetry $C_1 = C_3$ at $\mu = \frac{1}{2}$. In fact, from Eq. (54) we have

$$C_1 = 2\Omega(x_1, 0) = 2\Omega(x_3, 0) = C_3$$

since

$$\begin{aligned} 2\,\Omega(x_1, 0) &= x_1^2 + \frac{1}{4} + \frac{2}{1-2x_1} - \frac{2}{1+2x_1}, \\ 2\,\Omega(x_3, 0) &= x_3^2 + \frac{1}{4} + \frac{2}{2x_3-1} + \frac{2}{2x_3+1}, \end{aligned} \tag{64}$$

and $x_3 = -x_1$.

The last part of Proposition (7), concerning the maximum value of C_2 occurring at $x_2 = 0$ and correspondingly at $\mu = \frac{1}{2}$, is shown as follows. The total derivative of $\frac{1}{2}C_2 = \Omega(x_2, 0)$ with respect to the mass parameter is

$$\frac{d\Omega(x_2, 0)}{d\mu} = \frac{\partial\Omega(x_2, 0)}{\partial\mu} + \frac{\partial\Omega(x_2, 0)}{\partial x_2}\frac{dx_2}{d\mu}, \tag{65}$$

where

$$\frac{\partial \Omega(x_2, 0)}{\partial x_2} = 0 \tag{66}$$

and

$$\frac{\partial \Omega(x_2, 0)}{\partial \mu} = \tfrac{1}{2}(r_2^2 - r_1^2) + \frac{1}{r_2} - \frac{1}{r_1} + (1-\mu)\left(r_1 - \frac{1}{r_1^2}\right)\frac{\partial r_1}{\partial \mu} + \mu\left(r_2 - \frac{1}{r_2^2}\right)\frac{\partial r_2}{\partial \mu}. \tag{67}$$

Equation (66) states that x_2 is the abscissa of a collinear equilibrium point and Eq. (67) follows from the original definition of the function Ω. In Region 2 we have $r_1 = \mu - x$ and $r_2 = 1 - \mu + x$; Eq. (67) therefore can be written as

$$\frac{\partial \Omega}{\partial \mu} = \tfrac{1}{2}(r_2^2 - r_1^2) + \frac{1}{r_2} - \frac{1}{r_1} \tag{68}$$

since, according to Eq. (58),

$$\frac{d\Omega(x_2, 0)}{dx_2} = (1-\mu)\left(\frac{1}{r_1^2} - r_1\right) + \mu\left(r_2 - \frac{1}{r_2^2}\right) = 0.$$

Finally we have

$$\frac{\partial \Omega(x_2, 0)}{\partial \mu} = (1 - 2r_1)\frac{r_1 - r_1^2 - 2}{2r_1(1-r_1)} = f(r_1), \tag{69}$$

where the relation $r_1 + r_2 = 1$, valid in the second region, was utilized and where, for $0 \leqslant \mu \leqslant \frac{1}{2}$, we have $\frac{1}{2} \leqslant r_1 \leqslant 1$.

The only real root of the equation $f(r_1) = 0$ is $r_1 = \frac{1}{2}$, since $r_1 - r_1^2 - 2 < 0$ everywhere. The function $f(r_1)$ changes sign at $r_1 = \frac{1}{2}$, from positive to negative with decreasing r_1, that is, with increasing μ. Consequently the function $\Omega(x_2, 0)$ has a maximum at $\mu = \frac{1}{2}$. This maximum value can be computed from Eq. (54):

$$\Omega(x_2, 0) = \frac{x_2^2}{2} + \tfrac{1}{2}\mu(1-\mu) + \frac{1-\mu}{\mu - x_2} + \frac{\mu}{1 - \mu + x_2}, \tag{70}$$

and consequently, when $\mu = \frac{1}{2}$ and $x_2 = 0$, we have $\Omega(0, 0) = 2.125$ and $(C_2)_{\max} = 4.25$.

4.6.8 Properties of the functions $C_i(\mu)$

We first show that the functions $\Omega[x_2(\mu), 0]$ and $\Omega[x_3(\mu), 0]$ increase monotonically; consequently, the same is true for the functions $C_i(\mu)$ for $i = 2, 3$.

Returning to the function $f(r_1)$ defined by Eq. (69) we observe that it is everywhere positive and finite in the range $0 < \mu < \frac{1}{2}$, or equivalently, when $\frac{1}{2} < r_1 < 1$, since $1 - 2r_1 < 0$, $r_1 - r_1^2 - 2 < 0$, $r_1 > 0$, and $1 - r_1 > 0$. Consequently C_2 increases monotonically with μ. At the upper and lower values of the inequalities we have $f(r_1) \to \infty$ as $\mu \to 0$, $r_1 \to 1$, and $f(r_1) = 0$ when $\mu = \frac{1}{2}$, $r = \frac{1}{2}$.

The statement that $\Omega(x_3, 0)$ is a monotone increasing function is verified by evaluating again Eq. (68), but this time with $r_1 + 1 = r_2$. This gives

$$\frac{\partial \Omega(x_3, 0)}{\partial \mu} = \frac{r_1(1 + r_1)(1 + 2r_1) - 2}{2r_1(1 + r_1)} = g(r_1). \tag{71}$$

Since $r_1 = x_3 - \mu$ and $x_3 = 1$ when $\mu = 0$, we have $r_1 = 1$ associated with the value $\mu = 0$. Similarly since $x_3 > 1.1$ when $\mu = \frac{1}{2}$, we have $r_1 > 0.6$ associated with the value $\mu = \frac{1}{2}$. Therefore in the range $0 \leqslant \mu \leqslant \frac{1}{2}$ we have $0.6 < r_1 \leqslant 1$. The denominator of the function $g(r_1)$ defined by Eq. (71) is always positive and its numerator is a monotone increasing function of r_1 in the range $0.6 < r_2 \leqslant 1$. At $r_1 = 0.6$ this numerator is positive; therefore, in the whole range, $g(r_1)$ is positive and consequently $\Omega(x_3, 0)$ is a monotonically increasing function of μ.

To determine the maximum value of $\Omega(x_1, 0)$ as a function of μ, we use Eq. (68) with $r_1 = 1 + r_2$. This gives

$$\frac{\partial \Omega(x_1, 0)}{\partial \mu} = \frac{2 - r_2(1 + r_2)(1 + 2r_2)}{2r_2(1 + r_2)} = -g(r_2). \tag{72}$$

Since now (in the first range, on the left side of m_2) $r_2 = \mu - x_1 - 1$, we have, for $\mu = 0$, $x_1 = -1$ and $r_2 = 0$ and, for $\mu = \frac{1}{2}$, $x_1 > -1.2$ and $r_2 < 0.7$. Consequently the $0 \leqslant \mu \leqslant \frac{1}{2}$ range is associated with the $0 \leqslant r_2 < 0.7$ range for r_2. The cubic expression in the numerator of $-g(r_2)$ in Eq. (72) has only one positive root, $r_2 = 0.583156$. The corresponding value of μ can be obtained from Eq. (57):

$$\mu - 1 - r_2 + \frac{1-\mu}{(1 + r_2)^2} + \frac{\mu}{r_2^2} = 0,$$

which gives

$$\mu = \frac{r_2^3(r_2^2 + 3r_2 + 3)}{r_2^4 + 2r_2^3 + r_2^2 + 2r_2 + 1}. \tag{73}$$

Substituting the numerical value given for r_2 results in $\mu = 0.334364$.

The maximum value of $\Omega(x_1, 0)$ can be computed from Eq. (54) by substituting the critical values of r_2 and μ.

The values used in the preceding derivations, $x_3 > 1.1$ and $x_1 > -1.2$, are applicable when $\mu = \frac{1}{2}$. The proof is obtained by referring to Eqs. (18) and (32) of Section 4.4. In either case the equation to determine the location of the first or the third collinear point when $\mu = \frac{1}{2}$ becomes

$$F(\xi) = \xi^5 + 2.5\xi^4 + 2\xi^3 - 0.5\xi^2 - \xi - 0.5 = 0.$$

For L_1 we have $-x_1 = \xi + \frac{1}{2}$, and for L_3 we use $x_3 = \xi + \frac{1}{2}$. Thus, the previously quoted two inequalities may be written as one inequality, $0.6 < \xi < 0.7$. Now we note that the equation $F(\xi) = 0$ has only one positive root and this is between 0.6 and 0.7 since $F(0.6) < 0$ and $F(0.7) > 0$.

4.6.9 Properties of the distances between the primaries and the equilibrium points

First we show the monotonic property of the distances.

Let $r_i(L_j)$ be the distance between the ith primary and the jth collinear equilibrium point. Then $r_2(L_1) = \mu - x_1 - 1$, $r_2(L_2) = x_2 + 1 - \mu$, and $r_1(L_3) = x_3 - \mu$. In terms of these quantities the first part of Proposition (9) says that the $i = 2$, $j = 1, 2$ combinations give monotonically increasing functions of μ, while the $i = 1$, $j = 3$ combination results in a monotonically decreasing function.

In order to show these results, first eliminate r_1 and x from Eq. (57) by $r_1 = 1 + r_2$ and $x = \mu - 1 - r_2$. This gives an identity in μ valid in the first range, or left of m_2. Note that here $r_2 = r_2(\mu)$. The equation obtained for r_2 is

$$\mu - 1 - r_2 + \frac{1-\mu}{(1+r_2)^2} + \frac{\mu}{r_2^2} = 0.$$

From this $r_2' = dr_2/d\mu$ may be computed by differentiation with respect to μ:

$$r_2' = \frac{1 + 1/r_2^2 - 1/(1+r_2)^2}{1 + 2(1-\mu)/(1+r_2)^3 + 2\mu/r_2^3}. \tag{73a}$$

Since $r_2 > 0$, $(1 + r_2)^{-2} < r_2^{-2}$ and so $r_2' > 0$. Consequently the function $r_2(\mu)$ monotonically increases.

To prove that the second combination ($i = j = 2$) gives also a monotonically increasing function, eliminate r_1 and x from Eq. (58) by $r_1 = 1 - r_2$ and $x = r_2 - 1 + \mu$. The equation obtained for r_2 is

$$r_2 - 1 + \mu + \frac{1-\mu}{(1-r_2)^2} - \frac{\mu}{r_2^2} = 0,$$

from which by differentiation we have

$$r_2' = \frac{-1 + 1/r_2^2 + 1/(1-r_2)^2}{1 + 2(1-\mu)/(1-r_2)^3 + 2\mu/r_2^3}. \tag{73b}$$

Since $0 < r_2 < \frac{1}{2}$ the numerator is larger than $+7$ and therefore $r_2' > 0$. Consequently the function $r_2(\mu)$ increases monotonically.

Finally to verify that the function defined by the third combination, $r_1(L_3)$, decreases monotonically, consider Eq. (59) with $x = r_1 + \mu$ and with $r_2 = 1 + r_1$:

$$r_1 + \mu - \frac{1-\mu}{r_1^2} - \frac{\mu}{(1+r_1)^2} = 0.$$

By differentiation we obtain

$$r_1' = \frac{1/(1+r_1)^2 - 1/r_1^2 - 1}{1 + 2(1-\mu)/r_1^3 + 2\mu/(1+r_1)^3}. \tag{73c}$$

From this it follows by the same argument as presented in the case of $r_2(L_1)$ that $r_1' < 0$. Consequently the function $r_1(\mu)$ decreases monotonically.

Now we turn our attention to the second part of Proposition (9), concerning the inequalities $r_2(L_1) \leqslant r_1(L_3)$, $r_2(L_2) \leqslant r_1(L_2)$, and $r_1(L_2) < r_1(L_3)$ for $0 < \mu \leqslant \frac{1}{2}$. First we note that $r_1(L_2) = r_2(L_2) = \frac{1}{2}$ when $\mu = \frac{1}{2}$ and that $r_1(L_2) + r_2(L_2) = 1$ for $0 \leqslant \mu \leqslant \frac{1}{2}$. The second of the three inequalities to be proved may be written as $r_2(L_2) \leqslant 1 - r_2(L_2)$ or $r_2(L_2) \leqslant \frac{1}{2}$. But we have shown in this section that $r_2(L_2)$ increases monotonically as μ increases from 0 to $\frac{1}{2}$, consequently $r_2(L_2)$ has its largest value at $\mu = \frac{1}{2}$. Q.E.D.

The verification of the first inequality follows a similar pattern. First we note that $r_1(L_3) = r_2(L_1)$ when $\mu = \frac{1}{2}$. But we have shown that $r_2(L_1)$ increases and $r_1(L_3)$ decreases monotonically with increasing μ. The two quantities in question are equal at $\mu = \frac{1}{2}$, and $r_1(L_3)$ increases while $r_2(L_1)$ decreases as $\frac{1}{2} \to \mu \to 0$; consequently, in this range $r_1(L_3) \geqslant r_2(L_1)$. Q.E.D.

The proof of the third inequality $r_1(L_2) \leqslant r_1(L_3)$ is only slightly more cumbersome. Note first that the equality sign corresponds to $\mu = 0$, in which case $r_1(L_2) = r_1(L_3) = 1$. When $\mu = \frac{1}{2}$ the inequality is certainly correct since $r_1(L_2) = \frac{1}{2}$, while $r_1(L_3) > \frac{1}{2}$, as was shown at the end of Section 4.6.8 in a different form ($0.6 < \xi = r_1(L_3) < 0.7$). So the inequality to be verified is correct in the range $0 < \mu \leqslant \frac{1}{2}$, unless in this range there is a value of μ for which $r_1(L_2) = r_1(L_3)$. That this is not the case follows from Eqs. (58) and (59). Writing $r_2 = 1 - r_1$ and $x = \mu - r_1$, in Eq. (58), we obtain

$$\mu - r_1 + \frac{1-\mu}{r_1^2} - \frac{\mu}{(1-r_1)^2} = 0.$$

On the other hand, if we write $r_2 = 1 + r_1$ and $x = \mu + r_1$, Eq. (59) becomes

$$\mu + r_1 - \frac{1-\mu}{r_1^2} - \frac{\mu}{(1+r_1)^2} = 0.$$

In the previous two equations we have $r_1 = r_1(L_2)$ and $r_1 = r_1(L_3)$, respectively. Solving these equations for μ and equating the results gives

$$\mu = \frac{(r_1^3 - 1)(1-r_1)^2}{(r_1^2 - 1)(1-r_1)^2 - r_1^2} = -\frac{(r_1^3 - 1)(1+r_1)^2}{(r_1^2+1)(1+r_1)^2 - r_1^2}$$

or

$$1 + r_1^2 = (1 - r_1^2)^2,$$

the only positive root of which is $r_1 = 3^{1/2}$. Since $r_1 < 0.7$, we conclude that $r_1(L_2) \neq r_1(L_3)$ for $0 < \mu \leqslant \frac{1}{2}$ and consequently $r_1(L_2) \leqslant r_1(L_3)$. Q.E.D.

Note that all other possible combinations of inequalities between the distances $r_i(L_j)$ may be reduced to the three basic ones discussed in this section.

4.6.10 Properties of the distances between the center of mass and the equilibrium points

The previous proposition accounted for the distances monotonically increasing with μ between m_2 and the first, as well as between m_2 and the second equilibrium points. The distance between the third equilibrium point and m_1 on the other hand decreases monotonically with increasing μ.

The distances between the center of mass (origin of the coordinate system) and the equilibrium points are the subjects of the present section. It will be shown that the distance between L_1 and the origin, $-x_1(\mu)$, has a maximum in the range $0 \leqslant \mu \leqslant \frac{1}{2}$. The distance between L_2 and the origin, $-x_2(\mu)$, decreases monotonically and the distance between the origin, and the third equilibrium point $x_3(\mu)$ increases monotonically.

We begin with the last statement and show that $dx_3(\mu)/d\mu > 0$. Since in the third range $x_3 = r_1 + \mu$, the sign of $r_1' + 1$ is to be evaluated using the previously found expression for r_1' in Section 4.6.9, Eq. (73c). From this we obtain

$$r_1' + 1 = \frac{1/(1+r_1)^2 + [2(1-\mu) - r_1]/r_1^3 + 2\mu/(1+r_1)^3}{1 + 2(1-\mu)/r_1^3 + 2\mu/(1+r_1)^3},$$

where all terms are positive, and consequently the function $x_3(\mu)$ increases monotonically. Note that $r_1 < 1$ and $1 < 2(1-\mu) < 2$.

The second part of Proposition (10) refers to the distance between the origin and L_2. Since in the second range we have $x_2 = r_2 - 1 + \mu$, the derivative in question becomes $dx_2/d\mu = r_2' + 1$. It was shown in Section 4.6.9 that $r_2' > 0$; consequently, the derivative $dx_2/d\mu > 0$ and the function $-x_2(\mu)$ decreases monotonically, see Eq. (73b).

The distance between the center of mass of the primaries and the first equilibrium point is not a monotonic function of the mass parameter. To find the maximum of this distance $-x_1$ consider in the first range $x_1 = \mu - 1 - r_2$ from which $-x_1' = r_2' - 1$.

Equation (73a) of Section 4.6.9 gives an equation for r_2', from which the condition $x_1' = 0$ may be written as

$$r_2^3(3 - 2\mu + r_2) + (r_2 + 1)^3(2\mu - r_2) = 0$$

or as

$$\mu = \frac{3r_2^2 + r_2}{2(3r_2^2 + 3r_2 + 1)}.$$

Since r_2 is connected with μ by Eq. (73a) of Section 4.6.9, we have

$$\mu = \frac{r_2^3(r_2^2 + 3r_2 + 3)}{r_2^4 + 2r_2^3 + r_2^2 + 2r_2 + 1},$$

which is equivalent to the condition $\Omega_x = 0$ and to Eq. (73). The last two equations furnish a sixth-order equation for r_2,

$$6r_2^6 + 21r_2^5 + 31r_2^4 + 19r_2^3 - r_2^2 - 5r_2 - 1 = 0,$$

the only positive root of which is $r_2 = 0.450574$. The corresponding value of the mass parameter is $\mu_0 = 0.0178944$, and consequently $(x_1)_{\min} = \mu_0 - 1 - r_2 = -1.271630$. The sufficiency condition for the minimum is that $r_2'' < 0$, which may be seen to be satisfied by differentiating Eq. (73a) of Section 4.6.9 with respect to μ.

This completes the proof and discussion of the ten propositions presented regarding the properties of the function $\Omega(x, y)$.

4.7 Regions of motion

4.7.1 The problem of two bodies

As mentioned before the curves $\Omega(x, y) = \text{const}$ represent various regions of possible motion since this essentially positive function is connected with an essentially positive quantity, the square of the relative velocity, through the Jacobian integral.

Consider the function

$$\Omega(x, y) = \tfrac{1}{2}[(1 - \mu)r_1^2 + \mu r_2^2] + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} \tag{74}$$

in the case when $\mu = 0$. We have

$$\Omega(x, y) = \tfrac{1}{2}r_1^2 + \frac{1}{r_1}, \tag{75}$$

where $r_1 = r$ is the distance between the third particle and the primary with unit mass, located at the origin of the coordinate system. The curves $\Omega(x, y) = \text{const}$ are concentric circles around the origin.

The Jacobian integral $v^2 = 2\Omega(x, y) - C$ becomes

$$v^2 = r^2 + 2/r - C. \tag{76}$$

Consider now the motion with initial conditions $v = v_0$ and $r = r_0$. The Jacobian constant of this motion from Eq. (76) becomes

$$C_0 = r_0^2 + 2/r_0 - v_0^2. \tag{77}$$

The radius of the circle of zero velocity belonging to this C_0 value is given by the solution of the equation

$$C_0 = r^2 + 2/r = r_z^2 + 2/r_z, \tag{78}$$

which follows from Eq. (76). This equation, when solved for r (if there is a real positive solution), gives the radius of the circle of zero velocity belonging to the selected initial conditions. Calling this radius r_z, we have, from Eqs. (77) and (78),

$$r_z^2 + 2/r_z = r_0^2 + 2/r_0 - v_0^2, \tag{79}$$

which is the relation between the initial conditions (subscripts 0) and the radius of the circle of zero velocity (subscript z).

The Jacobian integral, Eq. (76), connects the variables (in this case r and v) for a given value of C (in this case C_0) at any time by

$$v^2 = r^2 + 2/r - C_0 \tag{80}$$

or

$$v^2 = r^2 + 2/r - r_z^2 - 2/r_z. \tag{81}$$

Here r is any distance from the origin and r_z is the radius of the curve of zero velocity. Since $v^2 \geqslant 0$, we must have

$$r^2 + 2/r - (r_z^2 + 2/r_z) \geqslant 0, \tag{82}$$

where the equality holds when $r = r_z$; that is, $v = 0$ or, in other words, when the particle is on its own curve of zero velocity. Those values of r for which Inequality (82) is satisfied will be associated with real values of the velocity. To find the ranges of possible motion the solution of a cubic equation is required:

$$r_z^3 - C_0 r_z + 2 = 0. \tag{83}$$

To clarify matters let the initial conditions be $r_0 = 2$, $v_0 = 1$. This means that the motion starts at any point from the circumference of a circle of radius 2 with unit velocity in any direction; see Fig. 4.8.

This initial condition gives a Jacobian constant of $C_0 = 4$ using Eq. (77). The radius of the circle of zero velocity is given by the solution of

$$r_z^3 - 4r_z + 2 = 0, \tag{84}$$

that is, approximately $(r_z)_{1,2,3} = 1.67$, 0.54, and -2.21. The two positive values (r_l and r_s) correspond to the two radii of the two curves of zero velocity belonging to $C_0 = 4$; see Fig. 4.8.

Figure 4.9 shows the function $C_0(r_z) = r_z^2 + 2/r_z$ and the two radii, the smaller and larger denoted by r_s and r_l, respectively.

Now we return to the Jacobian integral [Eq. (80)], which in our case is

$$v^2 = r^2 + 2/r - 4. \tag{85}$$

Observe that when $r = r_l = 1.67$, then $v = 0$ as expected since r is the radius of a curve of zero velocity. The velocity at the beginning of the motion (at $r_0 = 2$) is $v_0 = 1$. When the velocity is computed from Eq. (85) at any point for which $r^2 + 2/r$ is smaller than 4 the result will be imaginary. This means that the particle (with such initial

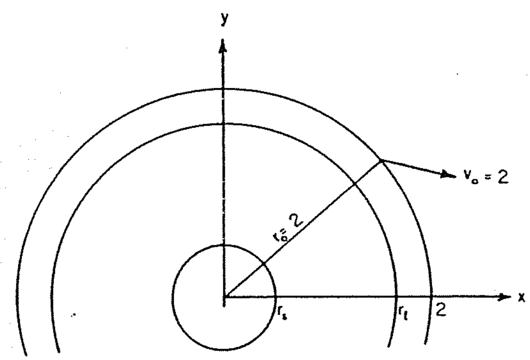

FIG. 4.8. Curves of zero velocity and initial conditions for the problem of two bodies.

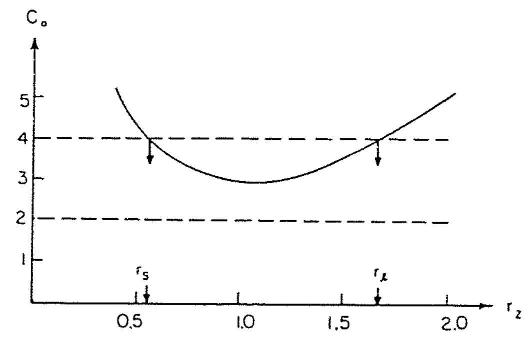

FIG. 4.9. Jacobian constant versus radius of the circle of zero velocity for the problem of two bodies.

conditions as result in $C_0 = 4$) can never find itself in a region for which $r^2 + 2/r < 4$. A plot of $r^2 + 2/r$ in Fig. 4.9 (with $r = r_z$) shows that $r \geqslant r_l = 1.67$ in order to have $r^2 + 2/r \geqslant 4$, i.e., to have a real value for the velocity. In other words, the particle cannot penetrate the circle of radius r_l and the motion must take place outside the circle of zero velocity with radius r_l.

Now consider another particle with initial conditions $r_0 = \frac{1}{2}$, $v_0 = \frac{1}{2}$. The Jacobian constant is again $C_0 = 4$, which can be computed using Eq. (77). If this particle approaches the circle with radius $r = r_s = 0.54$, its velocity becomes zero as Fig. 4.9 shows, and when the velocity is computed at any point for which $r^2 + 2/r$ is smaller than 4, the velocity becomes imaginary according to Eq. (85). This means that the particle starting inside the circle of zero velocity with radius r_s will stay inside this circle.

The important and simple idea is that any moving particle has its own Jacobian constant, therefore its own curves of zero velocity and consequently its own regions of permissible motion. We have seen in the previous example that particles outside the circle with radius r_l and inside the circle with radius r_s will not penetrate these curves of zero velocity. It is fallacious to think of particles in the annulus between these circles in the region $r_s < r < r_l$ since for such values of r the expression $r^2 + 2/r$ is always less than $C_0 = 4$, as Fig. 4.9 shows. There is no real velocity therefore which could serve as an initial condition for particles in the annulus. In other words, one cannot find particles for which the value of the Jacobian constant is 4 *inside* the annulus.

On the other hand, one may consider a particle with $r_0 = 1$ and with any initial velocity, say $v_0 = 1$, giving $C_0 = 2$. This particle is inside the previously established annulus but that annulus has no bearing on

its motion since the value of the Jacobian constant now is 2. As a matter of fact the cubic

$$r_z^3 - 2r_z + 2 = 0$$

[see Eq. (83) with $C_0 = 2$] has no positive roots (only one negative real root exists which is of no interest since $r_z > 0$). This is shown also in Fig. 4.9 where the $C_0 = 2$ line does not intersect the $C_0(r_z)$ curve since $(C_0)_{\min} = 3$. In this case there are no circles of zero velocity. The Jacobian integral is

$$v^2 = r^2 + 2/r - 2, \tag{86}$$

and since for any $r \geqslant 0$ the expression $r^2 + 2/r > 2$, the particle can attain any position with real velocity.

Only elliptic motion is possible inside the circle with radius r_s when $C_0 = 4$ since with $\mu = 0$ the physical situation corresponds to the two-body problem in a rotating coordinate system. Since the motion is bounded, it is elliptic. In considerable detail, such motions will be treated in Section 8.5.1. At this point the two-body problem is used to illustrate the meaning of the various regions of possible motions and their connection with the curves of zero velocity.

A short statement will summarize the results and will conclude this section. The Jacobian integral $v^2 = 2\Omega - C$ for the special problem,

$$\Omega(r) = r^2/2 + 1/r,$$

under consideration is

$$v^2 = r^2 + 2/r - C. \tag{87}$$

For given initial conditions v_0, r_0 the Jacobian constant becomes

$$C_0 = r_0^2 + 2/r_0 - v_0^2. \tag{88}$$

The radii of the circles of zero velocity belonging to this value of the Jacobian constant may be computed from

$$r_z^2 + 2/r_z - C_0 = 0. \tag{89}$$

The Jacobian integral may be written in the following four forms:

$$v^2 = r^2 + 2/r - (r_z^2 + 2/r_z), \tag{90}$$

$$v^2 - v_0^2 = r^2 + 2/r - (r_0^2 + 2/r_0), \tag{91}$$

$$v^2 = 2\,\Omega(r) - 2\,\Omega(r_z), \tag{92}$$

or finally

$$v^2 - v_0^2 = 2\,\Omega(r) - 2\,\Omega(r_0). \tag{93}$$

Motion is possible in regions where

$$\Omega(r) > \Omega(r_z).$$

On a circle of zero velocity ($r = r_z$) Eq. (92) gives $v = 0$. Motion is possible inside ($r < r_z$) or outside ($r > r_z$) of the circle of radius r_z, depending on the sign of $d\Omega/dr$ at $r = r_z$. If this derivative is positive, then $\Omega(r)$ increases as r increases (from $r = r_z = r_i$ to $r = r$), and therefore Ω is larger *outside* the curve of zero velocity ($r > r_z$) than *on* the curve $[\Omega(r) > \Omega(r_z)]$. Since this is equivalent to $v^2 > 0$, motion will be possible outside. On the other hand, when $(d\Omega/dr)_{r=r_z} < 0$, then for $r < r_z$ we have $\Omega(r) > \Omega(r_z)$ and motion is possible inside. These two cases are shown on Fig. 4.9 for r_s and r_i.

If there is no curve of zero velocity, motion is possible everywhere. This occurs when the initial conditions result in a value of C_0 such that $2(\Omega)_{\min} > C_0$.

From Eqs. (89) and (90) we have

$$v^2 = 2\,\Omega(r) - C_0\,; \tag{94}$$

therefore, when the above inequality is satisfied, $v^2 > 0$, and motion is possible independently of the value of r (everywhere).

When $d\Omega/dr = 0$ on the curve of zero velocity, a special and often simple investigation clears up the situation. For instance, in the above case $d\Omega/dr = 0$ implies $r = 1$. If $r = 1 = r_z$ is the curve of zero velocity, then $C_0 = 3$ which corresponds to a minimum of 2Ω and the previously mentioned annulus shrinks to the circumference of the unit circle. Motion is possible everywhere, but particles which reach this curve of zero velocity will have to be specially investigated for the stability of their subsequent motion.

The next section will apply these concepts to the restricted problem.

4.7.2 Regions of possible motions in the restricted problem

The behavior of the function

$$\Omega = \tfrac{1}{2}[(1-\mu)r_1^2 + \mu r_2^2] + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \tag{95}$$

in the range $0 \leqslant \mu \leqslant \frac{1}{2}$ is the key to finding the regions of motion. The curves of zero velocity which are defined by $2\Omega - C = 0$ (or the level curves of the function Ω, or the equipotential curves) may be shown on the x, y plane for various values of μ and these sets of curves (each set corresponding to one fixed value of μ) may be used to represent the function $\Omega(x, y)$. Another way to obtain a representation is to consider a set of three-dimensional plots of the function $z = \Omega(x, y)$ in an x, y, z coordinate system. (The section of this surface with the plane $y = 0$ for various values of μ is shown on Fig. 4.7, for instance.) Such a three-dimensional representation will show two peaks at m_1 and m_2, where $\Omega(x, y) \to \infty$. The two lowest areas, symmetrically located with respect to the x axis at L_4 and L_5, are separated by the two infinitely high peaks and by three passage ways at L_1, L_2, and L_3. Outside the L_1, L_3, L_4, and L_5 regions, as r_1 and r_2 approach infinity, $\Omega \to \infty$, once again. From this presentation, the equipotential curves may be visualized. For instance, when Ω is sufficiently large, a plane $z =$ const will intersect only the two peaks corresponding to m_1 and m_2 and the outside ridge corresponding to r_1, $r_2 \to \infty$. Furthermore when $\Omega < \frac{3}{2}$, $C < 3$, and curves of zero velocity do not exist—or, in other words, the plane $z =$ const is below the minima at L_4 and L_5; no equipotential lines exist.

A systematic discussion of the lines of zero velocity follows using the Jacobian constant as the parameter, rather than $\Omega = C/2$.

(i) *Outer and inner ovals* ($C_2 < C$). As $r_1, r_2 \to \infty$ or $r_1 \to 0$ or $r_2 \to 0$, we have $\Omega \to \infty$; therefore, the lines of zero velocity form three branches for large values of C. These are treated in the following three subsections: (i.1), (i.2), and (i.3).

(i.1) *Ovals of zero velocity around the primary with the larger mass.* Consider first $r_1 \to 0$. Since at the same time $r_2 \to 1$, in the equation $C = 2\Omega$, the term $2(1-\mu)/r_1$ dominates giving

$$C \cong 2(1-\mu)/r_1\,.$$

The curves of zero velocity therefore are ovals around m_1. The larger C becomes the smaller are the effects of the neglected terms and the approximation

$$r_z = r_1 = 2(1-\mu)/C \tag{96}$$

becomes better; the shape of the curves of zero velocity approaches a circle. Also, as C increases r_1 decreases; therefore, the larger the value of C, the smaller the ovals become. Since

$$\frac{d\Omega}{dr_1} \cong -\frac{1-\mu}{r_1^2} < 0,$$

motion is possible inside the oval of zero velocity. This follows from the summary at the end of the previous section, Section (4.7.1). Another reason can be presented as follows:

Let v_0 and $r_1 = r_0$ be the initial conditions such that the corresponding value of the Jacobian constant

$$C_0 \cong 2(1 - \mu)/r_0 - v_0^2 \tag{97}$$

is approximately correct. The curve of zero velocity for this value of $C = C_0$ is approximately a circle with radius $2(1 - \mu)/C_0 = r_z$. Note that r_0 and r_z are measured from m_1, that is from the location of the larger primary with mass $1 - \mu$, and that $r_z > r_0$ since from Eq. (97)

$$v_0^2 = 2(1 - \mu)(1/r_0 - 1/r_z) > 0.$$

Select now any other point located at a distance r_0' from the larger primary. The velocity v_0' of the particle (with Jacobian constant C_0) is

$$v_0'^2 = 2(1 - \mu)/r_0' - C_0$$

or

$$v_0'^2 = 2(1 - \mu)(1/r_0' - 1/r_z). \tag{98}$$

From this it follows that as long as the particle is inside the circle of zero velocity, $r_0' < r_2$, v_0' is real and motion is possible. When $r_0' = r_z$, $v_0' = 0$, and when $r_0' > r_z$, v_0' is imaginary. The particle therefore cannot cross to the other side of the curve of zero velocity.

(i.2) *Ovals of zero velocity around the primary with the smaller mass.* When $r_2 \ll 1$ and $r_1 \cong 1$, the curves of zero velocity are ovals around the primary with the smaller mass. The radii of the approximately circular zero velocity curves are

$$r_z = r_2 \cong 2\mu/C. \tag{99}$$

A comparison with Eq. (96) shows that the radius of the approximate curve of zero velocity for a given (large) C surrounding the primary with the larger mass (r_1) is larger than the one which encircles the other primary (r_2); in fact,

$$0 < r_1 - r_2 \cong 2/C < \tfrac{2}{3}$$

since curves of zero velocity do not exist for $C < 3$. The approximate ratio of the radii is $(1-\mu)/\mu$. Therefore for small μ we have $r_2 \cong \mu r_1$.

(i.3) *Ovals of zero velocity enclosing both primaries.* When both radial distances (r_1 and r_2) increase far beyond the equilibrium points ($r_1, r_2 \gg 1$), again large values of C can be obtained. In fact, by writing $r_1 = r_2 = r$, we have from Eq. (95)

$$C \cong r^2.$$

This equation expresses the physical situation when—because of the large distance—the gravity effects are neglected in comparison to the centrifugal force (the potential of which is r^2). The Jacobian integral is

$$v^2 \cong r^2 - C$$

and the curves of zero velocity are given by

$$C^{1/2} \cong r_z;$$

i.e., they are approximately circles.

Note that this radius is always larger than the previously obtained value, $2(1 - \mu)/C$, since comparison is only meaningful when $C > 3$.

The ovals of zero velocity expand as C increases, therefore motion is possible *outside* these limiting curves.

The following general remarks regarding the curves described under Case (i) may be made. Figure 4.10 shows the circular approximation for $\mu = 0.2$ and $C = 4$. The radii are $2\mu/C = 0.1$, $2(1 - \mu)/C = 0.4$, and $C^{1/2} = 2$.

It is of considerable interest to estimate the distortion of these circles as the effect of the perturbation introduced by the other primary. In what follows, detailed attention will be given to the distortion of the oval of zero velocity surrounding the primary with the larger mass. The perturbative effects on the other ovals of zero velocity can be analyzed by the same method.

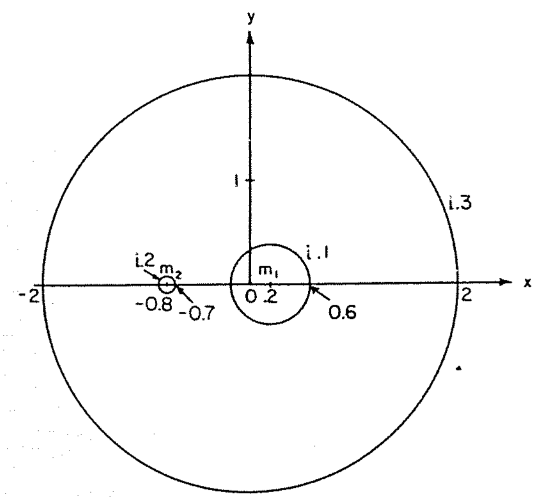

FIG. 4.10. Circular approximation for the curves of zero velocity for $C = 4$, $\mu = 0.2$.

For convenience' sake a polar coordinate system centered at the primary with the larger mass is introduced as shown in Fig. 4.11.

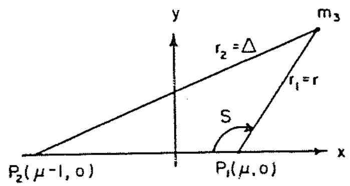

FIG. 4.11. Polar coordinates r and S.

In this system, using Eq. (95), the equation of the curves of zero velocity becomes

$$C = r^2 + 2/r + 2\mu(\tfrac{1}{2} - r\cos S - 1/r + 1/\Delta), \tag{100}$$

where

$$\Delta^{-1} = (r^2 + 1 - 2r\cos S)^{-1/2} = \sum_{n=0}^{\infty} r^n P_n(\cos S). \tag{101}$$

When curves of zero velocity are studied around the primary with the larger mass (when $r < 1$) the above expansion is convergent. The Legendre polynomials are defined by

$$P_n(x) = \frac{(2n-1)!!}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4} - \cdots\right]. \tag{102}$$

When $S = 0$, $x = 1$ and $P_n(1) = 1$, and when $S = \pi$, $x = -1$ and $P_n(-1) = (-1)^n$. Furthermore, at $S = \pi/2$, $x = 0$, $P_{2n+1}(0) = 0$, and

$$P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!}.$$

In these equations

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1,$$
$$(2n)!! = 2n(2n-2)(2n-4)\cdots 4\cdot 2,$$

and

$$0! = 0!! = (-1)!!$$

Maintaining quadratic but not higher-order terms in the expansion, we have from Eq. (100)

$$C = r^2 + 2/r + \mu(3 - 2/r + 3r^2\cos^2 S - r^2), \tag{103}$$

which, for small values of r ($r \ll 1$), adequately represents the curve of zero velocity.

In order to show some general properties of the ovals of zero velocity around the primary with the larger mass, the function $r(S)$ will be evaluated at four values, $S = 0$, $\pi/2$, π, and $3\pi/2$. Since the curves of zero velocity are symmetric with respect to the x axis, i.e., with respect to the line $S = 0$, we have $r(\pi/2) = r(3\pi/2)$. For given values of C and μ Eq. (100), which is generally valid, furnishes the value of r belonging to any given angle S, provided C is selected such that ovals of zero velocity exist around the primaries. This approach requires the solution of eigth-order algebraic equations for r. Of course Eq. (103) can also be used for small values of r, giving a cubic relation.

An alternate approach is as follows. Knowing that the curves of zero velocity are circles for $\mu = 0$, it might be expected that, when $\mu \neq 0$ but $\mu \ll \frac{1}{2}$, these circles would be distorted. If this is actually so, then we can write $r = r_0 + \rho(S)$ and substitute this expression in Eq. (100), neglecting second and higher powers of the function $\rho(S)$. In this way we obtain

$$\rho(S) = \frac{C - \left(r_0^2 + \frac{2}{r_0}\right) + 2\mu\left(r_0\cos S + \frac{1}{r_0} - \frac{1}{2} - \frac{1}{\Delta_0}\right)}{-2\left(\frac{1}{r_0^2} - r_0\right) + 2\mu\left(\frac{1}{r_0^2} - \cos S - \frac{r_0 - \cos S}{\Delta_0^3}\right)}, \tag{104}$$

where

$$\Delta_0 = (1 + r_0^2 - 2r_0\cos S)^{1/2}.$$

From this equation $\rho(S) < 0$. This can be seen by using a C such that

$$C = r_0^2 + 2/r_0$$

and considering that since $r_0 \ll 1$ we have

$$1/r_0^2 - r_0 > 0.$$

Furthermore, the two expressions in Eq. (104) multiplied by 2μ are both positive and the one in the denominator can be neglected.

Using Eq. (104), one obtains

$$\rho(\pi/2) < \rho(\pi) < \rho(0) < 0$$

and consequently

$$r(\pi/2) < r(\pi) < r(0) < r_0, \tag{105}$$

as shown on Fig. 4.12.

Similar analysis can be performed for the outer curve which was mentioned under (i.3).

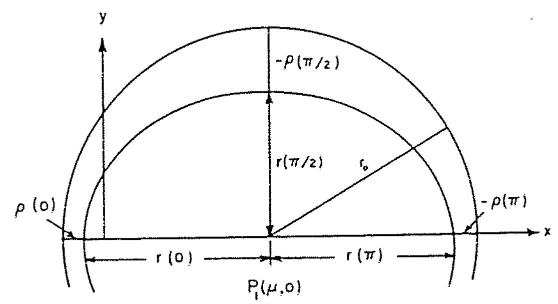

FIG. 4.12. Perturbation of the oval of zero velocity around the larger primary.

If the value of the Jacobian constant decreases, the size of the zero-velocity ovals surrounding m_1 and m_2 increases while the outer oval shrinks. Figures 4.13(a), (b), and (c) show the process for $\mu = 0.3$, 0.1, and 0.01. The values of the Jacobian constants are shown in the figures and also in the legends. The smallest values of C on the three figures are selected so that they are slightly higher than the corresponding values of C_2. Note that $C_2(0.3) = 4.130149$, $C_2(0.1) = 3.686953$, and $C_2(0.01) = 3.177541$. In Fig. 4.13(c) only one oval of zero velocity is

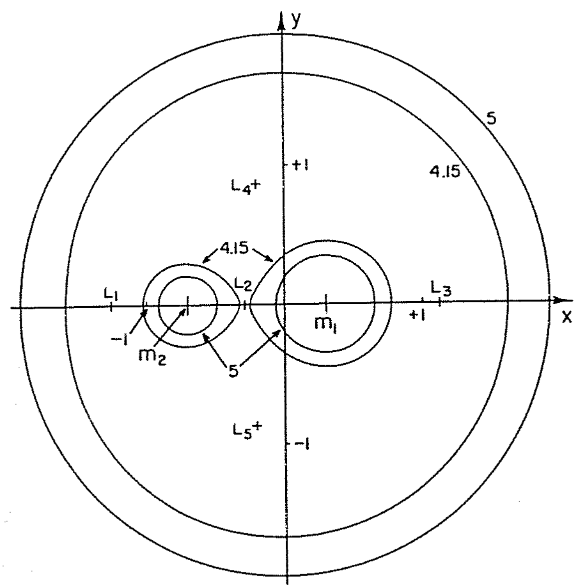

FIG. 4.13(a). Curves of zero velocity for $\mu = 0.3$ with $C = 5$ and 4.15.

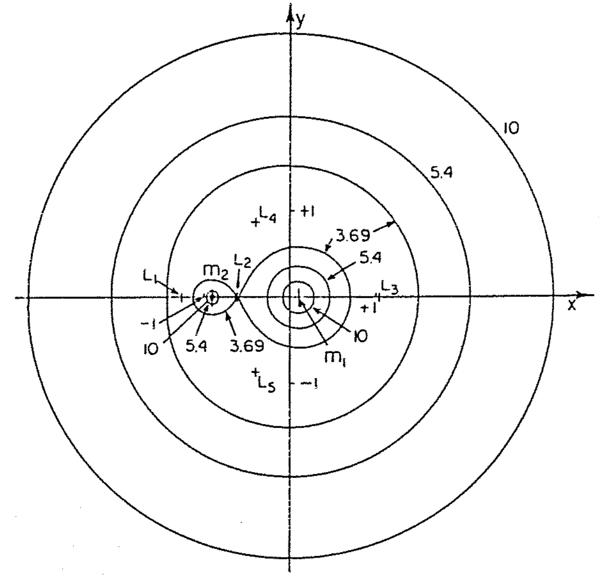

FIG. 4.13(b). Curves of zero velocity for $\mu = 0.1$ with $C = 10$, 5.4, and 3.69.

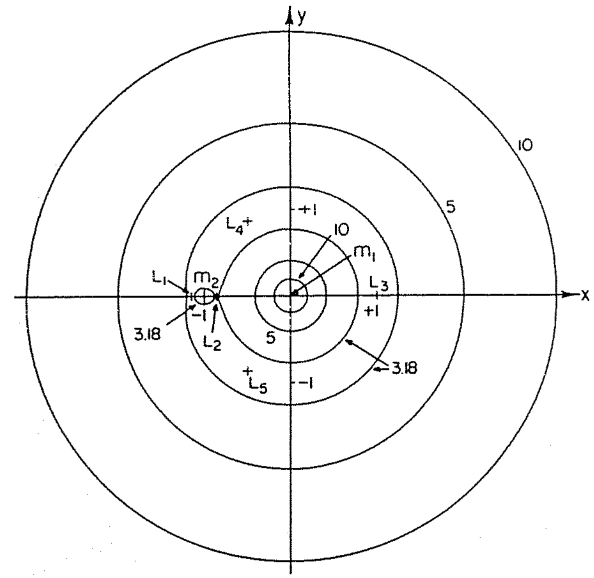

FIG. 4.13(c). Curves of zero velocity for $\mu = 0.01$ with $C = 10$, 5, and 3.18.

shown around m_2, that which corresponds to the smallest value of C, since curves with the larger values of C shrink to m_2 on the scale of the drawing.

Recalling that, for any positive μ not greater than $\frac{1}{2}$, we have

$$\Omega(L_2) > \Omega(L_1) > \Omega(L_3),$$

and therefore, as the value of the Jacobian constant is reduced from $C > 2\Omega(L_2)$, the first encountered critical value of C is $2\Omega(L_2) = C_2$. The curve of zero velocity associated with this value of the Jacobian constant has two branches. The outer oval shrinks and at C_2 maintains its previous character; it encloses both primaries as well as all the equilibrium points. (Were it not to include any one of the libration points it would have had to go through—in the shrinking process—a point whose associated Jacobian constant is smaller than C_2.) The inside ovals enlarge, meet at the point L_2, and form a figure eight. In this process the ovals never enclose L_1 and L_3, since the corresponding values of the Jacobian constant, C_1 and C_3, are smaller than C_2. Figure 4.14(a) shows the process with $C = 4.5$ and $C_2 = 4.130149$, using $\mu = 0.3$. The outside ovals are closer to L_1 than to L_3, and the larger ovals around the larger mass (m_1) become the larger half of the resulting figure-eight shape at C_2.

The computation of the intersections of the ovals and of the figure-eight curve with the x axis facilitates the construction of the curves of zero velocity.

The intersections on the right side of m_1 with the x axis, shown on

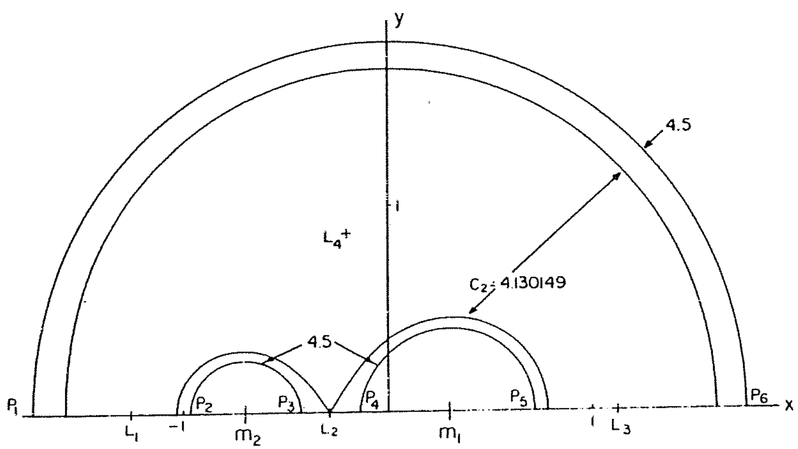

FIG. 4.14(a). Curves of zero velocity for $\mu = 0.3$ with $C = 4.5$ and $C_2 = 4.130149$.

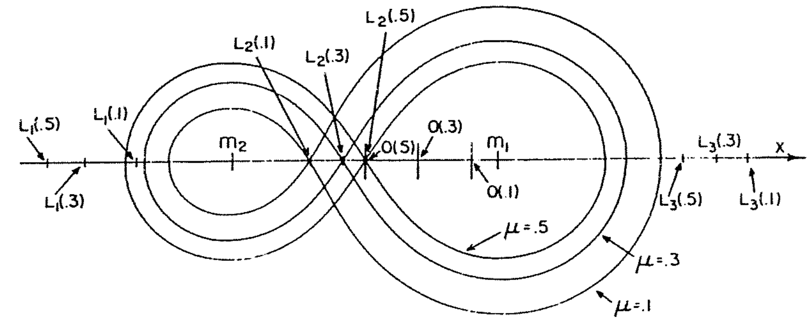

FIG. 4.14(b). Curves of zero velocity for $\mu = 0.5$, 0.3, and 0.1 with $C = C_2$.

Fig. 4.14(a), can be obtained as follows. The Jacobian constant is given by Eq. (95) as

$$C = r_1^2 + \mu(r_2^2 - r_1^2) + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} \tag{106}$$

and, since $r_2 = r_1 + 1$, we have

$$C = r_1^2 + \mu(2r_1 + 1) + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{1 + r_1}. \tag{107}$$

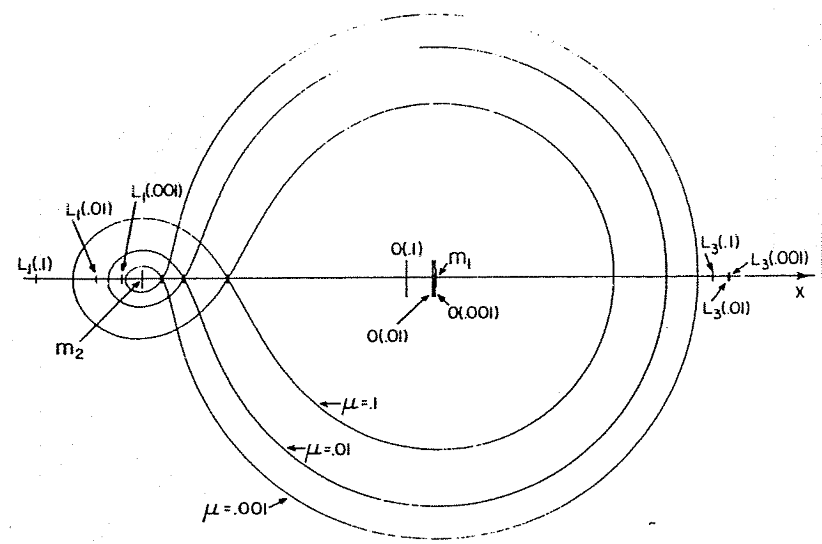

FIG. 4.14(c). Curves of zero velocity for $\mu = 0.1$, 0.01, and 0.001 with $C = C_2$.

This fourth-order equation for r_1 has two positive roots for large values of C, one smaller and one larger than $x_3 - \mu$, since L_3 separates the inner and outer curvers of zero velocity.

The intersection of the figure-eight curve with the x axis is obtained when $C = C_2$ is substituted in Eq. (107).

Intersections on the left of m_2 are obtained also from Eq. (106) but now $r_1 = 1 + r_2$ and consequently

$$C = (1 + r_2)^2 - \mu(1 + 2r_2) + \frac{2(1-\mu)}{1 + r_2} + \frac{\mu}{r_2}. \tag{107a}$$

This fourth-order equation for r_2 has two changes of sign and therefore two positive roots for large values of C. One root satisfies the inequality $\overline{L_1 m_2} < r_2$, the other, $r_2 < \overline{L_1 m_2}$. At $C = C_1$, these two roots become equal, as we shall see later.

Figures 4.14(b) and 4.14(c) show the development of the figure-eight curves of zero velocity as the value of the mass parameter decreases from $\mu = 0.5$ to $\mu = 0.001$. The locations of the collinear libration points are also shown with μ as the argument. On the figures the locations of the primaries are fixed and the origin of the coordinate system ($\mathcal{O}$) is moved to the center of mass at every value of μ. The values of the Jacobian constants are $C_2(0.5) = 4.25$, $C_2(0.3) = 4.1301495$, $C_2(0.1) = 3.6869531$, $C_2(0.01) = 3.177541$, and $C_2(0.001) = 3.040948$. The outer ovals are not shown.

Motion is still possible either inside the figure-eight curve or outside the outer oval.

Further reduction of the value of C changes the figure-eight curve of zero velocity into a pear-shaped figure since the curve will not cross the x axis at L_2 anymore; in fact, there is no crossing between m_1 and m_2. The figure-eight shape therefore separates two different types of curves of zero velocity: it ends Case (i) with one outer and two inner ovals and is the starting point for (ii), with inner pear shapes and outer ovals.

(ii) *Outer ovals and inner pear shapes* ($C_1 < C < C_2$). The first critical value of $C = C_2$ is reached as C is decreased from $+\infty$. With further decrease the value of $C = C_1 < C_2$ is obtained. This process starts with the figure eight and ends with contact at L_1 between the inner and outer areas. Figure 4.15 shows the curves of zero velocity for C_1 and C_2.

When $C_1 < C < C_2$ the curves of zero velocity constitute two branches. The first branch is the dumbbell or pear-shaped figure inside of which motion is possible. This curve encloses m_1, m_2, and L_2 but L_1 and L_3

are outside. When $C = C_2 - \varepsilon$, where $\varepsilon > 0$ and arbitrarily small, the figure-eight shape opens up at L_2 as shown on Fig. 4.16 and forms the dumbbells for $C_1 < C < C_2$. On Fig. 4.16 the values $\epsilon = C_2 - 4 = 0.1301495$ and $\mu = 0.3$ are used.

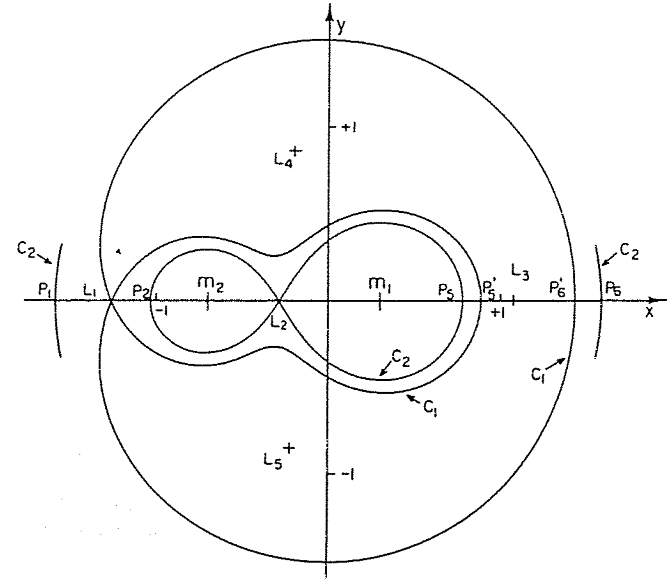

FIG. 4.15. Curves of zero velocity for $\mu = 0.3$ with $C = C_2$ and $C = C_1$.

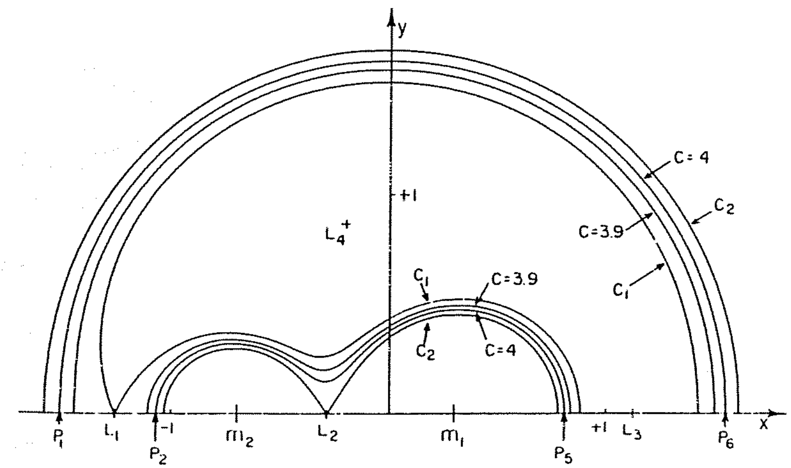

FIG. 4.16. Curves of zero velocity for $\mu = 0.3$ with $C = C_2$, 4, 3.9, and C_1.

Similarly when $C = C_1 + \varepsilon'$ the outer curve of zero velocity does not pass through L_1 but includes all masses and libration points. On Fig. 4.16 we have $\varepsilon' = 3.9 - C_1 = 0.1335870$. When C decreases from C_2 to C_1 the outer ovals retain their general shape and only show an indentation near L_1. Motion is always possible outside these ovals.

(iii) *Horseshoe-shaped curves* ($C_3 < C < C_1$). As the value of C is reduced further from $C = C_1$, the point L_1 is not on the curve of zero velocity any more and there is only one branch of the curve. The cusp at L_1, which exists when $C = C_1$, disappears as the curve does not intersect the x axis any more between $-\infty$ and m_1. Figure 4.17 shows a horseshoe-shaped curve which encloses only L_3, L_4, and L_5. The intersections with the x axis near L_3 move closer to L_3 as $C \to C_3$ and, when $C = C_3$, points A, A'—and points B, B'—join at L_3 forming a cusp. Figure 4.18 shows the two limiting situations, with the curves of zero velocity for C_1 and C_3. The series of horseshoe-shaped curves begin at C_1 when a cusp is formed at L_1 and end with C_3 when the cusp is at L_3. Motion is possible everywhere outside the area enclosed by the horseshoes.

(iv) *Tadpole-shaped curves* ($3 = C_4 = C_5 < C < C_3$). With further reduction of the value of the Jacobian constant from C_3, the cusp disappears as the curve leaves L_3. The series of curves of zero velocity form two branches, one enclosing L_4 and the other L_5. These curves

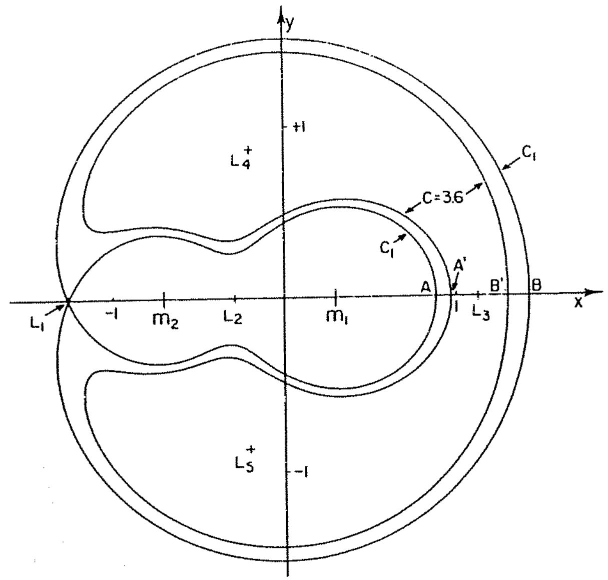

FIG. 4.17. Curves of zero velocity for $\mu = 0.3$ with $C = C_1$ and 3.6.

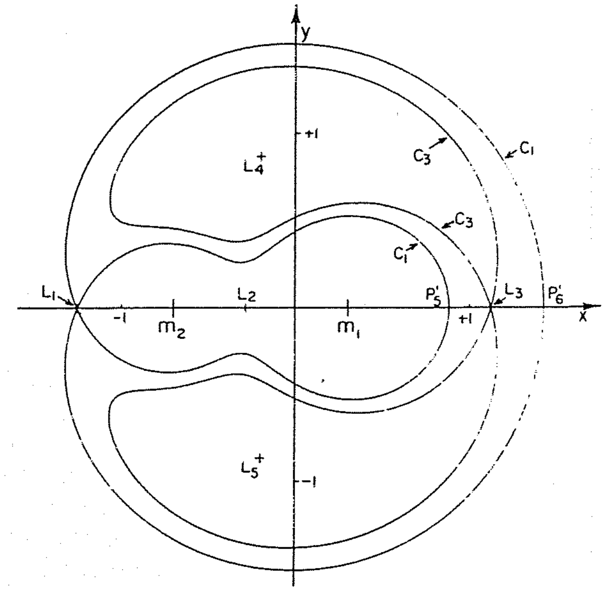

FIG. 4.18. Curves of zero velocity for $\mu = 0.3$ with $C = C_1$ and C_3.

shrink to the points L_4 and L_5 as C approaches 3. The approximate shape for $C = 3 + \varepsilon$ (where ε is an arbitrary small positive quantity) is elliptic with the center of the ellipse located at L_4 (L_5). To show this, we introduce the variables (ξ, η) in place of (x, y) by the equations

$$\xi = x + \tfrac{1}{2} - \mu, \qquad \eta = y - \tfrac{1}{2}\, 3^{1/2}. \tag{108}$$

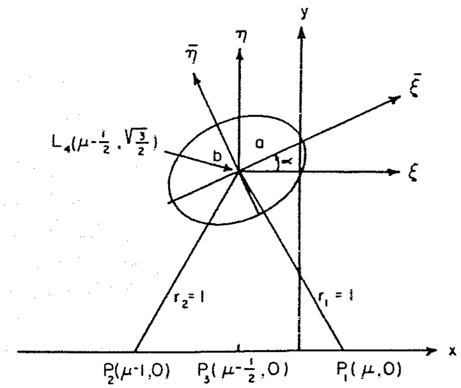

FIG. 4.19. Elliptic approximation to the curves of zero velocity around L_4.

These equations represent a translation of the x, y coordinate system to the equilateral libration point L_4, as shown on Fig. 4.19.

Applying the transformation (108) to Eq. (95) and neglecting terms containing the third or higher powers of ξ and η, one obtains

$$C = \frac{3}{4}\xi^2 + \frac{9}{4}\eta^2 - \frac{3 \cdot 3^{1/2}}{2}(1 - 2\mu)\xi\eta + 3. \tag{109}$$

The determinant

$$\begin{vmatrix} \frac{3}{4} & -\frac{3 \cdot 3^{1/2}}{4}(1 - 2\mu) \\ -\frac{3 \cdot 3^{1/2}}{4}(1 - 2\mu) & \frac{9}{4} \end{vmatrix} = \frac{27}{4}\mu(1 - \mu) > 0 \tag{110}$$

for $0 < \mu \leqslant \frac{1}{2}$; therefore, the curve is an ellipse.

The lengths and directions of the axes of the ellipse follow from the characteristic roots:

$$\lambda_{1,2} = \tfrac{3}{2}\{1 \pm [1 - 3\mu(1 - \mu)]^{1/2}\}. \tag{111}$$

Equation (111) shows that both λ are real and positive and that, for small values of μ (i.e., when μ^2 is neglected), we have

$$\lambda_1 \cong 3 - 9\mu/4 \quad \text{and} \quad \lambda_2 \cong 9\mu/4. \tag{112}$$

The direction of the major axis (α) is given by

$$\tan 2\alpha = 3^{1/2}(1 - 2\mu). \tag{113}$$

The lengths of the semiminor and major axes are given by

$$b = \frac{(C - 3)^{1/2}}{\lambda_1}, \qquad a = \frac{(C - 3)^{1/2}}{\lambda_2}. \tag{114}$$

Figure 4.19 shows the general arrangement. Note that the orientation and the eccentricity of the ellipse depend only on the value of μ, while the size of the ellipse depends also on the value of the Jacobian constant.

Figure 4.20 shows the family belonging to (iv). The curve marked C_3 represents the beginning and the points L_4 and L_5 the end of this series of curves of zero velocity. For $\mu = 0.3$, the value of the Jacobian constant at point L_3 is $C_3 = 3.5013501$. The elliptic approximation to the curve of zero velocity associated with $C = 3.05$ and shown on Fig. 4.20 has

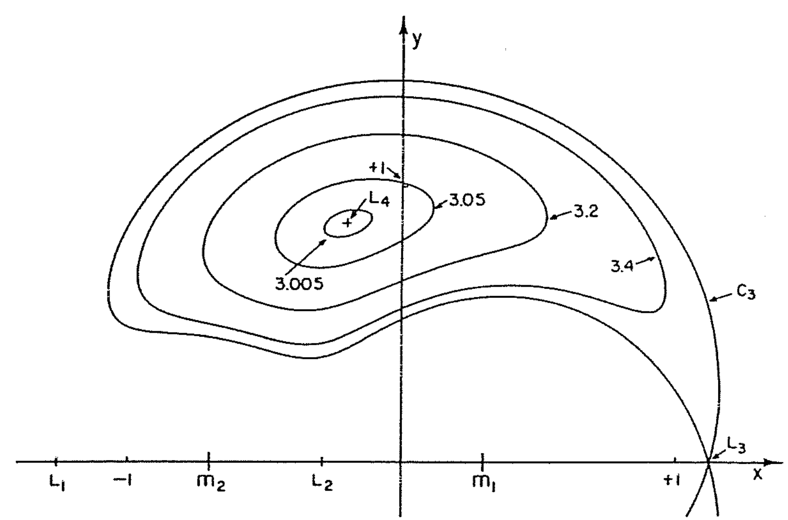

FIG. 4.20. Curves of zero velocity for $\mu = 0.3$ with $C = C_3$, 3.4, 3.2, 3.05, 3.005, and C_4.

the following characteristics: $a = 0.293$, $b = 0.144$, eccentricity $e \cong 0.87$, and $\alpha = 17^\circ.4$. Motion is possible everywhere outside the tadpole-like areas.

This concludes the detailed discussion of the various classes [(i)–(iv)] of curves of zero velocity. The following remarks are more general.

The construction of the curves of zero velocity is facilitated by the knowledge of their intersections with the x axis. The three double points of intersection, where the slopes are finite, are known since the location of these collinear points can be found in Appendixes I, II, and III. All other intersections with the x axis occur perpendicularly. The location of these intersections at a given value of μ for various values of C can be found by evaluating the function $2\Omega(x, 0) = C$ at various values of x. The inverse function, $x = x(C)$, can be obtained (since Ω_x is zero only at the libration points) either numerically or graphically. The prototype of Fig. 4.7 is drawn in Fig. 4.21 showing the function $C(x)$ [instead of the function $\Omega(x, 0)$] for the purpose of rediscussion. The figure corresponds to $\mu = 0.4$.

It can be seen that when $C > C_2$ there are six intersections between the x axis and the curves of zero velocity since there are six intersections between the curves $C(x)$ and a horizontal line drawn with a value of C higher than C_2. These six intersections, $Q_1, \ldots, Q_6$, correspond to points $P_1, P_2, \ldots, P_6$ shown in Fig. 4.14(a). When $C = C_2$, points Q_3 and Q_4 will coincide and we have five points, Q_1', Q_2', Q_{34}', Q_5', and Q_6', corresponding to P_1, P_2, L_2, P_5, and P_6 shown in Fig. 4.15.

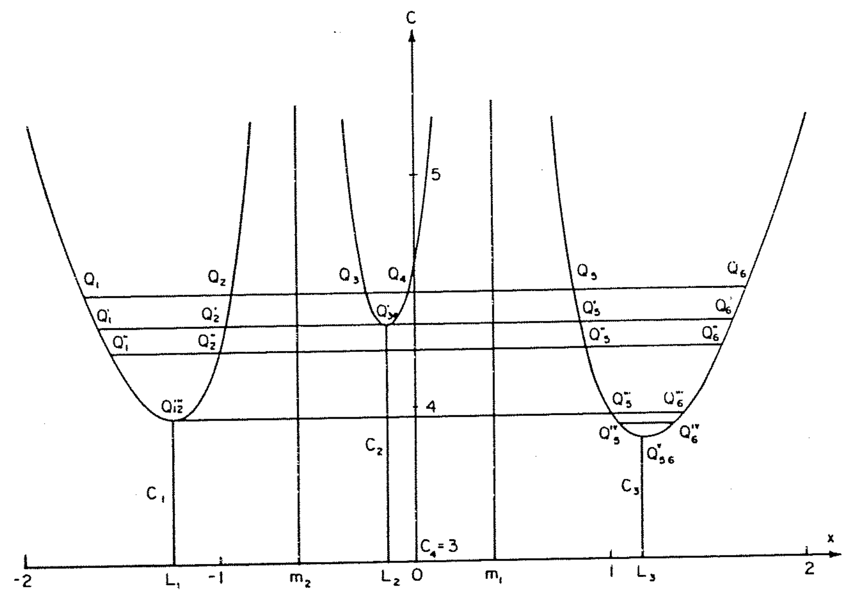

FIG. 4.21. The function $C = 2\Omega(x, 0)$.

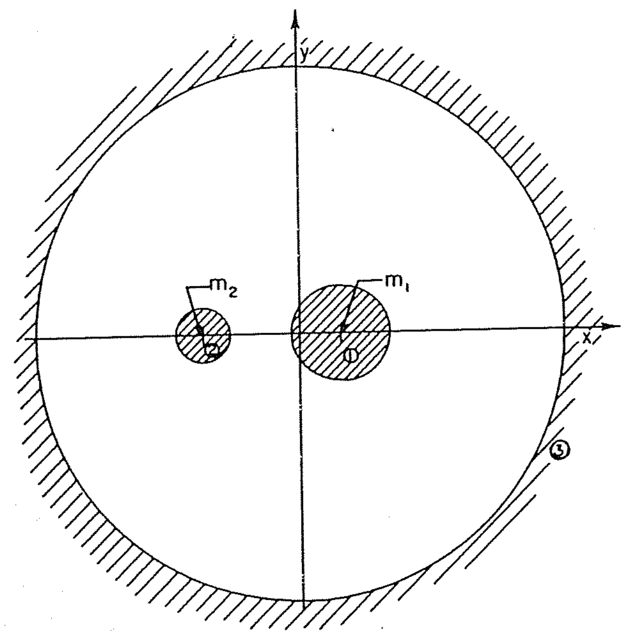

FIG. 4.22. General shape of permissible areas for $C > C_2$. Curves correspond to $\mu = 0.3$, $C = 5$.

Lowering C to a value between C_1 and C_2, four intersections may be obtained (see Fig. 4.21): Q_1'', Q_2'', Q_5'', and Q_6'', corresponding to the points P_1, P_2, P_5, and P_6 of Fig. 4.16. When $C = C_1$, the points Q_1'' and Q_2'' coincide and the three intersections become Q_{12}''', Q_5''', and Q_6''', corresponding to points L_1, P_5', and P_6' of Fig. 4.15 and of Fig. 4.18.

When $C_3 < C < C_1$, the two intersections of Fig. 4.21, the points Q_5^{IV} and Q_6^{IV}, correspond to points A' and B' of Fig. 4.17. These points coincide when $C = C_3$ forming Q_{56}^{V} on Fig. 4.21 and the only intersection on Fig. 4.20 marked by L_3.

Smaller values of C (when $C < C_3$) do not give intersections, as may be seen from both Figs. 4.20 and 4.21.

In summary, then, we can distinguish four ranges of the value of the Jacobian constant, with four distinct types of curves of zero velocity, corresponding to four different types of areas of possible motion.

(i) For large values of the Jacobian constant, $C > C_2$, the permissible areas are shown schematically on Fig. 4.22 by shading. There is no communication between the areas marked by 1, 2, and 3.

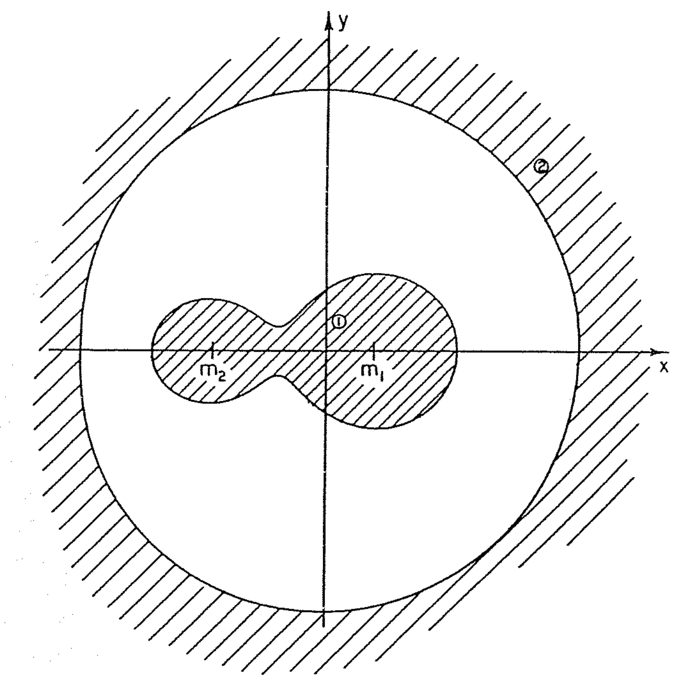

FIG. 4.23. General shape of permissible areas for $C_1 < C < C_2$. Curves correspond to $\mu = 0.3$, $C = 4$.

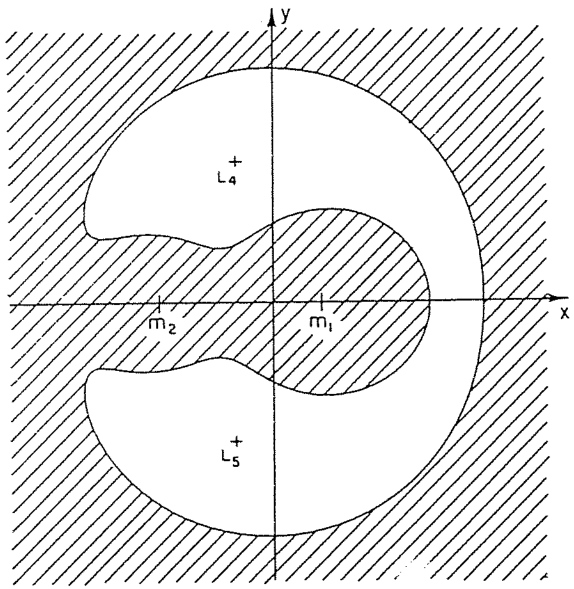

FIG. 4.24. General shape of permissible area for $C_3 < C < C_1$. Curve corresponds to $\mu = 0.3$, $C = 3.6$.

(ii) When the value of the Jacobian constant is between C_1 and C_2 the picture changes into Fig. 4.23. Communication or particle exchange between the neighborhoods of the two primaries is now possible but the third body still remains either outside the large oval (2) or inside the dumbbell-shaped area (1).

(iii) The next step in the development occurs when the value of the Jacobian constant is between C_3 and C_1. The permissible area is again shaded in Fig. 4.24. Exchange between the inner and outer areas is now possible and it occurs on the side of the primary with the smaller mass.

(iv) The final step is when the value of the Jacobian constant is between C_3 and 3, or $3 = C_4 < C < C_3$. The forbidden areas surrounding L_4 and L_5 are shrinking as $C \to 3$ and motion is possible everywhere when $C \leqslant 3$. Figure 4.25 shows the shaded area in which motion is possible.

4.7.3 The effect of the mass parameter on the curves of zero velocity

Figures 4.26(a) to 4.26(c) show the effect of the mass parameter on complete sets of curves of zero velocity. It can be seen that the topological

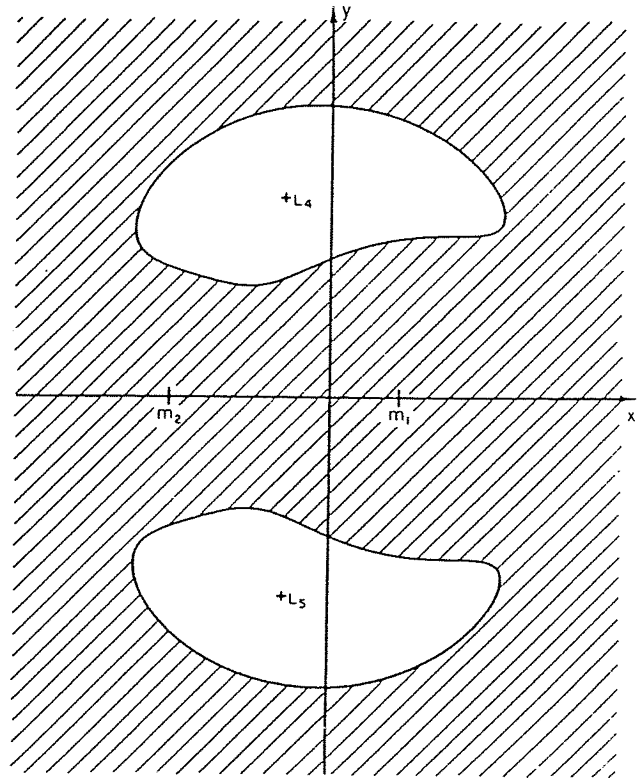

FIG. 4.25. General shape of permissible area for $C_4 < C < C_3$. Curves correspond to $\mu = 0.3$, $C = 3.3$.

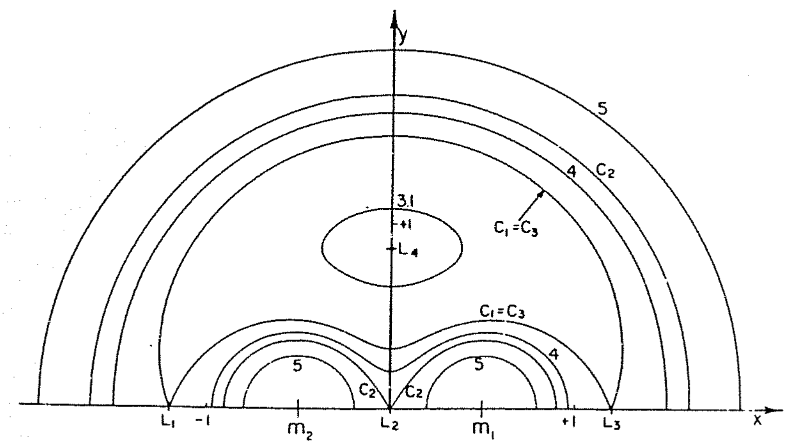

FIG. 4.26(a). Curves of zero velocity for the Copenhagen problem, $\mu = 0.5$.

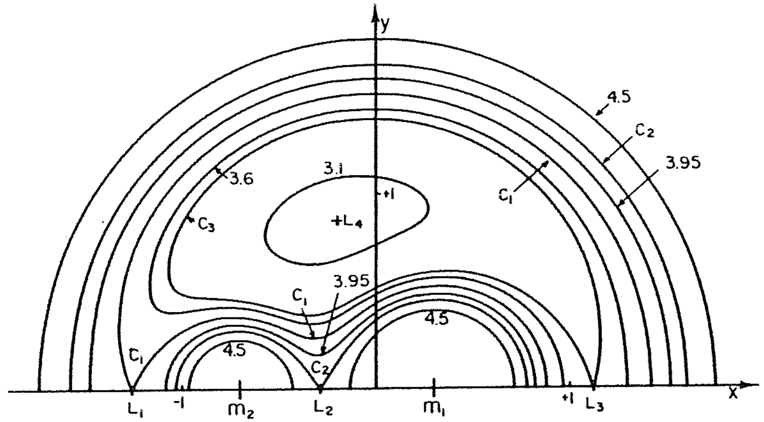

FIG. 4.26(b). Curves of zero velocity for $\mu = 0.3$.

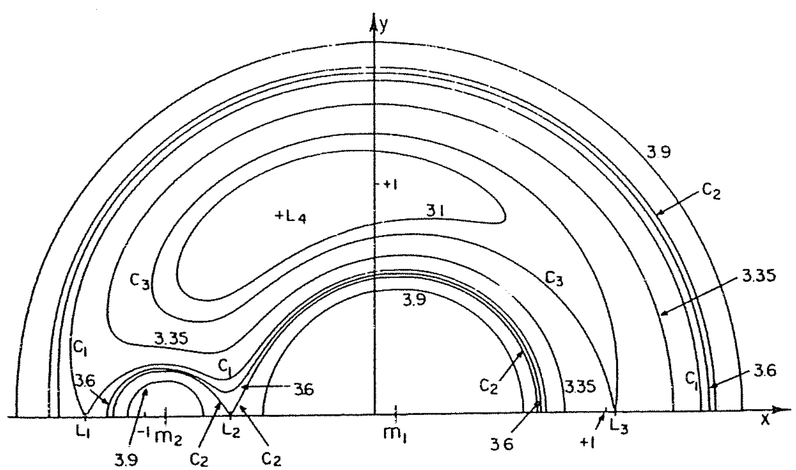

FIG. 4.26(c). Curves of zero velocity for Darwin's problem ($\mu = 1/11$).

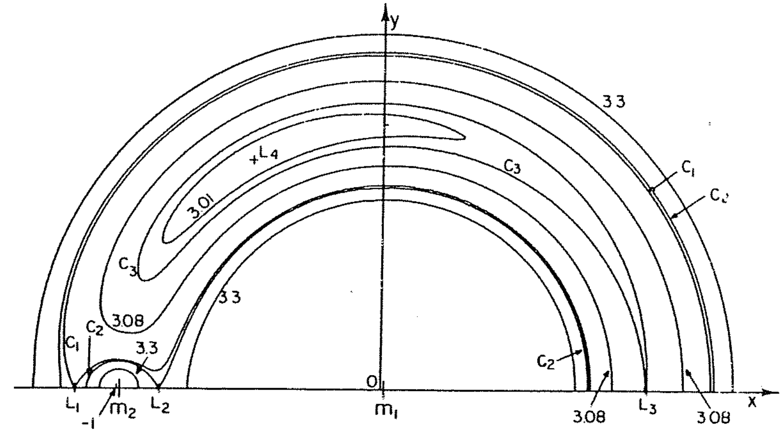

FIG. 4.26(d). Curves of zero velocity for the earth-moon problem ($\mu = 0.012141$)

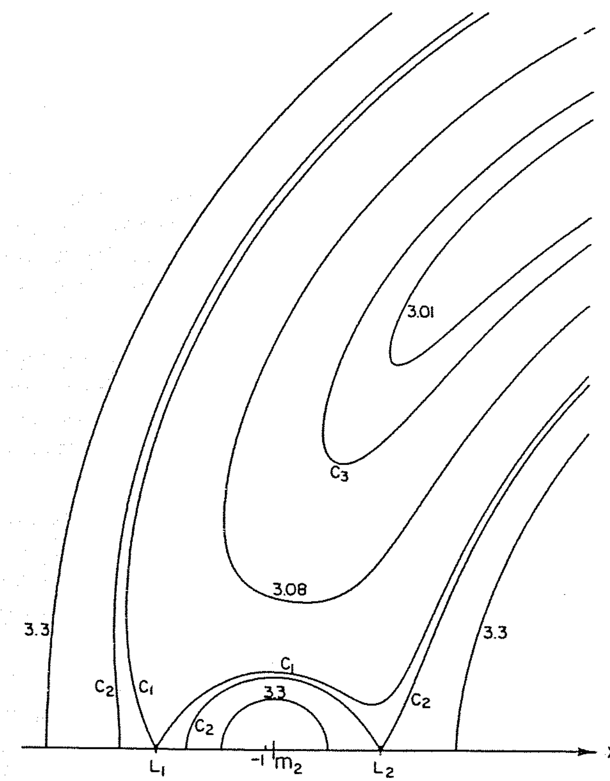

FIG. 4.26(e). Curves of zero velocity for the earth-moon problem ($\mu = 0.012141$) enlarged around the moon.

character of the curves does not change as μ varies from $\sim$0.01 to 0.5. Table I describes these figures in more detail.

TABLE I

Characteristics of the Curves of Zero Velocity Shown in Fig. 4.26

Figure No.	μ	C	C_2	C	C_1	C	C_3	C
4.26(a)	0.5	5	4.25	4	3.7067961	—	3.7067961	3.1
4.26(b)	0.3	4.5	4.1301495	3.95	3.7664130	3.6	3.5013501	3.1
4.26(c)	0.090909	3.9	3.6529163	3.6	3.5341816	3.35	3.1732222	3.1
4.26(d), (e)	0.012141	3.3	3.2002462	—	3.1840783	3.08	3.0241311	3.01

In Fig. 4.27 the *forbidden* regions are shaded. The curves of zero velocity are replaced by circles which are their topological equivalents. The triply connected domain [Fig. 4.27(a)] occurs at large values of the Jacobian constant ($C > C_2$), corresponding to our previously discussed Class (i). Figure 4.27(b) corresponds to Class (ii) and Fig. 4.27(c) to Class (iii). Note that this description is valid for any positive μ less than 0.5.

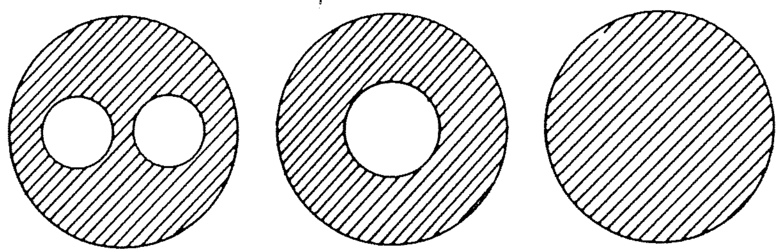

Fig. 4.27. (a) Triply connected forbidden region, $C_2 < C$. (b) Doubly connected forbidden region, $C_1 < C < C_2$. (c) Simply connected forbidden region, $C_3 < C < C_1$.

The participation of the libration points is also of interest in the process of the establishment of the different shapes of the curves of zero velocity. The forbidden region of Class (i) contains all five equilibrium points. The number of enclosed libration points is reduced to four in the case of Class (ii), since L_2 is in a permissible region. Class (iii) contains only three such points, L_3, L_4, and L_5.

The cut which reduces the triple connectivity of Fig. 4.27(a) to the double connectivity of Fig. 4.27(b) is between the circles drawn around m_1 and m_2. The next cut, which gives a simply connected domain, is, of course, between the inside and outside circles.

4.7.4 Curves of zero velocity in the regularized system

The curves of zero velocity may be transformed from the x, y system to the regularized u, v system. The result is a set of curves in the u, v plane, each member associated with a given value of the Jacobian constant, C:

$$2\Omega[x(u, v), y(u, v)] = C.$$

Figures 4.28(a)–(d) show the curves of zero velocity in the u, v system using Levi-Civita's, Thiele's, Birkhoff's, and Lemaître's transformations. The transformations are performed from the x, y plane in the first three cases and from the q_1, q_2 plane, introduced in Section 3.5, in the fourth case. The transformation equations are given in Table II, along with

TABLE II

Description of the Curves of Zero Velocity in the Regularized System Shown in Figs. 4.28

Figure No.	Transformation	Equation	C
4.28(a)	Levi-Civita	$z = \mu + w^2$	4.5, C_2, 3.95, C_1, 3.63, C_3, 3.2
4.28(b)	Thiele	$z = \mu - \cos^2 \frac{w}{2}$	5, C_2, 3.95, C_1, 3.63, C_3, 3.25
4.28(c)	Birkhoff	$z = \frac{w^2 + \mu(1-\mu)}{2w + 1 - 2\mu}$	4.6, C_2, C_1, C_3, 3.25
4.28(d)	Lemaître	$q = -\frac{1}{4}\left(w^2 + \frac{1}{w^2}\right)$	4.6, C_2, C_1, C_3, 3.25

the other characteristics of Figs. 4.28. The value of the mass parameter is $\mu = 0.3$ for all figures.

The following few remarks may be added to this subject.

The Jacobian integral in the system u, v is

$$u'^2 + v'^2 = 2\Omega^*(u, v),$$

where

$$\Omega^*(u, v) = [\Omega(u, v) - C/2] \, |f'(w)|^2$$

as was shown in Chapter 3.

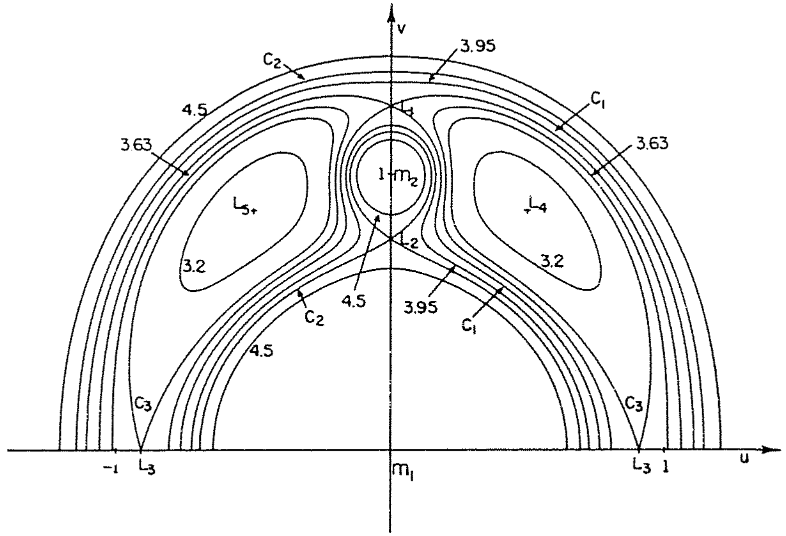

FIG. 4.28(a). Levi-Civita's transformation of curves of zero velocity, $\mu = 0.3$.

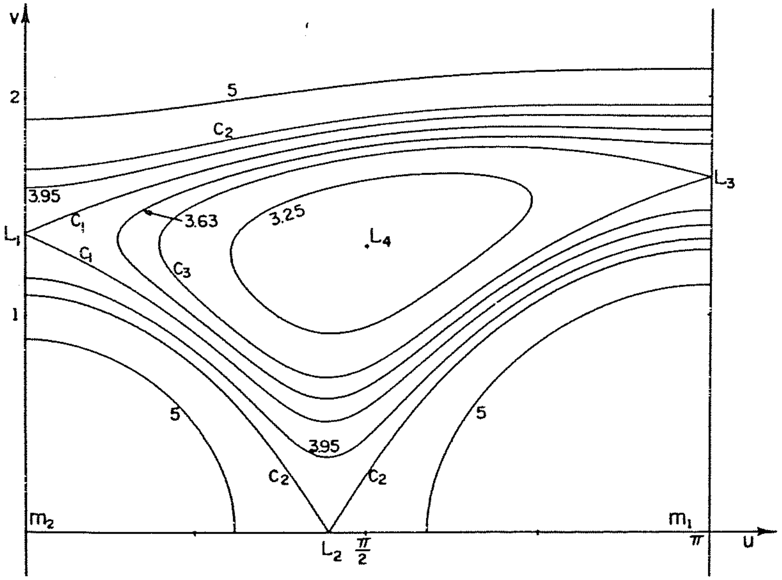

FIG. 4.28(b). Thiele's transformation of curves of zero velocity, $\mu = 0.3$.

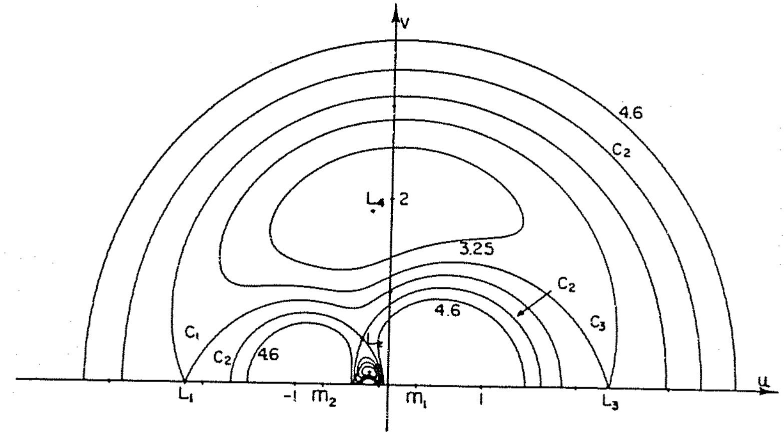

FIG. 4.28(c). Birkhoff's transformation of curves of zero velocity, $\mu = 0.3$.

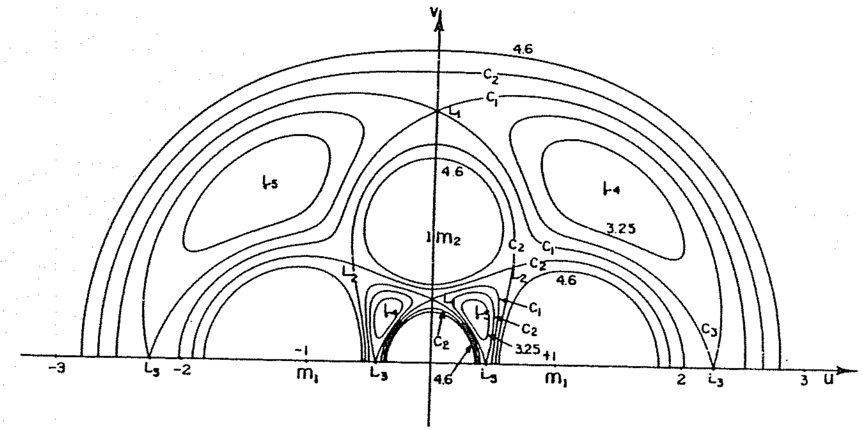

FIG. 4.28(d). Lemaître's transformation of curves of zero velocity, $\mu = 0.3$.

The curves of zero velocity in the u,v system, as defined by the equation

$$\Omega(u, v) = C/2,$$

therefore, are indeed the curves along which

$$\Omega^*(u, v) = 0$$

and also

$$u'^2 + v'^2 = 0.$$

So curves of zero velocity in the x, y system, along which $\dot{x}^2 + \dot{y}^2 = 0$, when transformed by the transformation $z = f(w)$, become curves of zero velocity in the u, v system, along which $u'^2 + v'^2 = 0$.

On the other hand, while the curves

$$2\Omega(x, y) = C$$

are curves of zero velocity in the x, y system, the curves

$$2\Omega^*(u, v) = C^*$$

are *not* curves of zero velocity, unless $C^* = 0$. The equation of these latter curves contains two parameters, C and C^*, since

$$2\Omega^*(u, v) = [2\Omega(u, v) - C] \, | f'(w) |^2 = C^*.$$

Therefore, for a given value of the Jacobian constant C, a set of curves satisfying the equation

$$2\Omega^*(u, v) = C^*$$

may be constructed, with varying values of C^*.

The curves

$$2\Omega(x, y) = C$$

are isotachs (lines of constant speed) in the x, y system but the curves

$$2\Omega(u, v) = C$$

are not isotachs in the u, v system. The isotachs in the u, v system are given by

$$2\Omega^*(u, v) = C^*.$$

We might use at this time the concept of a generalized potential function $\Omega(x, y)$ since the equations of motion may be written as

$$\ddot{x}_i + \sum_{j=1}^{2} a_{ij} \dot{x}_j = \frac{\partial \, \Omega(x_1, x_2)}{\partial x_i},$$

where $i, j = 1, 2$ and $x = x_1$, $y = x_2$, with $a_{ij} = \text{const}$.

The regularizing transformation gives

$$u_i'' + \sum_{j=1}^{2} a_{ij}(u_1, u_2) \, u_j' = \frac{\partial \, \Omega^*(u_1, u_2)}{\partial u_i},$$

where $i, j = 1, 2$ and $u_1 = u$, $u_2 = v$.

If we now consider $\Omega(x, y)$ and $\Omega^*(u, v)$ the potentials, we see that the curves

$$2\Omega(x, y) = C$$

are equipotential lines in the x, y system and the curves

$$2\Omega^*(u, v) = C^*$$

are equipotential lines in the u, v system. The curves

$$2\Omega(u, v) = C,$$

however, are not equipotential lines in the u, v system.

In summary, therefore, we have the following properties. The curves $2\Omega(x, y) = C$ are equipotential lines, isotachs and curves of zero velocity in the x, y system; the curves $2\Omega^*(u, v) = C^*$ are equipotential lines and isotachs in the u, v system; and finally the curves $2\Omega(u, v) = C$ are curves of zero velocity in the u, v system.

4.7.5 Additional properties of the curves of zero velocity

(A) As the first important property of the curves of zero velocity, we show that the third particle, if placed with zero velocity on a curve of zero velocity, will leave along the normal to the curve (Fig. 4.29). Assume that the initial conditions of the particle, x_0, y_0, $\dot{x}_0 = \dot{y}_0 = 0$, are chosen so that $P_0(x_0, y_0)$ is neither one of the libration points nor does it coincide with the primaries. In the first case, $P_0 \equiv L_i$, it is known that the particle stays at its starting point, and in the second case the velocity is not finite and so the motion cannot be started with zero velocity. Expansions in power series around the starting point are

$$x = x_0 + \dot{x}_0 t + \ddot{x}_0 \frac{t^2}{2!} + \dddot{x}_0 \frac{t^3}{3!} + \cdots \tag{115}$$

and

$$y = y_0 + \dot{y}_0 t + \ddot{y}_0 \frac{t^2}{2!} + \dddot{y}_0 \frac{t^3}{3!} + \cdots. \tag{116}$$

The coefficients can be determined using the equations of motion,

$$\ddot{x} = 2\dot{y} + \Omega_x \tag{117}$$

and

$$\ddot{y} = -2\dot{x} + \Omega_y. \tag{118}$$

Since x_0 and y_0 are given and $\dot{x}_0 = \dot{y}_0 = 0$, the first coefficients to be determined are $\ddot{x}_0$ and $\ddot{y}_0$. From Eqs. (117) and (118) we have

$$\ddot{x}_0 = \Omega_x(x_0, y_0) \neq 0$$

and

$$\ddot{y}_0 = \Omega_y(x_0, y_0) \neq 0.$$

The next coefficient can be obtained from the equation of motion by differentiation. From Eq. (117) we have

$$\dddot{x} = 2\ddot{y} + \Omega_{xx}\dot{x} + \Omega_{xy}\dot{y},$$

which, using Eq. (118), becomes

$$\dddot{x} = 2\Omega_y + (\Omega_{xx} - 4)\dot{x} + \Omega_{xy}\dot{y}. \tag{119}$$

At point $P_0(x_0, y_0)$, we have

$$\dddot{x}_0 = 2\Omega_y(x_0, y_0)$$

since $\dot{x}_0 = \dot{y}_0 = 0$.
Similarly

$$\dddot{y} = -2\Omega_x + \Omega_{xy}\dot{x} + (\Omega_{yy} - 4)\dot{y} \tag{120}$$

and

$$\dddot{y}_0 = -2\Omega_x(x_0, y_0).$$

Substituting the values found for $\ddot{x}_0$, $\ddot{y}_0$, $\dddot{x}_0$, and $\dddot{y}_0$ into the series given by Eqs. (115) and (116) we obtain

$$\begin{aligned} x &= x_0 + \Omega_x(x_0, y_0)\frac{t^2}{2} + \Omega_y(x_0, y_0)\frac{t^3}{3} + \cdots, \\ y &= y_0 + \Omega_y(x_0, y_0)\frac{t^2}{2} - \Omega_x(x_0, y_0)\frac{t^3}{3} + \cdots, \end{aligned} \tag{121}$$

and so the tangent to the orbit is

$$\left(\frac{dy}{dx}\right)_0 = \frac{\Omega_y(x_0, y_0)t - \Omega_x(x_0, y_0)t^2 + \mathcal{O}(t^3)}{\Omega_x(x_0, y_0)t + \Omega_y(x_0, y_0)t^2 + \mathcal{O}(t^3)}. \tag{122}$$

As $t \to 0$, we have from Eq. (122)

$$\left(\frac{dy}{dx}\right)_0 = \frac{\Omega_y(x_0, y_0)}{\Omega_x(x_0, y_0)}. \tag{123}$$

The tangent to the curve of zero velocity

$$2\Omega(x, y) = C$$

at a regular point $P_0(x_0, y_0)$ is computed from

$$d\Omega = \Omega_x\, dx + \Omega_y\, dy = 0,$$

and consequently it becomes

$$\left(\frac{dy}{dx}\right)_z = -\frac{\Omega_x(x_0, y_0)}{\Omega_y(x_0, y_0)}. \tag{124}$$

The tangent to the orbit, Ω_y/Ω_x, therefore is normal to the tangent of the curve of zero velocity, $-\Omega_x/\Omega_y$, as stated.

Note that the cubic terms in Eqs. (121) are related to the tangent of the curve of zero velocity at P_0. In fact, introducing the two-dimensional position vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j},$$

the vector normal to the curve of zero velocity,

$$\mathbf{n} = \Omega_x\mathbf{i} + \Omega_y\mathbf{j} = \operatorname{grad}\Omega,$$

and the vector tangent to the curve of zero velocity,

$$\mathbf{s} = \Omega_y\mathbf{i} - \Omega_x\mathbf{j},$$

Eqs. (121) can be written as

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{n}\frac{t^2}{2} + \mathbf{s}\frac{t^3}{3} + \cdots. \tag{125}$$

This shows, once again, that the motion starts along the normal ($\mathbf{n}t^2/2$) and that the transverse component ($\mathbf{s}t^3/3$) becomes significant as t increases. The effect of the term $\mathbf{s}t^3/3$ depends on the sign of t, since $\mathbf{s}$ is fixed at P_0. This means that if at t_0 the particle was on the curve of zero velocity at point $P_0(\mathbf{r}_0)$, then the transverse component changes sign before and after t_0, forming a cusp at P_0; see Fig. 4.29.

(B) The second remark about curves of zero velocity is related to their similarities to orbits. Such similarities will be pointed out in the next chapter, where it will be shown that the solutions of the linearized equations of motion around the equilateral points, for $\mu < \mu_0 = 0.03852...$, are ellipses with axes parallel to the axes found for the linearized curves of zero velocity in this chapter. The eccentricities of the two sets of ellipses are not the same, however. As another example regarding the similarities of orbits and curves of zero velocity, consider orbits around and close to the primary with the larger mass. These orbits are slightly distorted circles when suitable initial conditions are chosen. The curves of zero velocity around the same primary are also slightly distorted circles, and both the curves of zero velocity and the orbits become circles when $\mu = 0$ and the other primary has no effect. The situation is the same when orbits and curves of zero velocity are compared for $r \gg 1$, for large values of C associated with the outside ovals. To give precise analytical meaning to these similarities we first establish a condition which the dynamical system must satisfy in order to identify curves of zero velocity with orbits. Then this condition is applied to the problem of two bodies and the expected result is derived according to which curves of zero velocity are orbits since circles with their centers at the primary with unit mass are possible orbits as well as curves of zero velocity. After this, the criterion is applied to the restricted problem. It is found that there are no curves of zero velocity which are orbits in the restricted problem. Nevertheless, it is shown how certain curves of zero velocity may approximate certain orbits.

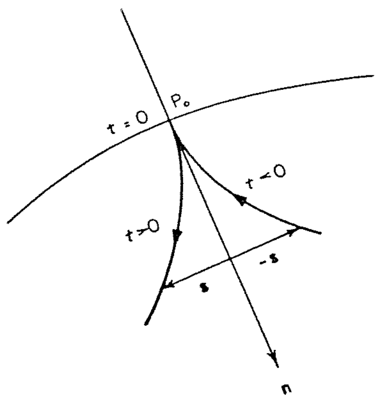

FIG. 4.29. Cusp at the curve of zero velocity.

If the third particle is to move along a curve of zero velocity,

$$\mathbf{v} \cdot \operatorname{grad} \Omega = 0 \tag{126}$$

or

$$\dot{x}\Omega_x + \dot{y}\Omega_y = 0. \tag{127}$$

This last equation is called the "tangency requirement" and it is meaningful everywhere except at the two primaries and the five equilibrium points, which are, once again, excluded. Let the Jacobian constant of the curve of zero velocity on which the third body is to move be C_1,

$$2\Omega - C_1 = 0, \tag{128}$$

and let C_0 be the Jacobian constant of the particle,

$$\dot{x}^2 + \dot{y}^2 = |\mathbf{v}|^2 = 2\Omega - C_0 = k^2. \tag{129}$$

Along the curve of zero velocity Eq. (128) is valid, so from Eq. (129) we have $C_1 - C_0 = k^2$. For a particle which is placed on its own Hill curve, $k^2 = 0$; i.e., if the Jacobian constant of the particle is C_1 and it is placed on the curve $2\Omega = C_1$, its velocity will be zero. Considering a particle with Jacobian constant $C_0 < C_1$ and placing it on a curve $2\Omega = C_1$ its velocity will be $k = \pm(C_1 - C_0)^{1/2}$.

The tangency requirement, Eq. (127), and the Jacobian integral, Eq. (129), can be solved for $\dot{x}$ and $\dot{y}$ which in turn can be substituted into Eq. (117) giving

$$F^4 \pm 2kF^3 + k^2 G = 0, \tag{130}$$

where

$$F^2 = (\partial\Omega/\partial x)^2 + (\partial\Omega/\partial y)^2,$$

$$G = \frac{\partial^2\Omega}{\partial x^2}\left(\frac{\partial\Omega}{\partial y}\right)^2 - 2\frac{\partial^2\Omega}{\partial x\,\partial y}\frac{\partial\Omega}{\partial x}\frac{\partial\Omega}{\partial y} + \frac{\partial^2\Omega}{\partial y^2}\left(\frac{\partial\Omega}{\partial x}\right)^2,$$

and the $\pm$signs correspond to direct or retrograde orbits.

Equation (130) is our result and it represents a condition on Ω. Those Hill curves for which Ω satisfies Eq. (130) represent orbits. The closed curves of Hill along which Ω satisfies Eq. (130) correspond to periodic orbits. Note that after $\Omega(x, y)$ is substituted into Eq. (130) an equation, $g(x, y, k) = 0$, is obtained. This equation, together with Hill's curve $\Omega(x, y) = \text{const}$, results in a constant value for k. This can be expressed by requiring that, from the equation $g(x, y, k) = 0$, equation $k = f(\Omega)$ should follow. The well-known condition for this is the vanishing of the Jacobian determinant

$$\partial(g, \Omega)/\partial(x, y) = 0. \tag{131}$$

Equation (130) in polar coordinates becomes

$$\frac{\Omega_r^2}{r}\left(\Omega_r + \frac{\Omega_{\varphi\varphi}}{r}\right) + \left(\frac{\Omega_\varphi}{r}\right)^2 \Omega_{rr} + \frac{2\Omega_r\Omega_\varphi}{r^2}\left(\frac{\Omega_\varphi}{r} - \Omega_{r\varphi}\right)$$
$$\pm \frac{2}{k}\left(\Omega_r^2 + \frac{\Omega_\varphi^2}{r^2}\right)^{3/2} + \frac{1}{k^2}\left(\Omega_r^2 + \frac{\Omega_\varphi^2}{r^2}\right)^2 = 0, \tag{132}$$

where

$$\Omega = \frac{1}{2}\left(r^2 + \frac{2}{r}\right) + \mu\left(\frac{1}{2} - r\cos\varphi - \frac{1}{r} + \frac{1}{\Delta}\right), \tag{133}$$

$r = r_1$, $\Delta = r_2$, and φ is the angle between r and the $-x$ direction, facing Δ. (See Fig. 4.11 with $\varphi = S$.) Note that the origin of the polar coordinate system is at $x = \mu$, $y = 0$.

For motion in the vicinity of $1 - \mu$, we have $r \ll 1$ and

$$\frac{1}{\Delta} = \sum_{n=0}^{\infty} r^n P_n(\cos\varphi). \tag{134}$$

If $r \gg 1$, we have

$$\frac{1}{\Delta} = \sum_{n=0}^{\infty} r^{-(n+1)} P_n(\cos\varphi), \tag{135}$$

P_n representing Legendre polynomials in both equations.

The case of a central Newtonian force field in a uniformly rotating (synodical) coordinate system is obtained from the general formula by writing $\mu = 0$:

$$\Omega = \tfrac{1}{2}(r^2 + 2/r). \tag{136}$$

The curves of zero velocity will be orbits if the function $\Omega(r)$ satisfies Eq. (132), which becomes

$$\frac{\Omega_r^3}{r} \pm \frac{2}{k}\Omega_r^3 + \frac{1}{k^2}\Omega_r^4 = 0$$

or, for $\Omega_r = r - 1/r^2 \neq 0$, i.e., $r \neq 1$,

$$\frac{1}{r} \pm \frac{2}{k} + \frac{\Omega_r}{k^2} = 0. \tag{137}$$

Substituting Eq. (136) in Eq. (137) gives

$$r^3 \pm 2kr^2 + k^2 r - 1 = 0, \tag{138}$$

where, by simple inspection and without use of the Jacobian determinant, we observe that along the lines of zero velocity Eq. (138) is satisfied. The curves of Hill are circles since $2\Omega - C_1$ gives

$$r^2 + 2/r = C_1, \tag{139}$$

and Eq. (138) results in constant values of k along circles.

The force function Ω when $\mu \neq 0$ can be written in the convenient form

$$\Omega = U(r) + \mu V(r, \varphi), \tag{140}$$

where

$$U(r) = \tfrac{1}{2}(r^2 + 2/r) \tag{141}$$

corresponds to the problem of two bodies in synodic coordinates, and

$$V(r, \varphi) = \tfrac{1}{2} - r\cos\varphi - 1/r + 1/\Delta \tag{142}$$

represents the perturbation.

The substitution of Eq. (140) into Eq. (132) gives the partial differential equation which the function U and V must satisfy in order to have curves of zero velocity as orbits. If in the process of substitution we omit terms of the order μ^2, we see that the second term of Eq. (132),

$$(\Omega_\varphi/r)^2\Omega_{rr} = (\mu^2/r^2)(U_{rr} + \mu V_{rr})V_\varphi^2 \cong 0,$$

and the third term,

$$\frac{2\Omega_r\Omega_\varphi}{r^2}\left(\frac{\Omega_\varphi}{r} - \Omega_{r\varphi}\right) = \mu^2 \frac{2V_\varphi(U_r + \mu V_r)}{r^2}\left(\frac{V_\varphi}{r} - V_{r\varphi}\right) \cong 0,$$

will be omitted entirely and the only contributions will come from the remaining terms:

$$\frac{\Omega_r^2}{r}\left(\Omega_r + \frac{\Omega_{\varphi\varphi}}{r}\right) \pm \frac{2}{k}\Omega_r^3 + \frac{1}{k^2}\Omega_r^4 = 0, \tag{143}$$

where the terms $\Omega_\varphi^2 = \mu^2 V_\varphi^2$ were omitted.

If the $\Omega_r = 0$ solution is ignored, then Eq. (143) becomes

$$\Omega_r\left(\frac{1}{r} \pm \frac{2}{k} + \frac{\Omega_r}{k^2}\right) + \frac{\Omega_{\varphi\varphi}}{r^2} = 0. \tag{144}$$

Although the term Ω_r^2 in this equation still contains a term of order μ^2, Eq. (144) is given to show its relation to the corresponding unperturbed Eq. (137). The perturbation manifests itself in the function $\Omega_{\varphi\varphi}r^{-2}$ when Eqs. (137) and (144) are compared.

Performing the elimination of the last remaining term of the order of μ^2, we obtain

$$AU' + \mu\left[V_r\left(A + \frac{U'}{k^2}\right) + \frac{V_{\varphi\varphi}}{r^2}\right] = 0, \tag{145}$$

where

$$A(r) = \frac{1}{r} \pm \frac{2}{k} + \frac{U'}{k^2} \tag{146}$$

and $U' = dU(r)/dr$.

Note that the function $A(r)$ is identical with the left side of Eq. (137) since for $\mu = 0$ we have $\Omega(r) = U(r)$.

The curves of zero velocity are orbits if the function $\Omega(r, \varphi)$ satisfies either the solution $\Omega_r = 0$ or Eq. (145) with constant k. To perform this check we use the Jacobian condition [Eq. (131)] with

$$\Omega = U + \mu V$$

and with either

$$g = \Omega_r \tag{147}$$

or

$$g = AU' + \mu\left[V_r\left(A + \frac{U'}{k^2}\right) + \frac{V_{\varphi\varphi}}{r^2}\right]. \tag{148}$$

Equation (131) with the function (147) gives

$$\Omega_{r\varphi}\Omega_r - \Omega_{rr}\Omega_\varphi = 0,$$

or

$$\mu V_{r\varphi}(U' + \mu V_r) - \mu V_\varphi(U'' + \mu V_{rr}) = 0,$$

and, omitting terms of order μ^2, we have

$$V_{r\varphi}U' - V_\varphi U'' = 0. \tag{149}$$

Equation (148), after neglecting terms of the order μ^2, gives

$$U'\left[V_{r\varphi}\left(A + \frac{U'}{k^2}\right) + \frac{V_{\varphi\varphi\varphi}}{r^2}\right] - V_\varphi(U'A)' = 0. \tag{150}$$

Close to the primary, with mass $1 - \mu$, we have from Eq. (134)

$$V = \frac{1}{2} - rc - \frac{1}{r} + \sum_{n=0}^{\infty} r^n P_n(c), \tag{151}$$

where $c = \cos\varphi$.

Substitution of the functions $U(r)$ and $V(r, c)$ either in Eq. (149) or in Eq. (150) gives essentially the same result. Using Eq. (149) we obtain

$$-3 + 3P_1' + 4rP_2' + 5r^2P_3' + 6r^3P_4' + \sum_{n=4}^{\infty} r^n[(n+3)P'_{n+1} - (n-3)P'_{n-2}] = 0. \tag{152}$$

Interpretation of this result is that (since $P_1' = 1$) as $r \to 0$ the curve of zero velocity approaches the orbit because Eq. (152) and so also Eq. (149) is satisfied. Also note that, if the Legendre expansion given by Eq. (134) is terminated with $n = 1$, that is, with the first power of r,

Eq. (152) is satisfied since only the first two terms remain. In other words, for small μ, using the linear expansion of $1/\Delta$, the curves of Hill are orbits if terms of order μ^2 are neglected.

For the outer oval the Legendre expansion now has the form of Eq. (135), and Eq. (149) becomes

$$-3 + 3P_1' + \frac{4P_2'}{r} + \frac{5P_3'}{r^2} + \frac{6P_4'}{r^3} + \sum_{n=4}^{\infty} \frac{1}{r^n}[(n+3)P'_{n-1} - (n-3)P'_{n-2}]. \tag{153}$$

The above equation is identical with Eq. (152) if $1/r$ is replaced by r. The conclusions follow from the discussion of Eq. (152). For large values of r when the Legendre expansion is

$$\frac{1}{\Delta} \cong \frac{P_0}{r} + \frac{P_1}{r^2},$$

the curves of zero velocity are orbits if terms of $\mathcal{O}(\mu^2)$ are neglected. Note that the Legendre expansion in the planetary case ($r \gg 1$) omits terms of order r^{-3} and in the satellite case ($r \ll 1$) terms of order r^2.

Equation (150), similarly to the satellite case, does not contribute additional information to the planetary case.

(C) We conclude this section on the curves of zero velocity with Fig. 4.30. This sketch, originally prepared by Deprit, shows part

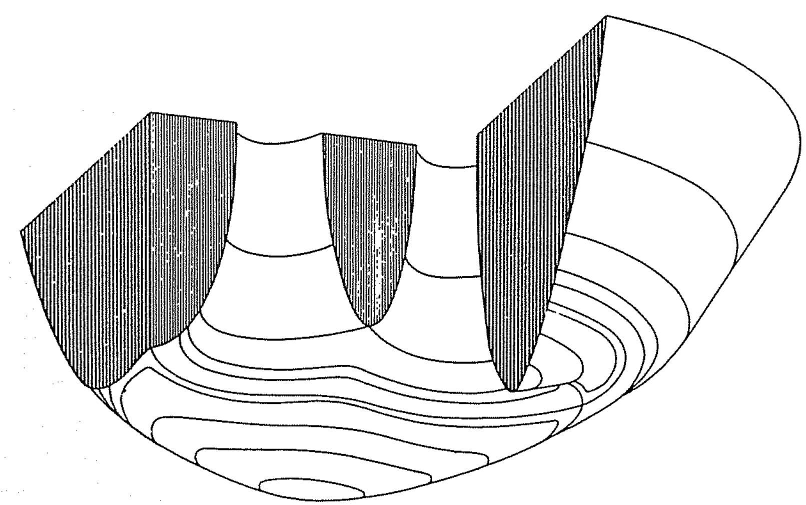

FIG. 4.30. Three-dimensional representation of the curves of zero velocity (Deprit, 1962, Ref. 36).

of the three-dimensional representation mentioned in Section 4.7.2. One of the triangular points is at the bottom and the two equal masses are located between the three hatched areas.

4.8 Applications

4.8.1 Motion of the moon

Probably the best-known application of the curves of zero velocity is that by Hill, who introduced the method to treat the problem of the motion of the moon by means of a modified restricted problem (see Chapter 10).

The basic idea in applying the contents of this chapter to the motion of the moon is as follows. If the assumptions of the restricted problem are applicable to the dynamical system formed by the sun, earth, and the moon, the Jacobian constant for the motion of the moon may be computed. Then by constructing the curves of zero velocity corresponding to this value of the Jacobian constant, one may determine for some of the orbits whether or not they are bounded.

The Jacobian constant for the moon is given by

$$C = 2\Omega - v^2,$$

where v is the velocity of the moon relative to the rotating coordinate system. This latter system rotates with the mean motion of the sun-earth system, i.e., with $n_\oplus = 1.99095 \times 10^{-7}$ rad/sec. The earth-moon system rotates with $n_☾ = 2.6617 \times 10^{-6}$ rad/sec. The mass parameter for the sun-earth system is

$$\mu = \frac{m_\oplus + m_☾}{m_\odot + m_\oplus + m_☾} = 3.03591 \times 10^{-6}.$$

The mean distance between the sun and the earth is $l = 149{,}525 \times 10^6$ meters and the mean distance between the earth and the moon is $d_{\oplus ☾} = 384.4 \times 10^6$ meters. The distance of the sun from the center of mass of the system is $a = 453{,}944$ meters, which follows from the above data and from the equation $a = l\mu$. The distance of the earth from the origin of the rotating coordinate system is $b = l(1 - \mu) = 149{,}524.546056 \times 10^6$ meters. The dimensionless mean distance of the earth from the moon is 2.570808×10^{-3}. The location of the second collinear libration point of the sun-earth system can be read from Appendix IIE as $x_2 = -0.989991$, and, since the dimensionless distance between the earth and origin of the coordinate system is

$1 - \mu = 0.99999696409$, the distance between the earth and L_2, in dimensionless units, becomes $\overline{EL_2} = 1 - \mu + x_2 = 0.01000596409$. Figure 4.31 shows the arrangement in the neighborhood of the earth, drawn to the proper scale.

The value of the Jacobian constant corresponding to L_2 can also be read from Appendix IIE as $C_2 = 3.0009000092671$. The location of the intersection (A) of the curve of zero velocity passing through L_2 with the negative x axis can be computed from

$$f(r) = r^4 + (3 - 2\mu)r^3 - (C_2 + 3\mu - 3)r^2 - (C_2 + \mu - 3)r + 2\mu = 0,$$

where $r = \overline{EA}$, or $r = \overline{EA'}$, in dimensionless units. The two positive roots of this equation $(r_A, r_{A'})$ correspond to points A and A', to the intersections of the inner and outer ovals with the negative x axis. The root $r_A = 0.00941$ furnishes the location of A. Since $f(r) \to 0$ as $r \to \pm\infty$ and since there are only two positive roots according to the

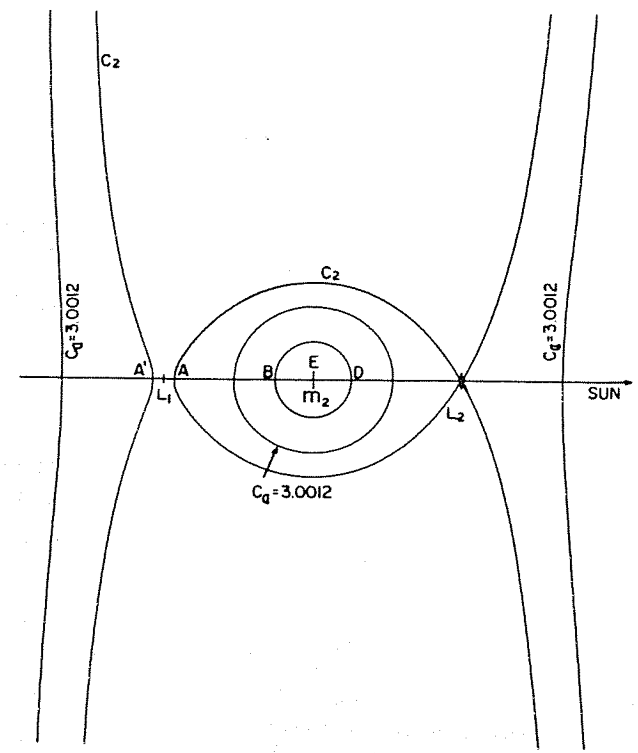

FIG. 4.31. Curves of zero velocity for the sun-earth system in the neighborhood of the earth, $\mu = 3.03591 \times 10^{-6}$.

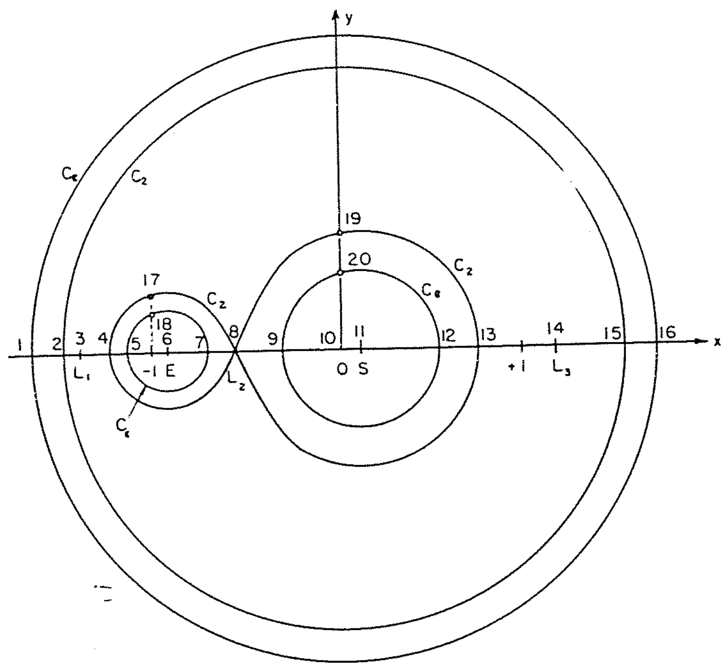

FIG. 4.32. Schematic curves of zero velocity for the sun-earth system, $\mu = 3.03591 \times 10^{-6}$.

sign rule and since finally $f'(r_A) < 0$, the other root $r_{A'}$ is larger than r_A as expected; in fact, $r_{A'} = 0.01076$.

The orbital velocity of the moon relative to the rotating system is is $v_\leftmoon = n_\leftmoon d_{\omega\leftmoon} = 1.023$ km/sec, and the dimensionless relative velocity of the moon is 0.03436.

The Jacobian constant for the moon can now be computed at various points on its orbit from

$$C_\leftmoon = (1-\mu)r_1^2 + \mu r_2^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - v^2.$$

For instance at point B we have $r_1 = 1 + r_2$, $r_2 = \overline{EB} = 2.570808 \times 10^{-3}$, which data give $C_\leftmoon{}^B = 3.0012 > 3.0009 = C_2$, and so the moon's Jacobian constant $(C_\leftmoon{}^B)$ is larger than the Jacobian constant corresponding to the limiting value (C_2). The moon therefore stays inside the oval of zero velocity which in turn is inside the limiting figure-eight curve of zero velocity and does not escape from the earth.

At point D we have $r_1 = 1 - r_2$, $r_2 = \overline{EB}$, and so

$$C_\leftmoon{}^D = C_\leftmoon{}^B - 7 \times 10^{-8} > 3.0009 = C_2,$$

from which we can conclude that either point B or point D can be used to evaluate $C_\leftmoon$, provided a circular motion of the moon with constant velocity is accepted.

It is known that C is increasing as we approach the point representing the earth (E), moving from the limiting curve of zero velocity (C_2) inside to the direction of E. Therefore the curve of zero velocity corresponding to the moon's motion ($C_\leftmoon$) will be inside the limiting curve (C_2). On the other hand the curve $C_\leftmoon$ will contain the moon's orbit since motion is only possible inside the curve of zero velocity for this case. These curves of zero velocity and special points are shown on Fig. 4.31.

Figure 4.32 shows in a highly schematic and necessarily distorted fashion both sides of the curves of zero velocity corresponding to $\mu = 3.03591 \times 10^{-6}$.

Because of the possible interest in additional details of the critical curves of zero velocity for the sun-earth system, the tabulated coordinates associated with the schematic Fig. 4.3_ are given in Table III. The error of the values given for the abscissas x is less than 5×10^{-11}.

TABLE III

COORDINATE VALUES FOR FIG. 4.32

Point	x	Remark
1	−1.01682 57807	$C_\leftmoon$
2	−1.01076 03350	C_2
3	−1.01007 01938	L_1
4	−1.00941 04109	C_2
5	−1.00545 76402	$C_\leftmoon$
6	−0.99999 69641	Earth
7	−0.99453 24684	$C_\leftmoon$
8	−0.98999 09310	L_2
9	−0.98340 58399	$C_\leftmoon$
10	0.00000 00000	$\mathcal{O}$
11	0.00000 30359	Sun
12	0.98025 15691	$C_\leftmoon$
13	0.98283 80084	C_2
14	1.00000 12650	L_3
15	1.01736 31718	C_2
16	1.02001 44609	$C_\leftmoon$

Point	x	y	Remark
17	−1	0.66777	C_2
18	−1	0.50554	$C_\leftmoon$
19	0	0.98280	C_2
20	0	0.98210	$C_\leftmoon$

4.8.2 Applications to cosmogony and to binary stars

Consider the sun and the earth as the primaries and allow all the assumptions of the restricted problem to be acceptable. The existence of periodic orbits around the sun belonging to Poincaré's *première sorte* is well established with a value of the undisturbed semimajor axis (i.e., radius) that places the orbit inside the zero velocity oval around the sun. An estimate of this semimajor axis is $a_1 < \mu - x_2$. The closer the orbit is to the figure-eight Hill curve which goes through L_2, the larger are the earth-produced perturbations which are experienced by the third body. As long as the Jacobian constant C of the third particle is greater than C_2 and its initial conditions satisfy the requirements for periodicity, the body continues to move around the sun and performs a planetary motion. Let us now decrease the Jacobian constant of the particle so that $C_1 < C < C_2$. The physical meaning of this step is that the third body is *not* confined to its motion around the sun, but it is allowed to become a satellite of the second primary, i.e., of the earth. This does not mean that it will become permanently a satellite of the earth since its Jacobian constant is such that it is *not* confined to the oval of zero velocity around the earth. The curve of zero velocity associated with C, between C_1 and C_2, is of an hour-glass (pear) shape with a very small oval around the earth, with its neck close to L_2 and with a large oval around the sun. If C is equal to C_2 the neck is closed. If C is slightly smaller than C_2, the neck is open and particles can pass through it; communication between the sun and the earth is established and exchanges between planetary and satellite orbits are possible. If a particle originally in a planetary orbit around the sun possesses enough relative energy (has a low enough value of C) to pass through the opening at L_2 it may start revolving around the earth in an orbit which may be represented by the semimajor axis $a_s < 1 - \mu + x_2$. If there is any mechanism in the system which leads to or allows dissipation of energy at this stage, then the corresponding curve of zero velocity closes around the earth and the particle will remain in its satellite orbit. The dissipating mechanism may be, for instance, tidal friction which may operate after the particle becomes a satellite of the earth. The eccentricity of the orbits of the primaries might also lead to "capture." While this effect is not nearly so plausible as the first described, it would solve the problem without the introduction of an actual physical dissipative mechanism. The Jacobian "constant" in the elliptic case is of course time dependent; nevertheless, the Jacobian integral has some practical applications and a quasi-steady approach might give interesting results. The variable curves of zero velocity associated with the time-dependent Jacobian "constant" in a quasi-steady sense can be followed using the appendixes of this chapter.

The applications to stellar dynamics, especially to problems of close binary systems, are discussed in Chapter 10 in connection with the three-dimensional aspects of the restricted problem. At this point only the possible applications of Propositions (8) and (10) of Sections 4.6.8 and 4.6.10 are mentioned.

Concerning the case when the first Lagrangian point is at a maximum distance from the mass center [see the third part of Proposition (10)] it might be expected that if the phenomena taking place follow the dynamical assumptions of the restricted problem, then particle exchange between the primaries and the outside world might be minimized. This might follow from the fact that the pear-shape curves of zero velocity, mentioned before which enclose both primaries, open up for communication with the outside *always* at the first libration point as the Jacobian constant decreases from C_2 to C_1, provided $\mu < \frac{1}{2}$.

Regarding the third part of Proposition (8), the following considerations might apply. It is to be understood that particles associated with a Jacobian constant less than 3 possess enough relative energy to reach every point in the plane of motion of the primaries. Increasing C from its minimum value of 3 means that particles with smaller and smaller relative energy content are being considered. The larger the value of C_1 is, at which communication occurs between the primaries and their "outside," the more particles will participate in the exchange, since a larger C_1 means lower relative energy, and there are more particles which have at least a specified low-energy content than a certain given higher energy content. This means that the value $\mu_0 = 0.334364$ of the mass parameter corresponding to maximum C_1 should result in maximum particle participation in the exchange.

4.8.3 Application to space mechanics

The difference between the values of the Jacobian constant at L_2 and L_1 corresponds to a range of the relative energy of particles, namely, $C_1 < C < C_2$. The total variation of the value of C in the range is $\Delta_1 = C_2 - C_1$. This represents the excess of the relative energy which particles require (in order to establish communication with the outside) above the energy level needed for the previously mentioned planet-satellite exchange. In other words, particles with very high values of C have low relative energy levels, and they either move around one of the primaries or move far outside of the system. Considering particles with somewhat smaller values of C, say $C = C_2$ (i.e., somewhat higher relative energy levels), one reaches the boundary case corresponding to the figure-eight zero-velocity curve passing through L_2. All particles with Jacobian constants between $C = \infty$ and $C = C_2$

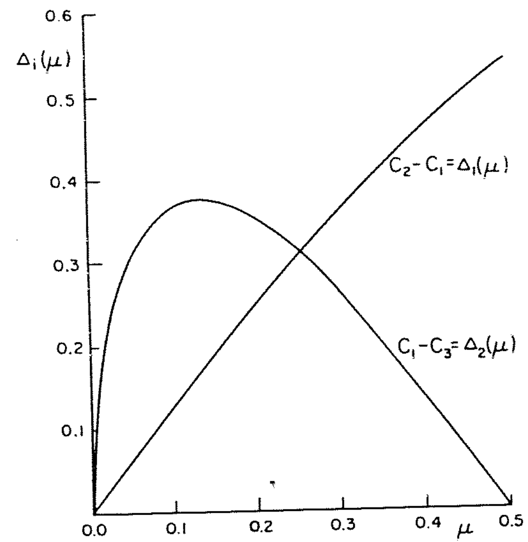

FIG. 4.33. Differences in the Jacobian constants Δ_i versus the mass parameter μ.

(between the corresponding relative energy levels) either will remain inside the above-mentioned figure-eight boundary curve or will stay outside a large oval which encloses both primaries. Those particles, for which $C_1 \leqslant C < C_2$, possess enough relative energy to communicate between the neighborhood of m_1 and that of m_2. If $C < C_1$ the particles can leave the system, while if $C > C_2$ the corresponding particles cannot change position from the vicinity of m_1 to that of m_2. Therefore the range $C_2 - C_1 = \Delta_1$ corresponds to those particles which have enough relative energy to travel between the primaries but do not have enough energy to leave the system.

The quantity $\Delta_2 = C_1 - C_3$ corresponds to those particles which have enough energy to leave (or enter) the system through the opening neck at L_1 but not enough to leave or enter at L_3.

Figure 4.33 shows Δ_1 and Δ_2 as functions of μ. The maximum of Δ_2 occurs at $\mu = 0.136017$ and it is $(\Delta_2)_{\max} = 0.376796$, corresponding to $x_1 = -1.268623$ and $x_3 = 1.056525$.

Example. Show that the preceding critical value of μ maximizes the function $\Delta_2(\mu) = 2[\Omega(x_1, 0) - \Omega(x_3, 0)]$. Hint: follow essentially the same method as was used to find μ for $(C_1)_{\max}$ in Section 4.6.8.

Consider now the Jacobian integral in the form $v = (2\Omega - C)^{1/2}$ and evaluate the velocity difference

$$\Delta v = \Delta\Omega/v - \Delta C/2v.$$

The function $\Omega(x, y)$ is constant within a very close approximation on a small circle with the center at either of the primaries; therefore, for particles which start their orbits at, say, $h = 100$–200 km above the earth's surface,

$$\Delta v \cong -\Delta C/2v. \tag{154}$$

At such a value of h we have

$$r_1 = (h + R)/l \quad \text{and} \quad 1 - (h + R)/l < r_2 < 1 + (h + R)/l,$$

where $l = 384{,}329$ km and $R = 6378$ km. Therefore, $r_1 = 0.01698545$,

$$0.98301455 < r_2 < 1.01698545,$$

and

$$58.17694385 < \Omega < 58.17694398,$$

representing a change in Ω of 2 parts in 10^9.

The variation of the initial magnitude of the velocity, therefore, is proportional to the change in C according to Eq. (154). The quantity Δ_1 is a measure of the allowable variation in velocity for particles which are to travel from m_1 to m_2 but are not to leave the system. A large value of Δ_1 corresponds to large variation of velocity, while a small value of Δ_1, which occurs at a small value of the mass parameter, i.e., $\mu \sim 0$, indicates extreme sensitivity to velocity "errors" if particles are required to reach the general area of the earth-moon system.

The maximum value of Δ_2 similarly represents the largest allowable velocity errors for particles which are to leave (or enter) the system through the opening at L_1 but not through the neck at L_3.

4.9 Notes

The term "phase space" is due to Gibbs [1, Vol. 2, pp. 6, 10]. Equations (2), when applied to the restricted problem, imply the existence of a "second integral" (in addition to the Jacobian integral). The references ([10–13]) mentioned in Section 2.10, therefore, are applicable here also.

The necessary condition mentioned in Section 4.1 for the existence of a second integral of the restricted problem has been treated experimentally by Hénon and by Bozis (see references [20] and [21] in Section 2.11).

The curves of zero velocity were introduced by Hill [2] and used by him in connection with what now is known as Hill's problem. Application of the concept to the restricted problem is due to him and to Bohlin [3].

The example and several notions in Section 4.2 are from Birkhoff's paper [4, p. 15].

The introduction of the equilibrium points as the singularities of the manifold of the states of motion may be considered as a sophisticated approach. A more elementary and direct method is to construct a free-body diagram (in the rotating system) and find points where the resultant force vector is zero. Neither the terminology nor the numbering of these points is uniform. They are most frequently called stationary, equilibrium, Laplacian, Lagrangian, Eulerian, or libration points, or centers of libration, and are numbered from left to right, from right to left, or starting with the middle one. Early references are by Lagrange [5, Vol. 6, p. 229], Laplace [6, Vol. 4, Book 10, p. 304], and Routh [7]. For the precise chronological details, see Section 5.7.

In Section 4.3, the concept of singularities of the manifold of the states of motion in regularized coordinates is employed. The regularization—as in Chapter 3—is the elimination of the singularities from the equations of motion. The singularities of the Jacobian function, $f(x, y, \dot{x}, \dot{y}) = \dot{x}^2 + \dot{y}^2 - 2\Omega + C$ or $F(u, v, u', v') = u'^2 + v'^2 - 2\Omega^*$, on the other hand, are the libration points, in the x, y *or* in the u, v system; cf. Birkhoff [4, p. 15].

The references for the three sets of limits given for the locations of the libration points are Wintner [8, p. 361] for (16a), Martin [9] for (16b), and Szebehely and Williams [10] for (16c).

Section 4.4 treats the problem of the actual evaluation of the locations of the collinear libration points, which amounts to the solution of fifth-order algebraic equations. The $(\mu/3)^{1/3}$ starting value goes back to Laplace [6], is used by Charlier [11, Vol. 2, p. 110] and by Moulton [12, p. 292], while the value recommended in the text is used by Plummer [13, p. 240] and by Brouwer and Clemence [14, p. 261]. A comprehensive discussion is in a paper by Szebehely and Williams [10]. Regarding the results of the computations see, in addition to [10] and [11], papers by Kuiper [15], Kopal [16], Kuiper and Johnson [17], Abhyankar [18], Goudas [19], and Deprit [20]. A short table also appears in Wintner's book [8], offering numerical results obtained by Rosenthal [21]. Note Wintner's slight correction of Rosenthal's presentation on one hand, and his (Wintner's) erroneous reference [8, p. 438] to Rosenthal's work on the other. The expansion formulas up to the sixth order for the location of the first and second collinear points given in Section 4.4 [Part (A), Eq. (22), and Part (B), Eq. (27)] may be found in [20]. The coefficients of the series for L_3, given in Section 4.4, Part (C), by Eq. (34), increase after the eighth power. Nevertheless, when the series is written in powers of μ (instead of ν) the terms decrease faster than the geometric series. The first two terms are given by Brouwer and Clemence [14].

The total computational time for Tables I, II, and III was 17 sec on an IBM 7094 computer.

Section 4.6 gives a comprehensive picture of the function $\Omega(x, y)$. This is important not only regarding the properties of the curves of zero velocity as shown in Section 4.7, but it also can be considered the basis of the restricted problem. This section begins with a list of properties which is followed by the detailed proofs. With such a separation the reader may simply use the list without involving himself with the verifications. Equation (39) is essentially due to Plummer [13, p. 238]. The relation between the dynamical problem, Dupin's indicatrix, and the curvature of the $z = \Omega(x, y)$ surface is discussed by Plummer [22, 23] with exceptional clarity. At the collinear libration centers the surface is "anticlastic" and at L_4 and L_5 it is "synclastic" according to Plummer's terminology. In the same paper [23], Plummer offers a proof of the inequality $\Omega_{xx}(L_i)\Omega_{yy}(L_i) - \Omega_{xy}^2(L_i) < 0$ for $i = 1, 2, 3$. References for the various derivations are Martin [9] and Wintner [8]; the maximum of the function $| x_1(\mu)|$ is given by Szebehely and Williams [10].

In 1931, in the same paper Martin [9] offers an erroneous theorem according to which the distances $r_2(L_2)$ and $r_2(L_1)$ change their relative magnitude, $r_1 \gtreqless r_2$, the change occurring at $\mu = \mu^* > \frac{1}{2}$. The correct relation is $r_2(L_2) \leqslant r_2(L_1)$ as follows from the third inequality of the second part of our Proposition (9), Section 4.6.9. The error was accepted by Rosenthal [24] in 1931, discovered by Markov [25] in 1933, and corrected by Rein [26] in 1936. Wintner [8, p. 363] in 1941 and Egorov [27, p. 123] in 1957 showed also that the inequality does not change direction.

Another error, this time committed by Wintner [8, p. 362] regarding the proof of the sign of $\Omega_{yy}(x_i, 0)$, was avoided by Plummer [13, p. 242] and corrected by Deprit [20]. The text follows Plummer's derivation in Section 4.6.2 regarding the critical Eq. (48). Another error by Wintner [8, p. 364] regarding the proof of the third inequality discussed in our Section 4.6.9, Part 2, is corrected in the text.

Sections 4.7.1 and 4.7.2 are rather detailed: first, because the curves of zero velocity offer one of the few tools with which information about the restricted problem can be obtained, and second, because teaching experience shows that the transfer of the simple ideas involved in this subject into applications turns out to be a major problem.

There seems to be no generally accepted terminology to describe the shapes of the curves of zero velocity, excepting the term "oval" as applied to the distorted circles discussed in Section 4.7.2(i). When the inner ovals join and form the inner "pear shapes" or "dumbbells" of Section 4.7.2(ii), the terminology becomes quite free. Dumbbells are symmetric with respect to two perpendicular axes, while the curves of zero velocity have only one axis of symmetry (unless $\mu = \frac{1}{2}$). The German candy "Katzenzunge" approximates the shape better than a dumbbell does, but it also has two axes of symmetry. The same applies to the curve called lemniscate, the equation of which, incidentally, is not the same as that of a zero-velocity curve. Pear shape might be an acceptable description. The shapes of the horseshoe curves of Section 4.7.2(iii) seem to correspond to their descriptive names, but the next class [Section 4.7.2(iv)] could be described by "bean shaped" equally well. The terminology "tadpole-like" shapes was adopted from Danby [28, p. 192].

Section 4.7.3 shows several sets of curves of zero velocity. A more complete set for $\mu = \frac{1}{2}$ is given by Strömgren [29], including curves drawn for sixteen different values of the Jacobian constant. Attention is directed to the curves belonging to $C = 3.1$ in Figs. 4.26(a)–(d) and to the curve of Fig. 4.26(e) corresponding to $C = 3.01$. None of these "tadpole-like" curves has a cusp. It is an almost common error in the literature to draw these curves with one or two cusps at the ends. The reason for this is a superficial modification of Darwin's [30, Vol. 4, p. 12] original drawing which was clearly the source of many of these erroneous figures. Darwin's figure shows the curve of zero velocity belonging to C_3, which, of course, has a cusp. Any new cusp on curves of zero velocity for a value of the Jacobian constant $3 < C < C_3$ would correspond to the existence of a sixth libration point, as H. Pollard pointed out.

Section 4.7.4 is of interest also to the subject treated in Chapter 3. The two basically different possibilities for introducing curves of zero velocity in the regularized system are given by Szebehely and Pierce [30a]. A slightly different version of Fig. 4.28(b), showing the curves of zero velocity in the Thiele system, is also given by Burrau and Strömgren [31].

The property of the curves of zero velocity (Section 4.7.5), that motion from them commences in a normal direction, is standard textbook material. Generalization of this property to more complex dynamical systems is given by Szebehely [32], who also discusses the relation between curves of zero velocity and orbits in the restricted problem [33]. Applications of the isotach concept to more general dynamical system are described by Nahon [34] and Szebehely [32]. Attention is called to

Moisseiev's major memoir presenting qualitative results obtained by an ingenious combination of curves of zero velocity and curvature properties of orbits [35]. The three-dimensional representation of the function $z = \Omega(x, y)$ shown as Fig. 4.30 is from Deprit's [36] paper on Lagrange's problem, to be discussed in Section 10.5.1.

The average dimensionless radius of the curve of zero velocity belonging to the moon is obtained from Table III of Section 4.8.1 as $\frac{1}{2}[|x_5| - |x_7|] \cong 0.0055$, which may be compared with 109.695 equatorial earth radii given by Hill [2] and by Brouwer and Clemence [14, p. 206] and with 0.00506 given by See [37, Vol. 2, p. 227]. This last reference also gives the Jacobian constants for other satellites in the solar system.

Applications to cosmogony and to binary systems are given in [38] and in the previously quoted references, especially [10, 15–17]. Attention is especially directed to [38] where several sets of curves of zero velocity are shown from $\mu = 0.3$ to $\mu = 0.5$. The original reference to these curves is an unpublished thesis [39]. A more detailed discussion of Kopal's contribution and the applications of the restricted problem to binary systems may be found in Chapter 10 in connection with the three-dimensional aspects of the restricted problem. Application of curves of zero velocity to solar system cosmogony are given by Huang [40] and applications to the limiting radius of a cluster are shown by Mineur [41]. Jean's treatise [42], especially pp. 243 and 396, is another standard reference.

Applications to space mechanics, in addition to those discussed in Section 4.8.3, can be found in Egorov's previously mentioned paper [26].

The difficulties involved in the construction of equipotential curves depend of course on the nature of the function. In our case various methods are available such as described by Charlier [11], by Moulton [12] and by Hewison [39]. Computer programs also exist which take as input the value of the mass parameter and the values of the Jacobian constants and with these produce the desired curves of zero velocity. Such programs have been prepared by Lawson *et al.* [43] and by M. Standish. Figures 4.13–4.18, 4.26, and 4.31 were prepared by M. Standish and Figs. 4.28(a)–(d) by D. Pierce at Yale University.

4.10 References

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Appendix I. First collinear equilibrium point

A. First collinear equilibrium point for $0 \leq \mu \leq 0.09$

μ	x_1	C_1	$(\Omega_{xx})_1$	$(\Omega_{yy})_1$
0.0	−1.0	3.0	9.0	−3.0
0.000001	−1.00694 86021	3.00042 90104	8.91755 48240	−2.95877 74120
0.000002	−1.00875 91723	3.00067 95007	8.89636 93726	−2.94818 46863
0.000003	−1.01003 02284	3.00088 90094	8.88156 70477	−2.94078 35238
0.000004	−1.01104 26656	3.00107 56194	8.86981 72863	−2.93490 86432
0.000005	−1.01189 79954	3.00124 68340	8.85991 86180	−2.92995 93090
0.000006	−1.01264 59173	3.00140 66842	8.85128 36074	−2.92564 18037
0.000007	−1.01331 49243	3.00155 76514	8.84357 58475	−2.92178 79237
0.000008	−1.01392 30521	3.00170 14000	8.83658 26095	−2.91829 13047
0.000009	−1.01448 25253	3.00183 91148	8.83015 98291	−2.91507 99146
0.00001	−1.01500 20578	3.00197 16774	8.82420 48987	−2.91210 24494
0.00002	−1.01891 81309	3.00311 47532	8.77960 13943	−2.88980 06971
0.00003	−1.02166 80892	3.00406 76167	8.74856 91883	−2.87428 45941
0.00004	−1.02385 87664	3.00491 41992	8.72401 26370	−2.86200 63185
0.00005	−1.02570 95531	3.00568 93267	8.70337 63627	−2.85168 81814
0.00006	−1.02732 78936	3.00641 17087	8.68541 25564	−2.84270 62782
0.00007	−1.02877 54028	3.00709 28620	8.66940 73933	−2.83470 36967
0.00008	−1.03009 10901	3.00774 05120	8.65490 99005	−2.82745 49502
0.00009	−1.03130 14017	3.00836 01581	8.64161 49227	−2.82080 74613
0.0001	−1.03242 51917	3.00895 58909	8.62930 52813	−2.81465 26406
0.0002	−1.04089 03393	3.01406 59688	8.53760 86790	−2.76880 43395
0.0003	−1.04682 58214	3.01829 32324	8.47431 93223	−2.73715 96612
0.0004	−1.05154 67105	3.02202 72196	8.42451 87049	−2.71225 93524
0.0005	−1.05552 88539	3.02542 95601	8.38285 40631	−2.69142 70316
0.0006	−1.05900 54417	3.02858 70839	8.34671 89285	−2.67335 94642
0.0007	−1.06211 02699	3.03155 32409	8.31462 57209	−2.65731 28605
0.0008	−1.06492 80850	3.03436 38832	8.28563 65445	−2.64281 82723
0.0009	−1.06751 63637	3.03704 45331	8.25911 79178	−2.62955 89589
0.001	−1.06991 60980	3.03961 41747	8.23461 97385	−2.61730 98693
0.002	−1.08786 44705	3.06137 69572	8.05368 47916	−2.52684 23958
0.003	−1.10028 53684	3.07903 96976	7.93024 15555	−2.46512 07777
0.004	−1.11004 69762	3.09441 02292	7.83381 98114	−2.41690 99057
0.005	−1.11818 88558	3.10823 87169	7.75357 44201	−2.37678 72100
0.006	−1.12522 10700	3.12092 81938	7.68425 62089	−2.34212 81044
0.007	−1.13143 65633	3.13272 67788	7.62288 42279	−2.31144 21140
0.008	−1.13702 10432	3.14380 09159	7.56758 71078	−2.28379 35539
0.009	−1.14210 05525	3.15426 91082	7.51710 56878	−2.25855 28439
0.01	−1.14676 50421	3.16421 95085	7.47054 83838	−2.23527 41919
0.02	−1.18007 79046	3.24533 32901	7.12820 00816	−2.06410 00408
0.03	−1.20119 12467	3.30719 55159	6.89503 66342	−1.94751 83171
0.04	−1.21643 05676	3.35821 71744	6.71241 56068	−1.85620 78034
0.05	−1.22809 36671	3.40189 41382	6.55976 95755	−1.77988 47877
0.06	−1.23730 95359	3.44010 76312	6.42724 52090	−1.71362 26045
0.07	−1.24472 72109	3.47401 76444	6.30929 09932	−1.65464 54966
0.08	−1.25075 84229	3.50440 35577	6.20244 40152	−1.60122 20076
0.09	−1.25568 27620	3.53182 11319	6.10438 65433	−1.55219 32716

B. First collinear equilibrium point for $0.1 \leq \mu \leq 0.5$

μ	x_1	C_1	$(\Omega_{xx})_1$	$(\Omega_{yy})_1$
0.10	−1.25969 98329	3.55668 44258	6.01348 47506	−1.50674 23753
0.11	−1.26295 77220	3.57931 24945	5.92854 00754	−1.46427 00377
0.12	−1.26556 96953	3.59995 78458	5.84864 49981	−1.42432 24990
0.13	−1.26762 45599	3.61882 47116	5.77309 44478	−1.38654 72239
0.14	−1.26919 33923	3.63608 12814	5.70132 88069	−1.35066 44034
0.15	−1.27033 40733	3.65186 81838	5.63289 58218	−1.31644 79109
0.16	−1.27109 44417	3.66630 45400	5.56742 43242	−1.28371 21621
0.17	−1.27151 45468	3.67949 23912	5.50460 56000	−1.25230 28000
0.18	−1.27162 82945	3.69152 00093	5.44417 98678	−1.22208 99339
0.19	−1.27146 46743	3.70246 44158	5.38592 62632	−1.19296 31316
0.20	−1.27104 86907	3.71239 33329	5.32965 52862	−1.16482 76431
0.21	−1.27040 20808	3.72136 67142	5.27520 30130	−1.13760 15065
0.22	−1.26954 38749	3.72943 79637	5.22242 65988	−1.11121 32994
0.23	−1.26849 08403	3.73665 49155	5.17120 07368	−1.08560 03684
0.24	−1.26725 78366	3.74306 06316	5.12141 48389	−1.06070 74195
0.25	−1.26585 81025	3.74869 40562	5.07297 07673	−1.03648 53837
0.26	−1.26430 34900	3.75359 05580	5.02578 09912	−1.01289 04956
0.27	−1.26260 46576	3.75778 23828	4.97976 70766	−0.98988 35383
0.28	−1.26077 12295	3.76129 90337	4.93485 84385	−0.96742 92192
0.29	−1.25881 19301	3.76416 75942	4.89099 13027	−0.94549 56513
0.30	−1.25673 46958	3.76641 30018	4.84810 78353	−0.92405 39176
0.31	−1.25454 67701	3.76805 82834	4.80615 54090	−0.90307 77045
0.32	−1.25225 47850	3.76912 47586	4.76508 59806	−0.88254 29903
0.33	−1.24986 48298	3.76963 22140	4.72485 55597	−0.86242 77798
0.34	−1.24738 25111	3.76959 90568	4.68542 37533	−0.84271 18767
0.35	−1.24481 30042	3.76904 24477	4.64675 33739	−0.82337 66869
0.36	−1.24216 10976	3.76797 84178	4.60881 00992	−0.80440 50496
0.37	−1.23943 12324	3.76642 19719	4.57156 21774	−0.78578 10887
0.38	−1.23662 75362	3.76438 71792	4.53498 01694	−0.76749 00847
0.39	−1.23375 38528	3.76188 72546	4.49903 67228	−0.74951 83614
0.40	−1.23081 37694	3.75893 46301	4.46370 63739	−0.73185 31869
0.41	−1.22781 06389	3.75554 10193	4.42896 53721	−0.71448 26861
0.42	−1.22474 76016	3.75171 74748	4.39479 15253	−0.69739 57626
0.43	−1.22162 76031	3.74747 44396	4.36116 40619	−0.68058 20309
0.44	−1.21845 34109	3.74282 17939	4.32806 35085	−0.66403 17543
0.45	−1.21522 76291	3.73776 88967	4.29547 15806	−0.64773 57903
0.46	−1.21195 27117	3.73232 46239	4.26337 10841	−0.63168 55421
0.47	−1.20863 09744	3.72649 74026	4.23174 58280	−0.61587 29140
0.48	−1.20526 46055	3.72029 52423	4.20058 05447	−0.60029 02723
0.49	−1.20185 56751	3.71372 57637	4.16986 08190	−0.58493 04095
0.50	−1.19840 61446	3.70679 62241	4.13957 30236	−0.56978 65118

C. First collinear equilibrium point for the sun-Jupiter system

μ	x_1	C_1	$(\Omega_{xx})_1$	$(\Omega_{yy})_1$
0.00085	−1.06624 80527	3.03571 91482	8.27209 98930	−2.63604 99465
0.00086	−1.06650 56873	3.03598 65411	8.26946 08824	−2.63473 04412
0.00087	−1.06676 12977	3.03625 27583	8.26684 35953	−2.63342 17976
0.00088	−1.06701 49226	3.03651 78177	8.26424 76096	−2.63212 38048
0.00089	−1.06726 65992	3.03678 17370	8.26167 25160	−2.63083 62580
0.00090	−1.06751 63637	3.03704 45331	8.25911 79178	−2.62955 89589
0.00091	−1.06776 42513	3.03730 62227	8.25658 34297	−2.62829 17149
0.00092	−1.06801 02960	3.03756 68219	8.25406 86782	−2.62703 43391
0.00093	−1.06825 45311	3.03782 63464	8.25157 33004	−2.62578 66502
0.00094	−1.06849 69886	3.03808 48118	8.24909 69438	−2.62454 84719
0.00095	−1.06873 76999	3.03834 22329	8.24663 92662	−2.62331 96331
0.00096	−1.06897 66953	3.03859 86244	8.24419 99350	−2.62209 99675
0.00097	−1.06921 40045	3.03885 40006	8.24177 86267	−2.62088 93134
0.00098	−1.06944 96561	3.03910 83753	8.23937 50271	−2.61968 75135
0.00099	−1.06968 36782	3.03936 17622	8.23698 88302	−2.61849 44151
0.00100	−1.06991 60980	3.03961 41747	8.23461 97385	−2.61730 98693
0.00101	−1.07014 69420	3.03986 56255	8.23226 74626	−2.61613 37313
0.00102	−1.07037 62359	3.04011 61275	8.22993 17206	−2.61496 58603
0.00103	−1.07060 40050	3.04036 56929	8.22761 22380	−2.61380 61190
0.00104	−1.07083 02737	3.04061 43339	8.22530 87476	−2.61265 43738
0.00105	−1.07105 50659	3.04086 20624	8.22302 09890	−2.61151 04945

D. First collinear equilibrium point for the earth-moon system

μ	x_1	C_1	$(\Omega_{xx})_1$	$(\Omega_{yy})_1$
0.0110	−1.15108 12284	3.17371 99416	7.42725 86457	−2.21362 93229
0.0111	−1.15149 57620	3.17464 73998	7.42308 91100	−2.21154 45550
0.0112	−1.15190 73758	3.17557 09499	7.41894 67443	−2.20947 33721
0.0113	−1.15231 61127	3.17649 06380	7.41483 11298	−2.20741 55649
0.0114	−1.15272 20148	3.17740 65094	7.41074 18580	−2.20537 09290
0.0115	−1.15312 51232	3.17831 86084	7.40667 85300	−2.20333 92650
0.0116	−1.15352 54778	3.17922 69783	7.40264 07562	−2.20132 03781
0.0117	−1.15392 31179	3.18013 16618	7.39862 81563	−2.19931 40781
0.0118	−1.15431 80818	3.18103 27005	7.39464 03587	−2.19732 01794
0.0119	−1.15471 04069	3.18193 01353	7.39067 70005	−2.19533 85003
0.0120	−1.15510 01298	3.18282 40063	7.38673 77270	−2.19336 88635
0.0121	−1.15548 72863	3.18371 43528	7.38282 21914	−2.19141 10957
0.0122	−1.15587 19114	3.18460 12132	7.37893 00549	−2.18946 50274
0.0123	−1.15625 40393	3.18548 46255	7.37506 09860	−2.18753 04930
0.0124	−1.15663 37037	3.18636 46266	7.37121 46609	−2.18560 73304
0.0125	−1.15701 09371	3.18724 12531	7.36739 07625	−2.18369 53812
0.0126	−1.15738 57719	3.18811 45405	7.36358 89808	−2.18179 44904
0.0127	−1.15775 82393	3.18898 45240	7.35980 90123	−2.17990 45062
0.0128	−1.15812 83702	3.18985 12379	7.35605 05604	−2.17802 52802
0.0129	−1.15849 61947	3.19071 47160	7.35231 33343	−2.17615 66671
0.0130	−1.15886 17423	3.19157 49916	7.34859 70496	−2.17429 85248

E. First collinear equilibrium point for the bodies of the solar system

		μ	x_1	C_1	$(\Omega_{xx})_1$	$(\Omega_{yy})_1$
MERCURY	and Sun	0.00000 01667	−1.00382 03944	3.00013 04262	8.95444 25439	−2.97722 12719
VENUS	and Sun	0.00000 24510	−1.00937 50347	3.00077 75604	8.88919 00644	−2.94459 50322
EARTH-MOON	and Sun	0.00000 30359	−1.01007 01938	3.00089 60448	8.88110 25512	−2.94055 12756
MARS	and Sun	0.00000 03233	−1.00476 57798	3.00020 26108	8.94325 59307	−2.97162 79654
JUPITER	and Sun	0.00095 38754	−1.06883 05211	3.03844 17152	8.24569 17789	−2.62284 58895
SATURN	and Sun	0.00028 55022	−1.04605 72665	3.01771 63648	8.48247 06964	−2.74123 53482
URANUS	and Sun	0.00004 37254	−1.02458 08107	3.00521 00988	8.71594 99842	−2.85797 49921
NEPTUNE	and Sun	0.00005 17732	−1.02601 13038	3.00582 08726	8.70002 12333	−2.85001 06166
PLUTO	and Sun	0.00000 27778	−1.00977 55079	3.00084 48097	8.88452 88544	−2.94226 44272
MOON	and Earth	0.01214 09319	−1.15564 50248	3.18407 77812	7.38122 62548	−2.19061 31274
MOON	and Earth	0.01215 06683	−1.15568 24834	3.18416 41431	7.38084 72091	−2.19042 36045
CRITICAL Value		0.03852 08965	−1.21443 88479	3.35119 28604	6.73721 23782	−1.86860 61891
DARWIN'S Value		0.09090 90909	−1.25608 29085	3.53418 16095	6.09584 76445	−1.54792 38223

Appendix II. Second collinear equilibrium point

A. Second collinear equilibrium point for $0 \leqslant \mu \leqslant 0.09$

μ	x_2	C_2	$(\Omega_{xx})_2$	$(\Omega_{yy})_2$
0.0	−1.0	3.0	9.0	−3.0
0.000001	−0.99308 14476	3.00043 03438	9.08397 16228	−3.04198 58114
0.000002	−0.99128 77034	3.00068 21674	9.10604 88029	−3.05302 44015
0.000003	−0.99003 04373	3.00089 30094	9.12159 71552	−3.06079 85776
0.000004	−0.98903 00939	3.00108 09528	9.13401 15347	−3.06700 57673
0.000005	−0.98818 57174	3.00125 35007	9.14452 00888	−3.07226 00444
0.000006	−0.98744 79071	3.00141 46843	9.15372 45842	−3.07686 22921
0.000007	−0.98678 83534	3.00156 69849	9.16197 02500	−3.08098 51250
0.000008	−0.98618 91443	3.00171 20668	9.16947 57324	−3.08473 78662
0.000009	−0.98563 81428	3.00185 11151	9.17638 93082	−3.08819 46541
0.00001	−0.98512 67004	3.00198 50110	9.18281 67185	−3.09140 83592
0.00002	−0.98127 80036	3.00314 14208	9.23149 61258	−3.11574 80629
0.00003	−0.97858 13298	3.00410 76184	9.26592 81541	−3.13296 40771
0.00004	−0.97643 60630	3.00496 75351	9.29350 68111	−3.14675 34055
0.00005	−0.97462 53925	3.00575 59971	9.31691 12686	−3.15845 56343
0.00006	−0.97304 32603	3.00649 17136	9.33745 58374	−3.16872 79187
0.00007	−0.97162 89044	3.00718 62017	9.35589 52773	−3.17794 76386
0.00008	−0.97034 38880	3.00784 71864	9.37270 80178	−3.18635 40089
0.00009	−0.96916 21723	3.00848 01674	9.38821 87594	−3.19410 93797
0.0001	−0.96806 52061	3.00908 92351	9.40265 92182	−3.20132 96091
0.0002	−0.95980 55124	3.01433 26664	9.51265 49481	−3.25632 74740
0.0003	−0.95401 00579	3.01869 32881	9.59110 91041	−3.29555 45520
0.0004	−0.94939 26354	3.02256 06364	9.65432 39990	−3.32716 19995
0.0005	−0.94548 93755	3.02609 63398	9.70822 64274	−3.35411 32137
0.0006	−0.94207 35233	3.02938 72276	9.75573 11694	−3.37786 55847
0.0007	−0.93901 52447	3.03248 67492	9.79851 55561	−3.39925 77780
0.0008	−0.93623 24504	3.03543 07559	9.83764 47366	−3.41882 23683
0.0009	−0.93366 95555	3.03824 47700	9.87384 27683	−3.43692 13842
0.001	−0.93128 69755	3.04094 77750	9.90762 71128	−3.45381 35564
0.002	−0.91321 85315	3.06404 41038	10.16749 90219	−3.58374 95110
0.003	−0.90037 36546	3.08304 01336	10.35538 12003	−3.67769 06002
0.004	−0.89001 82302	3.09974 36141	10.50814 78604	−3.75407 39302
0.005	−0.88116 83389	3.11490 46584	10.63929 87962	−3.81964 93981
0.006	−0.87334 38158	3.12892 62629	10.75550 82246	−3.87775 41123
0.007	−0.86627 01236	3.14205 65184	10.86063 37495	−3.93031 68747
0.008	−0.85977 39031	3.15446 18462	10.95713 44284	−3.97856 72142
0.009	−0.85373 79033	3.16626 07308	11.04668 75821	−4.02334 37910
0.01	−0.84807 87130	3.17754 13092	11.13049 56811	−4.06524 78406
0.02	−0.80346 56289	3.27192 61343	11.78073 32923	−4.39036 66462
0.03	−0.76964 34855	3.34699 84410	12.25416 74123	−4.62708 37062
0.04	−0.74090 98429	3.41116 43846	12.63893 54980	−4.81946 77490
0.05	−0.71522 53504	3.46791 63874	12.96782 36802	−4.98391 18401
0.06	−0.69159 97593	3.51913 28424	13.25724 25141	−5.12862 12570
0.07	−0.66946 93430	3.56597 12421	13.51677 65874	−5.25838 82937
0.08	−0.64847 95064	3.60920 86620	13.75259 18652	−5.37629 59326
0.09	−0.62839 12557	3.64939 86768	13.96892 86982	−5.48446 43491

B. Second collinear equilibrium point for $0.1 \leqslant \mu \leqslant 0.5$

μ	x_2	C_2	$(\Omega_{xx})_2$	$(\Omega_{yy})_2$
0.10	−0.60903 51100	3.68695 32299	14.16884 95090	−5.58442 47545
0.11	−0.59028 60914	3.72218 93018	14.35465 05198	−5.67732 52599
0.12	−0.57204 91314	3.75535 73453	14.52810 52543	−5.76405 26271
0.13	−0.55425 00599	3.78665 95409	14.69061 68094	−5.84530 84047
0.14	−0.53682 97798	3.81626 20189	14.84331 74522	−5.92165 87261
0.15	−0.51974 03585	3.84430 33313	14.98713 62140	−5.99356 81070
0.16	−0.50294 23195	3.87090 04974	15.12284 62095	−6.06142 31047
0.17	−0.48640 27162	3.89615 34255	15.25109 86455	−6.12554 93228
0.18	−0.47009 37289	3.92014 82166	15.37244 78252	−6.18622 39126
0.19	−0.45399 16226	3.94295 96796	15.48736 98971	−6.24368 49485
0.20	−0.43807 59585	3.96465 32763	15.59627 71610	−6.29813 85805
0.21	−0.42232 89887	3.98528 66491	15.69952 91513	−6.34976 45757
0.22	−0.40673 51836	4.00491 08343	15.79744 13417	−6.39872 06709
0.23	−0.39128 08599	4.02357 12389	15.89029 20662	−6.44514 60331
0.24	−0.37595 38833	4.04130 84335	15.97832 80826	−6.48916 40413
0.25	−0.36074 34284	4.05815 88029	16.06176 90892	−6.53088 45446
0.26	−0.34563 97833	4.07415 50839	16.14081 14252	−6.57040 57126
0.27	−0.33063 41893	4.08932 68140	16.21563 11261	−6.60781 55631
0.28	−0.31571 87081	4.10370 07075	16.28638 64673	−6.64319 32336
0.29	−0.30088 61102	4.11730 09732	16.35322 00931	−6.67661 00465
0.30	−0.28612 97821	4.13014 95841	16.41626 08128	−6.70813 04064
0.31	−0.27144 36469	4.14226 65080	16.47562 51216	−6.73781 25608
0.32	−0.25682 20976	4.15366 99049	16.53141 84965	−6.76570 92483
0.33	−0.24225 99384	4.16437 62970	16.58373 65049	−6.79186 82525
0.34	−0.22775 23360	4.17440 07157	16.63266 57565	−6.81633 28782
0.35	−0.21329 47756	4.18375 68291	16.67828 47237	−6.83914 23618
0.36	−0.19888 30237	4.19245 70525	16.72066 44508	−6.86033 22254
0.37	−0.18451 30950	4.20051 26446	16.75986 91675	−6.87993 45838
0.38	−0.17018 12230	4.20793 37919	16.79595 68219	−6.89797 84110
0.39	−0.15588 38344	4.21472 96813	16.82897 95424	−6.91448 97712
0.40	−0.14161 75256	4.22090 85646	16.85898 40391	−6.92949 20195
0.41	−0.12737 90424	4.22647 78131	16.88601 19515	−6.94300 59757
0.42	−0.11316 52610	4.23144 39667	16.91010 01497	−6.95505 00748
0.43	−0.09897 31706	4.23581 27751	16.93128 09935	−6.96564 04967
0.44	−0.08479 98581	4.23958 92333	16.94958 25544	−6.97479 12772
0.45	−0.07064 24934	4.24277 76128	16.96502 88039	−6.98251 44020
0.46	−0.05649 83159	4.24538 14864	16.97763 97719	−6.98881 98860
0.47	−0.04236 46216	4.24740 37498	16.98743 16758	−6.99371 58379
0.48	−0.02823 87514	4.24884 66380	16.99441 70245	−6.99720 85122
0.49	−0.01411 80789	4.24971 17384	16.99860 46973	−6.99930 23487
0.50	0.0	4.25	17.0	−7.0

C. Second collinear equilibrium point for the sun-Jupiter system

μ	x_2	C_2	$(\Omega_{xx})_2$	$(\Omega_{yy})_2$
0.00085	−0.93492 62821	3.03685 27030	9.85607 40205	−3.42803 70103
0.00086	−0.93467 11362	3.03713 34324	9.85967 85901	−3.42983 92950
0.00087	−0.93441 79271	3.03741 29860	9.86325 72710	−3.43162 86355
0.00088	−0.93416 66178	3.03769 13819	9.86681 05491	−3.43340 52745
0.00089	−0.93391 71723	3.03796 86375	9.87033 88957	−3.43516 94479
0.00090	−0.93366 95555	3.03824 47700	9.87384 27683	−3.43692 13842
0.00091	−0.93342 37337	3.03851 97960	9.87732 26108	−3.43866 13054
0.00092	−0.93317 96739	3.03879 37315	9.88077 88543	−3.44038 94271
0.00093	−0.93293 73441	3.03906 65925	9.88421 19176	−3.44210 59588
0.00094	−0.93269 67133	3.03933 83942	9.88762 22077	−3.44381 11038
0.00095	−0.93245 77513	3.03960 91516	9.89101 01200	−3.44550 50600
0.00096	−0.93222 04288	3.03987 88794	9.89437 60391	−3.44718 80195
0.00097	−0.93198 47173	3.04014 75919	9.89772 03389	−3.44886 01695
0.00098	−0.93175 05892	3.04041 53030	9.90104 33834	−3.45052 16917
0.00099	−0.93151 80173	3.04068 20263	9.90434 55265	−3.45217 27632
0.00100	−0.93128 69755	3.04094 77750	9.90762 71128	−3.45381 35564
0.00101	−0.93105 74382	3.04121 25621	9.91088 84780	−3.45544 42390
0.00102	−0.93082 93806	3.04147 64004	9.91412 99488	−3.45706 49744
0.00103	−0.93060 27785	3.04173 93021	9.91735 18435	−3.45867 59217
0.00104	−0.93037 76081	3.04200 12794	9.92055 44723	−3.46027 72361
0.00105	−0.93015 38467	3.04226 23441	9.92373 81376	−3.46186 90688

D. Second collinear equilibrium point for the earth-moon system

μ	x_2	C_2	$(\Omega_{xx})_2$	$(\Omega_{yy})_2$
0.0110	−0.84273 46477	3.18837 13920	11.20945 43422	−4.10472 71711
0.0111	−0.84221 56618	3.18943 17856	11.21711 13886	−4.10855 56943
0.0112	−0.84169 93078	3.19048 82657	11.22472 75303	−4.11236 37651
0.0113	−0.84118 55462	3.19154 08784	11.23230 33486	−4.11615 16743
0.0114	−0.84067 43384	3.19258 96690	11.23983 94114	−4.11991 97057
0.0115	−0.84016 56465	3.19363 46816	11.24733 62736	−4.12366 81368
0.0116	−0.83965 94336	3.19467 59598	11.25479 44777	−4.12739 72389
0.0117	−0.83915 56638	3.19571 35460	11.26221 45538	−4.13110 72769
0.0118	−0.83865 43019	3.19674 74820	11.26959 70202	−4.13479 85101
0.0119	−0.83815 53134	3.19777 78086	11.27694 23838	−4.13847 11919
0.0120	−0.83765 86648	3.19880 45659	11.28425 11404	−4.14212 55702
0.0121	−0.83716 43231	3.19982 77931	11.29152 37750	−4.14576 18875
0.0122	−0.83667 22562	3.20084 75288	11.29876 07622	−4.14938 03811
0.0123	−0.83618 24327	3.20186 38108	11.30596 25664	−4.15298 12832
0.0124	−0.83569 48218	3.20287 66761	11.31312 96422	−4.15656 48211
0.0125	−0.83520 93934	3.20388 61611	11.32026 24346	−4.16013 12173
0.0126	−0.83472 61180	3.20489 23016	11.32736 13794	−4.16368 06897
0.0127	−0.83424 49669	3.20589 51325	11.33442 69032	−4.16721 34516
0.0128	−0.83376 59118	3.20689 46883	11.34145 94241	−4.17072 97121
0.0129	−0.83328 89250	3.20789 10027	11.34845 93515	−4.17422 96758
0.0130	−0.83281 39796	3.20888 41090	11.35542 70867	−4.17771 35433

E. Second collinear equilibrium point for the bodies of the solar system

	μ	x_2	C_2	$(\Omega_{xx})_2$	$(\Omega_{yy})_2$
MERCURY and Sun	0.00000 01667	−0.99618 89786	3.00013 06484	9.04602 13700	−3.02301 06850
VENUS and Sun	0.00000 24510	−0.99067 83249	3.00078 08284	9.11357 72660	−3.05678 86330
EARTH-MOON and Sun	0.00000 30359	−0.98999 09310	3.00090 00927	9.12208 67063	−3.06104 33531
MARS and Sun	0.00000 03233	−0.99524 86696	3.00020 30418	9.05746 48073	−3.02873 24037
JUPITER and Sun	0.00095 38754	−0.93236 55874	3.03971 38022	9.89231 71231	−3.44615 85616
SATURN and Sun	0.00028 55022	−0.95476 09796	3.01809 70862	9.58088 69137	−3.29044 34568
URANUS and Sun	0.00004 37254	−0.97572 94921	3.00526 84024	9.30262 60717	−3.15131 30358
NEPTUNE and Sun	0.00005 17732	−0.97433 03181	3.00588 99075	9.32073 62883	−3.16036 81442
PLUTO and Sun	0.00000 27778	−0.99028 22680	3.00084 85134	9.11847 79472	−3.05923 89736
MOON and Earth	0.01214 09319	−0.83696 26374	3.20024 56106	11.29449 02969	−4.14724 51484
MOON and Earth	0.01215 06683	−0.83691 47190	3.20034 49097	11.29519 50582	−4.14759 75291
CRITICAL Value	0.03852 08965	−0.74493 51184	3.40220 01842	12.58607 90893	−4.79303 95446
DARWIN'S Value	0.09090 90909	−0.62660 34962	3.65291 62912	13.98774 25747	−5.49387 12874

Appendix III. Third collinear equilibrium point

A. Third collinear equilibrium point for $0 \leqslant \mu \leqslant 0.09$

μ	x_3	C_3	$(\Omega_{xx})_3$	$(\Omega_{yy})_3$
0.0	1.0	3.0	3.0	0.0
0.000001	1.00000 04167	3.00000 20000	3.00000 17500	−0.00000 08750
0.000002	1.00000 08333	3.00000 40000	3.00000 35000	−0.00000 17500
0.000003	1.00000 12500	3.00000 60000	3.00000 52500	−0.00000 26250
0.000004	1.00000 16667	3.00000 80000	3.00000 70000	−0.00000 35000
0.000005	1.00000 20833	3.00001 00000	3.00000 87500	−0.00000 43750
0.000006	1.00000 25000	3.00001 20000	3.00001 05000	−0.00000 52500
0.000007	1.00000 29167	3.00001 39999	3.00001 22500	−0.00000 61250
0.000008	1.00000 33333	3.00001 59999	3.00001 40001	−0.00000 70000
0.000009	1.00000 37500	3.00001 79999	3.00001 57501	−0.00000 78750
0.00001	1.00000 41667	3.00001 99999	3.00001 75001	−0.00000 87500
0.00002	1.00000 83333	3.00003 99996	3.00003 50003	−0.00001 75002
0.00003	1.00001 25000	3.00005 99991	3.00005 25007	−0.00002 62504
0.00004	1.00001 66667	3.00007 99984	3.00007 00013	−0.00003 50006
0.00005	1.00002 08333	3.00009 99974	3.00008 75020	−0.00004 37510
0.00006	1.00002 50000	3.00011 99963	3.00010 50029	−0.00005 25014
0.00007	1.00002 91667	3.00013 99950	3.00012 25039	−0.00006 12520
0.00008	1.00003 33333	3.00015 99935	3.00014 00051	−0.00007 00026
0.00009	1.00003 75000	3.00017 99917	3.00015 75065	−0.00007 87532
0.0001	1.00004 16667	3.00019 99898	3.00017 50080	−0.00008 75040
0.0002	1.00008 33333	3.00039 99592	3.00035 00321	−0.00017 50160
0.0003	1.00012 50000	3.00059 99081	3.00052 50722	−0.00026 25361
0.0004	1.00016 66667	3.00079 98367	3.00070 01284	−0.00035 00642
0.0005	1.00020 83333	3.00099 97448	3.00087 52006	−0.00043 76003
0.0006	1.00025 00000	3.00119 96325	3.00105 02888	−0.00052 51444
0.0007	1.00029 16666	3.00139 94997	3.00122 53931	−0.00061 26966
0.0008	1.00033 33333	3.00159 93466	3.00140 05135	−0.00070 02568
0.0009	1.00037 50000	3.00179 91730	3.00157 56499	−0.00078 78250
0.001	1.00041 66666	3.00199 89790	3.00175 08024	−0.00087 54012
0.002	1.00083 33329	3.00399 59151	3.00350 32111	−0.00175 16055
0.003	1.00124 99985	3.00599 08071	3.00525 72280	−0.00262 86140
0.004	1.00166 66632	3.00798 36539	3.00701 28552	−0.00350 64276
0.005	1.00208 33265	3.00997 44543	3.00877 00948	−0.00438 50474
0.006	1.00249 99882	3.01196 32069	3.01052 89488	−0.00526 44744
0.007	1.00291 66479	3.01394 99107	3.01228 94193	−0.00614 47097
0.008	1.00333 33054	3.01593 45645	3.01405 15084	−0.00702 57542
0.009	1.00374 99602	3.01791 71669	3.01581 52182	−0.00790 76091
0.01	1.00416 66120	3.01989 77168	3.01758 05507	−0.00879 02754
0.02	1.00833 28934	3.03959 00559	3.03532 35899	−0.01766 17949
0.03	1.01249 85063	3.05907 57745	3.05323 12375	−0.02661 56188
0.04	1.01666 31048	3.07835 35936	3.07130 56686	−0.03565 28343
0.05	1.02082 63343	3.09742 21971	3.08954 91149	−0.04477 45575
0.06	1.02498 78317	3.11628 02303	3.10796 38672	−0.05398 19336
0.07	1.02914 72245	3.13492 62991	3.12655 22780	−0.06327 61390
0.08	1.03330 41310	3.15335 89680	3.14531 67637	−0.07265 83818
0.09	1.03745 81596	3.17157 67591	3.16425 98073	−0.08212 99036

B. Third collinear equilibrium point for $0.1 \leqslant \mu \leqslant 0.5$

μ	x_3	C_3	$(\Omega_{xx})_3$	$(\Omega_{yy})_3$
0.10	1.04160 89086	3.18957 81504	3.18338 39613	−0.09169 19806
0.11	1.04575 59655	3.20736 15747	3.20269 18504	−0.10134 59252
0.12	1.04989 89069	3.22492 54170	3.22218 61747	−0.11109 30874
0.13	1.05403 72981	3.24226 80137	3.24186 97130	−0.12093 48565
0.14	1.05817 06923	3.25938 76506	3.26174 53259	−0.13087 26630
0.15	1.06229 86302	3.27628 25607	3.28181 59598	−0.14090 79799
0.16	1.06642 06396	3.29295 09222	3.30208 46501	−0.15104 23250
0.17	1.07053 62350	3.30939 08571	3.32255 45259	−0.16127 72630
0.18	1.07464 49164	3.32560 04280	3.34322 88138	−0.17161 44069
0.19	1.07874 61695	3.34157 76366	3.36411 08423	−0.18205 54212
0.20	1.08283 94642	3.35732 04210	3.38520 40469	−0.19260 20235
0.21	1.08692 42547	3.37282 66529	3.40651 19747	−0.20325 59873
0.22	1.09099 99781	3.38809 41353	3.42803 82900	−0.21401 91450
0.23	1.09506 60540	3.40312 05993	3.44978 67797	−0.22489 33899
0.24	1.09912 18836	3.41790 37015	3.47176 13597	−0.23588 06799
0.25	1.10316 68488	3.43244 10203	3.49396 60807	−0.24698 30403
0.26	1.10720 03111	3.44673 00530	3.51640 51352	−0.25820 25676
0.27	1.11122 16109	3.46076 82121	3.53908 28650	−0.26954 14325
0.28	1.11523 00660	3.47455 28213	3.56200 37681	−0.28100 18841
0.29	1.11922 49711	3.48808 11121	3.58517 25078	−0.29258 62539
0.30	1.12320 55959	3.50135 02189	3.60859 39204	−0.30429 69602
0.31	1.12717 11843	3.51435 71749	3.63227 30255	−0.31613 65128
0.32	1.13112 09529	3.52709 89072	3.65621 50350	−0.32810 75175
0.33	1.13505 40893	3.53957 22321	3.68042 53645	−0.34021 26822
0.34	1.13896 97508	3.55177 38489	3.70490 96440	−0.35245 48220
0.35	1.14286 70625	3.56370 03353	3.72967 37306	−0.36483 68653
0.36	1.14674 51157	3.57534 81400	3.75472 37215	−0.37736 18607
0.37	1.15060 29658	3.58671 35774	3.78006 59678	−0.39003 29839
0.38	1.15443 96301	3.59779 28197	3.80570 70899	−0.40285 35449
0.39	1.15825 40857	3.60858 18898	3.83165 39935	−0.41582 69968
0.40	1.16204 52673	3.61907 66536	3.85791 38876	−0.42895 69438
0.41	1.16581 20640	3.62927 28109	3.88449 43030	−0.44224 71515
0.42	1.16955 33169	3.63916 58866	3.91140 31130	−0.45570 15565
0.43	1.17326 78160	3.64875 12206	3.93864 85555	−0.46932 42777
0.44	1.17695 42968	3.65802 39575	3.96623 92569	−0.48311 96284
0.45	1.18061 14366	3.66697 90348	3.99418 42584	−0.49709 21292
0.46	1.18423 78509	3.67561 11710	4.02249 30442	−0.51124 65221
0.47	1.18783 20889	3.68391 48519	4.05117 55722	−0.52558 77861
0.48	1.19139 26290	3.69188 43166	4.08024 23079	−0.54012 11539
0.49	1.19491 78738	3.69951 35416	4.10970 42605	−0.55485 21302
0.50	1.19840 61446	3.70679 62241	4.13957 30236	−0.56978 65118

C. Third collinear equilibrium point for the sun-Jupiter system

μ	x_3	C_3	$(\Omega_{xx})_3$	$(\Omega_{yy})_3$
0.00085	1.00035 41666	3.00169 92623	3.00148 80797	−0.00074 40399
0.00086	1.00035 83333	3.00171 92449	3.00150 55934	−0.00075 27967
0.00087	1.00036 25000	3.00173 92272	3.00152 31073	−0.00076 15537
0.00088	1.00036 66666	3.00175 92093	3.00154 06214	−0.00077 03107
0.00089	1.00037 08333	3.00177 91913	3.00155 81356	−0.00077 90678
0.00090	1.00037 50000	3.00179 91730	3.00157 56499	−0.00078 78250
0.00091	1.00037 91666	3.00181 91545	3.00159 31645	−0.00079 65822
0.00092	1.00038 33333	3.00183 91358	3.00161 06791	−0.00080 53396
0.00093	1.00038 75000	3.00185 91169	3.00162 81940	−0.00081 40970
0.00094	1.00039 16666	3.00187 90978	3.00164 57090	−0.00082 28545
0.00095	1.00039 58333	3.00189 90785	3.00166 32242	−0.00083 16121
0.00096	1.00040 00000	3.00191 90590	3.00168 07395	−0.00084 03698
0.00097	1.00040 41666	3.00193 90393	3.00169 82550	−0.00084 91275
0.00098	1.00040 83333	3.00195 90194	3.00171 57706	−0.00085 78853
0.00099	1.00041 24999	3.00197 89993	3.00173 32865	−0.00086 66432
0.00100	1.00041 66666	3.00199 89790	3.00175 08024	−0.00087 54012
0.00101	1.00042 08333	3.00201 89584	3.00176 83186	−0.00088 41593
0.00102	1.00042 49999	3.00203 89377	3.00178 58348	−0.00089 29174
0.00103	1.00042 91666	3.00205 89168	3.00180 33513	−0.00090 16757
0.00104	1.00043 33333	3.00207 88956	3.00182 08679	−0.00091 04340
0.00105	1.00043 74999	3.00209 88743	3.00183 83847	−0.00091 91923

D. Third collinear equilibrium point for the earth-moon system

μ	x_3	C_3	$(\Omega_{xx})_3$	$(\Omega_{yy})_3$
0.0110	1.00458 32605	3.02187 62129	3.01934 75081	−0.00967 37541
0.0111	1.00462 49252	3.02207 39495	3.01952 42933	−0.00976 21466
0.0112	1.00466 65898	3.02227 16656	3.01970 10948	−0.00985 05474
0.0113	1.00470 82544	3.02246 93611	3.01987 79125	−0.00993 89562
0.0114	1.00474 99189	3.02266 70360	3.02005 47465	−0.01002 73732
0.0115	1.00479 15834	3.02286 46904	3.02023 15968	−0.01011 57984
0.0116	1.00483 32479	3.02306 23243	3.02040 84633	−0.01020 42317
0.0117	1.00487 49124	3.02325 99376	3.02058 53462	−0.01029 26731
0.0118	1.00491 65767	3.02345 75303	3.02076 22453	−0.01038 11227
0.0119	1.00495 82411	3.02365 51025	3.02093 91607	−0.01046 95804
0.0120	1.00499 99054	3.02385 26541	3.02111 60924	−0.01055 80462
0.0121	1.00504 15697	3.02405 01851	3.02129 30404	−0.01064 65202
0.0122	1.00508 32339	3.02424 76956	3.02147 00047	−0.01073 50023
0.0123	1.00512 48981	3.02444 51855	3.02164 69853	−0.01082 34926
0.0124	1.00516 65623	3.02464 26549	3.02182 39822	−0.01091 19911
0.0125	1.00520 82264	3.02484 01036	3.02200 09953	−0.01100 04977
0.0126	1.00524 98905	3.02503 75319	3.02217 80248	−0.01108 90124
0.0127	1.00529 15545	3.02523 49395	3.02235 50706	−0.01117 75353
0.0128	1.00533 32185	3.02543 23266	3.02253 21327	−0.01126 60663
0.0129	1.00537 48824	3.02562 96931	3.02270 92111	-0.01135 46055
0.0130	1.00541 65463	3.02582 70390	3.02288 63058	−0.01144 31529

E. Third collinear equilibrium point for the bodies of the solar system

		μ	x_3	C_3	$(\Omega_{xx})_3$	$(\Omega_{yy})_3$
MERCURY	and Sun	0.00000 01667	1.00000 00694	3.00000 03333	3.00000 02917	−0.00000 01458
VENUS	and Sun	0.00000 24510	1.00000 10212	3.00000 49019	3.00000 42892	−0.00000 21446
EARTH-MOON	and Sun	0.00000 30359	1.00000 12650	3.00000 60718	3.00000 53128	−0.00000 26564
MARS	and Sun	0.00000 03233	1.00000 01347	3.00000 06465	3.00000 05657	−0.00000 02829
JUPITER	and Sun	0.00095 38754	1.00039 74480	3.00190 68217	3.00167 00120	−0.00083 50060
SATURN	and Sun	0.00028 55022	1.00011 89592	3.00057 09212	3.00049 96942	−0.00024 98471
URANUS	and Sun	0.00004 37254	1.00001 82189	3.00008 74489	3.00007 65210	−0.00003 82605
NEPTUNE	and Sun	0.00005 17732	1.00002 15722	3.00010 35437	3.00009 06053	−0.00004 53027
PLUTO	and Sun	0.00000 27778	1.00000 11574	3.00000 55555	3.00000 48611	−0.00000 24306
MOON	and Earth	0.01214 09319	1.00505 86237	3.02413 10325	3.02136 54734	−0.01068 27367
MOON	and Earth	0.01215 06683	1.00506 26803	3.02415 02629	3.02138 27031	−0.01069 13516
CRITICAL Value		0.03852 08965	1.01604 71952	3.07551 53565	3.06862 16769	−0.03431 08385
DARWIN'S Value		0.09090 90909	1.03783 56421	3.17322 22145	3.16599 08439	−0.08299 54220

Appendix IV. Values of the Jacobian constant

A. Values of the Jacobian constant on the x axis $C = 2\Omega(x, 0)$, for $\mu = 0.000001$ and $\mu = 0.00001$

$\mu = 0.000001$				$\mu = 0.00001$			
x	C	x	C	x	C	x	C
		0.00	$\sim 2 \times 10^6$			0.00	$\sim 2 \times 10^5$
−3.0	9.66666 77778	0.05	40.00326 29200	−3.0	9.66667 77777	0.05	40.01013 05680
−2.9	9.09965 62976	0.10	20.01018 28200	−2.9	9.09966 64239	0.10	20.01182 83619
−2.8	8.55428 68560	0.15	13.35591 16285	−2.8	8.55429 71314	0.15	13.35661 63306
−2.7	8.03074 19021	0.20	10.04004 26669	−2.7	8.03075 23544	0.20	10.04042 66867
−2.6	7.52923 19541	0.25	8.06252 66001	−2.6	7.52924 26182	0.25	8.06276 60096
−2.5	7.05000 12133	0.30	6.75668 47607	−2.5	7.05001 21332	0.30	6.75684 76120
−2.4	6.59333 45813	0.35	5.83679 88080	−2.4	6.59334 58133	0.35	5.83691 66546
−2.3	6.15956 65082	0.40	5.16000 99286	−2.3	6.15957 81255	0.40	5.16009 92876
−2.2	5.74909 22534	0.45	4.64695 22559	−2.2	5.74910 43524	0.45	4.64702 25597
−2.1	5.36238 23647	0.50	4.25000 63333	−2.1	5.36239 50750	0.50	4.25006 33341
−2.0	5.00000 15000	0.55	3.93886 89019	−2.0	5.00001 49998	0.55	3.93891 62922
−1.9	4.66263 31945	0.60	3.69333 78056	−1.9	4.66264 77344	0.60	3.69337 80559
−1.8	4.35111 28827	0.65	3.49942 69459	−1.8	4.35112 88268	0.65	3.49946 17664
−1.7	4.06647 25769	0.70	3.34714 62581	−1.7	4.06649 04741	0.70	3.34717 68669
−1.6	3.81000 23021	0.75	3.22916 96984	−1.6	3.81002 30203	0.75	3.22919 69842
−1.5	3.58333 61111	0.80	3.14000 27361	−1.5	3.58336 11104	0.80	3.14002 73612
−1.4	3.38857 49796	0.85	3.07544 36728	−1.4	3.38860 69376	0.85	3.07546 61395
−1.3	3.22846 64832	0.90	3.03222 45218	−1.3	3.22851 09841	0.90	3.03224 52177
−1.2	3.10667 46111	0.95	3.00776 52943	−1.2	3.10674 61063	0.95	3.00778 45223
−1.1	3.02819 93469	1.00	3.00000 20000	−1.1	3.02835 70877	1.00	3.00001 99999
−1.00	4.99999 70000	1.1	3.02818 36053	−1.00	4.99997 00003	1.1	3.02819 96890
−0.95	3.00779 98374	1.2	3.10666 82980	−0.95	3.00813 00250	1.2	3.10668 29797
−0.90	3.03223 85311	1.3	3.22846 30530	−0.90	3.03238 53291	1.3	3.22847 66837
−0.85	3.07545 03888	1.4	3.38857 28537	−0.85	3.07553 33081	1.4	3.38858 56802
−0.80	3.14000 53751	1.5	3.58333 46889	−0.80	3.14005 37556	1.5	3.58334 68888
−0.75	3.22916 94445	1.6	3.81000 13005	−0.75	3.22919 44484	1.6	3.81001 30047
−0.70	3.34714 35851	1.7	4.06647 18445	−0.70	3.34715 01392	1.7	4.06648 31513
−0.65	3.49942 19806	1.8	4.35111 23316	−0.65	3.49941 21160	1.8	4.35112 33156
−0.60	3.69333 04445	1.9	4.66263 27700	−0.60	3.69330 44471	1.9	4.66264 34892
−0.55	3.93885 88329	2.0	5.00000 11667	−0.55	3.93881 56042	2.0	5.00001 16666
−0.50	4.24999 30000	2.1	5.36238 20987	−0.50	4.24993 00031	2.1	5.36239 24152
−0.45	4.64693 47599	2.2	5.74909 20382	−0.45	4.64684 76019	2.2	5.74910 22003
−0.40	5.15998 68334	2.3	6.15956 63320	−0.40	5.15986 83382	2.3	6.15957 63630
−0.35	5.83676 77505	2.4	6.59333 44355	−0.35	5.83660 60820	2.4	6.59334 43545
−0.30	6.75664 16350	2.5	7.05000 10914	−0.30	6.75641 63591	2.5	7.05001 09142
−0.25	8.06246 36668	2.6	7.52923 18514	−0.25	8.06213 66829	2.6	7.52924 15910
−0.20	10.03994 35003	2.7	8.03074 18149	−0.20	10.03943 50302	2.7	8.03075 14821
−0.15	13.35573 44647	2.8	8.55428 67814	−0.15	13.35484 47088	2.8	8.55429 63855
−0.10	20.00978 32244	2.9	9.09965 62334	−0.10	20.00783 24423	2.9	9.09966 57821
−0.05	40.00166 31221	3.0	9.66666 77222	−0.05	39.99413 27324	3.0	9.66667 72221

B. Values of the Jacobian constant on the x axis, $C = 2\Omega(x, 0)$, for $\mu = 0.0001$ and $\mu = 0.001$

$\mu = 0.0001$				$\mu = 0.001$			
x	C	x	C	x	C	x	C
		0.00	$\sim 2 \times 10^4$			0.00	$\sim 2 \times 10^3$
−3.0	9.66677 77657	0.05	40.07894 27889	−3.0	9.66777 65742	0.05	40.78091 57818
−2.9	9.09976 76765	0.10	20.02829 98427	−2.9	9.10077 91027	0.10	20.19463 70180
−2.8	8.55439 98739	0.15	13.36366 78475	−2.8	8.55542 61668	0.15	13.43463 56172
−2.7	8.03085 68657	0.20	10.04426 86716	−2.7	8.03190 08064	0.20	10.08286 80617
−2.6	7.52934 92466	0.25	8.06516 09632	−2.6	7.53041 43092	0.25	8.08919 66666
−2.5	7.05012 13189	0.30	6.75847 65889	−2.5	7.05121 18929	0.30	6.77481 28934
−2.4	6.59345 81196	0.35	5.83809 53912	−2.4	6.59457 98212	0.35	5.84990 99465
−2.3	6.15969 42834	0.40	5.16099 30449	−2.3	6.16085 44016	0.40	5.16994 73896
−2.2	5.74922 53265	0.45	4.64772 57056	−2.2	5.75043 34748	0.45	4.65476 79037
−2.1	5.36252 21612	0.50	4.25063 34122	−2.1	5.36379 12563	0.50	4.25634 12388
−2.0	5.00014 99775	0.55	3.93939 02417	−2.0	5.00149 77516	0.55	3.94413 44179
−1.9	4.66279 31101	0.60	3.69378 05904	−1.9	4.66424 45317	0.60	3.69780 90472
−1.8	4.35128 82400	0.65	3.49980 99924	−1.8	4.35287 95546	0.65	3.50329 42908
−1.7	4.06666 94115	0.70	3.34748 29676	−1.7	4.06845 52417	0.70	3.35054 52617
−1.6	3.81023 01555	0.75	3.22946 98497	−1.6	3.81229 68058	0.75	3.23219 92529
−1.5	3.58361 10359	0.80	3.14027 36151	−1.5	3.58610 36076	0.80	3.14273 65110
−1.4	3.38892 64131	0.85	3.07569 08078	−1.4	3.39211 07282	0.85	3.07793 75562
−1.3	3.22895 58046	0.90	3.03245 21750	−1.3	3.23338 52554	0.90	3.03452 15963
−1.2	3.10746 06268	0.95	3.00797 67988	−1.2	3.11456 29040	0.95	3.00989 92452
−1.1	3.02993 27343	1.00	3.00019 99950	−1.1	3.04551 48767	1.00	3.00199 95003
−1.00	4.99970 00300	1.1	3.02836 05204	−1.00	4.99700 29960	1.1	3.02996 82132
−0.95	3.01143 90764	1.2	3.10682 97898	−0.95	3.04526 16988	1.2	3.10829 71615
−0.90	3.03385 51305	1.3	3.22861 29831	−0.90	3.04873 70912	1.3	3.22997 51790
−0.85	3.07636 33388	1.4	3.38871 39361	−0.85	3.08474 79057	1.4	3.38999 56543
−0.80	3.14053 80606	1.5	3.58346 88791	−0.80	3.14543 12737	1.5	3.58468 79125
−0.75	3.22944 48375	1.6	3.81013 00381	−0.75	3.23198 38582	1.6	3.81129 94836
−0.70	3.34721 59577	1.7	4.06659 62093	−0.70	3.34790 19432	1.7	4.06772 58892
−0.65	3.49931 37139	1.8	4.35123 31468	−0.65	3.49835 40830	1.8	4.35233 05504
−0.60	3.69304 47076	1.9	4.66275 06727	−0.60	3.69047 07659	1.9	4.66382 15945
−0.55	3.93838 35625	2.0	5.00011 66564	−0.55	3.93408 76443	2.0	5.00116 56388
−0.50	4.24930 03100	2.1	5.36249 55716	−0.50	4.24303 09681	2.1	5.36352 62180
−0.45	4.64597 63563	2.2	5.74920 38120	−0.45	4.63729 71979	2.2	5.75021 90112
−0.40	5.15868 38163	2.3	6.15967 66638	−0.40	5.14688 15391	2.3	6.16067 87545
−0.35	5.83498 99919	2.4	6.59344 35355	−0.35	5.81888 83564	2.4	6.59443 44279
−0.30	6.75416 44855	2.5	7.05010 91326	−0.30	6.73173 39845	2.5	7.05109 03998
−0.25	8.05886 82916	2.6	7.52933 89772	−0.25	8.02632 85895	2.6	7.53031 19242
−0.20	10.03435 30198	2.7	8.03084 81453	−0.20	9.98380 06364	2.7	8.03181 38623
−0.15	13.34595 32332	2.8	8.55439 24172	−0.15	13.25764 27921	2.8	8.55535 18209
−0.10	19.98834 42149	2.9	9.09976 12599	−0.10	19.79540 19119	2.9	9.10071 51255
−0.05	39.91897 82031	3.0	9.66677 22120	−0.05	39.18207 70698	3.0	9.66772 11991

C. Values of the Jacobian constant on the x axis, $C = 2\Omega(x, 0)$, for $\mu = 0.01$ and $\mu = 0.1$

$\mu = 0.01$				$\mu = 0.1$			
x	C	x	C	x	C	x	C
		0.00	~ 2 × 10²			0.00	18.31222 22222
−3.0	9.67765 75577	0.05	49.53163 07692	−3.0	9.76588 32565	0.05	36.30302 63158
−2.9	9.11078 35753	0.10	22.03824 86239	−2.9	9.20000 00000	0.10	∞
−2.8	8.56557 60583	0.15	14.19280 10025	−2.8	8.65595 28131	0.15	36.30297 61905
−2.7	8.04222 32127	0.20	10.48775 93543	−2.7	8.13396 82540	0.20	18.31181 81818
−2.6	7.54094 30499	0.25	8.33852 90323	−2.6	7.63431 37255	0.25	12.32641 30435
−2.5	7.06198 96546	0.30	6.94299 00829	−2.5	7.15730 76923	0.30	9.34666 66667
−2.4	6.60566 11606	0.35	5.97085 47849	−2.4	6.70333 33333	0.35	7.57250 00000
−2.3	6.17231 00327	0.40	5.26121 15661	−2.3	6.27285 71429	0.40	6.40384 61538
−2.2	5.76235 65274	0.45	4.72628 88889	−2.2	5.86645 48495	0.45	5.58350 52910
−2.1	5.37630 66436	0.50	4.31413 91453	−2.1	5.48484 84848	0.50	4.98285 71429
−2.0	5.01477 66071	0.55	3.99205 36797	−2.0	5.12896 10390	0.55	4.53043 10345
−1.9	4.67852 72366	0.60	3.73841 08197	−1.9	4.80000 00000	0.60	4.18333 33333
−1.8	4.36851 40100	0.65	3.53834 51220	−1.8	4.49959 06433	0.65	3.91425 95308
−1.7	4.08596 37509	0.70	3.38129 95369	−1.7	4.23000 00000	0.70	3.70500 00000
−1.6	3.83250 05498	0.75	3.25956 99285	−1.6	3.99453 78151	0.75	3.54294 28904
−1.5	3.61037 39644	0.80	3.16740 22983	−1.5	3.79833 33333	0.80	3.41907 56303
−1.4	3.42293 58070	0.85	3.10041 24224	−1.4	3.65000 00000	0.85	3.32678 57143
−1.3	3.27586 65107	0.90	3.05520 11117	−1.3	3.56571 42857	0.90	3.26111 11111
−1.2	3.18150 17316	0.95	3.02909 22571	−1.2	3.58128 20513	0.95	3.21825 51669
−1.1	3.18550 19656	1.00	3.01995 02513	−1.1	3.80000 00000	1.00	3.19526 31579
−1.00	4.97029 60396	1.1	3.04598 31395	−1.00	4.72636 36364	1.1	3.20000 00000
−0.95	3.47490 00000	1.2	3.12289 79663	−0.95	6.70678 57143	1.2	3.26160 17316
−0.90	3.21794 63980	1.3	3.24351 73454	−0.90	∞	1.3	3.37090 90909
−0.85	3.17758 27243	1.4	3.40272 86325	−0.85	6.70723 68421	1.4	3.52157 19064
−0.80	3.19960 76023	1.5	3.59679 11889	−0.80	4.73000 00000	1.5	3.70904 76190
−0.75	3.26099 64912	1.6	3.82290 50266	−0.75	4.10348 03922	1.6	3.93000 00000
−0.70	3.35759 79116	1.7	4.07893 25774	−0.70	3.83000 00000	1.7	4.18192 30769
−0.65	3.49122 35294	1.8	4.36321 37102	−0.65	3.71250 00000	1.8	4.46289 76035
−0.60	3.66708 36906	1.9	4.67443 94628	−0.60	3.68809 52381	1.9	4.77142 85714
−0.55	3.89356 88312	2.0	5.01156 38376	−0.55	3.73315 93407	2.0	5.10633 39383
−0.50	4.18306 92677	2.1	5.37374 09130	−0.50	3.84000 00000	2.1	5.46666 66667
−0.45	4.55378 48631	2.2	5.76027 91815	−0.45	4.00967 17172	2.2	5.85165 89862
−0.40	5.03306 65978	2.3	6.17060 78483	−0.40	4.25000 00000	2.3	6.26068 18182
−0.35	5.66365 00000	2.4	6.60425 15879	−0.35	4.57613 63636	2.4	6.69321 47563
−0.30	6.51598 22814	2.5	7.06081 13819	−0.30	5.01333 33333	2.5	7.14882 35294
−0.25	7.71481 16424	2.6	7.53994 97951	−0.25	5.60304 94505	2.6	7.62714 28571
−0.20	9.50378 78843	2.7	8.04137 95338	−0.20	6.41571 42857	2.7	8.12786 32479
−0.15	12.43120 95238	2.8	8.56485 44642	−0.15	7.57916 66667	2.8	8.65072 07207
−0.10	18.04237 19101	2.9	9.11016 24954	−0.10	9.35000 00000	2.9	9.19548 87218
−0.05	33.03367 65957	3.0	9.67711 98892	−0.05	12.32779 41176	3.0	9.76197 17065

D. Values of the Jacobian constant on the x axis, $C = 2\Omega(x, 0)$, for $\mu = 0.2$ and $\mu = 0.3$

$\mu = 0.2$				$\mu = 0.3$			
x	C	x	C	x	C	x	C
		0.00	8.66000 00000			0.00	5.73380 95238
−3.0	9.84181 81818	0.05	11.29975 49020	−3.0	9.89511 19895	0.05	6.61250 00000
−2.9	9.27660 52227	0.10	16.61444 44444	−2.9	9.33022 72727	0.10	7.97000 00000
−2.8	8.73333 33333	0.15	32.60355 26316	−2.8	8.78732 71889	0.15	10.27171 56863
−2.7	8.21225 04537	0.20	∞	−2.7	8.26666 66667	0.20	14.91666 66667
−2.6	7.71365 07937	0.25	32.60345 23810	−2.6	7.76854 80944	0.25	28.90407 89474
−2.5	7.23788 67102	0.30	16.61363 63636	−2.5	7.29333 33333	0.30	∞
−2.4	6.78538 46154	0.35	11.29699 27536	−2.4	6.84145 96950	0.35	28.90392 85714
−2.3	6.35666 66667	0.40	8.65333 33333	−2.3	6.41346 15385	0.40	14.91545 45455
−2.2	5.95238 09524	0.45	7.08250 00000	−2.2	6.01000 00000	0.45	10.26757 24638
−2.1	5.57334 44816	0.50	6.05102 56410	−2.1	5.63190 47619	0.50	7.96000 00000
−2.0	5.22060 60606	0.55	5.33022 48677	−2.0	5.28023 41137	0.55	6.59250 00000
−1.9	4.89554 11255	0.60	4.80571 42857	−1.9	4.95636 36364	0.60	5.69820 51282
−1.8	4.60000 00000	0.65	4.41391 76245	−1.8	4.66212 12121	0.65	5.07694 44444
−1.7	4.33654 97076	0.70	4.11666 66667	−1.7	4.40000 00000	0.70	4.62857 14286
−1.6	4.10888 88889	0.75	3.88965 54252	−1.6	4.17350 87719	0.75	4.29740 42146
−1.5	3.92260 50420	0.80	3.71666 66667	−1.5	3.98777 77778	0.80	4.05000 00000
−1.4	3.78666 66667	0.85	3.58646 27040	−1.4	3.85067 22689	0.85	3.86505 13196
−1.3	3.71666 66667	0.90	3.49100 84034	−1.3	3.77500 00000	0.90	3.72833 33333
−1.2	3.74285 71429	0.95	3.42440 47619	−1.2	3.78333 33333	0.95	3.62998 25175
−1.1	3.93410 25641	1.00	3.38222 22222	−1.1	3.92000 00000	1.00	3.56294 11765
−1.00	4.49333 33333	1.1	3.35830 40936	−1.00	4.28692 30769	1.1	3.50333 33333
−0.95	5.12047 10145	1.2	3.40000 00000	−0.95	4.63250 00000	1.2	3.52134 50292
−0.90	6.42454 54545	1.3	3.49502 16450	−0.90	5.18666 66667	1.3	3.60000 00000
−0.85	10.40630 95238	1.4	3.63515 15152	−0.85	6.14989 13043	1.4	3.72844 15584
−0.80	∞	1.5	3.81468 22742	−0.80	8.12272 72727	1.5	3.89939 39394
−0.75	10.40671 05263	1.6	4.02952 38095	−0.75	14.10583 33333	1.6	4.10779 26421
−0.70	6.42777 77778	1.7	4.27666 66667	−0.70	∞	1.7	4.35000 00000
−0.65	5.13151 96078	1.8	4.55384 61538	−0.65	14.10618 42105	1.8	4.62333 33333
−0.60	4.52000 00000	1.9	4.85932 46187	−0.60	8.12555 55556	1.9	4.92576 92308
−0.55	4.19583 33333	2.0	5.19174 60317	−0.55	6.15955 88235	2.0	5.25575 16340
−0.50	4.02904 76190	2.1	5.55003 62976	−0.50	5.21000 00000	2.1	5.61206 34921
−0.45	3.96689 56044	2.2	5.93333 33333	−0.45	4.67916 66667	2.2	5.99373 86570
−0.40	3.98666 66667	2.3	6.34093 70200	−0.40	4.37000 00000	2.3	6.40000 00000
−0.35	4.08047 97980	2.4	6.77227 27273	−0.35	4.20063 18681	2.4	6.83021 50538
−0.30	4.25000 00000	2.5	7.22686 42951	−0.30	4.13333 33333	2.5	7.28386 36364
−0.25	4.50532 82828	2.6	7.70431 37255	−0.25	4.15128 78788	2.6	7.76051 38340
−0.20	4.86666 66667	2.7	8.20428 57143	−0.20	4.25000 00000	2.7	8.25980 39216
−0.15	5.36931 31868	2.8	8.72649 57265	−0.15	4.43452 02020	2.8	8.78142 85714
−0.10	6.07476 19048	2.9	9.27070 07007	−0.10	4.72000 00000	2.9	9.32512 82051
−0.05	7.09583 33333	3.0	9.83669 17293	−0.05	5.13557 69231	3.0	9.89068 06807

E. Values of the Jacobian constant on the x axis, $C = 2\Omega(x, 0)$, for $\mu = 0.4$ and $\mu = 0.5$

$\mu = 0.4$				$\mu = 0.5$	
x	C	x	C	x	C
		0.00	4.57333 33333	0.00	4.25000 00000
−3.0	9.92627 45098	0.05	4.90184 06593	0.05	4.29290 40404
−2.9	9.36146 24506	0.10	5.39285 71429	0.10	4.42666 66667
−2.8	8.81863 63636	0.15	6.12916 66667	0.15	4.66810 43956
−2.7	8.29804 91551	0.20	7.28000 00000	0.20	5.05190 47619
−2.6	7.80000 00000	0.25	9.24367 64706	0.25	5.64583 33333
−2.5	7.32484 57350	0.30	13.21888 88889	0.30	6.59000 00000
−2.4	6.87301 58730	0.35	25.20460 52632	0.35	8.21563 72549
−2.3	6.44503 26797	0.40	∞	0.40	11.52111 11111
−2.2	6.04153 84615	0.45	25.20440 47619	0.45	21.50513 15789
−2.1	5.66333 33333	0.50	13.21727 27273	0.50	∞
−2.0	5.31142 85714	0.55	9.23815 21739	0.55	21.50488 09524
−1.9	4.98712 37458	0.60	7.26666 66667	0.60	11.51909 09091
−1.8	4.69212 12121	0.65	6.10250 00000	0.65	8.20873 18841
−1.7	4.42870 12987	0.70	5.34538 46154	0.70	6.57333 33333
−1.6	4.20000 00000	0.75	4.82366 40212	0.75	5.61250 00000
−1.5	4.01046 78363	0.80	4.45142 85714	0.80	4.99256 41026
−1.4	3.86666 66667	0.85	4.18089 08046	0.85	4.57038 35979
−1.3	3.77873 94958	0.90	3.98333 33333	0.90	4.27428 57143
−1.2	3.76333 33333	0.95	3.84044 72141	0.95	4.06437 73946
−1.1	3.85000 00000	1.00	3.74000 00000	1.00	3.91666 66667
−1.00	4.09714 28571	1.1	3.63487 39496	1.1	3.75166 66667
−0.95	4.31710 31746	1.2	3.62444 44444	1.2	3.70680 67227
−0.90	4.63974 35897	1.3	3.68438 59649	1.3	3.74555 55556
−0.85	5.12250 00000	1.4	3.80000 00000	1.4	3.84742 69006
−0.80	5.88000 00000	1.5	3.96186 14719	1.5	4.00000 00000
−0.75	7.17931 15942	1.6	4.16363 63636	1.6	4.19528 13853
−0.70	9.82090 90909	1.7	4.40090 30100	1.7	4.42787 87879
−0.65	17.80535 71429	1.8	4.67047 61905	1.8	4.69401 33779
−0.60	∞	1.9	4.97000 00000	1.9	4.99095 23810
−0.55	17.80565 78947	2.0	5.29769 23077	2.0	5.31666 66667
−0.50	9.82333 33333	2.1	5.65217 86492	2.1	5.66961 53846
−0.45	7.18759 80392	2.2	6.03238 09524	2.2	6.04860 56645
−0.40	5.90000 00000	2.3	6.43744 10163	2.3	6.45269 84127
−0.35	5.16250 00000	2.4	6.86666 66667	2.4	6.88114 33757
−0.30	4.71095 23810	2.5	7.31949 30876	2.5	7.33333 33333
−0.25	4.43436 81319	2.6	7.79545 45455	2.6	7.80877 11214
−0.20	4.28000 00000	2.7	8.29416 33729	2.7	8.30704 54545
−0.15	4.22209 59596	2.8	8.81529 41176	2.8	8.82781 29117
−0.10	4.25000 00000	2.9	9.35857 14286	2.9	9.37078 43137
−0.05	4.36371 21212	3.0	9.92376 06838	3.0	9.93571 42857

Chapter **5**

Motion near the Equilibrium Points

5.1 Introduction

Particles placed at the five equilibrium points are stationary; these five points represent five particular solutions of the restricted problem. These Lagrangian points can be obtained by searching for solutions of the form $x =$ const, $y =$ const. The differential equations of motion require for this case that $\Omega_x = \Omega_y = 0$, which relations form the basis of the analysis presented in Chapter 4.

From a physical point of view the five equilibrium solutions represent points where the forces acting on the third body in the rotating system are balanced. Since the solutions are stationary, there is no motion relative to the rotating system, and only the gravitational and the centrifugal forces are to be considered. Drawing simple figures, also called free-body diagrams, for the equilateral and collinear configurations, the reader can verify some of the results of the previous chapter, without any information about the intricacies of the singularities of the manifold of the state of motion.

Not only are these equilibrium points solutions of the equations of motion but near these points other families of solutions exist. The present chapter discusses these solutions.

The existence of the libration points is generally considered an example of the importance of mathematical predictions. Lagrange showed in 1772 the existence of such solutions for the restricted problem formed by the sun-Jupiter system, and the discovery of their physical realization, the Trojan group of asteroids, began only in 1906 with the first-seen member of this group, 588 Achilles. The list of the fourteen Trojan asteroids which have been discovered and whose orbital elements have been computed and published as of 1965 is given in Section 5.6.1 in Table I.

In the 1960's, with increased interest in space explorations, the question of the existence of such points with respect to other primaries, especially for the earth-moon system, arises quite naturally. It is interesting to remark, in advance of any of the detailed results of this chapter, that Lagrange's discovery regarding the triangular libration points is generally applicable in the solar system. All the planets, when forming restricted problems of three bodies with the sun, may produce stable stationary solutions and so does the earth-moon system. The discovery of the physical realization of many of these solutions is still being awaited. The astronomical interest in these solutions is coupled with other possibilities. If there are stable stationary solutions for various primary combinations, from a practical point of view, then placing observational platforms at these points becomes feasible.

There are but few chapters in a work on the restricted problem which assume higher significance and more general interest than the chapter on the libration points. The nonlinear aspects of the problem appeal to the mathematician, the potentialities of space research which open up by placing instruments at the libration points offer new avenues for the physicist, and the station-keeping problems fascinate the guidance engineer.

The treatment which follows consists of four major parts. After the general ideas are outlined regarding the stability of linear and nonlinear systems in Section 5.2, the linearized equations are solved and motion and stability in the neighborhood of the equilibrium points are discussed in Sections 5.3 and 5.4. The nonlinear extension of these results is given in Section 5.5. The fourth and last part deals with applications, in Section 5.6.

5.2 Stability of linear and nonlinear systems

This introductory section outlines some of the mathematical ideas and theorems which will be needed. The linear and nonlinear problems are centered on the behavior of the dynamical system at and in the neighborhood of the equilibrium solutions.

The process of linearization presents no difficulty at the libration points, neither does the analysis of the stability of the linear system lead to any complications, in principle. The well-known theory of the stability of these linear differential equations is reviewed shortly in order to make

their application to the nonlinear system more precise. The interpretation of Liapunov's ideas and the work by LaSalle and Lefschetz, Cesari, and Coddington and Levinson is followed in parts and the anticipated nonlinear problems are discussed.

Section 5.2.1 introduces the basic and general ideas of stability and Section 5.2.2 is devoted to the problems of stability of the equilibrium solutions. The specific equations of the restricted problem are linearized in the neighborhood of the libration points in Section 5.2.3.

5.2.1 Liapunov's and orbital stability

The geometrical picture of the problem of stability of orbits may be formulated following Synge. Figure 5.1 shows the original, A, and the disturbed, A', orbits, or in general any two adjacent orbits we wish to compare and correlate.

The vector representing the deviation at time t is $\boldsymbol{\rho}$ if P and P' show the locations of the particle at the same time on the two orbits A and A'. This is an *isochronous correspondence* between the two orbits, with the distance between the corresponding points being

$$|\boldsymbol{\rho}| = PP'.$$

The vector $\boldsymbol{\xi}$ is the normal deviation. The distance

$$|\boldsymbol{\xi}| = PQ'$$

is not the distance between points corresponding to the same time. We refer to a *normal correspondence* when using $\boldsymbol{\xi}$.

The relation between these two descriptions is

$$\boldsymbol{\rho} = \boldsymbol{\xi} + \vartheta \mathbf{v},$$

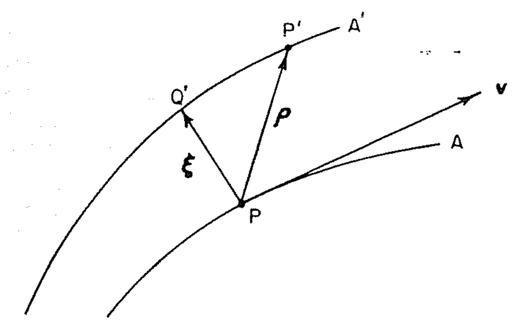

FIG. 5.1. Isochronous and normal correspondence.

where $\mathbf{v}$ is the velocity vector of the particle at point P and

$$\vartheta = \frac{\boldsymbol{\rho} \cdot \mathbf{v}}{|\mathbf{v}|^2}.$$

This last equation is obtained by forming the scalar product of $\mathbf{v}$ with both sides of the previous equation, remembering that

$$\mathbf{v} \cdot \boldsymbol{\xi} = 0.$$

The infinitesimal isochronous vector $\boldsymbol{\rho}$ satisfies the equation of deviation

$$\frac{\delta^2 \rho^i}{\delta t^2} + R^i_{jkl} v^j \rho^k v^l = Q^i_{,j} \rho^j,$$

where R is the curvature tensor given by

$$R^i_{jkl} = \frac{\partial}{\partial q^k} \left\{ {i \atop jl} \right\} - \frac{\partial}{\partial q^l} \left\{ {i \atop jk} \right\}$$
$$+ \left\{ {m \atop jl} \right\} \left\{ {i \atop mk} \right\} - \left\{ {m \atop jk} \right\} \left\{ {i \atop ml} \right\},$$

and $Q^i_{,j}$ is the covariant derivative of the contravariant force Q^i with respect to the coordinate q^j:

$$Q^i_{,j} = \frac{\partial Q^i}{\partial q^j} + \left\{ {i \atop kj} \right\} Q^k.$$

The notation used for Christoffel's symbol is

$$\left\{ {i \atop jk} \right\} = \frac{g^{il}}{2} \left(\frac{\partial g_{jl}}{\partial q^k} + \frac{\partial g_{kl}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^l} \right),$$

where the metric tensor g_{ij} gives the kinematical line element ds as

$$ds^2 = 2T\, dt^2 = g_{ij}\, dq^i\, dq^j,$$

with the kinetic energy T.

The equation of deviation for the infinitesimal normal vector $\boldsymbol{\xi}$ is

$$\frac{\delta^2 \xi^i}{\delta t^2} + \frac{d^2 \vartheta}{dt^2} v^i + 2 \frac{d\vartheta}{dt} Q^i + R^i_{jkl} v^j \xi^k v^l = Q^i_{,j} \xi^j.$$

The concept of isochronous correspondence leads to *Liapunov's stability*, while normal correspondence is related to orbital stability.

The solution $\psi(t)$ defined for $t \geqslant t_0$ of

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t)$$

is stable (Liapunov) if, given any $\epsilon > 0$, there exists a $\vartheta > 0$ such that any solution $\varphi(t)$ satisfying

$$|\varphi(t_0) - \psi(t_0)| < \delta$$

satisfies

$$|\varphi(t) - \psi(t)| < \epsilon$$

for $t > t_0$.

The difference $\varphi - \psi$ corresponds to the vector $\boldsymbol{\rho}$ in Fig. 5.1 since Liapunov's stability is based on an isochronous evaluation of the deviations. Note also that this definition of stability is the same as requiring uniform continuity of $\psi(t)$ with respect to the initial position $\psi(t_0)$.

The solution $\psi(t)$ is asymptotically stable if

$$|\varphi(t) - \psi(t)| \to 0$$

as $t \to \infty$.

We speak about conditional stability when the stability conditions are satisfied only by the solutions of a given manifold. The concept of complete stability is associated with asymptotic stability in the large; consequently, linear systems which are asymptotically stable enjoy complete stability.

The condition for *orbital stability* is associated with the normal correspondence as follows.

A periodic solution $\mathbf{p}(t)$ is orbitally stable if given any $\epsilon > 0$ there exists a $\delta > 0$ such that any solution $\varphi(t)$ for which

$$|\varphi(t_1) - \mathbf{p}(t_0)| < \delta$$

satisfies

$$|\varphi(t) - \mathbf{p}(t + c)| < \epsilon$$

for $t > t_0$.

Asymptotic stability is obtained when the last inequality is replaced by

$$|\varphi(t) - \mathbf{p}(t + c)| \to 0$$

as $t \to \infty$. In other words the periodic orbit possesses asymptotic orbital stability if an orbit which comes near to a point of the periodic orbit tends to it as $t \to \infty$.

5.2.2 Stability of equilibrium solutions

5.2.2.1 *Definition of the equilibrium solutions and of their stability.* If a dynamical system is in a state of equilibrium, it remains in that state as $t \to \infty$, from a mathematical point of view. A real system, on the other hand, is subject to disturbances; therefore, it is of considerable practical interest to inquire as to its response to these disturbances. Stability of the dynamical system describes its behavior to the disturbances. The motion which remains in the small neighborhood of the equilibrium point after it is disturbed is termed stable. A more precise definition is given after introducing the concept of an equilibrium solution.

Consider a system of ordinary differential equations,

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}),$$

which possesses an equilibrium point at $\mathbf{x} = \mathbf{a}$. Here the n-vector $\mathbf{x}$ has the components $x_1, \ldots, x_n$ and the components of the n-vector $\mathbf{X}$ are $X_1, \ldots, X_n$. Assume that any of the standard theorems of existence, such as the Cauchy–Lipschitz theorem, is satisfied.

A solution $\mathbf{x}(t)$ is called an equilibrium solution if the path in the $\mathbf{x}$ space is represented by a single point $\mathbf{a}$, where $\mathbf{x}(t_0) = \mathbf{a}$. This point is obtained from the equation $\mathbf{X}(\mathbf{x}) = 0$, that is, from $\dot{\mathbf{x}} = 0$, and it is called an equilibrium, stationary, or critical point.

The definition of a stable equilibrium solution (at the right, that is, for $t_0 \leqslant t < +\infty$) is as follows:

The solution $\mathbf{x} = \mathbf{a}$ or the point $\mathbf{a}$ is stable if for a given $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that, when the disturbances satisfy

$$|\,\mathbf{x}(t_0) - \mathbf{a}\,| \leqslant \delta,$$

then, for all $t > t_0$,

$$|\,\mathbf{x}(t) - \mathbf{a}\,| < \epsilon.$$

Otherwise the equilibrium solution $\mathbf{x} = \mathbf{a}$ is unstable. When the disturbed solution approaches the equilibrium point as $t \to \infty$ the solution is stable; in fact, it is called asymptotically stable. The formal condition of asymptotic stability is

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{a}.$$

Asymptotic stability is often not distinguished from plain stability; a regrettable negligence since using Cesari's pun, "plain stability is very labile."

The equilibrium solution is called completely stable (as in the general case) when it is asymptotically stable in the large, that is, in the domain where such initial conditions are located which result in asymptotic stability.

5.2.2.2 *Linearization and stability of the linear system.* Consider again the autonomous system of differential equations

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}).$$

If this equation may be written in the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{f}(\mathbf{x}),$$

then the linearized system of equations is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}.$$

Here $\mathbf{A}$ is a constant matrix and $\mathbf{f}(\mathbf{x})$ is a vector function such that

$$\frac{|\,\mathbf{f}(\mathbf{x})|}{|\,\mathbf{x}\,|} \to 0 \quad \text{as} \quad |\,\mathbf{x}\,| \to 0$$

for $t \geqslant 0$.

This requirement on $\mathbf{f}(\mathbf{x})$ is satisfied, for instance, when the components of $\mathbf{f}(\mathbf{x})$ are convergent power series beginning with second-order terms in $\mathbf{x}$.

Another notation is

$$|\,\mathbf{f}(\mathbf{x})| = \mathcal{O}(|\,\mathbf{x}\,|) \quad \text{as} \quad |\,\mathbf{x}\,| \to 0,$$

or there exists a function $\eta(\sigma) > 0$, $\eta(\sigma) \to 0$ as $\sigma \to 0$, such that

$$|\,\mathbf{f}(\mathbf{x})| \leqslant |\,\mathbf{x}\,|\,\eta(|\,\mathbf{x}\,|).$$

The stability properties for $t \geqslant t_0$ of the linearized system may be stated simply:

(A) For complex roots of the characteristic equation of $\mathbf{A}$ we have the following properties:

(a) When the characteristic roots all have negative real parts the equilibrium point is asymptotically stable. This is true also when some of the roots are multiple.

(b) When some or all of the characteristic roots have positive real parts the equilibrium point is unstable. This is true also when some of the roots are multiple.

(B) For *pure imaginary* roots the motion is oscillatory and the solution is stable though it is not asymptotically stable. If there are multiple roots the solution contains mixed (periodic and secular) terms and the equilibrium point is unstable.

(C) If the roots are real and all negative the solution is stable; if any of the roots are positive the point is unstable. These statements are also true for multiple roots.

A dynamical system with two degrees of freedom is described in the four-dimensional phase space. The dimensionality of this description may be reduced to three by studying the three-dimensional subspace of any isoenergetic hypersurface. A dynamical system with a single degree of freedom, on the other hand, may be represented in the phase space of two dimensions. Such a description leads to the classification of equilibrium points as follows. When both characteristic roots are real and of the same sign the point is called a node, which is stable or unstable according to whether the roots are negative or positive. When one root is positive and the other negative we have instability at the saddle point. If the roots are purely imaginary, the stable point of equilibrium is called a vortex. Finally if two roots are conjugate complex with negative (positive) real parts, then the stable (unstable) point is called a focus or spiral point.

We note that asymptotic stability of a linear system is always in the large since when it exists it exists for any initial condition. This means that an asymptotically stable critical point of a linear system is always completely stable. This is certainly not true for a nonlinear system where the size of the region of complete stability must be determined from the nonlinearities.

5.2.2.3 *Stability of the nonlinear system.* Under certain conditions the stability of a nonlinear system may be decided using the results obtained from the stability investigation of the linearized system. In other words, the characteristic roots of the linearized system may be used to study the stability of the nonlinear system.

By means of Liapunov's second method it may be shown what often is called Perron's theorem. A sufficient condition for an equilibrium point of the nonlinear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{f}(\mathbf{x})$$

to be asymptotically stable is that the characteristic roots of the linearized system all have negative real parts. If any of the real parts is positive, the equilibrium point is unstable. If some of the characteristic roots are zeros or pure imaginary, we are concerned with the so-called critical cases, and predictions as to the stability condition of the nonlinear system, from investigation of the linear part, may not be made directly.

The case of pure imaginary roots is of great practical interest. The terminology "infinitesimally stable" to describe the stability condition of the nonlinear system is misleading but probably still far better than the erroneous transfer of the stability conditions from the linear to the nonlinear system.

If k of the n roots have negative real parts and $n - k$ have real parts positive, then the solution is unstable but it is conditionally asymptotically stable.

If the linearized system given in the previous section is described by a matrix $\mathbf{A}$, elements of which are periodic, then the preceding results are still valid, and so the system is asymptotically stable if all the characteristic exponents have negative real parts. The system is unstable if at least one of the characteristic exponents has its real part positive.

The problem of stability of the equilibrium points is connected with a constant matrix $\mathbf{A}$, while the question of stability of periodic orbits around the equilibrium points necessitates the study of a periodic matrix $\mathbf{A}(t)$.

The potential energy of the dynamical system may also be used to establish a condition for stability; it has been known since Lagrange that when the potential energy has an isolated minimum at an equilibrium point then this is a stable point. If at a critical point the potential energy is not a minimum, the point is unstable (under some restrictions).

Other definitions of stability will not receive systematic reviews here, nevertheless, several may be mentioned. Some of these will be discussed in Chapter 8 since their application to the theory of periodic motions is more significant than to equilibrium solutions.

We begin with Hagihara's formulation of the problem of stability in celestial mechanics: "What is the interval of time at the end of which the solar system deviates from the present configuration by a previously assigned small amount?" The reader is not surprised to learn that present-day mathematics hardly enables us to answer this global question since much simpler questions of stability remain unanswered.

Poisson's stability requires that the system return infinitely often arbitrarily near to its original position, the intervening oscillation being of any magnitude. This and Poincaré's and Birkhoff's related recurrence and cycle theorems (see Section 8.4) exclude, for instance, secular perturbations in the mean distances of the planets from the sun if the system is to be stable.

Another point of view is that a stable system is confined to a certain region of the phase space or of the configuration space. Consequently

Hill's demonstration of the moon's orbit being inside of its closed curve of zero velocity indicates stability.

Stability of a system of n bodies according to Laplace is associated with no escapes and no collisions. Secularly stable systems are those which have oscillatory solutions and consequently pure imaginary roots of the characteristic equations plus damping which reduces the oscillations. To these we may add the definition of stability of equilibrium points or of steady motions by means of impulsive disturbances given by Klein and Sommerfield, Birkhoff's trigonometric and perturbative stability (the latter corresponding to Liapunov's stability for linear systems with constant coefficients, and to complete stability in Liapunov's sense if the system is stable to all orders), and the fashionable problems of structural stability.

The importance of nonlinear stability investigations cannot be overemphasized. Many examples exist in the literature in which the necessity of investigations in the nonlinear range has been made abundantly clear.

Lefschetz claims that in determining "practical" stability one must be concerned with complete stability or asymptotic stability in the large. This is a trivial problem for the linear system and a major problem for the nonlinear case. Another point to be concerned with is the relation of boundedness to stability; two independent concepts for nonlinear systems but closely related for linear systems.

5.2.3 The variational equations

The two second-order differential equations of motion

$$\ddot{x} - 2\dot{y} = \partial\Omega/\partial x$$

and

$$\ddot{y} + 2\dot{x} = \partial\Omega/\partial y$$

may be written as a system of four first-order equations when the following variables are introduced:

$$x_1 = x, \qquad x_3 = \dot{x},$$
$$x_2 = y, \qquad x_4 = \dot{y}.$$

The new system is

$$\dot{x}_1 = x_3, \qquad \dot{x}_2 = x_4,$$
$$\dot{x}_3 = 2x_4 + \frac{\partial\Omega(x_1, x_2)}{\partial x_1}, \qquad \dot{x}_4 = -2x_3 + \frac{\partial\Omega(x_1, x_2)}{\partial x_2},$$

corresponding to the equation

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$$

of the previous section (Section 5.2.2).

The equilibrium solutions are obtained by solving the following four equations for the components of the four-vector $\mathbf{x}$:

$$x_3 = 0, \qquad x_4 = 0,$$
$$2x_4 + \frac{\partial\Omega(x_1, x_2)}{\partial x_1} = 0, \qquad -2x_3 + \frac{\partial\Omega(x_1, x_2)}{\partial x_2} = 0.$$

The solutions are of course the five equilibrium points, since the first two equations give zeros for the velocity components

$$x_3 = \dot{x} = 0 \quad \text{and} \quad x_4 = \dot{y} = 0,$$

and the last two equations give zeros for the components of the gradient,

$$\partial\Omega/\partial x = \partial\Omega/\partial y = 0.$$

In order to study the motion near any of the equilibrium points $L(a, b)$ we write

$$x = a + \xi \quad \text{and} \quad y = b + \eta, \tag{1}$$

where ξ and η are coordinates relative to L. For the collinear points $L_{1,2,3}$ we have $a = x_{1,2,3}$, $b = 0$, and the coordinates of the triangular equilibrium points $L_{4,5}$ are $a = \mu - \frac{1}{2}$, $b = \pm 3^{1/2}/2$.

The function Ω may be expanded around L, giving

$$\Omega = \Omega(a, b) + \Omega_x(a, b)\,\xi + \Omega_y(a, b)\,\eta + \frac{1}{2!}\Omega_{xx}(a, b)\,\xi^2 + \Omega_{xy}(a, b)\,\xi\eta$$
$$+ \frac{1}{2!}\Omega_{yy}(a, b)\,\eta^2 + \mathcal{O}(3). \tag{2}$$

The differential equations of motion

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x, \\ \ddot{y} + 2\dot{x} &= \Omega_y \end{aligned} \tag{3}$$

become

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \Omega_{xx}(a, b)\,\xi + \Omega_{xy}(a, b)\,\eta + \mathcal{O}(2), \\ \ddot{\eta} + 2\dot{\xi} &= \Omega_{xy}(a, b)\,\xi + \Omega_{yy}(a, b)\,\eta + \mathcal{O}(2). \end{aligned} \tag{4}$$

In Eqs. (2) and (4) the symbols $\mathcal{O}(2)$ and $\mathcal{O}(3)$ stand for second- and higher-order and third- and higher-order terms in ξ and η. If these are omitted Eqs. (4) become the *variational equations* (more precisely the linear variational equations) and ξ and η are called the variations. Note that Eqs. (4) are obtained from Eqs. (3) using Eq. (2), and the fact that, since $L(a, b)$ is an equilibrium point, $\Omega_x(a, b) = \Omega_y(a, b) = 0$.

The characteristic roots of (4) are obtained from

$$\begin{vmatrix} \lambda^2 - \Omega^0_{xx} & -2\lambda - \Omega_{xy} \\ 2\lambda - \Omega^0_{xy} & \lambda^2 - \Omega^0_{yy} \end{vmatrix} = 0, \tag{5}$$

where the partial derivatives are to be computed at the equilibrium point,

$$\Omega^0_{\alpha\beta} = \Omega_{\alpha\beta}(a, b),$$

and α and β stand for x or y.

The fourth-order equation for λ is

$$\lambda^4 + (4 - \Omega^0_{xx} - \Omega^0_{yy})\,\lambda^2 + \Omega^0_{xx}\Omega^0_{yy} - (\Omega^0_{xy})^2 = 0. \tag{6}$$

The two second-order differential equations (4) may be transformed into a set of first-order equations by writing

$$x_1 = \xi, \qquad x_3 = \dot{\xi},$$
$$x_2 = \eta, \qquad x_4 = \dot{\eta}.$$

In this way the variational equations become

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$$

where the matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{B} & \mathbf{C} \end{pmatrix},$$

with

$$\mathbf{B} = \begin{pmatrix} \Omega^0_{xx} & \Omega^0_{xy} \\ \Omega^0_{xy} & \Omega^0_{yy} \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

5.3 Motion around the collinear points

It was shown in Section 4.6.2 that at the three collinear equilibrium points for $0 < \mu < \frac{1}{2}$

$$\Omega_{xy} = 0, \qquad \Omega_{xx} > 0, \qquad \Omega_{yy} < 0.$$

Consequently,

$$\Omega_{xx}\Omega_{yy} - \Omega^2_{xy} < 0.$$

The characteristic equation (6) may be written in the form

$$\Lambda^2 + 2\beta_1\Lambda - \beta_2^2 = 0, \tag{7}$$

where

$$\beta_1 = 2 - \frac{\Omega^0_{xx} + \Omega^0_{yy}}{2},$$
$$\beta_2^2 = -\Omega^0_{xx}\Omega^0_{yy} > 0, \tag{8}$$

and

$$\lambda = \pm\Lambda^{1/2}$$

Note that the two (real) roots of Eq. (7) are of opposite sign:

$$\Lambda_1 = -\beta_1 + (\beta_1^2 + \beta_2^2)^{1/2} > 0,$$
$$\Lambda_2 = -\beta_1 - (\beta_1^2 + \beta_2^2)^{1/2} < 0. \tag{9}$$

From these

$$\lambda_{1,2} = \pm\Lambda_1^{1/2}$$

and

$$\lambda_{3,4} = \pm\Lambda_2^{1/2}.$$

Consequently λ_1 and λ_2 are real and λ_3 and λ_4 are imaginary.

Since the solution of the variational equations is of the form

$$\xi = \sum_{i=1}^{4} A_i\, e^{\lambda_i t},$$
$$\eta = \sum_{i=1}^{4} B_i\, e^{\lambda_i t}, \tag{10}$$

in general there will always be one term which gives unbounded values for ξ and η as $t \to \infty$ (corresponding to $\lambda_1 > 0$). The solution is unstable.

The coefficients $A_1, \ldots, A_4$, $B_1, \ldots, B_4$ are not independent since from Eq. (5) we have

$$(\lambda_i^2 - \Omega^0_{xx})\, A_i = (2\lambda_i + \Omega^0_{xy})\, B_i, \tag{11}$$

and consequently the four initial conditions associated with Eqs. (4), say

$\xi(t_0)$, $\eta(t_0)$, $\dot\xi(t_0)$, and $\dot\eta(t_0)$, will completely determine the eight coefficients (A_i, B_i). In fact, at the collinear points Ω^0_{xy} being zero we have

$$B_i = \frac{\lambda_i^2 - \Omega^0_{xx}}{2\lambda_i} A_i = \alpha_i A_i, \tag{12}$$

and so

$$\begin{aligned} \xi_0 &= \xi(t_0) = \sum A_i\, e^{\lambda_i t_0}, \\ \dot\xi_0 &= \dot\xi(t_0) = \sum \lambda_i A_i\, e^{\lambda_i t_0}, \\ \eta_0 &= \eta(t_0) = \sum \alpha_i A_i\, e^{\lambda_i t_0}, \\ \dot\eta_0 &= \dot\eta(t_0) = \sum \alpha_i \lambda_i A_i\, e^{\lambda_i t_0}. \end{aligned} \tag{13}$$

The coefficients may be expressed as functions of the initial conditions since the determinant Δ of the system (13) is not zero. To show this compute

$$\Delta = e^{(\lambda_1 + \cdots + \lambda_4)t_0} \begin{vmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_1\lambda_1 & \alpha_2\lambda_2 & \alpha_3\lambda_3 & \alpha_4\lambda_4 \end{vmatrix} \tag{14}$$

or since

$$\sum \lambda_i = 0, \quad \lambda_2 = -\lambda_1, \quad \lambda_4 = -\lambda_3, \quad \alpha_2 = -\alpha_1, \quad \alpha_4 = -\alpha_3,$$

we have

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & -\lambda_1 & \lambda_3 & -\lambda_3 \\ \alpha_1 & -\alpha_1 & \alpha_3 & -\alpha_3 \\ \alpha_1\lambda_1 & \alpha_1\lambda_1 & \alpha_3\lambda_3 & \alpha_3\lambda_3 \end{vmatrix} = -\left(\frac{\Omega^0_{xx}}{\Omega^0_{yy}}\right)^{1/2} (\lambda_1^2 - \lambda_3^2)^2. \tag{15}$$

From this it follows that $\Delta \neq 0$ since, as we have seen in Chapter 4, $\Omega^0_{xx} \neq 0$ and $\lambda_1^2 = \Lambda_1 \neq \Lambda_2 = \lambda_3^2$. The latter becomes clear considering Eq. (9), according to which $\Lambda_1 = \Lambda_2$ if $\beta_1^2 + \beta_2^2 = 0$, but β_2 and β_1 are real and $\beta_2 \neq 0$ (for $\mu \neq 0$), consequently $\Lambda_1 \neq \Lambda_2$.

The inversion of Eqs. (13) gives the coefficients

$$\begin{aligned} A_1 &= \frac{e^{-\lambda_1 t_0}}{\lambda_1^2 - \lambda_3^2}\,[-\xi_0\alpha_3\lambda_3 - \dot\xi_0\alpha_3\delta + \eta_0\lambda_3\delta + \dot\eta_0], \\ A_2 &= \frac{e^{\lambda_1 t_0}}{\lambda_1^2 - \lambda_3^2}\,[-\xi_0\alpha_3\lambda_3 + \dot\xi_0\alpha_3\delta - \eta_0\lambda_3\delta + \dot\eta_0], \\ A_3 &= \frac{e^{-\lambda_3 t_0}}{\lambda_1^2 - \lambda_3^2}\,[\xi_0\alpha_1\lambda_1 + \dot\xi_0\alpha_1\delta - \eta_0\lambda_1\delta - \dot\eta_0], \\ A_4 &= \frac{e^{\lambda_3 t_0}}{\lambda_1^2 - \lambda_3^2}\,[\xi_0\alpha_1\lambda_1 - \dot\xi_0\alpha_1\delta + \eta_0\lambda_1\delta - \dot\eta_0], \end{aligned} \tag{16}$$

where

$$\delta = (\Omega^0_{yy}/\Omega^0_{xx})^{1/2}. \tag{17}$$

The coefficients A_1 and A_2 are associated with the real exponents (λ_1 and λ_2); consequently, the first two terms on the right side of Eqs. (10) in the solution represent exponential increase and decay with time. Selecting initial conditions such that $A_1 = A_2 = 0$, a particular solution containing only sine and cosine functions of the time is obtained. This solution will contain only two arbitrary constants, A_3 and A_4, and so the four initial conditions cannot be chosen arbitrarily.

When $A_1 = A_2 = 0$ we have, from Eqs. (10),

$$\begin{aligned} \xi &= A_3\, e^{\lambda_3 t} + A_4\, e^{-\lambda_3 t}, \\ \eta &= A_3\alpha_3\, e^{\lambda_3 t} - A_4\alpha_3\, e^{-\lambda_3 t}. \end{aligned} \tag{18}$$

Writing ξ_0 for $\xi(t_0)$ and η_0 for $\eta(t_0)$ as before, we may evaluate A_3 and A_4 as functions of λ_3, t_0, α_3, ξ_0, and η_0. Substituting these results back into Eqs. (18) we obtain

$$\begin{aligned} \xi &= \xi_0 \cos s(t - t_0) + (\eta_0/\beta_3) \sin s(t - t_0), \\ \eta &= \eta_0 \cos s(t - t_0) - \beta_3\xi_0 \sin s(t - t_0), \end{aligned} \tag{19}$$

where the real quantities s and β_3 are defined by

$$s = +[\beta_1 + (\beta_1^2 + \beta_2^2)^{1/2}]^{1/2} \quad \text{or} \quad \lambda_3 = is, \tag{20}$$

and

$$\beta_3 = \frac{s^2 + \Omega^0_{xx}}{2s} \quad \text{or} \quad \alpha_3 = i\beta_3. \tag{21}$$

From Eqs. (19) one obtains after some simple computations that

$$\dot\xi_0 = \eta_0 s/\beta_3 \quad \text{and} \quad \dot\eta_0 = -\beta_3\xi_0 s, \tag{22}$$

which means that once the initial conditions ξ_0, η_0 are selected, the corresponding initial velocities cannot be chosen at will.

Example. Derive Eqs. (22) from Eqs. (16), when $A_1 = A_2 = 0$.

The orbit is an ellipse as can be seen when Eqs. (18) are solved for $A_3 e^{\lambda_3 t}$ and $A_4 e^{-\lambda_3 t}$. In this way the time can be eliminated, giving

$$\xi^2 + \eta^2/\beta_3^2 = \xi_0^2 + \eta_0^2/\beta_3^2. \tag{23}$$

Here $\beta_3^2 > 3$, which can be seen as follows. From Eq. (21)

$$\beta_3 = \tfrac{1}{2}(s + \Omega^0_{xx}/s),$$

where $\Omega^0_{xx} > 3$ and $s > 0$. Consequently

$$\beta_3 > \tfrac{1}{2}(s + 3/s).$$

The minimum of the right side is $3^{1/2}$ and it occurs at $s = 3^{1/2}$, and so $\beta_3 > 3^{1/2}$, and in this way our proposition is proved.

Since $\beta_3^2 > 1$, the semimajor axis of the orbit is parallel to the y axis and its semiminor axis is always the ξ (or x) axis. The center of the ellipse is located at the collinear libration point. The eccentricity of the orbit is $e = (1 - \beta_3^{-2})^{1/2}$. The motion is periodic with respect to the rotating frame of reference with the synodic period $T = 2\pi/s$.

The motion along the orbit is retrograde since at $\xi = \xi_0$, $\eta = \eta_0 = 0$, Eqs. (22) give $\dot{\xi}_0 = 0$ and $\dot{\eta}_0 = -\beta_3\xi_0 s$. Here $\beta_3 s > 0$, so sign $\dot{\eta}_0 = -$ sign ξ_0. Therefore for $\xi_0 \gtreqless 0$ we have $\dot{\eta}_0 \lesseqgtr 0$, where the upper signs refer to the right side of the collinear libration point and the lower signs to the left side. The velocity therefore is directed downward in the first case and upward in the second, as shown in Fig. 5.2.

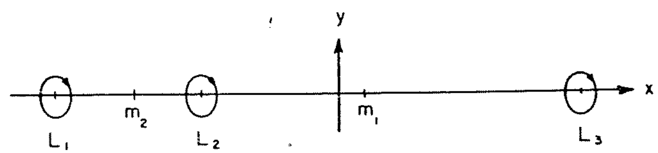

FIG. 5.2. Periodic orbits around the collinear points.

According to Section 4.6.2, at the first two collinear libration points $\Omega_{xx} = 9$ and $\Omega_{yy} = -3$ as the mass par eter approaches zero. Consequently at $\mu = 0$ we have

$$s = (2 \cdot 7^{1/2} - 1)^{1/2} = 2.07159\,42224,$$

$$\lambda_1 = (2 \cdot 7^{1/2} + 1)^{1/2} = 2.50828\,67902,$$

$$\beta_3 = (2 \cdot 7^{1/2} + 5)^{1/2} = 3.20803\,71915,$$

$$e = (2/3)^{1/2}(4 - 7^{1/2})^{1/2} = 0.95017\,49625.$$

At the third collinear point $\Omega_{xx} = 3$ and $\Omega_{yy} = 0$ as $\mu \to 0$, so $s = 1$, $\lambda_1 = 0$, $\beta_3 = 2$, and

$$e = 3^{1/2}/2 = 0.86602\,54038.$$

For the case of symmetry regarding the primaries, $\mu = \frac{1}{2}$, we have at the second collinear point $\Omega_{xx} = 17$ and $\Omega_{yy} = -7$. Consequently

$$s = (8 \cdot 2^{1/2} - 3)^{1/2} = 2.88335\,02214,$$

$$\lambda_1 = (8 \cdot 2^{1/2} + 3)^{1/2} = 3.78334\,62040,$$

$$\beta_3 = \left(\frac{67 + 48 \cdot 2^{1/2}}{7}\right)^{1/2} = 4.38963\,47228,$$

$$e = (2/119)(3357)^{1/2} = 0.97377\,51890.$$

The roots of the characteristic equation [Eq. (6)] are given in Appendix I. Sections A–F of Appendix I present the values of λ_1 and s for various values of μ at the three collinear libration points. Sections A–C contain information for $0 \leqslant \mu \leqslant 0.5$; Section D shows those values of μ which cluster around Jupiter's mass parameter (0.00095); Section E gives results clustering around the value of the mass ratio of the earth-moon system (0.012); and Section F contains some of the astronomically significant special cases. The four columns correspond to the values of μ, L_1, L_2, and L_3. There are two entries at every combination (μ, L_i), the first giving λ_1, the second the value of s. For instance, at $\mu = 0.1$ we have $\lambda_1 = 1.8094550539$ and $s = 1.6635459768$ at the first libration point (L_1). The corresponding value of $\Omega^0_{xx} = 6.0134847506$ can be obtained from Section B of Appendix I of Chapter 4. Computation of $\beta_3 = 2.63920267$ and $e = 0.92543667$ gives no difficulties. Assuming a semiminor axis of $\xi_0 = 0.01$, we have $\dot{\eta}_0 = -0.04390435$ and the length of the semimajor axis becomes $\beta_3\xi_0 = 0.02639203$. The synodic period is 3.77698326. The Jacobian constant for such a particle can be computed from

$$C = 2\Omega(x_0, 0) - \dot{\eta}_0^2,$$

where $x_0 = x_{L_1} + \xi_0 = -1.2496998$. This gives $2\Omega(x_0, 0) = 3.55730$ and $C = 3.55537$. Note that from Section B of Appendix I, Chapter 4, the value $C_1 = 3.556684$ corresponds to zero velocity at point L_1 for $\mu = 0.1$. The preceding value of $2\Omega(x_0, 0) = C^*$ represents the value of the Jacobian constant for a particle at x_0 with zero velocity. The curve of zero velocity going through this point $(x_0, 0)$ will correspond therefore to $C^* = 3.55730$. Since $C^* > C_1$, the zero velocity curve passing through the point $(x_0, 0)$ belongs to the dumbbell series, as expected. The Jacobian constant of the particle C, on the other hand, is smaller than either C^* or C_1, and the curve of zero velocity of the particle belongs to the horseshoe series since $C_1 > C > C_3 = 3.189578$. The orbit of the particle is in a zone of permissible motion between the two prongs of its horseshoe-shaped curve of zero velocity. In short $C^* > C_1 > C > C_3$ or $3.55730 > 3.556684 > 3.555370 > 3.189578$.

Example 1. Show that $s(L_3) < s(L_1) \leq s(L_2)$, where $s(L_i)$ is the mean motion, as defined by Eq. (20), at the collinear libration point L_i. The equality occurs at $\mu = \frac{1}{2}$.

Example 2. Find the value of the Jacobian constant C for a linearized orbit with initial conditions $\xi = \xi_0$, $\eta = \eta_0 = 0$, $\dot{\xi} = \dot{\xi}_0 = 0$, $\dot{\eta} = \dot{\eta}_0 = -\xi_0\beta_3 s$, and construct the curves $C = C(\xi_0)$ for L_1, L_2, L_3 with $\mu = 0.1$.

The influence of the real roots $(\lambda_{1,2})$ is to destroy the previously discussed periodic motions. The positive real root (λ_1) results in an unbounded motion while $\lambda_2 < 0$ will not influence stability since it produces a shrinking orbit. The selection $A_3 = A_4 = 0$ in Eqs. (10), for instance, gives

$$\begin{aligned} \xi &= \xi_0 \cosh \lambda_1(t - t_0) + \frac{\eta_0}{\alpha_1} \sinh \lambda_1(t - t_0), \\ \eta &= \eta_0 \cosh \lambda_1(t - t_0) + \xi_0\alpha_1 \sinh \lambda_1(t - t_0), \end{aligned} \tag{24}$$

which equations are similar to Eqs. (19). Elimination of the time shows that the orbits are hyperbolas with the axes parallel to the coordinate system ξ, η.

Combination of the cases giving solutions containing trigonometric and hyperbolic functions, simultaneously, may also be established.

It is essential to keep in mind that the results in this section follow from a linearization process and that those results which refer to unbounded motions require proper interpretations. The nonlinear aspects are discussed in Section 5.5; at this point only the limitations of the results of the linearization process are outlined:

(a) Inasmuch as one of the roots of the characteristic equation (6) is a real positive number for any $0 < \mu < \frac{1}{2}$, we may conclude that the collinear libration points constitute unstable solutions. This result is also valid when linearization is not performed and the inclusion of nonlinear terms will not alter the unstable character of these solutions. We shall return to this question in Section 5.5 in considerable detail.

The existence of initial conditions giving trigonometric functions as solutions [Eqs. (19)] means that the collinear libration points, while unstable, possess conditional stability in the linear sense.

(b) The unstable collinear libration points themselves represent solutions of the restricted problem in the exact sense, and these stationary solutions also satisfy the differential equations of the nonlinear problem. The solutions given in this section which represent motion in the *neighborhood* of the straight-line configurations are valid only for the linearized problem. The existence of periodic orbits, similar to those described by Eqs. (19), may be shown also in the nonlinear case for any value of μ by analytic continuation. In Section 5.5 the elliptic orbits given by Eqs. (19) are used as the basic solutions to generate periodic orbits around the collinear points. The initial conditions for these orbits in the linear case are ξ_0 and η_0 which, by means of Eqs. (22), immediately specify $\dot{\xi}_0$ and $\dot{\eta}_0$. The nonlinear case does not allow the derivation of such relations; in fact, we only know that Eqs. (22) do *not* hold when the higher-order terms in the function Ω are retained. Nevertheless, the set of initial conditions ξ_0, η_0, $\dot{\xi}_0$, and $\dot{\eta}_0$ given by Eqs. (22) furnish the starting point of an iteration which leads to the proper initial condition for the nonlinear case. The closer the initial point (ξ_0, η_0) is to the points L_i, or the smaller ξ_0 and η_0 are, the better a differential correction scheme will work to furnish the initial conditions desired for the establishment of periodic orbits in the nonlinear problem.

The previously described generation of three families of periodic orbits (around L_1, L_2, L_3) requires further comments since such families will encounter singular points. This circumstance may be seen when ellipse-like ovals are drawn around, say, L_1 with increasing dimension. As the orbits approach the primary located at m_2 collision occurs. When the size is further increased, loops appear around m_2. These orbits show little resemblance to the elliptic orbits with which the family was started. In Chapter 9, which treats the quantitative aspects of the restricted problem, this question will be discussed in further detail.

(c) The solutions of the linearized equations corresponding to the real characteristic root λ_1 may also be used to generate orbits for the nonlinear problem. These are hyperbolic orbits in the vicinity of the libration points. Orbits, which are approximately straight lines in the neighborhoods of the points L_i, may also be established as degenerate cases of hyperbolic arcs. Such orbits will also be discussed in Chapter 9.

5.4 Motion around the equilateral points

The characteristic equation (6) for the points L_4 and L_5 can be written immediately since the values of the second-order derivatives have been given in Section 4.6 as follows:

$$\Omega_{xx}(L_{4,5}) = \frac{3}{4}, \qquad \Omega_{xy}(L_4) = +\frac{3 \cdot 3^{1/2}}{2}(\mu - \tfrac{1}{2}),$$

$$\Omega_{xy}(L_5) = -\frac{3 \cdot 3^{1/2}}{2}(\mu - \tfrac{1}{2}), \qquad \Omega_{yy}(L_{4,5}) = \frac{9}{4}.$$

The characteristic equation therefore becomes

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1 - \mu) = 0. \tag{25}$$

Introducing again $\Lambda = \lambda^2$ as before, we have

$$\Lambda^2 + \Lambda + \frac{27}{4}\mu(1 - \mu) = 0, \tag{26}$$

the solution of which is

$$\Lambda_{1,2} = \tfrac{1}{2}\{-1 \pm [1 - 27\mu(1 - \mu)]^{1/2}\}. \tag{27}$$

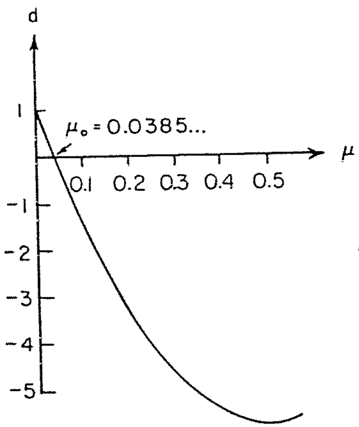

FIG. 5.3. Discriminant at the triangular equilibrium points.

Consequently the roots $\lambda_1 = +\Lambda_1^{1/2}$, $\lambda_2 = -\Lambda_1^{1/2}$, $\lambda_3 = +\Lambda_2^{1/2}$, and $\lambda_4 = -\Lambda_2^{1/2}$ depend in a simple manner on the value of the mass parameter. Three cases are distinguished regarding the sign of the discriminant which controls the nature of the roots. Figure 5.3 shows the discriminant $d = 1 - 27\mu(1 - \mu)$ and the critical value of the mass parameter. The roots of the equation $d = 0$ are

$$\mu_0 = \tfrac{1}{2}[1 - (69)^{1/2}/9]$$
$$= 0.03852\,08965\,04551...$$

and $\mu_1 = 1 - \mu_0$. Another expression is $\mu_0 = \frac{1}{2}[1 - (23/27)^{1/2}]$ with which we have $\gamma_0 = 1 - 2\mu_0 = (23/27)^{1/2}$.

In what follows the three regions of the values of μ will be treated separately. First, when $0 \leqslant \mu < \mu_0$ the motion is bounded and it is the superposition of two harmonic oscillations with different frequencies (see Section 5.4.1). We have stability in the linear sense. In the range $\mu_0 < \mu \leqslant 0.5$ the motion is unstable (see Section 5.4.2). At the value $\mu = \mu_0$ the solution contains secular terms (see Section 5.4.3).

5.4.1 Stable solutions of the linearized equations

In the first case $0 < 1 - 27\mu(1 - \mu) \leqslant 1$, or $\mu_0 > \mu \geqslant 0$. Consequently $-\frac{1}{2} < \Lambda_1 \leqslant 0$ and $-\frac{1}{2} > \Lambda_2 \geqslant -1$; that is, both values

of Λ are negative. The four roots of the characteristic equation are all pure imaginary:

$$\lambda_{1,2} = \pm i(-\Lambda_1)^{1/2} = \pm is_1 \quad \text{and} \quad \lambda_{3,4} = \pm i(-\Lambda_2)^{1/2} = \pm is_2 .$$

The solution is

$$\begin{aligned} \xi &= C_1 \cos s_1 t + S_1 \sin s_1 t + C_2 \cos s_2 t + S_2 \sin s_2 t, \\ \eta &= \bar{C}_1 \cos s_1 t + \bar{S}_1 \sin s_1 t + \bar{C}_2 \cos s_2 t + \bar{S}_2 \sin s_2 t, \end{aligned} \tag{28}$$

where

$$\begin{aligned} \bar{C}_i &= \Gamma_i(2s_i S_i - \Omega^0_{xy} C_i), \\ \bar{S}_i &= -\Gamma_i(2s_i C_i + \Omega^0_{xy} S_i), \end{aligned} \tag{29}$$

and

$$\Gamma_i = \frac{s_i^2 + \Omega^0_{xx}}{4s_i^2 + (\Omega^0_{xy})^2} = \frac{1}{s_i^2 + \Omega^0_{yy}} > 0. \tag{30}$$

Figure 5.4 shows the variation of the angular frequencies or mean motions s_1, s_2 with the mass parameter. For $\mu \cong 0$ we have

$$s_1 \cong \left(\frac{27}{4}\mu\right)^{1/2} = (6.75\mu)^{1/2} \quad \text{and} \quad s_2 \cong 1 - \frac{27}{8}\mu = 1 - 3.375\mu.$$

The curve $s_1(\mu)$ is perpendicular to the μ axis at $\mu = 0$, and both curves $s_1(\mu)$ and $s_2(\mu)$ possess infinite slopes at $\mu = \mu_0$, where $s_1(\mu_0) = s_2(\mu_0) = 1/2^{1/2}$. Note that $s_1 \leqslant s_2$.

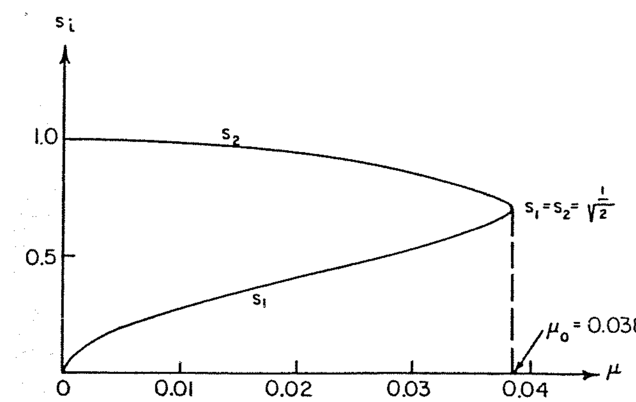

FIG. 5.4. Mean motions at the triangular libration points.

Section A of Appendix II gives the values s_1 and s_2 in the range $0 \leq \mu \leq \mu_0$. Sections B and C of Appendix II show s_1 and s_2 for values of μ clustering around the mass ratios of Jupiter-sun and moon-earth. Section D contains s_1 and s_2 for the astronomically interesting values of the mass parameter.

In the preceding solution [Eq. (28)] we recognize short- and long-period terms corresponding to large (s_2) and small (s_1) values of the mean motion s. Therefore, terms associated with s_2 are called "short-period terms" and the terms with the coefficients C_1, S_1, $\bar{C}_1$, and $\bar{S}_1$ are the long-period terms. For instance, for $\mu = 0.001$ we have $s_1 = 0.0824$ and $s_2 = 0.9966$, which correspond to periods of $T_1 = 2\pi/s_1 = 76.4$ and $T_2 = 2\pi/s_2 = 6.3$.

In fact for the small values of μ occurring in the planetary system (μ is of the order of 10^{-3} to 10^{-6}), s_1 is between 0.002 and 0.08. The corresponding long periods are between 12 and 500 times the period of the revolution of the primaries. The short period is approximately the same as the period of the revolution of the primaries since $s_2 \cong 1$.

At the critical value of $\mu = \mu_0$ the periods of the short- and long-period motions are equal. This critical period is approximately 41.4 % longer than the period of revolution of the primaries.

Either the short- or the long-period terms can be eliminated from the solution by properly selected initial conditions. The four initial conditions (ξ_0, η_0, $\dot{\xi}_0$, and $\dot{\eta}_0$ *at* $t = 0$) are linearly related to the four independent coefficients appearing in the solution. Therefore establishing the desired initial conditions presents no difficulty. For instance, if short-period terms are to be eliminated, the following four equations are to be satisfied: $C_2 = S_2 = \bar{C}_2 = \bar{S}_2 = 0$. From Eqs. (29) it can be seen that the last two of these four equations are satisfied when the first two are. The relations between the initial conditions and the coefficients of Eqs. (28) are

$$\begin{aligned} \xi_0 &= C_1 + C_2, & \eta_0 &= \bar{C}_1 + \bar{C}_2, \\ \dot{\xi}_0 &= S_1 s_1 + S_2 s_2, & \dot{\eta}_0 &= \bar{S}_1 s_1 + \bar{S}_2 s_2. \end{aligned} \tag{31}$$

Example. Show that the determinant of the system of linear Eqs. (31) for the determination of the $C_1, \ldots, \bar{S}_2$ coefficients from the given initial conditions $(\xi_0, \ldots, \dot{\eta}_0)$ is not zero if $\mu \neq \mu_0$.

The short-period terms will not occur in the solution if

$$\begin{aligned} \xi_0 &= C_1, & \eta_0 &= \bar{C}_1, \\ \dot{\xi}_0 &= S_1 s_1, & \dot{\eta}_0 &= \bar{S}_1 s_1. \end{aligned} \tag{32}$$

This means that for a given value of μ and for arbitrary selected values of ξ_0 and η_0 the first of Eqs. (29) furnishes S_1, and knowing S_1 the second of Eqs. (29) gives $\bar{S}_1$. In this way the four initial conditions are determined: ξ_0 and η_0 by arbitrary selection, $\dot{\xi}_0$ and $\dot{\eta}_0$ by computation using Eqs. (29) and (32). The result of this calculation is as follows:

$$\begin{aligned} \dot{\xi}_0 &= \tfrac{1}{2}(\xi_0 \Omega^0_{xy} + \eta_0/\Gamma_1), \\ \dot{\eta}_0 &= -\tfrac{1}{2}[\xi_0(s_1{}^2 + \Omega^0_{xx}) + \eta_0 \Omega^0_{xy}]. \end{aligned} \tag{33}$$

The orbit, after the elimination of either the short- *or* the long-period terms, becomes an ellipse which can be seen when Eqs. (28) are rewritten for the long-period solution:

$$\begin{aligned} \xi &= C_1 \cos s_1 t + S_1 \sin s_1 t, \\ \eta &= \bar{C}_1 \cos s_1 t + \bar{S}_1 \sin s_1 t, \end{aligned} \tag{34}$$

and when the time is eliminated. Multiplying the first of these equations by $\bar{C}_1$, the second equation by $(-C_1)$ and adding, we have

$$\xi \bar{C}_1 - \eta C_1 = (S_1 \bar{C}_1 - C_1 \bar{S}_1) \sin s_1 t. \tag{35}$$

Then multiplying the first of (34) by $\bar{S}_1$ and the second by (S_1) and again adding we obtain

$$\xi \bar{S}_1 - \eta S_1 = -(S_1 \bar{C}_1 - C_1 \bar{S}_1) \cos s_1 t. \tag{36}$$

After squaring and adding Eqs. (35) and (36), the trigonometric terms are eliminated and we obtain the desired result:

$$\xi^2(\bar{C}_1{}^2 + \bar{S}_1{}^2) + \eta^2(C_1{}^2 + S_1{}^2) - 2\xi\eta(C_1\bar{C}_1 + S_1\bar{S}_1) = (S_1\bar{C}_1 - C_1\bar{S}_1)^2 \tag{37}$$

or

$$\xi^2 \Gamma_1{}^2[4s_1{}^2 + (\Omega^0_{xy})^2] + \eta^2 + 2\xi\eta\Gamma_1\Omega^0_{xy} = 4s_1{}^2\Gamma_1{}^2(S_1{}^2 + C_1{}^2). \tag{38}$$

This is the equation of an ellipse, since

$$\begin{vmatrix} \Gamma_1{}^2[4s_1{}^2 + (\Omega^0_{xy})^2] & \Gamma_1\Omega^0_{xy} \\ \Gamma_1\Omega^0_{xy} & 1 \end{vmatrix} = 4s_1{}^2\Gamma_1{}^2 > 0. \tag{39}$$

The center of the ellipse is at the origin of the coordinate system ξ, η which coincides with L_4. The principal axes of the ellipse are rotated relative to the coordinate system ξ, η; the angle of rotation and the length of the minor and major axes may be found from Eq. (38). We delegate this exercise to an example and proceed in another way.

Observe that the equations of motion around the collinear points differ from those around the triangular points by the disappearance of the coupling terms on the right side of Eqs. (4). The reason for this is that $\Omega_{xy}(L_{1,2,3}) = 0$ and $\Omega_{xy}(L_{4,5}) \neq 0$. Comparison of Eq. (21) with (30) shows also the differences due to the term Ω_{xy}. The effect of Ω_{xy} can also be seen from Eq. (38), where this term is responsible for the rotation of the axes of the ellipse relative to the coordinate system ξ, η since Ω_{xy} appears as a factor of the bilinear term. These observations lead to the idea of rotating the coordinate system prior to writing the equations of motion, so that in the new system Ω_{xy} will not appear.

In other words, the expression for the function Ω around $L_{4,5}$ given by Eq. (2),

$$\Omega = \Omega^0 + \Omega^0_{xx}\frac{\xi^2}{2} + \Omega^0_{xy}\xi\eta + \Omega^0_{yy}\frac{\eta^2}{2} + \mathcal{O}(\xi^3, \eta^3), \tag{40}$$

is to be transformed into a system $\bar{\xi}, \bar{\eta}$, often called the normal coordinate system, where only quadratic terms appear without the bilinear $\xi\eta$ term. This will result in equations of motion similar in form to those which are applicable to the discussion of motion around the collinear points.

Such a transformation of coordinates has been performed in Chapter 4 when the curves of zero velocity were analyzed around L_4 and L_5. From Eq. (109) of Chapter 4 we have

$$\Omega = \frac{3}{8}\xi^2 + \frac{9}{8}\eta^2 - \frac{3 \cdot 3^{1/2}}{4}(1 - 2\mu)\xi\eta + \frac{3}{2}, \tag{41}$$

which corresponds to the preceding Eq. (40). The introduction of the variables $\bar{\xi}, \bar{\eta}$ by the transformation

$$\begin{aligned} \xi &= \bar{\xi}\cos\alpha - \bar{\eta}\sin\alpha, \\ \eta &= \bar{\xi}\sin\alpha + \bar{\eta}\cos\alpha \end{aligned} \tag{42}$$

is of course equivalent to a rotation of the coordinate system by the angle α. The new quadratic form becomes

$$\Omega = \frac{\lambda_2}{2}\bar{\xi}^2 + \frac{\lambda_1}{2}\bar{\eta}^2 + \frac{3}{2}, \tag{43}$$

where λ_1 and λ_2 are the roots of the characteristic equation

$$\lambda_{1,2} = \frac{3}{2}\{1 \pm [1 - 3\mu(1 - \mu)]^{1/2}\} \tag{44}$$

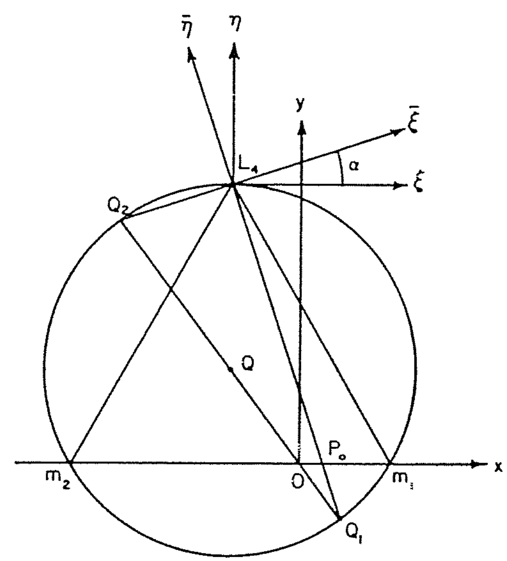

FIG. 5.5. Construction of the principal system of coordinates (Deprit, 1965, Ref. 39).

as given earlier by Eq. (111) in Chapter 4. The principal axes of the curves of zero velocity are oriented according to

$$\tan 2\alpha = 3^{1/2}(1 - 2\mu), \tag{45}$$

which equation corresponds to Eq. (113) of Chapter 4.

Figure 5.5 shows a simple construction of the principal directions of the curves of zero velocity or the normal coordinates of the linearized elliptic orbits. Point Q is the center of the equilateral triangle formed by m_1, m_2, and L_4. The intersections of the line $Q\mathcal{O}$ with the circle are points Q_1 and Q_2. The two principal directions, $\bar{\eta}$ and $\bar{\xi}$, are obtained by drawing the lines Q_1L_4 and Q_2L_4:

Example 1. Show that the $\bar{\eta}$ axis intersects the x axis at point $P_0(x_0, 0)$, where

$$x_0 = \frac{2\mu(1-\mu) - 1 + [1 - 3\mu(1-\mu)]^{1/2}}{1 - 2\mu},$$

and for small values of μ we have $x_0 = \mu/2 + \mathcal{O}(\mu^2)$; see Fig. 5.5.

Example 2. Verify the construction shown on Fig. 5.5.

The equations of motion become

$$\ddot{\bar{\xi}} - 2\dot{\bar{\eta}} = \bar{\lambda}_2 \bar{\xi}, \qquad \ddot{\bar{\eta}} + 2\dot{\bar{\xi}} = \bar{\lambda}_1 \bar{\eta}, \tag{46}$$

which equations are now of the same form as the equations of motion of the particle around the collinear points.

Equations (46) may be obtained in a variety of ways, depending on the individual taste of the reader. In what follows, a derivation using matrices is offered. Equations (42) may be written as $\boldsymbol{\zeta} = \mathbf{B}\bar{\boldsymbol{\zeta}}$, where the vectors $\zeta(\xi, \eta)$ and $\bar{\zeta}(\bar{\xi}, \bar{\eta})$ are connected by the rotation matrix $\mathbf{B}$. Equation (41) is simply $\Omega = \boldsymbol{\zeta}\mathbf{A}\boldsymbol{\zeta} + \frac{3}{2}$, where the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} \frac{3}{8} & \frac{-3 \cdot 3^{1/2}}{8}(1 - 2\mu) \\ \frac{-3 \cdot 3^{1/2}}{8}(1 - 2\mu) & \frac{9}{8} \end{pmatrix}. \tag{47}$$

With this notation the new quadratic form is

$$\Omega = \bar{\boldsymbol{\zeta}}\mathbf{B}^*\mathbf{A}\mathbf{B}\bar{\boldsymbol{\zeta}} + \tfrac{3}{2} = \bar{\boldsymbol{\zeta}}\mathbf{M}\bar{\boldsymbol{\zeta}} + \tfrac{3}{2}, \tag{48}$$

where $\mathbf{B}^*$ is the transpose of $\mathbf{B}$ and

$$\mathbf{M} = \begin{pmatrix} \bar{\lambda}_2/2 & 0 \\ 0 & \bar{\lambda}_1/2 \end{pmatrix}. \tag{49}$$

This gives immediately Eq. (43).

To obtain Eq. (46) we first write the equations of motion in the original system ξ, η as follows:

$$\ddot{\boldsymbol{\zeta}} + 2\mathbf{k} \times \dot{\boldsymbol{\zeta}} = \mathbf{grad}_\zeta\, \Omega = 2\mathbf{A}\boldsymbol{\zeta}. \tag{50}$$

We then transform it by means of $\boldsymbol{\zeta} = \mathbf{B}\bar{\boldsymbol{\zeta}}$. After substitution and multiplication by $\mathbf{B}^* = \mathbf{B}^{-1}$, we obtain

$$\ddot{\bar{\boldsymbol{\zeta}}} + 2\,\mathbf{B}^*(\mathbf{k} \times \mathbf{B}\dot{\bar{\boldsymbol{\zeta}}}) = 2(\mathbf{B}^*\mathbf{A}\mathbf{B})\bar{\boldsymbol{\zeta}} \tag{51}$$

or

$$\ddot{\bar{\boldsymbol{\zeta}}} + 2\mathbf{k} \times \dot{\bar{\boldsymbol{\zeta}}} = 2\mathbf{M}\bar{\boldsymbol{\zeta}} \tag{52}$$

since

$$\mathbf{B}^*(\mathbf{k} \times \mathbf{B}\mathbf{x}) = \mathbf{k} \times \mathbf{x}. \tag{53}$$

This last relation is valid for any vector $\mathbf{x}$ which is in the plane of ξ, η. Here $\mathbf{k}$ is a unit vector normal to this plane, so the $\mathbf{k} \times \mathbf{x}$ operation rotates the vector $\mathbf{x}$ by 90° counterclockwise. The $\mathbf{Bx}$ operation rotates

$\mathbf{x}$ by α, the $\mathbf{B}^*\mathbf{x} = \mathbf{B}^{-1}\mathbf{x}$ operation by $-\alpha$. Equation (53) therefore states the interchangeability of the order of rotations in a plane around a point.

Since Eq. (52) is identical with Eqs. (46), the proof is completed.

Solution of Eqs. (46) now proceeds the usual way. The characteristic roots are $\lambda_{1,2} = \pm\Lambda_1^{1/2}$ and $\lambda_{3,4} = \pm\Lambda_2^{1/2}$, where Λ_1 and Λ_2 are given by Eq. (27) at the beginning of Section 5.4. This fact should not be surprising since the eigenfrequencies of a dynamical system do not depend on the coordinate system used. Consequently the characteristic roots of Eqs. (46) are given by Eq. (25). This can be shown also formally by establishing the characteristic equation for the system of differential equations in the coordinate system $\bar{\xi}, \bar{\eta}$ and showing its equivalence with Eq. (25). From Eqs. (46) we have

$$\begin{vmatrix} \lambda^2 - \bar{\lambda}_2 & -2\lambda \\ 2\lambda & \lambda^2 - \bar{\lambda}_1 \end{vmatrix} = 0, \tag{54}$$

or

$$\lambda^4 + \lambda^2(4 - \bar{\lambda}_1 - \bar{\lambda}_2) + \bar{\lambda}_1\bar{\lambda}_2 = 0. \tag{55}$$

Here $\bar{\lambda}_1\bar{\lambda}_2 = \frac{27}{4}\mu(1 - \mu)$, since this is the value of the determinant given by Eq. (110) of Chapter 4. The same expression occurs also as the last term of Eq. (25). Furthermore $\bar{\lambda}_1 + \bar{\lambda}_2 = 3$, either from Eq. (111) of Chapter 4 or because $\Omega_{xx}(L_{4,5}) + \Omega_{yy}(L_{4,5}) = 3$. This completes the proof of the identity of Eqs. (25) and (55).

Therefore the solution of Eqs. (46) may be written similarly to Eqs. (28) in the form

$$\begin{aligned} \bar{\xi} &= A_1 \cos s_1 t + B_1 \sin s_1 t + A_2 \cos s_2 t + B_2 \sin s_2 t, \\ \bar{\eta} &= \bar{A}_1 \cos s_1 t + \bar{B}_1 \sin s_1 t + \bar{A}_2 \cos s_2 t + \bar{B}_2 \sin s_2 t, \end{aligned} \tag{56}$$

where the mean motions, s_1 and s_2, are the same as before. The relations between the coefficients $A_i, B_i, \bar{A}_i, \bar{B}_i$ are now simpler than those given by Eq. (29). We have

$$\begin{aligned} \bar{A}_i &= \bar{\alpha}_i B_i, \\ \bar{B}_i &= -\bar{\alpha}_i A_i, \end{aligned} \tag{57}$$

where

$$\bar{\alpha}_i = \frac{s_i^2 + \bar{\lambda}_2}{2s_i} = \frac{2s_i}{s_i^2 + \bar{\lambda}_1}. \tag{58}$$

The mean motions or frequencies can be obtained from $s_1 = \mp i\lambda_{1,2}$ and $s_2 = \mp i\lambda_{3,4}$, or $s_1 = +(-\Lambda_1)^{1/2}$ and $s_2 = +(-\Lambda_2)^{1/2}$. Note that s satisfies a modification of Eq. (55):

$$s^4 - s^2(4 - \bar{\lambda}_1 - \bar{\lambda}_2) + \bar{\lambda}_1\bar{\lambda}_2 = 0. \tag{59}$$

The long-period solution belonging to the eigenfrequency s_1 is

$$\begin{aligned} \bar{\xi} &= A_1 \cos s_1 t + B_1 \sin s_1 t, \\ \bar{\eta} &= \bar{\alpha}_1 B_1 \cos s_1 t - \bar{\alpha}_1 A_1 \sin s_1 t, \end{aligned} \tag{60}$$

or

$$\begin{aligned} \bar{\xi} &= \bar{\xi}_0 \cos s_1 t + \frac{\bar{\eta}_0}{\bar{\alpha}_1} \sin s_1 t, \\ \bar{\eta} &= \bar{\eta}_0 \cos s_1 t - \bar{\alpha}_1 \bar{\xi}_0 \sin s_1 t. \end{aligned} \tag{61}$$

The initial velocity components cannot be freely selected since Eqs. (60) represent a particular solution with only two arbitrary constants. The velocities at $t_0 = 0$ can be obtained from Eqs. (60):

$$\begin{aligned} \dot{\bar{\xi}}_0 &= \frac{\bar{\eta}_0 s_1}{\bar{\alpha}_1}, \\ \dot{\bar{\eta}}_0 &= -\bar{\xi}_0 \bar{\alpha}_1 s_1, \end{aligned} \tag{62}$$

which results are quite similar to Eqs. (22) and indicate again retrograde motions.

The elliptic orbit given by Eqs. (61) can be written as

$$\bar{\xi}^2 + \bar{\eta}^2/\bar{\alpha}_1^2 = \bar{\xi}_0^2 + \bar{\eta}_0^2/\bar{\alpha}_1^2. \tag{63}$$

Since $0 < \alpha_1^2 < 1$, the semimajor axis of the ellipse is along the $\bar{\xi}$ coordinate axis. To show that $\bar{\alpha}_1^2 < 1$ will be delegated to an example, since the procedure is quite similar to the one followed regarding Eq. (23).

Example. Show that $0 < \bar{\alpha}_i = s_i/2 + \lambda_2/2s_i < 1$.

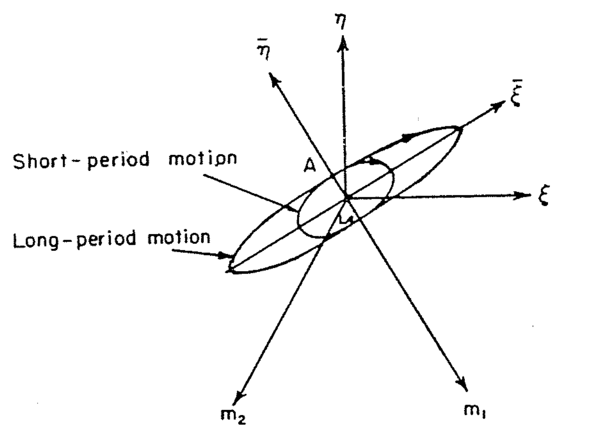

FIG. 5.6. The long-period and short-period motions around L_4.

The general situation is shown on Fig. 5.6, which may be compared to Fig. 4.19 of Chapter 4. The length of the semimajor axis is

$$\bar{a}_i = (\bar{\xi}_0^2 + \bar{\eta}_0^2/\bar{\alpha}_i^2)^{1/2}$$

and of the semiminor axis is

$$\bar{b}_i = (\bar{\alpha}_i^2 \bar{\xi}_0^2 + \bar{\eta}_0^2)^{1/2}.$$

The ratio of the lengths of the axes is $\bar{a}_i/\bar{b}_i = 1/\bar{\alpha}_i$ and the eccentricity of the orbit becomes $e_i = (1 - \bar{\alpha}_i^2)^{1/2}$. Note that e_i is independent of the initial conditions and depends only on the value of the mass parameter, μ.

The short-period terms are treated the same way using in the preceding equations $i = 2$. Since $\bar{\alpha}_2 > \bar{\alpha}_1$ we have $e_1 > e_2$; in fact, $e_1 \cong 1 - 3\mu/2$ and $e_2 \cong (3^{1/2}/2)(1 + 3\mu/8)$ with an error of less than 1 % in the range $0 < \mu < 0.03$.

The two orbits (with long and short periods) coincide at $\mu = \mu_0$ and the eccentricity becomes $e_1 = e_2 = [2(2^{1/2} - 1)]^{1/2}$. An equation for the eccentricities is obtained by substituting Eq. (58) into $e_i = (1 - \bar{\alpha}_i^2)^{1/2}$ giving

$$e_i = 2^{1/2} \sin \varepsilon_i,$$

where

$$\tan^2 \varepsilon_i = \frac{3[1 - 3\mu(1 - \mu)]^{1/2}}{4 \pm [1 - 27\mu(1 - \mu)]^{1/2}}.$$

Here the plus sign furnishes ε_2 and the minus gives ε_1. Figure 5.7 shows the dependence of the eccentricities on the mass parameter.

Recalling from Chapter 4 the similarity between certain curves of zero velocity and some orbits, it is of interest to compare the curves of zero velocity around L_4, as described in Section 4.7.2, Part (iv), with the elliptic orbits discussed in this section. Both results follow from linearization of the dynamical system. In Section 4.7.2 the potential function was limited to quadratic terms in Eq. (109), and in this section the resulting linear differential equations are solved. The orientation of the ellipses of zero velocity agree completely with the orientation of elliptic orbits. The eccentricities, however, differ. For small values of μ we have

$$\bar{\alpha}_1 \cong (3\mu)^{1/2}$$

and

$$\bar{\alpha}_2 \cong \frac{1}{4}\left(1 - \frac{9}{4}\mu\right)$$

for the orbits with the long- and short-period motions, respectively.

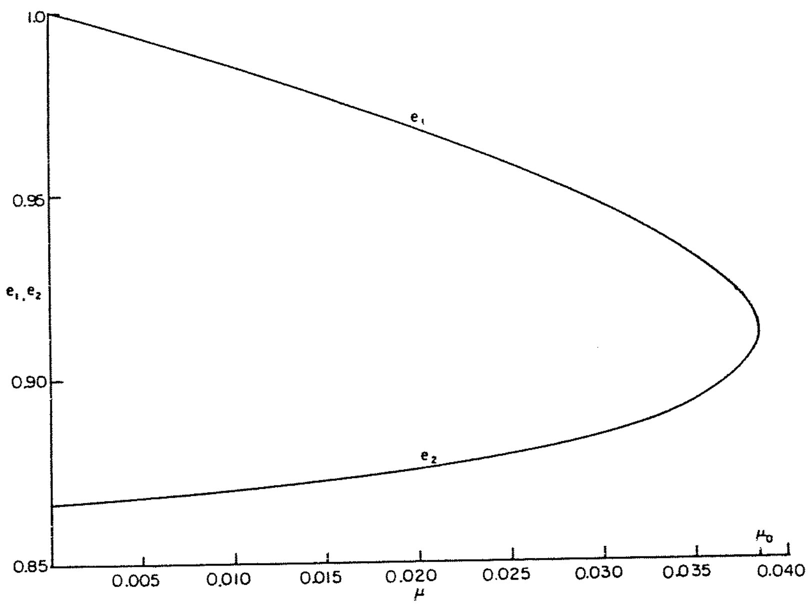

FIG. 5.7. Eccentricities of the long-period (e_1) and short-period (e_2) solutions of the linearized equations.

The parameter governing the eccentricity of the curves of zero velocity is $\bar{\alpha}_3$, which, from Eqs. (114) of Section 4.7.2, becomes

$$\bar{\alpha}_3 = b/a = \bar{\lambda}_2/\bar{\lambda}_1 \cong \tfrac{3}{4}\mu.$$

The eccentricities are given in all cases by $e_i = (1 - \bar{\alpha}_i^2)^{1/2}$ and consequently they may be ordered as follows: $e_3 > e_1 > e_2$. The eccentricity of the curves of zero velocity is larger than that of the orbits.

Example 1. Show that $C > 3$ for the long-period and $C < 3$ for the short-period linearized solutions around the triangular libration points. Furthermore show that $C = 3$ when $s_1 = s_2 = 1/2^{1/2}$.

Example 2. Show that the long-period orbit lies outside the curve of zero velocity, both corresponding to the same value of the Jacobian constant, $C > 3$, using the linearized system. Show also that the ratio of the areas of these ellipses is

$$\frac{A}{B} = \frac{s_2^2 - s_1^2}{2s_2} < 1,$$

where A is the area of the ellipse of the curve of zero velocity and B is the area of the long-period elliptic orbit.

Example 3. Show that $de/d\mu \to \infty$ as $\mu \to \mu_0$ for the long-period and short-period linearized solutions.

Since in the range $0 < \mu < \mu_0 = 0.03852$ the characteristic equation has pure imaginary roots the solution in general is composed of two harmonic oscillations with different frequencies. The linear system of differential equation (46) representing the motion therefore possesses bounded solutions for any initial conditions. The solution of the linear system shows stability; the behavior of the corresponding nonlinear system cannot be predicted since the pure imaginary roots establish what is generally called the critical condition, and further investigation is necessary.

5.4.2 Unstable solutions of the linearized equations

When $\mu_0 < \mu < \frac{1}{2}$ the discriminant of the characteristic equation is negative,

$$0 > d = 1 - 27\mu(1 - \mu) > -23/4,$$

and Eq. (27) becomes

$$\Lambda_{1,2} = \tfrac{1}{2}(-1 \pm i\delta), \tag{64}$$

where

$$0 < \delta = +d^{1/2} = [27\mu(1 - \mu) - 1]^{1/2} < \tfrac{1}{2}(23)^{1/2} = 2.3979157617.$$

The roots of the characteristic equation are, as before,

$$\lambda_{1,2} = \pm\Lambda_1^{1/2} \quad \text{and} \quad \lambda_{3,4} = \pm\Lambda_2^{1/2},$$

or

$$\begin{aligned}
\lambda_1 &= \frac{1}{2^{1/2}}(-1 + i\delta)^{1/2} = \alpha_1 + i\beta_1, \\
\lambda_2 &= -\frac{1}{2^{1/2}}(-1 + i\delta)^{1/2} = \alpha_2 + i\beta_2, \\
\lambda_3 &= \frac{1}{2^{1/2}}(-1 - i\delta)^{1/2} = \alpha_3 + i\beta_3, \\
\lambda_4 &= -\frac{1}{2^{1/2}}(-1 - i\delta)^{1/2} = \alpha_4 + i\beta_4.
\end{aligned} \tag{65}$$

The lengths of these roots are equal:

$$|\lambda| = |\lambda_{1,2,3,4}| = \frac{1}{2^{1/2}}[27\mu(1 - \mu)]^{1/4}. \tag{66}$$

From this it follows that, as $\mu_0 < \mu < \frac{1}{2}$, we have

$$1/2^{1/2} < |\lambda| < \tfrac{1}{2}(27)^{1/4} = 1.1397535285.$$

The principal argument θ of the first root is

$$\theta = \theta_1 = \arctan \frac{1 + (1 + \delta^2)^{1/2}}{\delta} \tag{67}$$

and so

$$\frac{\pi}{2} > \theta > \arctan \frac{3 \cdot 3^{1/2} + 2}{(23)^{1/2}} = 56^{\circ}.3187717113,$$

corresponding to $\mu_0 < \mu < \frac{1}{2}$.

The arguments of the four roots are related by

$$\theta = \theta_1 = \theta_2 - \pi = 2\pi - \theta_3 = \pi - \theta_4 .$$

The real and imaginary parts of the roots, α_i and β_i, are related by

$$\alpha = \alpha_1 = -\alpha_2 = \alpha_3 = -\alpha_4$$

and

$$\beta = \beta_1 = -\beta_2 = -\beta_3 = \beta_4 ,$$

where

$$\alpha = \frac{\delta}{2(1 + 2|\lambda|^2)^{1/2}}, \qquad \beta = \frac{(1 + 2|\lambda|^2)^{1/2}}{2}, \tag{68}$$

or

$$\alpha = \frac{\{[27\mu(1-\mu)]^{1/2} - 1\}^{1/2}}{2}$$

$$\beta = \frac{\{[27\mu(1-\mu)]^{1/2} + 1\}^{1/2}}{2}.$$

From this it follows that the real part of two of the characteristic roots are positive (and equal) and so the equilibrium point in this case is unstable. This result is also applicable to the nonlinear equations and the triangular libration points remain unstable when nonlinearities are included.

The limiting values of α and β corresponding to $\mu = \mu_0$ and to $\mu = \frac{1}{2}$ are

$$0 < \alpha < \left(\frac{23}{(2 + 3 \cdot 3^{1/2})8}\right)^{1/2} = 0.6320751956$$

and

$$\frac{1}{2^{1/2}} < \beta < \left(\frac{2 + 3 \cdot 3^{1/2}}{8}\right)^{1/2} = 0.9484297828.$$

The absolute value of the roots $|\lambda|$ given by Eq. (66), the principal argument θ given by Eq. (67), and the real and imaginary parts of the first root, α and β, given by Eqs. (68) are presented, in this order, in Appendix III.

The solution of the system of differential equations given by Eqs. (46) is

$$\bar{\xi} = \sum_{k=1}^{4} A_k e^{\alpha_k t} (\cos \beta_k t + i \sin \beta_k t),$$
$$\bar{\eta} = \sum_{k=1}^{4} B_k e^{\alpha_k t} (\cos \beta_k t + i \sin \beta_k t), \tag{69}$$

where

$$B_k = A_k \kappa_k$$

and

$$\kappa_k = \frac{\lambda_k^2 - \bar{\lambda}_2}{2\lambda_k} = \frac{2\lambda_k}{\bar{\lambda}_1 - \lambda_k^2},$$

or

$$\kappa_k = \frac{\alpha_k(|\lambda|^2 - \bar{\lambda}_2) + i\beta_k(|\lambda|^2 + \bar{\lambda}_2)}{2|\lambda|^2}. \tag{70}$$

Here $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are *not* the conjugate complex of $\lambda_{1,2}$ but the roots of the characteristic equation introduced first in Section 4.7.2, Part (iv), and given in this chapter in Section 5.4.1 as Eq. (44).

The solution given by Eqs. (69) may also be written as

$$\bar{\xi} = e^{\alpha t}(a_1 \cos \beta t + a_2 \sin \beta t) + e^{-\alpha t}(a_3 \cos \beta t + a_4 \sin \beta t),$$
$$\bar{\eta} = e^{\alpha t}(b_1 \cos \beta t + b_2 \sin \beta t) + e^{-\alpha t}(b_3 \cos \beta t + b_4 \sin \beta t), \tag{71}$$

where the new coefficients, $a_1, \ldots, a_4$, $b_1, \ldots, b_4$, are connected by the following set of equations:

$$b_1 = a_1\alpha' + a_2\beta',$$
$$b_2 = -a_1\beta' + a_2\alpha',$$
$$b_3 = -a_3\alpha' + a_4\beta',$$
$$b_4 = -a_3\beta' - a_4\alpha'. \tag{72}$$

Here

$$\alpha' = \frac{\alpha}{2\,|\lambda|^2}(|\lambda|^2 - \bar{\lambda}_2) \qquad \text{and} \qquad \beta' = \frac{\beta(|\lambda|^2 + \bar{\lambda}_2)}{2\,|\lambda|^2}. \tag{73}$$

The four initial conditions ξ_0, η_0, $\dot{\xi}_0$, and $\dot{\eta}_0$ are linear functions of the four constants $a_1, \ldots, a_4$. Selecting $a_1 = a_2 = 0$, the solution approaches the libration point asymptotically since from Eqs. (71) we have $\bar{\xi} \to 0$, $\bar{\eta} \to 0$ as $t \to \infty$. In general the solution is unstable and with arbitrary selected initial conditions the solution is unbounded.

5.4.3 Secular solutions of the linearized equations

When $\mu = \mu_0 = 0.0385208965\ldots$ we have $d = 1 - 27\mu(1 - \mu) = 0$; consequently, $\Lambda_{1,2} = -\frac{1}{2}$ and $\lambda_1 = \lambda_3 = i/2^{1/2}$, $\lambda_2 = \lambda_4 = -i/2^{1/2}$. The double roots give secular terms in the solution of Eqs. (46) and so the equilibrium point is unstable. The equations of motion may be treated also in the (ξ, η) system, which is parallel with the (x, y) system; nevertheless, some of the calculations are simpler when the same $(\bar{\xi}, \bar{\eta})$ system is used as in the previous two sections (5.4.1 and 5.4.2). The general form of the solution is

$$\xi = (a_1 + a_2 t)\cos t/2^{1/2} + (a_3 + a_4 t)\sin t/2^{1/2}$$

and

$$\eta = (b_1 + b_2 t)\cos t/2^{1/2} + (b_3 + b_4 t)\sin t/2^{1/2}. \tag{74}$$

The relations between a_i and b_i on one hand and between the initial conditions and these coefficients on the other hand depend on the choice of the coordinate system. Using the system ξ, η, we have

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \Omega^*_{xx}(L_4)\,\xi + \Omega^*_{xy}(L_4)\,\eta, \\ \ddot{\eta} + 2\dot{\xi} &= \Omega^*_{xy}(L_4)\,\xi + \Omega^*_{yy}(L_4)\,\eta, \end{aligned} \tag{75}$$

where the stars (*) signify that these partial derivatives are to be evaluated for $\mu = \mu_0 = 0.03852$. Since $\Omega_{xx}(L_4)$ and $\Omega_{yy}(L_4)$ are independent of the value of μ we have

$$\Omega^*_{xx}(L_4) = \Omega_{xx}(L_4) = \tfrac{3}{4}$$

and

$$\Omega^*_{yy}(L_4) = \Omega_{yy}(L_4) = \tfrac{9}{4}.$$

The mixed partial derivative is

$$\Omega_{xy}(L_4) = \frac{3 \cdot 3^{1/2}}{2}(\mu - \tfrac{1}{2}),$$

and since μ_0 is the solution of the equation

$$27\mu_0(1 - \mu_0) = 1,$$

we have

$$\Omega^*_{xy}(L_4) = -(23)^{1/2}/4.$$

Substituting Eqs. (74) into the equations of motion (75) and using the above values for the partial derivatives, the relations between the coefficients b_i and a_i may be established.

Working in the coordinate system $\bar{\xi}$, $\bar{\eta}$, which is lined up along the principal axes of the equipotential ellipses—or along the axes of the elliptic orbits discussed in 5.4.1—we have the solutions

$$\bar{\xi} = (\alpha_1 + \alpha_2 t)\cos t/2^{1/2} + (\alpha_3 + \alpha_4 t)\sin t/2^{1/2}$$

and

$$\bar{\eta} = (\beta_1 + \beta_2 t)\cos t/2^{1/2} + (\beta_3 + \beta_4 t)\sin t/2^{1/2}. \tag{76}$$

The differential equations of motion for this case are of the same form as Eqs. (46):

$$\begin{aligned} \ddot{\bar{\xi}} - 2\dot{\bar{\eta}} &= \bar{\lambda}_2{}^*\bar{\xi}, \\ \ddot{\bar{\eta}} + 2\dot{\bar{\xi}} &= \bar{\lambda}_1{}^*\bar{\eta}, \end{aligned} \tag{77}$$

where

$$\lambda^*_{1,2} = \tfrac{3}{2}\{1 \pm [1 - 3\mu_0(1 - \mu_0)]^{1/2}\} = \tfrac{3}{2} \pm 2^{1/2}. \tag{78}$$

The relations between the coefficients occurring in the solution (76) become

$$\begin{aligned} \beta_1 &= (\alpha_2 + \alpha_3)\,\bar{\gamma}, \\ \beta_2 &= \alpha_4\bar{\gamma}, \\ \beta_3 &= (\alpha_4 - \alpha_1)\,\bar{\gamma}, \\ \beta_4 &= -\alpha_2\bar{\gamma}, \end{aligned} \tag{79}$$

where $\bar{\gamma} = 2^{1/2} - 1$.

Finally, the initial conditions ($\bar{\xi}_0$, $\bar{\eta}_0$, $\dot{\bar{\xi}}_0$, $\dot{\bar{\eta}}_0$, $t_0 = 0$) are related to the coefficients of the solution (α_1, α_2, α_3, α_4) by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \bar{\gamma} & \bar{\gamma} & 0 \\ 0 & 1 & 1/2^{1/2} & 0 \\ -\bar{\gamma}/2^{1/2} & 0 & 0 & 1/2^{1/2} \end{pmatrix}, \tag{80}$$

inversion of which gives

$$\begin{aligned}
\alpha_1 &= \bar{\xi}_0 , \\
\alpha_2 &= -\frac{\bar{\eta}_0}{\bar{\gamma}^2} + \frac{2^{1/2}}{\bar{\gamma}} \dot{\xi}_0 , \\
\alpha_3 &= \frac{2^{1/2}}{\bar{\gamma}^2} \bar{\eta}_0 - \frac{2^{1/2}}{\bar{\gamma}} \dot{\xi}_0 , \\
\alpha_4 &= \bar{\gamma}\bar{\xi}_0 + 2^{1/2}\dot{\bar{\eta}}_0 .
\end{aligned} \tag{81}$$

Equations (81) show that, when the arbitrarily selected initial coordinates $\bar{\xi}_0$ and $\bar{\eta}_0$ are combined with special values of $\dot{\xi}_0$ and $\dot{\bar{\eta}}_0$, the secular terms may be eliminated. In fact with the special velocity components

$$\dot{\xi}_0 = \bar{\eta}_0/\bar{\gamma}2^{1/2} \quad \text{and} \quad \dot{\bar{\eta}}_0 = -\bar{\xi}_0\bar{\gamma}/2^{1/2}$$

we have the solution

$$\begin{aligned}
\bar{\xi} &= \bar{\xi}_0 \cos \frac{t}{2^{1/2}} + \frac{\bar{\eta}_0}{\bar{\gamma}} \sin \frac{t}{2^{1/2}} , \\
\bar{\eta} &= \bar{\eta}_0 \cos \frac{t}{2^{1/2}} - \bar{\xi}_0\bar{\gamma} \sin \frac{t}{2^{1/2}} ,
\end{aligned} \tag{82}$$

as expected when Eqs. (61) are recalled.

Example. Show that the Jacobian constant of the periodic solution [Eqs. (82)] of the linearized equations of motion around L_4 when $\mu = \mu_0$ is $C = 3$.

5.5 Nonlinear phenomena around the equilibrium points

5.5.1 Introduction

In the preceding two sections, 5.3 and 5.4, the linearized forms of the equations of motion were solved. The linearization process allows only first-order terms in the coordinates and velocities; consequently, meaningful results must be connected with small values of these variables. An expansion of Ω around the libration points in a Taylor series according to the powers of the "variations" may be ended with squares of the variables, giving linear differential equations of motion. Higher-order terms, when retained in the expansion, result in equations of the form

$$\begin{aligned}
\ddot{\xi} - 2\dot{\eta} - \Omega_{xx}(a, b)\, \xi - \Omega_{xy}(a, b)\, \eta &= X(\xi, \eta; \mu), \\
\ddot{\eta} + 2\dot{\xi} - \Omega_{xy}(a, b)\, \xi - \Omega_{yy}(a, b)\, \eta &= Y(\xi, \eta; \mu),
\end{aligned}$$

where the functions $X(\xi, \eta; \mu)$ and $Y(\xi, \eta; \mu)$ are polynomials of the form $a_2\xi^2 + b_2\eta^2 + c_2\xi\eta + \cdots$. Just as the second partial derivatives of the function Ω at the triangular libration points are either constants or linear functions of the mass parameter, the coefficients a_2, b_2, c_2,... in the preceding expansion are also either constants or depend linearly on μ, since these coefficients are higher-order partial derivatives of Ω.

When $X = 0$ and $Y = 0$, the linear system may be treated in a straightforward, if not always in a simple, manner. This analysis is contained in Section 5.3 regarding the collinear libration points, and in Section 5.4 the linearized motion around the triangular libration points is discussed. Several remarks, pertinent to the nonlinear problem, are made in these sections. In what follows, the nonlinear effects will be discussed in more detail. First, orbits at the collinear points are presented, second, orbits at the triangular libration points are treated, and finally the stability conditions are discussed.

This section (5.5) is closely related to and may be considered as an application of some of the basic ideas outlined in Section 5.2.

5.5.2 Collinear points

The general solution of the linearized equations of motion around the collinear points is unbounded because of the existence of a positive real root of the characteristic equation for any value of μ. Particular solutions around the collinear points may be established. These solutions exist for any value of μ and represent infinitesimal (linearized) periodic orbits corresponding to the imaginary root. The period of these orbits depends only on the value of μ and it is independent of the (infinitesimal) size of the orbit. This independence of the frequency and the amplitude is a characteristically linear phenomenon. When higher-order terms are included in the expansion, the mean motion changes with the size of the orbit. The existence of periodic orbits as solutions of the nonlinear equations follows from Horn's theorem by analytic continuation of the elliptic periodic orbits. The effect of the size of the orbit on its period also follows from Horn's theorem and it may be written as

$$T_i(\mu, \epsilon) = T_{i0}(\mu) + P_i(\mu, \epsilon), \tag{82a}$$

where the function P_i is analytic in ϵ and $P_i(\mu, \epsilon) \to 0$ as the orbital parameter $\epsilon \to 0$. The quantity ϵ may represent an initial condition, controlling the size of the orbit, as, for instance, in the set of equations [cf. Eqs. (22)],

$$\xi_0 = \epsilon, \qquad \eta_0 = 0, \qquad \dot{\xi}_0 = 0, \qquad \dot{\eta}_0 = -\epsilon\beta_3 s.$$

From Eqs. (19) we obtain

$$\xi = \epsilon \cos st \quad \text{and} \quad \eta = -\epsilon\beta_3 \sin st,$$

which shows that the orbit shrinks to the libration point as $\epsilon \to 0$. In the preceding equation (82a) for the period, the subscript i is 1, 2, or 3, according to which collinear point is being considered. The quantity $T_{i0}(\mu)$ is the period of the infinitesimal orbit and is given by $2\pi/s_i$, where the numerical value of s_i is given in Appendix I, Sections A to C.

Plummer's third-order theory shows the effect of the frequency on the amplitude, giving $P_i = K\epsilon^2$ with positive K. This is strictly valid only for $\epsilon \to 0$.

The analytic continuation indicates the existence of retrograde periodic orbits for all values of μ, at all three collinear libration points, for $\epsilon \neq 0$ but sufficiently small, considering the complete (nonlinear) equations of motion. The small parameter of the analytic continuation is *not* the mass parameter μ in the present case as it usually is in the study of periodic orbits in the restricted problem. Consequently such periodic orbits exist not only for small values of μ but also for $\mu = \frac{1}{2}$, as will be shown in considerable detail in Chapter 9. The "continuation-parameter" ϵ is related analytically by Wintner to the Jacobian constant using the Jacobian integral, and so C may also be used, in place of ϵ, in the formula for T_i by writing $\epsilon = (C_i - C)^{1/2}$.

Analytic continuation of the general solution containing trigonometric *and* hyperbolic functions, or of the solution containing only terms related to the real roots of the characteristic equation, is of considerable interest as shown by Deprit. In Section 9.4.2 orbits are shown which approach asymptotically, along a straight line, the collinear libration points with infinite period. These orbits in the neighborhood of the collinear libration points may be obtained from the solutions of the linearized equations.

5.5.3 Equilateral points

5.5.3.1 *Periodic orbits of finite amplitude for $\mu < \mu_0$.* The effect of nonlinearity on the motion around the triangular equilibrium points is more involved than around the collinear points because of the greater variety of forms of possible solutions of the corresponding linearized equations. Of the three general types of solution the librational motion, taking place in the range $0 < \mu < \mu_0$, is of the greater interest and it will be discussed first.

When $0 < \mu < \mu_0$ the linearized equations furnish oscillatory motion since the roots of the characteristic equation are pure imaginary numbers. For a given value of μ satisfying the preceding inequality there are two

periods, T_1 and T_2, associated with the two angular velocities, or mean motions s_1 and s_2. Since $s_1 < s_2$, $T_1 > T_2$, and consequently the solution corresponding to s_1 is called the long-period or librational motion. The linearized treatment gives infinitesimal orbits as solutions and these can be continued to orbits with finite size, similarly to the process described in connection with the collinear points. The periodic orbits of finite size which are generated from the infinitesimal long-period orbits and satisfy the complete, nonlinear equations of motion form the family of long-period orbits. The short-period family may be generated similarly from the solution of the linearized equation corresponding to s_2. Both families shrink to $L_{4,5}$ when the orbital parameter $\epsilon \to 0$.

Formally, we use Eqs. (61) with $\bar{\xi}_0 = \epsilon$, $\bar{\eta}_0 = 0$, and obtain

$$\bar{\xi} = \epsilon \cos s_i t + \mathcal{O}(\epsilon^2),$$
$$\bar{\eta} = -\bar{\alpha}_i \epsilon \sin s_i t + \mathcal{O}(\epsilon^2).$$

For the period we have, from Horn's theorem,

$$T_i(\mu, \epsilon) = 2\pi/s_i + P_i(\mu, \epsilon),$$

where $\bar{\alpha}_i$ and s_i are defined by Eqs. (58) and (59) and depend on the value of μ but not on ϵ. The function P_i has the property that $P_i(\mu, \epsilon) \to 0$ as $\epsilon \to 0$. The subscripts $i = 1$ and 2 correspond to long-period or short-period motion. A third-order theory by Pedersen, for instance, gives $P_i(\mu, \epsilon) = K\epsilon^2$, where $K = K(s_i)$ or, since $s_i = s_i(\mu)$, we have $K = K(\mu)$.

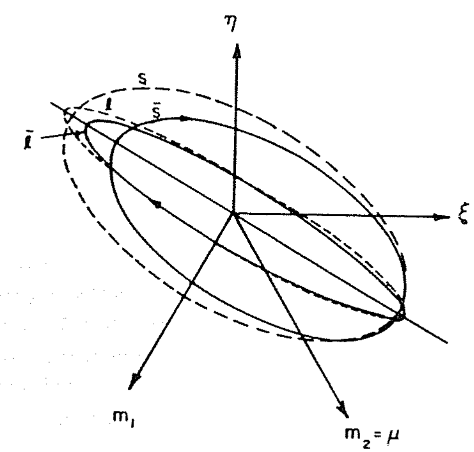

FIG. 5.8. Long-period, l, and short-period, s, generating orbits and corresponding actual solutions, $\bar{l}$ and $\bar{s}$, for $\mu = 0.01$ and $\epsilon_2 = \epsilon_s = 0.00111$, $\epsilon_1 = \epsilon_l = 0.00115$ (Moulton, 1920, Ref. 40).

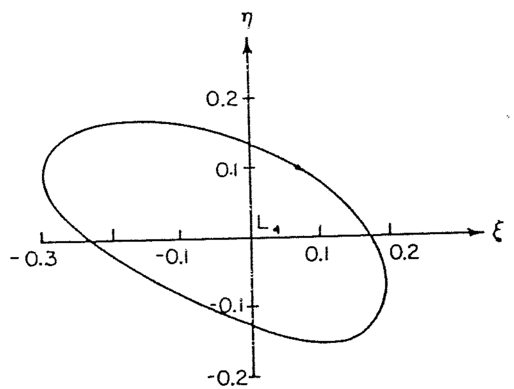

FIG. 5.9. Periodic orbit around L_4 with $\epsilon = 0.2$ and $\mu = 0.0369$. Continuation of the short-period infinitesimal orbit with $s_2 = 0.7746$ (Pedersen, 1935, Ref. 60).

Figure 5.8 shows generating ellipses representing the long-period and the short-period orbits at $\mu = 0.01$. The corresponding mean motions from Appendix II A become $s_1 = 0.26835$ and $s_2 = 0.96332$. Using the values $\epsilon_1 = 0.00115$ and $\epsilon_2 = 0.00111$, the resulting solutions of the nonlinear equations are also shown as obtained by Moulton.

Figures 5.9 and 5.10 by Pedersen show the continuation of infinitesimal periodic orbits with $s_2^2 = 0.6$ and $s_1^2 = 0.3$; the first corresponding to a member of the short-period family with $\mu = 0.0369185$, the second belonging to the long-period family with $\mu = 0.0321444$. The values of K are approximately -9 and $+27$, giving changes of the periods approximately $\Delta T_2 = -0.36$ and $\Delta T_1 = +1.08$. The linear theory gives $T_1 = 2\pi/s_1 = 11.45$ and $T_2 = 8.12$, so the changes in the periods due to the nonlinear effect are not negligible.

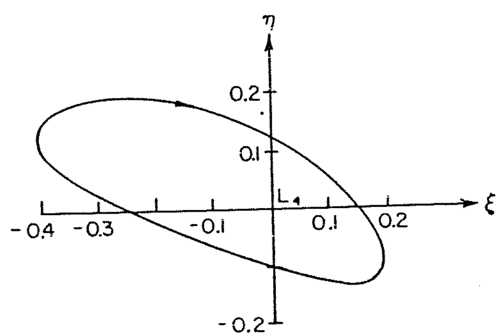

FIG. 5.10. Periodic orbit around L_4 with $\epsilon = 0.2$ and $\mu = 0.0322$. Continuation of the long-period infinitesimal orbit with $s_1 = 0.5477$ (Pedersen, 1935, Ref. 60).

The essential difference between the two families is that short-period orbits can be generated by analytic continuation from the infinitesimal periodic orbits at *any* value of the mass parameter while there are certain discreet, forbidden values of μ at which the generation of the long-period families is not possible. These forbidden values follow from Horn's theorem and were first established by Pedersen. They correspond to the case when the ratio of the periods of the infinitesimal long-period and short-period orbits, $T_1/T_2 = s_2/s_1$, is an integer. If this ratio is denoted by k, we have

$$k^2 = \frac{1 + [1 - 27\mu(1-\mu)]^{1/2}}{1 - [1 - 27\mu(1-\mu)]^{1/2}}$$

and

$$s_1 = 1/(1 + k^2)^{1/2}, \qquad s_2 = ks_1 .$$

The critical value of the mass parameter becomes

$$\mu = \frac{1}{2}\left[1 - \frac{(k^4 + 38k^2/27 + 1)^{1/2}}{1 + k^2}\right].$$

The existence of the critical values of μ is due to the vanishing of the determinant at $s_2/s_1 = k$ of the linear algebraic equations which are used to determine the Fourier coefficients of the solution of the nonlinear equations. For instance, when $k = 2$ we have $s_1 = 1/5^{1/2}$, $s_2 = 2/5^{1/2}$, and $\mu = 0.024294$. The preceding expressions for k^2 and μ show that the frequency of occurrence of the critical values of the mass parameter increases as $\mu \to 0$.

The families of the short-period orbits have the property that the periods of the members of any given family corresponding to a fixed value of μ decrease as the size of the orbits increases. The same negative characteristics of the relation between period and amplitude exists according to Pedersen's third-order theory for the long-period solution, provided $s_{10}^2 = 0.1188979 < s_1^2 < 0.2$. The period of the long-period solution increases with the size of the orbit when, for the generating solution, $0 < s_1^2 < s_{10}^2$ and $0.2 < s_1^2 < \frac{1}{2}$. The solutions of the linearized equations, of course, show no relation between frequency and amplitude; the previously listed nonlinear effect according to which the period influences the amplitude may be obtained from a third-order analysis, when terms of the third order regarding the size of the orbit, ξ^3 and η^3, are retained in the equations of motion.

For the sun-Jupiter system $s_1 = 0.080464$, and for the earth-moon system $s_1 = 0.29808$; consequently, the periods of the nonlinear long-period solutions increase with the size of the orbits for both systems. Numerical work along this line was performed by Rabe (see Section 9.6).

This concludes the discussion of the nonlinear aspects of the motion around the triangular libration points in the range of the mass parameter from $\mu = 0$ to $\mu = \mu_0$.

5.5.3.2 *Periodic orbits of finite amplitude for* $\mu > \mu_0$. For values of the mass parameter greater than μ_0 the roots of the characteristic equation have nonzero real parts and the orbits become spirals. Consequently no (infinitesimal) periodic orbits exist in the linear theory around $L_{4,5}$ when $\mu > \mu_0$. The spiral orbits asymptotically approach the triangular libration points or depart from them. Strömgren's numerical and graphical approach shows how the periodic orbits change into spirals as μ varies from $\mu < \mu_0$ to $\mu > \mu_0$.

The question naturally comes up whether the effect of nonlinearity will allow the existence of periodic orbits for $\mu > \mu_0$. In other words, while infinitesimal periodic orbits do not exist for $\mu > \mu_0$, do periodic orbits exist of finite size for the same values of μ? The answer is in the affirmative and we discuss first the continuation of the case $\mu = \mu_0$.

The special value of the mass parameter $\mu_0 = 0.03852$ gives the secular solution which, with properly selected initial conditions, may be changed into a periodic orbit as shown in Section 5.4.3, Eqs. (82). The same orbit and the same equations may be obtained as the limiting case for the librational motion, Eqs. (61). This limiting condition occurs when the long- and short-period orbits coincide, and the period of the motion is $2^{1/2}$ times the time of revolution of the primaries: $s_1 = s_2 = 1/2^{1/2}$. Both representations, Eqs. (61) and Eqs. (82), are expressed in the coordinate system $\bar{\xi}$, $\bar{\eta}$, the axes of which are parallel to the principal directions of the ellipses of zero velocity. The equations of the orbit may also be written in the system ξ, η which is parallel with the original coordinate system x, y; see Eqs. (34) and (74).

Let the initial conditions be $\xi_0 = \epsilon$, $\eta_0 = 0$, which give, using Eqs. (33), $\dot{\xi}_0 = -3(3^{1/2}/8)\,\epsilon\gamma$ and $\dot{\eta}_0 = -\epsilon(s_1^2 + \frac{3}{4})/2$. The mean motion $s_1 = 1/2^{1/2}$ and the value of $\gamma = (23/27)^{1/2}$ belonging to $\mu = \mu_0$ give

$$\xi = \epsilon \cos \frac{t}{2^{1/2}} - \frac{(46)^{1/2}}{8}\,\epsilon \sin \frac{t}{2^{1/2}},$$

$$\eta = -\frac{5 \cdot 2^{1/2}}{8}\,\epsilon \sin \frac{t}{2^{1/2}}.$$

These equations follow from Eqs. (32) and (34) or from Eqs. (74) with $a_2 = a_4 = b_1 = b_2 = b_4 = 0$, $a_1 = \epsilon$, $a_3 = -3 \cdot 3^{1/2}\epsilon\gamma/8s_1 = \epsilon(46)^{1/2}/8$, and $b_3 = -\frac{1}{2}(s_1 + 3/4s_1)\epsilon = -5 \cdot 2^{1/2}\epsilon/8$.

The periodic orbits given by the preceding equations exist only for infinitesimal values of ϵ and represent a solution of the linearized differential equations. If in the differential equations of motion higher-order terms are retained this solution may be continued to periodic orbits of finite size. Consider a given value of the mass parameter $\mu = \mu_0 + \delta$, where $\delta > 0$ is sufficiently small. Then for this fixed value of $\mu > \mu_0$ there is one periodic orbit of finite size with period $T_1 = T_2 = 2\pi 2^{1/2}$. The periodic orbits for various values of δ, all having the same period, $2\pi 2^{1/2}$, were called limiting orbits by E. W. Brown. They are generated from the limiting infinitesimal periodic orbit; see the preceding equations. The secular solution (see Section 5.4.3), when modified by using special initial conditions, becomes the limiting periodic orbit since it limits and unites the long-period and the short-period orbits. The same applies in the nonlinear case. When the value of the mass parameter is fixed and $\mu > \mu_0$ a family of long-period orbits with periods $T_l \neq 2\pi 2^{1/2}$ and showing a relation between amplitude and period is expected to exist. Similarly a family of short-period orbits may exist with periods $T_s \neq T_l$ which also vary with the amplitudes. As the periods of the short-period and long-period orbits for a fixed value of $\mu > \mu_0$ vary with varying amplitude, the value of $T_l = T_s = 2\pi 2^{1/2}$ is reached when these orbits coincide with each other as well as with the limiting orbit of finite amplitude.

Another cut of the manifold of periodic orbits may be discussed considering a fixed value of μ and a given value of the orbital parameter ϵ

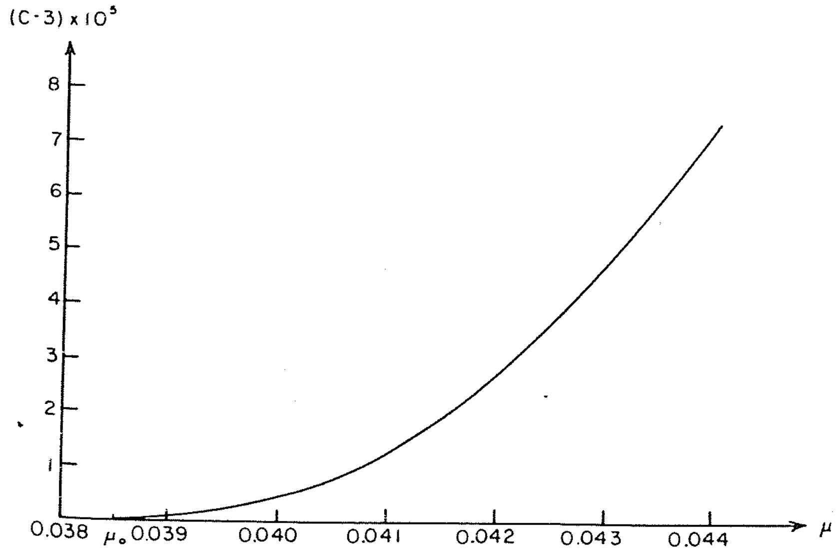

FIG. 5.11. Jacobian constant versus mass parameter for limiting orbits around the triangular libration point.

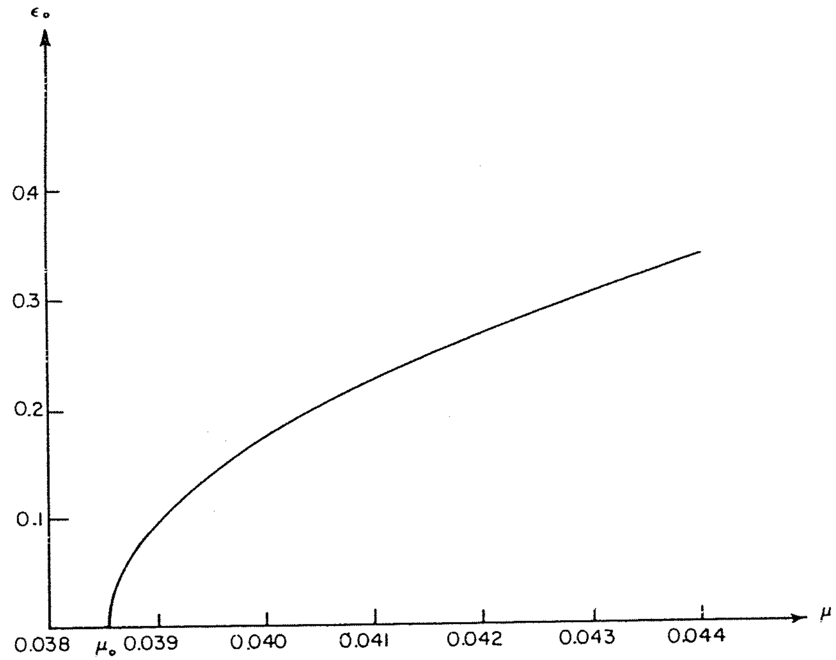

FIG. 5.12. Orbital parameter versus mass parameter for limiting orbits around the triangular libration point.

When $\mu < \mu_0$ for a given value of ϵ, there exist two periodic orbits, one generated from the long-period infinitesimal orbit and the other continued from the short-period infinitesimal orbit. The period of the long-period orbit of finite size is $T_l > T_1 = 2\pi/s_1$ and that of the short-period orbit of finite size is $T_s > T_2 = 2\pi/s_2$. (The exceptional cases mentioned previously are of course to be taken into consideration.) Similarly, when $\mu > \mu_0$ and again ϵ and μ are fixed, we may expect the existence of two periodic orbits of finite size, one with a longer period than the other. As the value of the orbital parameter ϵ changes, it will reach a critical value, say ϵ_0, at which value the two periodic orbits coincide and their periods become equal. In this way one finds that, if there is a limiting orbit belonging to the previously mentioned fixed value of μ, its period is $T = 2\pi 2^{1/2}$. Limiting orbits have been computed with high precision up to $\mu_l = 0.044$ by Deprit with his fourteenth-order theory. His results are shown in Figs. 5.11 and 5.12. The Jacobian constants of the limiting orbits increase with the value of the mass parameter as shown in Fig. 5.11. At $\mu = \mu_0$ the limiting orbit shrinks to the

libration point and the value of the Jacobian constant is 3. Figure 5.12 shows the value of the orbital parameter for the limiting orbits as a function of the mass parameter. At $\mu = \mu_0$, $\epsilon_0 = 0$. Figure 5.13 shows two limiting orbits belonging to various values of the mass parameter. The libration point to which the limiting orbit shrinks at $\mu = \mu_0$ has the coordinates $x = \frac{1}{2} - \mu_0 = 0.46148$ and $y = 3^{1/2}/2 = 0.86603$.

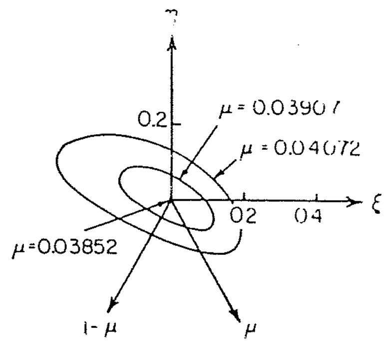

FIG. 5.13. Limiting orbits around the triangular libration point (Pedersen, 1933, Ref. 75).

As mentioned previously the idea of limiting orbits and the extension of periodic orbits for $\mu > \mu_0$ was first proposed by E. W. Brown. His original reasoning follows.

The characteristic equation (25) applicable to the triangular points becomes

$$s^4 - s^2 + \frac{27}{4}\mu(1 - \mu) = 0$$

when the substitution $\lambda = is$ is made. Infinitesimal periodic orbits, therefore, exist when the discriminant is positive or

$$0 < 1 - 27\mu(1 - \mu).$$

Brown proposes to study the equation

$$s^4 - s^2 + \frac{27}{4}\mu(1 - \mu) = k\epsilon^2,$$

where $k = k(\mu)$ and ϵ is proportional to the linear dimensions of the orbit. According to the new characteristic equation, we may have a positive value for the new discriminant,

$$0 < 1 - 27\mu(1 - \mu) + 4k\epsilon^2,$$

and consequently real values for s even when $\mu > \mu_0$, provided $\epsilon \neq 0$ is sufficiently large and $k > 0$. Noting that the term $k\epsilon^2$ appears in a higher than first-order theory, Brown concludes the existence of limiting orbits as described in Section 5.5.3.2.

5.5.4 Stability

The discussion of the stability of the equilibrium points and of the periodic orbits around them may be organized as follows. The first classification is according to the location of the libration point. Consequently we speak about collinear (1) and triangular (2) points. The second step is classification according to the value of the mass parameter. When $0 < \mu < \mu_0$ and the collinear points are considered we have Case 11. For the same range of the mass parameter, but for the triangular points, we have Case 21. When $\mu_0 < \mu < \frac{1}{2}$, the classification gives 12 and 22. Note that Cases 11 and 12 are the same since there exists no critical value of μ for the collinear points.

The next split occurs when we wish to distinguish between the libration point or an orbit around it. When the third digit is 1 we will refer to the equilibrium points and when it is 2 we speak about the periodic orbits around the points. In the treatment of the stability of the equilibrium points and orbits around them we distinguish between linear and nonlinear stability analyses. So if the fourth digit is 1 we refer to a linear treatment and if it is 2 we are concerned with a nonlinear analysis. The fifth and final classification is with respect to the size of the orbit, and we have 1 when the orbit is infinitesimal and 2 when its size is finite. By infinitesimal orbit we mean the solution to the linearized differential equations of motion.

The method of classification is shown schematically here:

1st digit: Configuration	2nd digit: Value of μ	3rd digit: Path
– Collinear : 1	– $(0 < \mu < \mu_0)$: 1	– Point : 1
– Triangular : 2	– $(\mu_0 < \mu < \frac{1}{2})$: 2	– Orbit : 2

4th digit: Method	5th digit: Size of orbit
– Linear : 1	– Infinitesimal : 1
– Nonlinear : 2	– Finite : 2

There are $2^5 = 32$ combinations; several of these, however, may be eliminated by the following considerations.

First note that, as already mentioned, cases described by numbers beginning with 11 and 12 are the same. This eliminates 8 cases, leaving 24. The second observation to be made is that when an equilibrium point is treated (as opposed to an orbit around it) we do not speak about the size of the orbit, so whenever the third digit is 1 the fifth digit loses its significance. This eliminates 6 cases leaving 18. Next, note that nonlinear stability analysis cannot be applied to an orbit of infinitesimal dimensions; consequently, all cases are eliminated for which the fourth digit is 2 and the last digit is 1, provided the third digit is 2, since we wish to apply this criterion to orbits rather than to points. This eliminates 3 more cases, leaving 15. Furthermore we know that infinitesimal periodic orbits do not exist around the triangular points when $\mu_0 < \mu < \frac{1}{2}$; consequently, Cases 22211 and 22221 may be eliminated. This reduces the number of cases to deal with to 14 since Case 22221 has already been eliminated because it deals with the nonlinear analysis of an infinitesimal orbit. Finally Cases 11211 and 21211 are eliminated. They refer to the linear stability analysis of infinitesimal periodic orbits around the collinear and the triangular points, respectively, for $0 < \mu < \mu_0$. As such, the conclusions regarding their stability are the same as the corresponding libration points. In other words, Cases 11111 and 11211 on one hand and Cases 21111 and 21211 on the other are identical from the point of view of stability.

The remaining 12 cases are 11111, 11121, 11212, 11222, 21111, 21121, 21212, 21222, 22111, 22121, 22212, 22222. The first four cases are concerned with the collinear, the rest with the equilateral configurations. A systematic discussion follows.

Case 11111. The stability analysis of the linear system regarding the collinear points results in instability for any value of the mass parameter as shown in Section 5.3. More precisely these points are unstable but are conditionally stable in the linear sense.

Case 11121. The nonlinear stability analysis of the collinear points shows also instability for any value of the mass parameter according to the discussion offered in Section 5.2.2.3. Conditional stability does not exist.

Case 11212. The linear stability analysis of periodic orbits of finite size around the collinear points shows instability for any value of the mass parameter according to Duboshin's work.

Case 11222. The nonlinear stability problem of periodic orbits of finite size around the collinear points has not been solved. Instability may be conjectured for any value of the mass parameter.

Case 21111. The triangular points are stable according to the linear stability analysis when $0 < \mu < \mu_0 = 0.03852$, as shown in Section 5.4.1.

Case 21121. The nonlinear stability analysis of the triangular points for the range $0 < \mu < \mu_0$ was performed by Leontovic who found stability.

Case 21212. Periodic orbits of finite size around the triangular points are stable in the linear sense in the range $0 < \mu < \mu_0$, as shown by Pedersen, Deprit, and Rabe, excepting the horseshoe orbits.

Case 21222. The stability problem in the nonlinear sense of periodic orbits of finite size around the triangular points for $0 < \mu < \mu_0$ is unsolved.

Case 22111. The triangular points are unstable in the range $\mu_0 < \mu < \frac{1}{2}$ according to the linear analysis as shown in Section 5.4.2.

Case 22121. The triangular points are unstable in the range $\mu_0 < \mu < \frac{1}{2}$ according to the nonlinear analysis based on the discussion of Section 5.2.2.3.

Case 22212. Periodic orbits of finite size (limiting orbits) around the triangular points when $\mu_0 < \mu < \mu_1 = 0.044$ are stable according to the linear analysis performed by Deprit.

Case 22222. The problem of nonlinear stability of periodic orbits of finite size (limiting orbits) around the triangular points when $\mu_0 < \mu < \frac{1}{2}$ is unsolved.

From this discussion the conclusion may be reached that the problem of the nonlinear stability of periodic orbits around the triangular points when the value of the mass parameter is smaller than the critical value ($\mu_0 = 0.03852$) is not solved. The corresponding linear problem leads to a critical case with pure imaginary characteristic roots, and extrapolation of the result of the stability investigation from the linear to the nonlinear case cannot be made directly. All other cases either are solved or are amenable to conjectures.

5.6 Applications

5.6.1 Trojans

The famous and classical application of Chapter 5 is the theory of the Trojan asteroids. These were predicted by Lagrange in 1772. Observational verification came in 1906 when the first member of the Trojan group, 588 Achilles, was discovered near the triangular libration point of the sun-Jupiter system.

Table I is an inventory of Trojans prepared from the 1965 list of minor planets. The symbol $\bar{n}$ stands for the mean motion per ephemeris day in seconds of arc and a is the semimajor axis in astronomical units. Note that for Jupiter $\bar{n} = 299''.13$ and $a = 5.2027$. The last column gives the number of the libration point. Entry 4 refers to L_4 or the triangular libration point with positive ordinate. A planet near this point follows Jupiter. The Trojans with $k = 5$ are ahead of Jupiter. In addition to these fourteen Trojans, thirteen more (faint) members of the group were discovered in 1965. The final cataloguing had not taken place by 1966.

TABLE I

TROJAN ASTEROIDS

No.	Name	$\bar{n}$	a	k
588	Achilles	298.264	5.2112	5
617	Partoclus	298.645	5.2068	4
624	Hector	306.170	5.1211	5
659	Nestor	296.080	5.2368	5
884	Priamus	297.814	5.2164	4
911	Agamemnon	305.120	5.1328	5
1143	Odysseus	300.443	5.1860	5
1172	Äneas	300.254	5.1881	4
1173	Anchises	308.455	5.0958	4
1208	Troilus	302.757	5.1595	4
1404	Ajax	302.437	5.1631	5
1437	Diomedes	304.209	5.1431	5
1583	Antilochus	292.784	5.2760	5
1647	Menelaus	297.306	5.2224	5

The importance of the deviations of the sun-Jupiter-asteroid system from the assumptions of the restricted problem depends on the questions to be studied. The eccentricity of Jupiter's orbit (approximately 0.05) and the perturbing effects of the other planets may be neglected at first in studying the stability of the triangular libration points.

The mass parameter of the sun-Jupiter system 0.000953875 is less than $\mu_0 = 0.03852$; therefore, harmonic motion around $L_{4,5}$ is possible. The eigenvalues of the characteristic equation give the two distinct frequencies [cf. Appendix IID] $s_1 = 0.080464$ and $s_2 = 0.996758$. Note that the approximate formula given in Section 5.4.1, $s_1 = (6.75\mu)^{1/2}$ and $s_2 = 1 - 3.375\mu$, gives the same numbers with an error less than one part in 300.

The dimensionless angular velocities or mean motions s_1 and s_2 give the dimensionless periods of oscillation $t_1 = 2\pi/s_1 = 78.090790$ and $t_2 = 6.303621$. The period of rotation of the coordinate system in dimensionless units is $t_0 = 2\pi$, since the angular velocity is unity. The period of rotation of the system in dimensional units is equal to the orbital period of revolution of Jupiter, $T_J = 11.862$ tropical years. The mean motion of the system is that of Jupiter: $n_J = 2\pi/T_J = 0.52977$ rad/yr. The general relation between dimensional time (t^*) and dimensionless time (t) is $t = n_J t^*$; consequently, the dimensional period of oscillation becomes

$$t_i^* = \frac{t_i}{n_J} = \frac{2\pi}{s_i}\frac{T_J}{2\pi} = \frac{T_J}{s_i}. \tag{83}$$

This equation gives $t_1^* = 147.4$ yr, and $t_2^* = 11.90$ yr. Observe that the short period is about the same as Jupiter's orbital period. The oscillation with the long period (t_1^*) is called *libration*.

The value of the mass parameter for the other planets is smaller than for Jupiter. The stability criterion for the triangular libration points, $\mu < \mu_0 = 0.03852$, is therefore satisfied for all planets when considered each with the sun forming restricted problems. Collections of asteroids, however, at the points L_4 and L_5 have been observed only for Jupiter, presumably since the conditions of the restricted problem are not so well satisfied for the other planets. The principal deviations from the assumptions of the restricted problem are the perturbations by Jupiter at the equilibrium points of the other planets and the nonzero eccentricities of the planetary orbits.

5.6.2 Equilateral libration points in the earth-moon system

It is of considerable interest to investigate the dynamics of particles around the equilateral libration points of the earth-moon system. The gravitational effect of the sun, as well as the eccentricity of the moon's orbit, will influence the stability of the triangular libration points and the orbit of the body with small mass around it. The mass parameter is $\mu = 0.012141$; therefore, the condition $\mu < \mu_0$ is satisfied. The sun, earth, moon, and the particle form a system of four bodies which is *restricted* in the sense that the orbits of the three primaries are not influenced by the fourth body, be it an artificial satellite, a space probe, or an asteroid. The location of the equilateral libration point in the earth-moon system has only geometric significance in the problem of four bodies since the existence of the sun's gravitational field will disturb the equilibrium of the forces acting on the fourth body. A systematic approach to the problem is as follows. First the assumptions of the restricted problem of three bodies are accepted and the linearized and nonlinear equations are studied in Section 5.6.2.1, Parts (A) and (B). Then the model is extended to include the sun in Section 5.6.2.2. Here a restricted problem of four bodies is considered first [Part (A)] and then the actual ephemerides are used [Part (B)].

5.6.2.1 *Linear and nonlinear effects in the restricted problem of three bodies.* (A) The linearized equations of the restricted problem and their solutions for the earth-moon case with $\mu = 0.012141$ are considered in this part and the effect of the sun is ignored. The dimensionless frequencies are

$$s_1 = 0.29808 \qquad \text{and} \qquad s_2 = 0.95454.$$

Using $T_☾ = 27^{d}.322$ as the orbital period of the moon, we have $t_1^* = 91^{d}.66$ and $t_2^* = 28^{d}.62$ for the long and short periods. Depending on the initial conditions the motion may contain both periods or only one. If the solution contains only the long-period motion, then the ratio of the length of the major and minor axes is $1/\bar{\alpha}_1 = 5.13557$. The ratio of the axes of the short-period motion becomes $1/\bar{\alpha}_2 = 2.03444$; both results are obtained from Eq. (58). The direction of the major axis is obtained from Eq. (45), which gives $\alpha = 29^{\circ}.6932$.

The intersection of the $\bar{\eta}$ axis with the x axis is at $x_0 = 0.005982$.

The motion which is the combination of the long- and short-period solutions is the general solution of the linearized problem. The coefficients $\bar{\alpha}_1$ and $\bar{\alpha}_2$ furnish the general solution as given by Eqs. (56). If at $t = 0$, $\bar{\xi}_0 = \bar{\eta}_0 = 0$ and also $\dot{\bar{\xi}}_0 = 0$, the solution becomes

$$\begin{aligned} \bar{\xi} &= \frac{2\dot{\bar{\eta}}_0}{s_2^2 - s_1^2}(\cos s_1 t - \cos s_2 t), \\ \bar{\eta} &= \frac{2\dot{\bar{\eta}}_0}{s_2^2 - s_1^2}(\bar{\alpha}_2 \sin s_2 t - \bar{\alpha}_1 \sin s_1 t). \end{aligned} \tag{84}$$

Note that for small values of t this solution is

$$\bar{\xi} \cong \dot{\bar{\eta}}_0 t^2, \qquad \bar{\eta} \cong \dot{\bar{\eta}}_0 t,$$

and after elimination of t we have

$$\bar{\xi} \cong \bar{\eta}^2/\dot{\bar{\eta}}_0 . \tag{85}$$

Positive $\dot{\bar{\eta}}_0$ takes the particle from the origin to the first quarter of the $\bar{\xi}, \bar{\eta}$ plane with $\bar{\xi} > 0$, $\bar{\eta} > 0$, and negative $\dot{\bar{\eta}}_0$ results in having the initial part of the orbit in the third quarter where $\bar{\xi} < 0, \bar{\eta} < 0$. Figure 5.14 shows the beginning of the orbit.

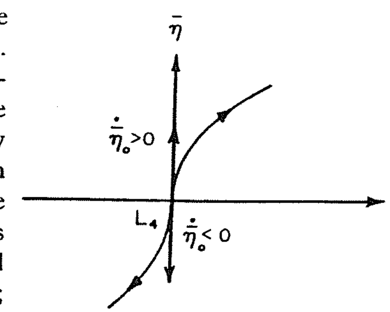

Fig. 5.14. Orbits leaving a triangular libration point.

Periodic solutions may be obtained when the initial conditions are selected so that only one frequency is present in the solution, as shown on Fig. 5.6. Periodic solutions may be obtained even when both frequencies are present in the solution, provided that the ratio s_1/s_2 is rational; $s_1/s_2 = n/m$ with n and m positive integers. For an arbitrary value of μ, we have

$$\frac{s_1}{s_2} = \frac{1 - [1 - 27\mu(1-\mu)]^{1/2}}{[27\mu(1-\mu)]^{1/2}}; \tag{86}$$

therefore, s_1/s_2 in general is irrational. On the other hand, the actual value of μ as a physical quantity is never given precisely in nature. Consequently, periodic orbits may always be obtained by allowing slight modifications of the value of μ. For instance, in the present example let

$$A = \frac{m^2 - n^2}{m^2 + n^2}. \tag{87}$$

Then the condition for having a rational number for s_1/s_2 is expressed by the equation

$$\mu = \frac{1 - [1 - 4/27(1 - A^2)]^{1/2}}{2}. \tag{88}$$

Since $s_2/s_1 = 3.2023 \cong 16/5$, we have $n = 5$, $m = 16$, $A^2 = 0.67579$, and $\mu = 0.01213$. Equations (87) and (88) follow from equating the right side of Eq. (86) with n/m and solving it for μ.

An orbit containing both frequencies and corresponding to Eqs. (84) is shown on Fig. 5.15. The initial conditions were listed previously, excepting the value of $\bar{\eta}_0$. Introducing script letters for the notation of new variables,

$$\mathscr{x} = \frac{\bar{\xi}(s_2^2 - s_1^2)}{2\bar{\eta}_0}, \qquad \text{and} \qquad \mathscr{y} = \frac{\bar{\eta}(s_2^2 - s_1^2)}{2\bar{\eta}_0},$$

we have, from Eq. (84),

$$\begin{aligned} \mathscr{x} &= \cos s_1 t - \cos s_2 t, \\ \mathscr{y} &= \bar{\alpha}_2 \sin s_2 t - \bar{\alpha}_1 \sin s_1 t. \end{aligned} \tag{89}$$

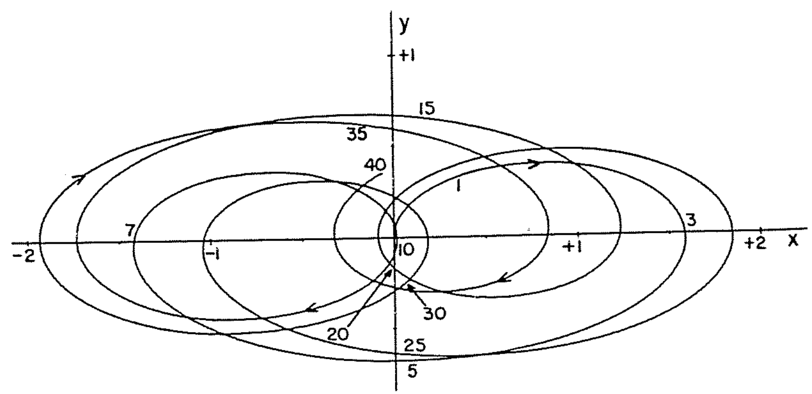

FIG. 5.15. Linearized solution around the earth-moon triangular libration point for $\frac{1}{2}$ year.

The range of motion is given by $-2 \leqslant \mathscr{x} \leqslant 2$ and $-(\bar{\alpha}_1 + \bar{\alpha}_2) = -0.6863 \leqslant \mathscr{y} \leqslant 0.6863 = \bar{\alpha}_1 + \bar{\alpha}_2$. The equation of the envelope may be approximated by

$$\mathscr{x}^2/4 + \mathscr{y}^2/0.4710 = 1, \tag{90}$$

since the semiminor and semimajor axes are 0.6863 and 2.

The orbit starts with $\dot{\mathscr{y}} > 0$ since from Eq. (89) we have $\dot{\mathscr{y}}(0) = \bar{\alpha}_2 s_2 - \bar{\alpha}_1 s_1 = \frac{1}{2}(s_2^2 - s_1^2) > 0$. The dimensionless time t is related to the actual time t^* through the mean motion of the moon $n_☾$:

$$t^* = \frac{t}{n_☾} = \frac{tT_☾}{2\pi} = \frac{27.322}{2\pi}t = 4.348t,$$

where t^* is in days since the period of the moon is $T_☾ = 27^{\mathrm{d}}.322$. The total real time corresponding to Fig. 5.15 is approximately one-half of a year. The numbers along the orbit represent the elapsed non-dimensional time.

The orbit is *not* periodic since s_1/s_2 is an irrational number. On the other hand, if $s_1 = 0.29808$ and $s_2 = 0.95454$ are accepted as the actual mean motions, then s_1/s_2 *is* a rational number and the orbit *does* become periodic. If the periods associated with s_1 and s_2 are $T_1 = 2\pi/s_1$ and $T_2 = 2\pi/s_2$ and if

$$s_2/s_1 = T_1/T_2 = m/n,$$

then the period of the resultant motion becomes

$$T^* = mT_2 = nT_1 = 2\pi m/s_2 = 2\pi n/s_1 = 2\pi/s^*, \tag{91}$$

where s^* is the mean motion of the resultant orbit. (These and the following considerations are also discussed from another point of view in Section 8.2.)

Equation (91) may now be used to evaluate T^* provided the values of m and n are found. If s_2/s_1 is approximated by a selected ratio, m/n, then of course $mT_2 \neq nT_1$ and recurrence occurs with an error ϵ. It is required that the inequality

$$| T^*/T_i - n_i | < \epsilon \tag{92}$$

be satisfied for every i, so in our case for $i = 1$ and 2.

Selecting $m = 16$, $n = 5$ we have

$$s_2/s_1 = 3.202294685 \qquad \text{and} \qquad m/n = 3.20.$$

The two values of s^* become

$$s_1^* = s_1/n = 0.059616 \qquad \text{and} \qquad s_2^* = s_2/m = 0.059659.$$

The corresponding periods become

$$T_1^* = 2\pi/s_1^* = 105.3943 \quad \text{and} \quad T_2^* = 105.3188.$$

The rounded mean value of these periods is $T^* = 105.36$ in dimensionless time units, which corresponds to $458^{\text{d}}\!.105$. The recurrence errors are

$$T^*/T_1 - n = -0.001625 \quad \text{and} \quad T^*/T_2 - m = 0.006271.$$

So the orbit is recurrent with a period of T^* and with an error of recurrence of $\epsilon < 0.0063$, or $4^{\text{h}}\!.32$, or 0.4 % of the period T^*. This error may be converted into a positional error:

$$\epsilon^* = \{[x(T^*) - x(0)]^2 + [y(T^*) - y(0)]^2\}^{1/2},$$

which for the present example gives $\epsilon^* < 0.021 = 8073$ km.

The recurrence error may be reduced when a better approximation for s_2/s_1 is used. Let, for instance, $m = 1601$ and $n = 500$, giving $m/n = 3.202$, $T^* = 10{,}538.94$ or $125^{\text{y}}\!.54$, and $\epsilon \cong 0.02$ % of the period.

The orbit will cover the area of the ellipse given by Eq. (90) uniformly if the ratio s_1/s_2 is an irrational number. In the actual computed case when s_1 and s_2 are approximated by 5 decimals the orbit is periodic even though the commensurability is of very high order, and consequently the orbit will not cover completely the inside of the region of motion as $t \to \infty$.

Figure 5.16 shows the development of the orbit as the time increases. The nondimensional time at the end of the orbit is $T = 110$, and

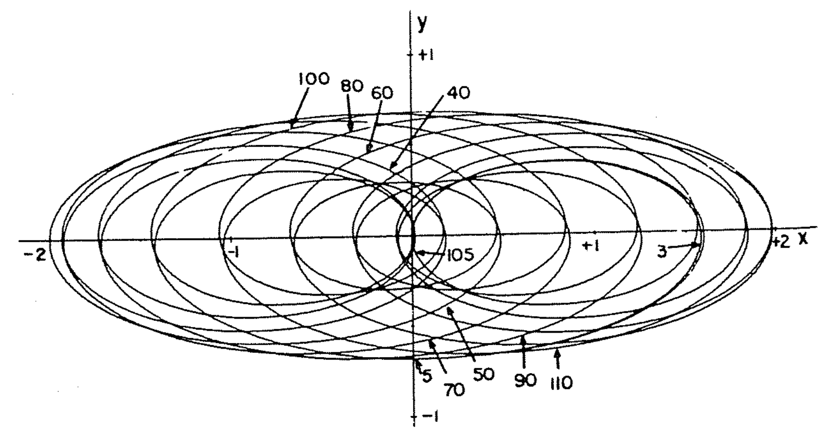

Fig. 5.16. Linearized solution around the earth-moon triangular libration point for one recurrency $\cong$ 1.3 years.

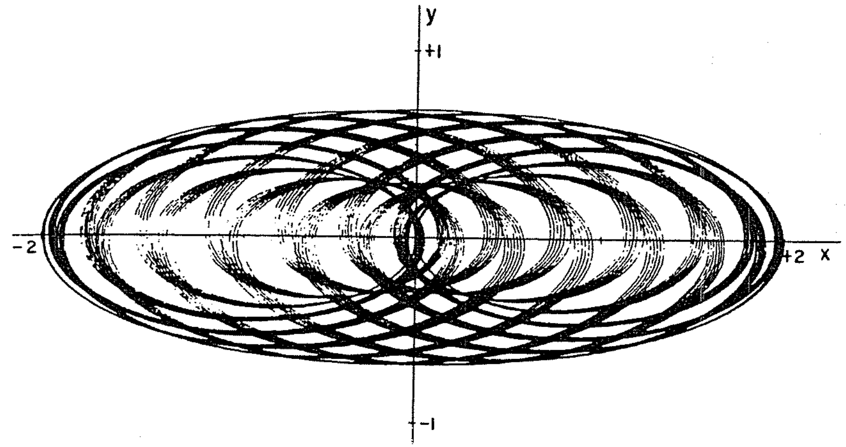

Fig. 5.17. Linearized solution around the earth-moon triangular libration point for five recurrencies $\cong$ 6.3 years.

therefore the figure shows the close recurrence at $T^* = 105.36$. The large features of the motion begin again at T^* even though the orbit does not repeat itself exactly. Figures 5.17 and 5.18 show the orbit up to $t \cong 500$ and $t \cong 1500$ corresponding to about 5 and 15 approximate recurrences or to 6.3 yr and to 19 yr.

This concludes the linear aspects of the problem of motion around the triangular libration points in the earth-moon field. It is to be re-

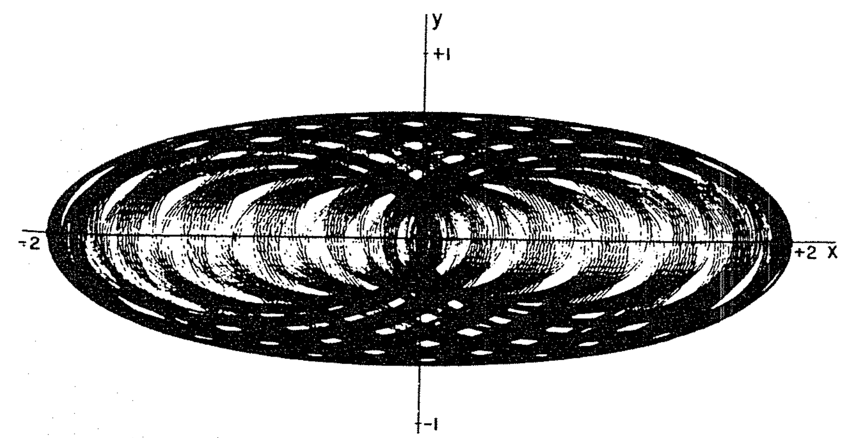

Fig. 5.18. Linearized solution around the earth-moon triangular libration point for fifteen recurrencies $\cong$ 19 years.

membered that the preceding remarks are pertinent and strictly valid only for the linearized equations of motion.

(B) The period of the long-period orbit increases and that of the short-period orbit decreases as the size of the orbit increases as discussed in Section 5.5.3.1, and consequently the ratio $s_2/s_1 = T_1/T_2$ increases from its value of 3.2023 corresponding to infinitesimal amplitudes. This is important since it means that the nonlinearity moves the commensurability farther away from the 3 : 1 ratio. The change in period may be estimated from the equation given by Pedersen and quoted in Section 5.5.3.1:

$$T_i = 2\pi/s_i + K(s_i)\,\epsilon^2,$$

where $K(s_1) \cong 13.9$ and $K(s_2) \cong -0.6$. Figure 5.19 shows the long-period solution for $\epsilon = 0.033$. This figure may be compared with Figure 5.8, where μ is close to the value applicable to the earth-moon case. Note that Fig. 5.8 was obtained by an analytic approximation, while Fig. 5.19 shows the results of numerical integration of the differential equations of motion. On Fig. 5.6, point A may be considered the starting point of the orbit obtained after linearization with $\bar{\xi}_0 = 0$, $\bar{\eta}_0 = \epsilon\bar{\alpha}_1$, $\dot{\bar{\xi}}_0 = \epsilon s_1$, $\dot{\bar{\eta}}_0 = 0$. Figure 5.19, when compared to Fig. 5.6, shows that the distortion is largest at the ends of the major axes since the initial conditions specify the variables at the ends of the minor axes and there should be no difference between the actual and the linearized solutions at point A. The solution of the nonlinear equations shows smaller values of $\bar{\eta}$ away from point A as if the major axis of the ellipse would bend in an arc around m_1. These comments become, of course, invalid as the size of the ellipse increases, and the nonlinear effects become more significant.

On the other hand, Fig. 5.19, after Michael, shows that the effects of nonlinearity are restricted to only minor modifications of the orbit, even when the total librational motion amounts to over 18,000 km.

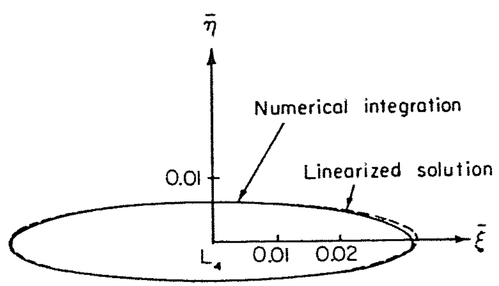

Fig. 5.19. Nonlinear effect in the restricted problem of three bodies on the long-period orbit around the earth-moon triangular libration point (Michael, 1963, Ref. 83).

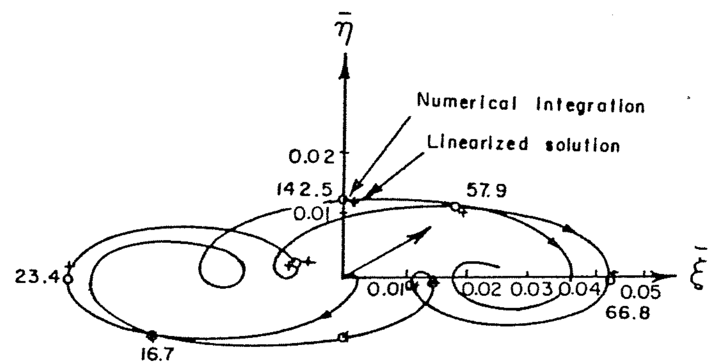

Fig. 5.20. Comparison of numerical integration and linearized solution when both frequencies are present in the restricted problem of three bodies. Times shown are in days (Michael, 1963, Ref. 83).

Generation of families of periodic orbits starting with small elliptic single-frequency periodic orbits and modifying the initial conditions to accommodate the nonlinearity present will be discussed in Chapter 9, devoted to the quantitative results in the restricted problem. At this point it suffices to mention that periodic orbits of large amplitudes can be established around the triangular libration points for the Trojan asteroids as well as for the earth-moon problem.

The effect of finite amplitude on orbits with both frequencies present is shown in Fig. 5.20 after Michael.

5.6.2.2 *Problems of four bodies.* (A) The introduction of a fourth body—in this case the sun—into the system may be done in several ways. The restricted problem of four bodies will now consist of the sun, the earth, and the moon as the three primaries, and the fourth body will be of infinitesimal mass, so as not to influence the motion of the primaries. In the restricted problem of three bodies the motion of the primaries satisfies precisely the equations of motion of the two-body problem. Consequently the logical generalization is to establish a solution of the problem of three bodies and find the motion of the fourth body in the field produced by the presumably known motion of the three primaries. Since no closed-form solution is known for the three-body problem of the sun, earth, and moon, such a generalization of the restricted problem of three bodies is rather difficult. Another possibility is to *assume* the motions of the three primaries and, without attempting to establish the exact solution of the equations governing these motions, accept an approximate solution. Such an approximation may be, for instance, that the mass center of the earth-moon system moves on an elliptic orbit around the sun, and the earth and the moon are also moving in elliptic orbits around their mass center. The plane of the orbit of the mass center of the earth-moon system, the plane of the ecliptic, is inclined relative to the plane containing the orbits of the earth and the moon. An admittedly simpler approximation is to assume circular orbits and zero inclination. Whatever the approximation, the idea is that as long as the orbits of the primaries are known functions of time the differential equation of the fourth body can be written as $\ddot{\mathbf{r}} = \mathbf{f}(\mathbf{r}, t)$. Furthermore, if the orbits of the three primaries may be expressed as relatively simple functions of the time, the solution of the differential equations of the fourth body may be approximated by analytical methods. If, on the other hand, the solution of the problem of three bodies forming the three primaries is available only in tabular form, then the equations of the fourth body may be treated by numerical integration. This latter approach corresponds to the last step in the previously outlined procedure; see Section 5.6.2.2, Part (B).

It is essential to notice that such restricted problems of four bodies face serious criticisms since they clearly neglect so-called indirect effects and consequently important secular perturbations may be neglected. The sun perturbs the fourth body of infinitesimal mass but it is not allowed—at least in the restricted four-body model—to influence the motion of the moon. To evaluate without detailed analysis the merits of such restricted four-body models is difficult since, in spite of the inconsistency of the dynamics involved, there is evidence that for certain problems this model may give reliable answers. For completeness' sake the major results of such a four-body approach are given in the following. The reader is warned to apply this model with caution.

Consider the artificial set of assumptions of circular motion of the center of mass of the earth-moon system around the sun, circular orbits of the earth and of the moon around their center of mass, and constant angle of inclination between the ecliptic and the earth-moon plane. Figure 5.21 (after Tapley) shows the orbit of the fourth body of infinitesimal mass, relative to the triangular libration point of the earth-moon system in the orbital plane of the earth and moon. The ξ axis is parallel with the line connecting the earth with the moon, and the η axis is perpendicular to the ξ axis and it is in the plane of the orbits of the earth and moon. The motion begins when the earth, the moon, and the sun are in the same line, in this order. Since L_4 is *not* an equilibrium point in this system, the fourth body experiences a resultant force and starts moving away from L_4 even when its initial velocity is zero. Figure 5.21 shows the motion for the first 250 days, and Fig. 5.22, also after Tapley, illustrates the envelopes of the displacement versus time curve. There is an indication that both the upper and lower envelopes show a periodicity of approximately 1600 days. This long-period variation of the displacement corresponds to a pulsating motion of the orbit and may be connected with a near commensurability in the system. Figure 5.21 was prepared from computations which used $T_S = 365^{\mathrm{d}}.287$ and $T_M = 27^{\mathrm{d}}.287$ for the period of the circular motion of the earth-moon barycenter around the sun and for the period of

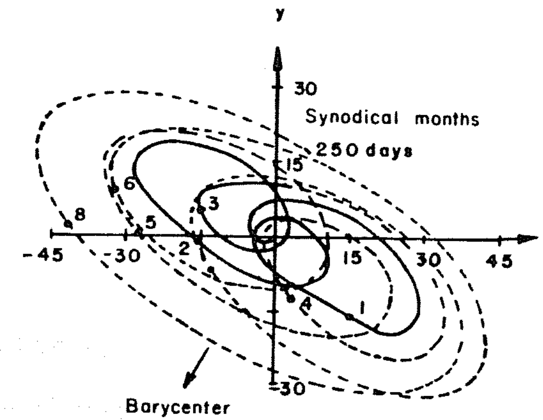

FIG. 5.21. Motion around L_4 in the restricted problem of four bodies; x and y in thousands of kilometers (Tapley, 1965, Ref. 89).

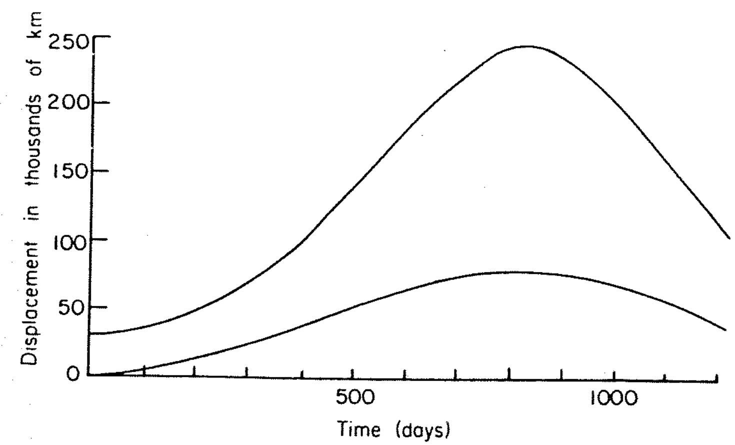

FIG. 5.22. Envelopes of the displacement versus time curve for motion around L_4 in the restricted problem of four bodies (Tapley, 1965, Ref. 89a).

the earth-moon system around its center of mass, respectively. The long-period motion of the linearized orbit around the triangular libration point has therefore the period of $T_1 = T_M/s_1 = 91^{\mathrm{d}}.537$, which gives the near-commensurability of $T_S/T_1 = 3.991$. The nonlinear effects introduce a change in T_1, amounting to $\Delta T_1 = 5^{\mathrm{d}}.32$, which may be computed from $\Delta T_1 = (T_M/2\pi)\, K(s_1)\, \epsilon^2$, using the previously mentioned value of $K = 13.86$, and the value of $\epsilon = 0.3$ as an acceptable average figure. The period of the long-period motion considering nonlinear effects becomes $\bar{T}_1 = 96^{\mathrm{d}}.86$, and the small divisor is $4\bar{T}_1 - T_S = 22^{\mathrm{d}}.15$. The period of the perturbation associated with this near-commensurability, therefore, becomes

$$\frac{\bar{T}_1 T_S}{4\bar{T}_1 - T_S} = 1597^{\mathrm{d}}.4. \tag{93}$$

The reader may compare the preceding considerations with the observations of Section 8.2, Part (D), about the great inequality. The commensurability of the periods of Jupiter and Saturn is near to 5 : 2 and $5T_{\jupiter} - 2T_{\saturn} = 144^{\mathrm{d}}.5$, giving

$$\frac{T_{\jupiter} T_{\saturn}}{5T_{\jupiter} - 2T_{\saturn}} = 918y.$$

The restricted problem of four bodies may also be linearized. The differential equations of motion contain in this case periodic forcing terms with angular velocity $\omega - \Omega$ and $2(\omega - \Omega)$, where ω and Ω are the mean motions of the moon and sun. The solution of the linearized

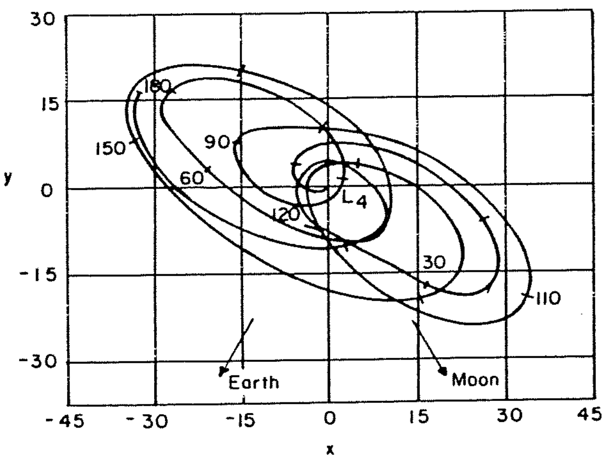

Fig. 5.23. Orbit of the linearized restricted problem of four bodies; x and y in thousands of kilometers (deVries, 1964, Ref. 88).

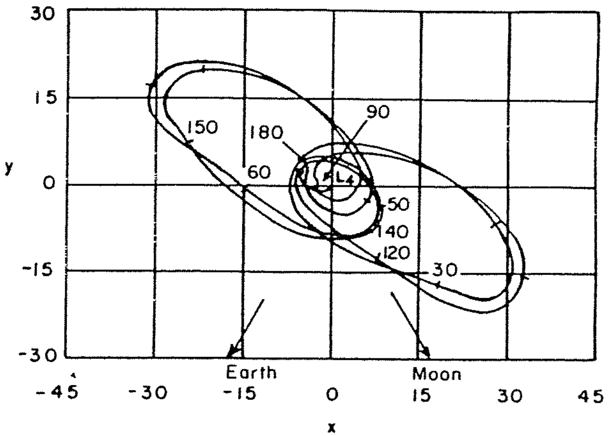

Fig. 5.24. Orbit of the complete restricted problem of four bodies; x and y in thousands of kilometers (deVries, 1964, Ref. 88).

equations is shown in Fig. 5.23, and the orbit obtained by numerical integration of the complete equations of the restricted problem of four bodies is given in Fig. 5.24; both orbits according to deVries. The initial configuration in both cases consists of conjunction when the sun, earth, and moon form a straight line (in this order). The initial conditions for the fourth body are those of the libration point of the restricted problem of three bodies. Figures 5.23 and 5.24 show remarkable agreement.

(B) The final step in the evolution process from the restricted problem of three bodies to the actual physical problem consists of using the ephemerides of the sun, earth, and moon to compute the motion of the fourth body around the triangular libration points. The possibility of a number of several intermediate steps between the restricted four-body model and the complete physical picture is to be kept in mind. For instance, the analytical approach to the problem of elliptic orbits of the primaries may be mentioned, since the existence of equilibrium points in this case is well established. This problem is discussed in Chapter 10. Another approximation with analytic feasibility is the replacement of the effect of the sun by a uniform ring which will give the desired secular effect.

The problem, which from a practical point of view is probably of primary importance, is the establishment of initial conditions which result in minimum displacement from the triangular libration point. It was mentioned before that the libration point as an equilibrium position does not exist generally when the assumptions of the restricted problem of three bodies are not valid and a particle placed at L_4—or for that matter, anywhere—will not preserve its zero relative velocity to the primaries. The question naturally comes up as to what initial conditions the third body should have—instead of the zero initial velocity required in the restricted problem—so that the amplitude of its motion be as small as possible, at least for a certain limited length of time. This is a problem, the solution of which may be approximated by a linearized four-body problem. The results of such an analysis can be improved by a process of trial and error and by numerical integration using the actual ephemerides of the participating bodies. An example calculated by Wolaver with starting date at the beginning of July 1964, for instance, shows that properly selected initial conditions reduce the envelope of the motion by 55 % during the first 200 days and by over 30 % for the next 275 days. These numbers refer to the comparison of the numerically integrated displacements relative to the instantaneous equilateral points when the fourth body of infinitesimal mass is released with zero velocity on one hand and when it is given a set of "proper" initial conditions on the other hand. The example demonstrates two facts. First, it shows that a linearized restricted four-body model might be used for approximating the result of certain problems, and the fact that it is not recommended as a basis of analytical perturbation studies does not necessarily mean that it should not be used judiciously. Second, the example shows the importance of initial conditions on the development of the actual motion. The set of initial conditions includes in this case also the calendar date of the beginning of the motion because of the importance of the initial sun-earth-moon configuration. Such is *not* the case when the restricted problem of three bodies is used as the model. Figure 5.25 shows the result of numerical integration using the ephemerides of the sun, earth, and moon.

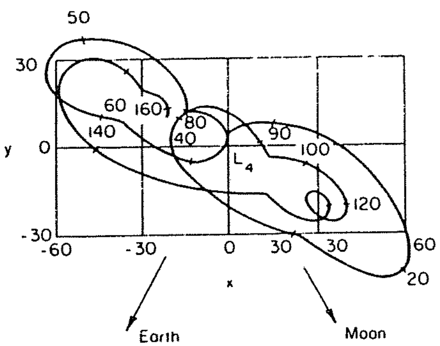

FIG. 5.25. Orbit using ephemeris values; x and y in thousands of kilometers (deVries, 1964, Ref. 88).

This concludes the discussion of the equilateral libration points in the earth-moon space excepting a few remarks which may be considered to be the physical consequences of the dynamics involved. While the sun-Jupiter system clearly possesses a collection of asteroids at the triangular libration points, the Trojans, the ability of the earth-moon system to collect debris or dust at the corresponding points is still in question. According to unconfirmed reports by Kordylewski, clouds (a magnitude or two fainter than the Gegenschein) in the vicinity of L_4 exist in the earth-moon space. The major perturbing effect on the Trojans is Saturn, while the stabilizing forces come from the sun and Jupiter. The major perturbation on the earth-moon libration clouds is the sun and the stabilizing effects are derived from the earth and the moon. This explains why the existence of accumulated material at L_4 or L_5 in the earth-moon system is not at all obvious. Bodies at the triangular libration points of sun-planet systems would face perturbations from Jupiter; therefore, it is not surprising that the only currently known material accumulation is confined to the sun-Jupiter system.

If the earth-moon system is able to maintain material at L_4 or L_5, then this material according to certain theories might be of lunar origin produced by meteoritic impacts on the moon. As meteors and meteoroids impact the lunar surface, dust and particles are ejected. Some of these ejected particles might actually congregate at the equilateral libration points. The origin of dust at L_4 or L_5 is not the earth since the atmosphere does not favor either frequent impacts or the departure of the dust. The possibility exists that the particles at L_4 and L_5, if any, are not from the moon but from elsewhere in the solar system. The highly speculative comment may be made to this assumption that the probability seems to be smaller for the capture of particles of general origin than of particles of lunar origin at the earth-moon libration points.

The constant of relative energy of particles captured at the triangular libration points must be high since these points are surrounded by zero velocity curves, penetration of which is forbidden by particles for which $C > 3$. Particles for which $C \leqslant 3$ may travel anywhere in the plane of motion, so there are no forbidden regions for them. Not only must the particles be able to enter the regions of high relative energy, but they also must arrive at low relative velocities in order to be captured. The orbit of the point L_4 in the restricted problem of three bodies is a circle of unit radius in the fixed coordinate system. The orbit of the moon in the same system and within the framework of the restricted problem is also a circle of radius $1 - \mu$, that is about the same as of L_4. Therefore it is not inconceivable that dust leaving the moon, because of meteoroid impact, acquires the proper velocity to be captured in the vicinity of the equilateral libration points.

In connection with this problem a paradox of the triangular libration points may be mentioned. Accepting the validity of the model of the restricted problem of three bodies, the triangular libration point is stable for sufficiently small values of μ, and a particle of infinitesimal mass at L_4 or L_5 will stay there. Since the particle has zero relative velocity at the libration points, it must arrive there with such a velocity. But this is impossible since if the third body can arrive at L_4 with zero velocity it also must be able to leave L_5 starting with zero velocity according to the symmetry properties to be shown in Section 8.6.3. This latter does not occur since L_5 is an equilibrium point. Asymptotic approaches to, or departures from, $L_{4,5}$ occur when $\mu > \mu_0$. Now, when $\mu < \mu_0$, the triangular libration points are stable but no particles can arrive there. When $\mu > \mu_0$, particles can have trajectories ending at $L_{4,5}$ but these points are unstable. The paradox is established by stating that no particles may arrive at the triangular libration points under stable conditions, only under unstable conditions, when, of course, small perturbations will clear away the particles. A part of the trivial solution of the paradox may be found in the fact that particles are not *at* but *near* the libration centers.

Another remark is related to the possibility of determining the mass ratio of the primaries by inverting the characteristic equation and solving it for the mass parameter. Accepting as a first approximation the linearized treatment and the assumptions of the restricted problem, the frequency of the motion of an artificial or natural body may be used to obtain the mass parameter of the primaries. If the third body has been placed in a single-frequency orbit and its observed period is τ, then $s = T_{☾}/\tau$ may be obtained very accurately and the earth-moon mass parameter is determined experimentally by using

$$\mu = \frac{1 - [1 - 16s^2(1 - s^2)/27]^{1/2}}{2}.$$

Harmonic analysis will give s_1 and s_2 of an orbit which has not been generated by special initial conditions, and μ can once again be determined provided observations are carried out for longer than 90 days.

5.6.3 Collinear points

The unstable nature of these points permits only a few applications; nevertheless, three aspects may be mentioned: the astronomical significance of motion around L_1, a possible interest in space exploration of motion around L_2, and finally motion around all three collinear points at high values of the mass parameter in the study of binaries.

The location of L_1 in the sun-earth system is given in Appendix IE of Chapter 4 as $x_1 = -1.01007019377695$, and, since $\mu = 3.0359 \times 10^{-6}$, the dimensionless distance between the earth and L_1 is $\mu - 1 - x_1 = 0.0100732336695$ au or 1.50696×10^6 km, which is the distance to the Gegenschein from the earth and is equivalent to a geocentric parallax of 15 min of arc. Presumably dust particles collect at this point and in spite of the unstable character of L_1 the particles remain for a short time if they arrive with the proper initial conditions.

Using $\Omega_{xx}(L_1) = 8.8811025512$ from Appendix IE of Chapter 4, and $s = 2.0570213486$ from Appendix IF of Chapter 5, the eccentricity of the orbits associated with the linearized Gegenschein theory becomes $e = 0.9495053$. If the period of the earth around the sun is $365^{\mathrm{d}}.25636$, then the orbital period of the trapped particles is $T = 177^{\mathrm{d}}.56578$.

Note that the corresponding values for the limiting case $\mu \to 0$ are $e = 0.9501749625$ and $T = 176^{\mathrm{d}}.31656$.

Motion around L_2 in the earth-moon system may be of considerable interest in space exploration. The location of L_2 is given by $x_2 = -0.836963$ in Appendix IIE of Chapter 4; consequently, the nondimensional distance between the moon and L_2 is $1 - \mu + x_2 = 0.150896$, using $\mu = 0.012141$. The point L_2 is therefore located 58,000 km from the moon and 326,400 km from the earth. It is not inconceivable that probes equipped with station keeping devices can be placed at this point in spite of the fact that the effect of the sun seems to be such that bounded orbits in the vicinity of L_2 do not exist. We mention that the three periods of the infinitesimal elliptic motion around L_1, L_2, and L_3 in the earth-moon system are $14^{\mathrm{d}}.67$, $11^{\mathrm{d}}.71$, and $27^{\mathrm{d}}.04$.

Finally, it is of interest to note that particles arriving with approximately the "proper" initial conditions at the collinear libration points show considerable inherent instability. Numerical results showing this will be discussed in more detail in Chapter 9, which deals with the quantitative aspects of the restricted problem; nevertheless, investigations pertinent in the case of $\mu = 0.1$ can be mentioned here. Such a value of the mass parameter is of interest in the study of close binary stars. The results show that even when the initial conditions are chosen according to the linearized requirements [see Eq. (22)] the particles perform only one or two oval orbits around L_1 or L_3 before they depart from the neighborhood of the collinear points. It is to be noted that periodic orbits may be obtained by slight modification of the initial conditions which follow from the linearized solution. Nevertheless, the point is that the orbits are very sensitive to changes in the initial conditions.

In conclusion it may be mentioned that both the explanation given for the Gegenschein and placing probes at unstable libration points have many well-founded objections. This leaves the major area of application of the collinear libration points to the field of stellar dynamics.

5.7 Notes

The linearized treatment occupying part of this chapter is based on the variational equations, also called "Jacobian equations," or, in some branches of applied mathematics, "perturbation equations." The first terminology is generally accepted in the literature of celestial mechanics, the second is not recommended, having the disadvantage that it is used also for other equations, and the third expression is considered definitely misleading. Wintner's general discussion [1, p. 65] of the concepts of linearization—in abstraction—is superior to his passages on the applications to the restricted problem [1, p. 370].

The details of linear analysis are given in the text. The temptation to omit all derivations and to refer to standard references, expecially to Whittaker's [2, pp. 177–186 or 427–429] treatments in addition to the previously mentioned passages by Wintner, was overcome by the realization that the application of general theorems to our relatively simple system is hardly justified. Instead of the admittedly beautiful general methods available, the opportunities offered by the special structure of our system are utilized.

Section 5.2.1 discusses stability according to Liapunov. For Wintner's [1, p. 98] discouraging remarks the reader will be compensated by studying one of the delightful short expositions either by La Salle and Lefschetz [3] or by Chetayev [4]. Cesari's comprehensive treatment [5] or one of the latest works by Saaty and Bram [6] are also strongly recommended. The general view on stability in celestial mechanics is documented by Hagihara [7]. The reader will find most helpful the texts by Coddington and Levinson [8] and by Synge [9].

The preferred references for the linearized treatment of motion around the triangular libration points (Section 5.4) are by Plummer [10] and by Martin [11], in this order; the first expositions, however, have been made by Gascheau [12], Routh [13], Gyldén [14], and Charlier [15]. The outstanding textbook discussions may be found in Brouwer's and Clemence's book [16, p. 262] and in Plummer's treatise [17, p. 241]. Charlier's [18, p. 117] detailed, Moulton's [19, p. 298] popular, Danby's [20, p. 194], and McCusky's [21, p. 115] concise treatments should also be mentioned. Pollard's discussion is especially pleasing [22, p. 83].

Plummer's [10] result related to the property of the surface $z = \Omega(x, y)$ and introduced in Section 4.6.2 may be recalled in relation to Section

5.2.3. The characteristic equation (6) may be expressed by the curvatures as

$$\lambda^4 + (4 - 2M)\lambda^2 + K = 0$$

and consequently

$$\lambda_1^2\lambda_3^2 = C_1C_2 \quad \text{and} \quad \lambda_1^2 + \lambda_3^2 = C_1 + C_2 - 4,$$

which last two equations relate the characteristic roots to the principal curvatures C_1, C_2. Here K and M are the Gaussian and the mean curvatures of the surface.

There are many examples for the determination of the orbits of Trojans mentioned in Sections 5.1 and 5.6.1. Here only Brouwer's work [23] on Achilles and Eckert's paper [24] on Hector are mentioned. The more general discussions of the group as such are by Brown [25], by Hertz [26], by Brown and Shook [27], and by Lovett [28, p. 263].

Lagrange's announcement (Sections 5.1 and 5.6.1) of the stationary solution may be found in his "Œuvres" [29, p. 272], but the original work is dated 1772, and it is published in the Prix de l'Académie; see the details of these references in Section 5.8 under [29].

Some references (see, for instance, Pollard's book [22, pp. 48, 50, 58]) speak about Lagrange's solution and Euler's solution when discussing the triangular and the collinear configurations of the general problem of three bodies and use the same terminology describing the stationary solutions of the restricted problem. I traced the origin of this question through Wintner [1, p. 284], through a remark made by Siegel [30, p. 75], and through the references mentioned in connection with Lagrange [27] to finally Euler's original (1765) paper [31]. Euler solved the problem of three bodies which are moving on a fixed straight line and maintain a constant ratio of their distances: $\overline{m_1m_2}/\overline{m_3m_2} = \text{const.}$ This ratio satisfies essentially the same fifth-order algebraic equation as mentioned in the text (cf. Eq. (15), Section 4.3) and constitutes the first particular solution of the problem of three bodies. Euler's solution may be regarded as a special case of the Lagrangian problem, and for this reason the terminology mentioned at the beginning of this paragraph is acceptable. On the other hand, the dynamics of Euler's problem is certainly not applicable to the restricted problem where two of the bodies are fixed and the line containing the three bodies is rotating. Consequently I prefer to speak about Lagrangian points exclusively in the restricted problem and do not follow the terminology of "Euler points" and "Lagrange points." [Note Ebert's [31a] paper on the generalization of Euler's problem and its relation to Contopoulos' method of finding the "third integral" (Section 2.10).]

Tables similar to those given in Appendices I, II, and III, but with fewer entries, have been given by Rosenthal [32] and by Deprit [33], among others. In the same paper Deprit [33] discusses the limiting values of the frequencies as $\mu \to 0$ at the collinear libration points (cf. Section 5.3). The values associated with the collinear points at $\mu = \frac{1}{2}$ as given in the text may be compared with Burrau's and Strömgren's results [34].

The linearized problem around the collinear libration points has been treated by Charlier [35] and by Plummer [36], in addition to the previously mentioned textbooks [1, 16, 17, 19–22].

The problem of singular points encountered by the families of periodic orbits around the collinear points has already been mentioned in Chapter 3 and will be taken up again from a numerical point of view in Chapter 9. The principal reference is by Burrau and Strömgren [34]. The solutions at the collinear points corresponding to the real characteristic root has been studied by Deprit and Henrard [37] and will be discussed in more detail in Chapter 9.

The introduction of the normal coordinates by a rotational matrix, given in Section 5.4.1, is discussed by Martin [11] and by Lanzano [38], among others. The construction shown in Fig. 5.5 appeared in a paper by Deprit and Delie [39]. The equations for the eccentricities of the elliptic orbits around $L_{4,5}$ are proved by Plummer [10], where also the two examples at the end of Section 5.4.1 are discussed. See also Martin's paper [11] and Moulton's book [40, p. 502].

The references for spiral motions around the triangular libration points (Section 5.4.2) are again by Martin [11] and Plummer [10] in addition to Strömgren's papers [41–43], the last two showing the transition from the $\mu > \mu_0$ case to the $\mu = \mu_0$ case.

The discussion of the nonlinear phenomena in Section 5.5 and the existence of the analytically continued periodic orbits is based on Wintner's paper [44] which in turn uses Horn's work [45]. Horn's theorem, finally, is related to Poincaré's theory of centers [46]. (Note once more that the analytic continuations discussed in Chapter 8 use the mass parameter μ as the small parameter and not the characteristic size of the orbit ϵ.) Other existence proofs by Moulton and his school [40] sometimes lack solid foundations according to Wintner [44], since after complex transformations and analytic continuations the final results do not always appear in the real domain. In fact, even Poincaré [47, Vol. 1, p. 160] seems to have forgotten about what Wintner calls the "Realitätsdiscussion" [44], in connection with the analytic continuation of the unstable periodic orbits around the single collinear libration point in Hill's problem. This is probably an example of genuine forgetfulness since Poincaré in Vol. 1 of his *Méthodes* [47] promises that such an investigation will be included in his Vol. 3.

The family of periodic orbits around the collinear libration points, mentioned in Section 5.5.2 and generated from infinitesimal elliptic orbits, was treated first numerically by Burrau [48] and analytically by Perchot and Mascart [49], following Tisserand's [50, Vol. 4, p. 492] recommendation and using Poincaré's [51] method. The family was also established numerically by Darwin [52] and analytically by Plummer [53] using a second-order theory, and again by Plummer [54] with his third-order theory. Several families with different values of μ are discussed in Chapter 9, where the principal references are given to the reports of the Copenhagen Observatory. Regarding the family around the external collinear points for $\mu = \frac{1}{2}$, see papers by Strömgren [55, 56], and regarding orbits around the inner libration point L_2 see Möller's reports [57, 58]. Particles moving around the collinear (and sometimes around the triangular) libration points are called oscillating satellites by Darwin and his followers. The activities of Moulton's school regarding the analytic continuation of infinitesimal periodic orbits around the collinear libration points are contained in Moulton's book [40, pp. 151–216].

The problem of analytic continuation of infinitesimal periodic orbits around the triangular libration points for $\mu < \mu_0$ (Section 5.5.3.1) has been studied by Moulton [40, pp. 299 and 501], Greaves [59], Pedersen [60], Siegel [61], Deprit and Delie [39], and Lanzano [38]. Figure 5.7 is from Mouton [40], and Figs. 5.8 and 5.9 are from Pedersen [60]. The equation for the change of period with amplitude, in its general form, is based on Horn's theorem [45] and in its specific form ($P = K\epsilon^2$) comes from Pedersen [60]. The critical values of the mass parameter follow directly from Horn's work [45] or from the Siegel–Liapunov theorem [30], and it is also established by Pedersen [60] and by Deprit and Delie [39]. Pedersen's theory [60] is of the third order, Deprit's and Delie's [39] of the fourteenth. The only value of $k = s_2/s_1$ which is investigated in any detail is $k = 2$, by Pedersen [62].

The existence proof of these orbits for $\epsilon \neq 0$ follows, of course, as mentioned previously, from Horn's theorem. The reference attributed to Moulton [40] was edited and partially written by him. The article in this work on the existence of these periodic orbits was prepared by Buck [63]. The other independent proof by Greaves [59] may have been completed prior to the appearance of Buck's work. Note that Buck discusses in great detail *seven* different types of generating solutions which are missing in Greaves' paper. Also note that neither Buck nor Greaves utilized Horn's theorem which, in spite of its publication in 1903, had not been applied to this problem until 1930 by Wintner [44]. See also Perron's [64] paper which gives existence proofs for all cases. These proofs are based also on Horn's work.

The changing character of the dependence of the period on the amplitude for the long-period solution, the sign of $dT_1/d\epsilon = 2K(s_1)$, is valid strictly for the $\epsilon \to 0$ limiting case and has been established by Pedersen [60].

The important special case of the Jupiter-sun triangular libration point was investigated by Brown [65], Thüring [66, 67], Deprit and Delie [39], Rabe [68], Willard [69], and Goodrich [70]; the last two are concerned with the short-period motion showing the negative characteristics for the amplitude-period relation.

Regarding the extension of these analytically continued orbits, examples are given in Chapter 9. Here only Brown's conjecture is mentioned according to which the long-period orbits become more and more elongated with increased value of ϵ and they stay close to the arc of the unit circle. When the orbit reaches the x axis it joins with its symmetric counterpart and horseshoe orbits develop. This conjecture seems to be valid as shown numerically by Rabe [68, 71] and Rabe and Schanzle [72] and is discussed in Chapter 9. Thüring on analytical [66] and numerical bases [67, 73] disagrees with the existence of these horseshoe orbits.

The concept of "limiting orbits" described in Section 5.5.3.2 for $\mu = \mu_0 + \delta$, with $\delta > 0$, was first suggested by Brown's third-order analysis [74], then established by Pedersen [75, 76] with his third-order theory. Deprit [77] also computed such orbits with his fourteenth-order expansions suggesting that his d'Alembert series cannot produce larger limiting orbits than the one corresponding to about $\mu = 0.044$. (For the applications of d'Alembert series to celestial mechanics, see [16, p. 79].) The question of existence of periodic orbits for $\mu = \frac{1}{2}$ around L_4 belonging to this family cannot apparently be settled with this method; see also Section 9.4.9. Figures 5.11 and 5.12, describing the characteristics of the limiting orbits, are based on Deprit's work [77], while Fig. 5.13 is from Pederson [76]. The third-order theory does not give a dependence of the Jacobian constant on the orbital parameter, but Deprit's higher-order theory shows this as $C = 3 + a\epsilon^4 + \cdots$, where $a > 0$.

The existence of two discrete families of periodic orbits for $\mu > \mu_0$ described in the text were conjectured by Brown [74]. While this paper is not overly explicit, self-explanatory, or simple, the basic idea is clearly present. The paper is one of Brown's typical significant contributions and, to pay tribute to his brilliance, his idea is outlined in Section 5.5.3.2.

The treatment of the stability questions in Section 5.5.4 may be divided into two parts. First, the results of linear stability analysis, which constitute routine matters even when applied to motions with finite amplitudes, may be settled. The second problem connected with the nonlinear stability aspects is considerably more complicated. The references for the stability analysis of the linearized equations are, of course, the same as the references for the solutions of the linearized equations [10–18, etc.]. The linearized stability analysis of orbits of finite size around the triangular libration points has been performed by Deprit and Delie [39] and Pedersen [60]. The same linearized treatment regarding periodic orbits of finite size around the collinear points was performed by Duboshin [78] showing instability and also demonstrating the existence of these orbits using Liapunov's method [79]. The nonlinear stability analysis referred to in the text regarding the triangular libration *point* is due to Leontovic [80]. It is based partly on Birkhoff's normalization procedure (see also Lanzano [38]) and partly on Arnol'd's theorem [81] described in Chapter 8. A discussion by Richards [82] that the triangular points are not asymptotically stable is closely related to the libration point paradox which is mentioned in Section 5.6.2.2 and which has evolved in the course of my seminars at Yale University. All these subjects are connected with the fact that if a motion around the triangular libration points is such that its Jacobian constant is larger than 3, then the motion must take place outside the curve of zero velocity which encloses the libration point. Therefore, the curve of zero velocity does not, in this case, offer an outside boundary for the orbit. If the Jacobian constant of the particle is less than 3, then there are no curves of zero velocity for the motion at all. The explanation given for the paradox may require familiarity with the symmetry property of Section 8.6.3.

Section 5.6.2.1 on applications emphasizes the earth-moon libration point problems. The measure of error in recurrence given by Eq. (92) and the definition of the recurrence time T^* as given by Eq. (91) are based on papers by Poincaré, Birkhoff, Chandrasekhar, and Frisch, all of which are mentioned in considerable detail in Section 8.10 as refs. [11, 12, 14, 15]. Figures 5.15–5.18 were prepared by D. Pierce.

Figures 5.19 and 5.20 of Section 5.6.2.1 are by Michael [83] whose work is of considerable interest also with respect to the question of how dust of lunar origin might arrive in the vicinity of the triangular libration points. The concept of the restricted four-body problem, with its advantages and pitfalls, is discussed by Huang [84], Danby [85], and Wolaver [86, 87]. The pro-arguments of Huang and the contra-arguments of Danby are well interpreted and utilized by Wolaver [86], who (based partly on deVries' work [88]) established initial conditions which result in minimum displacements. The circular restricted four-body problem was integrated for more than 2000 days by Tapley and Schultz [89, 89a]; from these references Figs. 5.21 and 5.22 follow. The proposed explanation of Tapley's period of 1600 days by means of the near commensurability, mentioned in Section 5.6.2.2, Part (A), is given by Szebehely [90]. The nonlinear amplitude effect on the period used to obtain T_1 in Eq. (93) is computed from Pedersen's paper [75]. For an alternate discussion of the question of resonance in the problem of four bodies, see Breakwell's and Pringle's [91] thorough paper. The linear equations for the restricted problem of four bodies are treated by deVries [88] from whose work Figs. 5.23, 5.24, and 5.25 originate. A comprehensive discussion with applications is offered by Steg and deVries [91a].

Reference should be made to Sehnal's paper [92] which is often quoted as the authoritative result regarding the fact that the sun destroys the stability of the earth-moon triangular libration points. It must be remarked following Wolaver [86] that the assumptions of Sehnal's paper may be considered too severe from a physical point of view.

The clouds [Section 5.6.2.2, Part (B)] in the vicinity of the earth-moon triangular libration points are described in several announcements listed in *Sky and Telescope* [93] as well as in an article by the discoverer, Kordylewski [94].

The idea to determine the earth-moon mass ratio from the mean motion of a librating artificial satellite seems to have been first proposed by Benedikt [95]. The terminology of "Selenoidal satellite" was introduced by him and Klemperer [96] for bodies near L_4. The reader might agree with Wolaver [87] who considers it fortunate that this terminology was not generally accepted.

The great sensitivity of the orbits to initial conditions near the collinear points (Section 5.6.2) was studied numerically by Abhyankar [97], whose results do not seem to support the explanation given by Gyldén [14] and Moulton [98] for the counter-glow or *Gegenschein*. The *coup de grâce* to the relation between L_1 and the *Gegenschein* is delivered on a theoretical basis by Moisseiev [99] suggesting either a careful rediscussion of the observational results or additional (possibly satellite-based) observations.

The nonexistence of bounded orbits at L_2 follows from an approximate evaluation of the effect of the sun in the restricted four-body problem. This approximate result is due to Richards [100].

Additional applications are mentioned by Szebehely and Williams [101], who also give tabulated and illustrated numerical results regarding the properties of the libration points. The stability of homogeneous ellipsoidal clusters is treated by Chandrasekhar [102, p. 221] using equations similar to Eq. (4) in Section 5.2.3. The application of Liapunov's work [79] to control problems should also be mentioned. Reference is made in addition to [3], only to Halkin's [103] major paper where the reader will find a systematic list of references as well as an exceedingly clearly defined set of concepts and statements.

The subject of orbit computation from the point of view of the astronomer is another area which is outside our scope but is essential, especially regarding the minor planets mentioned in this chapter. Only Herget's [104] volume is mentioned as one of the basic reference books.

An application of the concept of the curves of zero velocity and the Jacobian integral to the artificial satellite problem is discussed by O'Keefe [105] and by Poritsky [106]. The next step, using linearized analysis, quite similar to the methods of this chapter, is made by Blitzer [107] who studies the stability of the equilibrium solutions of the artificial satellite problem.

The fact that the linearized equations of the restricted problem around the equilibrium points are of the same form as the terminal guidance equations for rendezvous maneuvers in space dynamics has been recognized by many investigators. A thorough study of the rendezvous mission, including its analytical aspects, is available in Houbolt's paper [108]. Applications of the restricted problem to guidance from the point of view of the astronautical engineer is outside our scope. We remark, though, that basing guidance laws and equations on the (approximate) solutions of the restricted problem, instead of on the problem of two bodies, might open up new avenues for this field, which is so well represented in Battin's [109] book.

5.8 References

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Appendix I. Roots of the characteristic equation at the collinear points

A. Roots of the characteristic equation at the collinear points for $0 \leqslant \mu \leqslant 0.009$

μ		L_1	L_2	L_3
0.0	λ	2.50828 67902	2.50828 67902	0.0
	s	2.07159 42224	2.07159 42224	1.0
0.000001	λ	2.49169 87840	2.52506 75693	0.00162 018
	s	2.06150 07684	2.08182 14185	1.00000 087
0.000002	λ	2.48741 80143	2.52946 05820	0.00229 129
	s	2.05889 86598	2.08450 14835	1.00000 175
0.000003	λ	2.48442 25586	2.53254 97940	0.00280 624
	s	2.05707 84929	2.08638 68005	1.00000 262
0.000004	λ	2.48204 22021	2.53501 35773	0.00324 037
	s	2.05563 24696	2.08789 08184	1.00000 350
0.000005	λ	2.48003 50425	2.53709 72189	0.00362 284
	s	2.05441 34206	2.08916 30510	1.00000 437
0.000006	λ	2.47828 27591	2.53892 08579	0.00396 862
	s	2.05334 93688	2.09027 67354	1.00000 525
0.000007	λ	2.47671 75705	2.54055 33949	0.00428 660
	s	2.05239 90841	2.09127 38767	1.00000 612
0.000008	λ	2.47529 66024	2.54203 84552	0.00458 256
	s	2.05153 64888	2.09218 10729	1.00000 700
0.000009	λ	2.47399 08117	2.54340 56163	0.00486 054
	s	2.05074 39192	2.09301 63580	1.00000 787
0.00001	λ	2.47277 95083	2.54467 59659	0.00512 346
	s	2.05000 87920	2.09379 25906	1.00000 875
0.00002	λ	2.46368 72713	2.55427 63183	0.00724 564
	s	2.04449 36472	2.09966 17460	1.00001 750
0.00003	λ	2.45734 11991	2.56104 47030	0.00887 403
	s	2.04064 72440	2.10380 27221	1.00002 625
0.00004	λ	2.45230 74940	2.56645 27785	0.01024 681
	s	2.03759 80288	2.10711 33000	1.00003 500
0.00005	λ	2.44806 91990	2.57103 32121	0.01145 624
	s	2.03503 18479	2.10991 85158	1.00004 375
0.00006	λ	2.44437 36586	2.57504 70987	0.01254 963
	s	2.03279 51950	2.11237 77223	1.00005 249
0.00007	λ	2.44107 62324	2.57864 42740	0.01355 511
	s	2.03080 02057	2.11458 23827	1.00006 124
0.00008	λ	2.43808 54826	2.58191 96606	0.01449 097
	s	2.02899 13431	2.11659 04480	1.00006 999
0.00009	λ	2.43533 95019	2.58493 76466	0.01536 994
	s	2.02733 10110	2.11844 12329	1.00007 874

μ		L_1	L_2	L_3
0.0001	λ	2.43279 42078	2.58774 41597	0.01620 12783
	s	2.02579 24417	2.12016 27831	1.00008 74849
0.0002	λ	2.41374 74318	2.60902 09794	0.02291 12574
	s	2.01429 20158	2.13322 83040	1.00017 49395
0.0003	λ	2.40051 10155	2.62408 92785	0.02805 94539
	s	2.00631 34039	2.14249 62052	1.00026 23640
0.0004	λ	2.39004 28628	2.63616 68635	0.03239 91236
	s	2.00001 13834	2.14993 34251	1.00034 97584
0.0005	λ	2.38124 86747	2.64642 09381	0.03622 20450
	s	1.99472 25920	2.15625 38273	1.00043 71226
0.0006	λ	2.37359 45622	2.65542 45557	0.03967 78656
	s	1.99012 35343	2.16180 80365	1.00052 44568
0.0007	λ	2.36677 53289	2.66350 70206	0.04285 54863
	s	1.98602 93546	2.16679 76073	1.00061 17609
0.0008	λ	2.36059 82825	2.67087 71950	0.04581 28330
	s	1.98232 33791	2.17135 04144	1.00069 90351
0.0009	λ	2.35493 31094	2.67767 69033	0.04859 01427
	s	1.97892 67270	2.17555 33122	1.00078 62794
0.001	λ	2.34968 71129	2.68400 74035	0.05121 67129
	s	1.97578 33027	2.17946 83264	1.00087 34937
0.002	λ	2.31056 33638	2.73220 38755	0.07240 61496
	s	1.95239 86945	2.20934 11928	1.00174 40017
0.003	λ	2.28347 58257	2.76651 78860	0.08864 85267
	s	1.93626 98853	2.23067 94063	1.00261 15635
0.004	λ	2.26208 56013	2.79410 32959	0.10232 76487
	s	1.92356 99524	2.24787 43955	1.00347 62180
0.005	λ	2.24412 39065	2.81756 69425	0.11436 72690
	s	1.91293 09704	2.26252 82490	1.00433 80033
0.006	λ	2.22848 80206	2.83819 31077	0.12524 13520
	s	1.90368 87229	2.27543 09491	1.00519 69568
0.007	λ	2.21454 95492	2.85672 19227	0.13523 15797
	s	1.89546 50068	2.28703 80996	1.00605 31152
0.008	λ	2.20191 28184	2.87362 39921	0.14452 15666
	s	1.88802 18499	2.29763 95787	1.00690 65145
0.009	λ	2.19031 08884	2.88921 95860	0.15323 89103
	s	1.88119 88050	2.30743 27780	1.00775 71903

B. Roots of the characteristic equation at the collinear points for $0.01 \leqslant \mu \leqslant 0.39$

μ		L_1	L_2	L_3
0.01	λ	2.17955 42907	2.90373 78316	0.16147 65578
	s	1.87488 20534	2.31655 89900	1.00860 51771
0.02	λ	2.09870 62620	3.01397 61640	0.22767 88648
	s	1.82769 47047	2.38614 45202	1.01694 43793
0.03	λ	2.04174 17762	3.09174 10196	0.27810 16814
	s	1.79476 77185	2.43552 43431	1.02504 87434
0.04	λ	1.99594 36519	3.15351 53767	0.32035 50175
	s	1.76849 74578	2.47491 24191	1.03294 45788
0.05	λ	1.95680 43211	3.20536 57702	0.35740 17712
	s	1.74619 53967	2.50807 85236	1.04065 43464
0.06	λ	1.92214 80086	3.25030 34682	0.39076 81162
	s	1.72656 60609	2.53689 79834	1.04819 73989
0.07	λ	1.89074 14956	3.29007 46284	0.42136 39141
	s	1.70887 62116	2.56246 02957	1.05559 05499
0.08	λ	1.86181 16571	3.32579 53969	0.44978 36712
	s	1.69266 67241	2.58546 30320	1.06284 85165
0.09	λ	1.83483 87756	3.35822 88459	0.47644 12236
	s	1.67762 92977	2.60638 38230	1.06998 42690
0.10	λ	1.80945 50539	3.38792 30677	0.50163 83508
	s	1.66354 59768	2.62556 62167	1.07700 93102
0.11	λ	1.78538 97088	3.41528 69212	0.52560 31071
	s	1.65025 64572	2.64326 68224	1.08393 39007
0.12	λ	1.76243 70166	3.44063 46672	0.54851 27744
	s	1.63763 90746	2.65968 31175	1.09076 72420
0.13	λ	1.74043 66271	3.46421 37859	0.57050 83705
	s	1.62559 90985	2.67497 07942	1.09751 76282
0.14	λ	1.71926 08635	3.48622 31137	0.59170 42293
	s	1.61406 11864	2.68925 50777	1.10419 25702
0.15	λ	1.69880 62146	3.50682 51137	0.61219 45579
	s	1.60296 43302	2.70263 83907	1.11079 89003
0.16	λ	1.67898 74511	3.52615 44409	0.63205 80605
	s	1.59225 83644	2.71520 57080	1.11734 28600
0.17	λ	1.65973 34457	3.54432 40814	0.65136 12853
	s	1.58190 14858	2.72702 83957	1.12383 01745
0.18	λ	1.64098 41327	3.56142 98527	0.67016 10988
	s	1.57185 84510	2.73816 70298	1.13026 61153
0.19	λ	1.62268 82520	3.57755 37703	0.68850 65519
	s	1.56209 92387	2.74867 35039	1.13665 55551
0.20	λ	1.60480 16447	3.59276 66097	0.70644 03108
	s	1.55259 80406	2.75859 26360	1.14300 30137
0.21	λ	1.58728 59439	3.60712 98862	0.72399 97703
	s	1.54333 24857	2.76796 34101	1.14931 26990
0.22	λ	1.57010 75568	3.62069 74059	0.74121 79320
	s	1.53428 30380	2.77681 99499	1.15558 85417
0.23	λ	1.55323 68604	3.63351 64969	0.75812 41057
	s	1.52543 25210	2.78519 22914	1.16183 42265
0.24	λ	1.53664 75608	3.64562 89959	0.77474 44746
	s	1.51676 57389	2.79310 70037	1.16805 32185
0.25	λ	1.52031 61768	3.65707 20475	0.79110 25567
	s	1.50826 91715	2.80058 76911	1.17424 87875
0.26	λ	1.50422 16186	3.66787 87580	0.80721 95836
	s	1.49993 07258	2.80765 54046	1.18042 40291
0.27	λ	1.48834 48424	3.67807 87340	0.82311 48149
	s	1.49173 95321	2.81432 89804	1.18658 18834
0.28	λ	1.47266 85645	3.68769 85294	0.83880 58013
	s	1.48368 57759	2.82062 53226	1.19272 51520
0.29	λ	1.45717 70222	3.69676 20199	0.85430 86052
	s	1.47576 05574	2.82655 96376	1.19885 65131
0.30	λ	1.44185 57729	3.70529 07166	0.86963 79884
	s	1.46795 57732	2.83214 56333	1.20497 85354
0.31	λ	1.42669 15244	3.71330 40330	0.88480 75718
	s	1.46026 40176	2.83739 56863	1.21109 36902
0.32	λ	1.41167 19897	3.72081 95107	0.89982 99726
	s	1.45267 84972	2.84232 09852	1.21720 43633
0.33	λ	1.39678 57633	3.72785 30124	0.91471 69228
	s	1.44519 29590	2.84693 16517	1.22331 28654
0.34	λ	1.38202 22137	3.73441 88881	0.92947 93716
	s	1.43780 16284	2.85123 68463	1.22942 14413
0.35	λ	1.36737 13906	3.74053 01171	0.94412 75755
	s	1.43049 91551	2.85524 48573	1.23553 22794
0.36	λ	1.35282 39437	3.74619 84310	0.95867 11767
	s	1.42328 05673	2.85896 31790	1.24164 75201
0.37	λ	1.33837 10518	3.75143 44202	0.97311 92727
	s	1.41614 12301	2.86239 85791	1.24776 92635
0.38	λ	1.32400 43594	3.75624 76259	0.98748 04779
	s	1.40907 68109	2.86555 71563	1.25389 95771
0.39	λ	1.30971 59213	3.76064 66197	1.00176 29786
	s	1.40208 32476	2.86844 43914	1.26004 05027

C. Roots of the characteristic equation at the collinear points for $0.4 \leqslant \mu \leqslant 0.5$

μ		L_1	L_2	L_3
0.40	λ	1.29549 81524	3.76463 90730	1.01597 45821
	s	1.39515 67209	2.87106 51909	1.26619 40640
0.41	λ	1.28134 37826	3.76823 18171	1.03012 27616
	s	1.38829 36300	2.87342 39247	1.27236 22725
0.42	λ	1.26724 58167	3.77143 08944	1.04421 46964
	s	1.38149 05709	2.87552 44594	1.27854 71347
0.43	λ	1.25319 74968	3.77424 16036	1.05825 73089
	s	1.37474 43163	2.87737 01858	1.28475 06583
0.44	λ	1.23919 22682	3.77666 85375	1.07225 72986
	s	1.36805 17985	2.87896 40437	1.29097 48587
0.45	λ	1.22522 37483	3.77871 56153	1.08622 11738
	s	1.36141 00936	2.88030 85424	1.29722 17656
0.46	λ	1.21128 56964	3.78038 61097	1.10015 52811
	s	1.35481 64068	2.88140 57771	1.30349 34294
0.47	λ	1.19737 19866	3.78168 26692	1.11406 58331
	s	1.34826 80595	2.88225 74438	1.30979 19279
0.48	λ	1.18347 65813	3.78260 73359	1.12795 89349
	s	1.34176 24775	2.88286 48503	1.31611 93733
0.49	λ	1.16959 35054	3.78316 15589	1.14184 06094
	s	1.33529 71798	2.88322 89247	1.32247 79193
0.50	λ	1.15571 68222	3.78334 62040	1.15571 68222
	s	1.32886 97684	2.88335 02214	1.32886 97684

D. Roots of the characteristic equation at the collinear points for the sun-Jupiter system

μ		L_1	L_2	L_3
0.00085	λ	2.35770 81927	2.67434 12861	0.04722 19588
	s	1.98059 03099	2.17349 12708	1.00074 26610
0.00086	λ	2.35714 43378	2.67501 82916	0.04749 87562
	s	1.98025 22536	2.17390 97418	1.00075 13853
0.00087	λ	2.35658 49876	2.67569 02625	0.04777 39460
	s	1.97991 69190	2.17432 51241	1.00076 01092
0.00088	λ	2.35603 00552	2.67635 72937	0.04804 75560
	s	1.97958 42534	2.17473 74759	1.00076 88329
0.00089	λ	2.35547 94564	2.67701 94775	0.04831 96128
	s	1.97925 42059	2.17514 68536	1.00077 75563
0.00090	λ	2.35493 31094	2.67767 69033	0.04859 01427
	s	1.97892 67270	2.17555 33122	1.00078 62794
0.00091	λ	2.35439 09347	2.67832 96579	0.04885 91709
	s	1.97860 17686	2.17595 69047	1.00079 50022
0.00092	λ	2.35385 28555	2.67897 78258	0.04912 67222
	s	1.97827 92842	2.17635 76829	1.00080 37246
0.00093	λ	2.35331 87969	2.67962 14887	0.04939 28205
	s	1.97795 92285	2.17675 56969	1.00081 24468
0.00094	λ	2.35278 86862	2.68026 07263	0.04965 74892
	s	1.97764 15575	2.17715 09955	1.00082 11687
0.00095	λ	2.35226 24530	2.68089 56158	0.04992 07511
	s	1.97732 62286	2.17754 36260	1.00082 98903
0.00096	λ	2.35174 00286	2.68152 62325	0.05018 26283
	s	1.97701 32004	2.17793 36345	1.00083 86116
0.00097	λ	2.35122 13463	2.68215 26495	0.05044 31423
	s	1.97670 24323	2.17832 10658	1.00084 73326
0.00098	λ	2.35070 63414	2.68277 49378	0.05070 23141
	s	1.97639 38853	2.17870 59635	1.00085 60533
0.00099	λ	2.35019 49507	2.68339 31667	0.05096 01643
	s	1.97608 75211	2.17908 83699	1.00086 47736
0.00100	λ	2.34968 71129	2.68400 74035	0.05121 67129
	s	1.97578 33027	2.17946 83264	1.00087 34937
0.00101	λ	2.34918 27683	2.68461 77139	0.05147 19791
	s	1.97548 11940	2.17984 58731	1.00088 22135
0.00102	λ	2.34868 18588	2.68522 41615	0.05172 59822
	s	1.97518 11596	2.18022 10492	1.00089 09330
0.00103	λ	2.34818 43276	2.68582 68088	0.05197 87405
	s	1.97488 31655	2.18059 38927	1.00089 96522
0.00104	λ	2.34769 01197	2.68642 57161	0.05223 02721
	s	1.97458 71782	2.18096 44408	1.00090 83711
0.00105	λ	2.34719 91814	2.68702 09427	0.05248 05948
	s	1.97429 31653	2.18133 27297	1.00091 70897

E. Roots of the characteristic equation at the collinear points for the earth-moon system

μ		L_1	L_2	L_3
0.0110	λ	2.16950 31964	2.91734 90608	0.16930 47098
	s	1.86898 76395	2.32512 33025	1.00945 05093
0.0111	λ	2.16853 25639	2.91866 55963	0.17006 72117
	s	1.86841 88303	2.32595 20994	1.00953 48978
0.0112	λ	2.16756 78114	2.91997 45024	0.17082 62209
	s	1.86785 35394	2.32677 61666	1.00961 92601
0.0113	λ	2.16660 88495	2.92127 58895	0.17158 17840
	s	1.86729 17131	2.32759 55723	1.00970 35963
0.0114	λ	2.16565 55909	2.92256 98652	0.17233 39463
	s	1.86673 32990	2.32841 03829	1.00978 79063
0.0115	λ	2.16470 79504	2.92385 65347	0.17308 27525
	s	1.86617 82459	2.32922 06634	1.00987 21903
0.0116	λ	2.16376 58445	2.92513 60008	0.17382 82459
	s	1.86562 65038	2.33002 64772	1.00995 64483
0.0117	λ	2.16282 91920	2.92640 83640	0.17457 04693
	s	1.86507 80240	2.33082 78864	1.01004 06802
0.0118	λ	2.16189 79134	2.92767 37225	0.17530 94643
	s	1.86453 27588	2.33162 49517	1.01012 48861
0.0119	λ	2.16097 19311	2.92893 21722	0.17604 52717
	s	1.86399 06617	2.33241 77323	1.01020 90661
0.0120	λ	2.16005 11691	2.93018 38072	0.17677 79315
	s	1.86345 16870	2.33320 62862	1.01029 32202
0.0121	λ	2.15913 55533	2.93142 87192	0.17750 74827
	s	1.86291 57903	2.33399 06702	1.01037 73484
0.0122	λ	2.15822 50111	2.93266 69982	0.17823 39638
	s	1.86238 29282	2.33477 09397	1.01046 14508
0.0123	λ	2.15731 94716	2.93389 87319	0.17895 74123
	s	1.86185 30580	2.33554 71492	1.01054 55273
0.0124	λ	2.15641 88654	2.93512 40066	0.17967 78649
	s	1.86132 61381	2.33631 93516	1.01062 95781
0.0125	λ	2.15552 31246	2.93634 29063	0.18039 53577
	s	1.86080 21279	2.33708 75992	1.01071 36031
0.0126	λ	2.15463 21828	2.93755 55136	0.18110 99260
	s	1.86028 09875	2.33785 19427	1.01079 76023
0.0127	λ	2.15374 59750	2.93876 19092	0.18182 16045
	s	1.85976 26780	2.33861 24321	1.01088 15759
0.0128	λ	2.15286 44377	2.93996 21722	0.18253 04270
	s	1.85924 71613	2.33936 91162	1.01096 55239
0.0129	λ	2.15198 75084	2.94115 63800	0.18323 64270
	s	1.85873 43999	2.34012 20429	1.01104 94462
0.0130	λ	2.15111 51264	2.94234 46084	0.18393 96369
	s	1.85822 43573	2.34087 12590	1.01113 33429

F. Roots of the characteristic equation at the collinear points for the bodies of the solar system

		μ		L_1	L_2	L_3
MERCURY	and Sun	0.00000 01667	λ	2.49913 45093	2.51749 77806	0.00066 144
			s	2.06602 32389	2.07720 59094	1.00000 015
VENUS	and Sun	0.00000 24510	λ	2.48596 56423	2.53095 68491	0.00253 649
			s	2.05801 60695	2.08541 45725	1.00000 214
EARTH-MOON	and Sun	0.00000 30359	λ	2.48432 85016	2.53264 69976	0.00282 298
			s	2.05702 13486	2.08644 61319	1.00000 266
MARS	and Sun	0.00000 03233	λ	2.49688 19318	2.51978 28025	0.00092 117
			s	2.06465 28561	2.07859 87993	1.00000 028
JUPITER	and Sun	0.00095 38754	λ	2.35205 95465	2.68114 05054	0.05002 240
			s	1.97720 46482	2.17769 50769	1.00083 327
SATURN	and Sun	0.00028 55022	λ	2.40221 99948	2.62213 09485	0.02737 320
			s	2.00734 29092	2.14129 10251	1.00024 969
URANUS	and Sun	0.00004 37254	λ	2.45065 24720	2.56823 84787	0.01071 334
			s	2.03659 58230	2.10820 67848	1.00003 826
NEPTUNE	and Sun	0.00005 17732	λ	2.44737 94103	2.57178 10075	0.01165 761
			s	2.03461 43029	2.11037 66030	1.00004 530
PLUTO	and Sun	0.00000 27778	λ	2.48502 22165	2.53193 03650	0.00270 030
			s	2.05744 28277	2.08600 87247	1.00000 243
MOON	and Earth	0.01214 09319	λ	2.15876 22415	2.93193 63676	0.17780 521
			s	1.86269 73151	2.33431 05439	1.01041 178
MOON	and Earth	0.01215 06683	λ	2.15867 35676	2.93205 69574	0.17787 596
			s	1.86264 54216	2.33438 65302	1.01041 996
CRITICAL Value		0.03852 08965	λ	2.00222 63097	3.14510 18945	0.31448 249
			s	1.77209 02929	2.46953 97106	1.03178 893
DARWIN'S Value		0.09090 90909	λ	1.83247 01415	3.36103 45364	0.47878 835
			s	1.67631 23209	2.60819 51360	1.07062 732

Appendix II. Roots of the characteristic equation for stable solution at the equilateral points

A. Roots of the characteristic equation for stable solution at the equilateral points

μ	s_1	s_2
0.0	0.0	1.0
0.000001	0.00259 80837	0.99999 66250
0.000002	0.00367 42557	0.99999 32499
0.000003	0.00450 00388	0.99998 98748
0.000004	0.00519 62122	0.99998 64996
0.000005	0.00580 95585	0.99998 31244
0.000006	0.00636 40708	0.99997 97491
0.000007	0.00687 40019	0.99997 63738
0.000008	0.00734 86383	0.99997 29984
0.000009	0.00779 44303	0.99996 96230
0.00001	0.00821 60746	0.99996 62475
0.00002	0.01161 96183	0.99993 24900
0.00003	0.01423 14773	0.99989 87274
0.00004	0.01643 35673	0.99986 49598
0.00005	0.01837 38155	0.99983 11872
0.00006	0.02012 80858	0.99979 74096
0.00007	0.02174 14434	0.99976 36269
0.00008	0.02324 32500	0.99972 98392
0.00009	0.02465 38996	0.99969 60464
0.0001	0.02598 82406	0.99966 22486
0.0002	0.03676 35241	0.99932 39932
0.0003	0.04503 89536	0.99898 52315
0.0004	0.05202 15701	0.99864 59614
0.0005	0.05817 87690	0.99830 61809
0.0006	0.06375 01906	0.99796 58878
0.0007	0.06887 81528	0.99762 50799
0.0008	0.07365 53585	0.99728 37551
0.0009	0.07814 61824	0.99694 19111
0.001	0.08239 74830	0.99659 95459
0.002	0.11687 42240	0.99314 67242
0.003	0.14357 64432	0.98963 92297
0.004	0.16630 36511	0.98607 45893
0.005	0.18652 53691	0.98245 01446
0.006	0.20499 49422	0.97876 30324
0.007	0.22216 02669	0.97501 01619
0.008	0.23831 38726	0.97118 81888
0.009	0.25365 98348	0.96729 34861
0.01	0.26834 77485	0.96332 21091
0.02	0.39613 84405	0.91819 07950
0.03	0.51820 58086	0.85525 59500
0.038521	0.70710 67812	0.70710 67812

B. Roots of the characteristic equation at the equilateral points for the sun-Jupiter system

μ	s_1	s_2
0.00085	0.07593 33152	0.99711 28981
0.00086	0.07638 09123	0.99707 87112
0.00087	0.07682 59540	0.99704 45190
0.00088	0.07726 84845	0.99701 03216
0.00089	0.07770 85467	0.99697 61190
0.00090	0.07814 61824	0.99694 19111
0.00091	0.07858 14320	0.99690 76981
0.00092	0.07901 43352	0.99687 34799
0.00093	0.07944 49301	0.99683 92564
0.00094	0.07987 32542	0.99680 50277
0.00095	0.08029 93439	0.99677 07938
0.00096	0.08072 32345	0.99673 65547
0.00097	0.08114 49606	0.99670 23103
0.00098	0.08156 45556	0.99666 80607
0.00099	0.08198 20525	0.99663 38059
0.00100	0.08239 74830	0.99659 95459
0.00101	0.08281 08783	0.99656 52806
0.00102	0.08322 22688	0.99653 10100
0.00103	0.08363 16841	0.99649 67343
0.00104	0.08403 91529	0.99646 24533
0.00105	0.08444 47036	0.99642 81670

C. Roots of the characteristic equation at the equilateral points for the earth-moon system

μ	s_1	s_2
0.0110	0.28249 16547	0.95926 97561
0.0111	0.28387 97488	0.95885 98898
0.0112	0.28526 33937	0.95844 91620
0.0113	0.28664 26668	0.95803 75679
0.0114	0.28801 76439	0.95762 51024
0.0115	0.28938 83993	0.95721 17604
0.0116	0.29075 50060	0.95679 75368
0.0117	0.29211 75353	0.95638 24264
0.0118	0.29347 60572	0.95596 64240
0.0119	0.29483 06407	0.95554 95243
0.0120	0.29618 13529	0.95513 17219
0.0121	0.29752 82603	0.95471 30115
0.0122	0.29887 14276	0.95429 33877
0.0123	0.30021 09188	0.95387 28449
0.0124	0.30154 67965	0.95345 13776
0.0125	0.30287 91222	0.95302 89803
0.0126	0.30420 79564	0.95260 56473
0.0127	0.30553 33586	0.95218 13728
0.0128	0.30685 53872	0.95175 61512
0.0129	0.30817 40997	0.95132 99766
0.0130	0.30948 95525	0.95090 28430

D. Roots of the characteristic equation at the equilateral points for the bodies of the solar system

		μ	s_1		s_2
MERCURY	and Sun	0.00000 01667	0.00106 06606		0.99999 94375
VENUS	and Sun	0.00000 24510	0.00406 74698		0.99999 17278
EARTH-MOON	and Sun	0.00000 30359	0.00452 68887		0.99998 97536
MARS	and Sun	0.00000 03233	0.00147 71587		0.99999 89090
JUPITER	and Sun	0.00095 38754	0.08046 38757		0.99675 75255
SATURN	and Sun	0.00028 55022	0.04393 53588		0.99903 43759
URANUS	and Sun	0.00004 37254	0.01718 19885		0.99985 23787
NEPTUNE	and Sun	0.00005 17732	0.01869 68830		0.99982 51980
PLUTO	and Sun	0.00000 27778	0.00433 01556		0.99999 06248
MOON	and Earth	0.01214 09319	0.29807 84932		0.95454 13621
MOON	and Earth	0.01215 06683	0.29820 92835		0.95450 05098
CRITICAL	Value	0.03852 08965	0.70710 67812		0.70710 67812
DARWIN'S	Value	0.09090 90909	$\lvert\lambda\rvert =$ 0.86423 05228	$\alpha =$	0.35135 05348
			$\theta =$ 66.01187 83680	$\beta =$	0.78958 67263

Appendix III. Roots of the characteristic equation for unstable solution at the equilateral points

μ	$\|\lambda\|$	θ	α	β
0.04	0.71352 42690	84.57034 43660	0.06751 62294	0.71032 27726
0.05	0.75248 75963	76.00451 49541	0.18198 56899	0.73014 98417
0.06	0.78549 91496	72.06561 16729	0.24187 69460	0.74733 15576
0.07	0.81418 14256	69.48087 21935	0.28538 69249	0.76252 58664
0.08	0.83954 77398	67.59226 43731	0.32003 15668	0.77615 73318
0.09	0.86227 72315	66.12918 89470	0.34894 27058	0.78851 82382
0.10	0.88284 92869	64.95189 21062	0.37377 99242	0.79981 96245
0.11	0.90161 60127	63.97861 79990	0.39554 48358	0.81021 95487
0.12	0.91884 42941	63.15750 22064	0.41489 44666	0.81983 98736
0.13	0.93474 16503	62.45368 39790	0.43228 57578	0.82877 67953
0.14	0.94947 28812	61.84268 36650	0.44805 06400	0.83710 77446
0.15	0.96317 12027	61.30670 87595	0.46243 85179	0.84489 60782
0.16	0.97594 59370	60.83246 07819	0.47564 19199	0.85219 43651
0.17	0.98788 79695	60.40976 72709	0.48781 27922	0.85904 67509
0.18	0.99907 37138	60.03069 34269	0.49907 32840	0.86549 06948
0.19	1.00956 80506	59.68894 55681	0.50952 31343	0.87155 82737
0.20	1.01942 65469	59.37945 90020	0.51924 48770	0.87727 71753
0.21	1.02869 71573	59.09810 62608	0.52830 76005	0.88267 14682
0.22	1.03742 15473	58.84148 61136	0.53676 97211	0.88776 22055
0.23	1.04563 61359	58.60676 81152	0.54468 10666	0.89256 79046
0.24	1.05337 29257	58.39157 61636	0.55208 44684	0.89710 49327
0.25	1.06066 01718	58.19389 99806	0.55901 69944	0.90138 78189
0.26	1.06752 29247	58.01202 69214	0.56551 09171	0.90542 95099
0.27	1.07398 34755	57.84448 88140	0.57159 44828	0.90924 15811
0.28	1.08006 17231	57.69002 00645	0.57729 25280	0.91283 44115
0.29	1.08577 54802	57.54752 43142	0.58262 69790	0.91621 73305
0.30	1.09114 07288	57.41604 76646	0.58761 72606	0.91939 87410
0.31	1.09617 18356	57.29475 70008	0.59228 06316	0.92238 62242
0.32	1.10088 17342	57.18292 23108	0.59663 24633	0.92518 66278
0.33	1.10528 20800	57.07990 21674	0.60068 64725	0.92780 61426
0.34	1.10938 33827	56.98513 17334	0.60445 49155	0.93025 03668
0.35	1.11319 51191	56.89811 27974	0.60794 87532	0.93252 43625
0.36	1.11672 58309	56.81840 54561	0.61117 77898	0.93463 27036
0.37	1.11998 32087	56.74562 11421	0.61415 07909	0.93657 95182
0.38	1.12297 41642	56.67941 67589	0.61687 55845	0.93836 85239
0.39	1.12570 48928	56.61948 97336	0.61935 91469	0.94000 30600
0.40	1.12818 09279	56.56557 38365	0.62160 76762	0.94148 61141
0.41	1.13040 71877	56.51743 56451	0.62362 66551	0.94282 03461
0.42	1.13238 80149	56.47487 15543	0.62542 09048	0.94400 81081
0.43	1.13412 72122	56.43770 52547	0.62699 46306	0.94505 14625
0.44	1.13562 80710	56.40578 56129	0.62835 14604	0.94595 21964
0.45	1.13689 33967	56.37898 49030	0.62949 44779	0.94671 18346
0.46	1.13792 55298	56.35719 73467	0.63042 62492	0.94733 16503
0.47	1.13872 63626	56.34033 79283	0.63114 88449	0.94781 26737
0.48	1.13929 73533	56.32834 14592	0.63166 38581	0.94815 56990
0.49	1.13963 95360	56.32116 18691	0.63197 24171	0.94836 12898
0.50	1.13975 35285	56.31877 17113	0.63207 51956	0.94842 97828

Chapter 6

Hamiltonian Dynamics in the Extended Phase Space

6.1 Introduction

In this chapter those aspects of Hamiltonian dynamics are summarized which will be needed afterwards. Emphasis is placed on operations in the extended phase space—also called space of states and energy—since transformations required in the process of regularization will be performed in this space.

In what follows, this introductory section explains and attempts to justify this statement. First we describe configuration space and phase space. In the n-dimensional space of configurations or configuration space the coordinates of a point are the n generalized coordinates q_k of the dynamical system. In the $2n$-dimensional phase space the coordinates of a point are the n generalized coordinates q_k and the n generalized momenta p_k of the dynamical system. Points in the configuration space correspond to given configurations while points in the phase space correspond to definite states of the system.

Description of a dynamical system with n degrees of freedom in the n-dimensional configuration space and in the $2n$-dimensional phase space offers advantages in certain types of problems but is not sufficiently general for our purposes. The phase-space representation is associated with Hamiltonian dynamics, which in turn utilizes the concept of canonical transformations, unquestionably the most powerful tools in the arsenal of the formalistic approach to dynamics. The $2n$ coordinates of the phase space do not include the time; therefore, the technique of canonical transformations is not directly applicable to problems where transformation of the independent variable, in addition to the state variables, is necessary. If regularization of the restricted problem is to be performed, the transformation of the time is important, as was shown in Chapter 3. Therefore if the potentially most effective technique of canonical variables and contact transformations is to be employed in the regularization process of the restricted problem, then this technique must provide for the possibility of time transformations in the canonical framework.

The reader who wonders about the advisability of presenting the equations of motion and the process of regularization also in the canonical system of variables, in addition to the previously given forms, is reminded of the power of Hamiltonian dynamics. A purpose of the formalistic approach in dynamics is to furnish such formulations of dynamical problems as are "best" suited for analytical or numerical undertakings. Depending on the actual dynamical problem at hand, the selection of the "best" set of variables is of fundamental importance. The experience gained by following the derivations may enable the reader to discover the proper generating function for his problem.

Equations of motions and canonical transformations are discussed first, in Sections 6.2 and 6.3. The concept of the extended phase space is introduced and operations in this system are developed in Section 6.4. This is followed by examples (Section 6.5) which lead into our method of generalized time-transformations (Section 6.6).

6.2 Equations of motion

The configuration of a dynamical system with n degrees of freedom and n independent coordinates $(q_1, q_2, \ldots, q_n)$ is described at any time if the coordinates are known as functions of the time. The space formed by the q_i coordinates is called the configuration space since they represent the configuration of the system. The $2n$ initial conditions $q_i = q_i^0$ and $\dot{q}_i = \dot{q}_i^0$ at $t = t_0$ determine the solution in the form

$$q_i = q_i(t, q_i^0, \dot{q}_i^0).$$

The configuration space is n-dimensional; i.e., its dimensionality is the same as the number of degrees of freedom. Points in the configuration space do not correspond to the set of state variables $(q_i, \dot{q}_i)$; as a matter of fact a point of the configuration space is not uniquely related to any particular motion. The point $(q_1, q_2, \ldots, q_n)$ in the configuration space will not determine the behavior of the system, since for this

purpose $2n$ initial conditions are needed while a point in the configuration space corresponds to n initial conditions only. At one given point in the configuration space many orbits originate, each corresponding to a different set of the n initial velocities.

The configuration space is associated with Lagrangian dynamics where the equations of motion appear as n second-order differential equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad (i = 1, \ldots, n),$$

with $L = T - V$ as the kinetic potential or the Lagrangian function.

The Lagrangian equations can be considered also as the Euler–Lagrange solutions of a variational problem,

$$\delta \int_{t_1}^{t_2} L\, dt = 0, \tag{1}$$

which is known as Hamilton's principle for conservative systems and states that the motion of the system from time t_1 to t_2 is such that the line integral

$$\int_{t_1}^{t_2} L\, dt$$

is a local extremum for the path.

It is noted that points of the n-dimensional configuration space describe the system at one instant but its future behavior requires additional information. This is contained in the phase-space description where the $2n$ "coordinates" are the generalized coordinates and momenta, $q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n$. The latter are defined by

$$p_i = \frac{\partial L(q_j, \dot{q}_j, t)}{\partial \dot{q}_i}. \tag{2}$$

The Hamiltonian of a dynamical system is defined by

$$H(q, p, t) = \dot{q}_i p_i - L(q, \dot{q}, t), \tag{3}$$

where the generalized velocities are to be eliminated by solving Eq. (2) for $\dot{q}_i$:

$$\dot{q}_i = f_i(q, p, t).$$

This elimination can be performed in general since the kinetic energy is a positive definite form in the velocities and therefore

$$\det \frac{\partial^2 L}{\partial \dot{q}_i\, \partial \dot{q}_j} \neq 0.$$

Note that in Eq. (3) and later the summation convention is used, according to which a repeated index means summation with respect to that index,

$$\dot{q}_i p_i = \sum_{i=1}^{n} \dot{q}_i p_i .$$

Computing the total differential of H from Eq. (3),

$$dH = \dot{q}_i\, dp_i + p_i\, d\dot{q}_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial L}{\partial t} dt,$$

and observing that since $H = H(q_i, p_i, t)$,

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt,$$

we have

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}, \tag{4}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \tag{5}$$

and

$$\left(p_i - \frac{\partial L}{\partial \dot{q}_i}\right) d\dot{q}_i - \frac{\partial L}{\partial q_i} dq_i = \frac{\partial H}{\partial q_i} dq_i .$$

This last equation, on account of the Lagrangian equations of motion and of Eq. (2), becomes

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{6}$$

Equations (5) and (6) constitute a $2n$th-order system and they are called the canonical equations of Hamilton.

At this time we will show, as a corollary to Eq. (4), that

$$\frac{\partial H}{\partial t} = \frac{dH}{dt}, \tag{7}$$

which result will be needed in connection with our discussion of the extended phase space. The above relation, together with Eq. (4), becomes

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}.$$

The left side of this, using Eq. (3), is

$$\frac{dH}{dt} = p_i \ddot{q}_i + \dot{p}_i \dot{q}_i - \frac{\partial L}{\partial q_i} \dot{q}_i - \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i - \frac{\partial L}{\partial t}.$$

Here the first and fourth terms on the right cancel, because of Eq. (2), and the second and third terms cancel since

$$\dot{p}_i = \frac{\partial L}{\partial q_i}.$$

Therefore

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}.$$

The phase-space representation, therefore, led us to the variables q_i, p_i; $2n$ in number, and to $2n$ first-order differential equations of motion. A point in the $2n$-dimensional phase space (q_i^0, p_i^0) corresponds to a set of initial conditions, $q_i = q_i^0$, $p_i = p_i^0$, at $t = t_0$ which will, in general, determine the subsequent motion of the system. The $2n$ coordinates of the phase space offer more than the instantaneous configuration of the system; they represent its complete dynamical description. These coordinates are also called the state-of-motion variables and their number equals the order of the differential equations representing the motion of the dynamical system.

The variables q_i, p_i form a canonical system; the q_i's are called generalized coordinates and the p_i's are the conjugate (generalized) momenta.

6.3 Canonical transformation in the phase space

Let $q_1, \ldots, q_n, p_1, \ldots, p_n$ be a set of $2n$ canonical variables and $Q_1, \ldots, Q_n$, $P_1, \ldots, P_n$ another set such that

$$\begin{aligned} Q_1 &= Q_1(q_1, \ldots, q_n, p_1, \ldots, p_n, t), \\ &\vdots \\ Q_n &= Q_n(q_1, \ldots, q_n, p_1, \ldots, p_n, t), \\ P_1 &= P_1(q_1, \ldots, q_n, p_1, \ldots, p_n, t), \\ &\vdots \\ P_n &= P_n(q_1, \ldots, q_n, p_1, \ldots, p_n, t). \end{aligned} \tag{8}$$

If now the transformation given by Eqs. (8) is such that the new variables $Q_1, \ldots, Q_n$, $P_1, \ldots, P_n$ are again canonical, and with a new Hamiltonian $(\tilde{H})$ we have

$$\dot{Q}_i = \partial \tilde{H} / \partial P_i, \qquad \dot{P}_i = -\partial \tilde{H} / \partial Q_i, \tag{9}$$

then Eqs. (8) represent a canonical transformation.

One method of generating some but not all such transformations follows from Hamilton's principle in its modified form. Equations (1) and (3) give

$$\delta \int_{t_1}^{t_2} (p_i \dot{q}_i - H)\, dt = 0, \tag{10}$$

and after the transformation has been performed we have

$$\delta \int_{t_1}^{t_2} (P_i \dot{Q}_i - \tilde{H})\, dt = 0. \tag{11}$$

The difference between these variations is

$$\delta \int_{t_1}^{t_2} (p_i\, dq_i - P_i\, dQ_i - H + \tilde{H})\, dt = 0. \tag{12}$$

A way to relate the new and old variables is to have

$$p_i \dot{q}_i - P_i \dot{Q}_i - H + \tilde{H} = dW/dt, \tag{12a}$$

which satisfies Eq. (12) since

$$\delta \int_{t_1}^{t_2} \frac{dW}{dt}\, dt = \delta[W(t_2) - W(t_1)] = 0.$$

Note that the notation $W(t_i)$ here does not indicate that W depends only on t but that the value of the function W is to be computed at $t = t_i$.

Equation (12a) is often written in the form of differentials (with the summation convention still operating):

$$p_i\, dq_i - P_i\, dQ_i = dW + (H - \tilde{H})\, dt, \tag{12b}$$

where W is the "generating function," depending in principle on $(4n + 1)$ variables; i.e.,

$$W = W(q_i, Q_i, p_i, P_i, t).$$

Since Eqs. (8) represent $2n$ relations between the above-mentioned $(4n + 1)$ variables, the function W depends actually only on $2n$ independent variables and on the time. This fact immediately suggests four possibilities for W:

$$\begin{aligned} W_1 &= W_1(q, Q, t), \\ W_2 &= W_2(q, P, t), \\ W_3 &= W_3(p, Q, t), \\ W_4 &= W_4(p, P, t). \end{aligned} \tag{13}$$

For W_1 we have

$$dW_1 = \frac{\partial W_1}{\partial q_i}\, dq_i + \frac{\partial W_1}{\partial Q_i}\, dQ_i + \frac{\partial W_1}{\partial t}. \tag{14}$$

Substituting dW_1 as given by Eq. (14) into Eq. (12) for dW, we have

$$p_i\, dq_i - P_i\, dQ_i = \frac{\partial W_1}{\partial q_i}\, dq_i + \frac{\partial W_1}{\partial Q_i}\, dQ_i + \left(\frac{\partial W_1}{\partial t} + H - \tilde{H}\right) dt,$$

and therefore

$$p_i = \partial W_1/\partial q_i, \qquad P_i = -\partial W_1/\partial Q_i, \tag{15}$$

and

$$\tilde{H} = H + \partial W_1/\partial t. \tag{16}$$

The $(2n + 1)$ equations [Eqs. (15) and (16)] give the new canonical set of $2n$ variables and the new Hamiltonian. If the generating function is independent of the time then, of course,

$$\tilde{H} = H,$$

and the Hamiltonian is an invariant of the transformation.

Since $W_1 = W_1(q, Q, t)$, Eqs. (15) in general result in

$$p_i = p_i(q, Q, t) \tag{17}$$

and

$$P_i = P_i(q, Q, t), \tag{18}$$

after the partial derivatives are evaluated. It is, therefore, necessary to solve Eq. (17) for Q_i,

$$Q_i = Q_i(q, p, t), \tag{19}$$

and substitute these relations into Eq. (18),

$$P_i = P_i(q, p, t), \tag{20}$$

in order to obtain expressions of the new variables in terms of the old ones. A similar process is required for the new Hamiltonian since the result obtained from Eq. (16) in general is of the form

$$\tilde{H} = \tilde{H}(q, p, Q, t), \tag{21}$$

while the Hamiltonian canonical equations of motion (9) require the knowledge of

$$\tilde{H} = \tilde{H}(Q, P, t). \tag{22}$$

Inverting Eqs. (19) and (20) and substituting the results into Eq. (21) will produce the new Hamiltonian function in the desired form; see Eq. (22).

We assume that the Jacobian of the transformation and all four of its subdeterminants $\partial Q_i/\partial q_j, \ldots, \partial P_i/\partial p_j$ are not zero. The first condition, which can be expressed either by the first partial derivatives of the new variables with respect to the old ones or by the second partial derivatives of the function W, is satisfied and it can be shown that the Jacobian determinant of canonical transformations has the value of unity.

Similar discussion and derivation applies to the other three generating functions (W_2, W_3, and W_4) which can be shown to be related to W_1 by

$$\begin{aligned} W_2 &= W_1 + Q_i P_i, \\ W_3 &= W_1 - q_i p_i, \\ W_4 &= W_1 + Q_i P_i - q_i p_i. \end{aligned} \tag{22a}$$

TABLE I

SUMMARY OF CANONICAL TRANSFORMATIONS IN THE PHASE SPACE

	$q_i =$	$p_i =$	$Q_i =$	$P_i =$	$\tilde{H} - H =$
$W_1(q_i, Q_i, t)$		$\dfrac{\partial W_1}{\partial q_i}$		$-\dfrac{\partial W_1}{\partial Q_i}$	$\dfrac{\partial W_1}{\partial t}$
$W_2(q_i, P_i, t)$		$\dfrac{\partial W_2}{\partial q_i}$	$\dfrac{\partial W_2}{\partial P_i}$		$\dfrac{\partial W_2}{\partial t}$
$W_3(p_i, Q_i, t)$	$-\dfrac{\partial W_3}{\partial p_i}$			$-\dfrac{\partial W_3}{\partial Q_i}$	$\dfrac{\partial W_3}{\partial t}$
$W_4(p_i, P_i, t)$	$-\dfrac{\partial W_4}{\partial p_i}$		$\dfrac{\partial W_4}{\partial P_i}$		$\dfrac{\partial W_4}{\partial t}$

The transformation equations are collected in Table I. Using, for instance, the first two lines of this table we have

$$\begin{aligned} P_i &= -\partial W_1/\partial Q_i, & p_i &= \partial W_1/\partial q_i, \\ Q_i &= \partial W_2/\partial P_i, & p_i &= \partial W_2/\partial q_i. \end{aligned}$$

Taking derivatives of $W_2 = W_1 + Q_i P_i$, the reader can verify these relations.

Note that, for $r = 1, \ldots, 4$, we have

$$\tilde{H} = H + \partial W_r/\partial t,$$

and the relation between the new and the old Hamiltonians found for W_1 (Eq. 16) is applicable to all four generating functions.

Special and important transformations are furnished by bilinear generating functions. For the first generating function we have

$$W_1(q_i, Q_i, t) = q_i Q_i$$

which gives

$$Q_i = p_i, \qquad P_i = -q_i.$$

This shows that the new and old variables are "asymmetrically exchanged."

Specialization of the second generating function,

$$W_2(q_i, P_i, t) = q_i P_i,$$

results in an identity transformation,

$$Q_i = q_i, \qquad P_i = p_i.$$

The generating function $W_3(p_i, Q_i, t) = p_i Q_i$ gives the same result with negative signs. On the other hand note that if $W_3 = p_i Q_i$ is a generating function then clearly $W_3 = -p_i Q_i$ is also one. The latter gives again the identity transformation.

Finally the generating function $W_4(p_i, P_i, t) = p_i P_i$ is mentioned which furnishes again an asymmetric exchange.

After the above summary of Hamiltonian dynamics and canonical transformations in the phase space, we are ready to generalize these results to the so-called "space of states and energy" or "extended phase space."

6.4 The extended phase space

Introducing as the $(n + 1)$th generalized coordinate the time, it can be shown that its conjugate momentum becomes $-H$. The dynamical system with n degrees of freedom is now represented by $(2n + 2)$ variables in the extended phase space where the coordinates are

$$q_1, q_2, \ldots, q_n, q_{n+1}$$

and

$$p_1, p_2, \ldots, p_n, p_{n+1},$$

with $q_{n+1} = t$ and $p_{n+1} = -H$.

Canonical transformations will then include time transformations since the time in the extended phase space is one of the generalized coordinates.

Another notation may clarify the issue and will allow us to derive Hamilton's principle in the extended phase space. Consider a dynamical system of n degrees of freedom with generalized coordinates $q_1, \ldots, q_n$ and with momenta $p_1, \ldots, p_n$. Let

$$q_\rho = x_\rho, \quad p_\rho = y_\rho, \qquad \text{for} \quad \rho = 1, \ldots, n \tag{23a}$$

and

$$t = x_{n+1}, \quad -H = y_{n+1}. \tag{23b}$$

With this notation we introduce the $(n+1)$-vectors x_i and y_i as the frame of representation in the extended phase space ($i = 1, \ldots, n+1$) while preserving the n-vectors q_ρ and p_ρ for the phase space ($\rho = 1, \ldots, n$).
In the phase space, of course, Hamilton's principle applies and

$$\delta \int_{t_1}^{t_2} (\dot{q}_\rho p_\rho - H)\, dt = 0,$$

where the summation convention is again operative,

$$\dot{q}_\rho p_\rho = \sum_{\rho=1}^{n} \dot{q}_\rho p_\rho .$$

We introduce now a special parameter w so that the new $(n+1)$th coordinate $t = x_{n+1}$ and the independent variable $t = w$ may be distinguished. With this, Hamilton's principle can be written as

$$\delta \int_{w_1}^{w_2} \left(y_\rho \frac{dx_\rho}{dw} - H \frac{dt}{dw} \right) dw = 0.$$

Using Eqs. (23b) we have

$$\delta \int_{w_1}^{w_2} \sum_{i=1}^{n+1} y_i \frac{dx_i}{dw}\, dw = 0, \tag{24}$$

which is Hamilton's principle in the extended phase space.
Two remarks are in order. First, the reader's attention is directed to the simplicity and symmetry of this result—two characteristics of any operation within the framework of formalistic dynamics which are of paramount importance. Second, we wish to emphasize the importance of Eq. (24). The generalization of canonical transformations requires the reformulation and generalization of Hamilton's principle since the

concept of the generating function is introduced by means of Hamilton's principle in the phase space [Eq. (10)]. Generating functions in the extended phase space will be similarly introduced through Eq. (24), which corresponds to Eq. (10).
We will now show that the equations of motion in the extended phase space are

$$\frac{dx_i}{dw} = \frac{\partial \Gamma}{\partial y_i} \quad \text{and} \quad \frac{dy_i}{dw} = -\frac{\partial \Gamma}{\partial x_i}, \tag{25}$$

with $i = 1, 2, \ldots, n, n+1$, with $w = t$, and with

$$\Gamma = y_{n+1} + \gamma(x_1, \ldots, x_{n+1}, y_1, \ldots, y_n), \tag{26}$$

where

$$\gamma(q_1, \ldots, q_n, t, p_1, \ldots, p_n) = H. \tag{27}$$

Since $y_{n+1} = -H$, we see from Eq. (26) that the Hamiltonian in the extended phase space is zero, $\Gamma = 0$, and attention is directed to the fine distinction between H as a variable and H as the Hamiltonian function. This double role of H does not result in confusion in the applications, as some examples in the next section will clearly demonstrate.
The verification of Eq. (25) is simple for $i = n+1$, since for this case we have

$$\frac{dx_{n+1}}{dw} = \frac{\partial \Gamma}{\partial y_{n+1}} \quad \text{and} \quad \frac{dy_{n+1}}{dw} = -\frac{\partial \Gamma}{\partial x_{n+1}}$$

or

$$\frac{dt}{dw} = \frac{\partial \Gamma}{\partial y_{n+1}} = 1 \quad \text{and} \quad \frac{d(-H)}{dw} = -\frac{\partial H}{\partial t}.$$

Accordingly, $t = w + \text{const}$ and

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}, \tag{28}$$

which last equation was proved earlier; see Eq. (7). This completes the verification of Eq. (25) for $i = n+1$.
It remains to prove Eqs. (25) for $i = 1, \ldots, n$. The left side of the first equation is

$$\frac{dx_i}{dw} = \frac{dq_i}{dw} = \frac{dq_i}{dt},$$

and the right side is

$$\frac{\partial \Gamma}{\partial y_i} = \frac{\partial \Gamma}{\partial p_i} = \frac{\partial \gamma}{\partial p_i} = \frac{\partial H}{\partial p_i};$$

therefore, the first of Eqs. (25) for $i = 1, \ldots, n$ gives

$$\dot{q}_i = \partial H/\partial p_i ,$$

which is the first set of the Hamiltonian equations of motion [cf. Eq. (5)].

The left side of the second of Eqs. (25) is

$$\frac{dy_i}{dw} = \frac{dp_i}{dw} = \frac{dp_i}{dt},$$

and the right side is

$$-\frac{\partial \Gamma}{\partial x_i} = -\frac{\partial \Gamma}{\partial q_i} = -\frac{\partial \gamma}{\partial q_i} = -\frac{\partial H}{\partial q_i};$$

therefore, the second set of Eqs. (25) gives

$$\dot{p}_i = -\partial H/\partial q_i ,$$

which is the second set of the Hamiltonian equations of motion [cf. Eq. (6)].

This completes the verification of Eqs. (25).

Canonical transformations in the extended phase space are based on Eq. (24). The modified form of the Hamiltonian principle using the old variables $x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}$ is

$$\delta \int_{w_1}^{w_2} \left(y_i \frac{dx_i}{dw} \right) dw = 0, \tag{29}$$

which, after the canonical transformation is performed, becomes

$$\delta \int_{w_1}^{w_2} \left(Y_i \frac{dX_i}{dw} \right) dw = 0. \tag{30}$$

Note that two of the old variables had special meanings, $x_{n+1} = t$ and $y_{n+1} = -H$. After transformation, these special significances associated with the subscript $(n + 1)$ disappear since in general the transformation does not preserve the physical meaning of the variables. Also note that, while $x_{n+1} = t$ and consequently $w = t + \text{const}$ or $w = x_{n+1} + \text{const}$, the variable X_{n+1} does not, in general, have such a simple relation to the parameter w. The reason for introducing w as the new parameter in connection with the derivation of Eq. (24) is probably clearer now also. We will subject t to canonical transformation as the $(n + 1)$th coordinate while w of course will remain our parameter which itself is unchanged under the canonical transformation.

Subtracting Eq. (30) from Eq. (29) we obtain the basic formula for the generating function in the $(2n + 2)$-dimensional extended phase space, similarly to Eq. (12), which latter, however, was applicable to the original $2n$-dimensional phase space only. The result of the subtraction is

$$y_i \, dx_i - Y_i \, dX_i = dW, \tag{31}$$

where W in principle depends on the $(2n + 2)$ old variables, on the $(2n + 2)$ new variables, and on the parameter w, in total on $(4n + 5)$ variables. Due to the $(2n + 2)$ relations between the old and new variables, the number of independent variables in actuality is only $2n + 3$, which fact leads us—similarly to Eqs. (13)—to four generating functions, which are identical with those given in Eq. (13), except that for the independent variable t we write $t = w$. Also note that $i = 1, \ldots, (n + 1)$. Consequently Table I is also applicable in the extended phase space.

The examples which follow serve two purposes. They will facilitate understanding the application of the concept of extended phase space to the restricted problem and they will also clarify the above-described technique.

6.5 Examples of Hamiltonian dynamics in the extended phase space

Consider a dynamical system with two degrees of freedom. The Hamiltonian which is independent of the time is given in the four dimensional phase space by

$$H = H(q_1, q_2, p_1, p_2).$$

The equations of motion form a fourth-order system:

$$\dot{q}_i = \partial H/\partial p_i \quad \text{and} \quad \dot{p}_i = -\partial H/\partial q_i , \tag{32}$$

with $i = 1, 2$.

The canonical transformation to be performed will be a so-called extended point transformation, given by

$$W_3 = -[p_1 f_1(Q_1, Q_2) + p_2 f_2(Q_1, Q_2)] \tag{33}$$

which leads to

$$\begin{aligned} q_1 &= f_1(Q_1, Q_2), \qquad P_1 = p_1 \frac{\partial f_1}{\partial Q_1} + p_2 \frac{\partial f_2}{\partial Q_1}, \\ q_2 &= f_2(Q_1, Q_2), \qquad P_2 = p_1 \frac{\partial f_1}{\partial Q_2} + p_2 \frac{\partial f_2}{\partial Q_2}, \end{aligned} \tag{34}$$

and to

$$\bar{H} = H.$$

Note that the transformation equations (34) connect the new and the old generalized coordinates directly, without involving the momenta. The equations $q_i = f_i(Q)$, therefore, transform points of the plane (q_1, q_2) into points of the plane (Q_1, Q_2).

In the extended 6-dimensional phase space we have, for the same dynamical system,

$$\Gamma = p_3 + \gamma(q_1, q_2, p_1, p_2), \tag{35}$$

where the relation to the previous formulation is given by

$$p_3 = -H$$

and by

$$\gamma(q_1, q_2, p_1, p_2) = H(q_1, q_2, p_1, p_2).$$

Note that q_3 does not occur in Γ because the original Hamiltonian did not depend on the time explicitly.

The equations of motion in the extended phase space are

$$\frac{dq_i}{dw} = \frac{\partial \Gamma}{\partial p_i} \quad \text{and} \quad \frac{dp_i}{dw} = -\frac{\partial \Gamma}{\partial q_i}, \tag{36}$$

with $i = 1, 2, 3$.

For $i = 3$, we have

$$\frac{dq_3}{dw} = \frac{\partial \Gamma}{\partial p_3}, \quad \text{or} \quad \frac{dt}{dw} = 1,$$

and

$$\frac{dp_3}{dw} = 0, \quad \text{or} \quad p_3 = \text{const.}$$

We see that the fact that the original Hamiltonian did not contain the time explicitly appears as $p_3 = -H = \text{const.}$

The $i = 1, 2$ cases of Eqs. (36) are of course identical to the $i = 1, 2$ cases of Eqs. (32).

The canonical transformation in the extended phase space will now be a generalization of Eq. (33):

$$W_3 = -p_i f_i(Q_1, Q_2, Q_3), \tag{37}$$

where the summation is to be extended from 1 to 3.

Application of Table I gives

$$q_i = f_i(Q_1, Q_2, Q_3) \quad \text{and} \quad P_i = p_j\, \partial f_j/\partial Q_i, \tag{38}$$

where the summation convention is to be applied to j.

Denoting the partial derivates occurring in Eqs. (38) by a_{ij}, we have

$$P_i = a_{ij}(Q_1, Q_2, Q_3)\, p_j$$

or

$$p_i = b_{ij}(Q_1, Q_2, Q_3)\, P_j, \tag{39}$$

where the (b_{ij}) matrix is the inverse of $(a_{ij}) = (\partial f_j/\partial Q_i)$.

Since W_3 as given by Eq. (37) does not contain w,

$$\bar{\Gamma} = \Gamma,$$

and so

$$\bar{\Gamma} = p_3 + \gamma(q_1, q_2, p_1, p_2),$$

where the new variables will still have to be introduced. The result is

$$\bar{\Gamma} = b_{3j}P_j + \bar{\gamma}(Q_1, Q_2, Q_3, P_1, P_2, P_3), \tag{40}$$

where $\bar{\gamma}$ is obtained from γ by using the first set of Eqs. (38) and Eq. (39). The first term in Eq. (40) follows from Eq. (39), according to which

$$p_3 = b_{3j}P_j = b_{31}P_1 + b_{32}P_2 + b_{33}P_3.$$

Note that, while γ depends only on q_1, q_2, p_1, and p_2, after the transformation $\bar{\gamma}$ will depend on all six new variables.

The equations of motion in the transformed extended phase space are

$$\frac{dQ_i}{dw} = \frac{\partial \bar{\Gamma}}{\partial P_i} \quad \text{and} \quad \frac{dP_i}{dw} = -\frac{\partial \bar{\Gamma}}{\partial Q_i}, \tag{41}$$

which for $i = 3$ give

$$\frac{dQ_3}{dw} = b_{33}(Q_1, Q_2, Q_3) + \frac{\partial \bar{\gamma}}{\partial P_3} \tag{42}$$

and

$$\frac{dP_3}{dw} = -P_j \frac{\partial b_{3j}(Q_1, Q_2, Q_3)}{\partial Q_3} - \frac{\partial \bar{\gamma}}{\partial Q_3}. \tag{43}$$

Note that w is not transformed, so the equation $dw = dt$ is still valid and Eq. (42) relates the old time variable (t) to the variable Q_3. This may be considered the new time variable, but such terminology does not necessarily carry a physical meaning.

Three special cases of interest of this example are discussed in the following parts.

(A) If

$$\frac{\partial \tilde{\gamma}}{\partial P_3} = 0, \tag{44}$$

that is if the original Hamiltonian and the canonical transformation represented by Eq. (37) are such that the above relation is satisfied, then from Eq. (42) we have

$$dQ_3 = b_{33}(Q_1, Q_2, Q_3)\, dt, \tag{45}$$

which shows that the time transformation is affected by all three generalized coordinates.

(B) If the functions f_i governing the transformation [Eq. (37)] are further specified as

$$\begin{aligned} f_1 &= f_1(Q_1, Q_2), \\ f_2 &= f_2(Q_1, Q_2), \\ f_3 &= f_3(Q_3), \end{aligned} \tag{46}$$

we have

$$\begin{aligned} q_1 &= q_1(Q_1, Q_2), \\ q_2 &= q_2(Q_1, Q_2), \\ q_3 &= q_3(Q_3), \end{aligned} \tag{47}$$

and

$$\begin{aligned} P_1 &= p_1 \frac{\partial f_1}{\partial Q_1} + p_2 \frac{\partial f_2}{\partial Q_1}, \\ P_2 &= p_1 \frac{\partial f_1}{\partial Q_2} + p_2 \frac{\partial f_2}{\partial Q_2}, \\ P_3 &= p_3 \frac{\partial f_3}{\partial Q_3}. \end{aligned} \tag{48}$$

The first two equations of (47) and the first two of (48) are identical with the system of equations in the original phase space (34), and the set q_1, q_2, p_1, p_2 is transformed into the set Q_1, Q_2, P_1, P_2, without involving q_3, p_3, Q_3, and P_3.

The new Hamiltonian in the extended phase space now becomes

$$\tilde{\Gamma} = P_3 f_4(Q_3) + \tilde{\gamma}(Q_1, Q_2, P_1, P_2), \tag{49}$$

where

$$f_4(Q_3) = (df_3/dQ_3)^{-1}. \tag{50}$$

The equations of motion in the transformed extended phase space are given by Eqs. (41), which for $i = 3$ give

$$\frac{dQ_3}{dw} = f_4(Q_3) \quad \text{and} \quad \frac{dP_3}{dw} = -P_3 \frac{df_4}{dQ_3}. \tag{51}$$

The first of Eqs. (51) gives the transformation of the time and it is identical with the third equation of (47), since

$$w = \int \frac{dQ_3}{f_4(Q_3)},$$

or

$$t = f_3(Q_3).$$

The second of Eqs. (51) combined with the first one results in

$$\frac{dP_3}{dw} = -\frac{P_3}{f_4}\frac{df_4}{dw},$$

from which $P_3 f_4(Q_3) = \text{const}$, as expected since $P_3 f_4 = p_3 = -H$ and H does not contain the time explicitly.

Note that

$$P_3 = -H/f_4(Q_3)$$

is not constant, in spite of the fact that $p_3 = \text{const}$. In fact

$$P_3 = -H\, df_3/dQ_3;$$

that is, the new "momentum" (P_3) and the old "momentum" (p_3) are related through the time transformation, or through the old "coordinate" (q_3) and the new "coordinate" (Q_3). The quotation marks are used to emphasize the degree of generalization achieved since clearly the old "momentum" is the Hamiltonian and the old "coordinate" is the time.

(C) If a transformation affecting only the time is desired, we specialize the functions f_i further.

Let $f_1 = Q_1$, $f_2 = Q_2$, and $f_3 = \int f(Q_3)\, dQ_3$ with arbitrary $f(Q_3)$. The generating function of this transformation is

$$W_3 = -\left[p_1 Q_1 + p_2 Q_2 + p_3 \int f(Q_3)\, dQ_3\right], \tag{52}$$

which is the identity transformation for the variables possessing subscripts 1 and 2, since using the equations of the transformation (34) we have

$$q_1 = Q_1, \quad q_2 = Q_2,$$
$$p_1 = P_1, \quad p_2 = P_2.$$

For subscript 3 we obtain

$$q_3 = \int f(Q_3)\, dQ_3 \quad \text{and} \quad P_3 = p_3 f(Q_3).$$

So we note that the time is transformed according to

$$dt = f(Q_3)\, dQ_3. \tag{53}$$

The new Hamiltonian becomes

$$\tilde{\Gamma} = \frac{P_3}{f(Q_3)} + \gamma(Q_1, Q_2, P_1, P_2),$$

where the tilde over γ is omitted since for q_1, q_2, p_1, p_2 we have the identity transformation.

The equations of motion are again given by (41) and we have, for $i = 1$ and 2,

$$\frac{\partial \gamma}{\partial P_i} = \frac{dQ_i}{dw} \quad \text{and} \quad -\frac{\partial \gamma}{\partial Q_i} = \frac{dP_i}{dw} \tag{54}$$

and, for $i = 3$,

$$\frac{1}{f(Q_3)} = \frac{dQ_3}{dw} \quad \text{and} \quad -P_3 \frac{d}{dQ_3} \frac{1}{f(Q_3)} = \frac{dP_3}{dw}. \tag{55}$$

The first of Eqs. (55) is identical with Eq. (53) and the second shows that $p_3 = \text{const}$, as expected. More interesting are Eqs. (54). We note that $dw = dt$ and that the function γ is the original Hamiltonian. So Eqs. (54) can be written as

$$\frac{\partial H}{\partial p_i} = \frac{dq_i}{dt} \quad \text{and} \quad -\frac{\partial H}{\partial q_i} = \frac{dp_i}{dt}, \tag{56}$$

remembering that coordinates and momenta denoted by lower-case and by capital letters are equal for $i = 1, 2$.

By means of Eq. (53) the previous Hamiltonian set (56) can be changed into

$$\frac{\partial H^*}{\partial p_i} = \frac{dq_i}{dT} \quad \text{and} \quad -\frac{\partial H^*}{\partial q_i} = \frac{dp_i}{dT}, \tag{57}$$

where $H^* = H f(Q_3)$ and $Q_3 = T$ is the new time variable.

Equations (57) form a canonical set with a new Hamiltonian (H^*) and a new time variable (T).

Considering the original Hamiltonian as given by Eq. (35),

$$\Gamma = p_3 + \gamma(q_1, q_2, p_1, p_2),$$

and forming a new function Γ^* by multiplying Eq. (35) by $f(Q_3)$, we obtain

$$\Gamma^* = f(Q_3)\,\Gamma = p_3 f(Q_3) + f(Q_3)\,\gamma(q_1, q_2, p_1, p_2) \tag{58}$$

or

$$\Gamma^* = P_3 + f(Q_3)\,\gamma(Q_1, Q_2, P_1, P_2). \tag{59}$$

The equations of motion in the transformed extended phase space become, by analogy with Eqs. (57),

$$\frac{\partial \Gamma^*}{\partial P_i} = \frac{dQ_i}{dT} \quad \text{and} \quad \frac{\partial \Gamma^*}{\partial Q_i} = -\frac{dP_i}{dT}, \tag{60}$$

which give for $i = 3$

$$1 = 1 \quad \text{and} \quad -\gamma \frac{df}{dQ_3} = \frac{dP_3}{dT},$$

or

$$p_3 f(Q_3) = P_3.$$

For $i = 1$ and 2 Eqs. (57) are obtained again.

Note, therefore, that by changing Γ into $\Gamma^* = f(Q_3)\Gamma$ and introducing a new time variable, $Q_3 = T$, by $dt = f(Q_3)\, dQ_3$, the canonical forms of the equations are preserved.

6.6 Generalized time transformations

In the previous example [Part (C)] the time transformation involved only the new and old time variables. A necessary generalization for problems in regularization is to modify the equation for the transformation of the time to

$$dt = f(q_1, q_2)\, dT. \tag{61}$$

It will be shown that, if Γ is replaced by $\Gamma^* = f(q_1, q_2)\Gamma$, we obtain the following canonical set:

$$\frac{dq_i}{dT} = \frac{\partial \Gamma^*}{\partial p_i} \quad \text{and} \quad \frac{dp_i}{dT} = -\frac{\partial \Gamma^*}{\partial q_i}, \tag{62}$$

where, as before, the original variables (q_1, q_2, p_1, p_2) are not transformed.

The verification starts with $i = 3$:

$$\frac{dq_3}{dT} = f(q_1, q_2) \quad \text{or} \quad dt = f(q_1, q_2)\, dT$$

$$\frac{dp_3}{dT} = 0 \quad \text{or} \quad \frac{dp_3}{dt} f(q_1, q_2) = 0,$$

and so $p_3 = \text{const.}$

Here we used

$$\Gamma = p_3 + \gamma(q_1, q_2, p_1, p_2)$$

and

$$\Gamma^* = p_3 f(q_1, q_2) + f(q_1, q_2)\, \gamma(q_1, q_2, p_1, p_2).$$

For $i = 1$ or 2 we have, from the first of Eqs. (62),

$$\frac{dq_i}{dT} = f(q_1, q_2) \frac{\partial \gamma}{\partial p_i} \quad \text{or} \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i},$$

and from the second set of equations we obtain

$$\frac{dp_i}{dT} = -p_3 \frac{\partial f}{\partial q_i} - \frac{\partial f \gamma}{\partial q_i}$$

or

$$\frac{dp_i}{dT} = -(p_3 + \gamma) \frac{\partial f}{\partial q_i} - f \frac{\partial \gamma}{\partial q_i},$$

which may be written as

$$\frac{dp_i}{dT} = -f \frac{\partial \gamma}{\partial q_i},$$

since $p_3 + \gamma = \Gamma = 0$ along a solution.

Finally, the last equation can be changed into

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i},$$

which completes the proof.

Note that the above method cannot be used in general in the phase space and that the introduction of the concept of the extended phase space is essential. To show this we let the Hamiltonian of a dynamical

system with n degrees of freedom be $H(q, p)$. The canonical equations of motion are

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \tag{63}$$

with $i = 1, 2, \ldots, n$.

Let the transformation of the time be given by

$$dt = f(q_1, \ldots, q_n)\, dT. \tag{64}$$

For the purpose of introducing Eq. (64) into Eqs. (63) consider the notations

$$\dot{q}_i = q_i' f^{-1} \quad \text{and} \quad \dot{p}_i = p_i' f^{-1},$$

where dots denote derivatives with respect to t, primes denote derivatives with respect to T, and f^{-1} is the reciprocal of $f(q_1, \ldots, q_n)$.

Equations (63) become

$$q_i' = f \frac{\partial H}{\partial p_i} \quad \text{and} \quad p_i' = -f \frac{\partial H}{\partial q_i}. \tag{65}$$

To preserve the canonical character of these equations, we must have

$$q_i' = \frac{\partial H^*}{\partial p_i} \quad \text{and} \quad p_i' = -\frac{\partial H^*}{\partial q_i}. \tag{66}$$

The problem, therefore, is to find H^* satisfying

$$\frac{\partial H^*}{\partial p_i} = f \frac{\partial H}{\partial p_i} \quad \text{and} \quad \frac{\partial H^*}{\partial q_i} = f \frac{\partial H}{\partial q_i}. \tag{67}$$

Since

$$\frac{\partial^2 H^*}{\partial p_i\, \partial q_j} = \frac{\partial^2 H^*}{\partial q_j\, \partial p_i},$$

we have from Eqs. (67)

$$\frac{\partial}{\partial q_j}\left(f \frac{\partial H}{\partial p_i}\right) = \frac{\partial}{\partial p_i}\left(f \frac{\partial H}{\partial q_j}\right)$$

or

$$\frac{\partial f}{\partial q_j} \frac{\partial H}{\partial p_i} = \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_j}.$$

According to Eq. (64), f depends only on the q_i's; consequently $\partial f/\partial p_i = 0$ and

$$\frac{\partial f}{\partial q_j}\frac{\partial H}{\partial p_i} = 0.$$

From this it follows that

$$\frac{\partial f}{\partial q_j} = 0,$$

since

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \neq 0$$

in general.

So f is constant and the transformation of the time is linear—an uninteresting and trivial case. (We note that the case $f(p_1, \ldots, p_n)$ leads to a similar result.)

The important case, from the point of view of the applications, is the transformation of the time given by Eq. (61), for which, as shown previously, the phase-space presentation lacks generality.

We summarize our method as follows. If a given "extended phase-space Hamiltonian" Γ is transformed into $\Gamma^* = f(q_1, q_2)\,\Gamma$ and simultaneously a new time variable (T) is introduced by $dt = f(q_1, q_2)\,dT$, then the canonical character of the equations is preserved. The equations of motion in the original system are given by

$$\frac{dq_i}{dt} = \frac{\partial \Gamma}{\partial p_i} \quad \text{and} \quad \frac{dp_i}{dt} = -\frac{\partial \Gamma}{\partial q_i},$$

and in the transformed system by

$$\frac{dq_i}{dT} = \frac{\partial \Gamma^*}{\partial p_i} \quad \text{and} \quad \frac{dp_i}{dT} = -\frac{\partial \Gamma^*}{\partial q_i}.$$

6.7 Notes

The treatment offered in Chapter 6 is not intended to be complete; its sole purpose is to bring the concepts and methods of the extended phase space to the attention of the reader.

Textbooks and treatises on various levels are available. Whittaker's frequently quoted work [1] and Pars' book [2] are recommended as general references. The outstanding reference for Sections 6.1–6.3 is Goldstein's textbook [3], and for Sections 6.4–6.6 Synge's [4] article. For the application of Hamiltonian dynamics and of canonical transformations in celestial mechanics the basic reference is Brouwer and Clemence [5, p. 530]. Finally, for the analytically inclined reader, Wintner's [6] and Siegel's [6a] treatments are recommended.

The terminology "extended phase space" was introduced by Lanczos [7]. Synge [4, p. 143] prefers "space of states and energy."

The dependence of the Hamiltonian on the coordinates and momenta is indicated in the literature and in the text in a variety of ways: $H(q_1, q_2, \ldots, q_i, \ldots, q_n;\ p_1, p_2, \ldots, p_i, \ldots, p_n)$, or $H(q_i, p_i)$, or $H(q, p)$, or $H(\mathbf{q}, \mathbf{p})$. These forms are used alternatively, depending on the various formulations of the problems on hand.

The introduction of four types of generating functions in Section 6.3 is for sake of convenience only. Since Eqs. (22a) relate the generating functions W_2, W_3, and W_4 to W_1 it is sufficient to use only one of the four. This approach is accepted by Brouwer and Clemence [5].

Section 6.4 discusses the extended phase space and its variables, $2n + 2$ in number. The introductory nature of the presentation puts emphasis on ideas and methods to be utilized later, rather than on completeness. Derivation of the equations of motion (25) using the variables of the extended phase space may proceed from Eq. (24) subject to Eq. (26). The process of verification presented in the text, however, might prove more useful in familiarizing the reader with operations in the phase space.

The example connected with an extended point transformation proposed in its general form in Section 6.5 has been applied by Ollongren [8] to a problem in stellar dynamics: transformation of galactic orbits into Lissajous figures. In fact, canonical transformations capable of transforming the time as well as the coordinates and momenta have a much wider area of application than to perform the regularization of the Hamiltonian equations of motion of the restricted problem.

The end of part (B), Section 6.5, indicates the questionable significance of the terms "coordinates" and "momenta" when used in describing canonical variables after several transformations. This point of view is emphasized by Brouwer and Clemence [5]. The question will be taken up again in Section 7.7.3.

Section 6.6 offers the extension of canonical transformations to a form useful in transformations affecting regularizations. The majority of authors employing canonical transformations for transforming the independent variable seldom follow the formalism outlined in Section 6.6 and expressed in Eqs. (61) and (62). Nevertheless the basic idea is present in the writings of almost everyone from Levi-Civita [9] to Brouwer and Clemence [5]. The expression of this fundamental idea sometimes suffers from the *ad hoc* nature of the formulation, the best example of which may be seen in Birkhoff's footnote [10]. (Note that in the body of his paper Birkhoff does not treat the transformations of regularization by canonical methods.) Section 6.5 and its results are offered in the text in order to remove the need for special and *ad hoc* treatments.

For those interested in the original references, Hamilton's [11] basic papers are quoted in Section 6.8.

6.8 References

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Chapter 7

Canonical Transformations of the Restricted Problem

7.1 Introduction

The purpose of Chapter 6 was to summarize information regarding Hamiltonian dynamics and canonical transformations in the phase space and in the extended phase space. The present chapter will utilize this and apply it to the restricted problem.

It has been shown in Chapter 1 that a rotating coordinate system has considerable advantages over a fixed system. Selection of a suitable rotating system permits an integral of the motion to be established. In the framework of Hamiltonian dynamics the canonical transformation from the fixed to the rotating system eliminates the explicit appearance of the time in the Hamiltonian function which, therefore, becomes an integral. This chapter begins (Section 7.2) with a description of a canonical transformation from the sidereal to the synodic coordinate systems. The physical meaning of the two sets of position coordinates and their conjugate momenta and the relation between the Hamiltonian function and the Jacobian integral are discussed in Section 7.3. Canonical transformations resulting in a change of the origin of the coordinate system are given in Section 7.4 for the sake of completeness.

Section 7.5 is concerned with the canonical transformation from synodical rectangular Cartesian coordinates to polar coordinates and with the equations of motion in the polar system. Another canonical transformation from polar coordinates to Delaunay's elements is then found and the physical meaning of the new variables is discussed in Section 7.6. Several modifications of Delaunay's variables are presented in Section 7.7, using canonical transformations, and Poincaré's variables are derived.

The last part of this chapter deals with the process of regularization within the framework of canonical transformations. First, in Section 7.8.1, a regularizing transformation is performed for the problem of two bodies in the six-dimensional extended phase space. The treatment is kept on a general level and an example, following Levi-Civita, is worked out in detail. The restricted problem is then subjected to the process of canonical regularization and the results of Section 7.8.2 are correlated with those found in Chapter 3.

Inasmuch as some general-perturbation techniques use Hamiltonian dynamics and since regularization seems to be essential to the solution of many new problems in celestial mechanics, this chapter and especially Section 7.8.2 may be of importance when methods of general perturbations are applied to problems in space dynamics. In particular, an approach by general perturbations to the establishment of so-called interplanetary and lunar trajectories might benefit by such techniques.

Birkhoff spoke once about "an almost mysterious significance" which has been attributed to the Hamiltonian equations. The mathematical treatment of these equations involves difficulties due to the lack of flexibility of the underlying group of transformations, and consequently a formidable formal literature has grown up. This mathematical literature partly intended to serve the physicists who in turn became impressed with the mathematical developments. The applicability of these equations and of the variational principle associated with them is unquestionable in certain areas of physics. This utilization impressed mathematicians just as the mathematical developments impressed the physicists.

To judge whether this *circulus amatus* will benefit the studies of lunar and interplanetary trajectories is not possible yet due to the lack of applications.

7.2 Canonical transformation from sidereal to synodic coordinates

In Chapter 1, the fixed or sidereal coordinate system X, Y was introduced and the equations of motion of the third body in the restricted problem were given. These variables (X, Y) were made dimensionless in Eqs. (39) of Section 1.5 by introducing $\xi = X/l$ and $\eta = Y/l$. In order to comply with the conventional notation we write $\xi = q_1$,

$\eta = q_2$ and to find the conjugate momenta we compute the Lagrangian function:

$$L = \tfrac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \phi(q_1, q_2, t), \tag{1}$$

where the dimensionless form of the force function is

$$\phi = \frac{\mu_1}{\rho_1} + \frac{\mu_2}{\rho_2} \tag{2}$$

and where dots denote derivatives with respect to the dimensionless time, t.

From Eq. (1), using the definition of the momenta, $p_i = \partial L/\partial \dot{q}_i$, we have

$$p_1 = \dot{q}_1 \quad \text{and} \quad p_2 = \dot{q}_2 .$$

Note that the conjugate momenta are simply the velocity components relative to the fixed coordinate system.

The Hamiltonian function is defined by

$$H = \sum_{i=1}^{2} p_i \dot{q}_i - L,$$

which gives

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) - \phi(q_1, q_2, t). \tag{3}$$

The function $\phi(q_1, q_2, t)$ contains the time (t) explicitly through ρ_1 and ρ_2 [cf. Section 1.5, Eqs. (42)]; therefore, the Hamiltonian is not constant.

The Hamiltonian equations of motion in the sidereal coordinate system are

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

or

$$\dot{q}_1 = p_1, \qquad \dot{q}_2 = p_2, \tag{4}$$

and

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

or

$$\dot{p}_1 = \partial\phi/\partial q_1, \qquad \dot{p}_2 = \partial\phi/\partial q_2 . \tag{5}$$

Note that Eqs. (4) combined with Eqs. (5) are identical with Eqs. (40) of Section 1.5.

We apply now a bilinear canonical transformation of type W_3 with time-dependent coefficients representing a uniform rotation:

$$W_3(p_1, p_2, Q_1, Q_2, t) = -a_{ij}(t)\, p_i Q_j, \tag{6}$$

where the summation convention applies to repeated subscripts, and where

$$(a_{ij}) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \tag{7}$$

represents a uniform rotation with unit angular velocity. This matrix is identical to **A** in Eq. (25) of Section 1.4.

According to Table I of Section 6.3 the transformation equations are

$$q_i = -\partial W_3/\partial p_i, \qquad P_i = -\partial W_3/\partial Q_i,$$

and

$$\bar{H} = H + \partial W_3/\partial t.$$

Substituting here Eq. (6), we obtain

$$q_i = a_{ij} Q_j, \qquad P_i = a_{ji} p_j, \tag{8}$$

and

$$\bar{H} = \tfrac{1}{2}(P_1^2 + P_2^2) + Q_2 P_1 - Q_1 P_2 - \bar{F}(Q_1, Q_2). \tag{9}$$

Note that Eq. (6) may be arrived at by an inverse process. We wish to obtain expressions for ρ_1 and ρ_2 which do not depend explicitly on the time. This is the requirement we place on the transformation to be introduced. Since m_1 and m_2 are on a rotating straight line, ρ_1 and ρ_2 will not show explicit dependence on the time if this rotating line becomes one of the coordinate axes, say Q_1 in Fig. 7.1. The second new coordinate, Q_2, will form a right-handed Cartesian system with Q_1

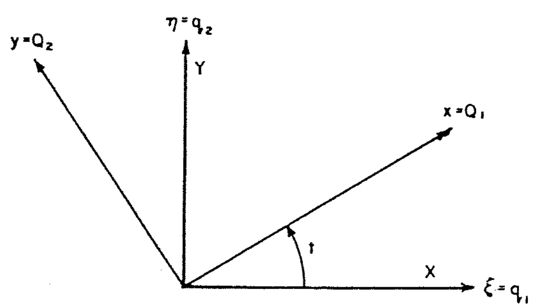

FIG. 7.1. Canonical variables in the synodic and in the sidereal coordinate systems.

and therefore the transformation equations become the well-known equations connecting a fixed q_1, q_2 with a rotating Q_1, Q_2 system:

$$q_1 = Q_1 \cos t - Q_2 \sin t,$$
$$q_2 = Q_1 \sin t + Q_2 \cos t$$

or

$$q_i = a_{ij} Q_j.$$

Since $q_i = -\partial W_3/\partial p_i$, the generating function is obtained by the method described in Section 1.7. This gives

$$W_3 = -\int a_{ij} Q_j \, dp_i + G(Q_i),$$

which, with the selection $G = 0$, is identical with Eq. (6).

The proof of Eqs. (8) requires the computation of the partial derivatives of the bilinear form W_3, which is performed after the dummy subscript i is changed to k in Eq. (6):

$$\frac{\partial}{\partial p_i} a_{kj} p_k Q_j = a_{kj} Q_j \delta_{ik} = a_{ij} Q_j$$

and similarly

$$\frac{\partial}{\partial Q_i} a_{kj} p_k Q_j = a_{kj} p_k \delta_{ij} = a_{ji} p_j.$$

Note that Eqs. (8) can also be written as

$$\mathbf{q} = \mathbf{AQ} \quad \text{and} \quad \mathbf{P} = \mathbf{A}^*\mathbf{p}, \tag{10}$$

where $\mathbf{A} = (a_{ij})$ and $\mathbf{A}^{-1} = \mathbf{A}^*$, since the inverse of an orthogonal matrix **A** is its transpose. Consequently Eqs. (8) can be inverted giving

$$\mathbf{Q} = \mathbf{A}^*\mathbf{q} \quad \text{and} \quad \mathbf{p} = \mathbf{AP}. \tag{11}$$

Verification of Eq. (9) requires the evaluation of

$$\frac{\partial}{\partial t}(a_{ij}) = \begin{pmatrix} -\sin t & -\cos t \\ \cos t & -\sin t \end{pmatrix},$$

from which it follows that

$$\mathbf{A}^* \frac{d\mathbf{A}}{dt} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

With this relation and using $\mathbf{p} = \mathbf{AP}$, we have

$$\frac{\partial W_3}{\partial t} = -\mathbf{p}\frac{d\mathbf{A}}{dt}\mathbf{Q} = -\mathbf{PA}^*\frac{d\mathbf{A}}{dt}\mathbf{Q}$$

or

$$\frac{\partial W_3}{\partial t} = Q_2P_1 - Q_1P_2 .$$

Therefore,

$$\bar{H} = \tfrac{1}{2}(p_1^2 + p_2^2) - \phi(q_1, q_2, t) + Q_2P_1 - Q_1P_2, \tag{12}$$

where the new variables are still to be introduced in the first and second terms on the right side. For this we first need to express $p_1^2 + p_2^2 = \mathbf{p}^2$ with the new variables using $\mathbf{p} = \mathbf{AP}$. We obtain

$$\mathbf{p}^2 = \mathbf{APAP} = \mathbf{PA^*AP} = \mathbf{P}^2; \tag{13}$$

that is, the length of the momentum vector is invariant in the transformation.

Similarly $\mathbf{q}^2 = \mathbf{Q}^2$, and therefore, from Eqs. (42) of Section 1.5, we obtain

$$\rho_1^2 = (q_1 - \mu_2 \cos t)^2 + (q_2 - \mu_2 \sin t)^2 = \mathbf{Q}^2 + \mu_2^2 - 2\mu_2 Q_1,$$
$$\rho_2^2 = (q_1 + \mu_1 \cos t)^2 + (q_2 + \mu_1 \sin t)^2 = \mathbf{Q}^2 + \mu_1^2 + 2\mu_1 Q_1,$$

or

$$\begin{aligned}\rho_1^2 &= Q_2^2 + (Q_1 - \mu_2)^2,\\ \rho_2^2 &= Q_2^2 + (Q_1 + \mu_1)^2,\end{aligned} \tag{14}$$

where $\mu_1 = 1 - \mu$ and $\mu_2 = \mu$.

Since ρ_1 and ρ_2 when expressed with the new variables do not contain the time explicitly, $\phi(q_1, q_2, t) = \bar{F}(Q_1, Q_2)$, and the new Hamiltonian $(\bar{H})$ is time independent.

For the sake of completeness and convenience the functions $\phi(q_1, q_2, t)$ and $\bar{F}(Q_1, Q_2)$ are here written out explicitly:

$$\begin{aligned}\phi(q_1, q_2, t) = \mu_1[(q_1 - \mu_2 \cos t)^2 + (q_2 - \mu_2 \sin t)^2]^{-1/2}\\ + \mu_2[(q_1 + \mu_1 \cos t)^2 + (q_2 + \mu_1 \sin t)^2]^{-1/2}\end{aligned} \tag{15}$$

and

$$\bar{F}(Q_1, Q_2) = \mu_1[(Q_1 - \mu_2)^2 + Q_2^2]^{-1/2} + \mu_2[(Q_1 + \mu_1)^2 + Q_2^2]^{-1/2}. \tag{16}$$

This completes the derivation of the new Hamiltonian and the equations of motion now can be established with

$$\bar{H} = \tfrac{1}{2}(P_1^2 + P_2^2) + Q_2P_1 - Q_1P_2 - \bar{F}(Q_1, Q_2)$$

as

$$\dot{Q}_1 = P_1 + Q_2, \qquad \dot{Q}_2 = P_2 - Q_1, \tag{17}$$

$$\dot{P}_1 = P_2 + \frac{\partial \bar{F}}{\partial Q_1}, \qquad \dot{P}_2 = -P_1 + \frac{\partial \bar{F}}{\partial Q_2}. \tag{18}$$

7.3 Interpretation of the new variables

The fixed or sidereal coordinate system ξ, η with the associated canonical variables q_1, q_2, p_1, p_2 possesses a Hamiltonian which depends on the time explicitly.

It was shown in Section 1.5 that introducing a uniformly rotating or synodic coordinate system x, y offers a time-independent Hamiltonian, which is closely related to the Jacobian integral. The generating function given by Eq. (6) performs the same transformation, and the sidereal system q_1, q_2 is transformed into the synodic system Q_1, Q_2. The momenta in the fixed system p_1, p_2 are simply the velocities, more precisely the velocity components of the third body relative to the sidereal system, along the axes ξ, η; see Eqs. (4). On the other hand, in the rotating (synodic) system Eqs. (17) show that $\dot{Q}_1$ and $\dot{Q}_2$ are the components of the relative velocity (with respect to the rotating system) along the rotating axes. The momenta P_1, P_2 are the components of the velocity (with respect to the fixed system) along the rotating axes.

No difficulty is encountered if it is realized that the velocity of the third body might be given relative to a rotating system or relative to a fixed system. In addition, a velocity vector can be resolved into components along the fixed or along the rotating axes. The four possibilities are:

(a) velocity relative to the fixed system, components along the fixed axes: $\dot{q}_1 = p_1, \dot{q}_2 = p_2$;

(b) velocity relative to the fixed system, components along the rotating axes: $P_1 = \dot{Q}_1 - Q_2, P_2 = \dot{Q}_2 + Q_1$;

(c) velocity relative to the rotating system, components along the fixed axes: $\dot{q}_1 + q_2, \dot{q}_2 - q_1$;

(d) velocity relative to the rotating system, components along the rotating axes $\dot{Q}_1, \dot{Q}_2$.

The vector formulation is

$$\mathbf{v} = \dot{\mathbf{r}} + \mathbf{k} \times \mathbf{r}, \tag{19}$$

where $\mathbf{v}$ is the velocity vector relative to the fixed system, $\dot{\mathbf{r}}$ is the velocity vector relative to the rotating system, $\mathbf{k}$ is a unit vector perpendicular to the plane (ξ, η) and forming a right-handed system, and $\mathbf{k} \times \mathbf{r}$ is the velocity vector of the rotating system at r.

This vector equation can be projected along the fixed axes, corresponding to (a) and (c), or along the rotating axes, giving results (b) and (d).

The magnitude of the velocity vector is an invariant of the transformation, so $\mathbf{p}^2 = \mathbf{P}^2$, as shown before; the "speed" relative to the fixed system is the same whether the components are projected along the fixed or along the rotating axes [cf. (a) and (b)]. The same is true for the velocities relative to the rotating system:

$$(\dot{q}_1 + q_2)^2 + (\dot{q}_2 - q_1)^2 = \dot{Q}_1^2 + \dot{Q}_2^2,$$

corresponding to (c) and (d). The details of the proof of the above equation are left to the reader with the remark that the right side can be written as

$$\dot{\mathbf{Q}}^2 = \left(\frac{d}{dt}\mathbf{A}^*\mathbf{q}\right)^2,$$

which in turn can be expanded and simplified by realizing that

$$\dot{\mathbf{A}}\dot{\mathbf{A}}^* = \mathbf{A}\mathbf{A}^* = \mathbf{I} \qquad \text{and} \qquad \mathbf{A}^*\dot{\mathbf{A}} = \dot{\mathbf{A}}\mathbf{A}^*.$$

The conjugate momenta belonging to q_1, q_2 are p_1, p_2, which in turn are equal to $\dot{q}_1, \dot{q}_2$. Therefore, in the fixed system, the velocities are the conjugate momenta of the coordinates. This is not true in the rotating system, where the conjugate momenta belonging to Q_1, Q_2 are P_1, P_2, which are not the velocities relative to the rotating system.

The Hamiltonian can be expressed in the system q_1, q_2, p_1, p_2 [Eq. (3)] as well as in the system Q_1, Q_2, P_1, P_2 [Eq. (9)]. The latter is generally called the Jacobian integral. Using the rotating rectangular coordinate system, the Jacobian integral might be represented in two forms, depending on whether the conjugate momenta P_1, P_2 or the velocity components $\dot{Q}_1, \dot{Q}_2$ are employed. Equation (9) uses canonical variables and represents the Hamiltonian $(\bar{H})$ as the sum of three terms. The first term is the kinetic energy (per unit mass) of the third body since $\mathbf{P}^2 = \dot{\mathbf{q}}^2$. The second term is $\partial W_3/\partial t$ and it corresponds to the angular momentum since

$$\frac{\partial W_3}{\partial t} = Q_2 P_1 - Q_1 P_2 = |\mathbf{P} \times \mathbf{Q}|.$$

The angular momentum in the fixed system is the same and the reader can show that

$$|\mathbf{P} \times \mathbf{Q}| = |\mathbf{p} \times \mathbf{q}|.$$

The third term of Eq. (9) is the force function.

Using Eqs. (17) the momenta P_1, P_2 may be expressed by the velocities $\dot{Q}_1, \dot{Q}_2$ and the Hamiltonian $(\bar{H})$ can be put in a form containing coordinates and velocities in the rotating system:

$$\bar{H} = \tfrac{1}{2}(\dot{Q}_1^2 + \dot{Q}_2^2) - \tfrac{1}{2}(Q_1^2 + Q_2^2) - \bar{F}(Q_1, Q_2) \tag{20}$$

or

$$\bar{H} = \tfrac{1}{2}\dot{\mathbf{Q}}^2 - \tfrac{1}{2}\mathbf{Q}^2 - \bar{F},$$

if vector notation is preferred.

The first term is half of the square of the velocity relative to the rotating system, the second term is the centrifugal potential, and the third term is the gravitational force function.

It is of interest to compare Eq. (20) with the Jacobian integral as presented by Eq. (54) of Section 1.5:

$$\dot{x}^2 + \dot{y}^2 = 2\Omega - C, \tag{21}$$

where $\dot{x} = \dot{Q}_1, \dot{y} = \dot{Q}_2$, and

$$\Omega = \tfrac{1}{2}\mu_1\mu_2 + \tfrac{1}{2}(Q_1^2 + Q_2^2) + \bar{F}. \tag{22}$$

Adding Eqs. (20) and (22) we obtain

$$\bar{H} + \Omega = \tfrac{1}{2}\mu_1\mu_2 + \tfrac{1}{2}(\dot{Q}_1^2 + \dot{Q}_2^2). \tag{23}$$

When the Hamiltonian is constant $(\bar{H}_0)$ we have from Eq. (23)

$$2\Omega - (\dot{Q}_1^2 + \dot{Q}_2^2) = \mu_1\mu_2 - 2\bar{H}_0 = C, \tag{24}$$

where C is the Jacobian constant as defined by Eq. (54) of Section 1.5. So the Jacobian integral and the equation $\bar{H}_0 = \text{const}$ are identical. Note also that the relation between the Hamiltonian constant and the Jacobian constant is

$$\bar{H}_0 = \frac{\mu_1\mu_2 - C}{2} = -\frac{\bar{C}}{2}, \tag{25}$$

where the last part of the equation comes from Eq. (55) of Section 1.5.

In cases when the dynamical system possesses a time-independent Hamiltonian, this Hamiltonian can play a dual role. It is a function of the appropriate coordinates and momenta and as such it is essential in formulating the canonical equations of motion. But it also represents an integral of the motion, in which case it becomes constant, similarly to the Jacobian constant in Eq. (21). In this sense we can compare the *constants* H_0 and C, giving Eq. (25), and the *functions* H and Ω, giving Eq. (23).

Finally, a comparison between the Hamiltonian equations of motion [Eqs. (17) and (18)] and the equations given in Section 1.5 for the motion in the synodical system [Eqs. (52)] shows that they can be written in the same form.

Solving Eqs. (17) for P_1 and P_2 and substituting the results in Eqs. (18) we obtain

$$\begin{aligned} \ddot{Q}_1 - 2\dot{Q}_2 &= Q_1 + \partial\tilde{F}/\partial Q_1 , \\ \ddot{Q}_2 + 2\dot{Q}_1 &= Q_2 + \partial\tilde{F}/\partial Q_2 . \end{aligned} \tag{26}$$

According to Eq. (22) we have

$$\partial\Omega/\partial Q_i = Q_i + \partial\tilde{F}/\partial Q_i , \tag{27}$$

and therefore Eqs. (26) become

$$\begin{aligned} \ddot{Q}_1 - 2\dot{Q}_2 &= \partial\Omega/\partial Q_1 , \\ \ddot{Q}_2 + 2\dot{Q}_1 &= \partial\Omega/\partial Q_2 , \end{aligned} \tag{28}$$

which are identical with Eqs. (52) of Section 1.5, since $Q_1 = x_1 Q_2 = y$.

7.4 Change of origin by means of canonical transformations

The origins of the coordinate systems Q_1, Q_2 and of q_1, q_2 are located at the center of mass of the primaries. It is often opportune to shift the origin to the location of one of the primaries; therefore, at this point canonical transformations will be introduced which execute such changes of the origin of the synodic system.

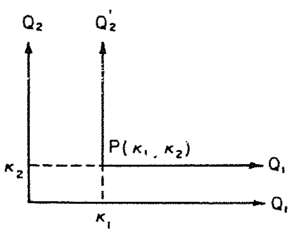

Fig. 7.2. Canonical variables when the origin is translated.

We start with the system Q_1, Q_2 and shift the origin to a point with coordinates κ_1, κ_2. Accordingly we have

$$Q_1 = Q_1' + \kappa_1 , \qquad Q_2 = Q_2' + \kappa_2 , \tag{29}$$

which equations define the system Q_1', Q_2' (see Fig. 7.2).

To find the canonical transformation which accomplishes this, we use $W_3(Q_1', Q_2', P_1, P_2)$, where primes denote the new variables.

According to Table I,

$$Q_i = -\partial W_3/\partial P_i , \qquad P_i' = -\partial W_3/\partial Q_i'. \tag{30}$$

Equations (29) in conjunction with the first equation of (30) give

$$Q_1' + \kappa_1 = -\partial W_3/\partial P_1 \quad \text{and} \quad Q_2' + \kappa_2 = -\partial W_3/\partial P_2 ,$$

from which by means of the method given in Section 1.7 we obtain

$$W_3 = -(Q_1' + \kappa_1) P_1 - (Q_2' + \kappa_2) P_2 + f(Q_1', Q_2'), \tag{31}$$

where f is an arbitary function, selection of which will influence the new conjugate momenta P_1', P_2' through the second set of Eqs. (30). Whatever selection is made for $f(Q_1', Q_2')$, it will not influence the shift of the origin since this selection does not effect the first set of Eqs. (30). The new conjugate momenta are

$$\begin{aligned} P_1' &= P_1 - \partial f/\partial Q_1', \\ P_2' &= P_2 - \partial f/\partial Q_2'. \end{aligned} \tag{32}$$

The new Hamiltonian (H') becomes, from Eq. (9),

$$\begin{aligned} \tilde{H}' = \frac{1}{2}\Big[\Big(P_1' + \frac{\partial f}{\partial Q_1'}\Big)^2 + \Big(P_2' + \frac{\partial f}{\partial Q_2'}\Big)^2\Big] + (Q_2' + \kappa_2)\Big(P_1' + \frac{\partial f}{\partial Q_1'}\Big) \\ - (Q_1' + \kappa_1)\Big(P_2' + \frac{\partial f}{\partial Q_2'}\Big) - \tilde{F}'(Q_1', Q_2'), \end{aligned} \tag{33}$$

and the equations of motion are

$$\dot{Q}_1' = Q_2' + \frac{\partial f}{\partial Q_1'} + \kappa_2 + P_1', \tag{34}$$

$$\dot{Q}_2' = -Q_1' + \frac{\partial f}{\partial Q_2'} - \kappa_1 + P_2', \tag{35}$$

$$\begin{aligned} \dot{P}_1' = &-\frac{\partial^2 f}{\partial Q_1'^2}\Big[Q_2' + \frac{\partial f}{\partial Q_1'} + \kappa_2 + P_1'\Big] \\ &+ \frac{\partial^2 f}{\partial Q_1' \partial Q_2'}\Big[Q_1' - \frac{\partial f}{\partial Q_2'} + \kappa_1 - P_2'\Big] \\ &+ \frac{\partial f}{\partial Q_2'} + P_2' + \frac{\partial \tilde{F}'}{\partial Q_1'} , \end{aligned} \tag{36}$$

and

$$\dot{P}_2' = \frac{\partial^2 f}{\partial Q_2'^2}\left[Q_1' - \frac{\partial f}{\partial Q_2'} + \kappa_1 - P_2'\right] - \frac{\partial^2 f}{\partial Q_1'\,\partial Q_2'}\left[Q_2' + \frac{\partial f}{\partial Q_1'} + \kappa_2 + P_1'\right] - \frac{\partial f}{\partial Q_1'} - P_1' + \frac{\partial \tilde{F}'}{\partial Q_2'}, \tag{37}$$

where $\tilde{F}'$ is $\tilde{F}$ expressed by means of the new variables Q_1', Q_2'.

The special case of $\kappa_1 = \mu$, $\kappa_2 = 0$, $f \equiv 0$ shifts the origin to the primary with mass $1 - \mu$. The previous equations reduce to

$$Q_1' = Q_1 - \mu, \qquad Q_2' = Q_2, \qquad P_1' = P_1, \qquad P_2' = P_2,$$

$$\tilde{H}' = \tfrac{1}{2}(P_1'^2 + P_2'^2) + Q_2'P_1' - (Q_1' + \mu)P_2' - \tilde{F}'(Q_1', Q_2'), \tag{38}$$

$$\dot{Q}_1' = Q_2' + P_1', \tag{39}$$

$$\dot{Q}_2' = -(Q_1' + \mu) + P_2', \tag{40}$$

$$\dot{P}_1' = P_2' + \partial\tilde{F}'/\partial Q_1', \tag{41}$$

$$\dot{P}_2' = -P_1' + \partial\tilde{F}'/\partial Q_2'. \tag{42}$$

Example. Show that Levi-Civita's variables correspond to the selection $\kappa_2 = 0$, $f = \kappa_1 Q_2'$. These variables shift the origin only along the axis Q_1 by κ_1 and result in the following set of equations for the motion:

$$\dot{Q}_1' = P_1' + Q_2', \qquad \dot{Q}_2' = P_2' - Q_1',$$
$$\dot{P}_1' = P_2' + \kappa_1 + \frac{\partial\tilde{F}'}{\partial Q_1'}, \qquad \dot{P}_2 = -P_1' + \frac{\partial\tilde{F}'}{\partial Q_2'}. \tag{43}$$

7.5 Transformation from rectangular synodic to polar coordinates

At the end of Section 7.2, the canonical equations of motion are given, using the set of variables Q_1, Q_2, P_1, P_2 [cf. Eqs. (17) and (18)]. The corresponding Hamiltonian is given by Eq. (9). The coordinates Q_1, Q_2 represent a uniformly rotating Cartesian rectangular system with origin at the center of mass of m_1 and m_2. We now return to this system and establish a canonical transformation to polar coordinates. The variables of the new system will be denoted by primes and the reader is advised to distinguish between these variables and the ones used in the previous section. Section 7.4 might be considered a useful and opportune insertion, not directly influencing the main stream of the transformations.

Instead of presenting directly the generating function, an instructive derivation is offered. Since the new coordinates are known in advance, generating functions will be found from the known relations between the old and the new coordinates. Expressions for the old coordinates Q_1, Q_2 in terms of the new ones Q_1', Q_2' are simpler than vice versa; therefore, the relations

$$Q_1 = Q_1(Q_1', Q_2'), \qquad Q_2 = Q_2(Q_1', Q_2')$$

will be utilized.

Table I of Section 6.3 immediately suggests the use of W_3 or W_4 as the transformations which furnish the old coordinates as partial derivatives of the generating function. Note however that W_4 is not appropriate since it contains only the momenta and therefore it does not allow the utilization of the relation between the coordinates.

After the selection of $W_3 = W_3(Q_1', Q_2', P_1, P_2)$ a simplification is introduced. As mentioned before, the sign of the generating function is of no consequence except when it contains the time explicitly; see Eqs. (15) and (16) of Chapter 6. Therefore in problems such as the present one, when $\partial W_3/\partial t = 0$, the equations

$$Q_i = \partial W_3/\partial P_i, \qquad P_i' = \partial W_3/\partial Q_i' \tag{44}$$

can be used instead of the ones given on the third line of Table I of Section 6.3.

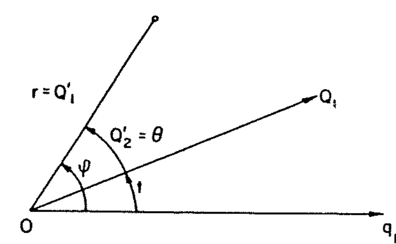

FIG. 7.3. Polar coordinates as canonical variables.

The new (polar) coordinates are introduced by $Q_1' = r$ and $Q_2' = \theta$. The radial coordinate Q_1' is the distance between the origin of the system Q_1, Q_2 and the third body, while the angular coordinate Q_2' is the angle between the (rotating) axis Q_1 and the radius vector. According to Fig. 7.3,

$$Q_1 = Q_1' \cos Q_2' \qquad \text{and} \qquad Q_2 = Q_1' \sin Q_2'. \tag{45}$$

Combining the first of Eqs. (44) for $i = 1, 2$ with Eqs. (45), two partial differential equations are obtained for the determination of W_3:

$$Q_1' \cos Q_2' = \frac{\partial W_3}{\partial P_1} \qquad \text{and} \qquad Q_1' \sin Q_2' = \frac{\partial W_3}{\partial P_2}. \tag{46}$$

The general solution of these equations is

$$W_3 = Q_1'(P_1 \cos Q_2' + P_2 \sin Q_2') + f(Q_1', Q_2'). \tag{47}$$

The new momenta are obtained from the second set of Eqs. (44):

$$\begin{aligned} P_1' &= P_1 \cos Q_2' + P_2 \sin Q_2' + \partial f/\partial Q_1', \\ P_2' &= Q_1'(-P_1 \sin Q_2' + P_2 \cos Q_2') + \partial f/\partial Q_2'. \end{aligned} \tag{48}$$

Introducing the vector $\mathbf{P}'$ with components P_1' and P_2'/Q_1' and the vector $\mathbf{P}$ with components P_1 and P_2 we can rewrite Eqs. (48) in the form

$$\mathbf{P}' = \mathbf{A}^*(Q_2')\,\mathbf{P} + \operatorname{grad} f, \tag{49}$$

where the matrix $\mathbf{A}(Q_2')$ is given by Eq. (7) with the argument being Q_2' instead of t and the components of the vector grad f are $\partial f/\partial Q_1'$ and $(\partial f/\partial Q_2')(Q_1')^{-1}$.

Inversion of Eq. (49) gives

$$\mathbf{P} = \mathbf{A}(Q_2')\,(\mathbf{P}' - \operatorname{grad} f), \tag{50}$$

and the new Hamiltonian ($\tilde{H}'$) now can be computed by means of Eqs. (45) and (50). Substitution into Eq. (9) gives

$$\tilde{H}' = \tfrac{1}{2}(P_1'^2 + P_2'^2/Q_1'^2) - P_2' - \tilde{F}'(Q_1', Q_2'), \tag{51}$$

where the arbitrary function introduced in Eq. (47) was chosen to be $f = \text{const}$.

Evaluation of the terms occurring in Eq. (9) proceeds as follows. Firstly

$$\mathbf{P}^2 = \mathbf{AP'AP'} = \mathbf{P'A^*AP'} = \mathbf{P}'^2 = P_1'^2 + P_2'^2/Q_1'^2.$$

The second and third terms in Eq. (9) form the negative angular momentum and as such it becomes $-P_2'$, as the reader may verify by direct substitution.

The equations of motion are

$$\dot{Q}_1' = P_1', \qquad \dot{Q}_2' = P_2'/Q_1'^2 - 1 \tag{52}$$

and

$$\dot{P}_1' = P_2'^2/Q_1'^3 + \partial\tilde{F}'/\partial Q_1', \qquad \dot{P}_2' = \partial\tilde{F}'/\partial Q_2'. \tag{53}$$

Knowing the physical meaning of Q_1' and Q_2', interpretation of P_1' and P_2' follows from Eq. (52). Since $Q_1' = r$, $P_1' = \dot{r}$, and so the first conjugate momentum becomes the radial velocity. Furthermore $Q_2' = \theta$, therefore, $P_2' = r^2(\dot{\theta} + 1)$ or

$$P_2' = r^2 \frac{d}{dt}(\theta + t) = r^2\dot{\varphi}, \tag{54}$$

and so the second conjugate momentum is the angular momentum. Note also that the previously introduced second component of $\mathbf{P}'$,

$$\frac{P_2'}{Q_1'} = r\frac{d}{dt}(\theta + t) = r\dot{\varphi}, \tag{55}$$

is the component of the velocity relative to the fixed system normal to r.

The force function $\tilde{F}'$ is evaluated from Eq. (16):

$$\tilde{F}' = \mu_1/r_1 + \mu_2/r_2, \tag{56}$$

where

$$\begin{aligned} r_1^2 &= Q_1'^2 + \mu_2^2 - 2\mu_2 Q_1' \cos Q_2', \\ r_2^2 &= Q_1'^2 + \mu_1^2 + 2\mu_1 Q_1' \cos Q_2'; \end{aligned} \tag{57}$$

see Fig. 7.4.

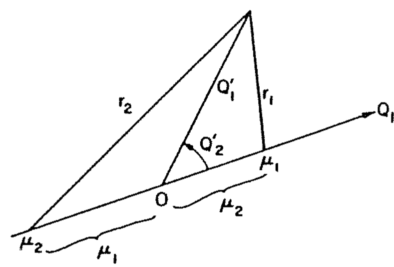

FIG. 7.4. Notation for the force function.

Note that, if $\mu = \mu_2 = 0$ and consequently $\mu_1 = 1$, the force function becomes

$$\tilde{F}' = 1/Q_1', \tag{58}$$

since $r_1 = r = Q_1'$, and observe that the problem is reduced to the case of a Newtonian central force.

Finally, it is remarked that if in Eq. (51) we substitute

$$\begin{aligned} P_1' &= \dot{r}, & P_2' &= r^2\frac{d}{dt}(\theta + t), \\ Q_1' &= r, & Q_2' &= \theta, \end{aligned} \tag{59}$$

we obtain

$$\tilde{H}' = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \tfrac{1}{2}r^2 - \tilde{F}'(r, \theta), \tag{60}$$

where the first term on the right side is the square of the velocity relative to the rotating system, i.e.,

$$\dot{r}^2 + r^2\dot{\theta}^2 = \dot{Q}_1^2 + \dot{Q}_2^2 = \dot{x}^2 + \dot{y}^2,$$

and the second term is

$$-\tfrac{1}{2}r^2 = -\tfrac{1}{2}(Q_1^2 + Q_2^2) = -\tfrac{1}{2}(x^2 + y^2).$$

Therefore Eq. (60) reduces to (20).

7.6 Transformation from polar coordinates to Delaunay's variables

The polar system established in Section 7.5 of this chapter was described by primed capital letters: Q_1', Q_2', P_1', P_2'. For the new variables, to be introduced in this section, lower-case primed letters will be used: q_1', q_2', p_1', p_2'. Table I, summarizing the notation, is furnished for the readers' convenience.

TABLE I

SUMMARY OF NOTATION

Symbols	Application
q_i, p_i	Fixed Cartesian rectangular coordinates
Q_i, P_i	Rotating Cartesian rectangular coordinates
Q_i', P_i'	Polar coordinates
q_i', p_i'	Delaunay's variables

The considerations with which Delaunay's variables will be introduced follow some of Poincaré's ideas. First the generating function will be derived, then the new variables will receive physical interpretation, and finally the equations of motion will be given. The transformation is defined by a generating function of the second type, i.e.,

$$W_2 = W_2(Q_1', Q_2', p_1', p_2'),$$

and, according to Table I of Section 6.3,

$$P_i' = \partial W_2/\partial Q_i' \tag{61}$$

and

$$q_i' = \partial W_2/\partial p_i'. \tag{62}$$

Assigning an invariant property to the angular momentum of the third particle we have $p_2' = P_2'$. Equation (61), with $i = 2$, gives

$$P_2' = \partial W_2/\partial Q_2' = p_2',$$

and so

$$W_2 = p_2'Q_2' + f(Q_1', p_1', p_2'), \tag{63}$$

which explains the selection of the generating function, W_2.

Furthermore, it is recalled that P_1' represents the radial velocity, $P_1' = \dot{r}$. In order to determine the function $f(Q_1', p_1', p_2')$ in the above equation, again Eq. (61) is used, now with $i = 1$. When Eq. (63) is substituted into Eq. (61), we obtain

$$P_1' = \dot{r} = \partial f/\partial Q_1'. \tag{64}$$

This equation becomes useful when $\dot{r}$ is expressed by means of Q_1', p_1', and p_2' since these are the variables of the function f. First it is shown that

$$\dot{r} = \left[-\frac{a(1-e^2)}{Q_1'^2} + \frac{2}{Q_1'} - \frac{1}{a}\right]^{1/2}, \tag{65}$$

where a and e are the semimajor axis and the eccentricity of an ellipse representing the motion of the third particle when it is acted upon only by a unit mass located at the origin, $\mu = \mu_2 = 0$, $\mu_1 = 1$. Equation (65) gives, therefore, $\dot{r}$ for the problem of two bodies but also for the restricted problem if a and e are considered variables.

The energy integral of this problem of two bodies in the sidereal system is

$$v^2 = 2/r - 1/a, \tag{66}$$

where v is the velocity relative to the fixed system. The square of this velocity can be expressed as

$$v^2 = \dot{r}^2 + r^2\dot{\varphi}^2, \tag{67}$$

where once again $\varphi = t + \theta$ is the polar angle with respect to the fixed axis q_1 (see Fig. 7.3).

The angular momentum of the same problem of two bodies is

$$r^2\dot{\varphi} = [a(1-e^2)]^{1/2}, \tag{68}$$

since at the perihelion

$$r = a(1-e) \quad \text{and} \quad v = r\dot{\varphi} = \frac{1}{a^{1/2}}\left(\frac{1+e}{1-e}\right)^{1/2}$$

from Eq. (66).

When the solution of Eq. (68) for $\dot{\varphi}$ is substituted into Eq. (67) the result becomes

$$v^2 = \dot{r}^2 + \frac{a(1-e^2)}{r^2} = \frac{2}{r} - \frac{1}{a}$$

from which Eq. (65) is obtained with the substitution $r = Q_1'$. This completes the proof of Eq. (65).

The next step is to write in Eq. (65) p_2' for $[a(1-e^2)]^{1/2}$, since both quantities represent the angular momentum, and p_1' for $a^{1/2}$, since this is the definition of Delaunay's variable p_1'. The function f is now expressed by Q_1', p_1', and p_2' as planned, and Eq. (64) becomes

$$\frac{\partial f}{\partial Q_1'} = \left(-\frac{p_2'^2}{Q_1'^2} + \frac{2}{Q_1'} - \frac{1}{p_1'^2}\right)^{1/2} \tag{69}$$

and so

$$f = \int \left(-\frac{p_2'^2}{Q_1'^2} + \frac{2}{Q_1'} - \frac{1}{p_1'^2}\right)^{1/2} dQ_1' + g(p_1', p_2'). \tag{70}$$

Returning now to Eq. (63) and to the problem of finding the generating function, we have

$$W_2 = p_2'Q_2' + I + g(p_1', p_2'), \tag{71}$$

where I stands for the quadrature appearing in Eq. (70), giving $f = I + g$.
Another way of writing I is

$$I = \int_z^{Q_1'} \left(-\frac{p_2'^2}{\xi^2} + \frac{2}{\xi} - \frac{1}{p_1'^2}\right)^{1/2} d\xi,$$

where ξ is a dummy variable and z is independent of Q_1' and Q_2', but otherwise arbitrary.
Note that

$$\frac{\partial I}{\partial Q_1'} = \left(-\frac{p_2'^2}{Q_1'^2} + \frac{2}{Q_1'} - \frac{1}{p_1'^2}\right)^{1/2},$$

since the derivative of an integral with respect to its upper limit is the integrand evaluated at the upper limit, according to the well-known relation

$$\frac{d}{dx}\int_z^x f(\xi)\, d\xi = f(x),$$

provided z is independent of x.
A recapitulation at this point shows that the term $p_2' Q_2'$ in the last expression (71) for W_2 gives the relation $p_2' = P_2'$ while the term I satisfies the requirement

$$\dot{r} = P_1' = \partial W_2/\partial Q_1'.$$

The function $g(p_1', p_2')$ is still to be selected and it will influence, according to Eqs. (62), the variables q_1', q_2'. Note that the new momenta p_1', p_2' are already available from the relations

$$p_2' = P_2' \tag{72}$$

and

$$P_1' = \left(-\frac{p_2'^2}{Q_1'^2} + \frac{2}{Q_1'} - \frac{1}{p_1'^2}\right)^{1/2}, \tag{73}$$

and so p_1' and p_2' may be expressed by means of the old variables Q_1', P_1', and P_2'.
The two additional Delaunay variables q_1', q_2' are given by Eqs. (62) and (71) in the following form:

$$q_1' = \frac{\partial I}{\partial p_1'} + \frac{\partial g}{\partial p_1'} \qquad \text{and} \qquad q_2' = Q_2' + \frac{\partial I}{\partial p_2'} + \frac{\partial g}{\partial p_2'}.$$

Instead of selecting the function g the same is accomplished by selecting the lower limit of the integral I as

$$z = z(p_1', p_2')$$

and letting $g(p_1', p_2') \equiv 0$. After I is evaluated and the limits are substituted, $I = I(Q_1', p_1', p_2')$ is obtained and W_2 is given by Eq. (71) with $g = 0$. Furthermore, denoting the integrand by $h(\xi, p_1', p_2')$ results in

$$I = \int_{z(p_1', p_2')}^{Q_1'} h(\xi, p_1', p_2')\, d\xi; \tag{74}$$

and letting

$$\int h(\xi, p_1', p_2')\, d\xi = \mathscr{H}(\xi, p_1', p_2') + C$$

gives

$$I = \mathscr{H}(Q_1', p_1', p_2') - \mathscr{H}[z(p_1', p_2'), p_1', p_2']. \tag{75}$$

Equation (75) shows that

$$h(Q_1', p_1', p_2') = P_1'$$

and that

$$\mathscr{H}(z, p_1', p_2') = -g(p_1', p_2'),$$

which connects the functions $\mathscr{H}$ and g.
Using now Eq. (74) to define I, we have

$$W_2 = p_2'Q_2' + I, \tag{76}$$

where only the lower limit $z(p_1', p_2')$ remains to be determined.
It will be shown that by selecting the perihelion point for the lower limit,

$$z = p_1'[p_1' - (p_1'^2 - p_2'^2)^{1/2}], \tag{77}$$

Eq. (76) gives the remaining two Delaunay variables, q_1' and q_2'. The proof is straightforward and consists of evaluating I and then computing its partial derivatives with respect to p_1' and p_2'.

Prior to this, however, it is of interest to observe that the substitutions $p_1' = a^{1/2}$ and $p_2' = [a(1-e^2)]^{1/2}$ into Eq. (77) give the previously mentioned physical meaning: $z = a(1-e)$. The integration, therefore, is to be performed from the perihelion to an arbitrary point on the ellipse in the problem of two bodies [see, for instance, Eq. (74)].

Equations (62) are now used to obtain q_1' and q_2', with the result

$$q_1' = \arccos\left\{\frac{1 - Q_1'/p_1'^2}{[1-(p_2'/p_1')^2]^{1/2}}\right\} - \frac{Q_1'}{p_1'}\left(-\frac{p_2'^2}{Q_1'^2} + \frac{2}{Q_1'} - \frac{1}{p_1'^2}\right) \tag{78}$$

and

$$q_2' = Q_2' - \arccos\frac{p_2'^2/Q_1' - 1}{[1-(p_2'/p_1')^2]^{1/2}}. \tag{79}$$

The interpretation of the new variables is as follows. Writing r for Q_1', a for $p_1'^2$, $a(1-e^2)$ for $p_2'^2$, and $\dot r$ for the last factor in Eq. (78) gives

$$q_1' = \arccos\frac{1 - r/a}{e} - \frac{r\dot r}{a^{1/2}}. \tag{80}$$

For a two-body motion, however,

$$\frac{r\dot r}{a^{1/2}} = e \sin u, \tag{81}$$

with u as the eccentric anomaly as will be shown later. Equation (80) gives

$$r = a[1 - e\cos(q_1' + e\sin u)], \tag{82}$$

which, when compared with

$$r = a(1 - e\cos u),$$

gives Kepler's equation

$$q_1' = u - e\sin u, \tag{83}$$

with the mean anomaly $q_1' = l$.

Therefore, it may be seen that Delaunay's first variable (q_1') is the mean anomaly for the problem of two bodies.

Making the required substitutions in Eq. (79) and using the variables r, a, e, and θ gives

$$q_2' = \theta - \arccos\frac{a(1-e^2)/r - 1}{e},$$

from which

$$r = \frac{a(1-e^2)}{1 + e\cos(\theta - q_2')}. \tag{84}$$

Comparing Eq. (84) with the expression for r in terms of the true anomaly (f),

$$r = \frac{a(1-e^2)}{1 + e\cos f},$$

gives

$$q_2' = \theta - f. \tag{85}$$

The second Delaunay variable q_2' is therefore the longitude of the apsidal line (or the argument of the perihelion) relative to the rotating system, and it is measured from the rotating Q_1 axis (see Fig. 7.5).

This completes the analysis regarding the use of the generating function W_2 as given by Eq. (76), excepting the proof of Eq. (81). For this purpose consider $r = a(1 - e\cos u)$, from which $\dot r = ae\dot u \sin u$. From Kepler's equation $l = \bar n(t - t_0) = u - e\sin u$, one obtains $\dot l = \bar n = \dot u(1 - e\cos u) = \dot u r/a$. Here $\bar n$ is the mean motion on the ellipse, $\bar n = a^{-3/2}$; consequently, $a^{-3/2} = \dot u r/a$ or $\dot u = (ra^{1/2})^{-1}$. Combining this last result with $\dot r = ae\dot u\sin u$ gives Eq. (81).

Before evaluating the new Hamiltonian, the results of this section are summarized. The generating function

$$W_2 = Q_2'p_2' + \int_{p_1'[p_1'-(p_1'^2-p_2'^2)^{1/2}]}^{Q_1'} \left(-\frac{p_2'^2}{\xi^2} + \frac{2}{\xi} - \frac{1}{p_1'^2}\right)^{1/2} d\xi \tag{86}$$

furnishes the canonical transformation from the variables in the polar coordinate system

$$Q_1' = r, \qquad Q_2' = \theta,$$

$$P_1' = \dot r, \qquad P_2' = r^2\frac{d}{dt}(t + \theta),$$

to Delaunay's system

$$q_1' = l, \qquad q_2' = \theta - f,$$

$$p_1' = a^{1/2}, \qquad p_2' = [a(1-e^2)]^{1/2}.$$

Here r is the radial coordinate, θ is the angle between r and the rotating axis on which the primaries are located (Q_1), l is the mean anomaly, f is the true anomaly, $(\theta - f)$ is the argument of the perihelion in the rotating system, and a and e are the semimajor axis and eccentricity of the ellipse which was described earlier. The special symbols used for Delaunay's variables are $q_1' = l$, $q_2' = \bar g$, $p_1' = L$, $p_2' = G$.

The new Hamiltonian is obtained from Eq. (51) by substitution of the new variables. Evaluating the first term in Eq. (51) by means of Eqs. (72) and (73) gives

$$\frac{1}{2}\left(P_1'^2 + \frac{P_2'^2}{Q_1'^2}\right) = \frac{1}{2}\left(\frac{2}{Q_1'} - \frac{1}{p_1'^2}\right),$$

and the new Hamiltonian ($\tilde{h}'$) becomes

$$\tilde{h}' = \frac{1}{2}\left(\frac{2}{Q_1'} - \frac{1}{p_1'^2}\right) - p_2' - \tilde{f}'. \tag{87}$$

Note that $\tilde{h}'$ still contains one of the old variables (Q_1') and $\tilde{f}'$ is still to be expressed by the new variables.

In fact

$$\tilde{f}' = \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} = \frac{1}{r_1} + \mu\left(\frac{1}{r_2} - \frac{1}{r_1}\right), \tag{88}$$

so when $\mu = 0, \tilde{f}' = 1/Q_1'$ [cf. Eq. (58)] and

$$\tilde{h}' = -1/2p_1'^2 - p_2'. \tag{89}$$

If $\mu \neq 0$, the formulation may be made similar by writing the force function as

$$\tilde{f}' = 1/Q_1' + R$$

and we obtain from Eq. (87)

$$\tilde{h}' = -1/2p_1'^2 - p_2' - R, \tag{90}$$

where R represents the deviation of $\tilde{f}' = 1/Q_1'$ from the actual $\tilde{f}'$. In other words R is the perturbation relative to the previously described problem of two bodies or central force problem,

$$R = \tilde{f}' - 1/Q_1'.$$

The equations of motion are

$$\dot{q}_i' = \partial\tilde{h}'/\partial p_i' \quad \text{and} \quad \dot{p}_i' = -\partial\tilde{h}'/\partial q_i', \tag{91}$$

which, using Eq. (90), become

$$\dot{q}_1' = 1/p_1'^3 - \partial R/\partial p_1', \quad \dot{q}_2' = -1 - \partial R/\partial p_2' \tag{92}$$

and

$$\dot{p}_1' = \partial R/\partial q_1', \quad \dot{p}_2' = \partial R/\partial q_2'. \tag{93}$$

For $\mu = 0$, $R = 0$, and the equations of motion [Eqs. (91) and (92)] become

$$\dot{q}_1' = 1/p_1'^3, \quad \dot{q}_2' = -1, \quad \dot{p}_1' = 0, \quad \dot{p}_2' = 0. \tag{94}$$

The first equation is equivalent to $\bar{n}a^{3/2} = 1$, since $\dot{q}_1' = \dot{l} = \bar{n}$, and, according to the third equation, $p_1' = a^{1/2} = \text{const}$. The second equation gives $q_2' = -t + \text{const}$, which says that the apsidal line in the rotating system rotates with unit angular velocity in the counter-clockwise direction, or in other words the perihelion is fixed relative to the fixed system.

Finally, the third and fourth equations state that a and e are constant. Together the four equations (94) describe elliptic motion for the problem of two bodies.

The set q_1', q_2', p_1', p_2' together with $\tilde{h}'$ is the final result of this section. This set of variables forms Delaunay's elements. However, it is important to realize that various combinations of these also may form canonical sets of variables. This subject is discussed in the next section.

7.7 Modifications of Delaunay's elements

7.7.1 Introduction

The notation

$$q_1' = l, \quad q_2' = \bar{g}, \quad p_1' = L, \quad p_2' = G \tag{95}$$

has been introduced before and now the simplifying notation $\tilde{h}' = H$ is added in the hope that the reader will not be confused by this. It is to be realized that the tilde, bar, prime, etc., as well as the use of capital and lower-case letters, are unsatisfactory when many canonical transformations are made successively. It is conventional in the literature to resort to the use of H for the Hamiltonian whatever the variables are—a questionable practice but one which seldom causes misunderstandings.

The Hamiltonian associated with the variables (95) is given by Eq. (90) and can now be written as

$$H = -1/2L^2 - G - R. \tag{96}$$

The equations of motions are

$$\frac{dl}{dt} = \frac{1}{L^3} - \frac{\partial R}{\partial L}, \quad \frac{d\bar{g}}{dt} = -1 - \frac{\partial R}{\partial G},$$
$$\frac{dL}{dt} = \frac{\partial R}{\partial l}, \quad \frac{dG}{dt} = \frac{\partial R}{\partial \bar{g}}, \tag{97}$$

which are, of course, equivalent to Eqs. (92) and (93). The reader's attention is directed to the importance of distinguishing between coordinates $(l, \bar{g})$ and their conjugate momenta (L, G), since a confusion may result in an error of sign in the equations of motion. This remark will receive further amplification later.

An impressive array of modifications and variations of Delaunay's variables is treated in the literature. In the following, those sets will be discussed which gained popularity.

7.7.2 Equations in the fixed system

Equations (97) refer to the rotating system since $\bar{g} = g_2' = \theta - f$ is the argument of the pericenter relative to the rotating system. As Fig. 7.5 shows, $g = \bar{g} + t$ is the argument of the perihelion measured from the fixed axis, q_1. If the use of g instead of $\bar{g}$ is preferred as a canonical variable, a generating function dependent upon the time also must be found to perform the transformation from the set $(q; p) = (l, \bar{g}; L, G)$ to the variables $(Q; P) = (l, g; L, G)$. Such a generating function may be written immediately with some experience, nevertheless in the following its derivation is offered as an example.

The generating function must furnish the identity transformation for all the variables except $\bar{g}$. Therefore, it is expected that a properly selected $W_2 = W_2(q_1, q_2, P_1, P_2, t)$ satisfies the requirements. From Table I of Section 6.3 we have

$$p_i = \partial W_2/\partial q_i, \qquad Q_i = \partial W_2/\partial P_i, \tag{98}$$

and

$$\tilde{H} = H + \partial W_2/\partial t. \tag{99}$$

Since $Q_2 = g = \bar{g} + t = q_2 + t$, Eqs. (98) give

$$\partial W_2/\partial P_2 = q_2 + t,$$

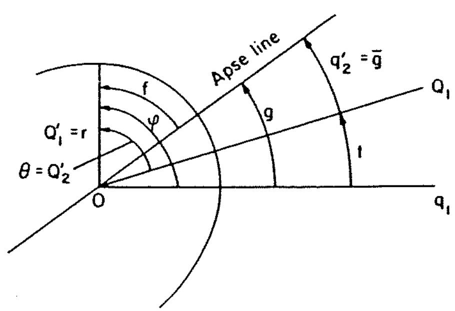

FIG. 7.5. Summary of the angular variables used.

or

$$W_2 = (q_2 + t)P_2 + \alpha(q_1, q_2, P_1, t).$$

Furthermore, since $Q_1 = l = q_1$ we have

$$\partial W_2/\partial P_1 = q_1 = \partial\alpha/\partial P_1,$$

from which

$$\alpha(q_1, q_2, P_1, t) = q_1 P_1 + \beta(q_1, q_2, t),$$

and so

$$W_2 = (q_2 + t)P_2 + q_1 P_1 + \beta(q_1, q_2, t).$$

In order to preserve the momenta, we have

$$p_1 = P_1 = \partial W_2/\partial q_1 = P_1 + \partial\beta/\partial q_1$$

and

$$p_2 = P_2 = \partial W_2/\partial q_2 = P_2 + \partial\beta/\partial q_2.$$

Therefore the function β can depend only on the time and, writing $\beta(0, 0, t) = \gamma(t)$, the generating function becomes

$$W_2 = (q_2 + t)P_2 + q_1 P_1 + \gamma(t),$$

with an unspecified function $\gamma(t)$.

The new Hamiltonian $(\tilde{H})$ is obtained from Eq. (99):

$$\tilde{H} = -1/2P_1^2 - P_2 - R + P_2 + d\gamma/dt,$$

since

$$\partial W_2/\partial t = P_2 + d\gamma(t)/dt.$$

In order to arrive at a Hamiltonian independent of the time, let $\gamma(t)$ be a constant. In this way we obtain

$$\tilde{H} = -1/2L^2 - R. \tag{100}$$

Note that the Hamiltonian is simpler than it was before the transformation since the term $-G$ is absent [cf. Eq. (96)]. The equations of motion with the new set l, g, L, G in the fixed coordinate system are

$$\begin{aligned} \dot{l} &= 1/L^3 - \partial R/\partial L, & \dot{g} &= -\partial R/\partial G, \\ \dot{L} &= \partial R/\partial l, & \dot{G} &= \partial R/\partial g. \end{aligned} \tag{101}$$

These equations reduce to $\dot{l} = L^{-3}$ and $\dot{g} = \dot{L} = \dot{G} = 0$ for the problem of two bodies represented in a fixed coordinate system. The first equation is Kepler's law and the rest show that the orientation of the apsidal line, the length of the semimajor axis, and the eccentricity are constant.

7.7.3 Interchange of coordinates and momenta

Equations (101) follow from the Hamiltonian equations of motion using $\bar{H}$ as given by Eq. (100). Introducing $F = -\bar{H}$ gives

$$\dot{l} = -\partial F/\partial L, \qquad \dot{L} = \partial F/\partial l,$$
$$\dot{g} = -\partial F/\partial G, \qquad \dot{G} = \partial F/\partial g, \tag{102}$$

which are canonical equations with F as the Hamiltonian, with (L, G) as "coordinates," and with (l, g) as "momenta."

In connection with Eqs. (97) the remark was made that the terminology must be carefully preserved regarding coordinates and momenta in order to avoid errors of sign in the equations of motion. This warning is not to be interpreted to mean that any special physical significance is associated with the words "coordinates" and "momenta."

In general the Hamiltonian equations and the canonical variables have the following property. With coordinates q_i, momenta p_i, and with H as the Hamiltonian we have

$$\dot{q}_i = \partial H/\partial p_i \quad \text{and} \quad \dot{p}_i = -\partial H/\partial q_i. \tag{102a}$$

On the other hand, if $F = -H$ we have

$$\dot{q}_i = -\partial F/\partial p_i \quad \text{and} \quad \dot{p}_i = \partial F/\partial q_i, \tag{102b}$$

which are canonical equations with F as the Hamiltonian, only if the q_i variables are regarded as momenta and the p_i's as generalized coordinates.

7.7.4 Linear combinations of Delaunay's elements

In the following modifications we will use as the starting point the system $q_1 = L$, $q_2 = G$, $P_1 = l$, $P_2 = g$ with the Hamiltonian,

$$F = 1/2L^2 + R.$$

Linear combinations of Delaunay's elements can be produced by $W_2 =$ type generating functions. Consider in general

$$W_2(q_i, P_i) = a_{ij}P_i q_j, \tag{103}$$

where the elements of the nonsingular matrix a_{ij} are constants. Then according to Table I of Section 6.3,

$$p_k = \partial W_2/\partial q_k = a_{ik}P_i,$$
$$Q_k = \partial W_2/\partial P_k = a_{ki}q_i, \tag{104}$$

and if the inverse of the matrix a_{ij} is b_{ij}, we have

$$P_i = b_{ik}p_k \quad \text{and} \quad Q_i = a_{ik}q_k.$$

We see that if the new coordinates are linear combinations of the old coordinates, then the new momenta will also be linearly related to the old momenta.

Three cases are of interest, corresponding to the matrices

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

(A) In the case of the first matrix, Eq. (103) becomes

$$W_2 = q_1(P_1 - P_2) + q_2P_2,$$

and so from Eqs. (104) we have

$$p_1 = P_1 - P_2, \quad p_2 = P_2, \quad Q_1 = q_1, \quad Q_2 = q_2 - q_1.$$

With $q_1 = L$, $q_2 = G$, $p_1 = l$, $p_2 = g$, the new canonical set becomes

$$Q_1 = L, \quad Q_2 = G - L, \quad P_1 = l + g, \quad P_2 = g.$$

(B) Applying the second matrix *to the above* result [Case (A)] gives a new set of canonical variables. With the generating function

$$W_2 = q_1P_1 - q_2P_2,$$

we get

$$p_1 = P_1, \quad p_2 = -P_2, \quad Q_1 = q_1, \quad Q_2 = -q_2.$$

The new variables, therefore, become

$$Q_1 = L, \quad Q_2 = L - G, \quad P_1 = l + g, \quad P_2 = -g.$$

(C) Finally, with the third matrix, we get as the new variables [from the new variables of Set (B)]

$$Q_1 = L - G, \quad Q_2 = G, \quad P_1 = l, \quad P_2 = l + g.$$

7.7.5 Poincaré's variables

(A) The generating function which introduces cyclic (or ignorable) coordinates when applied to the problem of the harmonic oscillator is

$$W_1(q, Q) = Cq^2 \cot Q.$$

A slight modification of this is

$$W_4(p_i, P_i) = \frac{P_1^2}{2} \cot p_1 + \frac{P_2^2}{2} \cot p_2 . \tag{105}$$

In celestial mechanics it is this generating function which offers a possibility of introducing Poincaré's variables. According to Table I of Section 6.3 the transformation equations are

$$q_i = -\partial W_4/\partial p_i \quad \text{and} \quad Q_i = \partial W_4/\partial P_i ,$$

which give (without summation)

$$q_i = \frac{P_i^2}{2} \frac{1}{\sin^2 p_i} \quad \text{and} \quad Q_i = P_i \cot p_i .$$

Equivalently, we have

$$Q_i = (2q_i)^{1/2} \cos p_i \quad \text{and} \quad P_i = (2q_i)^{1/2} \sin p_i .$$

Applying these results to the variables given in Case (C), Section 7.7.4, and considering as the old variables the set

$$q_1 = L - G, \quad q_2 = G, \quad p_1 = l, \quad p_2 = l + g,$$

we obtain

$$\begin{aligned} Q_1 &= [2(L - G)]^{1/2} \cos l, & Q_2 &= (2G)^{1/2} \cos(l + g), \\ P_1 &= [2(L - G)]^{1/2} \sin l, & P_2 &= (2G)^{1/2} \sin(l + g). \end{aligned} \tag{106}$$

Equations (106) define one combination of Poincaré's variables.

(B) Inasmuch as the generating function W_4 cannot be used to produce the identity transformation, it is often necessary to introduce Poincaré's variables by means of another generating function, such as

$$W_3(Q, p) = \frac{Q_1^2}{2} \tan p_1 + \frac{Q_2^2}{2} \tan p_2 .$$

Note that this generating function may be combined with an identity transformation and the result may be applied to the set of variables given in Section 7.7.4, Part (B). The set of variables introduced this way belongs partly to Poincaré's and partly to Delaunay's sets. In fact let

$$q_1 = L, \quad q_2 = L - G, \quad p_1 = l + g, \quad p_2 = -g,$$

and

$$W_3(Q, p) = Q_1 p_1 + \frac{Q_2^2}{2} \tan p_2 .$$

The first part of the function W_3 gives

$$q_1 = \partial W_3/\partial p_1 = Q_1 \quad \text{and} \quad P_1 = \partial W_2/\partial Q_1 = p_1 ,$$

while the second part results in

$$q_2 = \frac{Q_2^2}{2 \cos^2 p_2} \quad \text{and} \quad P_2 = Q_2 \tan p_2 ,$$

or

$$Q_2 = (2q_2)^{1/2} \cos p_2 \quad \text{and} \quad P_2 = (2q_2)^{1/2} \sin p_2 .$$

The new set of variables therefore becomes

$$\begin{aligned} Q_1 &= L, & Q_2 &= [2(L - G)]^{1/2} \cos g, \\ P_1 &= l + g, & P_2 &= -[2(L - G)]^{1/2} \sin g. \end{aligned}$$

Note that the minus signs were omitted in the formula for W_3 *and* in the equations of the transformation.

7.8 Regularization with canonical variables

7.8.1 The problem of two bodies

The regularization of the restricted problem of three bodies (the elimination of one or both singularities of the equations of motion) consists of two steps: a transformation of the coordinates *plus* the associated transformation of the time. These two steps will now be performed using a canonical transformation in the phase space for the coordinates and another transformation in the extended phase space to achieve the required transformation of the time.

This program will be executed, as an introductory exercise, for the problem of the Newtonian central force field (for the problem of two bodies) first. Using a fixed Cartesian rectangular coordinate system $(x, y) = (q_1, q_2)$, the problem of two bodies with properly selected units possesses the Hamiltonian

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) - \frac{1}{(q_1^2 + q_2^2)^{1/2}} , \tag{107}$$

where the conjugate momenta are

$$p_i = \dot{q}_i , \quad i = 1, 2. \tag{108}$$

The canonical equations of motion give, in addition to Eqs. (108),

$$\dot{p}_1 = -\frac{q_1}{(q_1^2 + q_2^2)^{3/2}}, \quad \dot{p}_2 = -\frac{q_2}{(q_1^2 + q_2^2)^{3/2}}. \tag{109}$$

The coordinate transformation mentioned in Section 6.5 [Eq. (33)] with a change of sign becomes

$$W_3(p_i, Q_i) = p_1 f(Q_1, Q_2) + p_2 g(Q_1, Q_2), \tag{110}$$

where f and g are, for the time being, arbitrary, excepting that they are conjugate harmonic functions and therefore satisfy the Cauchy–Riemann relations

$$\frac{\partial f}{\partial Q_1} = \frac{\partial g}{\partial Q_2} \quad \text{and} \quad \frac{\partial f}{\partial Q_2} = -\frac{\partial g}{\partial Q_1}. \tag{111}$$

Since W_3, as given by Eq. (110), does not contain the time explicitly, we may use

$$q_i = \partial W_3/\partial p_i \quad \text{and} \quad P_i = \partial W_3/\partial Q_i,$$

ignoring the negative signs shown in Table I of Section 6.3.

The transformation equations become

$$q_1 = f(Q_1, Q_2), \qquad q_2 = g(Q_1, Q_2), \tag{112}$$

$$\begin{aligned} P_1 &= p_1 \frac{\partial f}{\partial Q_1} + p_2 \frac{\partial g}{\partial Q_1}, \\ P_2 &= p_1 \frac{\partial f}{\partial Q_2} + p_2 \frac{\partial g}{\partial Q_2}. \end{aligned} \tag{113}$$

Introducing the notations

$$\partial f/\partial Q_1 = a_{11}, \qquad \partial g/\partial Q_1 = a_{12}$$

and utilizing Eqs. (111), Eqs. (113) become

$$\begin{aligned} P_1 &= a_{11}p_1 + a_{12}p_2, \\ P_2 &= -a_{12}p_1 + a_{11}p_2. \end{aligned}$$

This may be written as

$$\mathbf{P} = \mathbf{A}\mathbf{p}, \tag{114}$$

where the nonsingular matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ -a_{12} & a_{11} \end{pmatrix}$$

with

$$\mathbf{A}^{-1} = \mathbf{A}^*/D,$$

where

$$D(Q_1, Q_2) = \det \mathbf{A} = (\partial f/\partial Q_1)^2 + (\partial g/\partial Q_1)^2. \tag{115}$$

The inversion of Eq. (114) gives

$$\mathbf{p} = \mathbf{A}^*\mathbf{P}/D,$$

and therefore

$$\mathbf{p}^2 = \mathbf{P}^2/D.$$

The new Hamiltonian becomes

$$\bar{H} = \frac{1}{2}\frac{P_1^2 + P_2^2}{D} - \frac{1}{(f^2 + g^2)^{1/2}} \tag{116}$$

and the equations of motion are

$$\dot{Q}_1 = P_1/D, \qquad \dot{Q}_2 = P_2/D, \tag{117}$$

$$\begin{aligned} \dot{P}_1 &= \frac{1}{2}\frac{(P_1^2 + P_2^2)}{D^2}\frac{\partial D}{\partial Q_1} - \frac{1}{2}\frac{1}{(f^2 + g^2)^{3/2}}\frac{\partial}{\partial Q_1}(f^2 + g^2), \\ \dot{P}_2 &= \frac{1}{2}\frac{(P_1^2 + P_2^2)}{D^2}\frac{\partial D}{\partial Q_2} - \frac{1}{2}\frac{1}{(f^2 + g^2)^{3/2}}\frac{\partial}{\partial Q_2}(f^2 + g^2). \end{aligned} \tag{118}$$

The same results can be obtained by means of the Hamiltonian in the extended phase space,

$$\Gamma = P_3 + \frac{P_1^2 + P_2^2}{2D} - \frac{1}{(f^2 + g^2)^{1/2}}, \tag{119}$$

with the associated equations of motion (see Eqs. (25) of Section 6.4),

$$\frac{dQ_i}{dw} = \frac{\partial \Gamma}{\partial P_i} \quad \text{and} \quad \frac{dP_i}{dw} = -\frac{\partial \Gamma}{\partial Q_i}. \tag{120}$$

Substitution of the function Γ given by Eq. (119) into the first set of Eqs. (120) gives

$$\frac{dQ_1}{dw} = \frac{P_1}{D}, \qquad \frac{dQ_2}{dw} = \frac{P_2}{D}, \qquad \frac{dQ_3}{dw} = 1. \tag{121}$$

Since $Q_3 = t$, we have $w = t + \text{const}$ and

$$\frac{dQ_i}{dt} = \frac{P_i}{D} \qquad (i = 1, 2).$$

These equations are identical with Eqs. (117). The second set of Eqs. (120) leads to a similar conclusion, except the case $i = 3$, for which

$$\frac{dP_3}{dw} = -\frac{\partial \Gamma}{\partial Q_3}$$

or $\dot{P}_3 = 0$. This is expected since $P_3 = -\bar{H} = \text{const}$.

To perform a change of the time variable, the method outlined in Section 6.6 is followed. Let the transformation be given by

$$dt = D(Q_1, Q_2)\, d\tau, \tag{122}$$

where D is given by Eq. (115) and τ is the new time variable. Note that the transformed time T of Eq. (61) of Section 6.6 now becomes the regularized time (τ) of Chapter 3.

The new Hamiltonian function in the extended phase space becomes $\Gamma^* = D\Gamma$ or

$$\Gamma^* = DP_3 + \tfrac{1}{2}(P_1^2 + P_2^2) - \frac{D}{(f^2+g^2)^{1/2}}, \tag{123}$$

which, using Eqs. (62) of Section 6.6 gives the following equations of motion:

$$dQ_i/d\tau = P_i \tag{124}$$

and

$$\frac{dP_i}{d\tau} = -\frac{\partial}{\partial Q_i}\left[D\left(P_3 - \frac{1}{(f^2+g^2)^{1/2}}\right)\right], \tag{125}$$

with $i = 1, 2$ in both cases.

The equations of motion for $i = 3$ are

$$\frac{\partial \Gamma^*}{\partial P_3} = \frac{dQ_3}{d\tau} \quad \text{and} \quad \frac{\partial \Gamma^*}{\partial Q_3} = -\frac{dP_3}{d\tau}$$

or

$$D = dt/d\tau \quad \text{and} \quad P_3 = \text{const},$$

which results are in complete agreement with Eq. (122) and with the fact that $P_3 = -\bar{H} = \text{const}$.

Inasmuch as the singularity of the problem is associated with the term

$$\frac{D}{(f^2+g^2)^{1/2}} = \frac{D}{r}$$

in Eq. (125), the guide for selecting the functions f and g is that this expression be regular in the region of interest. Note that if $f + ig = \Phi$ with $i = (-1)^{1/2}$ is introduced, the above expression becomes

$$\frac{D}{(f^2+g^2)^{1/2}} = \frac{|\Phi'|^2}{|\Phi|},$$

where

$$\Phi' = \frac{\partial f}{\partial Q_1} + i\frac{\partial g}{\partial Q_1}.$$

Levi-Civita's selection is

$$\Phi = (Q_1 + iQ_2)^2,$$

which corresponds to Eq. (64) of Section 3.3.

This gives $D = 4(Q_1^2 + Q_2^2)$, $f^2 + g^2 = (Q_1^2 + Q_2^2)^2$, and so

$$\frac{D}{(f^2+g^2)^{1/2}} = 4.$$

The equations of motion follow from Eqs. (124) and (125):

$$\frac{dQ_i}{d\tau} = P_i, \qquad \frac{dP_i}{d\tau} = -P_3\frac{\partial D}{\partial Q_i}, \tag{126}$$

for $i = 1, 2$.

Substituting $-\bar{H}$ for P_3, evaluating the partial derivates, and denoting derivatives with respect to the new time variable by primes, we obtain

$$Q_1' = P_1, \qquad Q_2' = P_2, \qquad P_1' = 8\bar{H}Q_1, \qquad P_2' = 8\bar{H}Q_2. \tag{127}$$

The second-order differential equations for Q_1 and Q_2 become

$$Q_i'' = 8\bar{H}Q_i \qquad (i = 1, 2), \tag{128}$$

and we see that trigonometric or hyperbolic functions will furnish the solutions depending on the sign of $\bar{H}$. Elliptic (hyperbolic) motions correspond to $\bar{H} < 0$ ($\bar{H} > 0$) and the corresponding solutions are represented by trigonometric (hyperbolic or exponential) functions. The case $\bar{H} = 0$ gives linear relations for the functions $Q_i(\tau)$. Equations (128) are identical with Eq. (68) of Section 3.3 since $\Phi = w$, $Q_1 = u$, $Q_2 = v$, and $\bar{H} = -C/2$ for $\mu = \mu_2 = 0$, according to Eq. (25) of Section 7.3. The above comments about the elliptic and hyperbolic solutions are expressed by Eqs. (87), (88), and (89) of Section 3.3.

After the solution of Eqs. (128) is obtained in the form $Q_i = Q_i(\tau)$, it is to be substituted in the equation

$$dt = 4(Q_1^2 + Q_2^2)\, d\tau$$

to obtain the relation between t and τ. Once this is done, the functions $Q_1(t)$ and $Q_2(t)$ are determined and the final solution $q_1(t)$, $q_2(t)$ is obtained from the relations $q_1 = Q_1^2 - Q_2^2$, $q_2 = 2Q_1Q_2$.

7.8.2 Regularization of the restricted problem

The main purpose of this section is to show how regularization is performed within the framework of canonical variables utilizing the concept of the extended phase space. The discussion is conducted on a generalized level so that the reader may use this section as his starting point for the solution of new problems. The treatment begins with the Hamiltonian in the rotating rectangular Cartesian coordinate system, Eq. (9), but instead of capital, lower-case letters will be used preserving capital letters for the transformed coordinates. Accordingly

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + q_2 p_1 - q_1 p_2 - F(q_1, q_2), \tag{129}$$

where q_1 and q_2 are uniformly rotating Cartesian rectangular coordinates, p_1 and p_2 are the conjugate momenta, and

$$F(q_1, q_2) = \mu_1/r_1 + \mu_2/r_2,$$

with

$$r_1 = [(q_1 - \mu_2)^2 + q_2^2]^{1/2} \quad \text{and} \quad r_2 = [(q_1 + \mu_1)^2 + q_2^2]^{1/2} \tag{130}$$

[cf. Eqs. (16)].

First the coordinate transformation in the phase space is performed, then the transformation in the extended phase space is taken up.

The generating function to be used is the same as given by Eq. (110),

$$W_3 = p_1 f(Q_1, Q_2) + p_2 g(Q_1, Q_2),$$

and the equations of the transformation are again given by

$$q_i = \partial W_3/\partial p_i \quad \text{and} \quad P_i = \partial W_3/\partial Q_i \quad (i = 1, 2).$$

Consequently Eqs. (111)–(115) do not change and

$$\mathbf{p}^2 = \mathbf{P}^2/D$$

as before.

The new Hamiltonian requires the computation of the quantity $q_2 p_1 - p_2 q_1$, which was not present in the problem of two bodies. By substitution we obtain

$$q_2 p_1 - p_2 q_1 = \frac{1}{2D}\left[P_1 \frac{\partial}{\partial Q_2}(f^2 + g^2) - P_2 \frac{\partial}{\partial Q_1}(f^2 + g^2)\right].$$

Therefore, the new Hamiltonian using Eq. (129) becomes

$$\tilde{H} = \frac{1}{2D}\left[P_1^2 + P_2^2 + P_1 \frac{\partial}{\partial Q_2}(g^2 + f^2) - P_2 \frac{\partial}{\partial Q_1}(g^2 + f^2)\right] - \tilde{F}(Q_1, Q_2), \tag{131}$$

where in computing $\tilde{F}$ we write $f(Q_1, Q_2)$ in place of q_1 and $g(Q_1, Q_2)$ in place of q_2 in Eqs. (130).

The equations of motion are

$$\dot{Q}_1 = \frac{1}{2D}\left[2P_1 + \frac{\partial}{\partial Q_2}(f^2 + g^2)\right],$$
$$\dot{Q}_2 = \frac{1}{2D}\left[2P_2 - \frac{\partial}{\partial Q_1}(f^2 + g^2)\right], \tag{132}$$

$$\dot{P}_1 = -\partial \tilde{H}/\partial Q_1, \qquad \dot{P}_2 = -\partial \tilde{H}/\partial Q_2. \tag{133}$$

In the extended phase space the Hamiltonian function is

$$\Gamma = P_3 + \frac{1}{2D}\left[P_1^2 + P_2^2 + P_1 \frac{\partial}{\partial Q_2}(f^2 + g^2) - P_2 \frac{\partial}{\partial Q_1}(f^2 + g^2)\right] - \tilde{F}(Q_1, Q_2) \tag{134}$$

and the equations of motion are reducible essentially to the same form as Eqs. (132) and (133).

The new Hamiltonian function in the extended phase space is $\Gamma^* = D\Gamma$ or

$$\Gamma^* = DP_3 + \frac{1}{2}\left[P_1^2 + P_2^2 + P_1 \frac{\partial}{\partial Q_2}(f^2 + g^2) - P_2 \frac{\partial}{\partial Q_1}(f^2 + g^2)\right] - D\tilde{F}, \tag{135}$$

and with this the equations of motion using the new time variable (τ) can be established. The previously used relation between dt and $d\tau$ still applies, $dt = D\, d\tau$. Denoting derivatives with respect to $d\tau$ by primes, we have

$$Q_1' = P_1 + \frac{1}{2}\frac{\partial}{\partial Q_2}|\Phi|^2,$$
$$Q_2' = P_2 - \frac{1}{2}\frac{\partial}{\partial Q_1}|\Phi|^2, \tag{136}$$
$$Q_3' = D.$$

The last equation is $dt = D\, d\tau$, as expected, and

$$\Phi(Q_1, Q_2) = f(Q_1, Q_2) + i g(Q_1, Q_2),$$

as before.

The second set of the Hamiltonian equations of motion is

$$P_1' = -P_3 \frac{\partial D}{\partial Q_1} - \frac{1}{2}\left[P_1 \frac{\partial^2 |\Phi|^2}{\partial Q_1 \partial Q_2} - P_2 \frac{\partial^2 |\Phi|^2}{\partial Q_1^2}\right] + \frac{\partial}{\partial Q_1}(D\tilde{F}),$$
$$P_2' = -P_3 \frac{\partial D}{\partial Q_2} - \frac{1}{2}\left[P_1 \frac{\partial^2 |\Phi|^2}{\partial Q_2^2} - P_2 \frac{\partial^2 |\Phi|^2}{\partial Q_2 \partial Q_1}\right] + \frac{\partial}{\partial Q_2}(D\tilde{F}), \tag{137}$$
$$P_3' = 0.$$

The last equation expresses the fact that $\tilde{\Pi}$ is constant. The singularities appear in the last term of the first two equations of the system (137), similarly to the term $D(f^2 + g^2)^{-1/2}$ occurring in the problem of two bodies.

Equations (136) and (137) represent a sixth-order system. Omitting the last trivial equation in both groups, it can be shown that the remaining fourth-order system of equations is equivalent to Eq. (92′) of Section 3.4. This requires the computation of Q_1'' and Q_2'' from Eqs. (136), the substitution into the resulting equations the values of P_1' and P_2' as given by Eqs. (137), and the substitution of P_1 and P_2 from Eqs. (136). The two second-order equations obtained this way are

$$\begin{aligned} Q_1'' - 2DQ_2' &= \frac{\partial}{\partial Q_1} D(\tfrac{1}{2} \mid \Phi \mid^2 - P_3 + \tilde{F}), \\ Q_2'' + 2DQ_1' &= \frac{\partial}{\partial Q_2} D(\tfrac{1}{2} \mid \Phi \mid^2 - P_3 + \tilde{F}). \end{aligned} \tag{138}$$

In order to compare these equations with the corresponding single Eq. (92′) in Section 3.4, consider first Eq. (22) in the form

$$\tfrac{1}{2}(f^2 + g^2) + \tilde{F} = \Omega - \tfrac{1}{2}\mu_1\mu_2 ,$$

where instead of Q_1 and Q_2 occurring in Eq. (22) and representing Cartesian rectangular synodic coordinates, $q_1 = f$ and $q_2 = g$ were written. Since $\mid \Phi \mid^2 = f^2 + g^2$, the last equation becomes $\frac{1}{2} \mid \Phi \mid^2 + \tilde{F} = \Omega - \frac{1}{2}\mu_1\mu_2$. Subtracting from this Eq. (25) in the form

$$P_3 = -\tilde{H}_0 = \frac{C - \mu_1\mu_2}{2}$$

gives

$$\tfrac{1}{2} \mid \Phi \mid^2 - P_3 + \tilde{F} = \Omega - C/2. \tag{139}$$

The substitution of Eq. (139) into (138) gives

$$\begin{aligned} Q_1'' - 2DQ_2' &= \frac{\partial}{\partial Q_1} D(\Omega - C/2), \\ Q_2'' + 2DQ_1' &= \frac{\partial}{\partial Q_2} D(\Omega - C/2). \end{aligned} \tag{140}$$

The comparison with Eq. (92′) of Section 3.4 requires that $w = Q_1 + iQ_2$, and $D = \mid \Phi \mid^2 = \mid f'(w) \mid^2$. With these notations Eqs. (140) become

$$w'' + 2i \mid f'(w) \mid^2 w' = \text{grad}_w \mid f' \mid^2 (\Omega - C/2),$$

which is identical with Eq. (92′) of Section 3.4. Q.E.D.

This concludes the discussion of a general method to regularize the equations of motion within the framework of Hamiltonian dynamics.

7.9 Notes

Some of the canonical transformations presented in this Chapter are given by Whittaker [1, pp. 353–355], some by Brouwer and Clemence [2, pp. 539–540], and some by Poincaré [3].

The detailed description in Section 7.3 of the velocities and momenta in the sidereal and synodic coordinate systems was inspired by the many misleading statements and erroneous interpretations of these variables in the literature. Even Whittaker's [1, p. 354] usual clarity is subject to possible misinterpretations.

The derivation of Delaunay's variables given in Section 7.6 follows Poincaré [3] and Whittaker [1] (who refers to Poincaré) and it is a slight modification of the standard introduction of Delaunay's variables as given in several more recent works, such as by Smart [4], Goldstein [5], Brown and Shook [6], etc.

The introduction of F in place of H for the Hamiltonian in Section 7.7.3 avoids a possible misunderstanding since in the three-dimensional case Delaunay's third variable is generally denoted by H. To distinguish between "coordinates" and "momenta" is important only for the correctness of the arithmetic but it has little, if any, physical significance. This point, already mentioned in Sections 6.5, 6.7, and 7.7 is again made clear by Eqs. (102a) and (102b) in Section 7.7.3. (See also Brouwer and Clemence [2, p. 291].)

The three sets of Delaunay's variables derived in Section 7.7.4 are combinations of interest in celestial mechanics and correspond to some examples mentioned in [2, p. 540].

Sections 7.8.1 and 7.8.2 present canonical regularization in a systematic fashion. Several variations of this treatment may be designed; most importantly one may combine the two transformations (for the coordinates and for the time) into one [7].

The quotation attributed to Birkhoff [8] in the introductory section is from one of his papers delivered at a meeting of the American Mathematical Society in 1926.

A reference connecting the general perturbation methods of Chapter 8, the use of Delaunay's variables mentioned in this chapter, and an analytical approach to the "swing-by" trajectories of Chapter 10 is a paper by Hori [9].

7.10 References

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Chapter 8

Periodic Orbits

8.1 Introduction

The most suitable introduction to the subject of periodic orbits is probably a listing of reasons for their importance. Since often the motivation for studying the theoretical aspects of periodic orbits is unknown to the student, in what follows the *raison d'être* of the subject will be first outlined.

The rest of the introduction outlines the contents of Chapter 8.

Poincaré in the first volume of his "*Méthodes Nouvelles*" considers the subject of periodic orbits the only opening through which the reputably inaccessible problem of three bodies can be penetrated. Indeed, Poincaré's famous conjecture emphasizes the importance of periodic orbits. If a particular solution of the restricted problem is given, one can always find a periodic solution (with a period which might be very long) such that the difference between these two solutions is as small as desired for any given length of time. Schwarzschild's version of Poincaré's conjecture is equally interesting when put in the language of phase space. In an arbitrarily close neighborhood of any point in the phase space there is a point representing a periodic orbit. In other words, small modifications of any set of initial conditions will result in a periodic orbit—in general, with very long period. Either in the original form or in its modified version, Poincaré's conjecture suggests the use of periodic orbits as reference orbits. Actually, both Encke and Hill have done just this, the former using an ellipse, the latter the variational orbit as a reference orbit.

For nonintegrable dynamical systems it seems to be impossible to obtain complete information regarding *any* orbit unless it is either asymptotic, periodic, or almost periodic. There is no numerical method available which could handle the problem $t \to \infty$; consequently, if the behavior of a solution is unknown from other than numerical considerations as $t \to \infty$, it cannot be established. The long time behavior on the other hand is considered known if the motion is either periodic, almost periodic, or asymptotic. This comment refers to precise information regarding an orbit and not to statistical (e.g., ergodic) properties.

Another reason for interest in periodic orbits is that in the restricted problem several families of such orbits can be generated from solutions of the two-body problem, and by analytic continuation the existence of these periodic orbits can be shown. Also certain asymptotic and periodic solutions can be obtained from the linearized solutions discussed in connection with motion around the libration points.

Darwin's conjecture is that, in the classification of the totality of orbits, periodic orbits play an important role in that they separate the various classes of orbits. If there is any basis for this conjecture, it is certainly a strong justification for interest in periodic orbits.

It is important that a study of periodic orbits reduce the dimensionality of the problem in the phase space. Consider, for instance, orbits with initial conditions $x = x_0$, $y = 0$, $\dot{x} = 0$, and $\dot{y} = \dot{y}_0$, that is, orbits which intersect the x axis perpendicularly at $x = x_0$ with velocities normal to the x axis. The totality of such initial conditions corresponds to the points of the x_0, $\dot{y}_0$ plane. If simple periodic orbits are considered, a relation $\dot{y}_0 = \dot{y}_0(x_0)$ can be established (with possible branch points), and the totality of initial conditions will correspond now to the points of the $\dot{y}_0(x_0)$ curve.

The above five paragraphs containing five reasons are intended to furnish the motivation for studying periodic orbits within the framework of the restricted problem. The significance of periodic phenomena in general in celestial mechanics and in the physical sciences can be added to the above arguments.

This chapter also outlines the existence proofs of certain types of periodic orbits. Starting with the problem of two bodies a condition for the existence of periodic orbits in the rotating coordinate system is derived. Then the totality of (elliptic) motions around the primary with unit mass is established inside the zero velocity curve. The states of motion are represented by points inside and on a hollow cylinder. The descriptions of the motion on a torus, following Kolmogorov, and of a transformation of a ring into itself, following Birkhoff, are offered.

The ring transformation is continued from the two-body case to the

case $\mu \neq 0$. This furnishes the existence proof of those periodic orbits which Poincaré called "*première sorte*." These orbits are obtained by analytic continuation from the circular two-body orbits.

The representation of a dynamical problem as a transformation of a surface into itself was originated by Poincaré. Assume that a two-dimensional surface can be constructed in the state of motion manifold with the property that the streamlines mentioned in Chapter 2 cut the surface at least once within a fixed interval of time. Poincaré and Birkhoff study these intersections of the streamlines with the "surface of section" and find that, as the same streamline pierces the surface at various points with increasing time, the surface is transformed into itself. The number of degrees of freedom in the restricted problem is $n = 2$; the dimensionality of the state of motion manifold (or the order of the system) is $2n = 4$, which is reducible by the Jacobian integral to $2n - 1 = 3$. This three-dimensional state of motion manifold (existing for any fixed value of the Jacobian constant) possesses $2n - 2 = 2$-dimensional surfaces, some of which qualify as the above-mentioned surfaces of section. The fact that the restricted problem has two degrees of freedom is essential in these investigations. The three-dimensional variation of the restricted problem is associated with $n = 3$ and, since there is still only one integral, the surface of section becomes four dimensional.

Once the dynamical system is associated with the transformation (T) of a surface into itself, the properties of the dynamical system are represented by T. The periodicity of the dynamical system simply becomes the invariance of certain points of the surface under the transformation T.

The existence of solutions of the *deuxième sorte* is presented using Delaunay's variables. These solutions are generated by the elliptic solutions of the corresponding two-body problem. Periodic orbits of the *third kind* require the introduction of a third coordinate (in addition to x and y) and therefore will be discussed in Chapter 10.

The short sections (8.7) and (8.8) on Whittaker's condition for periodic orbits and on Poincaré's characteristic exponents conclude this chapter. The relation to Chapter 5 becomes strong and the reader's review of the problems of stability of linear and of nonlinear systems according to Liapunov as discussed in Section 5.2 and its subsections may be useful.

Inasmuch as the results of numerical orbit computations are described in Chapter 9, those periodic orbits which have been established by numerical methods are treated there. It is often difficult to decide whether the analytical or the numerical aspects dominate regarding certain questions related to periodic orbits. The reader will find the actual orbits discussed in Chapter 9, while the present chapter is reserved for more general remarks of analytical nature.

A treatment of periodic orbits which I would consider adequate is far beyond the scope of this work. The results of Poincaré, of Birkhoff, of the Russian school, of recent topological investigations, and of efforts to give the classical perturbation approaches of celestial mechanics precise analytic meaning, must all be discussed in considerable detail before their application to the restricted problem is treated. The idea of omitting Chapter 8 was tempting but I kept it to serve as an outline of another volume devoted to periodic orbits.

8.2 Definitions

(A) We speak of a *periodic motion* of a dynamical system when the same configuration is repeated at regular intervals of time. Included are motions which are repeated in a relative sense. In a fixed system of coordinates, for instance, the two-body problem will have a solution which is repeated after a period,

$$T_{\text{sid}} = 2\pi/n,$$

where n is the mean motion and *sid* refers to a *sidereal* system. Introduce now a rotating coordinate system with $T_{\text{sys}} = 2\pi$ as its rotational period and let *sys* refer to the rotating coordinate *system*. Periodic motion in this rotating system exists if $n = p/q$, where p and q are integers, or in other words when n is a rational number. The synodic path (the orbit in the rotating system) closes (starts repeating itself) after the lapse of $|p|$ sidereal periods ($|p| T_{\text{sid}} = 2\pi p/n = 2\pi q$) so

$$T_{\text{syn}} = 2\pi q,$$

where *syn* refers to a *synodic* coordinate system.

The two periods are equal when $2\pi q = 2\pi/n$, or when the mean motion (in the fixed system) is the reciprocal of an integer ($n = 1/q$).

The above remarks apply to elliptic orbits but not to circular motions which are always periodic in both systems. Again let n be the mean motion and $T^c_{\text{sid}} = 2\pi/n$, where the superscript c refers to circular motion. For a direct circular orbit the mean motion of the particle must be decreased by one in order to obtain its mean motion relative to the rotating system and so

$$T^c_{\text{syn}} = \frac{2\pi}{n-1},$$

where n can be an irrational or rational number. More will be said about these equations later; at this point only a simple example is mentioned which shows that periodicity is not necessarily an absolute (physical) property of the dynamical system. Elliptic motions in the two-body problem are always periodic in the fixed system but this is not necessarily the case in the rotating frame of reference. Circular motions on the other hand are always periodic in both systems. This difference is basically responsible for the important distinction which exists between Poincaré's periodic orbits of the first and second kind.

(B) Let the system of differential equations

$$\frac{dx_i}{dt} = X_i(x_1, \ldots, x_n) \qquad (i = 1, \ldots, n)$$

possess a particular solution, $x_i = \varphi_i(t)$. If the functions φ_i have the property that

$$\varphi_i(0) = \varphi_i(T),$$

in other words, when the initial value of φ_i is the same as its value at an epoch T and consequently $\varphi_i(t) = \varphi_i(t + T)$, then the functions φ_i are periodic in t with period T, and $x_i = \varphi_i(t)$ is a *periodic solution* of the differential equations.

(C) A continuous function $x(t)$ is *almost periodic* (Bohr) if, given $\epsilon > 0$, there exists an $E(\epsilon)$ such that for every real number α there is a T (translation number) which satisfies $\alpha \leqslant T \leqslant \alpha + E(\epsilon)$ and for which

$$|x(t + T) - x(t)| \leqslant \epsilon \qquad \text{for all} \quad t.$$

When $x(t)$ is a periodic function, $\epsilon = 0$ and T becomes the period. An almost periodic function, therefore, is "periodic" with an "error" ϵ.

Note that every almost periodic function $x(t)$ may be approximated uniformly by a trigonometric polynomial, or

$$\left| x(t) - \sum_n a_n e^{i\lambda_n t} \right| < \epsilon^*$$

for a given ϵ^* with a finite number of terms.

A special class of almost periodic functions is called *quasi-periodic* (Bohl). These possess only a finite number of basic frequencies (ω_i) and may be represented by

$$x(t) = X(\omega_1 t, \ldots, \omega_n t),$$

where X is a continuous periodic function of the arguments, $\omega_i t$.

(D) The problem of *small divisors* in celestial mechanics is known as the resonance phenomenon of general dynamics. The divisors in question are defined by the expression

$$\omega_i k_i = \sum_{i=1}^{n} \omega_i k_i ,$$

where the quantities ω_i are the mean motions, or natural frequencies, or eigenvalues of the dynamical system and the symbols k_i represent integers. The frequencies are not commensurable when $\omega_i k_i = 0$ if and only if $k_i = 0$ ($i = 1, \dots, n$). In other words, when no set of integers (different from zero) can be found so that the above sum becomes zero, we speak of incommensurable frequencies. This can be illustrated, for instance, by $\omega_1 = 1$, $\omega_2 = 2^{1/2}$, i.e., by an irrational value ω_2/ω_1. For large absolute values of k_1 and k_2 the quantity $k_1 + 2^{1/2}k_2$ can become arbitrarily small, but for small integers ($|k_1|$ and $|k_2|$) the sum $\omega_i k_i$ is not near zero.

If the ratios of the frequencies are rational, the sum $\omega_i k_i$ can become zero without $k_i = 0$ and we speak of commensurability. This sum appears in the expansion of the disturbing function in celestial mechanics:

$$\sum_{k_1, \dots, k_n} a_{k_1, \dots, k_n} e^{i(k_j \omega_j)},$$

which expression when integrated will introduce the denominators mentioned before.

An example is the "great inequality" related to the motion of Jupiter and Saturn. The orbital periods are $T_{♃} = 4332^{\mathrm{d}}58$ and $T_{♄} = 10{,}759^{\mathrm{d}}20$; consequently, $5T_{♃} - 2T_{♄} = 144^{\mathrm{d}}5$. The same near-commensurability can also be expressed in terms of frequencies or mean motions, since in a day Jupiter moves $\varphi_{♃} = 299''12893$ and Saturn $\varphi_{♄} = 120''45505$; consequently, $2\varphi_{♃} - 5\varphi_{♄} = 4''01739$. The dimensionless divisor has the same value, of course, whether it is computed in terms of mean motions or of periods:

$$\frac{5T_{♃} - 2T_{♄}}{(T_{♃} + T_{♄})/2} = 0.02.$$

The point of conjunction requires 918 years to complete a revolution and with this long period large perturbations are associated because of the appearance of a small divisor.

(E) The *torus concept* can be introduced by the mth-order system of differential equations

$$\dot{x}_i = f_i(x_1, \dots, x_m)$$

provided the following periodicity condition is satisfied.

Let α_i be constants and

$$f_i(x_1, \dots, x_j, \dots, x_m) = f_i(x_1 + \alpha_1, \dots, x_j + \alpha_j, \dots, x_m + \alpha_m)$$

for $i = 1, \dots, m$. If the x_i are angular variables, the α_i become their respective "periods" and the x_i have to be reduced mod α_i. The functions $x_i(t)$ representing the solution can be shown to lie in this case on a torus.

A simple example of importance in celestial mechanics is $\dot{x}_i = \beta_i =$ const, in which case $x_i = \beta_i t + x_i^0$. In a synodic coordinate system the argument of the perihelion (say x_1) rotates with constant angular velocity, $\beta_1 = 1$, and the mean anomaly (say x_2) is a linear function of the time ($\beta_2 = n$), both variables referring to a two-body elliptic orbit.

For rational values of the ratio β_1/β_2 the motion is periodic in the rotating system and the orbit on the torus closes upon itself. For irrational values, the curve, after a sufficiently large number of revolutions, approaches its initial point arbitrarily closely and we have a quasi-periodic motion.

While the trajectory of a periodic orbit on the torus does not cover the whole surface of the torus, the almost periodic solution will be everywhere dense, in fact it will be uniformly distributed. By this we mean that the time τ which the point describing the trajectory spends in a given region is proportional to the size of this region for large τ.

(F) A system is *recurrent* when a set of initial conditions repeats itself within an arbitrary small "error" of ϵ, infinitely often. Precise recurrence is associated with $\epsilon = 0$. If the phase space is bounded for a dynamical system which consists of material particles under the influence of forces depending only on the position coordinates, then Poincaré's cycle theorem holds, i.e., the system is recurrent.

If the angular variables have the frequencies ω_i, the condition for precise recurrence is the same as for commensurability, $\sum_{i=1}^{N} k_i \omega_i = 0$, $k_i \neq 0$. The recurrence time T^* in this case is

$$T^* = n_i 2\pi/\omega_i ,$$

where the n_i's are positive integers and

$$\omega_1 : \omega_2 : \dots : \omega_N = n_1 : n_2 : \dots : n_N .$$

When the recurrence is within an error of ϵ, we have

$$|(T^* \omega_i/2\pi) - n_i| < \epsilon,$$

which inequality is to be satisfied simultaneously for all i. For an example see Section 5.6.1 and the analysis of Figs. 5.14 and 5.15 of Chapter 5.

(G) *Asymptotic* orbits are defined by their property of approaching a given orbit more and more closely as $t \to \infty$ or $t \to -\infty$. (An example of the second kind is an unstable periodic orbit which when disturbed will deviate from its original shape and will approach it in "reverse".) The orbit which the asymptotic orbit approaches may be an equilibrium solution, i.e., a point into which the orbit spirals as $t \to \infty$. Examples are the asymptotic-periodic solutions discussed in Chapter 9 and the solutions associated with unstable Lagrangian points mentioned in Chapter 5.

(H) A dynamical system is *quasi-ergodic* if the orbit comes infinitely close to every point belonging to its energy surface. The orbit leaves its starting point in the phase space and its initial conditions determine a constant energy surface which goes through this starting point. The orbit will stay on this energy surface and if the system is quasi-ergodic it will cover the surface densely. The quasi-ergodic property is possessed by every dynamical system which satisfies the following ("normal") conditions (Hopf):

(a) the Hamiltonian H does not depend explicitly on time;
(b) $H = \sum \mu^i H_i(x, y)$;
(c) $H_i (i \neq 0)$ is periodic in x;
(d) H_0 is independent of x.

The concept is, of course, closely associated with Poincaré's integral invariants mentioned in Chapter 2 and with the end of Part (E) of this section.

8.3 Surface transformations and representation on a torus

In this section Moser's, Poincaré's, and Brouwer's mapping theorems and Kolmogorov's perturbation theorem will be described.

(A) *Moser's theorem* on the invariant curves of area-preserving mappings of an annulus may be arrived at as follows.

Consider polar coordinates r and φ with $0 < a_1 \leqslant r \leqslant a_2$, $0 \leqslant \varphi \leqslant 2\pi$, i.e., in a circular annulus (ring) of width $a_2 - a_1$. The simplest pure rotational mapping (corresponding to a rigid body rotation) is described by

$$r_1 = r, \qquad \varphi_1 = \varphi + f_0, \tag{1}$$

with $f_0 = \text{const}$. Every point is rotated by the same constant angle

f_0. A more interesting case is the "twist mapping" given by the transformation

$$r_1 = r, \qquad \varphi_1 = \varphi + f(r), \tag{2}$$

when the angle of rotation depends on the radial coordinate through the monotone function $f(r)$. Along any circle $r = r_0 = \text{const}$, the angle of rotation is the same, $\varphi_1 - \varphi = f(r_0) = \text{const}$. The mapping preserves circles and rotates them by an angle which increases or decreases with increasing radius as df/dr is positive or negative. The Jacobian of both mappings [Eqs. (1) and (2)] is 1, therefore the transformations are area preserving.

A modification of the twist mapping is

$$r_1 = r + g(r, \varphi), \qquad \varphi_1 = \varphi + f(r) + h(r, \varphi). \tag{3}$$

This transformation may be considered to be a perturbation of the previous mapping when the functions $|g|$ and $|h|$ are small. This will receive precise meaning in what follows. If in addition the functions g and h are periodic in φ with period 2π, we are considering a mapping of considerable importance in celestial mechanics. In Eqs. (3) the symbol r represents the semimajor axis and φ the argument of the perihelion of the unperturbed elliptic motion. The previously discussed twist mapping (2) has invariant curves which are circles and under certain conditions one should expect to find invariant curves in the above mapping (3) also. These curves should be close to the circles of the twist mapping since g and h are small perturbations.

Moser's theorem lists the condition under which the perturbed twist mapping has invariant curves and establishes a method of their construction. The analyticity of the functions f, g, and h is not required (in comparison with and as opposed to Kolmogorov's theorem); only the existence of a finite number of derivatives is assumed.

It is essential to assume that every closed curve (which is near a circle) and its image curve intersect in the mapping described by (3). Let, for some constant $c > 1$,

$$c^{-1} \leqslant df(r)/dr \leqslant c$$

and, for $\delta > 0$,

$$g| + |h| < \delta.$$

Furthermore we require that g and h have continuous derivatives up to order l and

$$|f|_l + |g|_l + |h|_l < c,$$

where the notation is explained as follows;

$$|F(r,\varphi)|_z = \sup |(\partial/\partial r)^\alpha (\partial/\partial\varphi)^\beta F(r,\varphi)|,$$

with $\alpha + \beta \leqslant z$.

The invariant curve is given by

$$\bar\varphi = \varphi + p(\varphi),$$
$$\bar r = r_0 + q(\varphi),$$

where the periodic functions p and q have z continuous derivatives and

$$|p|_z + |q|_z < \epsilon.$$

Note that $l = l(z)$ and $\delta = \delta(\epsilon, z, c)$; furthermore, the transformation of the invariant curve gives

$$\varphi_1 = \varphi + f(r_0),$$

where $f(r_0) = \omega$ is the rotation number and

$$f(a) + \epsilon < f(r_0) < f(b) - \epsilon,$$

with $b - a \geqslant 1$.

Finally it is important that $f(r_0)/2\pi$ cannot be closely approximated by rationals; i.e., for all integers n, m, and $n > 0$,

$$|n\omega - 2\pi m| \geqslant \epsilon n^{-3/2}.$$

For the proof and additional explanations of this theorem we refer the reader to the references.

(B) *Poincaré's theorem* of interest at this point is known as his last geometric theorem. A transformation (possessing an invariant integral) of a ring leaves two points invariant if the coefficients of rotation at the boundaries are of opposite signs. Consider a mapping of a ring into itself and let the ring be formed by two concentric circles. The transformation moves the points of the outside circle in a positive sense and those of the inside circle in a negative sense. The mapping has an invariant integral; for instance, it preserves areas. Then the theorem says that there are at least two fixed points of this transformation. The original theorem refers to circular rings and to the necessity of having an integral invariant. A specialization to area preservation from any integral invariant and a generalization to simply closed curves, one within the other, from circular boundaries is allowed.

Another fixed point theorem is *Brouwer's*, which may be stated as follows:

If $R: a_i \leqslant x_i \leqslant b_i$, $i = 1, 2, \dots, n$, is a closed interval of the Euclidean space and if $T: y = f(x)$, $x \in R$, is a continuous mapping of R into itself [i.e., if $f_i = f_i(x_1, \dots, x_n)$ and $a_i \leqslant f_i \leqslant b_i$ for all $x \in R$], then there is at least one point $x_0 \in R$ (fixed point) such that $Tx_0 = x_0$, i.e., $f(x_0) = x_0$.

(C) Introduction of Kolmogorov's theorem is accomplished through a Hamiltonian function which depends only on the generalized momenta, viz., $H = H(p_1, p_2, \dots, p_n)$. Then, the canonical equations of motion become

$$\dot p_i = -\partial H/\partial q_i = 0, \qquad p_i = \text{const} \tag{4}$$

and

$$\dot q_i = \partial H/\partial p_i = \omega_i(p_1, \dots, p_n) = \text{const} \qquad (i = 1, 2, \dots, n). \tag{5}$$

If the generalized coordinates q_i are angular variables mod 2π, the ω_i notation, signifying angular velocities, is used. The motion can be represented on an n-dimensional torus. Commensurability corresponds to satisfying the equation $\sum_{i=1}^{n} \omega_i k_i = 0$ with integer k_i's, at least one of which is not zero.

Consider now a "normal" dynamical system [see also Section 8.2, Part (H)] with the Hamiltonian

$$H(q_i, \dots, q_n, p_1, \dots, p_n) = H_0(p_1, \dots, p_n) + \mu H_1(q_1, \dots, q_n, p_1, \dots, p_n) + \cdots,$$

where μ is a (small) parameter and H_1 is periodic with respect to q_i;

$$H_1(q_1, \dots, q_n, p_1, \dots, p_n) = H_1(q_1 + 2\pi, \dots, q_n + 2\pi, p_1, \dots, p_n).$$

According to Kolmogorov's theorem if

$$\det \frac{\partial^2 H_0}{\partial p_i\, \partial p_j} \neq 0 \tag{6}$$

and if H is analytic, then the majority of the tori associated with the previously described case (i.e., when $H_0 = H$) do not vanish, and are only somewhat deformed as the result of the perturbation.

In fact, and more precisely, if H is analytic and H_0 is not degenerate in a domain G, then for any $\epsilon > 0$ there is an $M(\epsilon, G, H_0) > 0$ such that if $|H - H_0| \leqslant M$ then the motion has the following properties:

(a) there is a decomposition of the domain such that

$$G = G_1 \cup G_2, \qquad \mathrm{m}(G_2) \leqslant \epsilon\, \mathrm{m}(G_1),$$

with G_1 being invariant (i.e., it contains the orbit which started in it) and with m standing for the measure of G_i;

(b) G_1 consists of invariant n-dimensional analytic tori A_ω given by

$$p = p_\omega + f_\omega(Q),$$
$$q = Q + g_\omega(Q),$$

where f_ω and g_ω are analytic functions, periodic in $Q = (Q_1, ..., Q_n)$, and ω is a parameter belonging to the torus A_ω;

(c) the perturbations from the original tori $p = p_\omega$ are small,

$$|f_\omega| < \epsilon, \qquad |g_\omega| < \epsilon,$$

and the invariant tori A_ω differ little from the tori $p = p_\omega$;

(d) the motion on the tori is quasi-periodic with n frequencies $\omega_1, ..., \omega_n$, and is given by

$$\dot{Q} = \omega, \qquad \text{with} \quad \omega = (\partial H_0/\partial p)_{p_\omega},$$

provided

$$\left| \sum_{i=1}^{n} k_i \omega_i \right| \geqslant \frac{c}{(\sum_{i=1}^{n} |k_i|^{\alpha})}$$

for any set of integers $k_1, ..., k_n$, $\alpha > 0$, and $c = \text{const}$. This last condition corresponds to the "nonresonance" condition of Moser.

Kolmogorov's theorem is *not* directly applicable to the degenerate case, which is so important in celestial mechanics. When the motion of the unperturbed system H_0 is described by a smaller number of frequencies than the perturbed system H_1, we have

$$H_0 = H_0(p_1, ..., p_k)$$

and

$$H_1 = H_1(q_1, ..., q_n, p_1, ..., p_n)$$

with $n > k$ and

$$\det \frac{\partial^2 H_0}{\partial p_i \, \partial p_j} = 0.$$

Modifications of Kolmogorov's theorem are available for this so-called degenerate case.

The Hamiltonian of the restricted problem as given by either Eq. (90) of Section 7.6 or by Eq. (96) of Section 7.7.1 is

$$H = -1/2p_1^2 - p_2 + \mu H_1 + \mu^2 H_2 + \cdots$$

and so

$$H_0 = -1/2p_1^2 - p_2,$$

where $p_1 = L$ and $p_2 = G$ are Delaunay's variables.

The Hessian of H_0 is zero since H_0 is a linear function of one of its variables (p_2). Poincaré has shown that another Hamiltonian function can be constructed whose Hessian is different from zero. Introduce for this purpose the new Hamiltonian $\mathscr{H}'$ by

$$\mathscr{H}' = H^2/2h,$$

where $h = H$ is the energy integral. (Note that H does not depend explicitly on time, and h is a constant.)

The equations of motion using the new Hamiltonian become

$$\dot{q}_r = \partial\mathscr{H}'/\partial p_r \quad \text{and} \quad \dot{p}_r = -\partial\mathscr{H}'/\partial q_r \qquad (r = 1, 2)$$

as substitution into the original equations of motions

$$\dot{q}_r = \partial H/\partial p_r \quad \text{and} \quad \dot{p}_r = -\partial H/\partial q_r$$

shows.

The new Hamiltonian can be written as

$$\mathscr{H}' = \mathscr{H}_0' + \mu\mathscr{H}_1' + \mu^2\mathscr{H}_2' + \cdots.$$

The new unperturbed Hamiltonian is

$$\mathscr{H}_0' = \frac{1}{2h}\left(\frac{1}{4p_1^4} + \frac{p_2}{p_1^2} + p_2^2\right)$$

and its Hessian becomes

$$\frac{3}{2h^2 p_1^6}(1 + 2p_2 p_1^2) \neq 0.$$

(D) To comprehend the significance of Kolmogorov's theorem, Poincaré's comment may be recalled according to which the basic problem of dynamics is to find the effect of the perturbation (μH_1) on the behavior of the orbits as $t \to \infty$. The answer to this question cannot come from the questionably convergent series of standard perturbation methods. In what follows, a short outline of a perturbation theory is presented, allowing an evaluation of the basic difficulties and remedies. For clarification it should be stated that by perturbation we mean the deviation of a dynamical system from an *integrable* system. It would probably be better to refer to an *integrated* system since the question of integrability is a complex one. Also the integrability of a system can easily be destroyed by small perturbations; consequently, if the dynamical system is described only with a certain accuracy, one may not be able to distinguish between integrable and nonintegrable systems. In these terms Kolmogorov's theorem is of utmost importance since it states that certain invariant surfaces will exist in spite of small perturbations—or in spite of our limited knowledge regarding the exact description of the dynamical system.

Consider the Hamiltonian of a dynamical system with n degrees of freedom:

$$H(q, p) = H_0(p) + \mu H_1(q, p) + \mu^2 H_2(q, p) + \cdots,$$

where $p = p_i$ and $q = q_i$ stand for the n momenta and coordinates and represent n-vectors. Also $\mu \ll 1$ and Eq. (6) is valid.

We observe that, for $\mu = 0$, $H = H_0$ and so

$$\dot{q} = \partial H_0/\partial p = \omega(p) \qquad \text{and} \qquad \dot{p} = -\partial H_0/\partial q = 0.$$

The solution of the unperturbed equations is

$$q = \omega t + \alpha \qquad \text{and} \qquad p = \beta.$$

In celestial mechanics the function H is periodic in the coordinates, so for $k = 1, 2,\ldots, n$ we have

$$H_k(q, p) = H_k(q + 2\pi, p).$$

Poincaré's question as to how the terms

$$\sum_{k=1}^{\infty} \mu^k H_k$$

affect the behavior of the system as $t \to \infty$ may now be reformulated as follows: are the tori of the previously given solution,

$$q = \omega t + \alpha, \qquad p = \beta,$$

invariant? Since the components of p will not remain constant, may we expect that the solution will stay near the torus

$$p = \text{const}$$

when $\mu \neq 0$?

Consider a canonical transformation from the variables q, p to $q^{(1)}$, $p^{(1)}$ so that

$$H = H_0(p) + \mu H_1(p, q) + \cdots$$

becomes

$$H = H_0^{(1)}(p^{(1)}) + \mu^2 H_1^{(1)}(q^{(1)}, p^{(1)}) + \cdots.$$

The continuation of this process from

$$q^{(1)}, p^{(1)} \qquad \text{to} \qquad q^{(2)}, p^{(2)}$$

gives

$$H = H_0^{(2)}(p^{(2)}) + \mu^3 H_1^{(2)}(q^{(2)}, p^{(2)}) + \cdots,$$

and after ν steps we have

$$H = H_0^{(\nu)}(p^{(\nu)}) + \mu^{\nu+1} H_1^{(\nu)}(q^{(\nu)}, p^{(\nu)}) + \cdots.$$

In the limit, if it exists,

$$H = H_0^{(\infty)}(p^{(\infty)})$$

and the system may be integrated with $p^{(\infty)} = \text{const}$ and $q^{(\infty)} = \omega^{(\infty)} t + \alpha^{(\infty)}$.

In this process then the solution of the perturbed problem may deviate less and less from the solution of the problem represented by the Hamiltonian

$$H_0^{(\nu)}(p^{(\nu)})$$

because of the increasing power of μ occurring in the perturbation terms. Consider the νth step and write the equations of motion in the form

$$\dot{q}^{(\nu)} = \frac{\partial H_0^{(\nu)}}{\partial p^{(\nu)}} + \mu^{\nu+1} \frac{\partial H_1^{(\nu)}}{\partial p^{(\nu)}} + \cdots$$

or

$$\dot{q}^{(\nu)} = \omega^{(\nu)}(p^{(\nu)}) + \mu^{\nu+1} f^{(\nu)}(q^{(\nu)}, p^{(\nu)}) + \cdots$$

and

$$\dot{p}^{(\nu)} = -\frac{\partial H_0^{(\nu)}}{\partial q^{(\nu)}} - \mu^{\nu+1} \frac{\partial H_1^{(\nu)}}{\partial q^{(\nu)}} + \cdots$$

or

$$\dot{p}^{(\nu)} = -\mu^{\nu+1} g^{(\nu)}(q^{(\nu)}, p^{(\nu)}) + \cdots.$$

For $\mu = 0$ the solution is

$$\dot{q}^{(\nu)} = \omega^{(\nu)}(p^{(\nu)}) \qquad \text{and} \qquad \dot{p}^{(\nu)} = 0$$

or

$$q^{(\nu)} = \omega^{(\nu)} t + \alpha^{(\nu)} \qquad \text{and} \qquad p^{(\nu)} = \beta^{(\nu)},$$

where $\alpha^{(\nu)}$, $\beta^{(\nu)}$, and $\omega^{(\nu)}$ are constant vectors.

When $\mu \neq 0$ we have

$$q^{(\nu)} = \omega^{(\nu)}t + t\mu^{\nu+1}\bar{f}^{(\nu)}$$

and

$$p^{(\nu)} = -t\mu^{\nu+1}\bar{g}^{(\nu)},$$

where the symbols $\bar{f}$ and $\bar{g}$ stand for averages.

So during the time-interval t the motion deviates from the unperturbed quasi-periodic motion by an amount proportional to $t\mu^{\nu+1}$. The time interval for equal deviations is $t \sim \mu^{-(\nu+1)}$ or, in other words, as the order of approximation (ν) increases, t also increases since $\mu < 1$. The successive approximations consequently become "better" as ν increases.

The problem of small divisors may prevent the application of these ideas unless the effect of the increasing powers of the small parameter, μ^ν, compensates for the effect of the small denominators, ωk. To show this, consider a generating function

$$W_2(q, P) = qP + \mu S(q, P),$$

which gives the transformation equations

$$p = P + \mu\, \partial S/\partial q \qquad \text{and} \qquad Q = q + \mu\, \partial S/\partial P,$$

where p, q, P, Q are n-vectors.

Before the new variables are substituted into the expression for the Hamiltonian, two Taylor-series expansions will be given, using subscript notation this time, in order to avoid a possible misunderstanding because of the occurrence of second-order derivatives of a scalar with respect to a vector:

$$H_0(p_j) = H_0(P_j) + \mu \frac{\partial H_0}{\partial p_j}\frac{\partial S}{\partial q_j} + \frac{\mu^2}{2!}\frac{\partial^2 H_0}{\partial p_j\, \partial p_k}\frac{\partial S}{\partial q_j}\frac{\partial S}{\partial q_k} + \cdots$$

and

$$H_1(q_j, p_j) = \bar{H}_1(q_j, P_j) + \mu \frac{\partial H_1}{\partial p_j}\frac{\partial S}{\partial q_j} + \mu^2(\quad) + \cdots.$$

Now substituting these expressions into

$$H = H_0(p) + \mu H_1(p, q) + \mu^2 H_2(p, q) + \cdots$$

and returning to the vectorial notation, we have

$$H = H_0(P) + \mu\left[\frac{\partial H_0}{\partial p}\frac{\partial S}{\partial q} + \bar{H}_1(q, P)\right] + \mu^2\left[\frac{1}{2!}\frac{\partial S}{\partial q}\frac{\partial^2 H_0}{\partial p^2}\frac{\partial S}{\partial q} + \frac{\partial H_1}{\partial p}\frac{\partial S}{\partial q} + \bar{H}_2(q, P)\right] + \cdots,$$

where the gradient vectors $\partial f/\partial x$ and their derivatives $\partial^2 f/\partial x^2$ are defined by the results given in the subscript notation used in the preceding expansions with $f = H_0$, H_1, S, and $x = p, q$.

In order to eliminate the factor of the first power of μ in the expression for H we have

$$\frac{\partial H_0}{\partial p}\frac{\partial S}{\partial q} + \bar{H}_1 = 0 \qquad \text{or} \qquad \omega\frac{\partial S}{\partial q} + \bar{H}_1 = 0.$$

As remarked previously, H_k is a periodic function of q, therefore, the same is true for $\bar{H}_1(q, P)$:

$$\bar{H}_1 = \sum_{k_1} \cdots \sum_{k_n} h_{k_1,\ldots,k_n}(P)\, e^{i(kq)} = \sum_k h_k(P)\, e^{i(kq)},$$

where again the summation convention is operative, i.e., $k_\nu q_\nu = kq = \sum_{\nu=1}^{n} k_\nu q_\nu$. For the n summations indicated according to $k_1, \ldots, k_n$, we note that in general the summation extends from $-\infty$ to $+\infty$ on all indices and includes all integers different from 0 after the elimination of the secular terms.

The solution of the linear partial differential equation for S can now be written in the form

$$S = \sum \cdots \sum S_{k_1,\ldots,k_n}(P)\, e^{i(kq)},$$

where

$$S_{k_1,\ldots,k_n}(P) = \frac{i\, h_{k_1,\ldots,k_n}(P)}{k\omega},$$

which last equation is obtained by equating the factors of the various exponential terms in the partial differential equation, after substitution.

The small divisors occur when the eigenvalues or the mean motions of the unperturbed motion, ω, show near-commensurability.

When the ω_i are incommensurable, small divisors appear only for high order of approximation. Inasmuch as a set of rational numbers has measure zero, randomly selected ω_i will be incommensurable. Therefore, for all ω_i we have $\omega_i k_i \neq 0$ for all integers k_i except of the set of measure zero. This statement, of course, takes us again back to the nonresonance requirements of Moser and Kolmogorov.

The problem can be formulated as follows. The elimination of the term μH_1 from the series produces $\mu^2 H_1^{(1)}$; will $H_1^{(1)}$ be such that the method will show (quadratic) convergence? In other words will $H_1^{(1)}$ satisfy $|\mu^2 H_1^{(1)}| < \alpha M^2$ when $|\mu H_1| < M$? The answer to this question is affirmative and it assures that with accelerated convergence of the series $H^{(k)}$ the terms $\mu^k H^{(k)}$ will be convergent. This rapid convergence overcomes the influence of the small denominators and the series will converge for "almost all" ω_j.

8.4 Analytic continuation

The four most frequently used methods of proving the existence of periodic orbits are the method of analytic continuation, the process of equating Fourier coefficients, the application of fixed point theorems, and the method of power series. The idea of the first method is to use a known periodic solution and, by small changes of the parameters and of the initial conditions, continue the known solution. We speak about periodic *orbits* in celestial mechanics, about periodic *solutions* in the theory of differential equations, and about periodic *motions* when treating dynamical systems in general. The terminology is of course only incidental; the basic idea is that we continue analytically a known periodic solution based on the properties of the underlying differential equations and of the known solution. The method of equating the coefficients of equal frequency of trigonometric terms does not require the knowledge of a generating solution but it may lead to difficulties not only in practice but also in principle. This point will be taken up again in Chapter 10 in connection with Hill's problem. The third method does not require either the possible encounter with infinite determinants or the knowledge of a periodic orbit but it is based on the establishment of the equivalence of a surface transformation with the dynamical problem. This approach, originated by Poincaré, will be discussed further in this chapter. The fourth method was used in Chapter 5 to establish periodic orbits of finite size around the libration points and in that context it was referred to as Horn's theorem.

Regarding the first method, analytic continuation, consider a system of differential equations

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}, \mu),$$

where $\mathbf{x}$ and $\mathbf{X}$ are m vectors and μ is a scalar parameter. An alternate notation is

$$\dot{x}_i = X_i(x_1, x_2, \ldots, x_k, \ldots, x_m, \mu),$$

where $i = 1, 2, \ldots, m$.

The solution satisfying the initial conditions

$$\xi_1, \xi_2, \ldots, \xi_m$$

or simply $\boldsymbol{\xi}$, may be written as

$$\mathbf{x} = \mathbf{x}(t, \boldsymbol{\xi}, \mu)$$

with

$$\boldsymbol{\xi} = \mathbf{x}(0, \boldsymbol{\xi}, \mu).$$

Consider now a special solution:

$$\mathbf{x} = \mathbf{x}(t, \boldsymbol{\xi}^*, \mu^*).$$

A special selection of μ^* and ξ^* results in a periodic solution if

$$\mathbf{x}(t, \boldsymbol{\xi}^*, \mu^*) = \mathbf{x}(t + \tau^*, \boldsymbol{\xi}^*, \mu^*)$$

or, because of the uniqueness theorem of differential equations, it is sufficient and necessary that

$$\mathbf{x}(0, \boldsymbol{\xi}^*, \mu^*) = \mathbf{x}(\tau^*, \boldsymbol{\xi}^*, \mu^*) = \boldsymbol{\xi}^*.$$

In what follows the function $\mathbf{X}$ is required to be regular in that region of the $m + 1$ space where the solution $\mathbf{x}$ and μ^* are during $0 \leq t \leq \tau^*$. We also restrict the investigation to such generating solutions which are *not* equilibrium solutions.

The problem of analytic continuation is to find periodic solutions for different values of $\boldsymbol{\xi}$ and μ which are close to but not equal to $\boldsymbol{\xi}^*$ and μ^*. The condition which the generating solution

$$\mathbf{x} = \mathbf{x}(t, \boldsymbol{\xi}^*, \mu^*)$$

and the differential equation

$$\dot{\mathbf{x}} = \mathbf{X}$$

must satisfy, contains the answer to the possibility of performing analytic continuation and establishing in this way the existence of new periodic orbits.

What is pertinent here is connected with implicit functions and with their functional determinants. Let n analytic relations be represented by

$$\psi_i(y_k, \alpha) = 0,$$

where $i, k = 1, 2, \ldots, n$.

Let the solution of this system of equations for $\alpha = \bar{\alpha}$ be $y_k = \bar{y}_k$.

If, now, the functional determinant

$$D = \det(\partial\psi_i/\partial y_k),$$

evaluated at $\bar{\alpha}$ and $\bar{y}_k$, is not zero, then the solution exists in the neighborhood of $\bar{\alpha}$ and it may be represented by the series

$$y_k - \bar{y}_k = A_k(\alpha - \bar{\alpha}) + B_k(\alpha - \bar{\alpha})^2 + \cdots,$$

where

$$A_k, B_k, \ldots \text{ are constants.}$$

Instead of giving a proof, the above statement can be made plausible by writing for $\psi_i(y_k, \alpha)$ its Taylor-series expansion around $(\bar{y}_k, \bar{\alpha})$ as follows:

$$\psi_i(y_k, \alpha) = \psi_i(\bar{y}_k, \bar{\alpha}) + \sum_{k=1}^{n} \left(\frac{\partial \psi_i}{\partial y_k}\right)_{\bar{\alpha}, \bar{y}_k} (y_k - \bar{y}_k) + \left(\frac{\partial \psi_i}{\partial \alpha}\right)_{\bar{\alpha}, \bar{y}_k} (\alpha - \bar{\alpha}) + \cdots.$$

Solution for y_k of the linear system requires that the determinant formed of the coefficients

$$\left(\frac{\partial \psi_i}{\partial y_k}\right)_{\bar{\alpha}, \bar{y}_k}$$

be different from zero, as stated before. Once again it is pertinent to remember that analytic continuation refers to only such values of y_k, α which are so close to the values $\bar{y}_k$, $\bar{\alpha}$ that the above linear approximation is meaningful.

Now we return to the original equations and propose the following problem:

Find such periodic solutions of the differential equations of motion

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}, \mu)$$

which are close to the periodic solution

$$\mathbf{x} = \mathbf{x}(t, \boldsymbol{\xi}^*, \mu^*)$$

with period τ^*. The solutions we are seeking are for $\mu \neq \mu^*$, have the initial condition $\boldsymbol{\xi} \neq \boldsymbol{\xi}^*$, and their period is $\tau \neq \tau^*$.

Note that there are possible changes of the initial conditions which do not result in new orbits. If $\boldsymbol{\xi} \neq \boldsymbol{\xi}^*$ but $\boldsymbol{\xi}$ is another point on the orbit, then we change the epoch only but not the orbit. In order to exclude such degeneration, we fix one component of the vector $\boldsymbol{\xi}^*$, say $\xi_m{}^*$, and change all the others. In this way we have

$$\xi_m = \xi_m{}^*$$

and

$$\xi_k \neq \xi_k{}^*, \quad \text{for} \quad k = 1, \dots, m-1.$$

The generating orbit satisfies the equation

$$\mathbf{x}(\tau^*, \boldsymbol{\xi}^*, \mu^*) - \boldsymbol{\xi}^* = 0$$

and we have similarly for the new (generated) orbits

$$\boldsymbol{\varphi}(\tau, \boldsymbol{\xi}, \mu) \equiv \mathbf{x}(\tau, \boldsymbol{\xi}, \mu) - \boldsymbol{\xi} = 0.$$

These m equations contain $m + 1$ unknowns for a given μ: m components of $\boldsymbol{\xi}$ and the new period τ. The $(m + 1)$th equation is $\xi_m = \xi_m{}^*$.

The equations

$$\boldsymbol{\varphi}(\tau, \boldsymbol{\xi}, \mu) = 0$$

have a solution when $\mu = \mu^*$, which solution is $\boldsymbol{\xi} = \boldsymbol{\xi}^*$ and $\tau = \tau^*$. At this point we wish to see if these equations have solutions when μ is close to μ^* but $\mu \neq \mu^*$. These solutions of period τ we denote by $\boldsymbol{\xi}$.

The $m + 1$ equations in detailed form read

$$\begin{aligned}
\varphi_1(\tau, \xi_1, \xi_2, \dots, \xi_k, \dots, \xi_m, \mu) &= 0,\\
\varphi_2(\tau, \xi_1, \xi_2, \dots, \xi_k, \dots, \xi_m, \mu) &= 0,\\
&\vdots\\
\varphi_k(\tau, \xi_1, \xi_2, \dots, \xi_k, \dots, \xi_m, \mu) &= 0,\\
&\vdots\\
\varphi_m(\tau, \xi_1, \xi_2, \dots, \xi_k, \dots, \xi_m, \mu) &= 0,
\end{aligned}$$

and

$$\xi_m = \xi_m{}^*.$$

The functional determinant for this system is

$$D = \begin{vmatrix}
\dfrac{\partial \varphi_1}{\partial \xi_1} & \dfrac{\partial \varphi_1}{\partial \xi_2} & \cdots & \dfrac{\partial \varphi_1}{\partial \xi_{m-1}} & \dfrac{\partial \varphi_1}{\partial \tau}\\
\dfrac{\partial \varphi_2}{\partial \xi_1} & \dfrac{\partial \varphi_2}{\partial \xi_2} & \cdots & \dfrac{\partial \varphi_2}{\partial \xi_{m-1}} & \dfrac{\partial \varphi_2}{\partial \tau}\\
\vdots & \vdots & & \vdots & \vdots\\
\dfrac{\partial \varphi_m}{\partial \xi_1} & \dfrac{\partial \varphi_m}{\partial \xi_2} & \cdots & \dfrac{\partial \varphi_m}{\partial \xi_{m-1}} & \dfrac{\partial \varphi_m}{\partial \tau}
\end{vmatrix}$$

which is to be evaluated at $\tau = \tau^*$, $\boldsymbol{\xi} = \boldsymbol{\xi}^*$, and $\mu = \mu^*$.

The vector located in the last column of the determinant becomes

$$\frac{\partial \boldsymbol{\varphi}}{\partial \tau} = \dot{\mathbf{x}}(\tau, \boldsymbol{\xi}, \mu) = \mathbf{X}[\mathbf{x}(\tau, \boldsymbol{\xi}, \mu), \mu],$$

since the initial condition $\boldsymbol{\xi}$ does not depend on the time. At τ^*, $\boldsymbol{\xi}^*$, μ^* the vector $\partial \boldsymbol{\varphi}/\partial \tau$ is

$$\frac{\partial \boldsymbol{\varphi}}{\partial \tau} = \mathbf{X}(\boldsymbol{\xi}^*, \mu^*),$$

so the evaluation of the elements in the last column are simple.

The rest of the elements are obtained from

$$\frac{\partial \varphi_i}{\partial \xi_k} = \frac{\partial x_i}{\partial \xi_k} - \delta_{ik}$$

at τ^*, ξ^*, μ and therefore they are known since the dependence of the generating solution on the initial condition is known.

Consider once again the original differential equations of motion and differentiate both sides according to the initial condition. In this way we obtain

$$\frac{\partial \dot{x}_i}{\partial \xi_k} = \frac{\partial X_i}{\partial x_j}\frac{\partial x_j}{\partial \xi_k},$$

where the summation convention regarding j is to be remembered.

Since the initial condition and the time are independent variables, we have that

$$\frac{\partial}{\partial \xi_k}\frac{dx_i}{dt} = \frac{d}{dt}\frac{\partial x_i}{\partial \xi_k}.$$

So the elements in the first $m - 1$ columns of the functional determinant may be obtained by integrating the linear variational equations

$$\frac{d}{dt}\frac{\partial x_i}{\partial \xi_k} = \frac{\partial X_i}{\partial x_j}\frac{\partial x_j}{\partial \xi_k},$$

where

$$\frac{\partial X_i}{\partial x_j} = Y_{ij}[\mathbf{x}(t, \boldsymbol{\xi}^*, \mu^*), \mu^*]$$

is to be evaluated along the generating solution.

If $D \neq 0$, we can solve the system of m equations for the m unknowns and express the solution as

$$\xi_i - \xi_i^* = A_i(\mu - \mu^*) + B_i(\mu - \mu^*)^2 + \cdots,$$

for $i = 1, 2, \ldots, m - 1$,

$$\xi_m - \xi_m^* = 0,$$

$$\tau - \tau^* = A(\mu - \mu^*) + B(\mu - \mu^*)^2 + \cdots,$$

where A_i, B_i, A, and B are constant.

8.5 The problem of two bodies in a rotating coordinate system

8.5.1 Curves of zero velocity and elliptic orbits

Consider the motion of a particle when $\mu = 0$. The force function as shown in Section 4.7.1 becomes

$$\Omega = \tfrac{1}{2}r_1^2 + 1/r_1, \tag{7}$$

corresponding to Fig. 8.1, which shows one of the primaries with unit mass located at the origin, the other, with zero mass at $P_2(-1, 0)$. Therefore, $r_1 = r = (x^2 + y^2)^{1/2}$ and

$$\Omega = \tfrac{1}{2}r^2 + 1/r,$$
$$\Omega_x = x(1 - 1/r^3), \qquad \Omega_y = y(1 - 1/r^3). \tag{8}$$

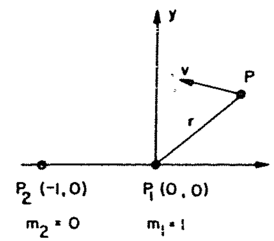


FIG. 8.1. Two-body problem in a rotating coordinate system.

The equations of motion are

$$\ddot{x} - 2\dot{y} = \Omega_x,$$
$$\ddot{y} + 2\dot{x} = \Omega_y, \tag{9}$$

and the Jacobian integral becomes

$$\dot{x}^2 + \dot{y}^2 = 2\Omega - C$$

or

$$v^2 = r^2 + 2/r - C \equiv U(r), \tag{10}$$

where v is the velocity relative to the synodic coordinate system, xy. The reader will identify Eq. (10) with Eq. (76) of Section 4.7.1.

The motion of P takes place on a conic section which is fixed in space. In the rotating coordinate system the conic section appears to rotate in the negative sense with unit angular velocity. Solution of Eqs. (9) can be obtained by transforming the well-known two-body solution to rotating coordinates. This, however, is of little interest at this point.

The Hill curves or curves of zero velocity are given by

$$rU = r^3 - Cr + 2 = 0, \tag{11}$$

which equation follows immediately from Eq. (10) when the $r = 0$ case is excluded. Figure 8.2 shows $rU(r)$ as a function of r for various values of C.

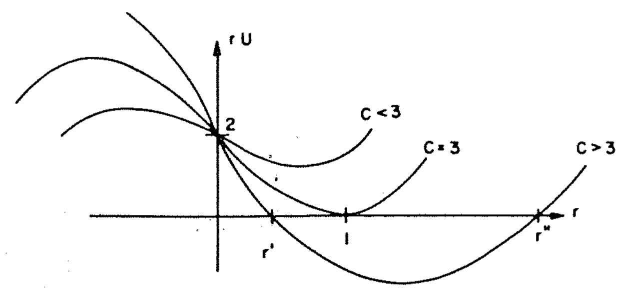


FIG. 8.2. Radii r' and r'' of the curves of zero velocity for $\mu = 0$.

When $C = 3$, $rU = (r - 1)^2(r + 2)$, so since $r > 0$ the curve of zero velocity is the unit circle. If $C < 3$, no curves of zero velocity exist in accordance with the previously found result, and when $C > 3$ the two circles of zero velocity have radii r' and r'' so that

$$0 < r' < 1 < r'' < C.$$

Example. Show that $r'' < C$ for $C > 3$.

As C decreases from $C > 3$ the two circles approach the unit circle and, at $C = 3$, $r' = r'' = 1$. Increasing C when $C \geqslant 3$ results in decreasing r', so inside a circle with radius r' motion is possible. The converse is true for the other circle which expands as C increases and therefore motion is possible outside of a circle with radius r''. The curves of zero velocity and the possible areas of motion are shown on Fig. 8.3.

The motion inside of the circle of radius r' in a fixed system is elliptic (or circular) since the motion is bounded. The focus of the ellipse is P_1 and the ellipse is completely known if its semimajor axis a, its eccentricity e (or its semiminor axis b), and the direction of its apsidal line are given. Since the particle is limited in its motion to the inside of the circle with radius r', we have $(1 + e)a < r' < 1$.

Figure 8.4 shows the fixed elliptic orbit and its relation to the rotating and to the fixed coordinate systems. The angle between the fixed and rotating systems is $(x, X) = t$, as before. The fixed location of the

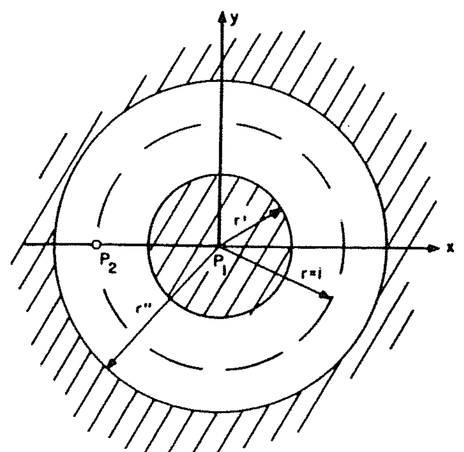


Fig. 8.3. Curves of zero velocity for $\mu = 0$, $C > 3$. The shaded areas show the regions of possible motion.

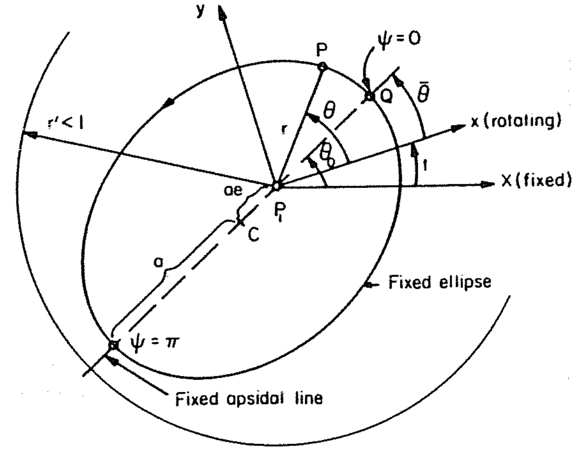


Fig. 8.4. Relations between the fixed and the rotating systems.

semimajor axis is given by θ_0 and the variable direction of the same line is $\bar{\theta}$. A comparison of Fig. 8.4 with Fig. 7.5 of Chapter 7 is of interest. The Delaunay variable g of Chapter 7 becomes the constant angle θ_0 and $\bar{g}$ is the linearly varying angle $\bar{\theta}$. The true longitude which is measured from the rotating axis is θ on both figures.

The arrow on the orbit indicates a direct orbit relative to the fixed system. If the particle is moving in the opposite direction, we speak of a retrograde orbit. If the motion is retrograde in the fixed system, it will also be retrograde relative to the rotating system. The invariance of the "direct" character of the orbit in the two systems requires proof. The mean motion (angular velocity) of the rotating system is 1. The mean motion of the particle in the fixed system is $n = a^{-3/2}$, since

$$n^2a^3 = 1.$$

But $a < 1$ so $n > 1$ if the motion takes place inside the zero velocity curve of radius $r' < 1$, and therefore the mean angular velocity of the particle is larger than the mean angular velocity of the rotating coordinate system. This result is identical with the statement that, if an orbit is a direct orbit in the fixed system, it will also be a direct orbit in the rotating system, if $a < 1$.

The question of the *instantaneous* orientation of orbits is handled differently from the above discussion of "motions in the average." The reader will find the evaluation of the sign of $d(\theta - \bar{\theta} - t)/dt = \dot{\theta}$ a simple task.

The rest of this section is devoted to the discussion of the totality of motions inside the circle $r' < 1$ and relations regarding all possible elliptic and circular motions in the rotating coordinate system will be established.

From the conservation of the angular momentum (or law of areas) we have

$$r^2 \frac{d}{dt}(\theta + t) = \pm[a(1 - e^2)]^{1/2}, \tag{12}$$

which may be shown as follows.

At the point Q on Fig. 8.4 we have $r_Q = a(1 - e)$ and in the fixed system the energy integral is

$$V^2 = 2/r - 1/a, \tag{13}$$

where V is the absolute value of the velocity vector in the fixed system.

The velocity at Q therefore becomes

$$V_Q = \pm\left(\frac{1 + e}{a(1 - e)}\right)^{1/2}. \tag{14}$$

The instantaneous angular velocity of the particle at Q in the fixed system is V_Q/r_Q and its angular momentum becomes

$$V_Q r_Q = \pm[a(1 - e^2)]^{1/2}.$$

The expression $r^2\, d(\theta + t)/dt$ is the angular momentum of the particle in the fixed system at point P since $(\theta + t - \theta_0)$ is the true anomaly in the fixed system and r is the position vector. So Eq. (12) expresses the conservation of angular momentum.

The plus sign indicates direct, the minus retrograde orbits, which convention will be maintained throughout the derivation.

It is observed at this point that at apogee (higher apsidal passage, aphelion, apocenter, and apofocus) and at point Q, perigee (lower apsidal passage, perihelion, pericenter, and perifocus), the velocity vector in the fixed and in the rotating coordinate system is perpendicular to the radius vector since the radial component of the velocity at these points is zero. Using this fact the magnitude of the velocity at Q relative to the rotating coordinate system can be computed:

$$v_Q = V_Q - r_Q, \tag{15}$$

or, using Eq. (14),

$$v_Q = \pm\left(\frac{1 + e}{a(1 - e)}\right)^{1/2} - a(1 - e). \tag{16}$$

Equation (15) follows when it is recognized that r_Q is the velocity of the rotating system (with unit angular velocity) at r_Q. It might also be mentioned that the velocity at Q relative to the rotating system is

$$v_Q = r_Q\left(\frac{d\theta}{dt}\right)_Q,$$

and relative to the fixed system

$$V_Q = r_Q\left[1 + \left(\frac{d\theta}{dt}\right)_Q\right] \quad \text{or} \quad V_Q = r_Q\left[\frac{d}{dt}(\theta + t)\right]_Q.$$

In order to relate the velocity at Q to the Jacobian constant, v_Q is evaluated from Eq. (10) by writing $r = r_Q = a(1 - e)$. This gives

$$v_Q{}^2 = a^2(1 - e)^2 + \frac{2}{a(1 - e)} - C. \tag{17}$$

Equating v_Q as given by Eqs. (16) and (17) we obtain

$$\pm a(1 - e^2)^{1/2} = \frac{C}{2} a^{1/2} - \frac{1}{2a^{1/2}} \tag{18}$$

or

$$\pm b = \frac{C}{2} a^{1/2} - \frac{1}{2a^{1/2}}, \tag{19}$$

since

$$b = a(1 - e^2)^{1/2}.$$

Note that Eq. (18) is actually the one used in Section 8.3(C) and originally introduced in Section 7.6 as Eq. (90). With

$$C = -2H_0, \quad p_1 = L = a^{1/2}, \quad p_2 = G = [a(1 - e^2)]^{1/2},$$

Eq. (18) becomes Eq. (90) of Section 7.6 after the latter is multiplied by $a^{1/2}$.

Equation (19) connects the lengths of the semimajor and minor axes for a given C and is shown in Fig. 8.5.

The curve $b = b(a)$ intersects the line $b = a$ at point N which therefore corresponds to a circular orbit. Since $b > 0$, the circular orbit with radius a_2 will be a direct orbit. Similarly, point M corresponds to a retrograde circular orbit with radius a_1. The part MN of the curve represents all possible elliptic motions within the circle with radius r' since between M and N, $b \leqslant a$. The intersection with the horizontal axis corresponds to a degenerate ellipse with $e = 1$. Note that b itself

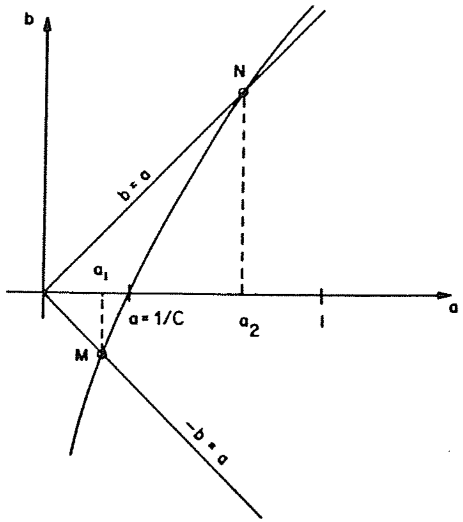


FIG. 8.5. Relation between the semimajor and minor axes (Birkhoff, 1915, Ref. 2).

is always positive; the sign on the right side of Eq. (19), however, is positive or negative, corresponding to direct or retrograde orbits.

The function $b(a)$ is monotone increasing and

$$\frac{db}{da} \to 0 \qquad \text{as} \qquad a \to \infty.$$

For any value of a for which $0 < a \leqslant 1$, we have

$$\frac{db}{da} = \frac{1}{4a^{1/2}}(C + 1/a) \geqslant 1,$$

since $C \geqslant 3$. Consequently, the curve $b(a)$ intersects the line $b = a$ only once in the region where a is between zero and unity.

From Eq. (19)

$$C = 1/a \pm 2b/a^{1/2}$$

and by the substitutions $b = a = a_2$ and $-b = a = a_1$ one obtains

$$C = 1/a_2 + 2a_2^{1/2}$$
$$C = 1/a_1 - 2a_1^{1/2}. \tag{20}$$

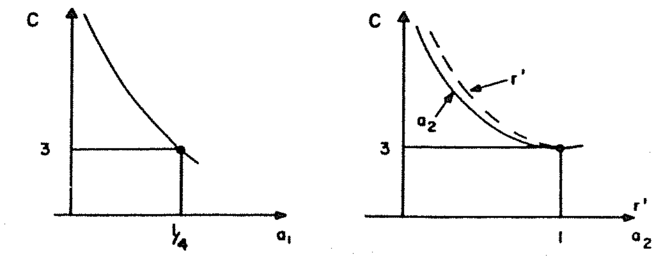


FIG. 8.6. Variation of the Jacobian constant with the radius of the retrograde (a_1) and of the direct (a_2) circular orbits.

From these equations Fig. 8.6 can be prepared which shows that as $3 \to C \to \infty$, $\frac{1}{4} \to a_1 \to 0$, $1 \to a_2 \to 0$, and, as shown before, $1 \to r' \to 0$. Note that for a given C value $r' > a_2 > a_1$.

The totality of motions is represented by the segment MN in Fig. 8.5 provided only the circular area $r' < 1$ is investigated. There are only two circular motions possible, all other motions being elliptic. Figure 8.7 shows these circular orbits.

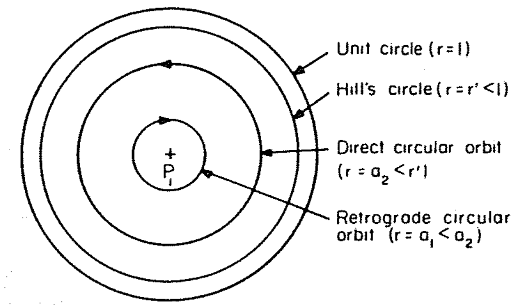


FIG. 8.7. Circular orbits for $\mu = 0$.

8.5.2 Representation of the states of motion

The quantities a, $\bar{\theta}$, and ψ are selected as the state of motion variables with which the totality of motions is represented (Fig. 8.4). The variable a has been defined previously, $\bar{\theta}$ is the longitude of the apsidal line with reference to the rotating x axis, and ψ is the mean anomaly. For a given C, the value of a will determine b and e, and the direction of the motion. The variable $\bar{\theta}$ determines the instantaneous position of the ellipse and ψ determines the location of the moving point on the ellipse. With the exceptions of $a = a_1$ or $a = a_2$ when $\bar{\theta}$ and ψ are indeterminate, a set of values $(a, \bar{\theta}, \psi)$ will determine a unique state of motion, once C is given.

Since $a_1 \leqslant a \leqslant a_2$, $0 \leqslant \psi \leqslant 2\pi$ and $0 \leqslant \bar{\theta} \leqslant 2\pi$, the states of motion may be represented by the points of a hollow cylinder (see Fig. 8.8). If C is given, a_1 and a_2 can be computed from Eqs. (20) and the cylinder can be constructed. Since ψ is measured along the axis of the cylinder, the anchor ring (bottom) represents states of motion for which $\psi = 0$ (perigee) and the top ring is associated with $\psi = 2\pi$. The length of the semimajor axis, a is measured along the radius and $\bar{\theta}$ is used as the polar angle. If an orbit from $\psi = 0$ to $\psi = 2\pi$ is followed, a will not change since the orbit is a given ellipse. On the other hand $\bar{\theta}$ will change from $\bar{\theta}$ to $\bar{\theta}'$ where $\bar{\theta}$ is the location of the apsidal line in the rotating system at the beginning of the motion (say, at perigee $\psi = 0$) and $\bar{\theta}'$ is the location of the apsidal line in the same system after the particle has completed one circuit on the ellipse ($\psi = 2\pi$). In Fig. 8.8 the points P and P' represent points on a given orbit. The anchor ring ($\psi = 0$) represents the totality of motions at perigee, the inside cylindrical surface ($a = a_1$) represents retrograde ($\bar{\theta} - \psi = \text{const}$), and the outside surface ($a = a_2$) direct circular orbits ($\bar{\theta} + \psi = \text{const}$).

Since the location of the particle on the ellipse is determined com-

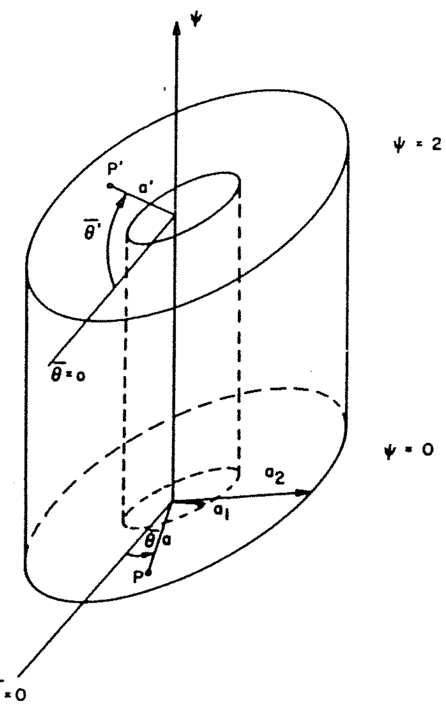


FIG. 8.8. Hollow cylinder representation of states of motion for $\mu = 0$.

pletely by the variable ψ, the sections $\psi = \pm 2k\pi$ ($k = 0, 1, \ldots$) of the hollow cylinder represent identical locations of the particle on its ellipse. In this sense a torus can be formed by joining the $\psi = 0$ and $\psi = 2\pi$ ends of the hollow cylinder. This torus has an inside hole with radius a_1. The orbit will intersect the $\psi = 0$ ring (which is now also the $\psi = 2\pi$ ring) after every circuit which the particle has made on its ellipse. This way, on a single ring, intersection points $P, P', P'', \ldots$ are obtained. These points will all lie on the same circle since the motion occurs on a given ellipse so that $a = a' = a'' = \cdots = \text{const}$. Only polar displacements ($\bar{\theta}$) occur between $P, P', P'', \ldots$, therefore the ring is transformed into itself as the motion takes place.

The time dependence of the variables a, $\bar{\theta}$, ψ for an orbit is given by

$$\begin{aligned} \bar{\theta} &= \theta_0 - t, \\ a &= a_0, \\ \psi &= a_0^{-3/2} t + \psi_0. \end{aligned} \tag{21}$$

The first equation follows from Fig. 8.4 which shows that $\theta_0 = \bar{\theta} + t$ is the position of the apsidal line relative to the fixed X axis and therefore it is constant. Also $a = a_0 = \text{const}$ for a given ellipse. The last equation of (21) is an immediate consequence of the fact that the mean anomaly is $\psi = n(t - t_0)$, where $n = a_0^{-3/2}$ is the mean motion.

The motion is represented by a spiral on a cylindrical surface whose radius is a_0. The constant pitch of the spiral is

$$\frac{d\psi}{d\theta} = \frac{d(a_0^{-3/2} t + \psi_0)}{-dt} = -a_0^{-3/2}.$$

8.5.3 Ring transformation

A more detailed inspection of the ring transformation leads to the establishment of a condition for periodic motion in the synodic system. Consider a point K on the ring $\psi = \psi_0$. The coordinates of K will determine the state of motion, or a set of initial conditions, or an orbit, depending on how the reader wishes to look at K. Let the coordinates of K be $\bar{\theta} = \theta_0$, $\psi = \psi_0$, $a = a_0$ and let $t = 0$ be the time when the particle is piercing the ring $\psi = \psi_0$. Let L be the point on the ring $\psi = \psi_0 + 2\pi$ where the particle intersects this ring. The time required to move from the plane ψ_0 to $\psi_0 + 2\pi$ is the same as the time required for a round trip on the fixed ellipse which is $2\pi/n = 2\pi a_0^{3/2}$. This follows also formally from Eqs. (21) by writing $\psi_0 + 2\pi = \psi$ in the third equation and solving for t. The $\bar{\theta}$ coordinate of L is obtained by realizing that one round trip requires the time $2\pi a_0^{3/2}$, during which time $\bar{\theta}$ changes to $\theta_0 - 2\pi a_0^{3/2}$. The third coordinate of L is invariant (a_0) since the size of the ellipse is not changing. So the coordinates of K at $t = 0$ are $K(a_0, \theta_0, \psi_0)$ and the coordinates of L at $t = 2\pi a_0^{3/2}$ are

$$L(a_0, \theta_0 - 2\pi a_0^{3/2}, \psi_0 + 2\pi).$$

In order to simplify the notation let the set $(a, \bar{\theta}, 0)$ represent K and $(a', \bar{\theta}', 2\pi)$ represent L. Then

$$a' = a, \qquad \bar{\theta}' = \bar{\theta} - 2\pi a^{3/2} \tag{22}$$

are the equations of the transformation (T) which take K into L. As mentioned before, this is a pure rotation in the negative direction. The amount of rotation is determined by the size of the ellipse, a. Various points of the ring $a_1 < a < a_2$ will be rotated by different amounts and the ring will be transformed into itself.

At this point we identify Eqs. (2) of Section 8.3, Part (A), with Eqs. (22). This requires the following substitutions: $r = a$, $r_1 = a'$, $\varphi = \bar{\theta}$, $\varphi_1 = \bar{\theta}'$, $f(r) = -2\pi a^{3/2}$. Perturbation takes Eqs. (22) into

$$a' = a + g(a, \bar{\theta}),$$
$$\bar{\theta}' = \bar{\theta} - 2\pi a^{3/2} + h(a, \bar{\theta}),$$

which are similar to Eqs. (3).

The transformation can be again applied to L in order to obtain M, etc. The coordinates of M represent the change occurring in the direction of the apsidal line ($\bar{\theta}$) after the second round trip of the particle on the ellipse has been completed. In general, starting with a point for which $\bar{\theta} = \bar{\theta}$ and $a = a$ when the particle is at perigee ($\psi = 0$), the transformation, T, will give

$$\bar{\theta}' = \bar{\theta} - 2\pi a^{3/2} = \bar{\theta}^{(1)}.$$

Applying T again we obtain

$$\bar{\theta}'' = \bar{\theta}' - 2\pi a^{3/2}$$

or

$$\bar{\theta}^{(2)} = \bar{\theta} - 4\pi a^{3/2}.$$

After the kth application of T we have

$$\bar{\theta}^{(k)} = \bar{\theta} - 2\pi k a^{3/2},$$

which is the angle between the apsidal line and the rotating x axis after the particle completed k circuits on its elliptic orbit. If during this time $\bar{\theta}$ decreased by 2π we have

$$\bar{\theta} - 2\pi l = \bar{\theta}^{(k)}$$

or

$$a^{-3/2} = k/l, \tag{23}$$

which is the necessary and sufficient condition for the existence of periodic orbits of this type. It is noted that while the ellipse made l revolutions (in the rotating system) the particle completed k circuits. The special situation when $l = 1 = k$, i.e., when the mean motion of the particle and that of the rotating system are equal, results in $a = 1$ which is not a permitted value within the circle $r' < 1$.

If the condition specified by Eq. (23) is met, i.e., if $a = (l/k)^{2/3}$, the orbit will be "closed" in the rotating system. Since $a < 1$, we observe that the ellipse makes fewer revolutions (l) relative to the rotating system than the number of complete round trips (k) made by the particle on its ellipse. The length of the semimajor axis must be the 2/3 power of a rational number, smaller than one.

The ratio k/l can also be looked at as the mean motion of the particle in the fixed system keeping in mind of course that the rotating system has unit mean motion.

The apsidal line regresses by $2\pi l$ after k circuits, so $2\pi l/k = 2\pi a^{3/2}$ is the amount of regression of the apsidal line after one single round trip, as shown previously. The progression of the apsidal line is $-2\pi a^{3/2}$ which quantity is often referred to as Poincaré's coefficient of rotation (σ).

Figure 8.9 shows the ring and the subsequent points obtained by the transformation, T.

8.5.4 Interpretation in the synodic system

The orbit in the fixed system is an ellipse but in the rotating system x, y it may become a more complicated curve as shown by the

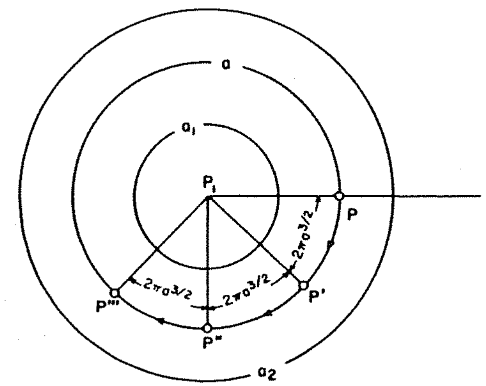


FIG. 8.9. The ring transformation: $P' = T(P)$, $P'' = T(P')$, $P''' = T(P'')$, ...

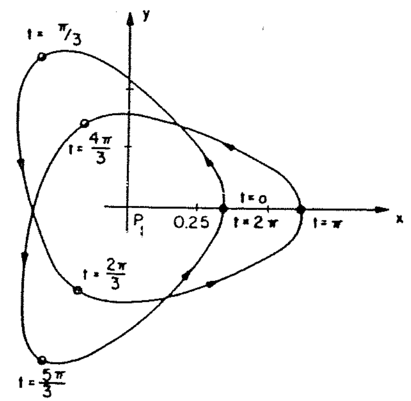


FIG. 8.10. Orbit in the rotating coordinate system: $\mu = 0$, $e = 0.3$, $a = 0.4807499$, $C = 3.4029327$, $k = 3$, $l = 1$.

example of Fig. 8.10 computed by Knowles. The reason for treating the orbit in the rotating frame is that for the restricted problem of three bodies the existence of the Jacobian integral depends on the use of such a system. The two-body problem treated in this chapter is unnecessarily complicated when a rotating frame is used; nevertheless, if the two-body results are to be "continued" to the restricted problem, then the use of the rotating frame is not only justified but necessary.

In Fig. 8.10 the particle starts its motion at $t = 0$, corresponding to the point of perigee $\psi = 0$ on its fixed elliptic orbit. At time $t = \pi/3$ the particle is at apogee $\psi = \pi$ and at $t = 2\pi/3$ it is back again at perigee $\psi = 2\pi$. The mean motion on the fixed ellipse is $n = 3$ and the length of the semimajor axis is $3^{-2/3}$. During one round trip (circuit) on the ellipse (which took $2\pi/3$ nondimensional time units) the apsidal line moved relative to the rotating axes by $-2\pi/3$ and the rotating axes moved by $2\pi/3$ relative to the fixed apsidal line. The perigee points occur at $t = 0, 2\pi/3, 4\pi/3$, and 2π, the apogees at $t = \pi/3$, $3\pi/3$, and $5\pi/3$, as shown in Fig. 8.10. At the first perigee point $t = 0$ the orbit is positively tangent to a circle going through this point with center at P_1 and directed as if it were a the direct orbit. (We speak of positive tangency of a curve to another if at a common point they have the same tangent and if the curves are described in the same sense.) At $t = 0$ the particle on the fixed ellipse is at perigee and in the fixed system its velocity vector is normal to the radius vector. The same is true for the orbit in the rotating system at perigee.

Note that if for the ellipse $e = 0.3$ is selected, the Jacobian constant can be computed using Eq. (18) with the positive sign and with

$a = (3)^{-2/3}$. This gives $C = 3.4029327$. In Chapter 9 this orbit will be continued for $\mu \neq 0$.

Returning now to the circle which is positively tangent to the orbit in the rotating frame at $t = 0$, we search for the next positive tangency between the orbit and the same circle. This occurs at $t = 2\pi/3$. The next positive tangency (after $t = 2\pi/3$) occurs at $t = 4\pi/3$. Between consecutive positive tangencies the particle makes a round trip on its ellipse, i.e., ψ changes from 0 to 2π. The location of the positive tangency point (occurring at $t = 2\pi/3$) is given by the angle $-2\pi/3$ (see Fig. 8.10), which is the amount of rotation of the apsidal line. In this way another determination of point L is possible when point K is given (cf. Fig. 8.3). In other words the ring transformation which takes P into P' and this into P'', etc., can be looked at as the series of points which have consecutively positive tangency with the same circle.

Consider now a series of orbits in the synodic system x, y with initial conditions $0 < x_0 < a_2$, $y_0 = 0$, $\dot{x}_0 = 0$, and $\dot{y}_0 > 0$, all belonging to a given value of C. These orbits will be direct orbits and they will display a perigee point at $t = 0$. (At $x_0 = a_1$ a direct elliptic orbit exists since at $x_0 = a_1$ the circular orbit would be retrograde.) At $x_0 = a_2$ we have a direct circular orbit. As x_0 increases from a_2 to r' the orbits $\dot{y}_0 > 0$ will display apogee points. The retrograde orbits ($\dot{y}_0 < 0$, $\dot{x}_0 = 0$, $y_0 = 0$) display perigees when x_0 is between 0 and a_1. So perigees are associated with direct orbits as $0 \leqslant x_0 \leqslant a_2$ and with retrograde orbits as $0 \leqslant x_0 \leqslant a_1$. This fact allows a construction shown in Fig. 8.11 where two series of auxiliary circles (these *all* are not orbits for a given C!) are drawn with P_1 as center in the plane x, y. The first series of circles is drawn in a retrograde sense with radii less than a_1, the second series in a direct sense with radii less than a_2. The double series of circles constitute a ring with two leaves in the plane x, y (see Fig. 8.11).

Note that a_1 and a_2 are *not* the semimajor axes of the ellipses described before but rather the radii of the retrograde and direct circular orbits belonging to the given value of C.

The states of motion corresponding to perigee ($\psi = 0$) are those for which the path becomes positively tangent to a member of this double series of circles. Let now point K represent a state of motion with given a and let it represent a perigee point. The previously discussed transformation, T, takes this point into L, where the orbit in the plane x, y will become again positively tangent to the same auxiliary circle. For any point K there is a positively tangent orbit to the auxiliary circle at K. The transformation which takes K into L represents a transformation of the ring into itself leaving radial distances unchanged and regressing each point by $2\pi a^{3/2}$ around P_1. The above construction was

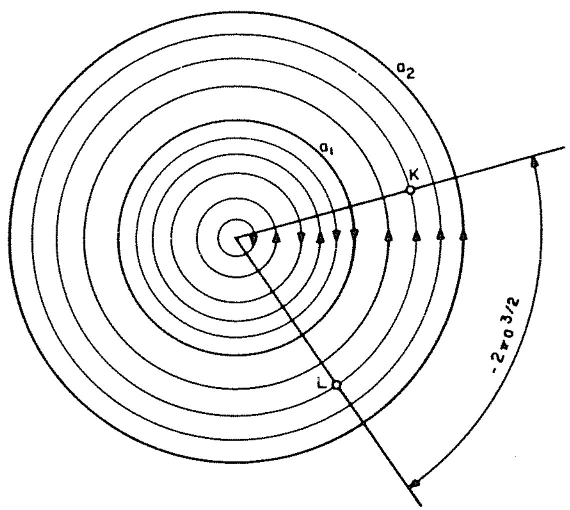


FIG. 8.11. Two-leafed ring of states of motion.

performed for one given C. For another value of C new values for a_1 and a_2 may be computed and the construction can again be performed. This means that the auxiliary circles of Fig. 8.11 can be looked at as the circles of radii a_1 and a_2 belonging to various values of C. This observation will prove to be useful later on.

In Section 8.5.2 it was proposed to connect the two ends of the hollow cylinder shown on Fig. 8.8 and this way to form a torus. The inside surface of this hollow torus corresponds to retrograde and the outside surface to direct circular orbits. Any other elliptic orbits will take place inside the torus, between the two surfaces mentioned previously.

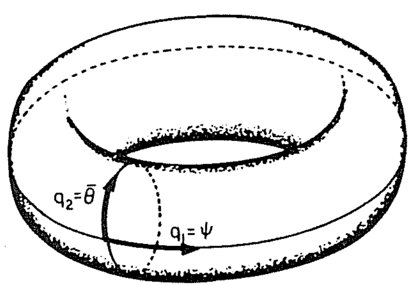


FIG. 8.12. Torus corresponding to the cylinder representation of the states of motion.

One may construct a simple torus (without a hole inside) and represent the elliptic motion on its surface as shown in Fig. 8.12. The previously described transformation of a ring into itself now can be represented by motion on a torus as follows (see also Sections 8.2 and 8.3).

As $q_1 = \psi$ increases from 0 to 2π corresponding to a complete circuit of the point on the ellipse and to $T = 2\pi/n$, the fixed line of apsides $q_2 = \bar{\theta}$ will change from θ_0 to $\theta_0 - 2\pi/n$, where $n = a^{-3/2}$. The angular velocities (mean motions) associated with the two angular motions q_1 and q_2 are

$$\omega_1 = \dot{q}_1 = n \qquad \text{and} \qquad \omega_2 = \dot{q}_2 = -1. \tag{24}$$

The condition for periodic motion appears on the torus as a spiral which closes upon itself. If one revolution along q_2 corresponds to one revolution along q_1 we speak of a simple periodic orbit. In general we deal with multiple periodic orbits when the particle completes k circuits on the ellipse while the ellipse makes l revolutions relative to the rotating system. If

$$\left|\frac{\omega_1}{\omega_2}\right| = \frac{k}{l} = n = a_0^{-3/2}$$

is a rational number, we have curves winding on the surface of the torus and closing. For irrational values of ω_1/ω_2 the curve will not close.

8.6 The restricted problem with small mass parameter

8.6.1 Periodic orbits of the first kind

The previous section established a ring transformation by which the totality of motions can be investigated and gave a condition for periodic orbits in the rotating frame. These results referred to $\mu = 0$, when one of the primaries had zero mass ($m_2 = 0$).

These investigations for $\mu = 0$ will be extended by the method of analytic continuation to $\mu \neq 0$. The main problem is to show that the circular orbits (a_1 and a_2) of the two-body problem become (noncircular) periodic orbits for $\mu \neq 0$. In this way the existence of direct and retrograde periodic orbits for $\mu \neq 0$ is shown (see Section 8.4).

These are the orbits which Poincaré called "première sorte." The orbits are generated from circular planar orbits ($e = 0$, $i = 0$).

The use of the method of analytic continuation to establish the existence of these orbits is synonymous with employing Poincaré's restricted problem, which treats "small" values of μ. If Poincaré's problem of small perturbation is looked upon as one end of a scale, the restricted problem with $\mu = \frac{1}{2}$ will be at the other end. Poincaré's problem is of primary interest in the dynamics of the solar system, while Strömgren's problem ($\mu = \frac{1}{2}$) is of importance primarily in stellar dynamics. The fact that certain families of periodic orbits have been established numerically for $\mu \sim 0$, $\mu = \frac{1}{2}$, as well as for the whole range of $0 < \mu < \frac{1}{2}$, will be discussed in Section 8.6.2 and also in Chapter 9. Here only Poincaré's problem will be treated.

The equations of motion for the $\mu \neq 0$ case may be written [see Eqs. (8), (9), and (10)] as

$$\begin{aligned} \ddot{x} - 2\dot{y} &= x(1 - 1/r^3) + f(\mu, x, y), \\ \ddot{y} + 2\dot{x} &= y(1 - 1/r^3) + g(\mu, x, y), \end{aligned} \tag{25}$$

and the Jacobian integral as

$$\dot{x}^2 + \dot{y}^2 = r^2 + 2/r - C + h(\mu, x, y), \tag{26}$$

where the functions f, g, and h approach zero as $\mu \to 0$. Note that singularities either have been eliminated from the equations or are excluded from the considerations.

A direct circular orbit as a solution will satisfy the above equations for $\mu = 0$ with the initial conditions:

$$x = \bar{x} = a_2, \qquad y = 0, \qquad \dot{x} = 0, \qquad \dot{y} = 1/a_2^{1/2} - a_2,$$

since $a_2^{-1/2}$ is the circular velocity in the fixed system and a_2 is the local velocity of the rotating frame of reference.

The Jacobian constant belonging to this solution is

$$C = 1/a_2 + 2a_2^{1/2}$$

according to Eq. (10) and the solution can be written in the rotating system as

$$x = a_2 \cos (n - 1)t \qquad \text{and} \qquad y = a_2 \sin (n - 1)t.$$

The solution for $\mu \neq 0$ in the rotating system becomes

$$x = x(\mu, \bar{x}, t), \qquad y = y(\mu, \bar{x}, t).$$

The conditions put on this solution for $\mu \neq 0$ are as follows: at $t = 0$, $x = \bar{x}$, $y = 0$, $\dot{y} > 0$, and $\dot{x} = 0$; at $t = \bar{t}$, $x = \bar{x}_1$, $y = 0$, $\dot{y} < 0$, and $\dot{x} = 0$. Here $\bar{t}$ is the time when the orbit crosses again the x axis. The solution of interest is therefore an orbit which, with $\mu \neq 0$, crosses the x axis perpendicularly twice, corresponding to the direct circular orbit (for $\mu = 0$) which crosses the x axis at $x = \pm a_2$ perpendicularly. Such a solution for $\mu \neq 0$ will be the analytic continuation of the upper half of a circular orbit (for $\mu = 0$). Since the equations of motion are not changed if x, y, t are replaced by $x, -y, -t$, one concludes that the establishment of the upper half ($y > 0$) of the orbit for $\mu \neq 0$ constitutes the existence of a complete symmetric periodic orbit.

To show the existence of the orbit specified by the conditions listed above, observe that if $\bar{x} = a_2$ and $\mu = 0$ the orbit will intersect the x axis twice perpendicularly ($\dot{x} = 0$ at the intersections). If $\mu \neq 0$ the value of $\dot{x}$ at the second crossing will depend on the initial value $x = \bar{x}$ and on μ. Denoting $\dot{x}$ at the second crossing by ψ, we have

$$\left(\frac{dx}{dt}\right)_{t=\bar{t}} = \psi(\mu, \bar{x}).$$

The function $\psi(\mu, \bar{x})$ is analytic since it is an analytic function of the angle between the x axis and the orbit, which in turn depends analytically on μ and $\bar{x}$.

Therefore the function $\psi(\mu, \bar{x})$ can be written as a power series:

$$\psi = \alpha_2(\bar{x} - a_2) + \beta_2\mu + \gamma_2(\bar{x} - a_2)^2 + \delta_2\mu(\bar{x} - a_2)^2 + \cdots,$$

where

$$\left[\frac{\partial\psi}{\partial\bar{x}}\right]_{\substack{\bar{x}=a_2 \\ \mu=0}} = \alpha_2 .$$

Consequently, the condition for finding the solution $\bar{x} = \bar{x}(\mu)$ to satisfy the equation $\psi(\mu, \bar{x}) = 0$, is $\alpha_2 \neq 0$ (cf. Section 8.4).

If μ is near zero, $\bar{x}$ will be near a_2, so

$$\left[\frac{\partial\psi}{\partial\bar{x}}\right]_{\substack{\bar{x}\cong a_2 \\ \mu\cong 0}} \cong \alpha_2$$

and so the two conditions for inverting the equation $\psi(\mu, \bar{x}) = 0$, $\partial\psi/\partial\bar{x} \neq 0$ and $\alpha_2 \neq 0$, are the same.

Before finding the conditions which lead to $\alpha_2 = 0$, note that at the second crossing $y(\mu, \bar{x}, \bar{t}) = 0$, from which $\bar{t} = \bar{t}(\mu, \bar{x})$ can be obtained if the function $y(\mu, \bar{x}, \bar{t}) = 0$ can be inverted. The condition for this is that $dy/dt \neq 0$ at $t = \bar{t}$, $x = \bar{x}$, and $\mu \neq 0$. To show that this condition is satisfied, first observe that the relation $y = y(0, a_2, t^*) = 0$ can be inverted since $dy/dt = -1/a_2^{1/2} + a_2 \neq 0$ at $t = t^*$, where t^* is the time for the second crossing with $\mu = 0$. This result $[\dot{y}(t^*) \neq 0]$ is, of course, expected since the orbit crosses the x axis perpendicularly. More formally we conclude that since $\bar{x}$ is near a_2 and μ is near zero, t^* will be near $\bar{t}$ and so

$$\left[\frac{dy}{dt}\right]_{\substack{t=\bar{t} \\ x=\bar{x} \\ \mu\neq 0}} \neq 0.$$

The $\alpha_2 = 0$ condition can be evaluated from two-body considerations as seen from the definition of α_2. The necessary and sufficient condition for $\alpha_2 = 0$ is that the time t^* for the particle (in its circular orbit from $t = 0$ to its second intersection with the x axis) be a multiple of the half period in the fixed plane. In other words, critical condition occurs when the period in the rotating plane is a multiple of the period in the fixed plane. This follows from

$$\alpha_2 = a_2(\dot{f} \sin f)_{t=t^*}, \tag{27}$$

where $f = \theta - \bar{\theta}$ is the true anomaly. From Eq. (27) $\alpha_2 = 0$ if $f = k\pi$.

Example. Derive Eq. (27).

For retrograde circular orbits the same considerations apply up to this point, and their existence is associated with the nonzero value of α_1 defined similarly to α_2.

Let now n be the mean motion of the particle relative to the fixed axes and consequently $n \mp 1$ relative to the synodic system. The upper sign corresponds to direct, the lower to retrograde circular motion. With this notation

$$t^* = \pi/(n \mp 1)$$

and the condition for $\alpha_2 = 0$ becomes $t^* = k\pi/n$, since the half period in the rotating system t^* is equal to the k multiple of the half period in the fixed system π/n. Another form is

$$n = \pm k/(k-1),$$

where $k > 1$ is an integer. Since $n > 0$, the lower sign is excluded, therefore, there are no critical values for retrograde circular orbits. The exceptional values of n become 2/1, 3/2, 4/3,..., the first representing the commensurability occurring for the Hecuba group of minor planets in the solar system.

The mean motion is related to the radius a_2 by Kepler's equation, $a_2 = n^{-2/3}$, and by Eq. (20) to the Jacobian constant,

$$C = \frac{2+n}{n^{1/3}}. \tag{28}$$

The function $C(n)$ monotonically increases when $1 \leqslant n \leqslant 2$ because its derivative is $(2/3)(n-1)n^{-4/3}$. Also $C(1) = C_{\min} = 3$ and $C(2) = C_{\max} = (32)^{1/3} = 3.174802104...$, when n is between 1 and 2.

The conclusion is reached, therefore, that retrograde circular orbits established for $\mu = 0$ can be continued for $\mu \neq 0$ and they give symmetric (with respect to the x axis) retrograde periodic orbits for any $C > 3$. Continuation of direct circular orbits established for $\mu = 0$ is possible when $C > (32)^{1/3}$ and these orbits become symmetric periodic direct orbits. When $3 \leqslant C \leqslant (32)^{1/3}$ analytic continuation of the direct circular orbits is possible for any C not one of the exceptional values.

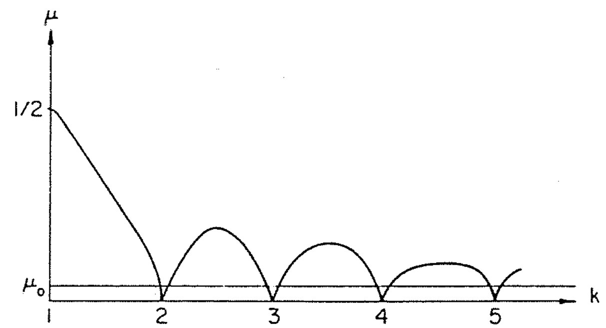
The existence of periodic orbits of the first kind for $\mu \neq 0$ generated from circular orbits with $\mu = 0$ has been shown by the above discussion.

Close similarity to these periodic orbits is shown by the ones which are outside the zero velocity circle of radii r'' (see Fig. 8.3). Again, first considering the $\mu = 0$ case, we have circular motion with radius $r > r'' > 1$ and with speed, relative to the fixed system, of $1/r^{1/2}$. The speed relative to the rotating frame becomes $v = \pm 1/r^{1/2} - r$, where the upper sign corresponds to direct and the lower to retrograde orbits in the fixed frame.

It is now recalled that the "inside" circular orbits ($r < r' < 1$) had the invariant property regarding the direction of motion going from a fixed to a rotating system. This is not the case for the orbits $r > r''$. Both the direct and retrograde orbits of the fixed system become synodically retrograde for the "outside" circular orbits. The invariance of the retrograde orbits does not require proof; the statement regarding the direct sidereal orbits is proved by realizing that $v = +1/r^{1/2} - r < 0$ for $r > 1$.

The sidereally retrograde orbits for the case $r > 1$ can be continued from $\mu = 0$ to $\mu \neq 0$ into symmetric periodic orbits the same way as was shown for the $r < 1$ retrograde orbits; i.e., the analytic continuation does not have exceptional values of C. The sidereal direct outside orbits show the same kind of singular behavior regarding the analytic continuation as the sidereal direct inside orbits and the exceptional values of n are $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$,... with the critical value of C being $C(\frac{1}{2}) = C_{\max} = (31.25)^{1/2} = 3.149802625...$.

The first critical value of $C = (32)^{1/3}$ can be associated with three values of μ, depending on which libration point is selected. If $C_1 = (32)^{1/3}$, $\mu_1 = 0.011116722$; if $C_2 = (32)^{1/3}$, $\mu_2 = 0.009753256$; and if $C_3 = (32)^{1/3}$, $\mu_3 = 0.091782889$. For the second critical C value, $C = (32.25)^{1/3}$, we have $\mu_1 = 0.008566815$, $\mu_2 = 0.007618194$, and $\mu_3 = 0.078061547$. These values are given only for general orientation. Inasmuch as the existence proof of the periodic orbits of the first kind is based on analytic continuations from the $\mu = 0$ case, the extent to which the circular orbits can be continued remains unknown. Large values of C are identical in effect with small values of μ since in both cases the perturbation caused by the second primary (of mass μ)



1	2	3	4	5	k
0	1	2	3	4	m
∞	2/1	3/2	4/3	5/4	n
∞	32	343/12	250/9	2197/80	C^3
∞	$\sim$3.175	$\sim$3.057	$\sim$3.028	$\sim$3.017	C
0	2π	4π	6π	8π	T^*
0	4	9/4	16/9	25/16	a_2^{-3}
0	$\sim$0.630	$\sim$0.763	$\sim$0.825	$\sim$0.862	a_2
	Hecuba group	Hilda group	Thule group		Note

FIG. 8.13. Analytic continuation as a function of mean motion, Jacobian constant, period, etc., for direct inside circular orbits.

is small. When C is large, i.e., $C \gg C_{\max}$, the motion takes place either close to the first primary ($C \cong 2/r$), or far from it ($C \sim r^2$); in both cases the perturbation becomes small.

The method of analytic continuation does not contain answers to these questions. The result furnished by analytic continuation is that, for nonexceptional values of C, direct orbits can be continued from $\mu = 0$ to $\mu \neq 0$. Figure 8.13 is an attempt to show schematically the allowable ranges of μ, following Wintner. At $k = 2$ analytic continuation is not possible as has been shown above, therefore the curve of Fig. 8.13 touches the $\mu = 0$ line. How high the value of μ is, at say $k = 2.5$, is not known and the curve of Fig. 8.13 was drawn to show only that "complete" continuation (to $\mu = \frac{1}{2}$) may not be expected. For a small value of $\mu = \mu_0$ the figure shows that in the neighborhood of the critical values of k the orbits can be continued. This may indicate that, while gaps exist at the critical values, their neighborhood might be populated with periodic orbits *provided* the physical model possesses a small enough value of μ. The tabulated values attached to the figure are computed from the following equations:

$$
\begin{aligned}
n &= k/(k-1), \quad & m &= k - 1; \quad & n &= 1 + 1/m, \\
C &= (2+n)/n^{1/3}, \quad & T^* &= 2\pi/(n-1), \quad & a_2 &= n^{-2/3}.
\end{aligned} \tag{29}
$$

Here n is the mean motion in the fixed system, k is a positive integer, m is Poincaré's characteristic number, C is the Jacobian constant, T^* is the period in the rotating system, and a_2 is the radius of a direct circular orbit. The values of $m = 1, 2, 3$ correspond to minor planets of the Hecuba, Hilda, and Thule types.

The 14 Trojan asteroids correspond in Fig. 8.13 to $n = 1$, $m \to \infty$, $k \to \infty$, $T^* \to \infty$, $C = 3$, and $a_2 = 1$. These were discussed in Chapter 5. The Hilda group contains 20 members, some of them with large eccentricities (0.25), while the Trojans all have $e < 0.15$. The members of the Hilda group have inclination smaller than 15°, while one of the Trojans is inclined at 35°.

Note that the scales of the various quantities indicated in Fig. 8.13 are not related linearly to the scale of k and also that the figure refers only to direct inside orbits. The retrograde inside orbits can be continued for any value of $C \geqslant 3$ and, similarly, sidereally retrograde outside orbits have no critical C values, consequently no figure similar to Fig. 8.13 exists. Construction of such a figure for sidereally direct outside orbits is left for the readers' imagination.

As an addendum to this section, Birkhoff's work on the existence of retrograde and direct periodic orbits for arbitrary value of the mass-parameter ($0 < \mu < \frac{1}{2}$) is outlined.

The two-leafed ring consisting of circular orbits for $\mu = 0$ can be analytically continued for small μ and the new ring is formed of symmetric periodic orbits of the first kind. The existence of direct and retrograde periodic orbits is established this way for small μ as was shown in Section 8.6.1. On the other hand, a study of the differential equations of motion allows the proof of a theorem by Birkhoff, valid for *arbitary* value of μ (i.e., without analytic continuation): For any value of C for which a closed oval of zero velocity about m_1 exists, there is at least one periodic orbit making a single *retrograde* circuit around m_1. This orbit is symmetric to the x axis and the tangents to the orbit are parallel to the y axis at the two points where the orbit intersects the x axis. There are two points on the orbit, symmetrically located to the x axis, where the tangents to the orbit are parallel to the x axis.

Birkhoff proceeds to show the existence of *direct* periodic orbits for arbitrary μ. Taking into consideration the fact that for arbitrary μ the existence of retrograde orbits can be shown, the idea seems to be most tempting to find a (ring) transformation which is not based on the existence of direct and retrograde circular orbits but only on the existence of the above retrograde circular motion. Taking such a (ring) transformation we could go from the $\mu = 0$ case to arbitrary μ since retrograde periodic orbits exist for both cases. Extracting the direct circular orbits from this new transformation for $\mu = 0$ and extracting from the corresponding transformation the direct periodic orbits for $\mu \neq 0$ will complete the existence proof of direct periodic orbits for arbitrary μ.

To establish the (ring) transformation based on the retrograde circular orbits alone (for $\mu = 0$) we subject the ring transformation described in Section 8.5.3 to the following transformation:

$$a^* = 1/a - 1/a_2,$$
$$\theta^* = \bar{\theta},$$
$$\psi^* = \psi - \bar{\theta}.$$

The set a^*, θ^*, ψ^* is now represented by a solid cylinder and the "ring" by a solid circle or disk. This disk is transformed into itself (just as the ring in Section 8.5.3) according to

$$(\theta^*)' = -2\pi[(a^* + 1/a_2)^{3/2} + 1]^{-1} + \theta^*$$

and

$$(a^*)' = a^*,$$

i.e., the transformation is a pure rotation.

The outside circle of the ring of Section 8.5.3 with radius a_2 represents a direct orbit for a given C. This circle is transformed by the above equations to the zero circle, while the inside circle of retrograde orbits (a_1) becomes the boundary of the new disk.

Now a correspondence between the retrograde orbit for $\mu = 0$ (circle with radius a_1) and the retrograde orbit for $\mu \neq 0$ is established. Also let the states of motion ($\mu \neq 0$) inside a closed oval correspond to the states of motion ($\mu = 0$) inside a circle. To do this one might subject to continuous deformation the region of motion for $\mu \neq 0$ (i.e., the inside of the zero-velocity oval) without altering the vicinity of the singular point P_1, so that the zero-velocity curve for $\mu \neq 0$ becomes the zero-velocity circle for $\mu = 0$. In the process the retrograde orbit ($\mu \neq 0$) is also deformed and it becomes the retrograde circle ($\mu = 0$). It is known that according to Brouwer's fixed point theorem the transformation of the disk into itself for $\mu \neq 0$ will possess an invariant point. This will correspond to the direct orbit whose existence was the subject of the above discussion. For various values of C the totality of direct orbits of this type is obtained.

8.6.2 Periodic orbits of the second kind

Poincaré defines periodic orbits of the second kind as those which are generated from elliptic two-body orbits in the plane of the primaries:

$e \neq 0$, $i = 0$. The same initial conditions can be used as for periodic orbits of the first kind, i.e., at $t = 0$, $y(0) = 0$, $\dot{x}(0) = 0$. Also at the second crossing of the x axis we have $y(T_{syn}/2) = 0$, $\dot{x}(T_{syn}/2) = 0$, and so the period in the rotating system is T_{syn}.

The system of variables to be used here was described in Section 7.7 by Eqs. (96) and (97). For $\mu = 0$ this gives $L = \text{const}$, $G = \text{const}$, $\dot{l} = L^{-3}$, and $\dot{\bar{g}} = -1$, or equivalently $L = L_0 = a^{1/2}$, $G = G_0 = [a(1 - e^2)]^{1/2}$, $l = t/L_0^3 + l(0)$, and $\bar{g} = -t + \bar{g}(0)$. The initial conditions in terms of the new variables are simply conditions for an apocenter passage:

$$\bar{g}(0) = -\pi, \qquad l(0) = \pi.$$

At the second crossing the time is $t = T_{syn}/2$. The period in the fixed system is $T_{sid} = 2\pi L_0^3$, since the mean motion is $a^{-3/2} = L_0^{-3}$. The orbit will be periodic also in the rotating system if $T_{syn} = pT_{sid} = 2\pi q$, or $a^{3/2} = q/p$, as discussed in Section 8.2. So a pericenter passage takes place at $t = \pi q$ and

$$l(\pi q) = \pi(1 + p), \qquad \bar{g}(\pi q) = -\pi(1 + q),$$

which follow from the substitution of the initial conditions ($t = 0$) and the conditions for the second crossing ($t = T_{syn}/2$) in the solution of the equations of motion for $\mu = 0$. (Note that $q/p = l/k$ in the notation of Section 8.5.3.)

The functional determinant $\partial(l, g)/\partial(t, L)$ at $t = T_{syn}/2$, and $\mu = 0$ becomes $-3\pi p a^{-1/2} \neq 0$; therefore, the inversion of the functions in question can be performed and the requirement of the implicit function theorem is satisfied.

Care must be taken to avoid collisions using the above variables. Inasmuch as the apo- and pericenters of the orbits in question are on the x axis, where the primaries are also located, when $\mu \neq 0$ but close to zero, a singularity appears close to the point $x = -1$, $y = 0$. Excluding certain points of the plane (e, a) for which the solution loses its analytic character presents no problem.

Limitation of the extent of the analytic continuation of the orbits of the second kind is slightly more complicated than in the case of the orbits of the first kind. The situation described in Fig. 8.13 must be extended to a three-dimensional plot; that is, for a given (n, e), or equivalently for a set (C, e), the continuation of the given two-body ellipse from $\mu = 0$ to $\mu \neq 0$ is to be investigated.

For a discussion of the considerable literature the reader is referred to the last section (8.9) of this chapter.

8.6.3 Symmetry property of solutions

The selection of the initial conditions for the periodic orbits of the first and second kind discussed in Sections 8.6.1 and 8.6.2 requires additional comments. The equations of motion of the restricted problem have the property that if $x = x(t)$, $y = y(t)$ is a particular solution, then $x = x(-t)$, $y = -y(-t)$ is also a solution. This follows from inspection of the six terms occurring in the differential equations

$$\ddot{x} - 2\dot{y} = \Omega_x,$$
$$\ddot{y} + 2\dot{x} = \Omega_y,$$

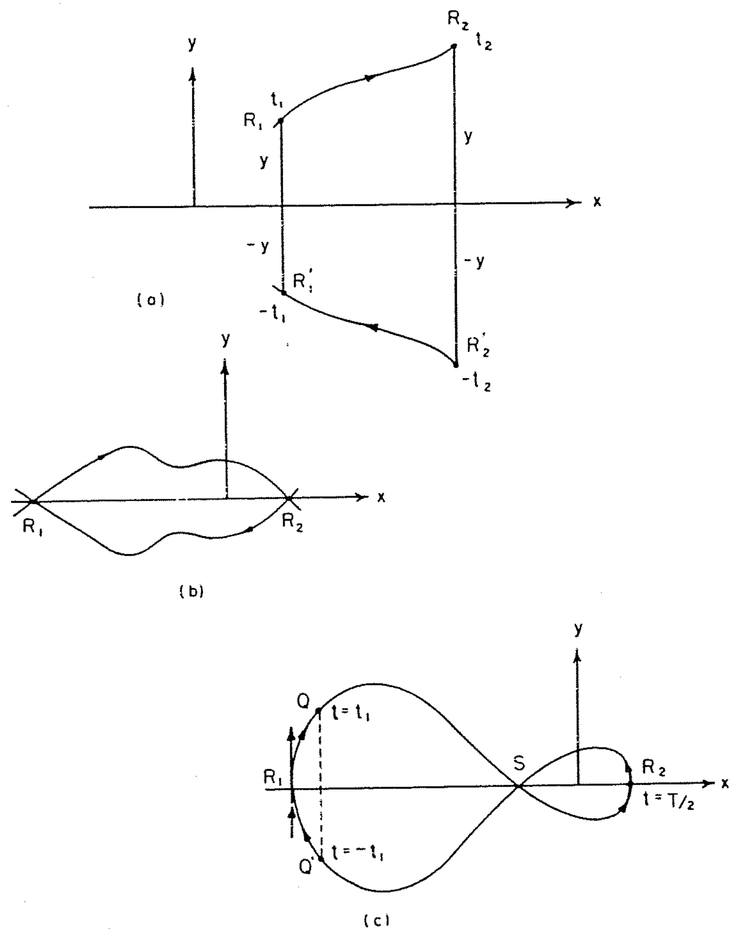


FIG. 8.14. Image trajectories.

where Ω depends on x and y^2 only, and consequently

$$\Omega(x, y) = \Omega(x, -y),$$
$$\Omega_x(x, y) = \Omega_x(x, -y),$$
$$\Omega_y(x, y) = -\Omega_y(x, -y).$$

Writing $-t$ for t, x for x, and $-y$ for y, $\ddot{x}$ becomes $\ddot{x}$, $\dot{y} \to \dot{y}$, $\Omega_x \to \Omega_x$, $\ddot{y} \to -\ddot{y}$, $\dot{x} \to -\dot{x}$, and $\Omega_y \to -\Omega_y$, which transitions verify the preceding statement.

The geometrical meaning of this is shown in Fig. 8.14(a). At $t = t_1$, the particle is at $R_1[x(t_1), y(t_1)]$, moving with $\dot{x} > 0$, $\dot{y} > 0$, as shown by the arrow, to point $R_2[x(t_2), y(t_2)]$. If this curve represents an orbit, i.e., a solution of the equations of motion, then its mirror image ($y \to -y$) with respect to the x axis is also a solution with $t \to -t$. On this second orbit the particle moves from $R_2'[x(-t_2) = x(t_2), \; -y(t_2) = y(-t_2)]$, with $\dot{x} < 0$, $\dot{y} > 0$ as shown by the arrow, to point $R_1'[x(-t_1) = x(t_1), \; -y(t_1) = y(-t_1)]$. Note that $\dot{x}$ changes sign going from the original curve to its image but $\dot{y}$ does not. Also $t_2 > t_1$ and so $-t_1 > -t_2$; the two curves are described in opposite sense.

One consequence of this symmetry property is that if R_1 and R_2 are on the x axis their mirror images will coincide with the originals [see Fig. 8.14(b)] and computation of a "return" orbit becomes superfluous since once a trajectory from R_1 to R_2 has been established it can be used for a trajectory from R_2 to R_1 also. Note that by this property of the solution all consecutive collision orbits computed and shown in Figs. 3.14, 3.15 and 9.35 can be reversed. The original orbits were established from the earth to the moon. Reflecting these orbits around the x axis they will represent trajectories from the moon to the earth, as pointed out by Miele.

Let now at point R_1 on the x axis [Fig. 8.14(c)] the velocity be perpendicular to the x axis. The initial conditions then at R_1 are $x < 0$, $y = 0$, $\dot{x} = 0$, $\dot{y} > 0$. There is another orbit, which is the mirror image of the orbit R_1Q, coming from below the axis of syzygies and occupying point Q' at $t = -t_1$. Any time the extension of the curve R_1Q intersects the x axis, the extension of its image curve (R_1Q') will also have a zero ordinate; see point S. (In fact we speak of extension in time of the R_1Q curve representing the orbit starting at $t = 0$ at point R_1, and correspondingly we have regression when point Q' is obtained.)

When the orbit intersects the x axis perpendicularly [see point R_2 on Fig. 8.14(c)], its image must do the same and a periodic orbit is established. If the intersection at S is perpendicular, a simple periodic orbit is established. The condition for the existence of a periodic orbit is therefore to have two perpendicular crossings. This is a sufficient condition but not a necessary one since periodic orbits are known to exist without perpendicular crossings and without symmetry about the x axis. If an orbit intersects the x axis at least twice perpendicularly, it must be a symmetric periodic orbit (with respect to the x axis). A periodic orbit, on the other hand, which is not symmetric to the x axis cannot have two perpendicular crossings. (See also Section 9.9.3.)

8.7 Whittaker's criterion for the existence of periodic orbits

Hamilton's principle [see Eq. (1), Section 6.2] according to which the expression

$$\int_{t_1}^{t_2} L\,dt$$

has a stationary value on an actual trajectory when applied to closed orbits, furnishes a condition for the existence of periodic orbits. When the kinetic potential (L) does not contain the time explicitly, Hamilton's principle can be written as

$$\delta \int_{t_1}^{t_2} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\,dt = 0. \tag{30}$$

The above integral is called the action and the principle of least action is applicable. In Eq. (30) the summation convention is operative. It can also be written as

$$\delta \int_{t_1}^{t_2} T\,dt = 0$$

if the potential function is independent of the velocities and the kinetic energy is a quadratic function of the velocity components $T = \frac{1}{2}a_{ij}\dot{q}_i\dot{q}_j$. Finally, changing the independent variable from the time to the arc length along the orbit by means of

$$ds = (a_{ij}\dot{q}_i\dot{q}_j)^{1/2}\,dt,$$

we have

$$\delta I = \delta \int_{s_1}^{s_2} T^{1/2}\,ds = \delta \int_{s_1}^{s_2} (\Omega - C/2)^{1/2}\,ds = 0 \tag{31}$$

as the final form of the variational principle applicable to conservative dynamical systems. Here Ω is the force function (negative potential)

and C is twice the negative value of the energy constant, so that the integral of energy becomes

$$\Omega - T = C/2.$$

To be specific, consider a conservative dynamical system with two degrees of freedom described by the equations of motion

$$\ddot{x} = \Omega_x \quad \text{and} \quad \ddot{y} = \Omega_y.$$

The energy integral is $\dot{x}^2 + \dot{y}^2 = 2\Omega - C$.

It may be shown that the expression determining the sign of δI becomes

$$W = \frac{2\Omega - C}{\rho} + \Omega_x \cos\gamma + \Omega_y \sin\gamma, \tag{32}$$

where ρ is the radius of curvature of a closed curve and γ is the angle between the normal of the curve and the x axis.

With Eq. (32) Whittaker now establishes a criterion for the existence of periodic orbits. Let one closed curve (A_1) be enclosed by another one (A_2) and evaluate W on both curves. If $W(A_1)$ and $W(A_2)$ are of opposite sign, and there is no singularity in the ring between A_1 and A_2, then there exists a periodic orbit in the ring and its energy constant is C.

Example. Show that with $\Omega = 1/r$, i.e., for the problem of two bodies,

$$W(A_1) = \frac{1}{r_1}\left(\frac{1}{r_1} - \frac{1}{r_0}\right)$$

and

$$W(A_2) = \frac{1}{r_2}\left(\frac{1}{r_2} - \frac{1}{r_0}\right),$$

where A_1 and A_2 are concentric circles of radii r_1 and r_2 with center at the origin and r_0 is the radius of the periodic circular orbit between A_1 and A_2 with $C = 1/r_0$.

Generalization of this method to the restricted problem results in a slightly different equation for W:

$$W = \frac{2\Omega - C}{\rho} + \Omega_x \cos\gamma + \Omega_y \sin\gamma + (2\Omega - C)^{1/2}. \tag{33}$$

The "potential function" in this case does depend on the velocities so both the derivation and the result are different for the restricted problem. The major deviation is that the principle of least action is formulated with a modified Lagrangian function (see Section 8.9 for the references).

8.8 Characteristic exponents

Let

$$\frac{dx_i}{dt} = X_i \qquad (i = 1, 2, \ldots, n) \tag{34}$$

represent a system of ordinary differential equations with the right-hand sides (X_i) being functions of $x_1, \ldots, x_n$ and either *not* containing the time explicitly or containing only periodic functions of t.

Let

$$x_i = \varphi_i(t) \tag{35}$$

represent a periodic solution. In order to establish the variational equations, let

$$x_i = \varphi_i(t) + \xi_i \,. \tag{36}$$

The variational equation belonging to Eq. (34) is

$$\frac{d\xi_i}{dt} = \sum_{j=1}^{n} \frac{\partial X_i}{\partial x_j} \xi_j \,, \tag{37}$$

where the coefficients

$$\partial X_i/\partial x_j$$

are evaluated by substituting $x_i = \varphi_i(t)$. Therefore Eq. (37) represents a system of linear differential equations with coefficients which are periodic functions of the time.

By the theory of Fuchs and Floquet, the general solution of Eq. (37) for n different α_j is

$$\xi_i = \sum_{j=1}^{n} S_{ij}(t)\, e^{\alpha_j t}, \tag{38}$$

where the α_j = constants are called the characteristic exponents of Poincaré and the periodic functions S_{ij} have the same period as $\varphi_i(t)$.

Note that if the variation is applied to a stationary solution ($x_i = x_i^0 =$ const), then $S_{ij} =$ const. The variational equation is always linear, its coefficients can be constant or can be functions of the independent variable, depending on whether the varied solution is stationary or not.

Consider now the initial conditions for the variation:

$$\xi_i(0) = \beta_i \,, \tag{39}$$

and consequently

$$x_i(0) = \varphi_i(0) + \beta_i.$$

A period later (at time T) the variations assume the value

$$\xi_i(T) = \beta_i + \psi_i \,. \tag{40}$$

The solution of the variational equations is not periodic in general and certainly not with a period T,

$$\xi_i(0) = \beta_i \neq \xi_i(T) = \beta_i + \psi_i \,;$$

nevertheless, a given set of the β_i's will determine the set of the ψ_i's since the β_i's determine the solution (38) of the set of linear differential equations in question. In fact, if $\beta_i = 0$ then $\psi_i = 0$, and the "generating solution" $[\varphi_i(t)]$ is not varied. An expansion of $\psi_i = \psi_i(\beta_j)$ around $\beta_j = 0$ gives

$$\psi_i = \sum_{j}^{n} \frac{\partial \psi_i}{\partial \beta_j} \beta_j + \sum_{j,k=1}^{n} \frac{\partial^2 \psi_i}{\partial \beta_j\, \partial \beta_k} \frac{\beta_j \beta_k}{2!} + \cdots. \tag{41}$$

This expansion is often referred to as Poincaré's lemma.

A method of determining the jth characteristic value is as follows. The particular solution corresponding to this characteristic value is

$$\xi_i = S_{ij}(t)\, e^{\alpha_j t} \tag{42}$$

for $i = 1, 2, \ldots, n$ and for a fixed value of j.

A relation between β_i and ψ_i can be established because $S_{ij}(t) = S_{ij}(t + T)$. In fact, from Eqs. (40) and (42) we have

$$\beta_i + \psi_i = \xi_i(T) = S_{ij}(T)\, e^{\alpha_j T} = \beta_i\, e^{\alpha_j T}, \tag{43}$$

since

$$S_{ij}(T) = S_{ij}(0) = \xi_i(0) = \beta_i$$

by Eqs. (39) and (42).

Substituting Eq. (41) into (43), the following system of equations is obtained for $i = 1, 2, \ldots, n$ and *for a fixed value of j*:

$$\beta_i(1 - e^{\alpha_j T}) + \sum_{k=1}^{n} \frac{\partial \psi_i}{\partial \beta_k} \beta_k = 0. \tag{44}$$

This system of homogeneous equations possesses solutions different from the trivial solution if the associated determinant is zero. The terms in the main diagonal (ith row and ith column) of the determinant in question are

$$\frac{\partial \psi_i}{\partial \beta_i} + 1 - e^{\alpha_j T},$$

where there is, of course, no summation over i. The terms in the ith row and kth column ($i \neq k$) are $\partial\psi_i/\partial\beta_k$.

The determinant, when expanded, leads to an nth-order algebraic equation for $e^{\alpha T}$ which gives the values of the characteristic exponents (since T is known) if the derivatives $\partial\psi_i/\partial\beta_k$ are known. If the generating solution is known only numerically the coefficients $\partial\psi_i/\partial\beta_k$ are obtained by numerical integration.

Example. Show that when the differential equations form a Hamiltonian system, i.e., when $x,..., x_{2n} = q_1,..., q_n, p_1,..., p_n$, the characteristic exponents of a periodic solution form pairs, which are equal in magnitude but opposite in sign.

When the system of equations is autonomous (the time does not appear in the functions X_i) then one of the characteristic exponents is zero, and when the system possesses a uniform integral then in general another characteristic exponent is zero.

The sign of the real part of the characteristic exponent determines the stability (in the linear sense) of the periodic solution. Since the roots occur in pairs for a Hamiltonian system, the existence of a characteristic exponent with nonzero real part immediately indicates instability. The necessary condition for stability of a periodic orbit is that all the characteristic exponents must be pure imaginary.

Poincaré's approach is, therefore, essentially a linear analysis. The dynamical system called the restricted problem of three bodies is a Hamiltonian system with two degrees of freedom. Consequently, the characteristic exponents are $(0, 0, \alpha, -\alpha)$. If the real part of α is not zero, the periodic orbit is unstable; a result which may be extended to the nonlinear system and so it is of importance. If the real part of α is zero, the solution of the variational equation may be expressed by trigonometric functions of the time. This result in the linear sense means stability, for the nonlinear system, however, it is of less significance and it is what is termed in the literature the critical case.

Even in this case one should not dismiss the general importance of the characteristic exponents. For a family of periodic orbits the characteristic exponent may serve as a parameter varying (continuously) during the development of the family. This application does not depend on whether the characteristic exponent has significance for the question of stability in the nonlinear sense or only for the linear system. Parameters describing a family of periodic orbits are needed for a systematic treatment of these families. To decide which parameters are superior requires a combined experimental (numerical) and analytic approach, and even after such a study shows the advantages of certain parameters over others in case of a given family, the results may not be readily transferred to another family of periodic orbits. Parameters with invariant properties are in general favored and unfortunately the characteristic exponents depend on the choice of the variables. On the other hand periodicity is not an invariant property with respect to coordinate transformations; consequently, the fact that the characteristic exponents may indicate (linear) stability or instability, depending on the choice of variables, does not necessarily negate their significance regarding classification of periodic orbits.

Another, unquestionably more concise but more restricted formulation of the variational problem of periodic solutions is to establish the equation of normal displacement and investigate the stability by means of this equation. As a matter of fact the equations of motion of the restricted problem can be transformed isoenergetically from the system x, y into the system u, v, as we have seen in Chapter 3. Applying the results obtained we have for the infinitesimal normal displacement δn the following variational equation:

$$\frac{d^2\delta n}{dt^2} + \Theta\,\delta n = 0, \tag{45}$$

where

$$\Theta = 1 + 3(\dot{z} + 1)^2 - \frac{\partial^2\Omega}{\partial n^2}, \tag{46}$$

or

$$\Theta = \frac{1}{v}\frac{d^2v}{dt^2} + 2 + 2(\dot{z} + 1)^2 - \Omega_{xx} - \Omega_{yy}. \tag{47}$$

In Eqs. (46) and (47) the symbol z represents the angle between the tangent to the orbit and the positive x axis (cf. Chapter 2), and $v^2 = \dot{x}^2 + \dot{y}^2$. The function Θ is to be evaluated along the orbit, consequently $\Theta = \Theta(t)$, and if the orbit under investigation is periodic, so will be $\Theta(t)$. In fact, in this case the general solution can be written [similarly to Eq. (38)] in the form

$$\delta n = S_1(t)\,e^{\alpha t} + S_2(t)\,e^{-\alpha t},$$

where $S_i(t)$ has the same period as $\Theta(t)$.

There are other methods available to study the stability of periodic orbits. This question is discussed again in Section 9.10, where a different method is described for the evaluation of α. This is based on the intersection of the orbit with the surface of section, and the stability of the fixed point of the transformation, representing the periodic orbit, is of central concern.

8.9 Notes

The pertinent references of this chapter are Poincaré's "Méthodes Nouvelles" [1] and Birkhoff's famous Rendiconti paper [2]. The latter presents the foundation to Klose's work [3] which the reader will find easy and enjoyable reading material. Birkhoff's paper is also the starting point of Merman's major memoir [4].

Poincaré's conjecture (Section 8.1) regarding the denseness of periodic orbits is in Vol. 1, p. 82, of his *Méthodes*, while Schwarzschild's version can be found in two of his papers [5]. Another probably more questionable conjecture of Darwin is in his famous first paper on periodic orbits [6]. While attractive, Darwin's conjecture regarding the organization of orbits by means of periodic orbits as class divisors suffers from lack of preciseness. How do we know which orbit—among the "periodic" orbits—is also periodic and which one is not? The situation is not identical with, but certainly shows similarities to, the original idea of Burrau, according to which classes of periodic orbits are separated by collision orbits. It is well established today that if Strömgren's classes were to be defined according to Burrau's principle then neither would Strömgren's termination principle be operative nor would the simplicity of his classification be available. More about this appears in Chapter 9.

The basic papers on almost-periodic functions [Section 8.2, Part (C)] are by H. Bohr [7] and on quasi-periodic functions by Bohl [7a]. An excellent review is available by Jessen [8].

The literature on the problem of small divisors [Section 8.2, Part (D)] still awaits an analytic evaluation and so it will not be reviewed here. The essential idea is stated by Poincaré as follows: "Difficulties in celestial mechanics due to the existence of small divisors and the associated approximate commensurabilities of the mean motions are connected with the actual nature of things and cannot be circumvented." This view is shared by Brouwer [9] and by Garfinkel [9a] who, in their now classical artificial satellite theory, and in other publications, also point out the physical importance, in addition to the mathematical significance, of small divisors. Whittaker's dissertation on the adelphic integral [10] is another clear light. The classical and often over-simplified presentation by Moulton, Brown, and others, emphasizing the "practical" aspects, such as the simultaneous appearances of high powers of the inclinations and of the eccentricities as small factors together with small divisors, is not helpful in showing convergence but is certainly useful in solving actual problems of astronomy. It is hoped that the presentation in this chapter [Section 8.3,P art (D)] of some ideas on perturbation puts this problem in a light equally acceptable to the astronomically and to the mathematically oriented reader.

Poincaré presents the recurrence theorem [Section 8.2, Part (F)] in two of his works [11], while Birkhoff [12] gives the precise definition. One of the basic papers is by Hopf [13]. Chandrasekhar [14] and Frisch [15] discuss the general and the more specific questions, respectively, of the fundamental relations between reversibility and recurrence. The proof that a mechanical normal system is, in general, quasi-ergodic [Section 8.2, part (H)] is given by Fermi [16]. Lord Kelvin's comments on these questions are strongly recommended [16a]. The question of *integrable* versus *integrated* dynamical systems is of utmost significance. Two of the primary contributors to the mathematical aspects of modern celestial mechanics in the 1960's, V. I. Arnol'd and J. Moser, discuss this matter which can be looked at as the fundamental problem involved in A. N. Kolmogorov's theorem [Section 8.3, Part (C)]. There are two papers by Kolmogorov announcing and explaining his theorem but not offering rigorous proof [17]. Several papers by Arnol'd [18] are of interest regarding Chapter 8. One of his major works [19], a 100-page article, is probably the most important, in addition to the publication of his proof of Kolmogorov's theorem [20] and his discourse on circles mapped into themselves [21].

Moser's [22] proof of his mapping theorem [Section 8.3, Part (A)] and his other papers [23] and [23a] demonstrate the mathematical and astronomical significance of his work; see as background material a fundamental paper by Denjoy [23b].

Poincaré's last geometric theorem [Section 8.3, Part (B)] appeared without proof [24]. Birkhoff has given a proof [25], and has made additional remarks [26] and an extension of Poincaré's theorem [27].

The ideas presented in Section 8.4 on analytic continuation follow Siegel's treatment [28, p. 118], which in turn is closely connected with Poincaré's discourse [1, Vol. 1, p. 79]. Charlier's approach [29, Vol. 2, p. 187] is also based on Poincaré's treatment. Siegel's work is preferred. For L. E. J. Brouwer's fixed point theorem see [30].

The two-body considerations of Section 8.5 follow Birkhoff's work [2]. The identification of Eq. (18) with the Hamiltonian [see Eq. (90), Section 7.6] is due to Dr. W. Jefferys (private communication). Figure 8.12 was prepared by D. Kimball of the Yale Observatory.

Orbits outside those closed curves of zero velocity which enclose both primaries are discussed by Koopman [31] and Coe [31a]. In Koopman's work a surface of section is introduced for these outside orbits, similar to the Birkhoff–Poincaré surface of section, applicable to orbits near and around one of the primaries. Since the curves of zero velocity give only a lower bound for the outside orbits, the corresponding surface of section extends to infinity. In the torus representation this results in a rather complex body, the shape of which was studied by Szebehely [32].

The basis of these considerations is contained in Birkhoff's book [12] and in one of his papers [33]. The pertinent passages in Poincaré's *Méthodes* are in Vol. 3, pp. 196 and 372.

Distinction between direct and retrograde motion is feasible only in extremely simple cases. Global characterization is possible in the average when the mean motion of the particle is compared with the motion of the synodic system (cf. Section 8.5.1). Local (i.e., in this case instantaneous) characterization of the direction of the motion is discussed at some extent by Wintner in his book [34, p. 224]. An extremely thorough presentation of the problem of two bodies and the critical values of the Jacobian constant (corresponding to Section 8.6.1) are also given in well-organized tabulated form in the same book [34, p. 232].

Wintner's paper [35] discusses the pertinent questions about Fig. 8.13 of Section 8.6.1. This rather lengthy paper structures the existing meager knowledge regarding the limitations of the method of analytic continuation. The curve of Fig. 8.13 must be obtained experimentally, i.e., by numerically establishing periodic orbits for increasing values of μ beginning with $\mu = 0$. It will be shown in Chapter 9 that the periodic orbit given in Fig. 8.10 (prepared by S. Knowles) can be continued to approximately $\mu = 0.24$, i.e., that the surface $\mu = \mu(C, e)$ above the C, e plane mentioned at the end of Section 8.6.2 has the coordinates $C \cong 3.4$, $e = 0.3$, $\mu \cong 0.24$. The corresponding presentation using $\mu = \mu(n, e)$ gives $\mu \cong 0.24$, $n = 3$, $e = 0.3$.

This paper by Wintner [35] attacks—among others—Poincaré and his theory of establishing various kinds of periodic orbits. While everything stated by Wintner is correct, I think his criticism may be interpreted as missing Poincaré's basic principle. As mentioned before, Poincaré's restricted problem is concerned only with small values of μ and with orbits which can be continued from two-body orbits. In other words, his classification does not necessarily have meaning when μ is not small.

The example of Section 8.6.1 regarding the derivation of Eq. (27) is worked out by Birkhoff [2, p. 33]; the reader, nevertheless, will probably be dissatisfied with Birkhoff's derivation and will try to improve it. See also Siegel [28, p. 128], and Wintner [34, p. 226].

The principal reference regarding the dynamics of the Trojans is Brown's paper [36]; the Hecuba group is discussed by Poincaré [35b, Vol. 1, p. 35] and his method is applied to the Hilda group by Schubart [38].

The existence proof given for periodic orbits of the second kind (Section 8.6.2) follows Barrar's article [39] employing Delaunay's variables. Arenstorf [40] gave a proof using Cartesian rectangular

coordinates, which approach allows the drawing of an intellectually most satisfying parallel between his proof of existence of the second kind periodic orbits and Birkhoff's proof for the first kind. In general, of course, it is known that the method of proof is directly responsible for the results obtained. The use of Delaunay variables establishes periodic orbits with the exception of collision orbits, while the use of rectangular coordinates excludes orbits for which $e = (1 - a^{-3})^{1/2}$, $a > 1$, as shown by Arenstorf. Both proofs are based on analytic continuation and therefore result in orbits with "sufficiently small" values of μ.

Previous attempts to show the existence of periodic orbits of the second kind failed because a different periodicity condition was used. The importance of using symmetric conditions was recognized by Birkhoff in his 1915 Rendiconti paper. Prior to this work Poincaré [1, Vol. 1, p. 139], Schwarzschild [5], and Charlier [29, Vol. 2, p. 215] attempted to use a different periodicity condition. Their error was pointed out by Stäckel [41] and Wintner [35].

The existence of periodic orbits which close only after several revolutions has been shown by Birkhoff [33] and Moser [42] using fixed point theorems.

Various, numerically established, families of periodic orbits are not taken up here but will be presented in detail in Chapter 9. Periodic orbits of modified forms of the restricted problem such as the three dimensional case, elliptic motion of the primaries, and Hill's problem will be discussed in Chapter 10.

A discussion of the symmetry property (Section 8.6.3), also known as the "image method," was given by Miele with applications to lunar trajectories [43]. The symmetry property of the differential equations of motion is known since the publication of Euler's second lunar theory (1772).

The proof of the criterion for the existence of periodic orbits [Eq. (32) of Section 8.7] is given by Whittaker [44, p. 387]; its extension to the restricted problem [Eq. (33)] is also given by him [44]. See also Tonelli's [45], Signorini's [46], Birkhoff's [47], and Plummer's [48, p. 248] comments on Whittaker's results. The simple proof of Eq. (30) of Section 8.7 is given in connection with Eq. (143) of Section 10.5.2, Part (D).

The problem of characteristic exponents (Section 8.8) is discussed by Whittaker [44, p. 397]. His presentation follows Poincaré's book [1, Vol. 1, p. 174] and mémoire [11, especially p. 88]. See also Ince [49, p. 382] and Cesari [49a, p. 66]. The example of Section 8.8 is from Poincaré [1, Vol. 1, p. 174].

The differential equations for the normal displacement [Eqs. (45)–(47)] are derived in several standard reference texts (see, e.g., Plummer

[48, p. 245]). The derivation alluded to in the text (Section 8.8) is given by Birkhoff [33]. Other forms are given by Message [50]. A discussion of the relation between the second-order equation for the normal displacement and the fourth-order variational equations is given by Wintner [51]. The variational equations using the Thiele–Burrau regularizing variables were given by Rosenthal [52]. The physical picture is, of course, weakened by the fact that the method of characteristic exponents and of normal displacements is inherently a linearized process, which furnishes often only necessary conditions for stability. The mathematical problem is the solution of Hill's differential equations. In actual problems the nature of the characteristic roots is of interest only and Floquet's method is available (see, e.g., Danby [53]). No attempt is made to review the problem of stability in celestial mechanics or of the solutions of the restricted problem. Only one of the key references (which contains several hundred additional ones) is mentioned here by Hagihara [54]. The reader's attention once again is directed to Section 5.2 where several theorems are given which are applicable to the stability problem of periodic orbits. (The references of Section 5.2 especially regarding nonlinear stability problems are also applicable.) The significance of characteristic exponents goes beyond the stability problem, and it is not inconceivable to expect that organization of families of periodic orbits is greatly enhanced by values of the characteristic exponents as shown by Hénon [55], who also settled the linear stability question of the Copenhagen problem (see Section 9.10).

Chapter 9 on the quantitative aspects of the restricted problem offers alternative methods to evaluate the stability characteristics of periodic orbits (see especially Sections 9.10 and 9.11).

As a general reference the reader will find Pars' [56, p. 604] passages useful.

Inasmuch as this chapter discusses in considerable detail the problem of two bodies, mention of the hodograph method should be made. The mapping of the configuration space into velocity space or in general into any other vector space is today considered the hodograph method. The hodograph itself may be defined as the map of the trajectory, most often in the velocity space. The originator of the idea seems to be Mobius [57, p. 36] who mentions it even before Hamilton [58]. Tisserand [59, Vol. 1, p. 96] discusses two-body hodograph equations in connection with Laplace's [60] integrals. Regarding applications to space dynamics Altman's first [61] and latest [62] papers, Pistiner's [63] work, and Eades' [64] report should be mentioned.

Applications of Poincaré's, Birkhoff's, Arnold's, and Moser's approaches to a variety of problems are offered in several papers by Diliberto, Kyner, and Freund, of which [65] and [66] are examples.

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Chapter 9

Numerical Explorations

9.1 Introduction

One of the most important modern trends in dynamics is the extensive use of high speed electronic computers as experimental tools. It seems to be proper to refer to *experiments* because of the similarity of the processes of computational dynamics to experiments in the physical sciences. Consider a complex dynamical system (say, the restricted problem) and investigate the behavior of this system, regarding periodic motion. If a family of periodic orbits of this system exists, then often numerical integration must be used to find the family when the system is of any difficulty. On the other hand there are large classes of periodic orbits which defy the application of presently existing analytical existence proofs. Nevertheless, one may be able to find orbits which, within the limitations of numerical accuracies, show periodicity. Such experimental findings then may suggest that analytical attempts be made to show the actual existence of these orbits.

The major opportunity offered by experimental dynamics is to show numerically certain properties of a given dynamical system. After this, an analytical treatment can follow to actually prove the existence of the experimentally found behavior.

Besides this power of discovery which numerical methods possess, they also serve immediate engineering needs. Systems engineering is concerned with the totality of possible orbits in order to be able to make the proper choice of a predesigned flight path. As long as closed form (and preferably simple) analytical expressions do not exist to represent the solution of complicated dynamical systems, the computer solution is the only existing approach which can provide an orbit for a certain mission.

The aim of dynamics is to characterize dynamical systems by describing the totality of mótions and discussing their properties; to fulfill this purpose, numerical integration is one of the powerful tools. In the early 1930's two classical analysts, Sundman and Wintner spoke strongly in favor of numerical experimentation in connection with the problem of three bodies, pointing out that past success and further progress are both associated with numerical work. But the most outstanding example showing the importance of numerical undertakings is Elis Strömgren's and the Copenhagen school's work performed between 1913 and 1939.

The presentation of the quantitative results of the restricted problem can be organized in several ways; a chronological structure, an organization with respect to the value of the mass parameter used, or an arrangement according to the nature of orbits obtained, are various possibilities. Nevertheless, the basic idea is the following. A point in the phase space with coordinates x_0, y_0, $\dot{x}_0$, $\dot{y}_0$ will determine an orbit (singularities can be considered to have been eliminated). Therefore, the ∞^4 orbits representing the totality of possible motions can be described and organized for a fixed value of μ. The state of knowledge in 1967 is far from such a level where this presentation would be feasible for any value of the mass parameter with the exception of $\mu = 0$. The actual situation is that almost all of those orbits for which analytical existence proofs have been given, have been found and established numerically (frequently before the proof was offered). This remark refers especially to orbits with special properties, such as periodic or asymptotic orbits, for it is well known that after the elimination of the singularities the existence of solutions in general is settled.

The classification of numerically established orbits can be made simple by considering only two types: one group with analytic origin, the other stemming from practical requirements. In the first group belong all periodic, almost periodic, and asymptotic orbits which have well defined analytic character. Most of these have been the subject of analytic existence proofs. The second group consists of special-purpose, nonperiodic orbits which satisfy certain special requirements, such as connecting two points in the x, y plane (this is referred to as the solution of the two point boundary value problem) or establishing a connection between two special orbits (i.e., offering a transfer orbit), etc.

Both types are affected by the value of the mass parameter. This effect in general is not known, nevertheless, two aspects may be of

importance. First, it is known that there are discrete values of the mass parameter at which either certain special orbits exist, or the character of a certain group of orbits changes. Examples are orbits connected with the libration points. Second, as was shown previously, analytic continuation shows the existence of certain types of periodic orbits for small values of μ. A number of classes of periodic orbits exist for $\mu = 0$ which persists for small values of μ and also for $\mu = \frac{1}{2}$. So the effect of μ on certain classes of orbits is only qualitative and the set of orbits shows topological invariance regarding changes in μ. This is the case in the majority of the classes of orbits established for $\mu = \frac{1}{2}$. Small-amplitude periodic orbits which exist around the triangular libration points for $\mu < \mu_0 = .0385\ldots$ will become asymptotic orbits for $\mu > \mu_0$, and their character will change completely. Consequently such orbits will not survive the continuation process. On the other hand the value of the investigations at $\mu = \frac{1}{2}$ is especially great since many of the astronomically important orbits can be established for all values of μ, from $\mu = 0$ to $\mu = \frac{1}{2}$.

A possible method of classifying orbits, therefore, is using the value of the mass parameter as the criterion. This principle has many practical merits since in actual cases the value of the mass parameter is given when the astronomical problem is defined. Such a classification is also natural from a scholastic point of view since most of the literature is comprised of papers containing established families for one fixed value of μ.

In what follows, special attention will be given to two classifying aspects. We will carefully distinguish between orbits of analytical and of practical origin (roughly speaking between periodic and asymptotic orbits on one hand and the so called single purpose orbits of engineering interest such as earth-to-moon trajectories on the other hand). Also attention will be given to the value of the mass parameter as a classifying criterion. A double classification results, which, most of the time, interestingly enough, is also consistent with a chronological organization. Upon accomplishing such a set of presentations for different and fixed values of the mass parameter, a cross-review will be offered to show the effect of μ on the various families.

A characteristic problem of nomenclature exists regarding various types of orbits. The use of the terms group, family, class, kind, etc. to describe a set of orbits with some common property is not at all uniform in the literature. Furthermore, inasmuch as the various terms in the international literature do not transliterate uniformly, the confusion is complete; *Art*, *Gattung*, *Klasse*, *Gruppe*, *sorte*, *genre*, *famille*, *espèce*, just to mention a few can be translated into each other and into many possible English equivalents. Instead of introducing one additional system of nomenclature, these terms are used interchangeably. Definitions of the members of certain "families" are often questionable and quite arbitrary, designed only to satisfy the discoverer's theorem about his orbits. The basic principle of course is simply that an ∞^5 number of orbits, corresponding to the four initial conditions and to the value of the mass ratio, is to be organized in a fashion which reveals their basic properties. This has not been successful up to now. Not even the totality of periodic orbits has as yet been described.

In such a state of affairs, the present chapter can show only glimpses of the totality of orbits and attempt to organize the existing incomplete material in a systematic fashion. If the reader finds some members of a certain group of orbits missing, if members of two groups overlap or form gaps, or if certain classifications seem artificial, in short, if the reader finds that the problem of presenting the totality of existing orbits has not been solved, he is right; furthermore, with his fresh discovery he should feel free either to establish a new family of orbits of his own or to attempt to reorganize the material.

In addition to the previously mentioned nonuniform terminology regarding the various categories of orbits, the highest possible variety exists in the literature regarding the choice of notation, of units, and of coordinate systems. Our response to the confusion in matters terminological will be a most liberal attitude; in matters graphical we will adhere to the original drawings since these originals will be found in the references. Finally, in matters analytical our response will be rigidity in following the notation and units selected and used throughout this book.

The reader may find that emphasis is placed on periodic orbits—once again. The reason is that when numerical integration is employed the motion is known only within the interval of time for which the calculation has been carried out—unless it should prove that the motion is periodic (or asymptotic). In this case orbits are known for all times, future and past.

9.2 Organization of existing numerical results

(A) The first complete set of numerical results was published by members of the Copenhagen Observatory under the direction of Elis Strömgren during the period 1913–1939. The mass parameter for this work has the value of $\frac{1}{2}$ (with a few exceptions); i.e., the masses of the primaries are equal. Prior to this undertaking Sir George Howard Darwin from 1897 to 1910 using $\mu = 1/11$ (or in fact in our notation $\mu = 10/11$, which change interchanges the primaries with respect to the mass center) computed several systematic but incomplete (in the

Strömgrenian sense) families of periodic orbits. About the same time (1900–1917) F.R. Moulton's school performed analytical and numerical work, using in the latter mostly $\mu = 0.5$ and $\mu = 0.2$.

The above three undertakings all use values of the mass parameter between approximately 0.1 and 0.5 and the majority of these computations took place prior to the 1930's. All three aimed at establishing sets of periodic orbits, mostly by numerical but also by analytical means. For these reasons the Strömgren–Darwin–Moulton results will form the first group of investigations to be presented. The reference to *the Copenhagen category* seems to be justified when discussing this group because of the overwhelming superiority regarding completeness of Strömgren's groups of orbits.

(B) The second category that we shall recognize is characterized by $\mu \cong 0.012$, i.e., by a value of the mass parameter an order of magnitude smaller than that used by the first group of investigators. The value of the mass parameter for the earth and the moon as primaries is

$$\mu = \frac{1}{82.302 \pm 0.005},$$

corresponding to $\mu \cong 0.01214$ to $\mu \cong 0.01215$.

The first systematic investigation of periodic orbits for the earth–moon system can be found in the Soviet literature by V.A. Egorov (1957). A more complete and in certain aspects more extensive work appeared later (1962) as a thesis at the University of Louvain by R. Broucke. Rather special types of periodic orbits were studied by R. Newton (1959), S. Huang (1962), and R. Arenstorf (1963).

The contributions, therefore, in the second category, are all concerned with the earth-moon model and correspondingly with a value of μ which is approximately 0.012. They establish (often incomplete) families of periodic orbits, and appear in the literature during the period 1957–1963. The orbits of this group are best represented by the description *periodic lunar orbits.*

(C) Periodic orbits around the triangular libration points form the third category. Such orbits as continuations of the infinitesimal elliptic orbits are discussed in considerable detail in Chapter 5. The two values of the mass parameter for which such families are treated here are $\mu \cong 0.01214$ and $\mu \cong 0.0009538$. The first corresponds to the earth-moon, the second to the sun-Jupiter system. The reader is referred to Chapter 5 for a comprehensive discussion and should consider this section as a collection of some examples.

(D) The fourth category contains nonperiodic orbits computed for the previously mentioned value of $\mu \cong 0.012$. The practical implications of these orbits are great in space activities. Even the most superficial review of all these orbits is inconceivable, not only because of their staggering number but also because of their rather *ad hoc* nature. Those investigations which resulted in *families* of orbits will be our primary subject. Our scope is painfully limited by this restriction, nevertheless, it is believed that no other approach is feasible. The previously mentioned work by V.A. Egorov (1957) is the first reference, followed by B. Thüring (1959), R. Buchheim (1959), K. Ehricke (1962), V. Szebehely et al (1964), and others. These orbits will be referred to as *Lunar trajectories* since either the whole family or at least some of its

TABLE I

SUMMARY OF QUANTITATIVE RESULTS

Reference title	Principal contributors	Dates	General description	μ	Section
Copenhagen category	Darwin Moulton Strömgren	1897–1910 1900–1917 1913–39–64[a]	Families of periodic orbits	1/11 1/5, 1/2 1/2	9.4
Periodic lunar orbits	Egorov Newton Broucke Huang Arenstorf	1957 1958 1962 1962 1963	Families of periodic orbits	~0.012	9.5
Motion around the triangular libration pts.	Rabe Rabe Deprit	1961 1962 1965	Families of periodic orbits	~0.00095 ~0.012	9.6
Lunar trajectories	Egorov Thüring Buchheim Ehricke Szebehely Pierce Standish	1957 1959 1959 1962 1964 1965	Families of special nonperiodic orbits	~0.012	9.7
Application to binary systems	Kuiper Kopal Abhyankar Gould	1941 1956 1959 1959	Families of nonperiodic orbits	0.1 to 0.5	9.8
Additional periodic orbits	Message Deprit Szebehely Knowles	1959 1965 1965	2 : 1 commensurability asymptotic-periodic 1 : 3 commensurability	~0.00095 several 0 to 0.24	9.9

[a] Regarding these dates, see the remarks in Section 9.11.

members can be used for earth-to-moon missions. One of the most significant contributions to this subject made by Hoelker is discussed in Chapter 10 since Hoelker's family is essentially three dimensional.

(E) The next category contains nonperiodic orbits computed for purposes of stellar dynamics. The values of the mass parameter used are in the same range as for the Copenhagen group, $0.1 \leqslant \mu \leqslant 0.5$, however, the orbits are not periodic and are highly specialized. Proponents of these undertakings are G.P. Kuiper (1941), Z. Kopal (1956), K.D. Abhyankar (1959), N. Gould (1959), and others. This section is entitled "*Applications to binary systems.*"

(F) The final category contains three special topics: asymmetric and symmetric periodic orbits in the sun-Jupiter system computed by Message (1959), asymptotic-periodic orbits at the collinear libration points established by Deprit and Henrard (1965), and a three-fold analytic continuation of a second-kind periodic orbit. These are presented under the title "*Additional periodic orbits.*"

Table I shows the summary of quantitative results. The dates shown in the table refer to publications by the principal contributors *and* by members of his school.

9.3 Relation between various systems

In this section the major computational systems are described and compared. The basis of these comparisons is the system introduced in Section 1.5 by Eqs. (52), (54), and (57). The systems compared with this basic, reference, or standard system, are the systems introduced by Birkhoff, Wintner, Darwin, Strömgren, Moulton, Broucke, Charlier, and Rabe.

In the *reference system* [Fig. 9.1(a)] the masses $1 - \mu$ and μ are placed at $P_1(\mu, 0)$ and at $P_2(\mu - 1, 0)$. The function $\bar{\Omega}$ is defined [see Eq. (47), Section 1.5] by

$$\bar{\Omega} = \frac{1}{2} r^2 + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}, \tag{1}$$

where

$$r^2 = x^2 + y^2,$$

$$r_1^2 = (x - \mu)^2 + y^2,$$

and

$$r_2^2 = (x - \mu + 1)^2 + y^2.$$

The function Ω is introduced by

$$\Omega = \bar{\Omega} + \tfrac{1}{2}\mu(1 - \mu), \tag{2}$$

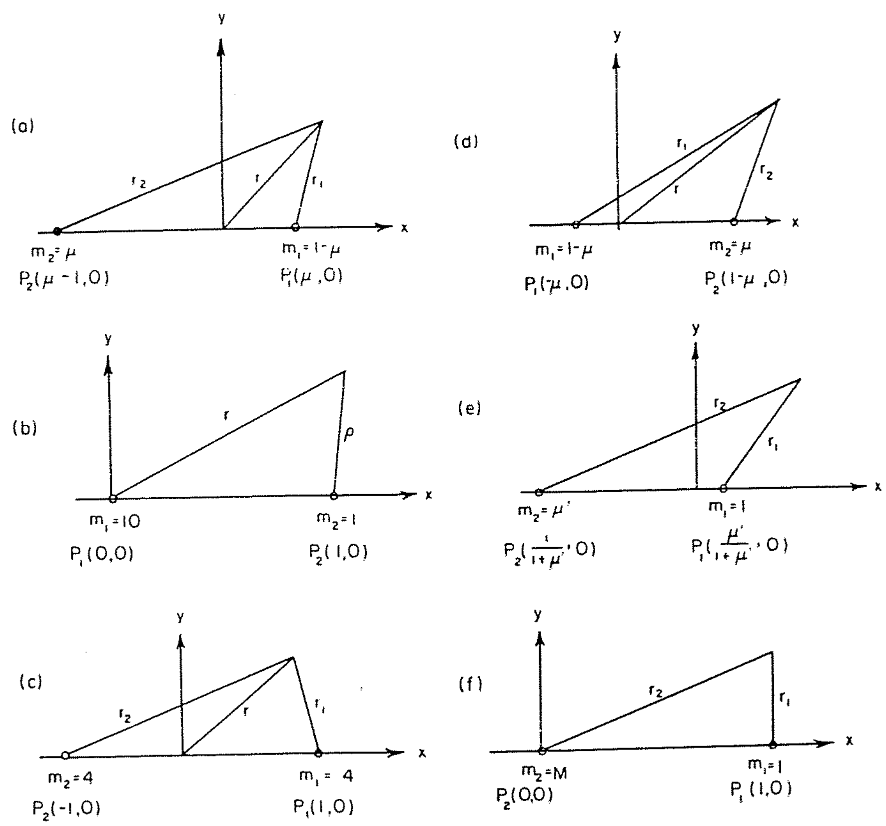

FIG. 9.1. Comparison of coordinate systems: (a) standard system, (b) Darwin's system, (c) Strömgren's system, (d) Moulton's system, (e) Charlier's systems, (f) Rabe's system.

consequently

$$\Omega = \tfrac{1}{2}[(1-\mu)\,r_1{}^2 + \mu r_2{}^2] + \frac{1-\mu}{r_1} + \frac{\mu}{r_2},$$

or

$$\Omega = (1-\mu)\left(\frac{1}{r_1} + \frac{r_1{}^2}{2}\right) + \mu\left(\frac{1}{r_2} + \frac{r_2{}^2}{2}\right).$$

The equations of motion are

$$\ddot{x} - 2\dot{y} = \Omega_x = \bar{\Omega}_x, \qquad \ddot{y} + 2\dot{x} = \Omega_y = \bar{\Omega}_y, \tag{3}$$

and the Jacobian integral is

$$\dot{x}^2 + \dot{y}^2 = 2\Omega - C, \tag{4}$$

which latter defines C as the Jacobian constant. Corresponding to $\bar{\Omega}$, we have $\bar{C}$ defined by

$$C = \bar{C} + \mu(1-\mu). \tag{5}$$

Note that $C(L_{4,5}) = 3$ and $\bar{C}(L_{4,5}) = 3 - \mu(1-\mu)$, giving a slight advantage of C over $\bar{C}$, the first being independent of the mass parameter at the triangular libration points.

At this point a comparison between our reference system on one hand and *Birkhoff's and Wintner's systems* on the other hand can be settled because our C is the Jacobian constant used by Birkhoff, $C = C_B$, and our $\bar{C}$ is Wintner's Jacobian constant, $\bar{C} = C_W$. So, for instance, $C_B(L_{4,5}) = 3$ and $C_W(L_{4,5}) = 3 - \mu(\mu - 1) \leqslant 3$. Also $C_B(L_{4,5}, \mu) = 3$, and $C_W(L_{4,5}, 0) = 3$. For the Copenhagen mass parameter ($\mu = \frac{1}{2}$) we have $C_B = 3$, $C_W = 11/4$, and for Darwin's value ($\mu = 1/11$ or $10/11$) we have $C_B = 3$, $C_W = 3 - 10/121$, all values computed at $L_{4,5}$. (At the collinear libration points the value of the Jacobian constant depends on μ, in general, for the location of these points depend themselves on the value of μ.)

At the origin

$$\bar{\Omega}(0, 0, \mu) = \frac{1-\mu}{\mu} + \frac{\mu}{1-\mu}$$

and so $\bar{\Omega}(0, 0, \frac{1}{2}) = 2$. When $\mu = \frac{1}{2}$ one of the collinear libration points (L_2) is at the origin, and since $2\bar{\Omega} = \bar{C}$, we have $\bar{C}(L_2, \frac{1}{2}) = 4$ and $C(L_2, \frac{1}{2}) = 17/4$.

The relation between Birkhoff's and Wintner's Jacobian constants is the same as between our C and $\bar{C}$, viz.,

$$C_B = C_W + \mu(1-\mu). \tag{6}$$

The origin of *Darwin's coordinate system* [Fig. 9.1(b)] coincides with the primary on the left which has the larger mass. This results in a translation of the coordinate system as well as an exchange of the roles of m_1 and m_2. The ratio of the masses in Darwin's work is 10, consequently $\mu = 10/11$ (and not $1/11$, because of the $m_1 \to m_2$ exchange). In Darwin's notation the Jacobian integral is

$$v^2 = 10(2/r + r^2) + (2/\rho + \rho^2) - C_D, \tag{7}$$

where v is the velocity relative to the rotating system, $r^2 = x^2 + y^2$, $\rho^2 = (x-1)^2 + y^2$, and C_D is Darwin's Jacobian constant. At the triangular libration points $C_D(L_{4,5}, 10/11) = 33$ and since $C(L_{4,5}) = 3$ due to the change in scale we have

$$C_D = 11C \quad \text{or} \quad C_D = 11\bar{C} + 10/11. \tag{8}$$

The Copenhagen system [Fig. 9.1(c)] is described by

$$\Omega = \tfrac{1}{2}r^2 + 4/r_1 + 4/r_2, \tag{9}$$

where $r^2 = x^2 + y^2$, $r_1^2 = (x + 1)^2 + y^2$, and $r_2^2 = (x - 1)^2 + y^2$. The primaries are located at $P_1(1, 0)$ and $P_2(-1, 0)$. The definition of the Copenhagen Jacobian constant C_K follows from

$$v^2 = 2\Omega - C_K, \tag{10}$$

and so $C_K(L_{4,5}) = 11$, for at L_4, $r = 3^{1/2}$ and $r_1 = r_2 = 2$.

On the other hand $\bar{C}(L_{4,5}, \mu) = 3 + \mu(\mu - 1)$, or $\bar{C}(L_{4,5}, \frac{1}{2}) = C_W(L_{4,5}, \frac{1}{2}) = 11/4$ and so

$$\bar{C} = C_K/4. \tag{11}$$

At the origin of the Copenhagen system (at L_2) we have $C_K = 16$, for here $r = 0$ and $r_1 = r_2 = 1$. The corresponding value for $\bar{C} = 4$ was found earlier, so $C_K = 4\bar{C}$ as expected.

Note that the Jacobian constant obeys the same scaling law as the square of the velocity, so once the question of using C or $\bar{C}$ as a basis of the comparison is decided, only the new length and time scales are to be determined in order to find the scaling law for the Jacobian constant. In the Copenhagen problem the length scale is multiplied by 2 when compared to our standard system and the time scale is the same, consequently, the corresponding Jacobian constants are related by a factor of 4. In Darwin's case the length scales are identical and the time scale changes by a factor of $n = (m_1 + m_2)^{1/2} = 11^{1/2}$, consequently, the Jacobian constants are related by a factor of 11. Using Kepler's equation, $n^2a^3 = k^2(m_1 + m_2)$, with $k^2 = 1$, the comparison amounts to the following. In our (standard) case $m_1 + m_2 = 1$, $n = 1$, $a = 1$, since the mean motion of the coordinate system is 1 and the separation of the primaries is 1. In the Copenhagen problem $m_1 + m_2 = 8$, $n = 1$, $a = 2$, and in Darwin's system $m_1 + m_2 = 11$, $a = 1$, $n = 11^{1/2}$. The corresponding velocities ($= na$) are 1, 2, and $11^{1/2}$.

Moulton interchanges the locations of m_1 and m_2 but otherwise uses our standard system [Fig. 9.1(d)]. Consequently $C_M = \bar{C}$.

Broucke's system [Fig. 9.1(d)] follows Moulton in placing the small mass to the right of the mass center (this is the opposite of the standard system) and uses the symbol E for the Jacobian constant. This results in

$$C_L = E = -\bar{C}/2, \tag{12}$$

because of his definition:

$$v^2 = r^2 + 2(1 - \mu)/r_1 + 2\mu/r_2 + 2E. \tag{13}$$

The next system to be discussed was introduced by *Charlier* [Fig. 9.1(e)]. It is believed that at this point the reader would think that this system would coincide with one of the previously mentioned ones. This however is not the case. Charlier selects for the masses of the primaries $m_1 = 1$ and $0 < m_2 = \mu' < 1$. (His original notation, $m_2 = \mu_1$ is slightly modified in order to avoid confusion.) The mean motion of his coordinate system is consequently $(1 + \mu')^{1/2}$ and since the separation of the masses is one, we have $1 + \mu'$ as the scale factor for the Jacobian constant. Charlier's definition of the Jacobian constant,

$$C_C = r_1^2 + 2/r_1 + \mu'(r_2^2 + 2/r_2), \tag{14}$$

corresponds to our standard C (as opposed to $\bar{C}$), therefore,

$$C = \frac{C_C}{1 + \mu'}. \tag{15}$$

With

$$\mu = \frac{\mu'}{1 + \mu'} \tag{16}$$

the transformation formula becomes

$$C = C_C(1 - \mu). \tag{17}$$

For instance, at $\mu' = 1/10$, which corresponds to Darwin's problem, we have $C = (10/11)C_C$ which gives $C_C = 3.3$ at L_4. Charlier's system is the generalization of Darwin's, and Eq. (8) compared with (17) gives $C_D = 10C_C$ when Charlier's Jacobian constant is applied for the $\mu' = 0.1$ case.

Rabe's Jacobian constant is the same as Charlier's regarding the use of $\mu' = M = 1/1047.355$, which corresponds to the sun-Jupiter system, but his system is different from all the previously mentioned ones since he places the smaller mass at the origin of the coordinate system and the larger mass at unit distance to the right of the origin [Fig. 9.1(*f*)]. The value of Rabe's Jacobian constant is related to C by

$$C_R = (1 + M)\,C, \tag{18}$$

which follows from Eq. (15). At the triangular libration point we have $C_R = 3.00286436$.

Table II summarizes the results.

TABLE II

SUMMARY OF SYSTEMS

Name of the system	Value of the mass parameter	Relation to C	Relation to $\bar{C}$
Standard	μ	$C = C$	$C = \bar{C} + \mu(1-\mu)$
Birkhoff's	μ	$C_B = C$	$C_B = \bar{C} + \mu(1-\mu)$
Wintner's	μ	$C_W = C + \mu(\mu - 1)$	$C_W = \bar{C}$
Darwin's	10/11	$C_D = 11C$	$C_D = 11\bar{C} + 10/11$
Strömgren's	1/2	$C_K = 4C - 1$	$C_K = 4\bar{C}$
Moulton's	1/5, 1/2	$C_M = C + \mu(\mu - 1)$	$C_M = \bar{C}$
Broucke's	0.012155099	$C_L = -\frac{1}{2}C + 0.0060036763$	$C_L = -\bar{C}/2$
Charlier's	μ	$C_C = C/(1-\mu)$	$C_C = \bar{C}/(1-\mu) + \mu$
Rabe's	0.0009538754	$C_R = 1.0009547861C$	$C_R = 1.0009547861\bar{C}$ $+0.0009538754$

The following remarks regarding the entries of Table II will explain their use. For a given value of μ a certain orbit in our standard system will be associated with a value of the Jacobian constant C or with $\bar{C}$. To compare this with an orbit computed in the Copenhagen system can be a confusing matter since the Jacobian constant of this latter orbit refers to $\mu = \frac{1}{2}$. An example will clarify matters. Consider the second collinear libration point, L_2. The value of the Jacobian constant in the Copenhagen system is $C_K = 16$, as seen before. The Jacobian constant in Darwin's system is $C_D = 40.18$, which value is given in Darwin's work and which the reader can verify. Now, it is recalled that the value of $C(L_2)$ depends on what value of μ is used to compute it since $C_{L_2} = C_{L_2}(\mu)$, as was discussed in considerable detail in Chapter 4. When $\mu = \frac{1}{2}$, $C(L_2) = 4.25$, and for $\mu = 10/11$ we have $C(L_2) = 3.652916$. These last two values of C for simplicity's sake will be denoted by $C_{1/2}$ and $C_{10/11}$. Similarly $\bar{C}_{1/2} = C_{1/2} - \frac{1}{4} = 4$ and $\bar{C}_{10/11} = C_{10/11} - 10/121 = 3.570271$, since for any μ, $\bar{C} = C - \mu(1-\mu)$. The value $C_K = 16$ can be obtained now from $\bar{C}_{1/2}$ using Eq. (11). From Table II we have $C_K = 4C_{1/2} - 1$, which gives the proper C_K value. Note that everywhere those values of C are used which correspond to $\mu = \frac{1}{2}$. Regarding Darwin's values we have, from Eq. (8), $C_D = 11C_{10/11} = 40.18$ and, from the table, $C_D = 11\bar{C}_{1/2} + 10/11 = 40.18$, as the reader can verify by substitution. (Note that Darwin's above-given value is correct only to two decimal figures.) The equations given in the text and in Table II are correct

provided they are used for the same value of μ. For instance the relation between the Copenhagen and the standard Jacobian constants is $C_K = 4\bar{C}$ according to Eq. (11). The proper interpretation of this equation is that $\bar{C}$ is to be computed at $\mu = \frac{1}{2}$, i.e.,

$$C_K = 4\bar{C}_{1/2} \tag{19}$$

and similarly

$$C_K = 4C_{1/2} - 1. \tag{20}$$

The corresponding equations for Darwin's Jacobian constant are

$$C_D = 11C_{10/11} \tag{21}$$

and

$$C_D = 11\bar{C}_{10/11} + 10/11. \tag{22}$$

To find a relation between C_K and C_D we combine Eqs. (19) and (22), obtaining

$$C_K = 4C_D/11 - 40/121 + 4(\bar{C}_{1/2} - \bar{C}_{10/11}) \tag{23}$$

or equivalently

$$C_K = 4C_D/11 - 1 + 4(C_{1/2} - C_{10/11}). \tag{24}$$

At the triangular libration points $C_{1/2} = C_{10/11} = 3$ and consequently from Eq. (24) we have

$$C_K = 4C_D/11 - 1, \tag{25}$$

which is satisfied by $C_K = 11$, $C_D = 33$. Furthermore $\bar{C}_{1/2}(L_{4,5}) = 3 - \frac{1}{4}$, $\bar{C}_{10/11}(L_{4,5}) = 3 - 10/121$, which values reduce Eq. (23) to (25). Note that if in Eqs. (19)–(22) the subscripts 1/2 and 10/11 are ignored, i.e., if $C_{1/2} = C_{10/11}$ and $\bar{C}_{1/2} = \bar{C}_{10/11}$ are assumed, then Eqs. (23) and (24) give contradictory and erroneous results.

9.4 The Copenhagen category

The work of the Copenhagen Observatory on the restricted problem uses unit value for the mass ratio. This results in a configuration which is symmetric with respect to the y axis. It is important to remark already here that this symmetry condition *in general* does not distinguish the basic shape of the families of orbits obtained with $\mu = \frac{1}{2}$ from those obtained with other values of μ. The significance of the fact that $\mu = \frac{1}{2} > \mu_0 = 0.0385...$ is unquestionably greater than the importance of the symmetry since infinitesimal periodic orbits around the triangular

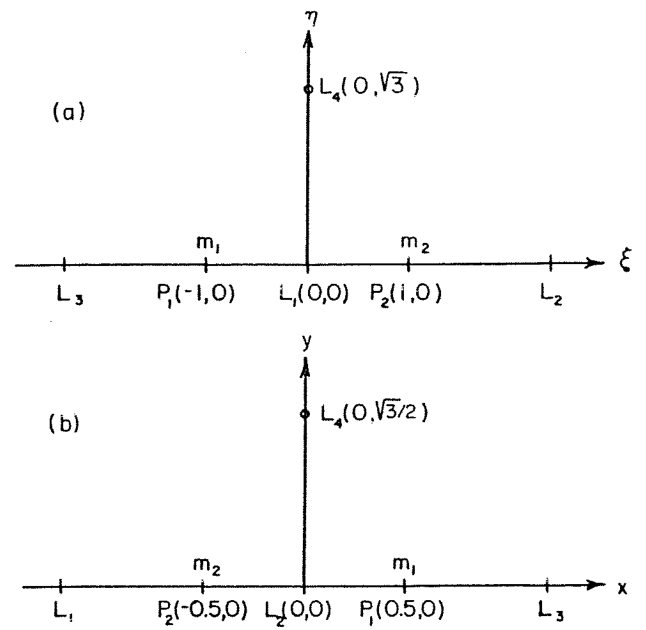

FIG. 9.2. (a) Strömgren's system, $\xi_{L_2} = 2.3968$. (b) Reference system for $\mu = \frac{1}{2}$, $x_{L_3} = 1.1984$.

libration points do not exist for $\mu = \frac{1}{2}$. Instead of these, the Copenhagen school finds asymptotic orbits approaching L_4 and L_5 and establishes families by means of these asymptotic orbits.

The most complete work in the range of $0.1 < \mu \leqslant \frac{1}{2}$ was performed by Elis Strömgren and his associates in Copenhagen; therefore, these results will be shown and discussed first. Darwin's and Moulton's contributions, as we will see, offer excellent commentaries and extensions of Strömgren's work but do not represent the completeness and precision achieved by the group at the Copenhagen Observatory.

The classification of Strömgren's families of orbits is based on the seven special points existing in the plane of the restricted problem: the five points of equilibrium and the two points where the primaries are located. The following remark will clarify the notation. Figure 9.2(a) and (b) compare the terminology of Strömgren with the standard notation used throughout this book. Note that the origin of Strömgren's coordinate system coincides with his first libration point (L_1) and the origin of the standard system coincides with the second libration point (L_2). Consequently the coordinates of L_1 in Strömgren's system are $\xi = \eta = 0$ while in the standard system they are $x = -1.1984$, $y = 0$. Furthermore, when Strömgren speaks about motions around

TABLE III

COMPARISON OF STRÖMGREN'S NOTATION WITH THE REFERENCE SYSTEM

	x	ξ	y	η	$\bar{C}$	C	C_K
m_1	0.5	−1	0	0	∞	∞	∞
m_2	−0.5	1	0	0	∞	∞	∞
L_1	−1.1984	0	0	0	3.4568	3.7068	16
L_2	0	2.3968	0	0	4.0000	4.2500	13.8272
L_3	+1.1984	−2.3968	0	0	3.4568	3.7068	13.8272
L_4	0	0	$\sqrt{3}/2$	$\sqrt{3}$	2.7500	3.0000	11
L_5	0	0	$-\sqrt{3}/2$	$-\sqrt{3}$	2.7500	3.0000	11

L_2, we refer to motion around L_3. Table III compares the two systems further. Inasmuch as in what follows the notation of the reference system is used [Fig. 9.1(a)], these above-mentioned translation devices [i.e., Table III and Figs. 9.2(a), 9.2(b)] become important only when the reader wishes to study the original papers prepared by the Copenhagen Observatory.

Due to the symmetry with respect to the y axis, motions associated with m_1 and L_3 are the mirror images of the motions taking place around m_2 and L_1. The expression "motion around" is widely and loosely used in the literature. When originally infinitesimal elliptic orbits *around* the collinear points become finite and in fact have large amplitudes, the word "around" becomes meaningless. The proper use of "around" is guaranteed when it is restricted to the generating orbits. In this sense we speak about seven special points and infinitesimal periodic generating orbits around them—if these orbits exist. Because of the symmetry conditions in the Copenhagen problem, we have L_1, m_2, L_2, (or L_2, m_1, L_3) and L_4 (or L_5) as the only centers. The last point (L_4) is not surrounded by infinitesimal periodic orbits for $\mu = \frac{1}{2}$, which fact leaves three special points. There are no direct infinitesimal periodic orbits around the collinear points, only retrograde orbits, and there are direct *and* retrograde periodic orbits around the primaries—which statements have been discussed and verified in some detail in Chapters 5 and 8. Consequently the following families of periodic orbits are established by Strömgren, using his classification and the previously defined meaning of "around":

(a) retrograde periodic orbits around L_3—direct orbits do not exist;
(b) retrograde periodic orbits around L_1—direct orbits do not exist;
(c) retrograde periodic orbits around L_2—direct orbits do not exist;
(d) periodic orbits around L_4—nonexistent for $\mu = \frac{1}{2}$;

(e) periodic orbits around L_5—nonexistent for $\mu = \frac{1}{2}$;
(f) retrograde periodic orbits around m_1;
(g) direct periodic orbits around m_1;
(h) retrograde periodic orbits around m_2;
(i) direct periodic orbits around m_2.

In the above list the direction of motion refers to the rotating coordinate system. The significantly different classes are (a), (c), (f), and (g), since (b) = (a), (h) = (f), and (i) = (g). Finally Classes (d) and (e) do not exist if the word "around" is properly interpreted. There will be more said about this later.

In addition to these four classes [(a), (c), (f), and (g)] Strömgren also established three classes with orbits around both primaries:

(k) periodic orbits around m_1 and m_2—motion is direct in the rotating system;
(l) periodic orbits around m_1 and m_2—motion is retrograde in the rotating system and direct in the fixed system;
(m) periodic orbits around m_1 and m_2—motion is retrograde in the rotating as well as in the fixed system.

Another family of periodic orbits found by Strömgren is not generated from either infinitesimal or two-body orbits:

(n) retrograde periodic orbits, asymmetric to the y axis; these orbits are related to orbits of Class (c) but are *not* generated from infinitesimal orbits around L_2.

While there are no infinitesimal periodic orbits around the triangular libration points for $\mu = \frac{1}{2}$, the existence of asymptotic orbits, spiraling in (and out of) these points, has been shown in Chapter 5 for this value of the mass parameter. Those asymptotic orbits which intersect the x axis perpendicularly are called asymptotic-periodic orbits by Strömgren, who found five such orbits. These orbits can be used to generate additional families of periodic orbits for which two of the above-mentioned asymptotic-periodic orbits are the limiting members. Such families therefore must number ten (or twenty, counting also the reflections with respect to the y axis). Two of these families are as follows:

(o) retrograde periodic orbits, which are *asymmetric* with respect to the y axis and which form a family limited by two asymptotic-periodic orbits;
(r) retrograde periodic orbits, which are *symmetric* with respect to the y axis and which form a family limited by two asymptotic-periodic orbits.

Asymptotic-periodic orbits approaching the triangular libration points in general intersect neither the x nor the y axis perpendicularly. The above-mentioned ten orbits intersect the x axis perpendicularly the first time, after leaving the libration point, i.e., after the first quarter of the total simple periodic orbit. There also exist simple periodic orbits which intersect the y axis perpendicularly after leaving the vicinity of the libration point, i.e., after the first half revolution. Seven such orbits which are not symmetric to the x axis but are symmetric to the y axis are known, or their number becomes 14 if L_4 *and* L_5 are considered separately, i.e., if the first seven orbits are reflected on the x axis.

In the following section several of these families will be discussed in a systematic fashion. The figures are all from the reports of the Copenhagen Observatory except when specially noted otherwise.

9.4.1 Retrograde periodic orbits around L_3; Class (a) of the Copenhagen category

This family of orbits is generated from the infinitesimal elliptic orbits around L_3. The value of the Jacobian constant for $\mu = \frac{1}{2}$ at $L_3(x = 1.1984)$ is $C = 3.7068$. Fig. 9.3(a) shows the retrograde orbits for $C \cong 3.71$, 3.55, 3.00, and 2.50, the last one corresponding to a collision orbit with m_1 located at $P_1(\frac{1}{2}, 0)$. This collision orbit was discussed in Chapter 3 on regulariza-

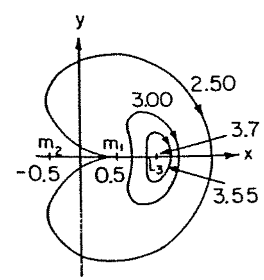

FIG. 9.3(a). Class (a) orbits of the Copenhagen category, $\mu = \frac{1}{2}$ for $C \cong 3.71$ to 2.50 (E. Strömgren, 1922, Ref. 19).

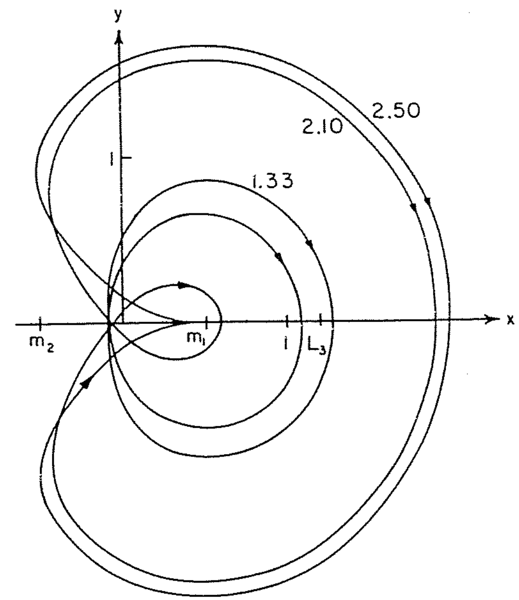

FIG. 9.3(b). Class (a) orbits of the Copenhagen category, $\mu = \frac{1}{2}$ for $C \cong 2.50$ to 1.33 (E. Strömgren, 1929, Ref. 22).

tion. It was established first by C. Burrau who considered it the end of this type of orbits, i.e., the termination of Class (a) orbits. The collision orbit actually can be continued into an orbit with a loop around m_1 (see Fig. 9.3(b), $C = 2.10$). Further reduction of C to say, 1.33 increases the size of the smaller (inner) loop and reduces the size of the larger (outer loop) until these two loops coincide and the single oval-shaped orbit intersects the x axis perpendicularly at L_3. This occurs at $C \cong 1.31$. The family is now continued by retracing the steps, back to the collision orbit and to the stationary point, L_3. In other words, upon establishing for $C = 1.33$ the double loop orbit, and for $C \cong 1.31$ a single loop orbit, then the inside loop

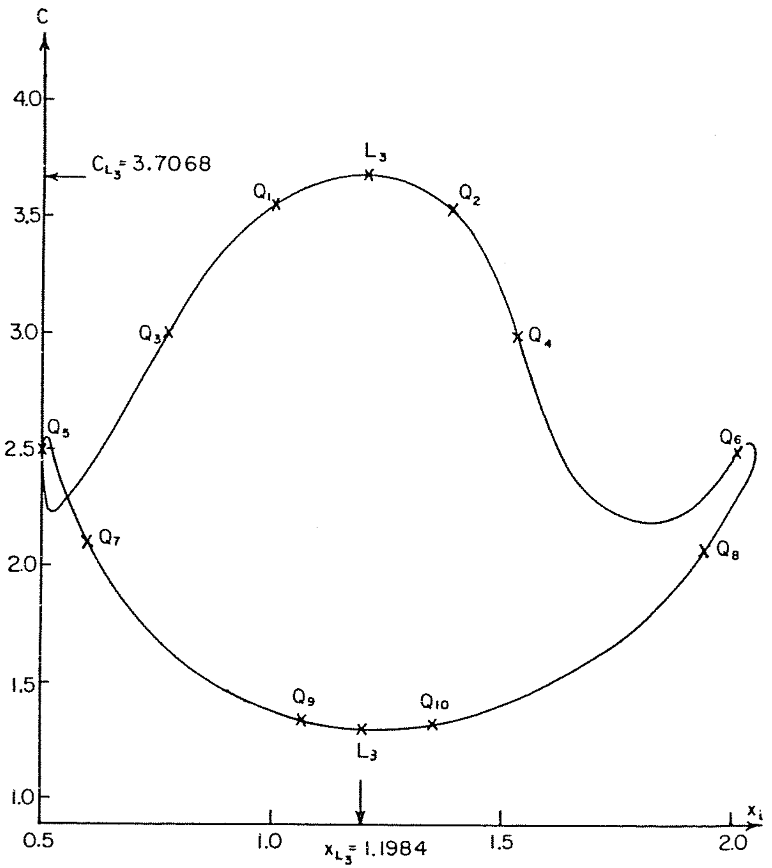

FIG. 9.3(c). Initial conditions for Class (a) orbits of the Copenhagen category, $\mu = \frac{1}{2}$.

becomes the outside loop and vice versa. The value $C = 1.31$ is the minimum value of C in the family as shown in Fig. 9.3(c). The points marked with crosses in this figure correspond to the orbits drawn in Figs. 9.3(a) and (b). The stationary solution (point L_3) is shown by the cross at the maximum of the curve $C = C(x)$. As C decreases to $C = 3.55$ the two perpendicular crossings of the orbit with the x axis are at $x_1 = 1$ and $x_2 = 1.37$; see points Q_1, Q_2. At $C \cong 3.00$ the intersections are at $x_1 \cong 0.76$ and $x_2 \cong 1.52$; see points Q_3, Q_4. The collision orbit (points Q_5 and Q_6) intersects at $x_1 = 0.5$ (i.e., at m_1) and at $x_2 \cong 2.01$. The first double loop orbit shown at $C = 2.10$ intersects the x axis at $x_1 \cong 0.6$ and $x_2 \cong 1.93$; see Q_7, Q_8. The second double loop orbit of Fig. 9.3(b) at $C = 1.33$ is associated with $x_1 \cong 1.07$ and $x_2 = 1.35$; see Q_9, Q_{10}. The two loops coincide and the orbit crosses the x axis at L_3, corresponding to the minimum point of the curve $C = C(x)$. Then Q_9 exchanges place with Q_{10}, Q_7 with Q_8, etc., and the representative points on the curve $C(x)$ move up and eventually reach the maximum point (L_3).

The family begins and ends at the external collinear libration point (L_3) and all its members are retrograde periodic orbits. The family is closed; i.e., beginning with any of its members, the set of orbits belonging to the family will lead us back to the orbit with which we started.

Class (b) becomes identical with Class (a) when x is replaced by $-x$.

9.4.2 Retrograde periodic orbits around L_2; Class (c) of the Copenhagen category

Similarly to Class (a), Class (c) begins with infinitesimal (elliptic) orbits around L_2 as discussed in Chapter 5. As the amplitude increases, these orbits [Fig. 9.4(a)], maintaining their symmetry with respect to both axes, will show consecutive collisions (Chapter 3) with both masses ($C = 2.432913$). In further developments this collision orbit is continued in looped orbits, which, when the amplitude is further increased, demonstrate again consecutive collisions and the process continues with the size of the orbits increasing from 0 (at L_2) to ∞. The linearized equations give 0.2278093... for the ratio of the minor and major axes of the infinitesimal elliptic orbits as $C \to C_2 = 4.25$, and this is the value of the Jacobian constant with which the family of orbits starts. The end of this class of orbits is reached when the size approaches infinity. The increase of size is monotonic with the decrease in the value of the Jacobian constant around $C = C_2$, but as the size increases the monotonic character disappears. Figure 9.4(a) shows the development of the family.

The orbits start at the second collinear libration point with initial

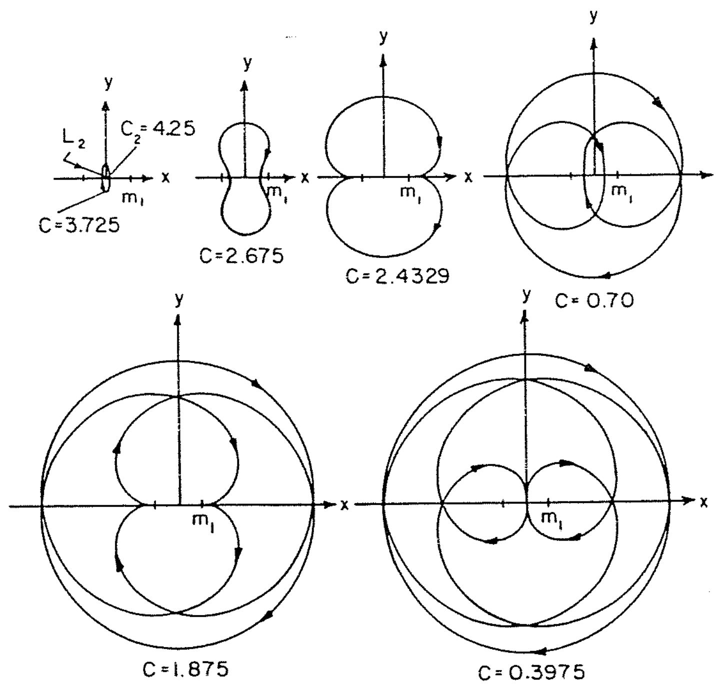

FIG. 9.4(a). Class (c) orbits of the Copenhagen category, $\mu = \frac{1}{2}$ (Möller, 1935, Ref. 22).

conditions $x_i = y_i = \dot{x}_i = \dot{y}_i = 0$. Then the initial point moves to $x_i > 0$, $y_i = 0$ with $\dot{x}_i = 0$, $\dot{y}_i < 0$. The initial conditions throughout the development of the whole class can be represented by a function

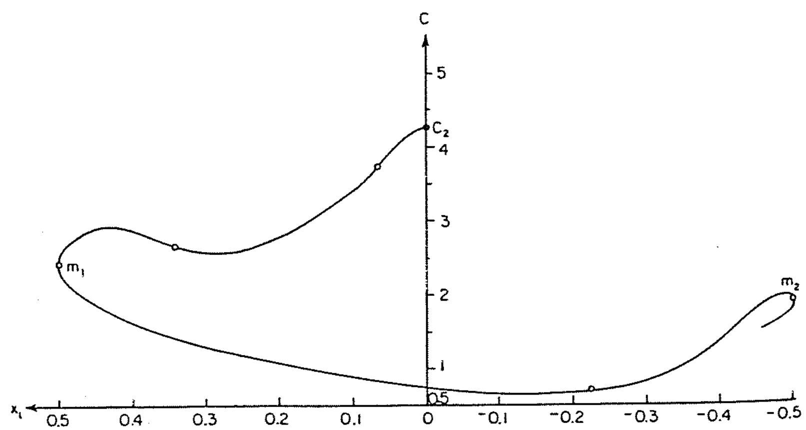

FIG. 9.4(b). Initial conditions for Class (c) orbits of the Copenhagen category, $\mu = \frac{1}{2}$.

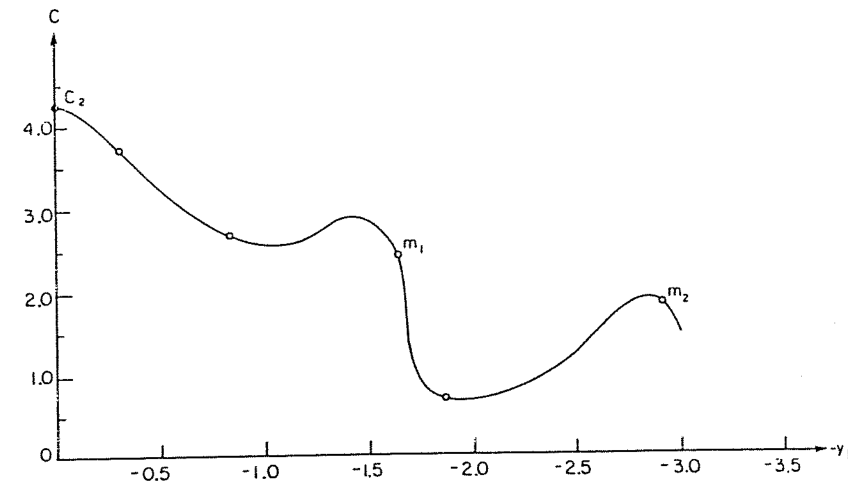

FIG. 9.4(c). Final conditions for Class (c) orbits of the Copenhagen category, $\mu = \frac{1}{2}$.

$C = C(x_i)$. This follows from the fact that for all members two of the initial conditions are always $y_i = 0$ and $\dot{x}_i = 0$, so knowing the values of x_i and C, the remaining initial condition ($\dot{y}_i$) is determined. Figure 9.4(b) shows the curve $C = C(x_i)$ in which the points L_2, m_1, and m_2 are marked to identify some of the initial conditions with the actual orbits shown in Fig. 9.4(a).

Figure 9.4(c) shows the "final" condition, i.e., the location of the perpendicular intersection of the orbit after the first $\frac{1}{4}$ revolution with the negative y axis (y_f). At this point $y_f \leqslant 0$, $x_f = 0$, $\dot{y}_f = 0$, $\dot{x}_f \leqslant 0$. Using the curve $C = C(y_f)$ the whole family can be generated from these "final" conditions when they are used as initial conditions. Note the monotone increasing trend of $|y_f|$ as the class develops.

9.4.3 Retrograde periodic orbits around m_1; Class (f) of the Copenhagen category

This family is generated from infinitesimal circular orbits around m_1; therefore, in a sense, the beginning of the family corresponds to the group of orbits that Poincaré called the first kind. The value of the Jacobian constant at the beginning of the family is $C_{m_1} \to \infty$, and as the oval-shaped orbit increases its size, the Jacobian constant decreases. The relation between the Jacobian constant and the radius of the retrograde *circular* orbit for the problem of two bodies can be established and this way approximate initial conditions can be computed at least for large values of C.

As the retrograde orbits are generated by moving the starting point farther to the right of m_1, the originally circular orbits become more and more distorted and at $C = 2.0444$ a collision orbit appears. This is followed by orbits which loop around m_2. The size of the loop increases and at $C = 1.74$ collision occurs with m_1. The process repeats itself, i.e., loop, collision, loop, collision, etc., while the size of the orbit increases without limit.

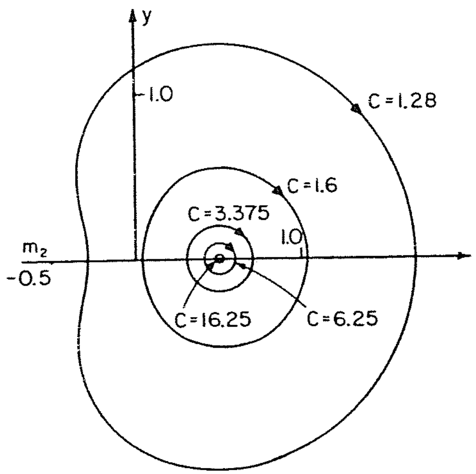

Fig. 9.5(a). Class (f) orbits of the Copenhagen category phase one, $\mu = \frac{1}{2}$ (Burrau and E. Strömgren, 1916, Ref. 22).

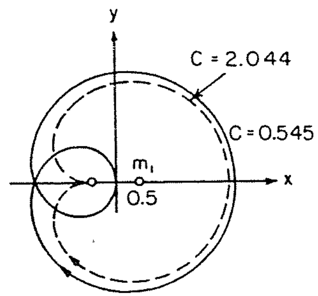

Fig. 9.5(b). Class (f) orbits of the Copenhagen category phase two, $\mu = \frac{1}{2}$ (Möller, 1935, Ref. 22).

Figures 9.5(a), (b), and (c) show the development and Figs. 9.5(d) and 9.5(e) give the initial and final conditions.

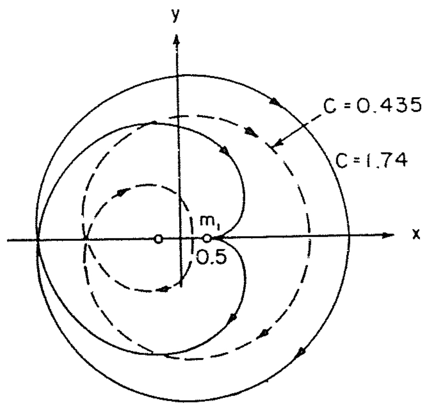

Fig. 9.5(c). Class (f) orbits of the Copenhagen category phase three, $\mu = \frac{1}{2}$ (Möller, 1935, Ref. 22).

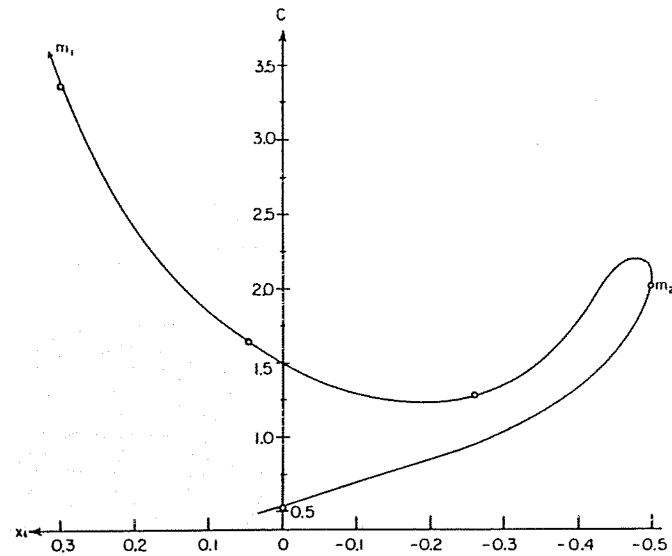

Fig. 9.5(d). Initial conditions for Class (f) orbits of the Copenhagen category, $\mu = \frac{1}{2}$.

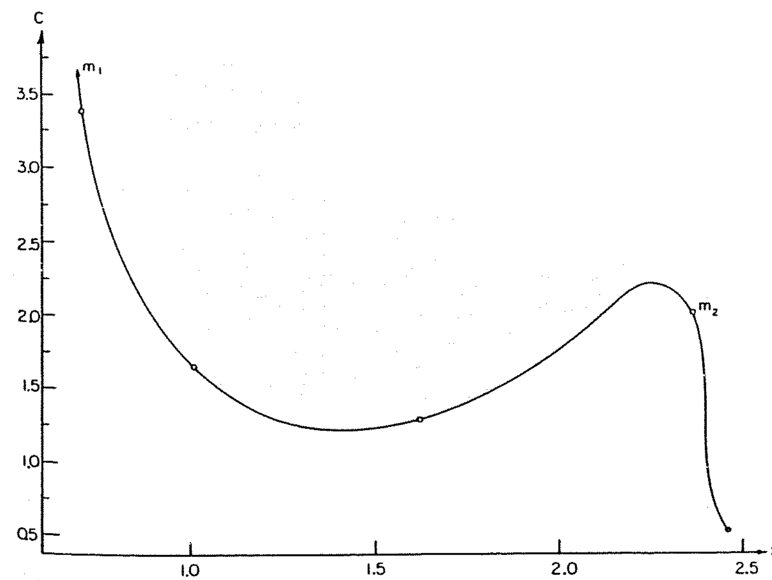

Fig. 9.5(e). Final conditions for Class (f) orbits of the Copenhagen category, $\mu = \frac{1}{2}$.

The initial conditions are $|x_i| \leqslant 0.5$, $\dot{x}_i = 0$, $y_i = 0$, $\dot{y}_i \neq 0$, and the family starts at the left of Figs. 9.5(d) and 9.5(e) asymptotically approaching the value $x_i = x_f = 0.5$ as $C \to \infty$. The final conditions are $x_f \geqslant 0.5$, $\dot{x}_f = 0$, $y_f = 0$, $\dot{y}_f \leqslant 0$.

9.4.4 Direct periodic orbits around m_1; Class (g) of the Copenhagen category

Quite similarly to the previous family, Class (g) is also generated from circular orbits around m_1 but the direction of the initial velocity vector is reversed and the family starts with direct circular orbits of small radii corresponding to high values of the Jacobian constant. The mean radii for $C = 50.25$, 16.25, 6.25 are $r \cong 0.0103$, 0.0347, and 0.118, respectively. The approximately circular orbits around m_1 are shown in Fig. 9.6(a).

Consider now these orbits starting from a point on the x axis for which $0 < x = x_i \leqslant 0.5$. Let the initial velocities be $\dot{y}_i < 0$ and $\dot{x}_i = 0$. As the initial point on the x axis moves to the left from m_1, a condition is reached, at $x_i \cong 0.081$, when the orbit does not turn around m_1

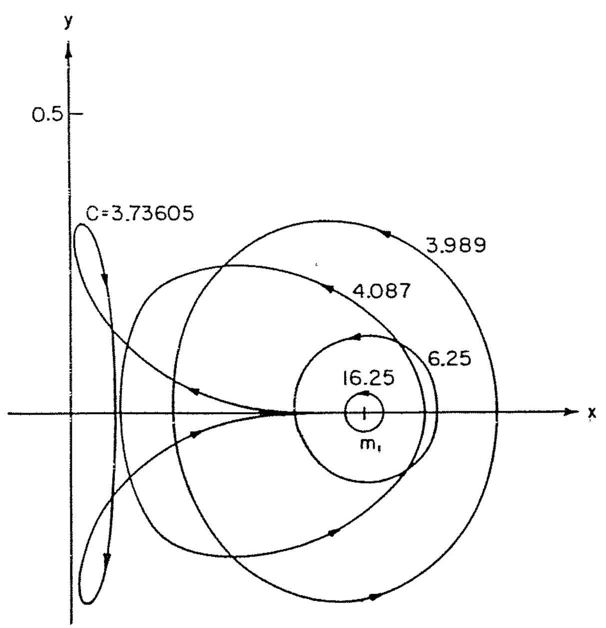

FIG. 9.6(a). Class (g) orbits of the Copenhagen category first phase, $\mu = \frac{1}{2}$ (Burrau and Strömgren, 1915, Ref. 19).

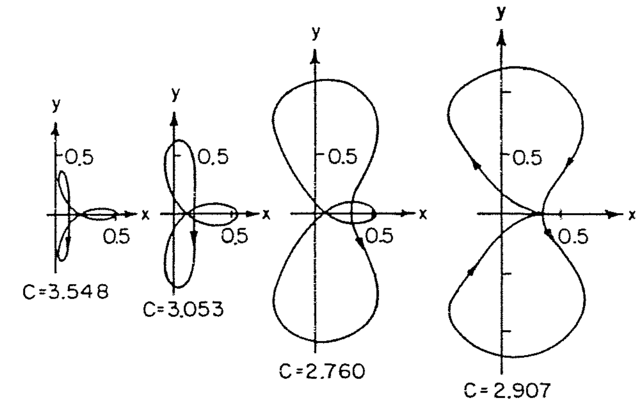

FIG. 9.6(b). Class (g) orbits of the Copenhagen category second phase, $\mu = \frac{1}{2}$ (E. Strömgren, 1922, Ref. 19).

anymore but a collision is established. The intersections with the x axis are given by $x_i = 0.382$, 0.181, 0.095, and 0.081 corresponding to $C = 16.25$, 3.989, 4.087, and 3.7361. This concludes the discussion of the first phase of Class (g) as shown in Fig. 9.6(a).

The collision orbit, which was the last member of the first phase, is followed by an orbit which loops around m_1 and is shown as the first orbit in Fig. 9.6(b). The initial conditions are still $\dot{y}_i < 0$, $\dot{x}_i = 0$, $y_i = 0$, $0 < x_i < 0.5$, but at the second perpendicular crossing we have $\dot{y}_f < 0$ while in the first phase we had $\dot{y}_f > 0$. The intersections with the x axis are $x_i \cong 0.094$, 0.182, 0.310, and 0.350 corresponding to $C = 3.548$, 3.053, 2.760, and 2.907. The last orbit of the second phase is a collision orbit again. The collision occurs as $x_f \to 0.5$ through the loop orbits shown in Fig. 9.6(b).

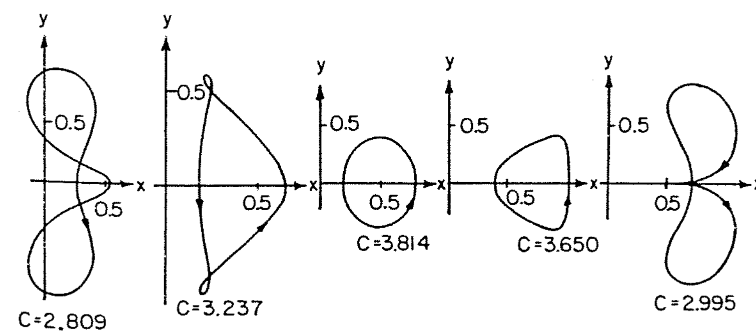

FIG. 9.6(c). Class (g) orbits of the Copenhagen category third phase, $\mu = \frac{1}{2}$ (E. Strömgren, 1922, Ref. 19).

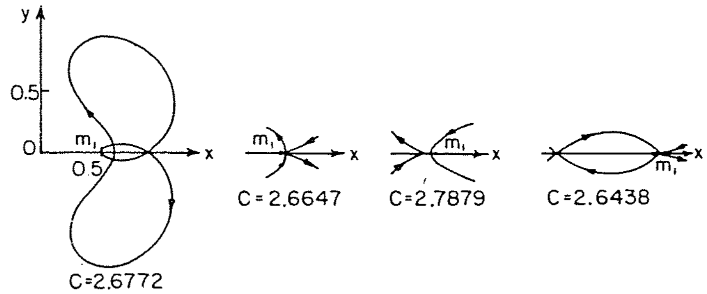

FIG. 9.6(d). Class (g) orbits of the Copenhagen category fourth phase, $\mu = \frac{1}{2}$ (E. Strömgren, 1922, Ref. 19).

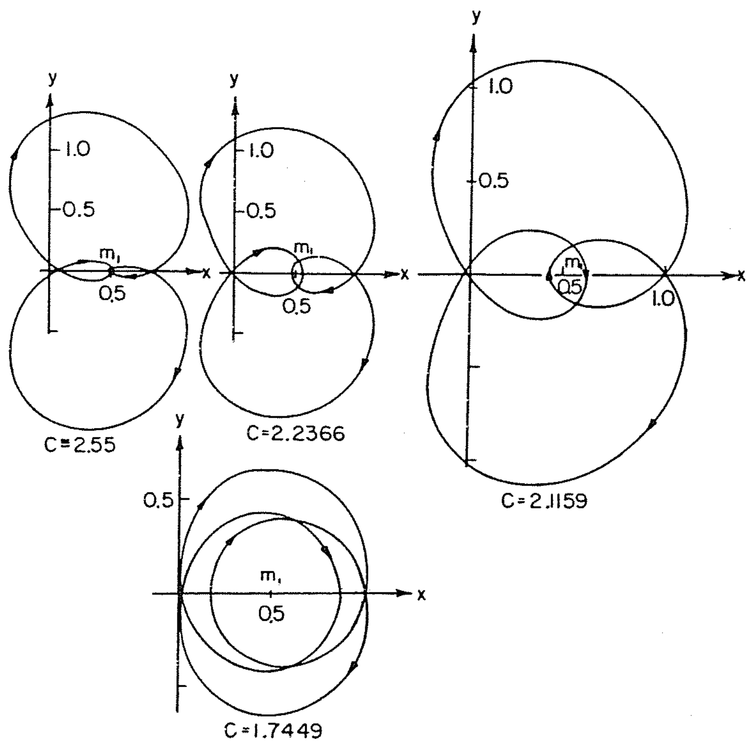

FIG. 9.6(e). Class (g) orbits of the Copenhagen category fifth phase, $\mu = \frac{1}{2}$ (for the first two orbits see E. Strömgren, 1922, Ref. 19).

The third phase [Fig. 9.6(c)] starts with an orbit *around* m_1 with $\dot{y}_I > 0$. Transition from the orbit at $C = 2.809$ to the one at $C = 3.814$ occurs by reducing the size of the loops above and below the x axis. Attention is directed to the third figure of the third phase. This orbit does *not* belong to the first phase but it is a member of the third phase. The next member shown ($C = 3.650$) is called the orbit of von Haerdtl and the last figure is again a collision orbit. Comparing the last members of Figs. 9.6(b) and 9.6(c) it is observed that the ejections are oppositely directed; i.e., the ejection in Phases 1 and 2 are toward m_2 and in Phase 3 it is away from m_2.

Some of the orbits have clearly defined directional properties; others

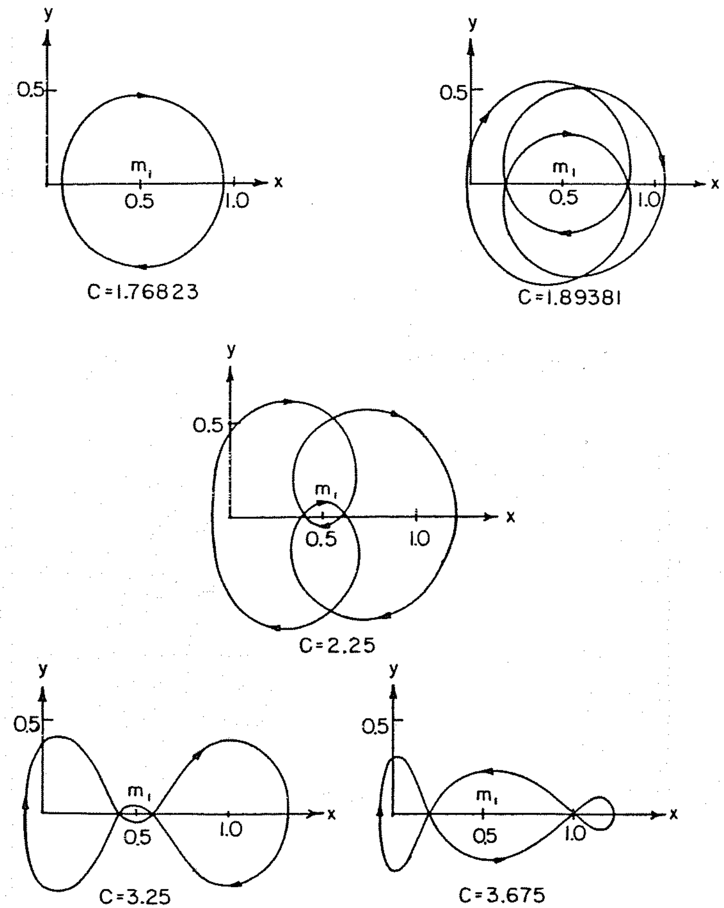

FIG. 9.6(f). Class (g) orbits of the Copenhagen category sixth phase, $\mu = \frac{1}{2}$.

show possible ambiguity. In Fig. 9.6(a) the curves belonging to $C = 16.25$, 6.25, 3.989, and 4.087 are direct orbits and the same applies to the curves of Fig. 9.6(c) with $C = 3.814$ and 3.650. The direction of the last member of Fig. 9.6(c), however, is not as clearly defined; in fact one can consider every collision orbit as a means of changing direction. In this sense Fig. 9.6(a) contains direct orbits, Phase 2 [Fig. 9.6(b)] consists of retrograde orbits, Phase 3 [Fig. 9.6(c)] has direct orbits, and the orbits following the last member of Phase 3 will have retrograde properties.

Comparison between the first few members (large values of C) of Classes (f) and (g) reveal that the development of Class (f) is slower;

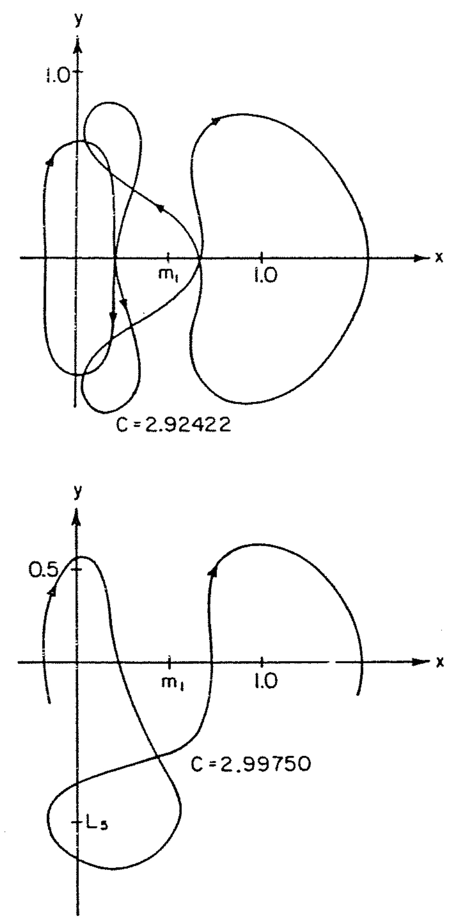

FIG. 9.6(g). Class (g) orbits of the Copenhagen category seventh phase, $\mu = \frac{1}{2}$.

i.e., for $\infty < C < 1.6$ the circular shapes are preserved for Class (f) while this is true for Class (g) only in the region $\infty < C < 6.25$.

Phase 4, Fig. 9.6(d), begins with a loop orbit as expected, since it follows the collision orbit shown as the last drawing of Fig. 9.6(c). This phase also consists of several collision orbits which are shown schematically. After the last member of this phase, the class continues by forming two loops as shown on the first orbit of Fig. 9.6(e). The inside loops now increase their size and the outside loops shrink as the value of the Jacobian constant decreases monotonically. The two sets of loops become indistinguishable as the sixth phase begins [Fig. 9.6(f)] and the role of the loops is now interchanged. In other words, the second orbit of Fig. 9.6(f) can be obtained from the last orbit of Fig. 9.6(e) by a 90° rotation and by a slight distortion. At this point the inside loops shrink and the outside loops expand while the Jacobian constant monotonically increases. In this way an orbit is obtained where the inside loops completely disappear ($C = 3.25$). The last member of this (sixth) phase shows that the middle part of the orbit increases its size. The last available, seventh phase is shown on Fig. 9.6(g). The Jacobian constant, after reaching a relative maximum of $C \cong 3.68$, starts decreasing to $C \cong 2.87$. The first orbit of Fig. 9.6(g) corresponds to a consequent increase of C which seems to approach 3. The second orbit of Fig. 9.6(g) seems to be a natural consequence but exact initial conditions are not available. The series of orbits of Fig. 9.6 are shown after Strömgren's drawings, excepting the orbits of the last phases which are Lieske's computational results. The termination of this class is not settled. We return to this question in Sections 9.4.10 and 9.11.

Classes (h) and (i) are similar to Classes (f) and (g) and the similarity becomes an identity for $\mu = \frac{1}{2}$ by interchanging m_1 and m_2. When $\mu \neq \frac{1}{2}$ a topological identity exists as will be shown later.

9.4.5 Synodically direct periodic orbits around m_1 and m_2; Class (k) of the Copenhagen category

The basic orbit in this class is shown in Fig. 9.7(a). Since the third body collides with both primaries, the Copenhagen literature refers to this as a double collision orbit, which in our terminology becomes an orbit with consecutive collisions. This orbit is clearly not a "perturbed two-body orbit" and its discovery by the Copenhagen group occurred when studying certain ejection orbits of other types. The intersection with the y axis occurs approximately at $y = 0.28$ and 0.34 and the Jacobian constant is $C = 3.669$. The symmetry of the Copenhagen problem ($\mu = \frac{1}{2}$) plays an important role in the existence of periodic orbits with consecutive collisions, since when $\mu \neq \frac{1}{2}$ such orbits are

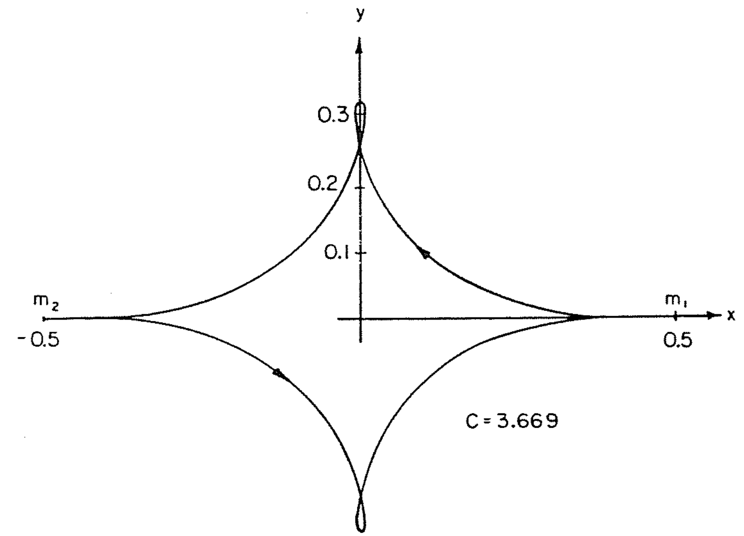

FIG. 9.7(a). Consecutive collision orbit in Class (k) of the Copenhagen category, $\mu = \frac{1}{2}$ (Burrau and E. Strömgren, 1917, Ref. 22).

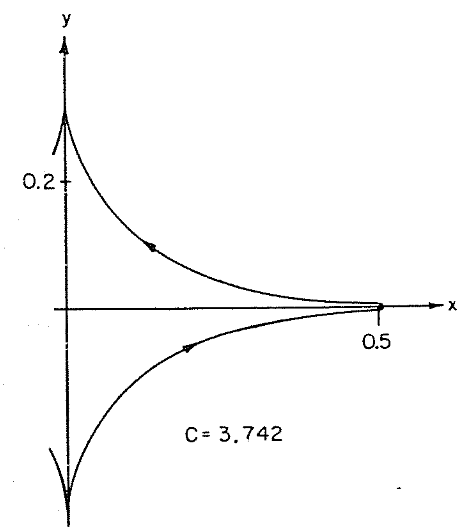

FIG. 9.7(b). Subclass (k_1) orbit in Class (k) of the Copenhagen category, $\mu = \frac{1}{2}$ (Burrau and E. Strömgren, 1917, Ref. 22).

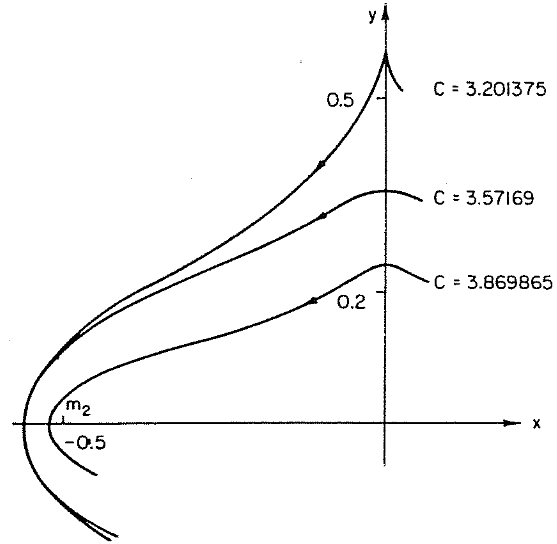

FIG. 9.7(c). Subclass (k_1) orbits in Class (k) of the Copenhagen category, $\mu = \frac{1}{2}$ (Lous, 1919, Ref. 22).

not known. Figure 9.4 of this chapter demonstrates also orbits with consecutive collisions ($C = 2.4329$ and 1.8750) belonging to Class (c). For the asymmetric mass distribution ($\mu \neq \frac{1}{2}$) none of these orbits has been found.

The orbit in Fig. 9.7(a) is basic to Class (k) since two subclasses can be generated by starting with this orbit. Subclass (k_1) intersects the x axis in the region $\frac{1}{2} < |x|$; in other words, the orbits belonging to this subclass do not cross the x axis between the primaries. Figure 9.7(b) shows a special member of this subclass at $C = 3.742$, forming a cusp at approximately 0.311 on the y axis corresponding to the point where the zero velocity curve (belonging to the same value of C) intersects the y axis. Further members of the same subclass are shown in Fig. 9.7(c). These are generated by moving the perpendicular intersection with the x

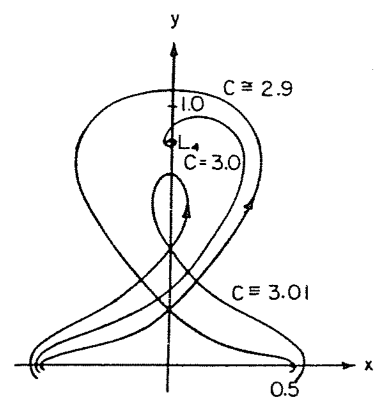

FIG. 9.7(d). Subclass (k_1) orbits in Class (k) of the Copenhagen category, $\mu = \frac{1}{2}$. Also shown is an orbit asymptotic to $L_4 (C = 3)$ (E. Strömgren, 1931, Ref. 22).

axis farther away from m_1 (or m_2). The approximate intersections with the x axis are 0.523 and 0.564, with the y axis at 0.243, 0.356, and 0.578, corresponding to $C \cong 3.87$, 3.57, and 3.20. The largest x_i value still giving periodic orbits is 0.572 and the corresponding value of C is 3.375. The curve $C \cong 3.20$ starts with a slightly smaller value of x_i and it *does not* form a cusp at the y axis.

The loop orbits, shown in Fig. 9.7(d), are the continuation of the orbits belonging to Subclass (k_1). The intersections of these loop orbits with the x axis are also limited by $\frac{1}{2} < | x_i | < 0.572$. Only two members are shown in Fig. 9.7(d), with $C \cong 2.9$ and 3.01 possessing maxima (at $x = 0$) at $y \cong 1.1$ and 0.75. The third curve ($C = 3$) shown in this figure is an asymptotic orbit which spirals out of (or into) the fourth libration point and intersects the x axis perpendicularly in the region mentioned before. This asymptotic curve, with a period approaching ∞, is the limiting orbit to which the loop orbits tend as the subclass develops. Subclass (k_1), therefore, is characterized by orbits completely outside of m_1 and m_2. It starts with a consecutive collision orbit [Fig. 9.7(a)] and it ends with an asymptotic orbit [Fig. 9.7(d)].

Returning now to the consecutive collision orbit [Fig. 9.7(a)] we generate another subclass (k_2) by continuing (changing) the cusps at m_1 and at m_2 into loops. (In fact the two possibilities of proceeding from collision orbits are in general either by introducing loops, which completely encircle m_1 or m_2, or by having orbits which go around the singularity, without completely encircling the primary.) These orbits have loops around m_1 and m_2, and also along the y axis, as shown in Fig. 9.7(e). This figure shows two orbits, similarly to the two orbits

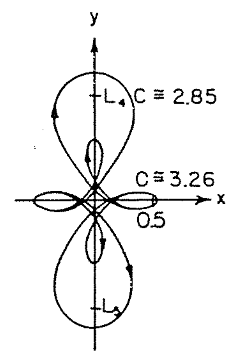

FIG. 9.7(e). Subclass (k_2) orbits in Class (k) of the Copenhagen category, $\mu = \frac{1}{2}$ (E. Strömgren, 1924, Ref. 22).

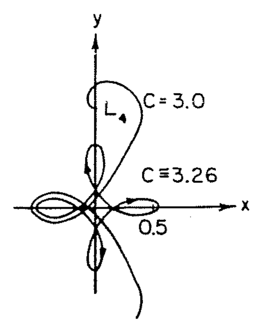

FIG. 9.7(f). Subclass (k_2) orbit ($C \cong 3.26$) in Class (k) of the Copenhagen category, $\mu = \frac{1}{2}$. Also shown is an orbit asymptotic to L_4($C = 3$) (E. Strömgren, 1924, Ref. 22).

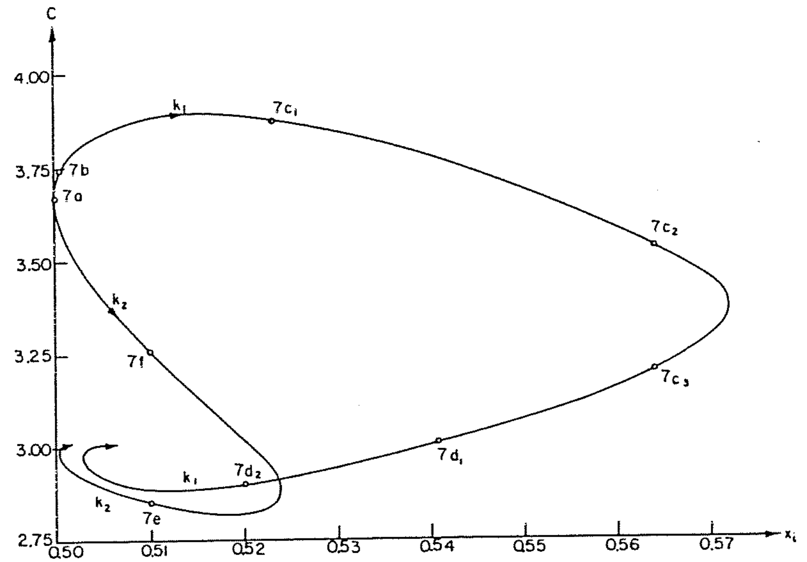

FIG. 9.7(g). Initial conditions for Class (k) orbits of the Copenhagen category, $\mu = \frac{1}{2}$.

shown in Fig. 9.7(d). Figure 9.7(f) shows also the asymptotic curve which spirals out of or into $L_{4,5}$ and which intersects the x axis perpendicularly after forming a loop around the primary. The orbits belonging to Subclass (k_2) start with the consecutive collision orbit [Fig. 9.7(a)] and end with the asymptotic orbit of Fig. 9.7(f) which has infinite period.

In a sense, therefore, Class (k) has a beginning and an end in the consecutive collision orbit of Fig. 9.7(a) since the two subclasses (k_1) and (k_2) are joined at their beginnings in this orbit and also are continuations of each other at the asymptotic orbits. Note, however, that (k_1) and (k_2) approach different asymptotic orbits.

Figures 9.7(g) and 9.7(h) show the initial and final conditions for Class (k). For identification purposes points on the curves are labeled with the corresponding figure numbers. The family starts with Fig. 9.7(a), i.e., with the consecutive collision orbit, and it develops into two branches (k_1) and (k_2) which are indicated by opposite arrows on Figs. 9.7(g) and 9.7(h). The final ordinate (y_f) becomes $3^{1/2}/2$ as Classes (k_1) and (k_2) approach their respective limiting asymptotic periodic orbits. Therefore the curve shown in Fig. 9.7(h) becomes closed when the branches marked (k_1) and (k_2) meet at L_4. The x_i values of the asymptotic periodic orbits to which Subclasses (k_1) and (k_2) approach are not identical and therefore Fig. 9.7(g) is not closed.

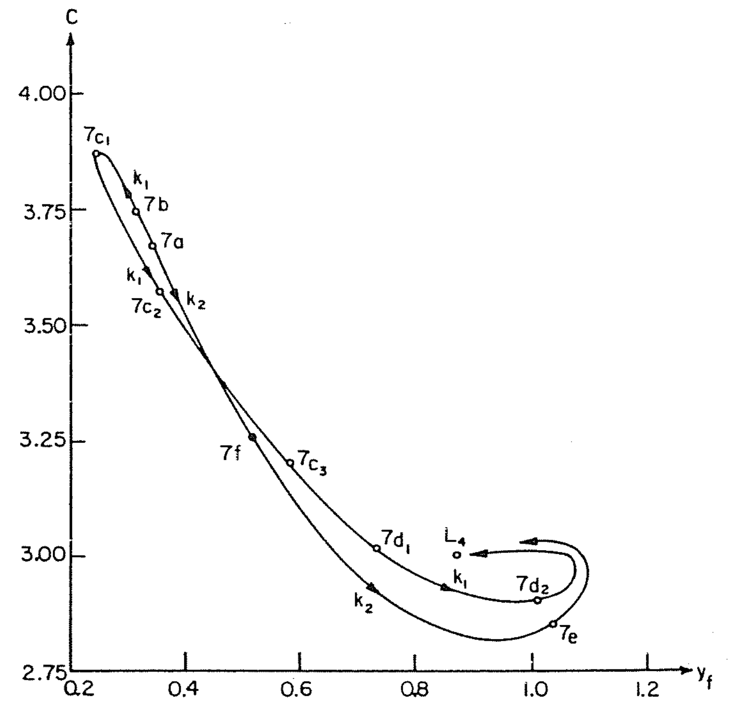

FIG. 9.7(h). Final conditions for Class (k) orbits of the Copenhagen category, $\mu = \frac{1}{2}$.

9.4.6 Synodically retrograde, sidereally direct periodic orbits around m_1 and m_2; Class (l) of the Copenhagen category

Consider circular orbits with radius r in the fixed (sidereal) system for the problem of two bodies. With the units used in this book we have $V = 1/r^{1/2}$ for the circular velocity, relative to this fixed system. The velocity of the rotating system at the distance r from the origin of the system normal to r is $+r$. Consequently the circular velocity relative to the rotating (synodic) system is

$$v = \pm 1/r^{1/2} - r,$$

where the $\pm$ sign refers to the direction of the velocity relative to the fixed system; the $+$ sign is associated with direct (counterclockwise) motion. For $r < 1$ we have seen that a sidereally direct motion is synodically also direct ($v = +1/r^{1/2} - r > 0$) and a sidereally retrograde motion is synodically also retrograde ($v = -1/r^{1/2} - r < 0$). For $r > 1$ the invariant behavior of the direction of the motion relative to fixed or rotating coordinate systems changes. Sidereally direct motion becomes synodically retrograde and sidereally retrograde motion preserves its direction as shown in Chapter 8.

Consequently it is possible to generate two families of periodic orbits by starting with circle-like orbits (with $r \gg 1$) which are either direct or retrograde in the fixed system. When sidereally direct orbits are used, the resulting retrograde orbits in the rotating system belong to Class (l). On the other hand when a family is generated from $r \gg 1$ circle-like orbits, which are sidereally retrograde, the resulting retrograde orbits in the rotating system belong to Class (m).

The generation of the first members of Class (l) with large values of r is as follows. Consider a point on the x axis at $x = x_i = r \gg 1$ and let the initial velocity at this point be $\dot{y} = \dot{y}_i = 1/r^{1/2} - r < 0$ and $\dot{x} = \dot{x}_i = 0$. The Jacobian constant is

$$C = 2\Omega - (\dot{y}_i)^2,$$

where

$$2\Omega = r^2 + \frac{1}{4} + \frac{1}{r - 0.5} + \frac{1}{r + 0.5}.$$

Substitution and rearrangement gives

$$C = 2r^{1/2} + \frac{4r^2 + 1}{(4r^2 - 1)\, r} + \frac{1}{4}.$$

This formula gives $C \to 2r^{1/2}$ as expected for $r \to \infty$. It furnishes for $r = 2.5$ the value of $C = 3.8456$ and the correct value is 3.859025. The discrepancy is due to the fact that the velocity used ($\dot{y}_i$) is for the problem of two bodies, and the preceding "correct" value of C is

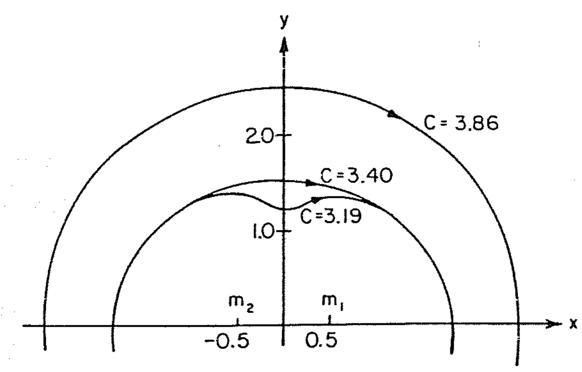

FIG. 9.8(a). Class (l) orbits of the Copenhagen category, $\mu = \frac{1}{2}$ (E. Strömgren and Fischer-Petersen, 1919, Ref. 22).

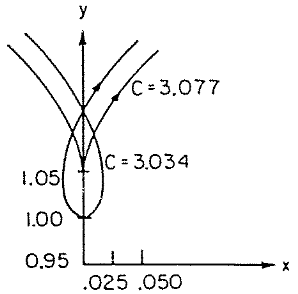

FIG. 9.8(b). Cusp and loop orbits of Class (I) in the Copenhagen category, $\mu = \frac{1}{2}$ (E. Strömgren and Fischer-Petersen, 1919, Ref. 22).

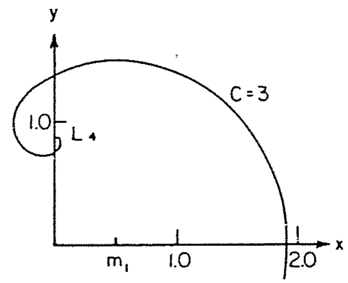

FIG. 9.8(c). Asymptotic-periodic orbit to L_4 of the Copenhagen category, $\mu = \frac{1}{2}$ (E. Strömgren, 1924, Ref. 22).

obtained for the restricted problem. In other words the value of $\dot{y}_i$ obtained from the two-body model is highly satisfactory as the basis of a differential correction procedure for arriving at periodic orbits of this type.

Figure 9.8(a) shows the orbit with the initial condition $x_i = 2.5$ ($C = 3.86$) and also another "circle-like" orbit at $C \cong 3.40$, which intersects the x axis at $x_i = 1.78$ and the y axis at $y = 1.5$. The third orbit shown in this figure has an indentation (preliminary to a cusp) at $|y_f| \cong 1.2$, a perpendicular intersection with the x axis at $x_i = 1.775$, and a Jacobian constant of $C = 3.19$. Figure 9.8(b) shows the continuation of the family regarding the behavior around $x \sim 0$. The cusp orbit is obtained at $C = 3.077$ and the cusp is located at $x = 0$, $|y_f| = 1.055$, while the perpendicular intersection with the x axis of this orbit is at $x_i \cong 1.8$. The cusp orbit is followed by a loop orbit as expected with the loop surrounding the cusp. The loop orbit shown in Fig. 9.8(b) intersects

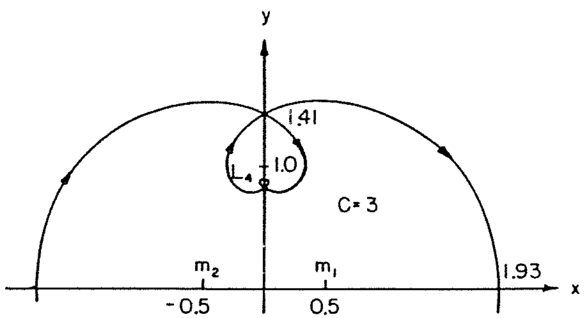

FIG. 9.8(d). Class (I) termination orbit of the Copenhagen category, $\mu = \frac{1}{2}$ (E. Strömgren, 1924, Ref. 22).

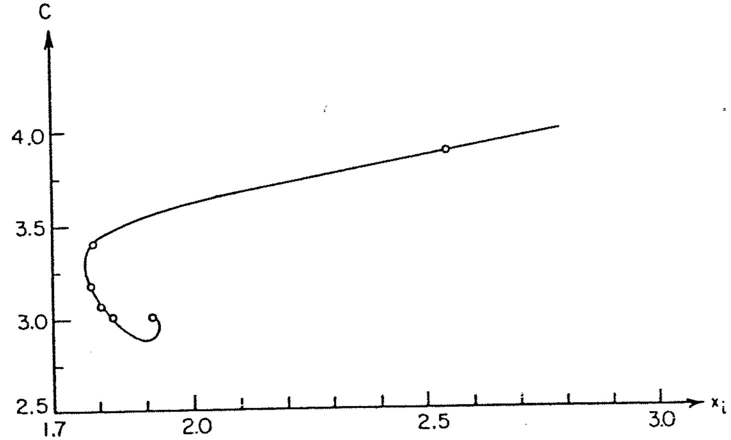

FIG. 9.8(e). Initial conditions for Class (I) orbits of the Copenhagen category, $\mu = \frac{1}{2}$.

the x axis perpendicularly at $x_i = 1.815$ and it has a Jacobian constant of $C = 3.034$.

Further development of this class is associated with the unstable character of the triangular libration point. The asymptotic orbit shown in Fig. 9.8(c) [quite similar to Figs. 9.7(d) and 9.7(f)] spirals into L_4 and intersects the x axis at $x_i \cong 1.92$ perpendicularly. This asymptotic orbit, because of its perpendicular intersection with the x axis, can form a periodic orbit (with infinite period) as shown in Fig. 9.8(d) and it becomes the termination of the class.

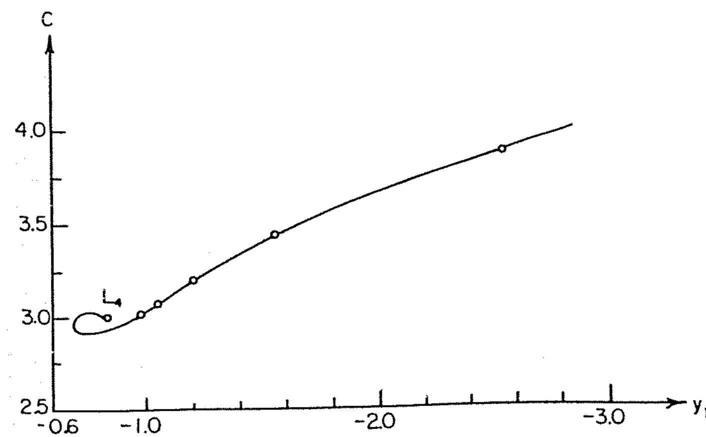

FIG. 9.8(f). Final conditions for Class (I) orbits of the Copenhagen category, $\mu = \frac{1}{2}$.

Figures 9.8(e) and 9.8(f) show the initial and final conditions for this class. The family begins at the right side of the figures with large values of x_i and $|y_f|$ and ends at $x_i \cong 1.92$ and $|y_f| = 3^{1/2}/2$ corresponding to point L_4. The points shown correspond to the orbits given in Figs. 9.8(a), (b), and (d).

9.4.7 Synodically and sidereally retrograde periodic orbits around m_1 and m_2; Class (m) of the Copenhagen category

The existence questions and methods of generation are the same for this class as for Class (l). The velocity relative to the rotating system is given in the limit by

$$v = -r - \frac{1}{r^{1/2}}$$

for circle-like orbits far from the primaries, i.e., for $r \gg 1$. For instance, at $r = 2.5$ the above equation gives $v = -3.1325$ and the actual initial velocity giving a periodic orbit with $x_i = 0$, $y_i = 2.5$ is $\dot{x}_i = +3.1388$. Computation of approximate values of the Jacobian constant follows the previously described method with the following result:

$$C = -2r^{1/2} + \frac{1}{4} - \frac{1}{r} + \frac{2}{(r^2 + \frac{1}{4})^{1/2}},$$

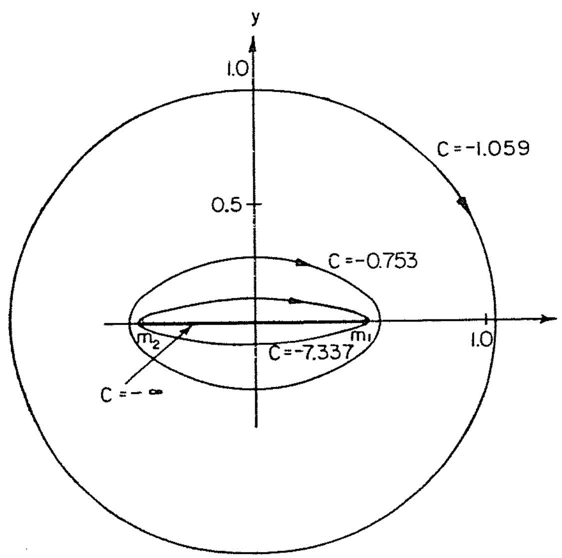

FIG. 9.9(a). Class (m) orbits of the Copenhagen category, $\mu = \frac{1}{2}$ (Möller, 1924, Ref. 22).

where r now is the distance between the origin and the intersection of the orbit with the y axis (y_i). For the example mentioned before $y_i = r = 2.5$ which gives $C = -2.53$ while the correct result is -2.567665. (Note the alternate method by starting on the y axis instead of on the x axis as for Class (l) when estimates for initial or final conditions are needed.)

For large values of r, $C \to -2r^{1/2}$. As the orbits shrink to the mass points and approach the straight line limiting orbit between the primaries, we have $C \to -\infty$. Since $C \to -\infty$ also as the size of the orbit increases, a maximum value of C is reached at $x_i \cong 0.61$, $y_f \cong 0.42$, which is $C_{\max} \cong -0.37$.

Three orbits belonging to this class are shown in Fig. 9.9(a); the corresponding C, x_i, and y_f values are $C = -1.059$, -0.753, -7.337; $x_i = 1.051$, 0.541, 0.5025, and $y_f = 1$, 0.266, 0.095.

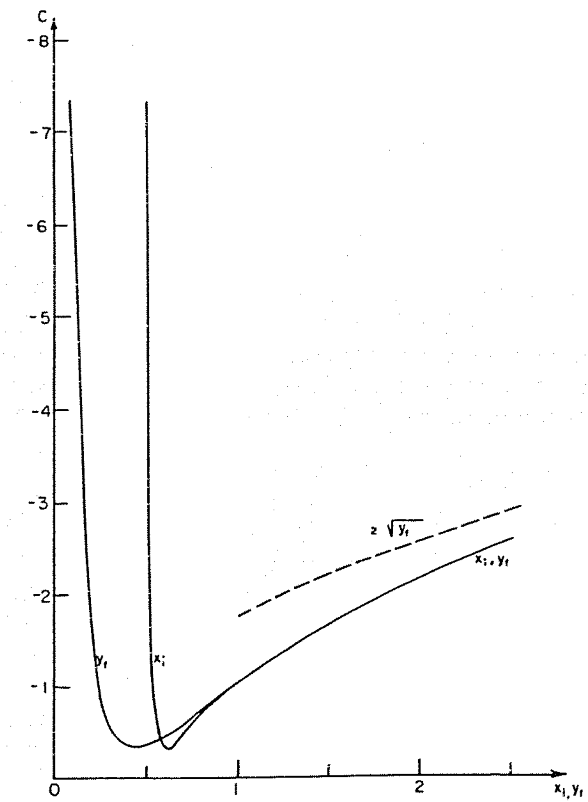

FIG. 9.9(b). Initial conditions for Class (m) orbits of the Copenhagen category, $\mu = \frac{1}{2}$.

The variation of the initial conditions with C is shown in Fig. 9.9(b). The curves merge as they approach the value $C = 2r^{1/2}$ asymptotically for large r. The curve $C(x_i)$ has a vertical asymptote at $x_i = 0.5$, where the primary is located, and the curve $C(y_f)$ has its vertical asymptote at $y_f = 0$, corresponding to the straight-line orbit.

9.4.8 Retrograde periodic orbits asymmetric to the y axis; Class (n) of the Copenhagen category

The symmetric periodic orbit with consecutive collisions of Class (c), shown in Fig. 9.4(a) and corresponding to $C = 2.4329$, can be used to give rise to a class of orbits, members of which are asymmetric with respect to the y axis. We do not refer to this class as one "around L_2" because first, this description is reserved for Class (c) and, second, there are no infinitesimal members of Class (n) "around L_2."

The class closes upon itself, i.e., its members can be arranged so that they are generated following a closed system. Figure 9.10(a) shows some of the characteristic members. Beginning with (a) the class starts out with a single-collision orbit. This is followed by (b) where the collision at m_1 changes into a close approach at m_1. The next member, shown in (c), is the mirror image of (b). This is followed by (d) which is the

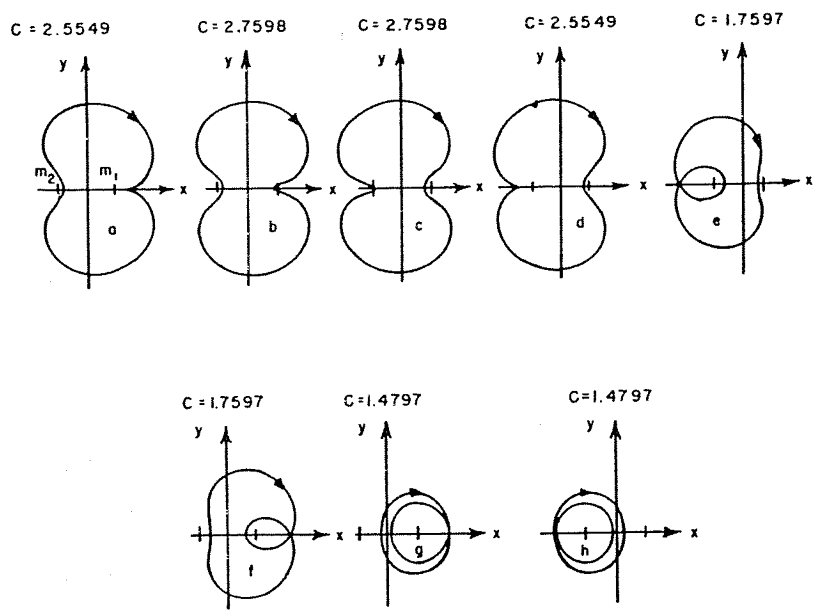

Fig. 9.10(a). Class (n) orbits of the Copenhagen category, $\mu = \frac{1}{2}$ (E. Strömgren, 1934, Ref. 22).

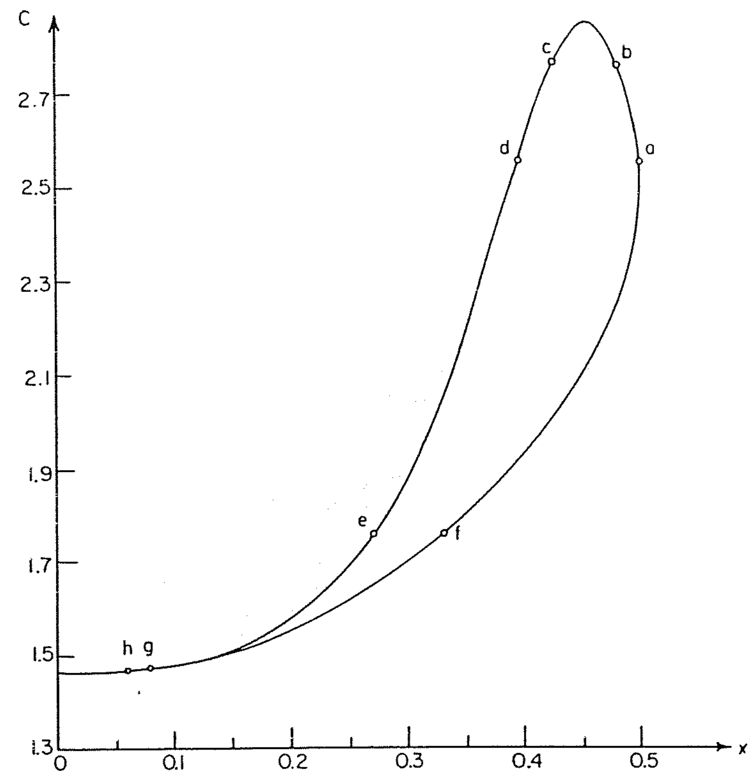

Fig. 9.10(b). Initial and final conditions of Class (n) orbits of the Copenhagen category, $\mu = \frac{1}{2}$.

mirror image of (a), i.e., it shows a collision with m_2. From (d), (e) is generated by changing the collision with m_2 into a loop orbit around m_2. As the loop increases and the orbit itself shrinks we obtain (h) as the next member. Now the loop becomes the orbit and the orbit becomes the loop, so after (h) the steps are retraced to (e), (d), (c), (b), and (a). At this point we proceed from (a) by forming a loop around m_1, instead of a close approach, as has been done before. Thus, (f) — instead of (b) — is obtained as the next member of the class. From (f), by increasing the size of the loop, (g) is obtained. Now the roles of the loop and of the orbit are again interchanged and the steps are retraced: (g) is followed by (f) and (a). This way the circle is closed with all members accounted for. The order is (a), (b), (c), (d), (e), (h), (h), (e), (d), (c), (b), (a), (f), (g), (g), (f), and then repeat from the beginning with (a).

Note that when (h), (h) [or (g), (g)] is listed, a simple periodic orbit (without a loop) is assumed to exist between the two identical (h) orbits in which the loop and the orbit coincide. The Jacobian constant for this orbit is approximately 1.4675, with $x_i = x_f \cong -0.012$.

The only consecutive collision orbit of this class is the previously mentioned symmetric orbit of Class (c), if we wish to include this member. There seem to be no asymmetric consecutive collision orbits.

The closed system of orbits belonging to Class (n) is represented by the closed curve shown in Fig. 9.10(b). The abscissa (x_i) shows the interchangeable initial and final values of the location of the perpendicular intersection of the orbits with the x axis. The letters associate some of the points of Fig. 9.10(b) with the orbits shown in Fig. 9.10(a).

9.4.9 Orbits asymptotically approaching the triangular libration points; additional classes ((o) and (r)) of periodic orbits in the Copenhagen category

In Section 5.4.2 spiral motions in the infinitesimal neighborhoods of the equilateral libration points were discussed in considerable detail. It was shown that for $\mu > \mu_0 = .03852...$ the solution of the linearized equations of motion can be written in the following form

$$x = x_0 + A_1 e^{\alpha t} \cos(\beta t + \gamma_1) + A_2 e^{-\alpha t} \cos(\beta t + \gamma_2),$$

$$y = y_0 + B_1 e^{\alpha t} \cos(\beta t + \delta_1) + B_2 e^{-\alpha t} \cos(\beta t + \delta_2).$$

Here x_0, y_0 are the coordinates of the libration point, A_1, A_2, B_1, B_2, γ_1, γ_2, δ_1, δ_2 are constants of integration (only four being independent), $\alpha = 0.63208$ and $\beta = 0.94843$ if $\mu = \frac{1}{2}$.

The special solutions of interest at this point are obtained when either $A_1 = B_1 = 0$ or $A_2 = B_2 = 0$, since in these cases the orbits spiral in or out of the triangular libration points with increasing time.

These orbits, originating from L_4 or L_5, when computed at large distances from the libration points will change their character since the spiral-type solutions are only valid in the (small) neighborhood of $L_{4,5}$. There are five orbits which intersect the x axis perpendicularly after leaving the neighborhood of L_4. The five orbits are shown in Fig. 9.11.

In connection with Class (k) the role of two of the five asymptotic orbits was displayed in Section 9.4.5. It was shown that Subclass (k$_1$) ends in the asymptotic orbit denoted by 1 in Fig. 9.11 and Subclass (k$_2$) ends in curve 2 of Fig. 9.11. Also, in Section 9.4.6, asymptotic orbit number 5 was shown to be the termination orbit of Class (l).

This leaves curves numbered 3 and 4 for the creation of new families.

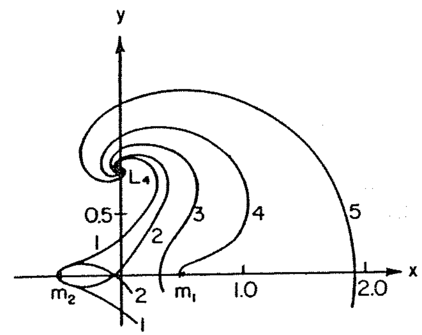

FIG. 9.11. Asymptotic-periodic orbits to L_4 of the Copenhagen category, $\mu = \frac{1}{2}$ (E. Strömgren, 1924, Ref. 22).

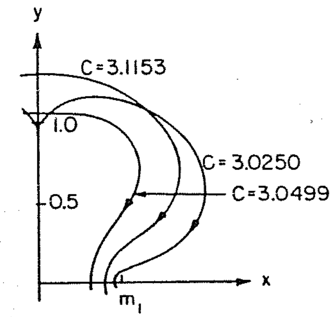

FIG. 9.12. Class (r) orbits of the Copenhagen category, $\mu = \frac{1}{2}$ (E. Strömgren, 1934, Ref. 22).

Symmetric (with respect to both axes) retrograde periodic orbits between curves 3 and 4 are called Class (r) in Strömgren's terminology, while asymmetric (with respect to the y axis) retrograde periodic orbits between the same asymptotic curves (3 and 4) are termed Class (o). Figure 9.12 shows three members of Class (r). The inside orbit ($C = 3.0499$) is close to curve 3 of Fig. 9.11 and the outside ($C = 3.0250$) approaches curve 4 of Fig. 9.11. An asymmetric class can be obtained by combining the right side of curve 3 with the left side of curve 4 and vice versa. Since these curves (3 and 4) are asymptotic to L_4 they can be considered as continuations of each other as shown on Fig. 9.13. This figure can now be used, together with

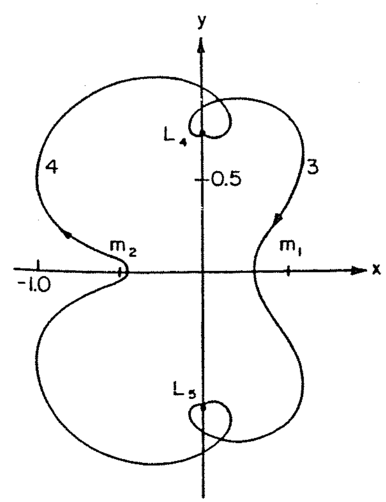

FIG. 9.13. Asymptotic-periodic orbits 3 and 4 of the Copenhagen category, $\mu = \frac{1}{2}$ (E. Strömgren, 1934, Ref. 22).

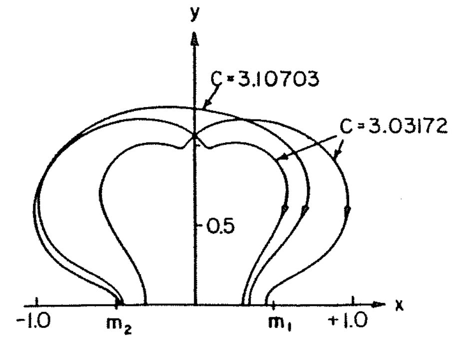

FIG. 9.14. Class (o) orbits of the Copenhagen category, $\mu = \frac{1}{2}$ (E. Strömgren, 1934, Ref. 22).

its mirror image (with respect to the y axis), to generate the asymmetric Class (o) mentioned before and shown in Fig. 9.14.

Other possible combinations of the five asymptotic orbits furnish further classes of symmetric and asymmetric orbits.

In Section 9.4 the concluding remarks were made about asymptotic orbits which intersect the y axis perpendicularly after a half revolution. These asymptotic-periodic orbits which are symmetric to the y axis but not to the x axis can also be used alone and in combinations to generate further classes of periodic orbits. A few examples are shown in Fig. 9.15.

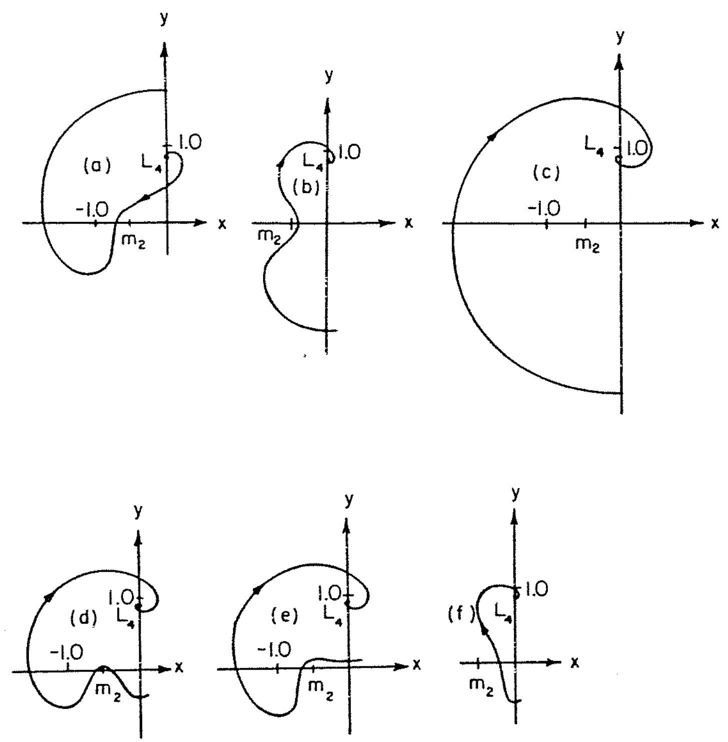

FIG. 9.15. Asymptotic-periodic orbits, asymmetric to the x axis in the Copenhagen category, $\mu = \frac{1}{2}$ (E. Strömgren, 1930, Ref. 22).

9.4.10 Strömgren's termination principle

The classes of orbits found by the Copenhagen school either close upon themselves (re-enter) or have a natural dynamic beginning and end. This natural beginning and termination can be either at one or at

both masses, or at one or two of the libration points, or, finally, at infinity.

Strictly speaking, the termination principle refers to only those classes which are generated by analytic continuation from either two-body or infinitesimal orbits. Classes like (k) and (n) are therefore excluded. In fact, as it was shown, both Classes (k) and (n) obey the principle, (k) with the special interpretation given in Section 9.4.5 and (n) by re-entering.

Another way to express the principle is to consider the sum of the maximum dimension of an orbit (D), its period (T), and the Jacobian constant (C). Let

$$m = D + T + |C|. \tag{26}$$

The natural end or beginning of a class now corresponds to $m \to \infty$ since at the primaries $C \to \infty$, at the triangular libration points $T \to \infty$, and when the orbits increase their size without limit, $D \to \infty$.

Classes (a) and (n) are closed, (c) begins at L_2 and ends at infinity, (l) and (m) begin at infinity and end at the triangular libration points and at the primaries, respectively, and Class (f) begins at m_1 and ends at infinity. Classes (k), (o), and (r) begin and end at the triangular libration points.

Class (c) beginning at L_2 does not satisfy the condition m since at L_2, $D = 0$, T is finite, and $C = C_2$ is also finite. Therefore this class is to be treated separately if the termination principle is expressed by Eq. (26).

A modified form of Eq. (26), suggested by Birkhoff, includes all classes investigated by the Copenhagen school. Let D again be maximum dimension of an orbit, T its period and d the minimum of all distances between the orbit and the primaries and libration points. Then at termination we have

$$\lim(D + T + 1/d) = \infty, \tag{27}$$

i.e., either the size or the period of the orbit becomes unbounded or the orbit approaches arbitrarily close the primaries or the libration points. This form of the termination principle can also be stated as follows: the members of a class either form a periodic set with respect to the Jacobian constant, i.e., the class is closed, or it is impossible to have a situation when at the beginning or at the end of a class

$$D \text{ and } T < K; \quad d > 1/K,$$

where $K > 0$ is finite.

Assume that a class terminates at a value of the Jacobian constant, $C = C_0$, and that $C = C(r)$ is an analytic function of a parameter r. At termination we have $C_0 = C(r_0)$ and Eq. (27) becomes

$$\lim_{r \to r_0}(D + T + 1/d) = \infty.$$

The termination principle can be proved by analytic continuation or by employing the method of the surface of sections discussed in Chapter 8; in both cases, well-known theorems of implicit functions are used.

Note that collision orbits do not, in general, terminate any class; in fact, the collision orbits occurring in various classes must be included as members of the classes in order to continue the class to its termination. [Class (m) is the only exception terminating in the straight-line consecutive collision orbit connecting the primaries (with $C \to -\infty, d \to 0$).] Class (a), originating at L_3 and closing upon itself, is an example in point since the inclusion of the collision orbit shown in Fig. 9.3(a) and marked with $C = 2.50$ (and its continuation with loop orbits) is essential to establish a self-closing class.

The two possible terminations of Class (g), to be mentioned in Section 9.11, satisfy the principle.

9.4.11 G. H. Darwin's periodic orbits

The families of periodic orbits computed by Darwin are discussed in the section dedicated to the Copenhagen category, in spite of the fact that the values of the mass parameter are different. Nevertheless, as will be shown, there are no essential differences between the established families.

The system used by Darwin was described in Fig. 9.1(b) of Section 9.3. The formula for converting Darwin's Jacobian constant to the value used in this work is $C = C_D/11$, as given by Eq. (8) of Section 9.3.

TABLE IV

DARWIN'S JACOBIAN CONSTANTS AND LOCATIONS OF THE COLLINEAR EQUILIBRIUM POINTS

L_i	C_D	C	r
L_1	38.875991	3.534181	1.346994
L_2	40.182076	3.652916	0.717512
L_3	34.905442	3.173222	0.946927
$L_{4,5}$	33.000000	3.000000	1.000000

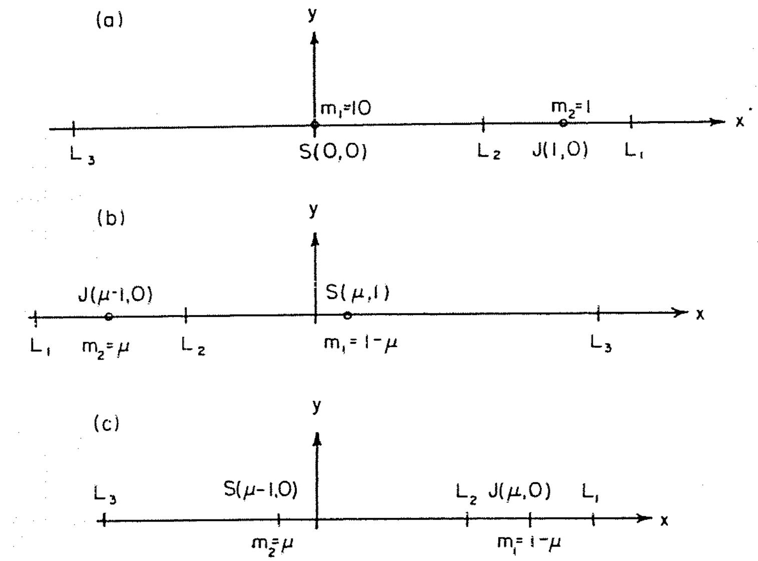

FIG. 9.16. (a) Darwin's system. (b) Reference system for $\mu = 1/11$. (c) Reference system for $\mu = 10/11$.

The values of the Jacobian constant at the libration points and the distance (r) of the libration points from the heavier primary are given in Table IV. Figures 9.16(a), (b), and (c) compare Darwin's and the reference systems.

As the figures show, Fig. 9.16(b) can be transformed into 9.16(c) by using $\mu = 10/11$ and then the latter can be transformed into 9.16(a) by translating the origin of the coordinate system to the location of the heavier mass $S(-1/11, 0)$. Instead of m_1 and m_2, Darwin uses the letters S and J for the heavier and lighter primary, corresponding to the sun and to Jupiter. Because of the difference between Darwin's system and the reference system regarding the location of the primaries, the proper value of the mass parameter is $\mu = 10/11$ when relations between the systems are to be established. Nevertheless, for basic symmetry reasons the $\mu = 1/11$ value is equally acceptable, especially if a mental $x \to -x$ transformation is executed.

(A) The majority of Darwin's work was concentrated on what, in view of the previous results, is known as *Class* (g) orbits. Due to the loss of Strömgren's symmetry condition, comparison of the Copenhagen Class (g) and Darwin's orbits requires care. Darwin studied direct periodic orbits around J, i.e., around the smaller primary, and established several families, all of which, as will be shown, can be considered as members of Class (g). Figure 9.17(a) is one of Darwin's original figures representing orbits which he called "Family A of satellites." It corresponds to Fig. 9.6(a) of Section 9.4.4. The collision orbit shown corresponds to $C_D = 38.85$ or $C_{10/11} = 3.53182$. Note that for the collision orbit of Fig. 9.6(c), $C_{1/2} = 3.73605$, i.e., $C_{1/2} > C_{10/11}$.

The collision orbit with J, in fact Darwin's whole family, is perturbed by S which has a mass ten times as large as J. In the case $\mu = \frac{1}{2}$ the relative perturbation is therefore smaller. Consequently one would

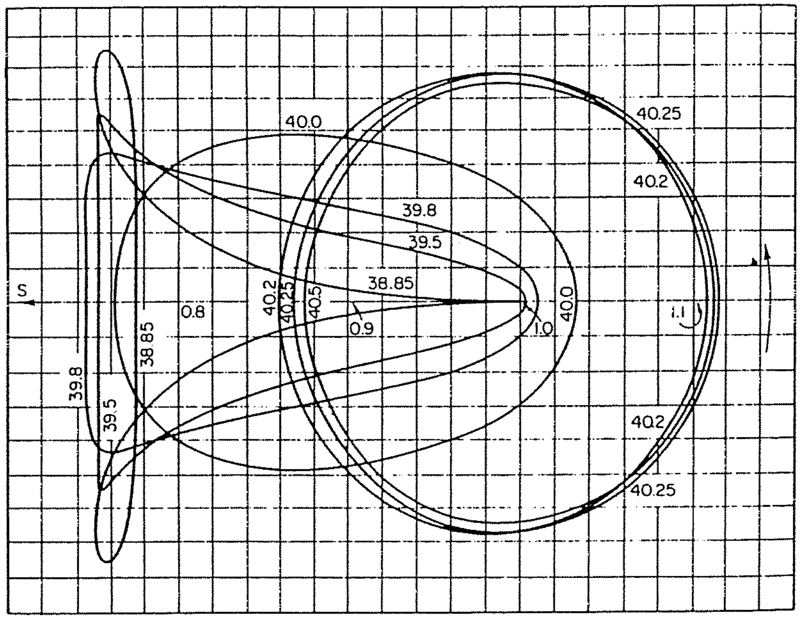

FIG. 9.17(a). Darwin's family A of satellites, $\mu = 10/11$ (Darwin, 1897, Ref. 2).

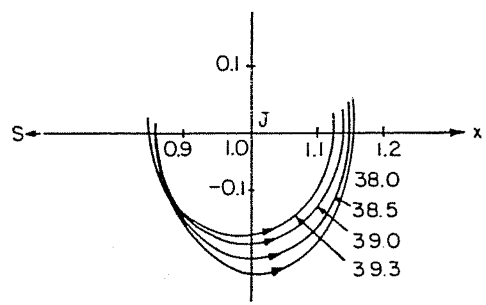

FIG. 9.17(b). Darwin's family B of satellites, $\mu = 10/11$ (Darwin, 1897, Ref. 2).

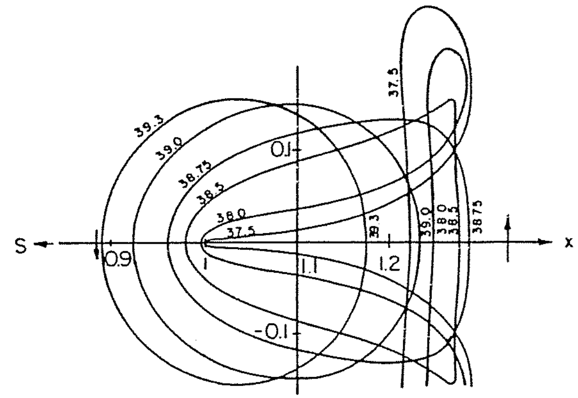

FIG. 9.17(c). Darwin's family C of satellites, $\mu = 10/11$ (Darwin, 1897, Ref. 2).

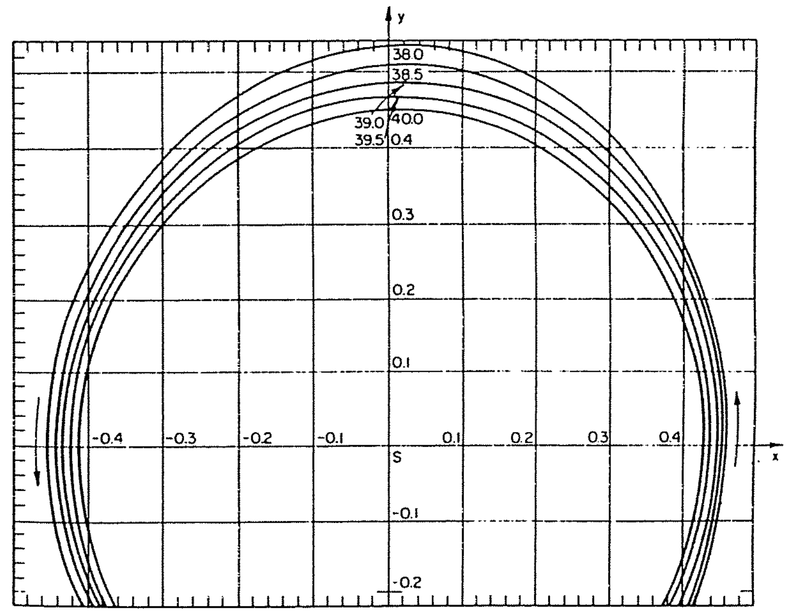

FIG. 9.17(d). Darwin's family A of planets, $\mu = 10/11$ (Darwin, 1897, Ref. 2).

expect that collision would be reached *sooner* (the family would develop *faster*) in Darwin's than in Strömgren's case. In other words the Jacobian constant of Darwin's collision orbit should be larger than Strömgren's. The fact that it is just the other way around shows that comparing values of the Jacobian constant of different values of μ is not necessarily meaningful and furthermore that the value of μ is not completely representative of the "magnitude of perturbations." Note another discrepancy between Fig. 9.6(a) and 9.17(a), namely, the variation of the value of the Jacobian constant. This is monotone decreasing for Darwin's family ($C_{10/11}$) as the group develops from the circle-like orbits to the collision orbit, while Class (g) shows a minimum value of $C_{1/2} = 3.989$ during the same development. It can be assumed that a minimum value of $C_{10/11}$ also exists, other than the one corresponding to the collision orbit.

Table V shows the values of the Jacobian constants of Fig. 9.17(a) in the reference system.

TABLE V

VALUES OF THE JACOBIAN CONSTANT IN FIG. 9.17(a)

$C_D =$ 40.5	40.25	40.2	40.0	39.8	39.5	38.85
$C_{10/11} =$ 3.682	3.659	3.655	3.636	3.618	3.591	3.532

Phase 2 of Strömgren's Class (g) is shown in Fig. 9.6(b). This is missing entirely in Darwin's work which continues in his terminology with the "Family B of satellites". This family begins with an orbit with $C_D = 39.3$ or $C_{10/11} = 3.573$ and its members have decreasing values of C. The next group of orbits ("Family C of satellites") starts out again with $C_D = 39.3$ and the values of the Jacobian constant of its members decrease again. Figures 9.17(b) and 9.17(c) by Darwin show Families B and C, which correspond to the third phase of Class (g) orbits [see Fig. 9.6(c)]. Note that the loop orbit of Fig. 9.17(c) with $C_D = 37.5$ or $C_{10/11} = 3.409$ corresponds to an orbit between the last two orbits of Figure 9.6(c). The collision orbit of Fig. 9.6(c) has $C_{1/2} = 2.995$ as its Jacobian constant while $C_{10/11}$ for collision, according to Fig. 9.17(c) must be less than $C_{10/11} = 3.409$. So one can conclude that Darwin's Families A, B, and C of satellites correspond to a part of Strömgren's Class (g). Tables VI and VII give those values of the Jacobian constants

TABLE VI

VALUES OF THE JACOBIAN CONSTANT IN FIG. 9.17(b)

C_D	39.3	39.0	38.5	38.0
$C_{10/11}$	3.573	3.545	3.500	3.455

TABLE VII

VALUES OF THE JACOBIAN CONSTANT IN FIG. 9.17(c)

C_D	39.3	39.0	38.75	38.5	38.0	37.5
$C_{10/11}$	3.573	3.545	3.523	3.500	3.455	3.409

in the standard system which are given in Fig. 9.17(c). It is seen that Darwin's "generator" orbit at $C_D = 39.3$ corresponds to, say, the second orbit of Fig. 9.6(c) (with $C_{1/2} = 3.814$). The value of the Jacobian constant decreases before and after this orbit in Class (g), similarly to Darwin's B and C satellites.

(B) Darwin also computed six direct periodic orbits around his larger primary corresponding to the beginning of Class (g). In Darwin's terminology these orbits belong to "Family A of planets." The perturbation is represented by J, and the orbits are ovals. For instance, at $C_D = 40$ or $C_{10/11} = 3.636$ the oval of the orbit intersects the y axis at ± 0.447 and the x axis at $+0.423$ and -0.414. Five of these orbits are shown in Fig. 9.17(d), after Darwin.

(C) Six orbits for what Darwin called *oscillating satellites* are also available. These are close to the linearized elliptic orbits around the collision libration points L_1 and L_2, corresponding to the beginning phases of Strömgren's classes (a) and (c).

(D) Orbits encircling both primaries and belonging to *Class* (l), i.e., direct relative to the fixed and retrograde with respect to the synodic system, are called, by Darwin, "Family D of superior planets." The smallest value of the Jacobian constant for which Darwin computed such an orbit is $C_D = 37$ or $C_{10/11} = 3.36$, and he found none of the members of the family showing Strömgren's interesting development [cf. Fig. 9.8(a)].

We conclude the review of Darwin's orbits with the remark that the orbits obtained by him and properly interpreted correlate with Strömgren's classification of orbits. This is significant since it means that the results obtained with $\mu = \frac{1}{2}$ are, qualitatively at least, identical with those obtained when $\mu = 10/11$, or $\mu = 1/11$. The meaning of "qualitative" is that first the shapes of the orbits are subject to topological transformations going from $\mu = \frac{1}{2}$ to $\mu = 10/11$. But the values of the Jacobian constants associated with the corresponding special orbits (e.g., collision orbits) in the two systems might be different also. Finally we observe that the development of a given class takes place at different rapidity for different values of the mass parameter.

9.4.12 Moulton's periodic orbits

The system in which the results of the Moulton school are presented corresponds to the interchange of the primaries and to the use of $\bar{C}$ as the Jacobian constant. In other words Moulton's Jacobian constant as defined by him is $C_M = \bar{C} = C - \mu(1 - \mu)$. In some of the numerical results this definition is changed and $C_M = C = \bar{C} + \mu(1 - \mu)$ is used by him. The orbits described in what follows have been checked in this regard and the values given in the text for the Jacobian constant have been made consistent.

(A) An interesting family of retrograde periodic orbits was established by Moulton encircling the triangular libration point L_4. These orbits

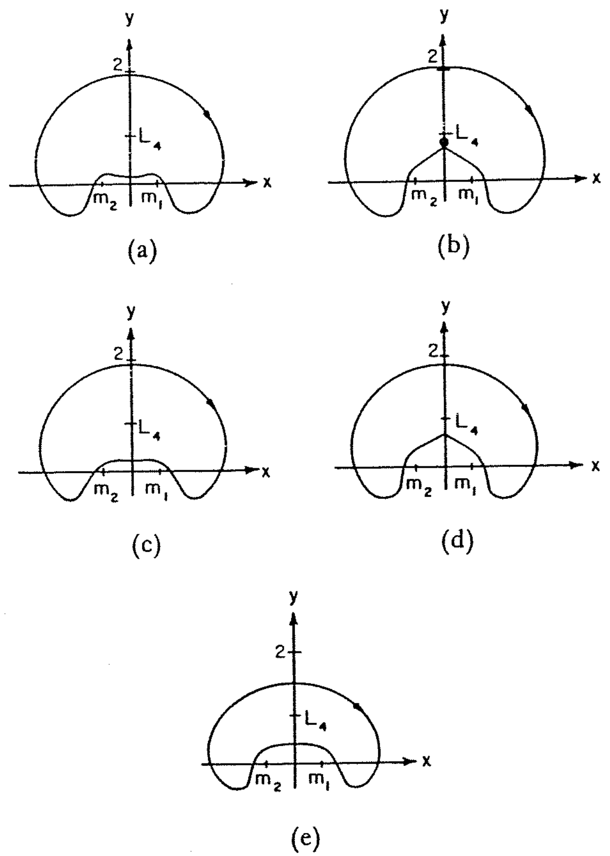

FIG. 9.18. Moulton's retrograde periodic orbits around L_4 for $\mu = \frac{1}{2}$ (Moulton, 1920, Ref. 1): (a) $C = 3.03$, (b) $C = 3.03$, (c) $C = 3.20$, (d) $C = 3.20$, (e) $C \cong 3.3284$.

are closely related to Strömgren's *asymptotic-periodic orbits* which are asymmetric to the x axis and are shown in Fig. 9.15. Figure 9.18 shows the orbits found by Moulton for $\mu = \frac{1}{2}$ and it also gives the values of the Jacobian constants in our standard system in order to avoid any misunderstandings. Figures 9.18(a), (c), and (e) show what Moulton considers as members of a class. Observe that the limiting orbits of this class may be asymptotic-periodic orbits similar to Fig. 9.15. In fact Moulton finds two periodic orbits for each given value of the Jacobian constant ($C = 3.03$ and 3.20), as shown in Figs. 9.18(a), (b), and 9.18(c), (d), until $C \cong 3.3284$ is reached. At this value of C the two families unite in Fig. 9.18(e). In view of the asymptotic-periodic orbits discussed in Section 9.4.9, a possible explanation is as follows. Consider figures similar to 9.15(a) to 9.15(f) as limiting orbits for classes. Then between two of these asymptotic-periodic orbits there exists a class, members of which are given in Figs. 9.18(a)–(e). In fact the class can be started with a figure similar to 9.15(a) and ended with a figure similar to 9.15(d). The order of the members is as follows: 9.15(a) (?), 9.18(b), 9.18(d), 9.18(e), 9.18(c), 9.18(a), and 9.15(d) (?). The "double periodic orbits" of Moulton [two orbits with the same value of C, e.g., Figs. 9.18(a) and 9.18(b)] are simply members of the same class at different states of development. The controversy regarding these orbits is discussed in Section 9.11.

(B) Moulton's (independent) discovery of *Class* (k) orbits [especially the one shown in Fig. 9.7(b)] will also be discussed in Section 9.11 since the several fascinating details would be distracting at this point. In addition to the consecutive collision orbit of Class (k), Moulton also found a number of *single collision orbits* for $\mu = \frac{1}{2}$, such as the one belonging to Class (a) [Fig. 9.3(a)], to Class (f) [Fig. 9.5(b)], and to Class (g) [Figs. 9.6(a) and the last of 9.6(c)].

(C) Two *single collision orbits* computed for $\mu = 4/5$ will be discussed in some detail since this contributes to the previously mentioned question of the rapidity with which families develop. First, the collision orbit of Class (a), shown in Fig. 9.3(a) and marked by the value of its Jacobian constant $C = 2.50$, is compared with Moulton's collision orbit shown in Fig. 9.19(a). Moulton's curve is identical in shape and direction (both orbits are retrograde) but its Jacobian constant is $C_{4/5} = 2.946$, i.e., $C_{1/2} < C_{4/5}$ for the same member of the same family. The relative perturbation is larger in Moulton's case than in Fig. 9.3(a); therefore, the reduction from $C_{1/2}(L_1) = 3.706796$ to the collision value ($C_{1/2} = 2.5$) is expected to occur more slowly than that from $C_{4/5}(L_1) = 3.712393$ to $C_{4/5} = 2.946$. The numerical results bear out these facts.

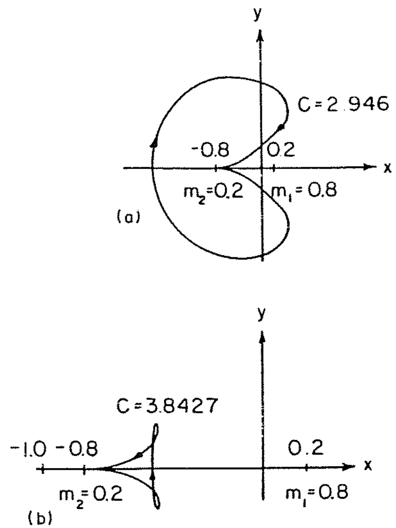

FIG. 9.19. (a) Moulton's closed orbit of ejection, $\mu = \frac{4}{5}$ (Moulton, 1920). (b) Moulton's closed orbit of ejection, $\mu = \frac{4}{5}$ (Moulton, 1920, Ref. 1).

The situation is quite different for the Class (g) collision orbit shown in Fig. 9.19(b). This is to be compared with Figs. 9.6(a) and 9.17(a), i.e., with Strömgren's and Darwin's results. Table VIII facilitates the comparison.

TABLE VIII

COMPARISON OF CLASS (g) COLLISION ORBITS FOR DIFFERENT VALUES OF THE MASS PARAMETER

μ	C	Figure number	m_p/m_c	d
1/2	3.73605	9.6(a)	1	0.425
10/11	3.53183	9.17(a)	10	0.225
4/5	3.8427	9.19(b)	4	0.3

All three orbits are direct periodic collision orbits, belonging to Class (g). In all three cases the third particle is ejected toward the other mass [i.e., not away from it as on the last one of Figs. 9.6(c) or on 9.17(c)]. The collision occurs in all three cases with the primary whose mass (m_c) is smaller than or equal to the mass of the perturbing primary (m_p). The table shows the ratios of the perturbing masses to the masses with which collision occurs (m_p/m_c), the Jacobian constants of the

collision orbits, and the distances between the location of the smaller mass and the point on the x axis where the orbit has a normal crossing (d). It is to be remembered that the orbits in Class (g) originate from the $C \to \infty$ orbits around the primary. Rapid development of a family therefore means that a collision orbit is reached at a relatively high value of the Jacobian constant. The fastest development is observed at $\mu = \frac{4}{5}$, then at $\mu = \frac{1}{2}$, and the slowest at $\mu = 10/11$. This trend cannot be explained by the magnitude of the relative perturbations as mentioned before.

In conclusion it is remarked once again that Strömgren's classification and general findings are able to accommodate Moulton's results when they are properly interpreted. The general topologically invariant nature of the orbits as μ varies, the termination principle, and the use of the Jacobian constant as a parameter (for a fixed value of the mass parameter) are all verified by the Strömgren–Darwin–Moulton findings—at least to the extent of the results presented.

9.5 Periodic lunar orbits

In this category belong those periodic orbits which were computed using the earth and the moon as the primaries.

The major results are presented in four sections. First, the effect of the change in the value of the mass parameter from $\mu = \frac{1}{2}$ to $\mu \cong 1/82.3$ is studied by comparing the orbits of the Copenhagen category with periodic lunar orbits in Sections 9.5.1 and 9.5.2. Then in Section 9.5.3 Egorov's periodic orbits are shown. The fourth section (9.5.4) discusses special periodic orbits already mentioned by Darwin but actually established only in the 1960's.

9.5.1 Motion around the collinear points

Strömgren's Class (a) orbits around L_3 and Class (b) orbits around L_2 [i.e., around the two libration points which are outside $\overline{m_1 m_2}$ in Fig. 9.20(a)] are symmetric with respect to the y axis. This is not the case when $\mu \neq \frac{1}{2}$; nevertheless, the general shape of the families is similar. Figure 9.20(a) shows the general arrangements used for the orbits of this section. The value of the mass parameter is $\mu = 0.0121550992 = 1/82.27$ or $m_\oplus/m_☾ = 81.27$.

Motion around L_3 is shown in Fig. 9.20(b) according to Broucke. The similarity between Fig. 9.20(b) and Figs. 9.3(a) and (b) will not escape the reader. In both cases the infinitesimal elliptic retrograde orbits around L_3 increase in size (with decreasing Jacobian constant) until a

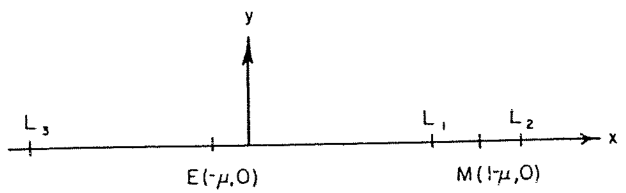

FIG. 9.20(a). The general arrangement for periodic lunar orbits.

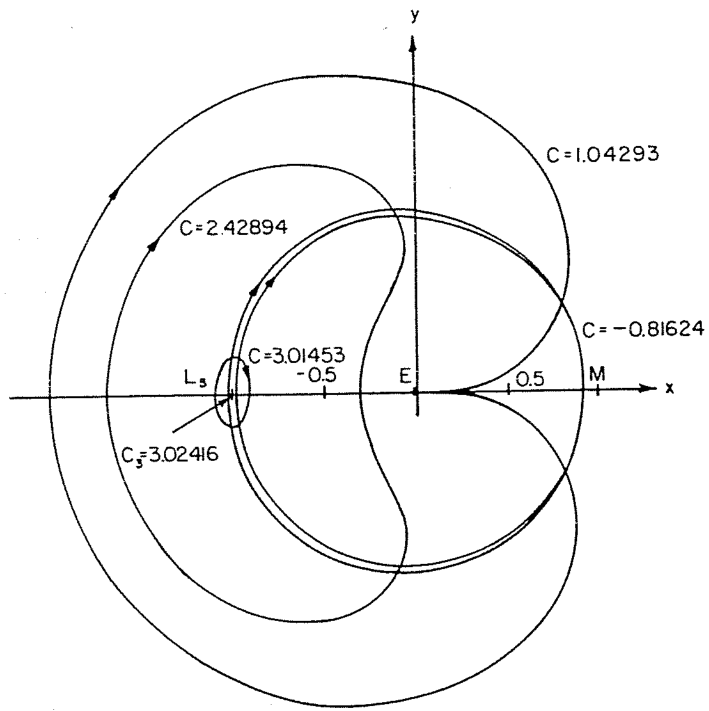

FIG. 9.20(b). Periodic lunar orbits, Class (b), $\mu = 1/82.27$ (Broucke, 1962, Ref. 4).

collision orbit is reached. After this a loop develops around m_1 in the Copenhagen problem and around the earth in the present case. The role of the loops interchange at approximately $C = -0.816$ and the family of orbits closes upon itself by reversing the trend. The invariancy of the general shape of the orbits and the construction of the family as μ changes from $\frac{1}{2}$ to $1/82.27$ for Class (a) is rather striking.

The corresponding family which can be generated from infinitesimal elliptic orbits around L_2 [Fig. 9.20(a)] contains a collision orbit also, at $C = 2.744564$.

The family of retrograde orbits around the libration point located between the primaries begins with infinitesimal elliptic orbits corresponding to Strömgren's Class (c). The members of the family, after the initial elliptic motion with small amplitude, show similarity to some of the members of Strömgren's asymmetric Class (n) because of the inherently asymmetric situation existing in the earth-moon system. Collision with E occurs during the development, with $C = 1.434059$. This collision orbit is followed by loop orbits. The end of this class can show two different developments, depending on the arrangements selected. First, when the appearance of the loops are considered according to Class (n), then, as the inside loop grows and the outside loop shrinks an oval orbit is reached. At this point the whole development is reversed, the role of the loops is interchanged, and we obtain a closed class. The second possible interpretation follows the development of Class (c) through repeated collisions with one and then with the other primary. Consequently loops develop and the size of the orbit increases to infinity.

9.5.2 Motion around the primaries

There are four different classes when $\mu \neq \frac{1}{2}$ which correspond to Strömgren's Classes (e), (f), (g), and (h) and are shown in Fig. 9.21 according to Broucke:

Class (e) contains retrograde orbits around E;
Class (f) contains retrograde orbits around M;
Class (g) contains direct orbits around M;
Class (h) contains direct orbits around E.

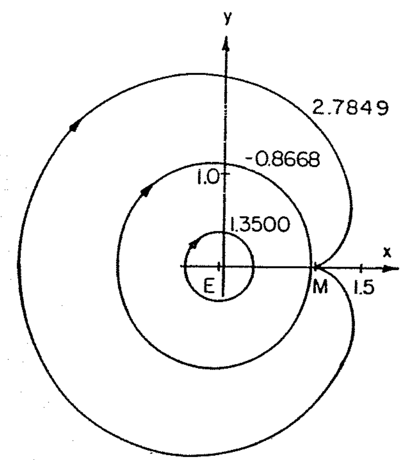

FIG. 9.21(a). Periodic lunar orbits, Class (e), $\mu = 1/82.27$ (Broucke, 1962, Ref. 4).

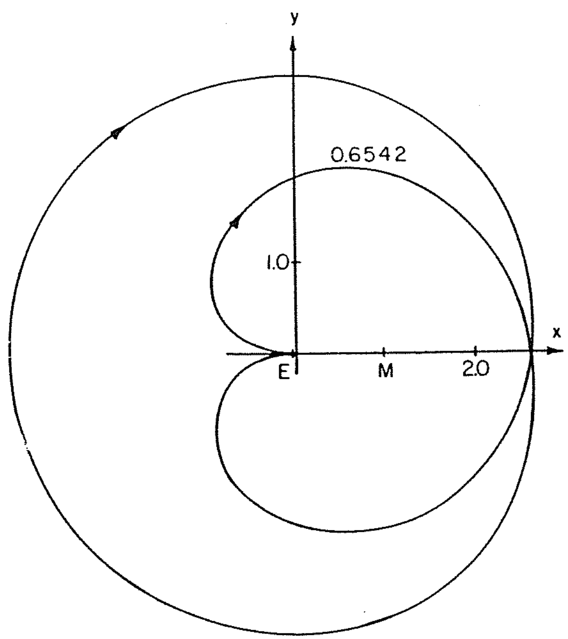

FIG. 9.21(b). Periodic lunar orbit, Class (e), $\mu = 1/82.27$ (Broucke, 1962, Ref. 4).

The retrograde orbits around E are shown in Figs. 9.21(a) and (b) and are to be compared with Figs. 9.5(a), (b), and (c) after reflecting the latter ones with respect to the y axis. For this reason these orbits form *Class* (*e*) instead of Class (f). The beginning of this class of orbits [in fact of all four classes (e), (f), (g), (h)] is an infinitesimal circular orbit with $C \to \infty$. As these ovals increase in size, a collision orbit with M is reached [Fig. 9.21(a)] corresponding to Fig. 9.5(b). Then the collision orbit becomes an orbit with a loop around M, which loop, as it expands, will cause collision with E [Fig. 9.21(b)]. This collision orbit is continued now with a loop around E, quite similarly to Fig. 9.5(c). The size of the orbits grows without limit as the family ends.

Retrograde orbits around M are shown in Figs. 9.21(c) and (d). These correspond to Figs. 9.5(a), (b), and (c) representing *Class* (*f*). Again, the two collision orbits (first with E, then with M) are shown in addition to a few oval orbits which correspond to the beginning of the family. The second collision orbit is followed by loop orbits and the family ends as the size grows to infinity.

Direct periodic orbits around M correspond to Strömgren's *Class* (*g*). Figure 9.21(e) shows the first and second phases, similarly to Figs. 9.6(a), and (b) of the Copenhagen category. Figure 9.21(f) corresponds to the third phase and to the first member of the fourth phase of Class (g) shown in Fig. 9.6(c). Continuation of Class (g) is demonstrated by orbits around E.

The direct orbits around E form a class which in the Copenhagen category is called *Class* (*h*) (which in turn is the mirror image of Class (g) with respect to the y axis). The beginning of the class is not shown (see direct orbits around M). Figure 9.21(g) shows what in Section 9.4.4 was referred to as the fifth phase [see Fig. 9.6(e)] of Class (g). Figure 9.21(h) corresponds to recognizable members of the sixth phase [Fig. 9.6(f)]. Finally Fig. 9.21(i) is to be compared with Fig. 9.6(g). The development shown in these figures associated with $\mu = 1/82.27$ is similar in every respect to the process described in connection with Class (g) of the Copenhagen category.

Regarding the motion around both primaries, Fig. 9.22(a) shows the similarity to Strömgren's Class (m) orbits [Fig. 9.9(a)]. The effect of the asymmetry of the primaries should be noticed, however, which results in close approaches to M while the perigee distance is still

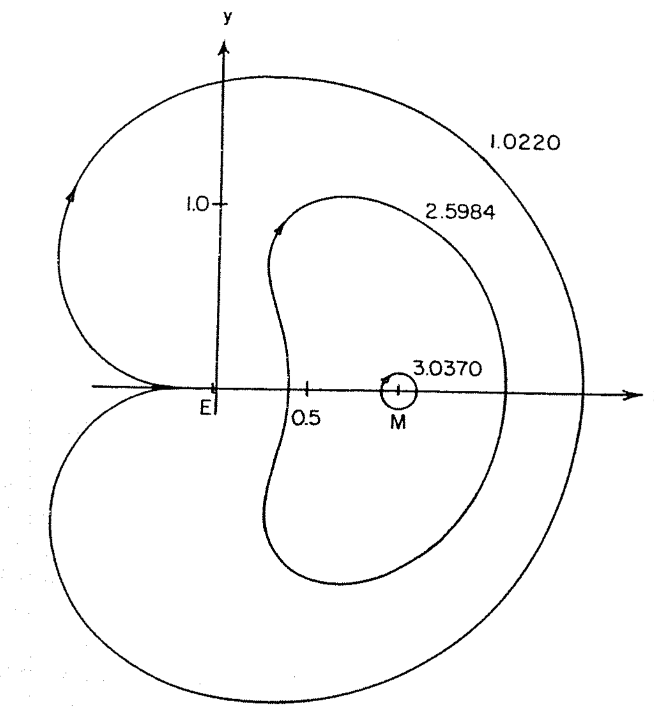

FIG. 9.21(c). Periodic lunar orbits, Class (f), $\mu = 1/82.27$ (Broucke, 1962, Ref. 4).

considerable. This is shown in Fig. 9.22(b) [corresponding partly to Fig. 9.9(b)]. The curve on the left shows the distance between the (perpendicular) intersection of the orbit with the negative x axis and the earth. The curve on the right gives the distances related to the moon.

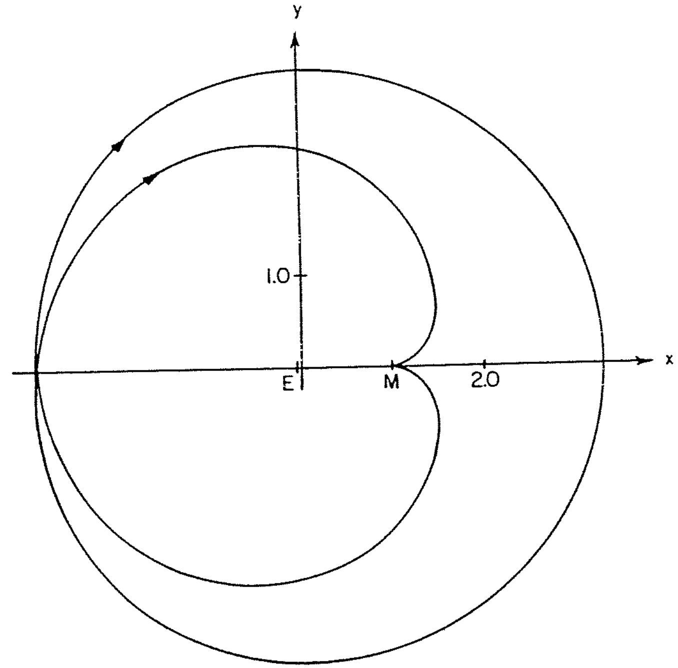

FIG. 9.21(d). Periodic lunar orbit, Class (f), $\mu = 1/82.27$ (Broucke, 1962, Ref. 4).

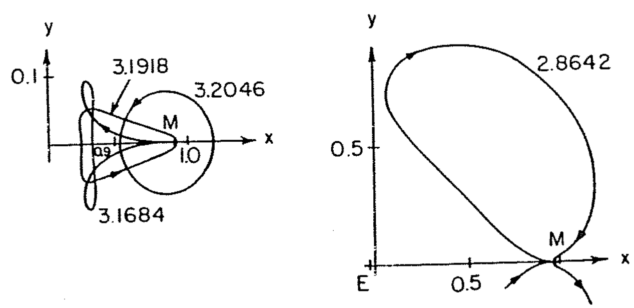

FIG. 9.21(e). Periodic lunar orbits, Class (g), $\mu = 1/82.27$ (Broucke, 1962, Ref. 4).

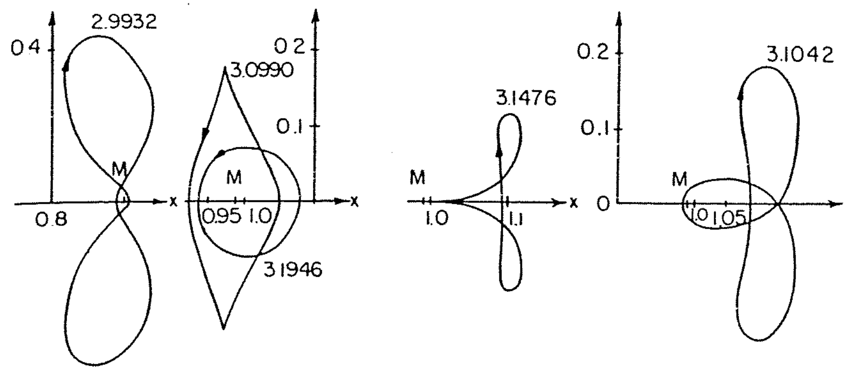

FIG. 9.21(f). Periodic lunar orbits, Class (g), $\mu = 1/82.27$ (Broucke, 1962, Ref. 4).

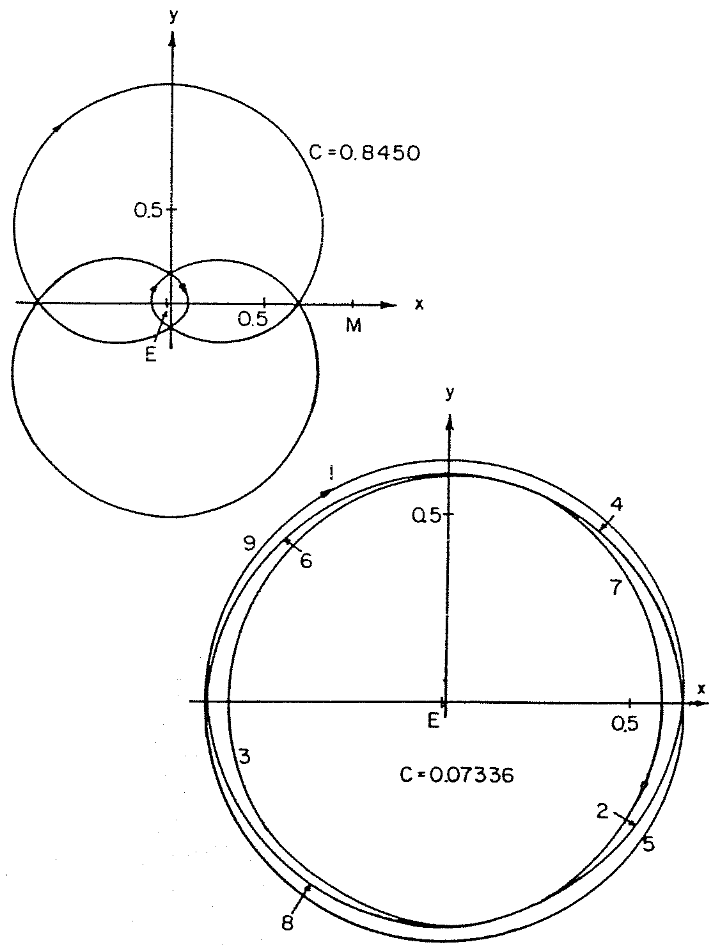

FIG. 9.21(g). Periodic lunar orbits, Class (h), $\mu = 1/82.27$ (Broucke, 1962, Ref. 4).

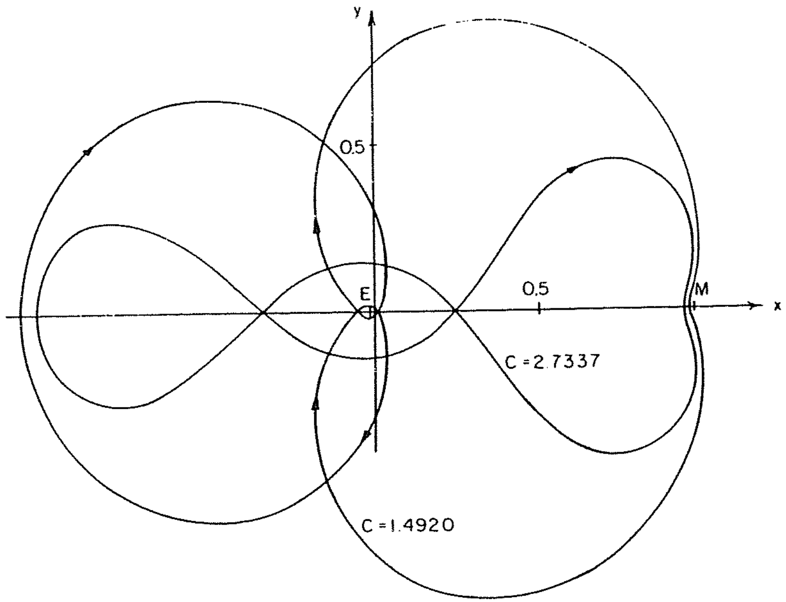

FIG. 9.21(h) Periodic lunar orbits, Class (h), $\mu = 1/82.27$ (Broucke, 1962, Ref. 4).

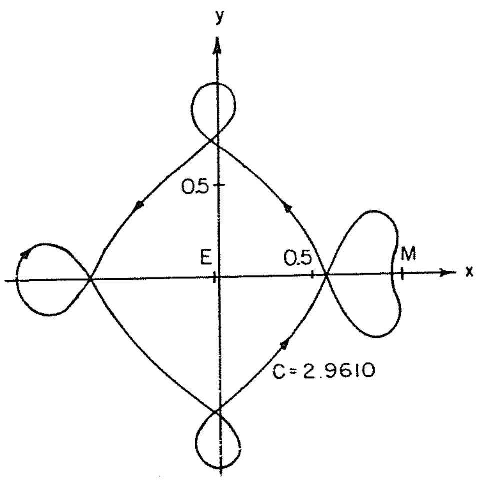

FIG. 9.21(i). Periodic lunar orbit, Class (h), $\mu = 1/82.27$ (Broucke, 1962, Ref. 4).

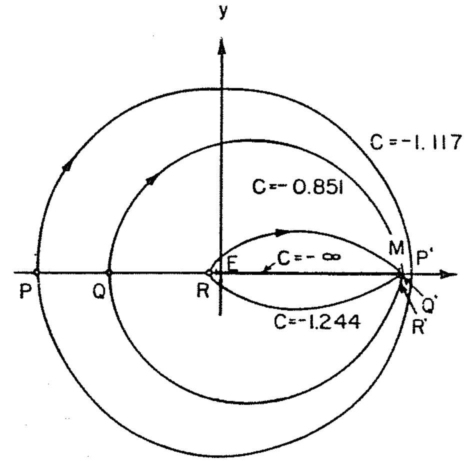

FIG. 9.22(a). Periodic lunar orbits, Class (m), $\mu = 1/82.27$ (Egorov, 1957, Ref. 3, and Broucke, 1962, Ref. 4).

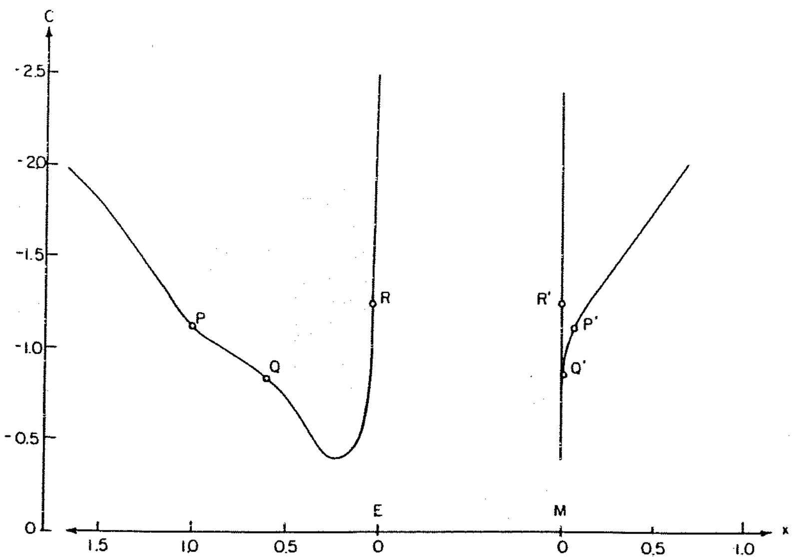

FIG. 9.22(b). Jacobian constant versus distances from M and E for Class (m) periodic lunar orbits.

TABLE IX

CLOSEST DISTANCES FROM EARTH AND MOON FOR CLASS (m) PERIODIC LUNAR ORBITS AS SHOWN IN FIGS. 9.22(a) AND (b)

Jacobian const. C	Dimensionless distance from E	Dimensionless distance from M	Distance in km from E	Distance in km from M
−1.9021	1.587845	0.614799	610367.9	236328.9
−1.7825	1.487845	0.515888	571927.9	198307.5
−1.6555	1.387845	0.417656	533487.9	160547.0
−1.5208	1.287845	0.320758	495047.9	123299.4
−1.3795	1.187845	0.226835	456607.9	87195.4
−1.2379	1.087845	0.140594	418167.9	54044.4
−1.1173 (P)	0.987845	0.074282	379727.8	28554.0
−1.0166	0.854463	0.032155	328455.8	12360.4
−0.8514 (Q)	0.609906	0.012155	234448.0	4672.4
−0.4485	0.238960	0.004155	91856.3	1597.2
−0.5401	0.087845	0.001746	33767.6	671.2
−1.2442 (R)	0.037845	0.000770	14547.6	296.0
−2.5016	0.017845	0.000349	6859.6	134.2

Table IX shows the numerical values associated with Figs. 9.22(a) and (b). Several aspects are of interest. First, there is a maximum value of the Jacobian constant ($C_{max} \cong -0.43$) similar to Class (m) of the Copenhagen category, where $C_{max} \cong -0.37$ was found. The asymmetric family described by Table IX begins with retrograde circular orbits of radii $\to \infty$ and $C = -\infty$. Then C increases to C_{max}, after which it again decreases and the family ends with the straight-line orbit connecting the primaries ($C \to -\infty$). The oval orbits suggest the establishment of "permanent earth-moon probes." The idea, while attractive, is not entirely practical, as the last two columns of Table IX show. These columns are obtained from the previous two columns by using the value $l = 384400.2$ km for the earth-moon distance. Considering that the equatorial radius of the earth is (6378.160 ± 0.02) km and that the moon's radius is 1738 km, one finds impact with the moon's surface of all orbits which come closer to the earth than approximately 100,000 km. That is, the smallest perigee obtainable is about 100,000 km which, considering the eccentricity of the moon's orbit and solar perturbations, is an estimate on the low side.

9.5.3 Egorov's periodic orbits

Earth-moon orbits corresponding to Strömgren's Classes (m) and (n) [or (c)] and some more complicated periodic orbits are the subjects of this section. All orbits have the common requirement that they pass

close to the earth and to the moon. Class (m) and its unsatisfactory behavior because of its large perigees were discussed in the previous section. Orbits which can be obtained as continuation of infinitesimal elliptic orbits around the libration point located between the primaries, because of the asymmetry of the masses can be considered belonging

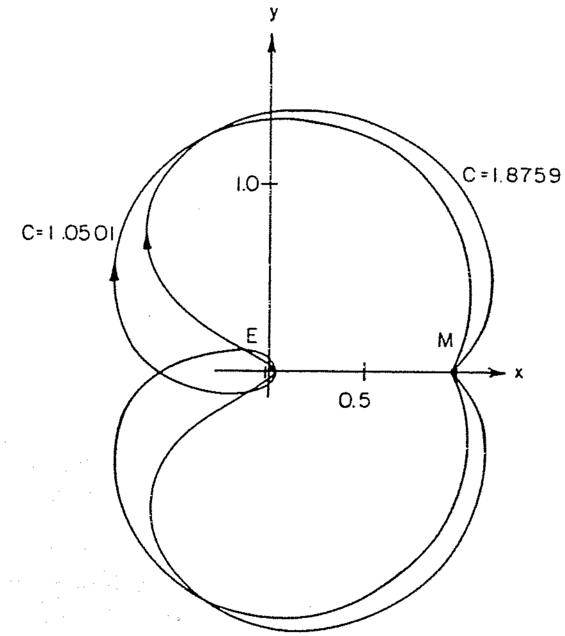

FIG. 9.23(a). Periodic lunar orbits, Class (c), $\mu = 1/82.27$ (Broucke, 1962, Ref. 4).

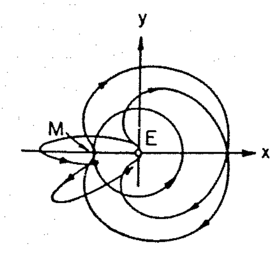

FIG. 9.23(b). Periodic consecutive close approach orbit in the earth-moon synodic and sideral systems (Egorov, 1957, Ref. 3).

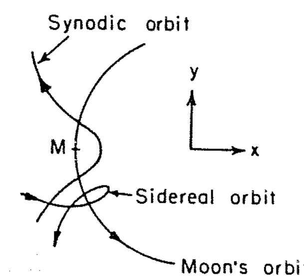

FIG. 9.23(c). Periodic consecutive close approach orbit enlarged (Egorov, 1957, Ref. 3).

to either Class (c) or (n). Figure 9.23(a) (computed by Broucke) shows two orbits of this class approaching the earth's center to 12,360 km (8516 km) and the moon's center to 1698 km (2258 km) having a Jacobian constant of $C = 1.876$ ($C = 1.05006$). Both orbits offer highly satisfactory reference trajectories for actual operations. Several more complicated periodic orbits with consecutive close approaches belonging basically to the same family were computed by Egorov. Figure 9.23(b) shows such an orbit in the rotating and in the fixed coordinate systems. The details regarding the close approach with the moon are shown in Fig. 9.23(c).

9.5.4 Special periodic orbits

This section discusses such periodic orbits in the field of the idealized earth-moon system which attempt to be useful for space research. As it was shown in connection with Fig. 9.22(a) the orbits belonging to Class (m) do not satisfy the practical requirements for a "permanent" earth-moon space probe. The orbits shown in Figs. 9.23(a), (b), and (c) might be more satisfactory; nevertheless, the practical significance of such orbits is always limited to their role as reference orbits. Keeping this restriction in mind other periodic orbits with close approaches to the moon can be found, and in this section several such families will be discussed.

(A) If the value of the Jacobian constant is limited by $C_1 \leqslant C < C_2$ periodic orbits which connect the general neighborhood of m_1 and m_2 may be searched for. Here, $C_1 = C(L_1)$ and $C_2 = C(L_2)$. If $C > C_2$ and the third body is initially close to, say, the earth (i.e., inside the curve of zero velocity belonging to C) it is not able to leave this closed oval and it remains a satellite of the earth. The same is true for a lunar satellite. The zero-velocity curve with the shape of a figure eight belonging to C_2 opens at L_2 if $C < C_2$, and so communication between m_1 and m_2 is established. If the zero velocity curve belonging to C_1 is to enclose the motion, then $C_1 \leqslant C$. The location of the collinear

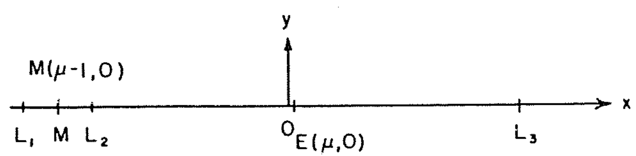

Fig. 9.24(a). Earth-moon system drawn approximately to scale, $\mu = 0.01214$.

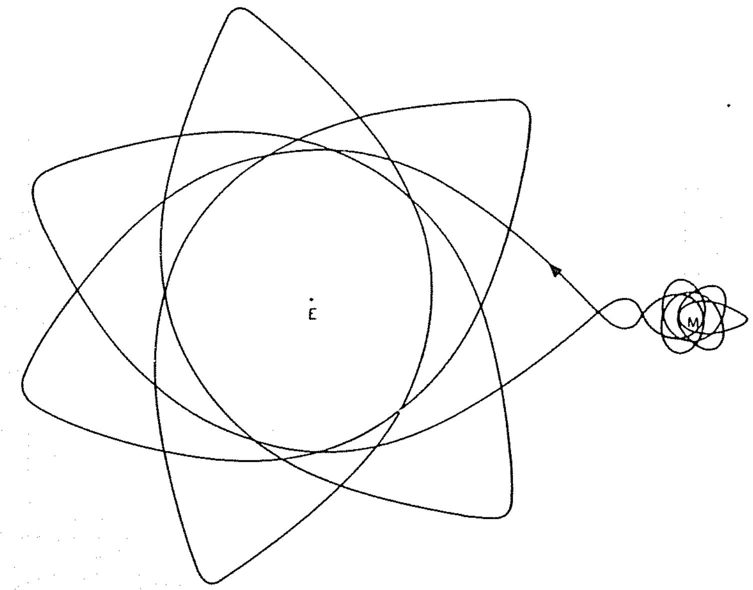

Fig. 9.24(b). Periodic orbit of an alternating satellite for earth and moon, $\mu = 1/81.45$ (Davidson, 1964, Ref. 37).

libration points and the primaries as well as the values of the critical Jacobian constant, as shown previously, depend on μ. In order to help visualization we use $\mu = 0.012141$, which gives $x_1 = -1.155645$, $x_2 = -0.836963$, $x_3 = 1.005059$, $C_1 = 3.1845078$, $C_2 = 3.200246$, and $C_3 = 3.024131$. Figure 9.24(a) shows the approximate situation.

Egorov's and Arenstorf's idea, to establish close-approach earth-moon periodic orbits, is now described. Consider a value of the Jacobian constant between C_1 and C_2. The curve of zero velocity encloses M, L_2, 0, and E and intersects the x axis between L_1 and M and also between E and L_3. Selecting now initial conditions such that $y \cong 0$, $x \cong x_2$, $\dot{y} \cong 0$, and $\dot{x} \cong 0$, we let the third body "squeeze" through the opening at L_2. The orbit with properly selected initial conditions leaves the neighborhood of L_2 toward the direction of point E, and the third body becomes a temporary satellite of the earth. Since $C < C_2$ the body can return to the opening at L_2 and can "squeeze" through toward M. It can now become a temporary satellite of the moon, can return to L_2, then to E, and the sequence may continue. In this way periodic orbits can be established with well-regulated close approaches. The importance of finding such a periodic orbit is more theoretical than practical since solar perturbations, oblateness, and eccentricity effects will destroy the fine balance of forces present in the restricted problem. Nevertheless, the existence of such periodic orbits has been established by Arenstorf analytically as well as numerically. One member of the family found by Davidson is shown in Fig. 9.24(b). Such orbits are understandably unstable in almost any sense; in fact, their sensitivity to the value of the mass parameter (μ) is such that in some cases transition from earth satellite to lunar satellite and vice versa might not occur if the value of μ is changed by a fraction of one per cent. The orbit is computed with $\mu = 0.012277471$ and $C = 3.185498720207$. The initial velocity components are $\dot{x}_0 = 0$, $\dot{y}_0 = 0.0086882909$, and the orbit starts on the right side of the moon at a distance of 0.162277471 on the x axis.

The libration points are at the distances of 0.168445637 and 0.151429938 on the right and left sides of the moon, respectively. The corresponding values of the Jacobian constant are $C_1 = 3.185285900531$ and $C_2 = 3.201635152079$. Consequently, $C_1 < C < C_2$.

Figure 9.24(c) shows a periodic orbit around the moon, i.e., around the smaller primary. The initial conditions in a moon-centered coordinate system are $x_0 = -0.120722529000$, $y_0 = 0$, $\dot{x}_0 = 0$, $\dot{y}_0 = -0.024511616261$. The existence of "second-kind" periodic

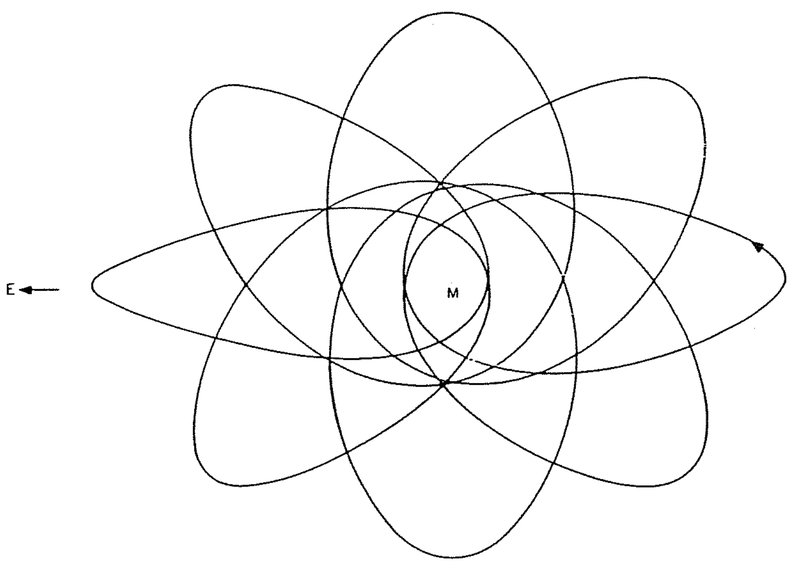

FIG. 9.24(c). Periodic orbit around the smaller primary—a lunar satellite, $\mu = 1/81.45$ (Arenstorf, 1966, Ref. 36).

orbits around the smaller primary is not obvious. The proof of existence is due to Arenstorf, whose orbit is shown in Fig. 9.24(c).

(B) Further groups of families with close approaches can be generated following the idea behind Poincaré's second kind of periodic orbits. Consider an elliptic orbit with the earth at its focus in the fixed coordinate system. Since the mass of the earth in our system is $1 - \mu$, the mean motion, n, on such an elliptic orbit is

$$n = (1 - \mu)^{1/2}/a^{3/2}$$

and its period becomes

$$T = \frac{2\pi}{(1 - \mu)^{1/2}} a^{3/2}.$$

The period of the moon in the fixed system is 2π. Since its mean motion is one, the condition for commensurability becomes

$$a = (l/k)^{2/3}(1 - \mu)^{1/3},$$

where l and k are integers with the same physical significance as in Chapter 8. While the ellipse makes l revolutions with respect to the rotating system, the third body completes k circuits on the ellipse. The ratio l/k is the ratio between the period of the particle relative to the fixed system and the period of the moon.

Selecting now $(1 + e)a \cong 1$, where e is the eccentricity of the elliptic orbit, one can establish an unperturbed (two-body) orbit with close approaches, especially if the quantity $a(1 - e)$, representing the distance of perigee, is kept sufficiently small. Table X gives the values of a corresponding to combinations of l and k. Approximate initial conditions (for the restricted problem) now can be computed and various periodic orbits with consecutive close approaches can be constructed.

TABLE X

SEMIMAJOR AXES FOR COMMENSURABILITY

	l		
k	1	2	3
1	0.9959	1.5809	2.0716
2	0.6274	—	1.3050
3	0.4788	0.7600	—
4	0.3952	—	0.8221
5	0.3406	0.5401	0.7085

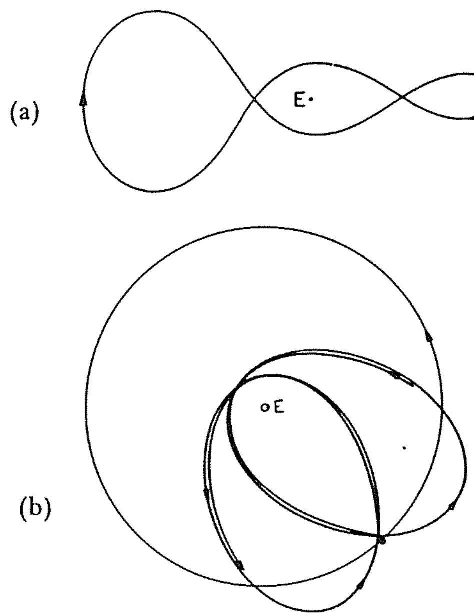

Fig. 9.25. (a) Consecutive close approach orbit with commensurability $\frac{1}{2}$ in the earth-moon synodic system (Arenstorf, 1963). (b) Consecutive close approach orbit with commensurability $\frac{1}{2}$ in the earth-moon sidereal system, $\mu = 1/81.45$ (Arenstorf, 1963, Ref. 36).

Figure 9.25(a) shows the combination $l = 1$, $k = 2$ in the rotating system, while Fig. 9.25(b) represents the same orbit in the fixed system, both figures after Arenstorf. The orbit is retrograde around the moon and direct around the earth. Figure 9.25(b) shows that the particle travels about twice around ($k = 2$) its elliptic orbit between two encounters with the moon ($l = 1$). The motion of the perigee (or the apsidal line) of the orbit in the fixed system is a three-body effect.

TABLE XI

Close Approaches with 2 : 3 Commensurability, Fig. 9.26.

Orbit number	Jacobian constant	Dimensionless distance from E	Dimensionless distance from M	Distance in km from E	Distance in km from M
1	2.8353	0.422555	0.012155	162430.1	4672.4
2	2.8550	0.433869	0.022155	166779.5	8516.4
3	2.7596	0.358754	0.112155	137905.1	43112.4
4	2.6374	0.191701	0.292156	73689.8	112304.6

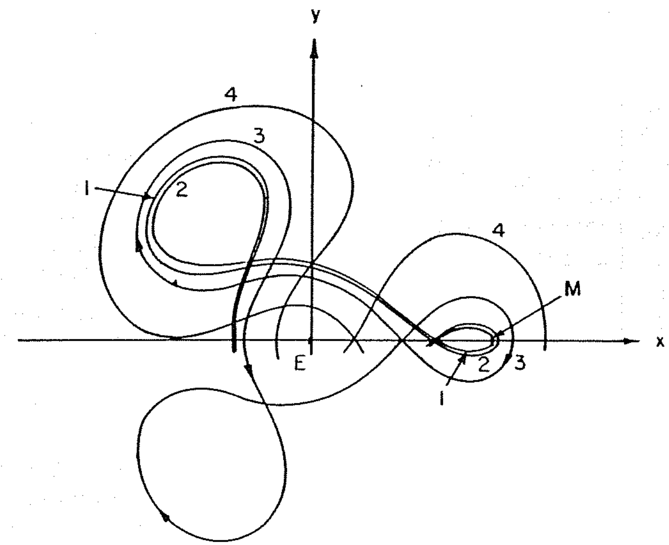

Fig. 9.26. Consecutive close approach orbits in the earth-moon system with commensurability $\frac{2}{3}$ (Broucke, 1962, Ref. 4, and Huang, 1962, Ref. 39).

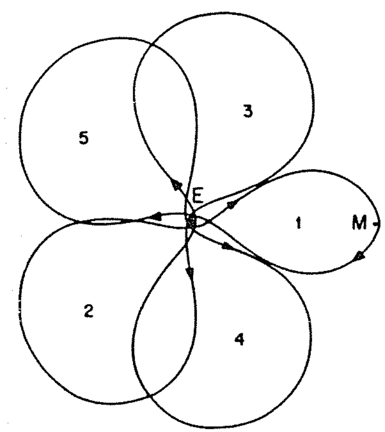

Fig. 9.27. Consecutive close approach orbit with commensurability $\frac{2}{5}$ in the synodic earth-moon system, $\mu = 1/81.45$ (Arenstorf, 1963, Ref. 36).

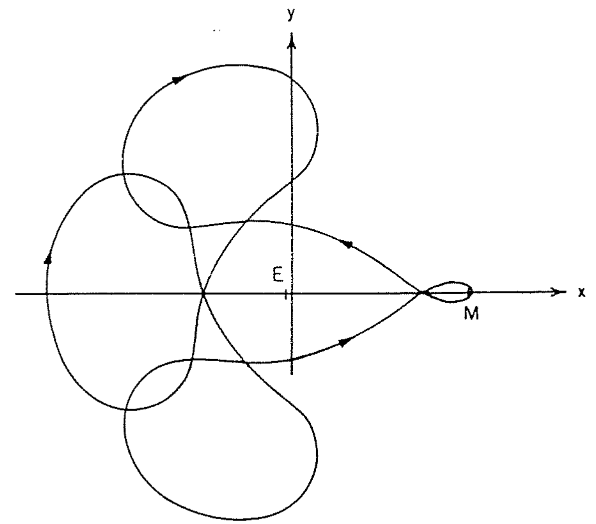

FIG. 9.28. Consecutive close approach orbit with commensurability $\frac{3}{4}$ in the synodic earth-moon system, $\mu = 1/82.27$ (Broucke, 1962, Ref. 4).

Figure 9.26, after Broucke and Huang, shows a family of orbits with $l = 2$, $k = 3$ in the synodic system. Of the four orbits shown only number 3 is drawn completely because of the symmetry condition with respect to the x axis. Table XI gives the pertinent numerical values.

Figure 9.27 shows the case when $l = 2$, $k = 5$ in the rotating system, according to Arenstorf.

Figure 9.28 corresponds to $l = 3$, $k = 4$ at $C = 2.7598$ and is Broucke's result. The closest distance to the moon is 1531 km, which would take the third body inside the moon without orbital correction.

It is remarked that while it is challenging to find families of periodic orbits for special purposes, their practical significance is limited to being reference orbits since it cannot be expected that such orbits will be followed for any length of time in the actual physical case because of the presence of perturbations (deviations from the restricted problem).

9.6 Motion around the triangular libration points

This section presents two examples of families of periodic orbits around the triangular libration points. The theory of these orbits was discussed in detail in Chapter 5. Here we show Horn's theorem and

analytic continuation at work. We follow Rabe's results using values of the mass ratio corresponding to the sun-Jupiter and to the earth-moon systems.

The large amplitude periodic orbits around the equilateral libration points discussed in Section 9.4.12 should be distinguished from the orbits of this section. The orbits discovered by Moulton and mentioned in Section 9.4.12 are not generated by the continuation of infinitesimal periodic orbits around L_4 and L_5, while the orbits of this section are produced this way. It is recalled that infinitesimal periodic orbits do not exist for $\mu > \mu_0 = 0.0385...$, but periodic orbits of finite dimensions *do* exist. Not only Moulton's orbits for $\mu = \frac{1}{2}$, but Pedersen's work discussed in Chapter 5 are recalled in this respect.

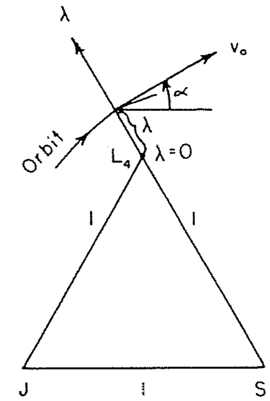

FIG. 9.29(a). Parameters of Rabe's family of periodic orbits.

Returning now to the first subject of this section, some of the numerically established periodic orbits for $\mu = 0.00095388$ will be discussed. The family found by Rabe is generated from the long-period infinitesimal periodic orbit by determining the initial velocity vectors at points on the line connecting the sun with L_4. The distance of the starting point along the line SL_4 is λ and consequently $\lambda = 0$ represents L_4. The Jacobian constant for this orbit is $C = 3$, its period is $147^y.40508$, and its amplitude (size) is zero. As λ increases the Jacobian constant increases monotonically and the successively generated orbits show some resemblance to the zero-velocity curves. This "resemblance" is not an identity as was discussed in Chapters 4 and 5. Figure 9.29(a) shows the general arrangement. The initial velocity vector v_0 is approximately perpendicular to the line SL_4, so $\cot \alpha \cong 3^{1/2}$, for the linearized problem. Once the starting point is given (by λ) and the value of the Jacobian constant is known, the angle α completely determines the orbit. Table XII gives the pertinent numerical data and Fig. 9.29(b) shows some of the orbits. The monotonically increasing values of the period and of the Jacobian constant are associated with increasing amplitudes up to the orbit for which $\lambda = 0.052$. This orbit intersects the x axis at the third collinear libration point, hence its Jacobian constant is $C = C_3 = 3.001906820$ and its period is not finite. In the sense of Strömgren's termination principle, Rabe's family can end with this orbit; nevertheless, the parameter λ can be increased further, as the last entry of the table [corresponding to the last orbit in Fig. 9.29(b)] shows. Further increase of λ to 0.07 indicates a decreasing trend for

TABLE XII

RABE'S FAMILY OF PERIODIC TROJANS, FIG. 9.29(b)
($\mu = 0.000953875$)

λ	C	$\cot \alpha$	Period (years)
0	3.000 000 000	—	147.40508
0.0025	3.000 004 615	1.7338	147.46400
0.0050	3.000 018 422	1.7340	147.64176
0.0075	3.000 041 330	1.7341	147.94105
0.0100	3.000 073 266	1.7343	148.36797
0.0125	3.000 114 141	1.7345	148.93130
0.0150	3.000 163 893	1.7346	149.64355
0.0175	3.000 222 427	1.7348	150.52097
0.0200	3.000 289 680	1.7349	151.58503
0.0225	3.000 365 581	1.7350	152.86343
0.0250	3.000 450 230	1.7350	154.39554
0.0275	3.000 542 822	1.7352	156.21419
0.0300	3.000 644 316	1.7351	158.40247
0.0400	3.001 136 376	1.7386	173.33071
0.0500	3.001 724 588	1.7620	222.35627
0.0600	3.002 443 050	1.7429	346.42089

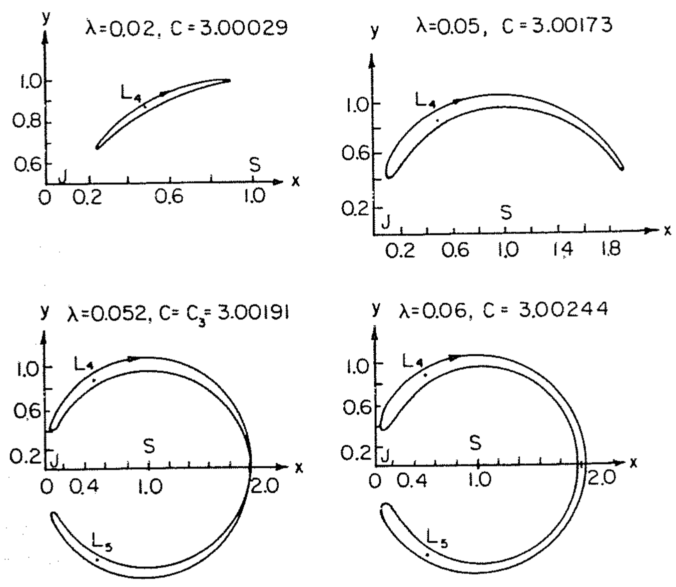

FIG. 9.29(b). Periodic orbits of Trojans, $\mu = 0.00095388$ (Rabe, 1961, Ref. 5).

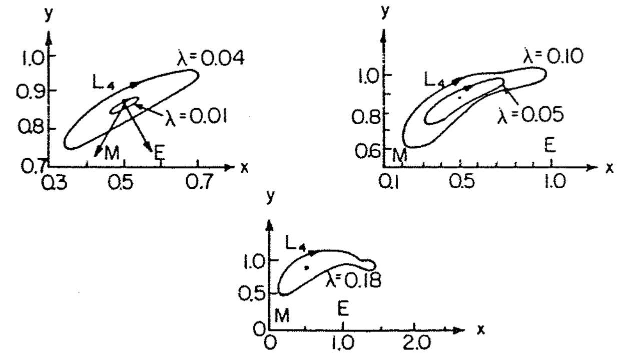

FIG. 9.29(c). Periodic libration orbits in the earth-moon system, $\mu = 0.0121396$ (Rabe and Schanzle, 1962, Ref. 6).

the period, in complete agreement with the fact that the period increases monotonically to ∞ which it approaches as $\lambda \to 0.052$ and $C \to C_3$. After this the other branch of the family starts with $C = C_3$, $T = \infty$, and, as λ increases from $\lambda = 0.052$, T decreases and C increases. The end of this second branch has not been determined by Rabe.

Periodic orbits around the equilateral libration point in the earth-moon system are shown in Fig. 9.29(c) and the corresponding numerical values are listed in Table XIII. Development of the family for the earth-moon case is much slower. For instance, around $\lambda = 0.05$ the

TABLE XIII

RABE'S FAMILY OF PERIODIC LIBRATION ORBITS IN THE EARTH-MOON SYSTEM, FIG. 9.29(c)
($\mu = 0.0121396054$)

λ	C	$\cot \alpha$	Period (days)
0	3.000 000 000	—	91.6563
0.01	3.000 062 316	1.7731	91.6858
0.02	3.000 248 460	1.8114	91.7438
0.03	3.000 553 364	1.8746	91.8215
0.04	3.000 952 074	1.9585	91.8945
0.05	3.001 396 914	2.0439	91.9423
0.06	3.001 841 721	2.1103	91.9605
0.07	3.002 260 499	2.1488	91.9568
0.08	3.002 644 726	2.1612	91.9437
0.09	3.002 995 923	2.1520	91.9357
0.10	3.003 322 446	2.1300	91.9506
0.12	3.003 979 371	2.0512	91.1627
0.14	3.004 936 230	1.9351	93.0180
0.16	3.006 787 044	1.7709	95.5634
0.18	3.009 785 995	1.4977	102.9439

Trojan orbit almost reaches the x axis while its earth-moon counterpart has a much smaller amplitude as a comparison of the corresponding orbits in Figs. 9.29(b) and (c) shows. Note, however, that the corresponding values of the Jacobian constant are approximately equal.

The previously discussed periodic orbits are generated from the so-called long-period infinitesimal orbits. It may be expected that similar families corresponding to the shorter periods occurring in the solution of the characteristic equation (cf. Chapter 5) exist. These families (not corresponding to members of the Trojan group) have been described in the literature too recently to be included here.

The accuracy of these numerically established periodic orbits can be represented by the residuals. These are the differences between the coordinates (in the phase space) at $t = 0$ and $t = T$. For the above-quoted results, with the exception of some of the motions with the largest amplitude, the residuals appear in the eighth decimal place for all coordinates of the phase space. This is a considerable improvement from the Moulton–Darwin–Strömgren results where residuals sometimes show up in the second decimal.

It is to be noted that the Jacobian integral used as a control of the computations is not as significant as the magnitude of the residuals. It is convenient to record the value of the Jacobian constant during the process of the numerical integration, and if it shows larger variations than, say, 10^{-5} (using our dimensionless units) then the computation may be suspected. But an eight-decimal accuracy of the Jacobian constant does not guarantee small residuals since it is not uncommon that an error of 10^{-9} in the Jacobian constant is associated with an error of 10^{-3} in the positions and velocities for certain orbits.

Inasmuch as residuals, as introduced above, are associated with the establishment of periodic orbits, their use for control of computations is definitely limited. Another control, which is rather meaningful for reversible dynamical systems and is applicable to any orbit, is the utilization of the reversibility principle. Assume that the numerical integration starts at $t = t_0$ and ends at $t = t_n$. At this point all velocity vectors of the system are reversed, i.e., $\mathbf{v}_i(t_n) \to -\mathbf{v}_i(t_n)$, and all position vectors are maintained. With this set $[\mathbf{r}_i(t_n), -\mathbf{v}_i(t_n)]$ *as initial conditions* the integration is started once again. At $t = 2(t_n - t_0)$ the integration is stopped and the differences in the coordinates

$$|\mathbf{r}_i[2(t_n - t_0)] - \mathbf{r}_i(t_0)|$$

and the corresponding velocity differences are evaluated. Note that the restricted problem is an irreversible dynamical system and so this test is applicable when the sense of rotation of the synodic system is also reversed.

9.7 Lunar trajectories

In this section families of trajectories are discussed which represent the motion of the infinitesimal particle under the influence of the earth and the moon. Since the assumptions of the restricted problem are not completely valid, the members of these families can be considered as reference orbits only. Computations of orbits of space vehicles connecting the general vicinity of the earth and of the moon must consider the effects of the earth's and moon's oblateness, the influence of the sun, and the effect of the eccentricity of the moon's orbit in addition to the possibility of operating on a three-dimensional trajectory. In spite of these additional considerations, which are all neglected in the model of the restricted problem, the families of lunar trajectories obtained by this model are of considerable help in general systems studies.

One of the most thorough organizations of families of lunar trajectories by Hoelker and Herring is an essentially three-dimensional study of the restricted problem; therefore, discussion is postponed to Section 10.2.4, Part C. The rather detailed classification of lunar impact trajectories and lunar circumnavigation trajectories by Egorov and Ehricke and the consecutive collision work referred to already in Section 3.11 form the basis of this section.

The numerous engineering aspects of the problem cannot be discussed here; nevertheless, one important consequence of using the restricted three-body model is to be brought out. Not considering the effects of the sun and of the moon's orbital eccentricity results in the supression of the significance of the firing time. This means that no calendar dates enter the considerations described herein. This unquestionably serious omission is also a significant simplification.

Some of the most important lunar missions are as follows: impact with the moon with arbitrary velocity, impact with zero (or small) velocity ("soft landing"), establishment of a lunar satellite, circumnavigation of the moon with return to the neighborhood of the earth, circumnavigation with return to the earth's atmosphere at a small (or zero) re-entry angle, fly-by operation and, finally, return from the moon to the earth with various constraints. Several of these missions require orbit modification in flight; i.e., the so-called free-flight (or "gravitational") trajectories cannot satisfy the mission requirements. From the point of view of this work and of the restricted problem, it is essential to show which of the above lunar missions can be performed without orbit correction. Then these missions are made the subject of intensive systematic study, and classes of trajectories as complete as possible are to be established. The modifications of the members of these classes according to the actual physical picture will offer the possibility for choosing the trajectory which satisfies other operational requirements. Up to this point the trajectory is still "free" and any propulsion to be used will be executed only for orbit correction and guidance purposes. Those missions which cannot be executed within the constraints of the restricted problem will require additional propulsion *ab initio*.

Impacts with unspecified landing velocity do not require additional propulsion; therefore, these trajectories can be handled within the framework of the restricted problem. It is advisable (if not mandatory) to use regularization for the computation of such families of orbits. Soft landing requires thrust, as can be shown by realizing that a particle with zero velocity will not leave the surface of the moon and consequently it cannot arrive there with zero velocity, due to the image trajectory principles discussed in Chapter 8.

The establishment of a "permanent" lunar satellite does not seem to be feasible without additional thrust for several theoretical and practical reasons. Within the constraints of the restricted problem, the fact that the velocity of arrival of the third body to the vicinity of the moon is around 3000 m/sec prevents the required "capture." The velocity of a satellite of the moon is approximately 1200–1700 m/sec between elevations of 150 to 1800 km; therefore, considering also the velocity of the moon ($\sim$1000 m/sec), a change in arrival velocity is necessary. Fesenkov attempted to show analytically the impossibility of such a capture, and while his work leaves some questions open, no capture orbit was found. Those trajectories which consist of several highly eccentric orbits around the earth and then elliptic orbits around the moon (see Egorov and Thüring) are not capture orbits since either collisions with one of the primaries or periodic repetions (see Arenstorf) of the above process can be expected from these already extremely unstable orbits.

The circumnavigation mission is possible within the framework of the restricted problem, but when flat re-entry conditions are required the free-flight corridors become narrow and the initial conditions so sensitive that the assumptions of the restricted problem without thrust make the validity of the computation highly questionable.

To study meaningful fly-by operations is feasible, but when so-called sling-shot or swing-by effects are to be established for interplanetary orbits, the investigations show again the limitations of the restricted problem. The reason for this is that the requirements call for very close approaches to the moon or to the planets which necessitates the inclusion of their orbital eccentricity.

Return trajectories can be studied using the restricted problem, especially in view of the image principle mentioned in Chapter 8

according to which every orbit computed from the earth to the moon furnishes also a return orbit by changing (t) to $(-t)$ and (y) to $(-y)$.

Of the several existing families of lunar trajectories six classes will be discussed for circumnavigation, four classes for lunar impact, and six classes for consecutive impact.

Figure 9.30, after Egorov, shows three basic members of the classes of circumnavigational trajectories. In all three cases the return orbit is a collision trajectory. The three figures show the synodic and the

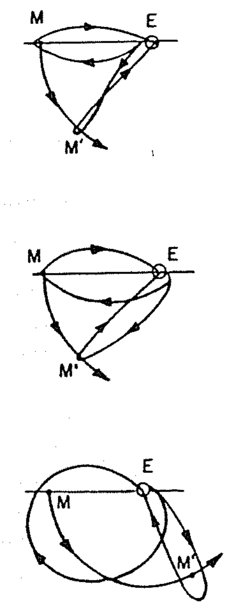

FIG. 9.30. Classes of trajectories for lunar circumnavigation (Egorov, 1957, Ref. 3).

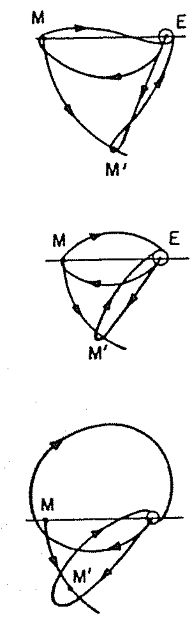

FIG. 9.31. Classes of trajectories for lunar circumnavigation with small angle of re-entry (Egorov, 1957, Ref. 3).

sidereal orbits as well as the (circular) motion of the moon. The orbits start at a fixed elevation above the earth's surface with a tangential velocity vector. Figure 9.31 shows three members of three classes of circumnavigating trajectories with small re-entry angles. In all three cases the initial velocity is radial; i.e., the orbit is a collision trajectory at the beginning.

Egorov's four classes of lunar impact trajectories are shown in Fig. 9.32. Again the sidereal (EM'), the synodic (EM), and the lunar (MM') orbits are shown. Figure 9.32(a) shows the class which is further treated in Fig. 9.33, drawn after Egorov. The three members of this class start

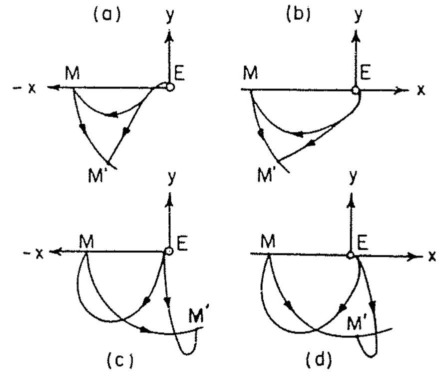

FIG. 9.32. The four classes of lunar impact trajectories (Egorov, 1957, Ref. 3).

at 200 km elevation with $\dot{y} = 0$, $\dot{x} < 0$ initial velocity. The three initial velocities are described by their deviations from the local parabolic velocity by $\Delta V_1 = +0.48251$ km/sec, $\Delta V_2 = -0.057828$ km/sec, and $\Delta V_3 = -0.092828$ km/sec.

The families of consecutive collision orbits have the common characteristics that all members go through both singularities. These orbits, computed by Pierce and Standish using Birkhoff's regularization, are mentioned in Section 3.5. The characteristics of the six families are given in Fig. 9.34. The letters a to f on the graph $C = C(\theta)$ correspond to the orbits which are shown in Figs. 9.35(a) to 9.35(f). A value of the function $C(\theta)$ completely represents a member of these classes, since the "firing angle" (θ), the associated Jacobian constant, and the given locations of E and M determine an orbit. Note that the initial and final velocities are both infinite.

These consecutive collision orbits can be organized into six classes,

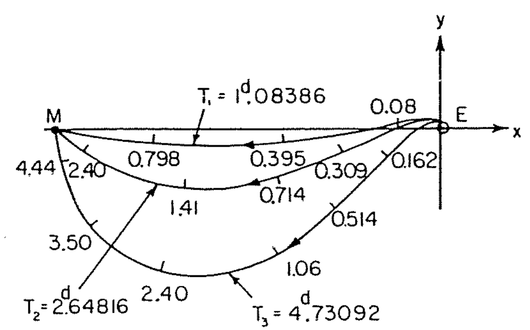

FIG. 9.33. Class (a) of lunar impact trajectories, $\mu = 1/82.47$ (Egorov, 1957, Ref. 3).

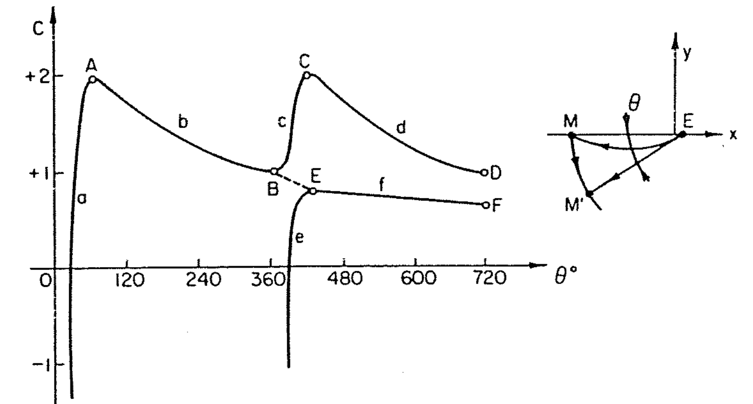

FIG. 9.34. Six classes of consecutive earth-moon collision orbits (Pierce and Standish 1965, Ref. 51).

denoted in what follows by (a), (b), (c), (d), (e), and (f). Another possible classification is into two families only: the first formed out of the previous classes (a), (b), (c), and (d); the second comprising (e) and (f).

The first class (a) begins with a straight-line orbit between the two primaries $\theta = 0$, and $C = -\infty$. As θ increases from 0° to about 60 the Jacobian constant increases (the relative energy decreases) to approximately $C = 2$. At these values (point A on Fig. 9.34) the type of orbits shown in Fig. 9.35(a) ceases to exist and the orbits change their character to what can best be described as "simple lofted trajectories." The dynamics involved is a straight-line departure (in the fixed system) from the earth to further than the moon's distance, prior to the arrival of the moon. As the vehicle falls back, the moon moves up (arrives) and collision takes place. Figure 9.35(b) shows the straight-line orbit (along the $+x$ axis) which turns around at $t = 8$ days and collides with the moon at $t = 13^d.67$ which is approximately half the moon's orbital period, without significant perturbations by the moon. This orbit shows similarities to a periodic orbit which approaches but does not collide with the moon, undergoes a collision at the earth, and is between the two orbits shown in Fig. 9.23(a). The Jacobian constant of this periodic orbit with the close approach is $C = 1.4340590$ and its half period is $T/2 = 3.138911$ in the dimensionless system. The consecutive collision orbit shown in Fig. 9.35(b) is *not* periodic, with $C = 1.43020205$.

Continuation of the family after point B in Fig. 9.34 involves large perturbations by the moon. Class (c) can be called the class of direct perturbed lofted trajectories. The ascending part of the orbit passes close to and behind the moon and is perturbed by it in the direct sense

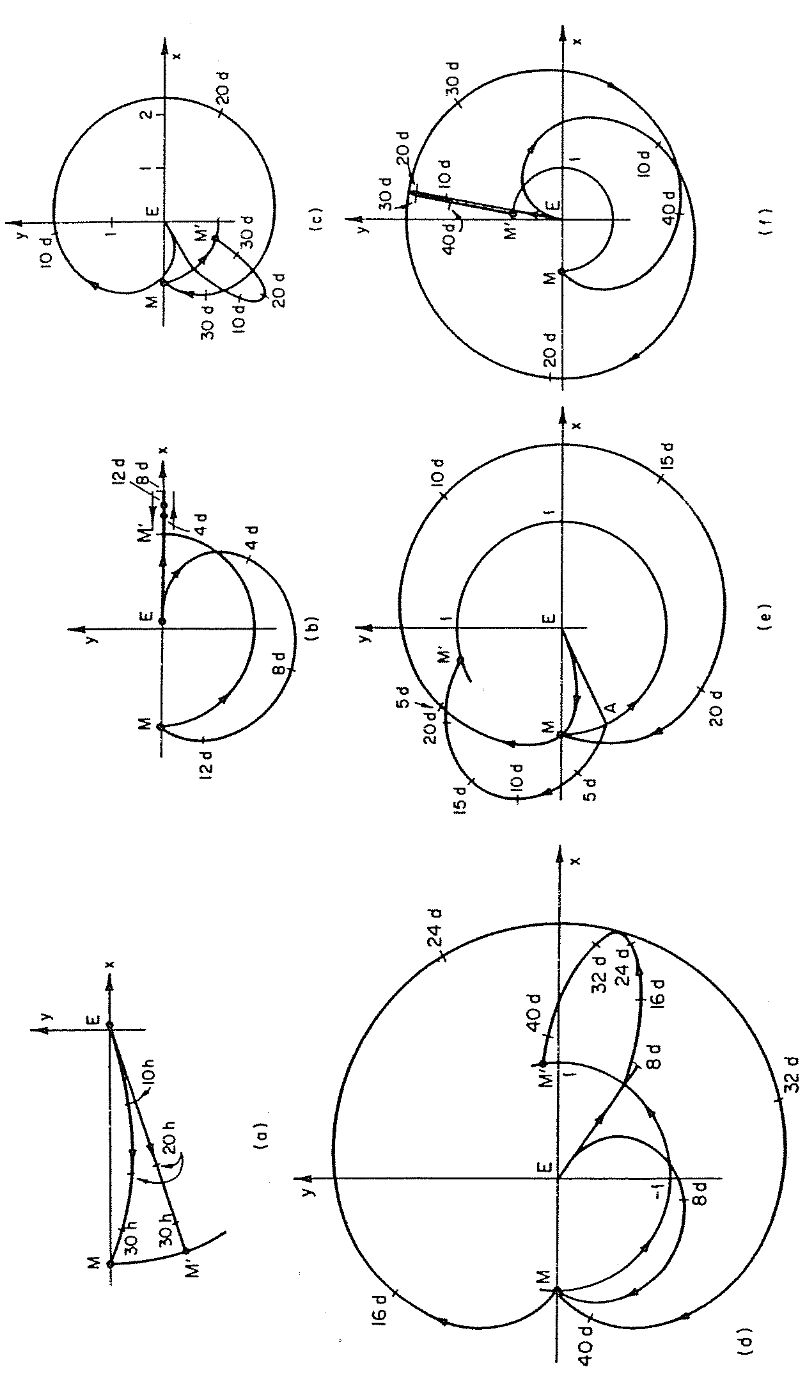

FIG. 9.35. (a) Class (a) of consecutive earth-moon collision orbit: $\theta = 20°$, $T = 35^d.85$, $C = -2.98147580$. (b) Class (b) of consecutive earth-moon collision orbit: $\theta = 180°$, $T = 13^d.67$, $C = 1.43020205$. (c) Class (c) consecutive earth-moon collision orbit: $\theta = 394°$, $T = 32^d.9$, $C = 1.5062879$. (d) Class (d) consecutive earth-moon collision orbit: $\theta = 504°$, $T = 41^d.5$, $C = 1.5857461$. (e) Class (e) consecutive earth-moon collision orbit: $\theta = 386°$, $T = 23^d.53$, $C = -0.4834287$. (f) Class (f) consecutive earth-moon collision orbit: $\theta = 620°$, $T = 47^d.1$, $C = 0.71521946$. For all orbits $\mu = 1/82.45$ (Pierce and Standish, 1965, Ref. 51)

[see Fig. 9.35(c)] so that, when it falls back, it collides with the moon. The end of this class is at approximately $C = 2$, again corresponding to point C in Fig. 9.34.

The next class (d) contains orbits of the same basic type, which are significantly perturbed by the moon in the direct sense. The difference between Classes (c) and (d) is in their first encounter with the moon. Class (d) orbits encircle the moon about 180° after the return from the end of a straight-line orbit beyond the orbit of the moon. This class is not continued beyond the point D in Fig. 9.34 because the complicated nature of the trajectories following this point does not allow practical applications.

Class (e) is obtained by "backward" continuation from Class (f). Therefore Class (f) is discussed first. Note that the previous four Classes, (a) to (d), follow from each other by systematically increasing the firing angle θ and finding the corresponding value for the Jacobian constant C. Class (f) is obtained by continuing Class (b) so that the vehicle leaves E on a straight-line orbit so far in front of the moon that it experiences almost no perturbations by the moon prior to collision with it. Figure 9.35(f) shows such an orbit with $\theta = 360 + 260 = 620°$. The vehicle has enough "relative energy" or low enough value of C to remain outside the orbit of the moon for over a month. Collision occurs after the moon makes approximately $1\frac{3}{4}$ revolutions.

Class (e) now is obtained by following decreasing C values instead of the path EB in Fig. 9.34. The result is a class of retrograde perturbed lofted trajectories, a member of which is shown in Fig. 9.35(e). The vehicle crosses the moon's orbit ahead of and close to the moon [see point A in Fig. 9.35(e)]. Then it is strongly perturbed in a retrograde sense and collides with the moon on an approximately geocentric elliptic orbit (see the path AM'). In the rotating system the vehicle follows the arc EM close to the moon and after one revolution collides with M.

9.8 Applications to binary systems

This section discusses nonperiodic orbits for the value of the mass parameter which was used in connection with the Copenhagen category, i.e., $0.1 \leqslant \mu \leqslant 0.5$. The nonperiodic orbits corresponding to $\mu \cong 1/82.27$ are the lunar trajectories, discussed in Section 9.7, while the nonperiodic orbits of the Copenhagen category receive applications in the study of binary stars.

The problem of classification of the computed orbits seems to be more difficult here than in connection with lunar trajectories because of the different nature of the physical problem. This manifests itself mostly by the longer duration of time of interest for the binary system problem than for lunar trajectories. The classes of orbits do not exist in the sense used previously (even in the rather lose form as when applied to lunar trajectories), but several significant conclusions regarding the expulsion and exchange of matter have been reached by integrating orbits within the framework of the restricted problem.

Orbits have been computed by Abhyankar starting in the vicinity of the external collinear libration points with initial velocities close to the values predicted by the linear theory (see Chapter 5) for $\mu = 0.1$. The shape of the orbits is in agreement with the unstable nature of the collinear libration points as seen in Fig. 9.36. All six trajectories originate at point $P(0.01, 0.01)$, the coordinates of which are relative to $L_1(1.2597, 0)$. Note that the primary with the larger mass (m_1) is to the left of the origin; consequently, L_1, which is outside of the masses $\overline{m_1 m_2}$, has a positive abscissa. The velocity deviations from the linearized

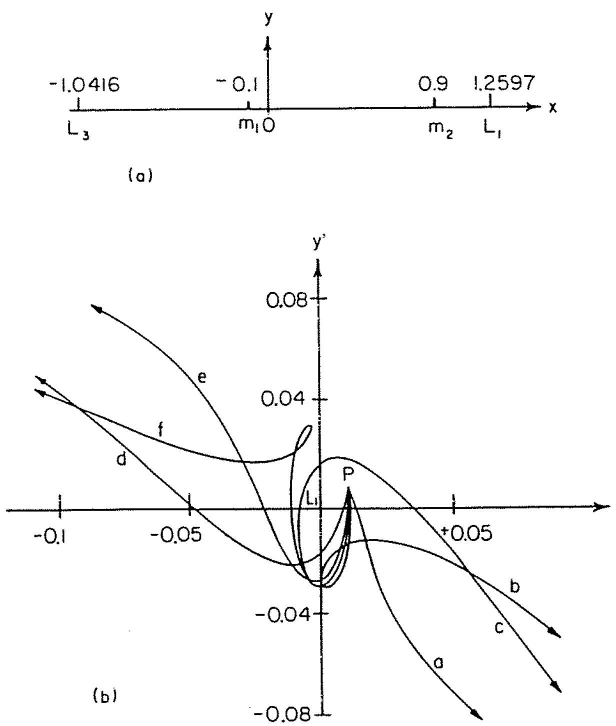

FIG. 9.36. Nonperiodic motion in the neighborhood of L_1 for $\mu = 0.1$ (Abhyankar, 1959, Ref. 52).

theory are $\mathbf{v}_a(0.01, 0.01)$, $\mathbf{v}_b(0.001, 0.001)$, $\mathbf{v}_c(0.0001, 0.0001)$, $\mathbf{v}_d = -\mathbf{v}_a$, $\mathbf{v}_e = -\mathbf{v}_b$, and $\mathbf{v}_f = -\mathbf{v}_c$. In other words, for instance, the trajectory of Fig. 9.36(b) marked e has the initial conditions $x'(0) = 0.01$, $y'(0) = 0.01$, $\Delta\dot{x}(0) = -0.001$, $\Delta\dot{y}(0) = -0.001$, where x' and y' measure the deviations from L_1 and $\Delta\dot{x}$ and $\Delta\dot{y}$ are the excess velocity components above the velocity which would give elliptic motion around L_1 using the linearized theory. Available numerical results indicate that, even when the initial velocity vector is selected according to the linear theory, the third body does *not* stay in the neighborhood around L_1 but departs after only one revolution.

The same result is valid qualitatively with respect to motion in the vicinity of the other external libration point L_3. So the conclusion might be reached that accumulation of material near these Lagrangian points is not favored by the dynamics of the situation. A focusing effect seems also to be operating according to which several of the computed orbits form rings around the primaries. The numerical results indicate that these conclusions might be extended up to $\mu = 0.5$.

Orbits with small initial velocity vectors along the positive x axis starting at L_1 have also been computed for $\mu = 0.4$ by Kuiper [for the notation, see Fig. 9.36(a)]. These orbits might spiral around both primaries first outward and then back again according to Kopal.

A study by Gould, restricted to radial initial velocities (regarding one of the primaries) and including L_2 as starting point, reveals that the resulting orbits are often funneled into a ring. This ring is either around one of the primaries, motion being direct, or around both primaries with retrograde motion. The large number of orbits computed for $0.167 < \mu < 0.5$ cannot be reproduced here and the reader's attention is directed to the last section of this chapter for the references.

Orbits starting with velocities different from zero at the collinear libration point L_2 have also been computed by Kopal for $0.17 < \mu < 0.5$ using initial velocity vectors *not* along the x axis. Most of the computed trajectories collide with the surface of one of the primaries.

9.9 Additional periodic orbits

This section treats three main subjects. The first topic (9.9.1) is an example regarding analytic continuations of an elliptic orbit from $\mu = 0$ to $\mu \neq 0$. The second subject (9.9.2) concerns asymptotic periodic orbits at the collinear libration points for certain particular values of the mass parameter. The third part (9.9.3) discusses families of symmetric and asymmetric periodic orbits with 2 : 1 commensurability ratio in the sun-Jupiter system. Analogous results found for the earth-moon system are also shown for comparison.

9.9.1 Analytic continuations of a periodic orbit of the second kind

Most of the orbits discussed in Section 9.5.4 are of the second kind in Poincaré's terminology. It is interesting to observe the limit of analytic continuation for such orbits as the mass parameter increases from $\mu = 0$. The example chosen was introduced already in Section 8.5.4 by Fig. 8.10. The following numerical values apply for $\mu = 0 : k = 3,\ l = 1,\ a = 0.480749844,\ e = 0.3,\ C = 3.4029327,$ $x(0) = a(1 + e) = 0.624974851$. Figure 9.37 shows once again the periodic orbit described by these data. This figure is identical with Fig. 8.10 of Chapter 8 and it is repeated here to facilitate comparison with the next three figures which were generated from the orbit shown in Fig. 9.37 by Knowles.

Continuation of the orbit $\mu = 0$ of Fig. 9.37 is possible along several different branches in the space of the parameters determining the orbit. Figures 9.38, 9.39, 9.40, and 9.41 show four of the possibilities. Keeping $\dot{y}(0)$ constant gives rise to a family shown in Fig. 9.38. Note the intersections of the orbits with the x axis and observe how $x(0)$—in fact the whole orbit—is moving to the right as μ increases. Another continuation of the orbit $\mu = 0$ is shown in Fig. 9.39. Here

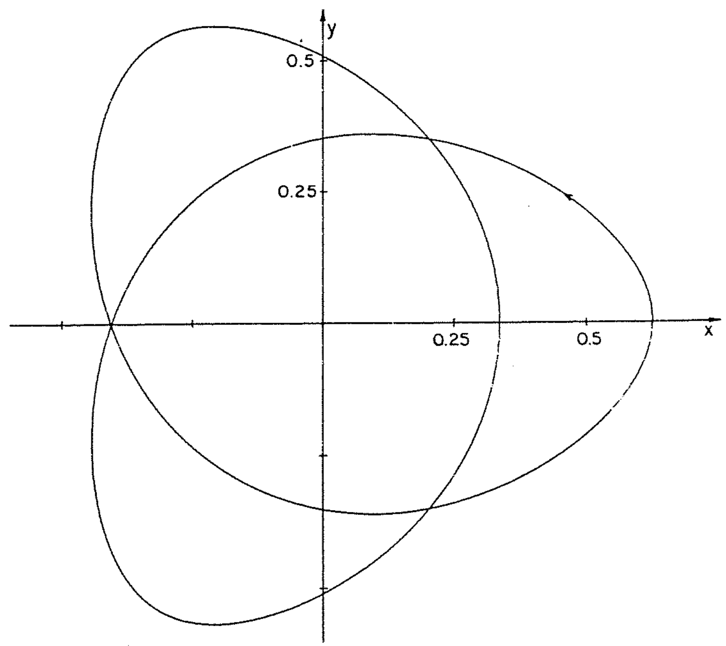

FIG. 9.37. Two-body orbit with commensurability $\frac{1}{3}$.

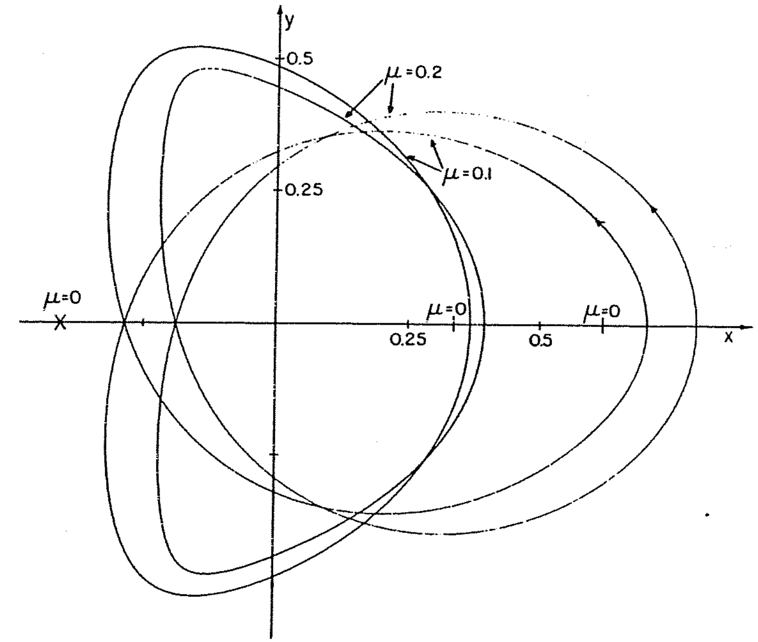

FIG. 9.38. Continuation of the orbit $\mu = 0$, commensurability $\frac{1}{3}$ with $\dot{y}(0) = \text{const.}$

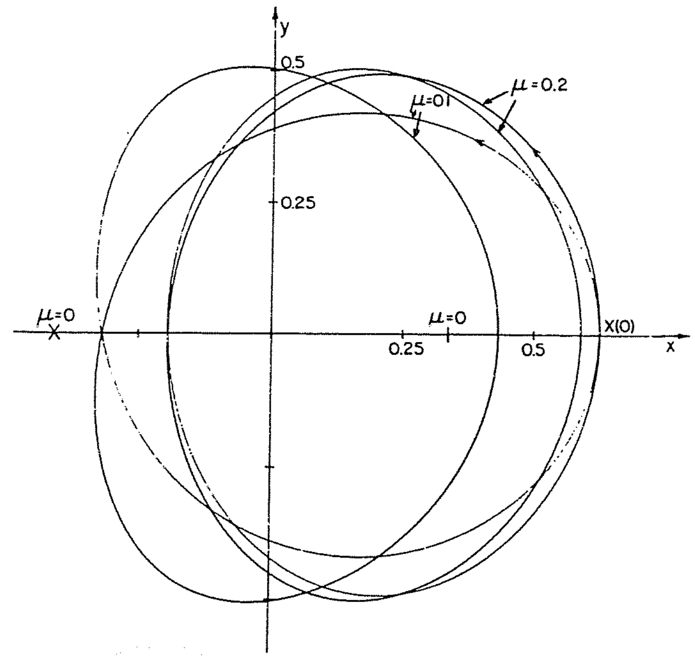

FIG. 9.39. Continuation of the orbit $\mu = 0$, commensurability $\frac{1}{3}$ with $x(0) = \text{const.}$

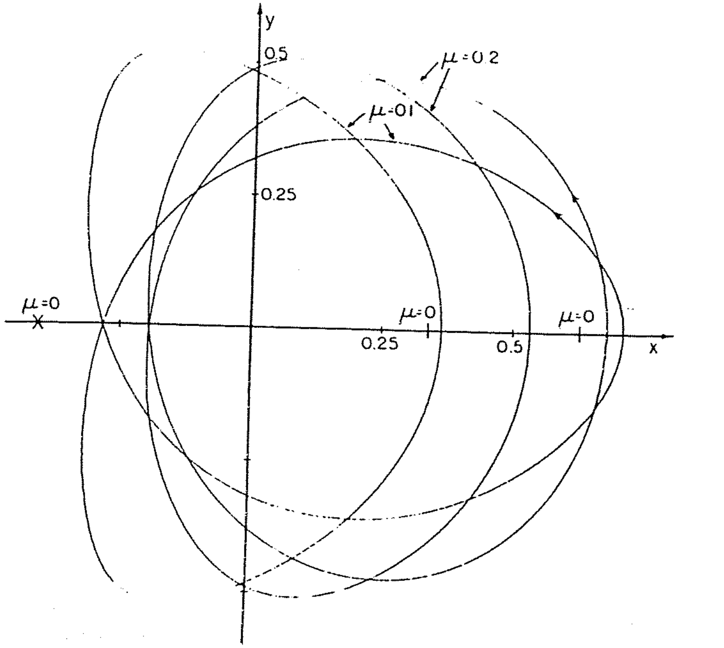

FIG. 9.40. Continuation of the orbit $\mu = 0$, commensurability $\frac{1}{3}$ with $\bar{C} = $ const.

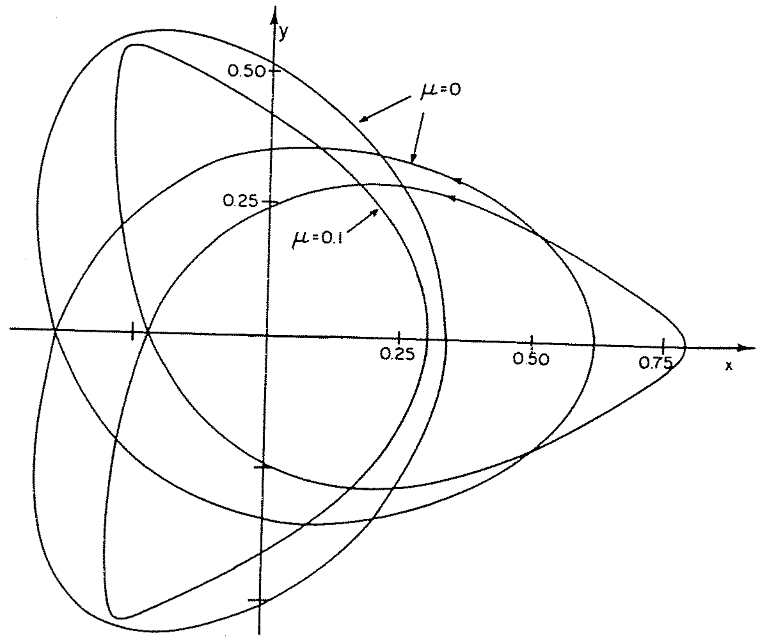

FIG. 9.41. Continuation of the orbit $\mu = 0$, commensurability $\frac{1}{3}$ with $C = $ const.

$x(0) = $ const, i.e., the members of the family start from the same point on the x axis where the generator orbit ($\mu = 0$) starts. The general trend of a shift to the right is again present but in addition a shrinking of the separation of the loops is observable. Note that when $\mu = 0$, $x(0) = a(1 + e)$, $x(T/2) = a(1 - e)$, consequently the separation is $2ae = 0.288449931$. For $\mu = 0.1$ this distance decreases to approximately 0.192 and it becomes 0.036 when $\mu = 0.2$, showing a trend toward a simple periodic orbit.

The third continuation is shown in Fig. 9.40. Here $\bar{C} = C - \mu(1 - \mu)$ is kept constant as the family develops. Note that $x(0)$ is not monotonic with μ this time, nevertheless $x(T/2)$ is. Another possibility of continuation is along the branch $C = $ const (see Fig. 9.41). Only the orbits for $\mu = 0$ and $\mu = 0.1$ are shown since the family develops additional loops soon after $\mu = 0.1$ and the nature and character of the orbits change strongly.

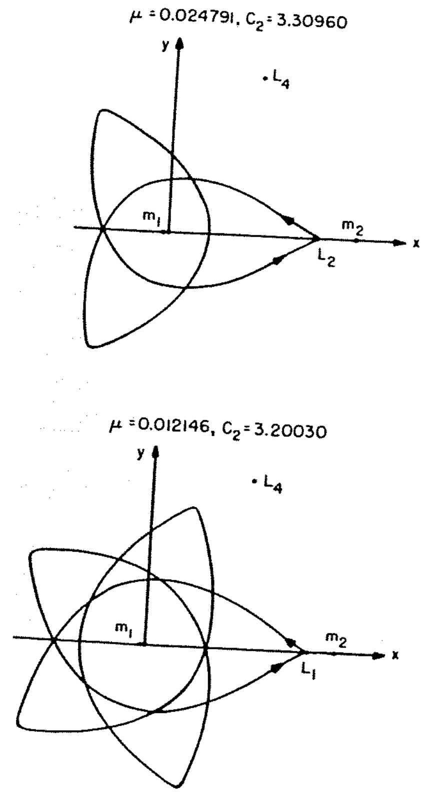

FIG. 9.42(a). Asymptotic-periodic orbits at L_2 (Deprit and Henrard, 1965, Ref. 54).

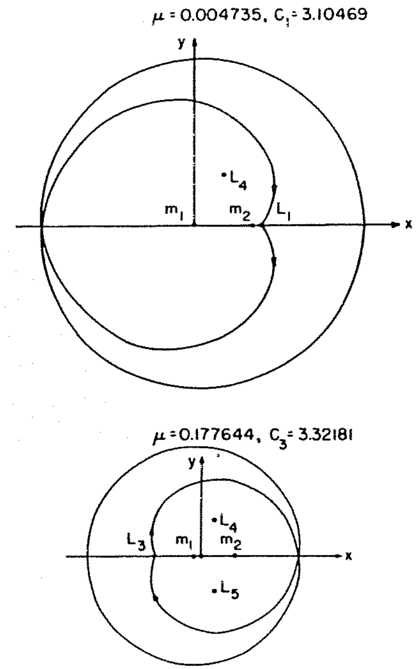

FIG. 9.42(b). Asymptotic-periodic orbits at L_1 and L_3 (Deprit and Henrard, 1965, Ref. 54).

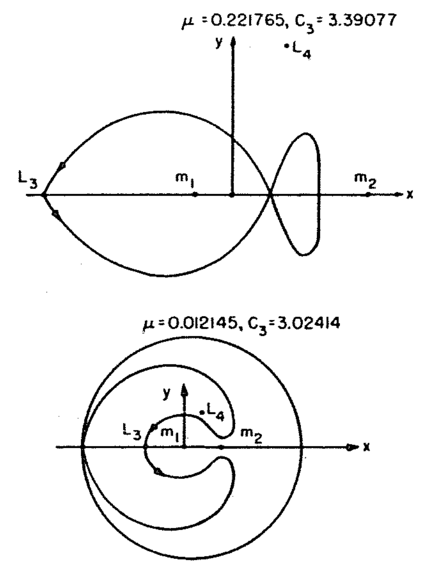

FIG. 9.42(c). Asymptotic-periodic orbits at L_3 (Deprit and Henrard, 1965, Ref. 54).

The above continuations are of interest, especially in view of the discourse presented in Sections 8.6.1 and 8.6.2. The limiting values of μ to which a given two-body orbit can be continued (keeping various parameters constant) are, in general, not known. The above example seems to indicate that, if $\mu > \mu_K \cong 0.24$, the family has a termination point. This means that one point of the surface $\mu = \mu(C, e)$ mentioned in Section 8.6.2 is given approximately by $\mu_K \cong 0.24$, $C \cong 3.4$, $e = 0.3$.

9.9.2 Asymptotic-periodic orbits at the collinear points

The second group of symmetric periodic orbits to be discussed in this section has the property that the period of its members is ∞, all orbits being asymptotic to one of the collinear libration points. Some of these orbits, computed by Deprit and Henrard $[C_{De} = \bar{C} = C - \mu(1 - \mu)]$, are shown in Figs. 9.42(a), (b), and (c). The orbits are generated with the real exponent obtained from the characteristic equations at the collinear libration points, discussed in Chapter 5. Figure 9.42(a) shows two orbits, both asymptotic to L_2. The remarkable closeness of the value of the mass parameter ($\mu = 0.012146$) of the second orbit to the earth-moon system suggests the possibility of using L_2 as a storage point in space operations since, with guidance in the actual case, particles might leave and return to L_2. The same reasoning might be applied to the second orbit of Fig. 9.42(c), since $\mu = 0.012145$.

The two orbits of Fig. 9.42(b) show symmetry in a certain sense. Note the similarity of these orbits to Strömgren's class (f) [Fig. 9.5(c)] and the change of the role of the collision with a primary into an asymptotic approach to a collinear libration point.

The first orbit of Fig. 9.42(c) is one of the few known figure eight periodic orbits which exist in the restricted problem.

9.9.3 Symmetric and asymmetric periodic orbits with commensurability 2 : 1

These retrograde orbits in the synodic system are sidereally direct and enclose both primaries; therefore, they correspond to Class (l) orbits of the Copenhagen category. Message generates these orbits from elliptic orbits with close to a commensurability 2 : 1 by varying the eccentricity of the generating ellipses. The near commensurability can be measured by the quantity δ according to the following relation:

$$\frac{k+1}{k} = \frac{n'}{n_0} + \frac{\delta}{k},$$

where $k \neq 0$ is a positive integer, n' and n_0 are the mean motions of Jupiter (i.e., of the synodic system) and of the third body (without

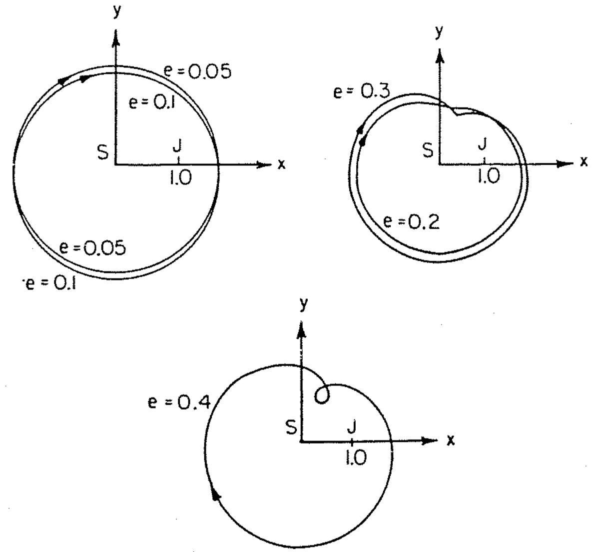

FIG. 9.43. Asymmetric periodic orbits of commensurability 2 : 1 for $\mu = 0.000954016$ (Message, 1959, Ref. 55).

perturbation), respectively, and δ is the measure of commensurability. For $k = 1$, $n' = 1$, we have

$$\delta = 2(n_0 - \tfrac{1}{2})/n_0 ,$$

which of course refers to the unperturbed orbit.

Figure 9.43 shows five members of a family of asymmetric periodic orbits for which the numerical information is contained in Table XIV.

TABLE XIV

MESSAGE'S ASYMMETRIC PERIODIC ORBITS OF COMMENSURABILITY 2 : 1, FOR $\mu = 0.000954016$

e	$n - \frac{1}{2}$	δ	C
0.05	0.001123	0.00450	3.14882
0.1	0.001077	0.00432	3.13938
0.2	0.000917	0.00368	3.10113
0.3	0.000671	0.00268	3.03574
0.4	0.000373	0.00149	2.94089

The eccentricity of the generating ellipse is e, the mean motion of the actual (perturbed) orbit is n, δ is the measure of the commensurability, and C is the Jacobian constant. Note that $C = -C_{Me} + \mu(1 - \mu)$. The formation of the cusp when $e \cong 0.3$ means that at the pericenter, or near to this point, the direct sidereal velocity becomes equal to the (direct) velocity of the system; consequently, the relative velocity is zero. The direct velocity of the body is smaller than the local frame-velocity at other parts of the orbit; consequently, the relative velocity is negative and the orbit is retrograde in the rotating system. (This question was mentioned already in Section 8.5.1.) The computations use $\mu = 0.000954016$ and therefore refer to the sun-Jupiter system. The family of asymmetric periodic orbits shown in Fig. 9.43 is related to Strömgren's class (l) but its generation is different since it is formed by perturbing a class of two-body orbits. In other words, every member of this family is obtained by analytic continuation from a two-body orbit with $e \neq 0$. Every member can be looked at as a generator of further classes of Poincaré's second-kind orbits which might be obtained for larger values of μ. It will not be surprising to see the establishment of such asymmetric families for other values of μ, especially with low eccentricity generating orbits.

In addition to these orbits, Message also established two other (symmetric) families for the same value of μ. Only one member of one

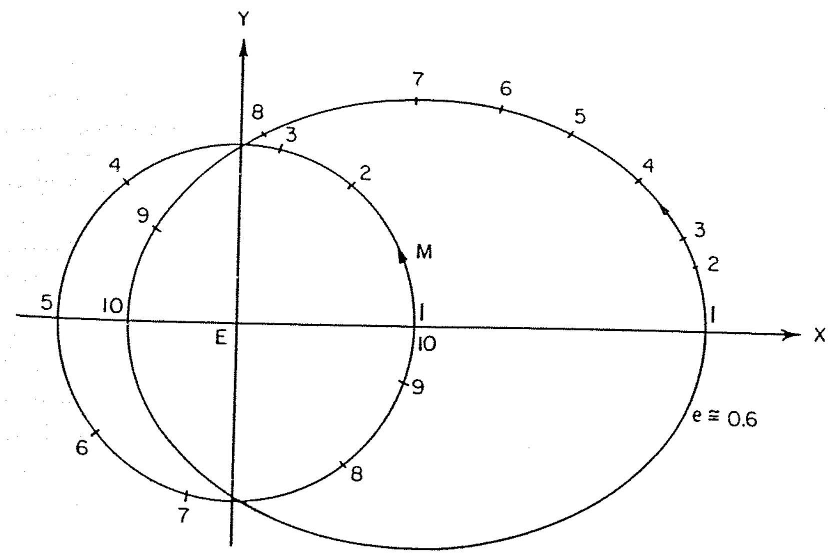

FIG. 9.44. Symmetric periodic lunar orbit of commensurability 2 : 1 in synodic coordinates for $\mu = 1/82.27$ (Broucke, 1962, Ref. 4).

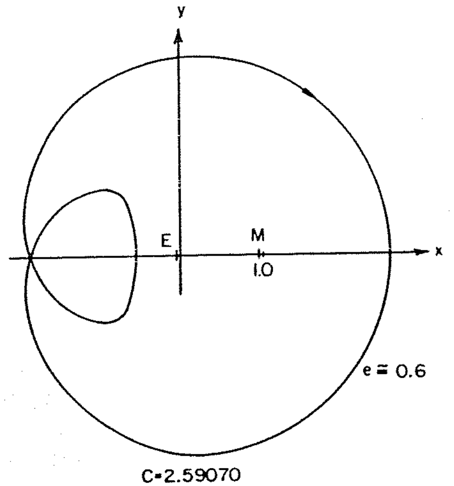

FIG. 9.45. Symmetric periodic lunar orbit of commensurability 2 : 1 in sidereal coordinates for $\mu = 1/82.27$ (Broucke, 1962, Ref. 4).

of these two classes is discussed here for its close relation to a Class (l) periodic lunar orbit. Figure 9.44 shows the (direct circular) orbit of the moon and also the direct elliptic orbit of the third body in a fixed coordinate system. The eccentricity is approximately 0.6, the mass parameter is $\mu = 0.0121551$, the Jacobian constant is $C = 2.59070$, and the 2 : 1 commensurability ratio can be observed following the time marks shown on the two orbits. Figure 9.45 shows the same orbit in the synodic system. This is to be compared with the orbit shown in Fig. 9.46 for which all the pertinent data are the same, except, of course, that the mass parameter now becomes $\mu = 0.000954016$ and the Jacobian constant is $C = 3.15505$. This example contributes additional information to the previously mentioned surface $\mu(C, e)$, cf. Section 9.9.1.

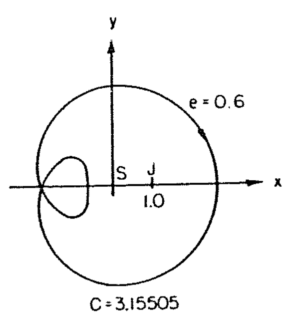

FIG. 9.46. Symmetric periodic orbit of commensurability 2 : 1 for $\mu = 0.000954016$ (Message, 1959, Ref. 55).

9.10 Stability

The details of numerical methods to establish periodic orbits and their stability lay outside our scope. The two operations which seem

to be unavoidable are numerical integration and differential correction. In Section 9.11 a few references are mentioned; here we describe, in what follows, two methods which may be considered new, both designed for obtaining periodic orbits as well as their stability parameters.

The standard methods to evaluate the stability of periodic orbits are discussed in Section 8.8. Both methods mentioned there furnish directly linear stability information. The evaluation of the characteristic exponents by the integration of the variational equations (first method) and the use of Darwin's equation of normal displacement (second method) are not reviewed here. The reader's attention is also directed to Chapter 5 where a general discussion of stability is offered. Huang's method proposed in 1963 and Hénon's work of 1965 are described in the following. The similarity of the methods will not escape the reader.

(A) Consider the set of initial conditions:

$$x(0) = x_0 \neq 0, \qquad y(0) = y_0 = 0,$$
$$\dot{x}(0) = \dot{x}_0 = 0, \qquad \dot{y}(0) = \dot{y}_0 \neq 0.$$

The simple periodic orbit with period T has the property that at $t = T/2$ it intersects again the x axis perpendicularly, $y(T/2) = 0$, $\dot{x}(T/2) = 0$. Huang proposes the following method for finding periodic orbits. Let $\dot{y}_0$ be a value of $\dot{y}$ which is close to the value giving a periodic orbit with the given value of x_0. Consider now a simple periodic orbit which, after the elapse of a period T, returns to $P_0(x_0, 0)$. If $\dot{y}_0$ is not the correct magnitude of the initial velocity in order to obtain a periodic orbit, after the elapse of approximately a period T, the particle will not return to P_0 but it will intersect the x axis at $P_1(x_1 \neq x_0, 0)$. The time interval between two successive crossings T_1 is not the period: $T_1 \neq T$. Consider now the successive values corresponding to repeated circuits

$$T_1, T_2, \ldots, T_n, \ldots.$$

The proposed measure of deviation of the orbit from a periodic orbit is

$$\Delta_{n+1,n} = T_{n+1} - T_n .$$

The suggested method of finding a periodic orbit is to find experimentally points on the curve

$$T_2 - T_1 = \Delta_{2,1} = f(\dot{y}_0)$$

and then find the root of the equation

$$f(\dot{y}_0) = 0.$$

For instance, the initial condition $x_0 = -0.43215$, combined with

$$\dot{y}_0 = \begin{cases} 2.2645059, \\ 2.2645060, \\ 2.2645061, \end{cases}$$

gives (retrograde) orbits with the previously mentioned time-intervals

$$\Delta_{2,1} = \begin{cases} 15 \times 10^{-6}, \\ < 0.5 \times 10^{-6}, \\ -17 \times 10^{-6}. \end{cases}$$

The corresponding values of T_1 are

$$T_1 = \begin{cases} 12.726637, \\ 12.726642, \\ 12.726648. \end{cases}$$

This example (given by Huang) uses a coordinate system which is identical to our standard system with the exception of the interchange of the location of the primaries. The mass ratio is $\mu = 0.01215$.

It is also suggested by Huang that a study of the sequence

$$\Delta_{2,1}, \Delta_{3,2}, \Delta_{4,3}, \ldots, \Delta_{n+1,n}, \ldots$$

may be informative regarding the stability of the periodic orbit. According to his definition a periodic orbit is unstable when $|\Delta_{n+1,n}|$ increases with n for orbits near the periodic orbit and an orbit is stable when members of the sequence computed for neighboring orbits decrease or oscillate without increasing their amplitude as n increases.

For instance, in the previous example the sequence obtained for the stable periodic orbit itself ($\dot{y}_0 = 2.2645060$) contains the following values of $\Delta_{n+1,n}$:

n	1	2	3	4	5	6	7	8	9	10
$10^6 \times \Delta_{n+1,n}$	<0.5	+2	−3	−1	+4	−3	+2	0	−3	−1

Computing a sequence for a nonperiodic orbit with initial conditions close to the periodic orbit we have the following table:

n	1	2	3	4	5	6	7	8	9	10
$10^6 \times \Delta_{n+1,n}$	−79	+82	−6	−77	+87	−13	−73	+89	−22	−66

This table may be obtained with the initial conditions $x_0 = -0.43215$, $\dot{y}_0 = 2.2645065$. The latter differs from the value of $\dot{y}_0$ which gives a periodic orbit by $\Delta\dot{y}_0 = 5 \times 10^{-7}$. Increasing the value of $\Delta\dot{y}_0$ to say 15×10^{-7}, the absolute values of the members of the sequence increase to an average figure of 200×10^{-6}. The sequence itself, belonging to a stable orbit, in all cases consists of members with approximately constant absolute value.

Huang's definition of stability and his method of computation, when applied to the periodic orbits described in Section 9.5.4 and shown in Fig. 9.26, verify the expectations. His direct orbits are unstable and the retrograde orbits are stable.

(B) Hénon studies the intersections of the orbits with Poincaré's surface of section instead of considering the complete trajectory in the phase space. We recall at this point that the Jacobian integral for a fixed value of C furnishes the value of $\dot{y}_0$ if x_0, y_0, and $\dot{x}_0$ are given. We investigate, therefore, the trajectory in the three-dimensional space $x, y, \dot{x}$; more specifically we study the intersections of the orbit with the $x, \dot{x}$ plane ($y = 0$), when $\dot{y} > 0$ for a fixed value of C.

The initial conditions are $x_0 \neq 0$, $\dot{x}_0 \neq 0$, $y_0 = 0$, $\dot{y}_0 > 0$. Note that the condition $\dot{x}_0 = 0$ is not required. For a given value of C the first intersection of the orbit with the $x, \dot{x}$ plane, for $\dot{y} > 0$, occurs at the beginning of the motion. When the simple periodic orbit intersects the x axis again, $T/2$ time has elapsed. At $t = T$, there is another intersection for which we have x_1, $\dot{x}_1$, $y_1 = 0$, and $\dot{y}_1 > 0$. If $x_0 = x_1$ and $\dot{x}_0 = \dot{x}_1$ we have a periodic orbit since $y_0 = y_1 = 0$ and $\dot{y}_0 = \dot{y}_1 > 0$ because the Jacobian constant was held fixed. In general,

$$x_1 = f(x_0, \dot{x}_0, C),$$
$$\dot{x}_1 = g(x_0, \dot{x}_0, C),$$

or in other words there is a mapping of the $x, \dot{x}$ plane into itself. The point x_0, $\dot{x}_0$ goes into x_1, $\dot{x}_1$. The fixed points of such a transformation represent periodic orbits, for which

$$x_0 = f(x_0, \dot{x}_0, C)$$

and

$$\dot{x}_0 = g(x_0, \dot{x}_0, C).$$

So to find periodic orbits we search for the fixed points of the above transformation by numerically integrating orbits and recording only the coordinates x_1 and $\dot{x}_1$ of the intersection of the orbit with the plane $y = 0$.

Orbits which are symmetric to the x axis have the initial conditions $x_0 \neq 0$, $\dot{x}_0 = 0$, $y_0 = 0$, $\dot{y}_0 \neq 0$, and consequently the search for invariant points is a one-dimensional process, along the x axis—again for a fixed value of C. The result is represented in the form of $C = C(x_0)$ curves as shown throughout this chapter. Hénon found 22 families of periodic orbits for the Copenhagen problem, with 7 new ones, and presented the corresponding characteristic curves [$C = C(x_0)$].

Turning now our attention to the question of stability we first mention Darwin's fundamental idea which led him to the equation of normal displacement given in Section 8.8. This idea is that the stability of periodic orbits may be studied by considering orbits near the periodic orbits with the same value of the Jacobian constant. Quite similarly we now inquire about the behavior of the neighborhood of the invariant point of the previously mentioned transformation. In other words, if the area preserving transformation at the fixed point is stable, the periodic orbit belonging to that fixed point is also stable.

Since the transformation in question is determined by the functions f and g, we have for an area-preserving transformation that

$$\frac{\partial(f, g)}{\partial(x_0, \dot{x}_0)} = 1. \tag{28}$$

Furthermore, from the symmetry property discussed in considerable detail in Chapter 8 we have the following two sets of equations describing the transformation:

$$x_1 = f(x_0, \dot{x}_0, C) \quad \text{and} \quad \dot{x}_1 = g(x_0, \dot{x}_0, C), \tag{29}$$

and

$$x_0 = f(x_1, -\dot{x}_1, C) \quad \text{and} \quad -\dot{x}_0 = y(x_1, -\dot{x}_1, C). \tag{30}$$

The second set is obtained from the first set by the following exchanges of variables:

$$x_1 \to x_0, \qquad \dot{x}_1 \to -\dot{x}_0, \qquad x_0 \to x_1, \qquad \dot{x}_0 \to -\dot{x}_1.$$

Considering now only isoenergetic changes we have, for the first set,

$$\begin{aligned} \Delta x_1 &= a\,\Delta x_0 + b\,\Delta \dot{x}_0, \\ \Delta \dot{x}_1 &= c\,\Delta x_0 + d\,\Delta \dot{x}_0, \end{aligned} \tag{31}$$

where a, b, c, and d are the appropriate partial derivatives of the functions f and g, their Jacobian determinant being unity on account of the area-preserving property of the transformation. So according to Eq. (28) we have

$$ad - bc = 1. \tag{32}$$

The eigenvalues of the transformation may be computed from the determinant

$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0 \tag{33}$$

or from the characteristic equation

$$\lambda^2 - (a + d)\lambda + 1 = 0, \tag{34}$$

where λ is related to Poincaré's characteristic exponent α by

$$\lambda = e^{\alpha T}.$$

The invariant point is unstable when

$$| a + d | > 2$$

since in this case the two roots are real and

$$\lambda_1 \lambda_2 = 1.$$

Consequently one characteristic exponent is always positive.

The two roots are complex conjugates when

$$| a + d | < 2,$$

in which case

$$| \lambda_1 | = | \lambda_2 | = 1$$

and both characteristic exponents are pure imaginary numbers. This is the case of stability in the linear sense.

For a symmetric simple periodic orbit the condition of symmetry gives, from Eqs. (30),

$$\begin{aligned} \Delta x_0 &= a\,\Delta x_1 - b\,\Delta \dot{x}_1, \\ \Delta \dot{x}_0 &= -c\,\Delta x_1 + d\,\Delta \dot{x}_1. \end{aligned} \tag{35}$$

If this set of equations is compared with the inversion of Eq. (31),

$$\begin{aligned} \Delta x_0 &= d\,\Delta x_1 - b\,\Delta \dot{x}_1, \\ \Delta \dot{x}_0 &= -c\,\Delta x_1 + a\,\Delta \dot{x}_1, \end{aligned} \tag{36}$$

we see that $a = d$.

The stability condition for symmetric simple periodic orbits, therefore, becomes that $| a | < 1$.

The quantity a may be computed from Eq. (31). We change the initial condition of a periodic orbit by $\Delta x_0 \neq 0$ and by $\Delta \dot{x}_0 = 0$. In this way we obtain

$$a = \Delta x_1 / \Delta x_0.$$

Computation of the other coefficients, b, c, and d, for symmetric simple periodic orbits is for control purposes only.

Hénon evaluated the linear stability characteristics of the families of periodic orbits of the Copenhagen category. Some of his major conclusions are:

(1) The classes of the Copenhagen problem may be organized in two groups according to the linear stability behavior. In the first group are Classes (f), (g), (l), and (m) and so this group contains the direct and retrograde satellite orbits and superior planetary orbits. In the second group we have all the other classes. Only the first group contains stable orbits of any appreciable number; the members of the second group are "almost all" unstable.

(2) Preparing plots of the relation $a = a(C)$, only the first group seems to show values of dC/da appreciably different from zero in the range of stability, $-1 < a < 1$.

(3) Collision orbits show no special behavior and they are found on the unstable parts of the curves $a(C)$.

(4) As an example, Fig. 9.47 shows the function $a(C)$ as determined by Hénon numerically for Class (l) of the Copenhagen category. These synodically retrograde orbits are direct orbits in the fixed system and are discussed in Section 9.4.6. The value of the Jacobian constant corresponding to point A is $C_A = 3.5526$ and that of B is $C_B = 3.678$. Orbits of this class are therefore stable when $C > C_A$.

(5) Figure 9.48 demonstrates the critical orbits of Classes (l) and (m) showing the corresponding ranges of stability. Orbits belonging to Class (m) (see Section 9.4.7) for which $C < C_{max} \cong -0.37$ are all stable if they are outside the cross-hatched area. Orbits belonging to

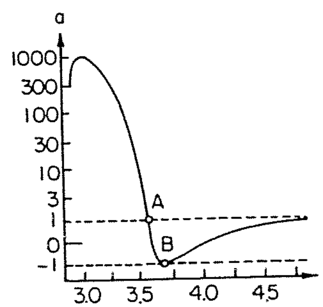

Fig. 9.47. Stability curve for Class (l) of the Copenhagen category, $\mu = \frac{1}{2}$ (Hénon, 1965, Ref. 21).

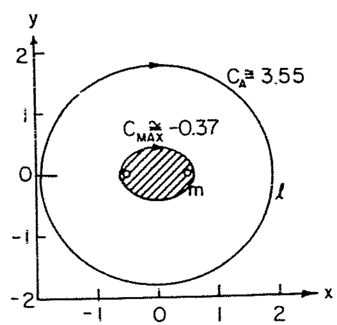

Fig. 9.48. Stability regions for Class (l) and (m) orbits of the Copenhagen category, $\mu = \frac{1}{2}$ (Hénon, 1965, Ref. 21).

Class (l) which are outside the curve drawn for $C = C_A$ are also all stable.

(6) Figure 9.49 shows the corresponding results for Classes (f) and (g) of the Copenhagen category. Those members of Class (g) for which $C > 3.99$ are all stable provided they belong to phase 1 as specified in Fig. 9.6(a). These direct orbits around m_1 are discussed in Section 9.4.4. The initial member of this group, for which $C = 3.99$, is marked by the letter g and is shown in Fig. 9.49. Information on the stability of Class (f) is also given in Fig. 9.49 by the curves marked f_1, f_2, and f_3. Orbits in the areas not cross-hatched are stable. The critical values of the Jacobian constant are: $C_1 = 1.4675$, $C_2 = 1.3094$, $C_3 = 1.2056$. Consequently, the direct orbits belonging to Class (f) in the Copenhagen category for which $C > C_1 = 1.4675$ are all stable. As C decreases a ring of instability develops between C_1 and C_2 which is followed by a ring of stability between C_2 and C_3. The members of Class (f), following the orbit with $C = C_3$, do not possess monotone decreasing values of C since the two-body effects do not dominate any more. The rest of the orbits are "almost all" unstable.

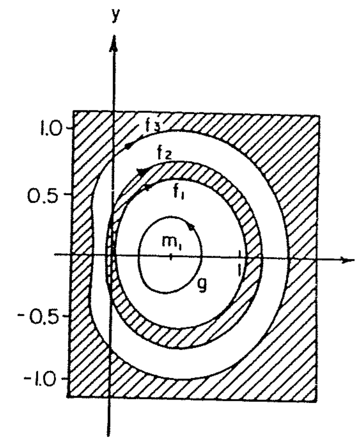

Fig. 9.49. Stability regions for Class (f) and (g) orbits of the Copenhagen category, $\mu = \frac{1}{2}$ (Hénon, 1965, Ref. 21).

Note that Darwin's stability index c is connected with Hénon's parameter a by the equation

$$a = \cos \pi c.$$

9.11 Notes

The pertinent references in the subject of numerical explorations may be divided into categories corresponding to the classification of the results. The primary publications pertaining to the *Copenhagen category*, Section 9.4, are the publications of the Copenhagen Observatory, Moulton's "Periodic Orbits" [1] and Darwin's papers on periodic orbits [2]. The second category on *periodic lunar orbits*, Section 9.5, is based on Egorov's [3] and on Broucke's [4] work. The section on the periodic *orbits around the triangular equilibrium points*, Section 9.6, uses Rabe's [5, 5a, 6] papers as the main sources. The principal references for Section 9.7 on *lunar trajectories* are by Egorov [3] and Ehricke [7]. Finally, the *applications to binary systems*, Section 9.8, is based on Kuiper's [8] and on Kopal's [9] contributions.

In addition to the above major references, several hundred articles and reports are available, many of which will be mentioned in the course of the comments and notes on this chapter.

Section 9.1 refers to two analysts defending numerical work. I believe this is important to emphasize since the anticomputer attitude is just as deplorable as the *computo ergo sum* approach in dynamics. Karl F. Sundman, in his lecture delivered at the 1930 meeting of the Astronomische Gesellschaft in Budapest, and Wintner [10] are the two outstanding analysts in celestial mechanics who express appreciation for numerical undertakings. In the same section the ***aim*** of dynamics is given following Birkhoff's [11] original idea.

It is interesting to read Wintner's [10, p. 324] concern over the confusion of terminology existing in the literature regarding families, kinds of orbits, groups, etc.

In Section 9.2 the mass parameter given for the earth-moon system seems to be the best value available (see [11a]). Fortunately the numerical value of μ is not critical, excepting some highly special and unstable orbits to be mentioned later. Investigators seem to feel free to select their own value of μ, an interesting choice being 0.0123456790 by Thüring [12]. The mass parameter for the sun-Jupiter system is

$$\mu = \frac{1}{1048.39 \pm 0.03}\,.$$

that is, between 0.0009538 and 0.0009539, according to Brouwer and Clemence [13].

Section 9.2 is concluded by Table I. This attempts to summarize those numerical explorations which are included in the text. The first date following Strömgren's name refers to the date of the first published paper dealing with the subject (1913). This is followed by the year when the last paper in the series of publications appeared (1939). The last date (1964) is the year of appearance of Bartlett's [14] paper in the Copenhagen series. No publication of the Copenhagen Observatory deals with the restricted problem between 1939 and 1964. Note that Bartlett's second paper [15], while published in the Copenhagen series, treats the restricted problem with arbitrary value of the mass parameter.

In Section 9.3, Eq. (7) is in Darwin's original form [2]. (See "Scientific Papers" Vol. 4, p. 4, Eqs. (1) and (2). In fact there are two equations there which are numbered 1; reference is made to the first one.) Equation (9) is given in the *Copenhagen Obs. Publ.* No. 100 [16] as Eq. (5), and Eq. (13) in Broucke's [4, p. 23] work as Eq. (10). Charlier's definition is in his "Die Mechanik des Himmels" [17, Vol. 2, p. 113]. The somewhat trivial question of relating the values of the Jacobian constants at different values of μ is expanded because of the repeated misunderstandings in the literature. Wintner [10, p. 359] writes $\tilde{C}_{1/2} = \tilde{C}_{10/11}$ and obtains

$$C_K = \frac{4C_D}{11} - \frac{40}{121} \cong \frac{4C_D}{11} - \frac{1}{3},$$

as the reader can verify referring to Eq. (23) in Section 9.3 and to Wintner's equation (26). After doing this, Wintner criticizes Fischer-Petersen [18] who writes $C_{1/2} = C_{10/11}$ and obtains

$$C_K = \frac{4}{11} C_D - 1,$$

which the reader may obtain from Eq. (24) of Section 9.3. This equation corresponds to Wintner's equation (27). Both results are erroneous except at the triangular libration points, where Fischer-Petersen's result is correct.

Section 9.4 is based on the 39 reports prepared and published by the Copenhagen Observatory. There are seven summary papers which are most pertinent, one by Fischer-Petersen [18], three by Strömgren [16, 19, 20], one by Bartlett [14], and two by Hénon [21].

TABLE XV

References for the Copenhagen Category

Section	Class	Copenhagen Observatory Publ. No.
9.41	(a)	14, 63
9.42	(c)	18, 60, 99
9.43	(f)	23, 99
9.44	(g)	21, 22, 27, 39
9.45	(k)	24, 26, 27, 32, 47, 80
9.46	(l)	30, 47, 69
9.47	(m)	44, 48
9.48	(n)	27, 39, 94
9.49	(o)	60, 97
9.49	(r)	60, 97
9.49	Asympt. orbits to $L_{4,5}$	47, 61, 64, 67
9.49	Asymm. orbits (Fig. 9.15)	47, 67, 89

In addition to these summary reports, many Copenhagen publications are on the restricted problem [22]. There are three papers concerning the Burrau–Thiele problem (discussed in Chapter 3) which were *not* published as *Copenhagen Observatory Publications*. These are related to Class (a) orbits of the Copenhagen category [23]. The main references according to the classes introduced in Section 9.4 are given in Table XV. The various Copenhagen reports appeared under the names of Strömgren, Burrau, Thiele, Möller, Fischer-Petersen, Lous, Lindow, Pedersen, Samter, Wintner, Rosenthal, Kristiansson, and Bartlett.

The last summary-type *Copenhagen Observatory Publication* by Bartlett [14] furnishes the initial conditions for over 800 periodic orbits of many members of almost every class, using the Thiele–Burrau regularized system. He discovers new classes as well. Some of the figures of this chapter, giving initial and final conditions, use Bartlett's [14] results—after retransformation—and some are based on the original papers by the Copenhagen school. Inasmuch as the transformation equations from the regularized to the physical coordinates are given in Chapter 3 as well as in Bartlett's paper, they will not be repeated here. An important contribution of this paper is the extension and continuation of Class (g) orbits of the Copenhagen category. Figures 9.6(f) and 9.6(g) have been obtained by integrating the equations of motion using Bartlett's (transformed) initial conditions. Both Strömgren and Bartlett express the opinion that this family ends at $L_{4,5}$. This was a conjecture on Strömgren's part and the second orbit of Fig. 6(g) is offered as supporting this idea. The orbits shown as the sixth and seventh phase of Class (g) were computed by J. Lieske. Strömgren's and Bartlett's conjecture regarding the termination of Class (g) is verified by Szebehely and Nacozy [24].

The tentative sentence of Section 9.4.8 regarding the nonexistence of asymmetric periodic consecutive collision orbits is related to Class (n) orbits of the Copenhagen category. The same statement appears in connection with Class (k) in Section 9.4.5, but with a slightly different connotation. Class (n) does not contain consecutive collision orbits which are asymmetric to the y axis. This *may* indicate that there are no consecutive collision orbits (at least of class n type) for $\mu \neq 1/2$. I have personal knowledge of many competent, diligent but unsuccessful attempts to find such orbits for $\mu \neq \frac{1}{2}$. Class (k) orbits do not appear as the result of an analytic continuation process. Consequently the existence of such orbits for $\mu \neq \frac{1}{2}$ is still in question. An article by Wintner [25] discusses the simpler subject of orbits with only one collision and corrects a few statements made by Moulton [1] which can be misunderstood. The question of consecutive periodic collision orbits for $\mu \neq \frac{1}{2}$ is still open.

The orbit in Fig. 9.6(c) referred to as v. Haerdtl's orbit is described in considerable detail by von Haerdtl [26]. See also an article on the same subject by Wintner [27] and by Coculesco [28].

Section 9.4.10 discusses the termination principle first announced by Strömgren in 1924 based purely on numerical results. Wintner's proof [10, 29] is lengthy and straightforward, while Birkhoff's sketchy proof is less than a page long and is not as easy to follow [30, p. 48].

Prior to Strömgren's discovery of the termination principle, Burrau [23] expressed a different opinion: "A mon sens, voici la véritable règle: les diverses classes des solutions sont séparées les unes des autres par des trajéctoires d'éjection."

The periodic orbits discussed in Section 9.4.11 were computed by Sir George Darwin (1845–1912), son of Charles Darwin. Attention is directed to his two major papers on periodic orbits [2]. No student of the restricted problem should miss these papers, the reading of which will fill the "expert" with humility and the beginner with enthusiasm. These papers are reprinted in the collection of his works entitled "Scientific Papers," Vol. 4, with a number of changes, the most important being the changes made in Fig. 1 of Plate IV. This remark is important since the literature refers to Darwin's papers as they are modified in the collection. An important contribution to Darwin's work is by Hough [31]. The student of Darwin's work will find a number of questionable drawings, conclusions and interpretations offered by him. These I left out of Section 9.4.11 since they would only confuse the reader. I especially refer to his figure 8 orbits which he probably constructed by combining two oval orbits, one around a primary and the other around a neighboring libration point.

Moulton's work mentioned in Section 9.4.12 is collected in his monumental book entitled "Periodic Orbits" [1], which consists mostly of the republication of his and his co-workers papers originally published between 1907 and 1912. The last chapter (Chapter 16) of the book is new and original and it is the most important for our purposes. The controversy regarding Moulton's discovery of retrograde periodic orbits around $L_{4,5}$ (shown in Fig. 9.18) between him on one side and Strömgren and Wintner on the other, can be settled at least partly by giving the correct values of the Jacobian constants and the proper interpretation of the results. Since Moulton gave the wrong value of C and the wrong interpretation, his orbits were not accepted. Strömgren [19, pp. 14–15] discusses this question and points out that infinitesimal periodic orbits do not exist around the triangular libration points for $\mu = \frac{1}{2}$ but orbits of finite size may. Then he says that Moulton's orbits are "not convincing" since they do not satisfy the Jacobian integral. Careful computation reveals that Moulton's results need slight revision but that the general shape of the curves given in the last chapter of his book [1] and reproduced in Section 9.4.12 as Fig. 9.18 is correct. Strömgren himself modified his 1922 views in this matter as his 1929 paper shows [32]. In fact the next year (1930) he [33] suggests the possibility of discovering Moulton's orbits, giving a reasoning close to the one offered in the text in Section 9.4.12(A). Furthermore note that Kristiansson [34] found a family of orbits for which Fig. 9.15(c) serves as limiting orbit twice; once as shown in the text and once as its mirror image with respect to the x axis. Regarding the values of the Jacobian constant given by Moulton, it is noted that in most cases the definition $C_M = \bar{C}$ applies while in a few cases the equation $C_M = C$ is valid. This happens with several of the orbits in question. The criticism offered by Wintner [10] does not seem to be convincing, especially in view of the interpretation given in Section 9.4.12(A). The controversy is settled by Szebehely [35] and Moulton's family is completed by Szebehely and Van Flandern [35a].

It is also of interest to note that Moulton found Strömgren's Class (k) and mentioned it in the last chapter of his book, published in 1920. The publication date of *Copenhagen Observatory Publication* No. 26, where the basic idea of Class (k) was born is 1917. This would give priority to the Copenhagen Observatory but Moulton remarks that the investigations reported in his last chapter were completed in 1917, therefore it is quite probable that Moulton's discovery is independent.

Sections 9.5.1 and 9.5.2 use, as mentioned before, the extensive work performed by Broucke [4], results of which are available only in the form of a dissertation.

I deviated from some of the interpretations given by Broucke, especially regarding Class (g) since I had more members of Strömgren's Class (g) available for comparison than he had. The remarks about Class (m), periodic lunar orbits, from the point of view of space dynamics can be found in Egorov's work [3].

The major contributor to Section 9.5.4 on special periodic orbits is Arenstorf [36]. Figure 9.24(b) was computed by Davidson [37]. The idea (without computation or analysis) of orbits like the one shown in Fig. 9.24(b) appears already in Darwin's first paper on periodic orbits [2, pp. 55–56]. Later Egorov [3] in 1958 and Thüring [12] in 1959 computed nonperiodic orbits passing through the neck of the zero-velocity curve at L_2 and revolving around E and M.

The chronology is quite similar in Section 9.5.4, Part (B), regarding the periodic orbits of the second kind with members having various commensurabilities. The first proposal of using such orbits for consecutive close approaches is in Newton's paper [38]. Then Huang [39] presented the same idea with extensive numerical backing (see, for instance, a member of Fig. 9.16.) Finally, Arenstorf [36] in 1963 gave an analytic proof and further numerical examples. Meanwhile Broucke [4] in 1962 computed orbits of this class (see Figs. 9.26 and 9.28) without elaborating on the advantages of consecutive close approaches.

Section 9.6 discusses Rabe's work. References [5, 5a, and 6] are mentioned at the beginning of Section 9.11. Prior to Rabe's work, Thüring [40, 40a, 40b] questioned the existence of such orbits. Wintner [41] proved by analytic continuation their existence (at least in the immediate neighborhood of $L_{4,5}$); see also Chapter 5. A reason for the discrepancy between Rabe and Thüring is explained in Rabe's work [5a]. Note that Wintner's proof is based on analytic continuation which assures the beginning of Rabe's family but the existence of large-amplitude periodic orbits still rests on Rabe's numerical proof. This is the reason for discussing questions of accuracy at this point and attempting to give an answer to the frequently asked question: what accuracy is required to obtain numerical proof of the existence of periodic orbits?

Numerical-integration methods, problems of error propagation, and related subjects cannot be discussed here. Brouwer's [42] evaluation of the accumulation of round-off errors is mentioned here because of its immediate applicability to orbit calculations. According to this the round-off error is proportional to $n^{1/2}$ for any element or in any direction except along the orbit, where the error in the mean longitude is proportional to $n^{3/2}$. Here n is the number of integration steps. The only other reference regarding numerical integration to be mentioned is Steffensen's method [42a], since it has been used with considerable success to integrate the differential equations of the restricted problem by several authors, one of the first being Rabe [5].

The reversibility question was also discussed in Chapter 8. At this point Baker's (Baker and Makemson [43, p. 339]) comments are referenced.

Rabe's orbits were recomputed by Deprit (Deprit and Delie [44]), who found that all Rabe's orbits were linearly stable except the horseshoe type shown in the last drawing in Fig. 9.29(b) corresponding to $\lambda = 0.06$. The families of short-period Trojan orbits have been computed by Goodrich [45].

To the references mentioned at the beginning of the section regarding lunar trajectories (Section 9.7), Buchheim's article [46] should be added with a warning regarding the topological liberties with which the zero-velocity curves are drawn.

The literature of capture is not reviewed here. Only Fezenkov's paper [47] mentioned in the text and a general excellent reference book by Leimanis and Minorsky [48] are listed. Inasmuch as most of the American literature is related to single, three-dimensional actual earth-moon trajectories rather than to families of orbits in the idealized restricted problem, this section makes heavy use of Egorov's [3] work. Excellent studies, for instance such as by Gapcynski and Woolston [49], which are connected to proposed Apollo type mission trajectories, had to be left out. Nevertheless the consecutive collision orbits described in this section (9.7) are based on Gapcynski's and Woolston's work [49]. The first orbit computed in Class (a) was obtained—in the first approximation—by slightly displacing one of the proposed Apollo trajectories and computing its equivalent in two dimensions. The references to the consecutive collision trajectories described here are two articles, the first by Szebehely *et al.* [50], and the second by Pierce and Standish [51].

References on orbits in binary systems (Section 9.8) in addition to Kopal's and Kuiper's papers are Abhyankar [52] and Gould [53]. The orbits obtained in connection with the analytic continuation studies shown in Section 9.91 were integrated numerically S. Knowles. The asymptotic-periodic orbits of Section 9.9.2 were computed by Deprit and Henrard [54]. The last part of the section on special orbits (9.9.3) reports on Message's work [55]. The asymmetric orbits are of real significance and novelty; the symmetric orbits show interesting similarity to the ones found by Broucke [4] at $\mu = 1/82.27$ and to the ones surmised (rather than computed) by G. Darwin for $\mu = 1/11$. A detailed study of asymptotic (nonperiodic) orbits at the collinear libration points is in Warren's article [56].

The actual numerical establishment of periodic orbits might follow a number of methods. In Section 9.10 only Huang's [39] and Hénon's [21] methods are discussed. The latter is based on Poincaré's and Birkhoff's ideas of the surfaces of section and on Siegel's work; the references to these are given in Chapter 8.

9.12 References

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Chapter 10

Modifications of the Restricted Problem

10.1 Introduction

The problem treated throughout this work and referred to as the restricted problem was defined in Section 1.2 as follows:

Two point masses m_1 and m_2 called the primaries revolve around their center of mass in circular orbits. In the plane of their motion moves a third body with infinitesimal mass, not influencing the motion of the primaries. Assuming Newtonian gravitational forces, find the behavior of the third body.

Any deviation from the above conditions results in problems which, with few exceptions, have not been considered up to now in this work. The number of possible modifications of the assumptions made in formulating the problem is large and it was outlined in Section 1.9. The present Chapter treats only three of the important modifications in detail. Two of these, the "three-dimensional" and the "elliptic" extensions, represent complications of both analytical and numerical nature. The third major modification to be discussed, Hill's problem, is a simplification of the restricted problem with far reaching applications to lunar theories.

In addition to these three modifications, several others will be mentioned briefly, such as Euler's and Lagrange's problems, the problem when other than Newtonian gravitational forces are acting, the problem when the masses of the primaries vary, etc.

The most immediate generalization of the restricted problem of three bodies is, of course, the general problem of three bodies. This latter problem was described in Section 1.8 and its extension to the general problem of n bodies was treated in Section 3.10, where the regularized equations of the n-body problem were given. Inasmuch as the present treatise concerns itself with the *restricted problem*, only few additional remarks will be made regarding the *general problem of three bodies*, although this is one of the most immediate and obvious generalizations.

The purpose of this last chapter is to complete the work by listing subjects peripheral to the main problem and by giving the pertinent references.

The treatment of topics in this chapter is more sketchy than in the previous ones and serves mostly to indicate the major features of the subjects discussed. The three major subjects mentioned in Sections 10.2, 10.3, and 10.4 may form the basis of other comprehensive treatises dedicated to these subjects alone.

10.2 The three-dimensional restricted problem

10.2.1 Formulation of the problem and equations of motion

(A) The primaries revolve with the plane x, y around their center of mass in circles. The bodies are on the rotating x axis, located at $P_1(\mu, 0, 0)$ and at $P_2(\mu - 1, 0, 0)$. The y axis is perpendicular to the x axis and it is in the plane of rotation. The plane x, y rotates with unit mean motion; the angular velocity vector is normal to the plane along the z axis, which forms a right-handed system with the coordinate axes x and y (see Fig. 10.1).

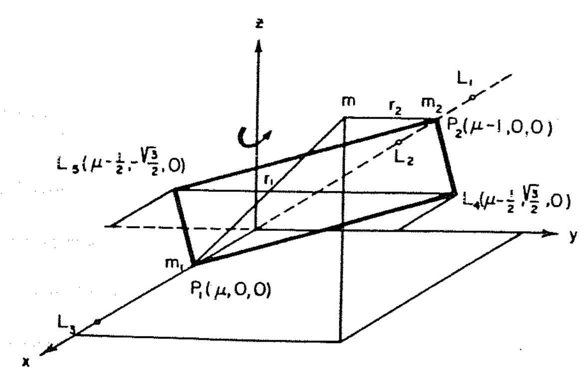


FIG. 10.1. Three-dimensional coordinate system and equilibrium points.

The equations of motion of the third particle are

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x, \\ \ddot{y} + 2\dot{x} &= \Omega_y, \\ \ddot{z} &= \Omega_z, \end{aligned} \tag{1}$$

where

$$\Omega = \tfrac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \tfrac{1}{2}\mu(1-\mu), \tag{2}$$

with

$$\begin{aligned} r_1^2 &= (x-\mu)^2 + y^2 + z^2, \\ r_2^2 &= (x-\mu+1)^2 + y^2 + z^2. \end{aligned} \tag{3}$$

Note that these definitions of r_1 and r_2 contain the coordinate z, so that now

$$\tfrac{1}{2}(x^2 + y^2) + \tfrac{1}{2}\mu(1-\mu) \neq \tfrac{1}{2}[(1-\mu)\, r_1^2 + \mu r_2^2].$$

Consequently, the elegant and symmetric form for the function Ω used in the planar case is not applicable when Eqs. (1) are written.

An alternate form of these equations is obtained if the definition given in Section 1.5 by Eq. (54) for the function Ω is retained. Denoting this by Ω', we have

$$\Omega' = \tfrac{1}{2}[(1-\mu)\, r_1^2 + \mu r_2^2] + \frac{1-\mu}{r_1} + \frac{\mu}{r_2},$$

which in the three-dimensional case becomes

$$\Omega' = \tfrac{1}{2}(x^2 + y^2 + z^2) + \tfrac{1}{2}\mu(1-\mu) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2},$$

where r_1^2 and r_2^2 are given by Eqs. (3).

Since Ω_z' in this case is

$$\Omega_z' = z + \Omega_z$$

while

$$\Omega_x = \Omega_x' \quad \text{and} \quad \Omega_y = \Omega_y',$$

the equations of motion (1) become

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x', \\ \ddot{y} + 2\dot{x} &= \Omega_x', \\ \ddot{z} + z &= \Omega_z'. \end{aligned} \tag{3'}$$

The Jacobian integral may be obtained, as in Section 1.4, by multiplying Eqs. (1) by $\dot{x}$, $\dot{y}$, and $\dot{z}$, adding the results, and integrating with respect to the time:

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2\,\Omega(x, y, z) - C. \tag{4}$$

(B) The Hamiltonian formulation for the three-dimensional case is equally simple. The equation corresponding to Section 7.2, Eq. (3), is

$$H = \tfrac{1}{2}(p_1^2 + p_2^2 + p_3^2) - \phi(q_1, q_2\ \ q_3, t),$$

where

$$p_1 = \dot{q}_1 = \dot{\xi}, \qquad p_2 = \dot{q}_2 = \dot{\eta}, \qquad p_3 = \dot{q}_3 = \dot{\zeta},$$

and

$$\phi = \mu_1/\rho_1 + \mu_2/\rho_2.$$

The coordinate system ξ, η, ζ is fixed and the variables are dimensionless. Note that the definitions of ρ_1 and ρ_2 are changed from those given for the two-dimensional case, since now

$$\begin{aligned} \rho_1^2 &= (q_1 - \mu_2 \cos t)^2 + (q_2 - \mu_2 \sin t)^2 + q_3^2, \\ \rho_2^2 &= (q_1 + \mu_1 \cos t)^2 + (q_2 + \mu_1 \sin t)^2 + q_3^2, \end{aligned}$$

where $\mu_1 = 1 - \mu$ and $\mu_2 = \mu$ (cf. Eq. (27) of Chapter 1).

In the synodic system the Hamiltonian is

$$\tilde{H} = \tfrac{1}{2}(P_1^2 + P_2^2 + P_3^2) + Q_2 P_1 - Q_1 P_2 - \tilde{F}(Q_1, Q_2, Q_3),$$

in complete agreement with Eq. (9) of Section 7.3.

The relation between the new (synodic) and previous (sidereal) coordinates and momenta is

$$\mathbf{Q} = \mathbf{A}^*\mathbf{q} \quad \text{and} \quad \mathbf{P} = \mathbf{A}^*\mathbf{p},$$

where $\mathbf{A}^*$ is the transpose of $\mathbf{A}$ and

$$\mathbf{A} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The distances occurring in the force function,

$$\tilde{F} = \mu_1/\rho_1 + \mu_2/\rho_2,$$

are

$$\begin{aligned} \rho_1^2 &= (Q_1 - \mu_2)^2 + Q_2^2 + Q_3^2, \\ \rho_2^2 &= (Q_1 + \mu_1)^2 + Q_2^2 + Q_3^2. \end{aligned}$$

The equations of motion are the same as in Section 7.2, Eqs. (17) and (18), with the addition of

$$\dot{Q}_3 = P_3 \quad \text{and} \quad \dot{P}_3 = \partial \bar{F}/\partial Q_3 .$$

(C) The three-dimensional Delaunay variables are $q_1 = L = a^{1/2}$, $q_2 = G = [a(1-e^2)]^{1/2}$, and $q_3 = H = [a(1-e^2)]^{1/2}\cos I$. The first two are the same as introduced in Sections 7.7.1, 7.7.2, and 7.7.3. The third generalized coordinate, q_3, is the projection of the angular momentum vector on the z axis and is therefore equal to the value of the integral of area in the plane x, y. The conjugate momenta are the mean anomaly l, the argument of perihelion g, and the longitude of the ascending node h. Note that $g = \varpi - \Omega$ is the angle from the ascending node to the perihelion since ϖ is the longitude of the perihelion and $\Omega = h$. For the two-dimensional arrangement, see Fig. 7.5, Section 7.7. The Hamiltonian function is

$$F = 1/2L^2 + R(L, G, H, l, g, h)$$

and the equations of motions are the same as Eq. (102) in Section 7.7.3 with the addition of

$$\dot{h} = -\partial F/\partial H \quad \text{and} \quad \dot{H} = \partial F/\partial h.$$

The linear combinations of Delaunay's variables discussed in Section 7.7.4 may be extended to the system with three degrees of freedom corresponding to the three-dimensional problem as follows. If Case (A) of Section 7.7.4 is generalized we obtain

$$Q_1 = L, \quad Q_2 = G - L, \quad Q_3 = H - G,$$

and

$$P_1 = l + g + h, \quad P_2 = g + h, \quad P_3 = h.$$

Generalization of Case (B) leads to

$$Q_1 = L, \quad Q_2 = L - G, \quad Q_3 = G - H,$$

and

$$P_1 = l + g + h, \quad P_2 = -(g + h), \quad P_3 = -h.$$

Finally Case (C) gives

$$Q_1 = L - G, \quad Q_2 = G - H, \quad Q_3 = H,$$

and

$$P_1 = l, \quad P_2 = l + g, \quad P_3 = l + g + h.$$

The three-dimensional versions of Poincaré's variables follow from Section 7.7.5. In Part (A) of Section 7.7.5, a transformation, W_4, is applied to the results of Case (C) of Section 7.7.4. The extension of this to the three-dimensional case gives

$$Q_1 = [2(L-G)]^{1/2}\cos l, \quad Q_2 = [2(G-H)]^{1/2}\cos(l+g), \quad Q_3 = H,$$

and

$$P_1 = [2(L-G)]^{1/2}\sin l, \quad P_2 = [2(G-H)]^{1/2}\sin(l+g), \quad P_3 = l + g + h.$$

Part (B) of Section 7.7.5 becomes

$$Q_1 = L, \quad Q_2 = [2(L-G)]^{1/2}\cos(g+h), \quad Q_3 = [2(G-H)]^{1/2}\cos h,$$

and

$$P_1 = l + g + h, \quad P_2 = -[2(L-G)]^{1/2}\sin(g+h),$$
$$P_3 = -[2(G-H)]^{1/2}\sin h.$$

10.2.2 Surfaces of zero velocity

10.2.2.1 *Singularities of the manifold.* The singularities of the manifold of the state of motion are obtained from the equations

$$\dot{x} = 0, \quad \dot{y} = 0, \quad \dot{z} = 0,$$
$$\Omega_x = 0, \quad \Omega_y = 0, \quad \Omega_z = 0.$$

The last one is

$$\Omega_z = -z\left(\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}\right) = 0,$$

where the second factor is positive for finite values of r_1 and r_2, therefore $z = 0$. This shows that the points of equilibrium are in the plane (x, y) and many of the previous results given in Chapter 4 are still valid. The same result also follows from a geometric consideration. The resultant force acting on a particle located at any of the points of equilibrium must be zero. The forces are the gravitational attractions directed toward the primaries and the centrifugal force which is perpendicular to the axis of rotation, z. These forces are coplanar if they are balanced, consequently the points of equilibrium must be in the xy plane.

The surfaces of zero velocity are defined by

$$2\,\Omega(x, y, z) = C \tag{5}$$

and, since $\Omega(x, y, z) = \Omega(x, \pm y, \pm z)$, the surfaces are symmetric with respect to the planes x, y and x, z. The intersection of the surfaces (5) with the plane x, y forms the curves of zero velocity discussed in Chapter 4. Therefore, the intersections of the surfaces of zero velocity with the planes y, z and z, x are required in addition, for a description of (5). In fact only the $z \geqslant 0$ half of the plane z, x and the $y \geqslant 0$, $z \geqslant 0$ quarter of the plane y, z are of interest because of the previously mentioned symmetry conditions.

10.2.2.2 *Surfaces of zero velocity for the problem of two bodies.* Prior to discussing the surfaces of zero velocity of the restricted problem it might prove to be instructive to survey these surfaces for the problem of two bodies in the usual rotating coordinate system. This model is obtained from Eqs. (2) and (3) by writing $\mu = 0$. The equation for the surfaces of zero velocity is

$$C = x^2 + y^2 + 2/r_1 , \tag{6}$$

where

$$r_1^2 = x^2 + y^2 + z^2. \tag{7}$$

An alternate form is

$$C = r^2 + \frac{2}{(z^2 + r^2)^{1/2}} , \tag{8}$$

where $r^2 = x^2 + y^2$ and the positive value of the square root is to be taken.

These surfaces are symmetric to the xy plane and also show rotational symmetry with respect to the z axis. Spherical symmetry is missing since the z axis is distinguished by being the axis of rotation. The intersection of the surfaces of zero velocity with the xy plane is given by

$$C = r^2 + 2/r. \tag{9}$$

When $C > 3$ there are two circles with radii r' and r'' for which $0 \leqslant r' \leqslant 1 \leqslant r'' \leqslant C$. These circles coincide at $C = 3$ and the common radius becomes $r' = r'' = 1$. This was discussed in detail in Section 8.5.1, where it was also shown that, when $C < 3$, the curves of zero velocity disappear from the xy plane. We also recall that

$$r' \geqslant 2/C \quad \text{and} \quad r'' \leqslant C^{1/2}, \tag{10}$$

where the equalities correspond to $C \to \infty$.

The intersection of the surfaces of zero velocity with the xz plane is given by

$$C = x^2 + \frac{2}{(x^2 + z^2)^{1/2}} . \tag{11}$$

For large values of C we have either large values of x in which $x \cong \pm C^{1/2}$ (first case), or small values of both x and z (second case). In the first case

$$| x | \leqslant C^{1/2},$$

where the equality sign corresponds to $z \to \infty$. This follows from Eq. (11). Note that solving Eq. (11) for $| x |$ and indicating that solution which is $\sim C^{1/2}$ by $x'(C, z)$ we have

$$x'(C, | z_1 |) < x'(C, | z_2 |)$$

if $| z_2 | > | z_1 |$.

The second case which gives large values for the Jacobian constant requires small values for x and z and so

$$(x^2 + z^2)^{1/2} = \frac{2}{C - x^2} \to \frac{2}{C} .$$

The corresponding curves are ovals around the origin for the second case and curves asymptotic to the $x = \pm C^{1/2}$ lines and intersecting the x axis at $\pm x'(C, 0)$ in the first case.

The intersection of the x and y axis with the surfaces occur when x and y satisfy the equations

$$C = x^2 + 2/x \tag{12}$$

and

$$C = y^2 + 2/y. \tag{13}$$

The intersection of the surface with the z axis occurs when $z' = \pm 2/C$. The solutions of Eqs. (12) and (13) correspond, of course, to the solutions of Eq. (9) and they become

$$x' = y' \geqslant 2/C \quad \text{and} \quad x'' = y'' \leqslant C^{1/2}.$$

So we conclude that the ovoid surrounding the unit mass at the origin of the coordinate system may be approximated by an oblate spheroid with the short axis along the z coordinate axis. The lengths of the three axes are $z' = 2/C \leqslant x' = y'$.

The other root of Eq. (12) corresponds to $x'' \leqslant C^{1/2}$. The outside cylinder intersects the xy plane in a circle of radius $r_0 < C^{1/2}$. The radius to which the cylinder asymptotically expands as $z \to \infty$ is $r_\infty = C^{1/2}$.

Figure 10.2 shows the curves of zero velocity in the xz plane for the problem of two bodies. Because of symmetry the same figure applies to the intersection of the surfaces of zero velocity with the yz plane.

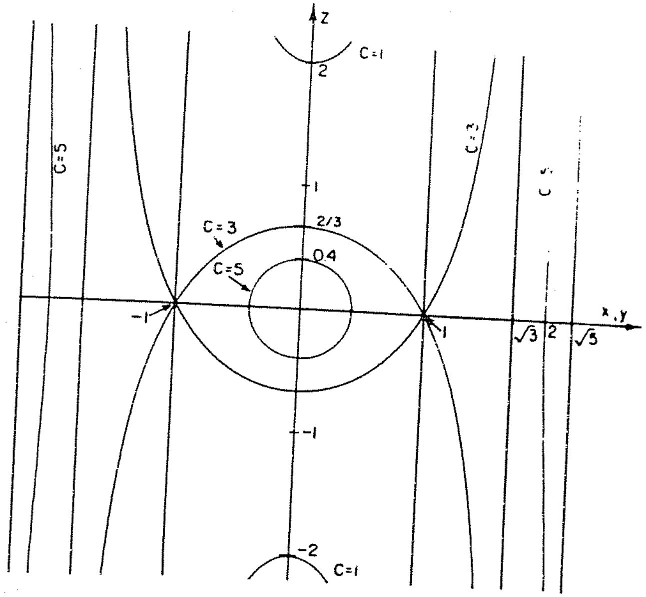


FIG. 10.2. Curves of zero velocity in the planes x, z and y, z for $\mu = 0$, $C = 5, 3$, and 1

On the curves the corresponding values of the Jacobian constant are shown. The curves marked $C = 5$ intersect the x and y axes at $x' = y' = \pm 0.414214$ and at $x'' = y'' = \pm 2$, while the intersection with the z axis is at $z' = \pm 2/C = \pm 0.4$. The vertical asymptotes are at $x_\infty = y_\infty = r_\infty = \pm 5^{1/2} = \pm 2.236068$. Figure 10.3 shows the circles of zero velocity in the xy plane. The radii for $C = 5$ are $r' = 0.414214$ and $r'' = 2$.

As the value of the Jacobian constant decreases from $C > 3$, the inner ovoid expands and the outer cylindrical surface shrinks. At $C = 3$ the intersection of the cylindrical surface with the xy plane and the ovoid's intersection with the x, y plane become the same circle with unit radius (see Fig. 10.3). The radius of the cylinder to which the surface of zero velocity is asymptotic is $\pm 3^{1/2}$. The intersection of the z axis with the ovoid is at $z = \pm \frac{2}{3}$ (see Fig. 10.2).

When $C < 3$, the equation

$$C = r^2 + 2/r$$

has no real positive roots, consequently the xy plane does not have common points with the surface of zero velocity. The intersections of the surface with the z axis are still at $z = \pm 2/C$ and the equation of the intersection of the surface with the z, x plane is still given by Eq. (11).

Solving Eq. (8) for z we obtain

$$z = \pm \left(\frac{4}{(C - r^2)^2} - r^2 \right)^{1/2}. \tag{14}$$

This surface approaches asymptotically the cylinder $r = C^{1/2}$. Figure 10.2 shows the intersection of the zero velocity surface with the z, x (or z, y) plane for $C = 1$. The asymptote is at $x_\infty = \pm 1$.

Motion is confined, for large values of the Jacobian constant, to the inside of the ovoid and to the outside of the cylinder, excluding the region between the ovoid and the cylinder. As $C < 3$ there is no boundary for the motion in the xy plane and the particle may move anywhere in this plane. The forbidden areas in the zx and yz planes are shown on Fig. 10.2. Outside of the surface [for values of z between the two values given by the right side of Eq. (14)] motion is allowed. As $C \to 0$, the intersection of the forbidden volume with the z axis moves to $z \pm \infty$. This concludes the discussion of the $\mu = 0$ case.

10.2.2.3 *Surfaces of zero velocity for the restricted problem.* In this section we study the equipotential surfaces

$$2\,\Omega(x, y, z) = C,$$

where the function Ω is defined by Eqs. (2) and (3) of Section 10.2.1.

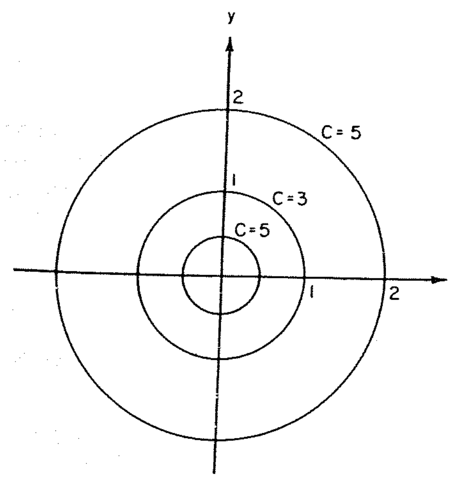


FIG. 10.3. Curves of zero velocity in the plane x, y for $\mu = 0$, $C = 5$ and 3.

Note first that, as $z \to \pm\infty$, $\Omega \to \frac{1}{2}(x^2 + y^2) + \frac{1}{2}\mu(1-\mu)$, and consequently $x^2 + y^2 \to C - \mu(1-\mu) = \bar{C}$. We see, therefore, that the surfaces of zero velocity are asymptotic to the circular cylinders with radii $\bar{C}$ as the surfaces $2\Omega = C$ depart more and more from the plane x, y.

A line parallel to the z axis pierces the zero velocity surface in two points or in no points. For, in this case, $x = x_0$, $y = y_0$ are fixed and the equation for C gives

$$C = \mu(1-\mu) + x_0^2 + y_0^2 + \frac{2(1-\mu)}{r_{10}} + \frac{2\mu}{r_{20}},$$

where

$$r_{10} = [(x_0 - \mu)^2 + y_0^2 + z^2]^{1/2}$$

and

$$r_{20} = [(x_0 - \mu + 1)^2 + y_0^2 + z^2]^{1/2}.$$

Consequently C becomes

$$C = C^* + \frac{2(1-\mu)}{(\alpha^2 + z^2)^{1/2}} + \frac{2\mu}{(\beta^2 + z^2)^{1/2}}, \tag{15}$$

where

$$C^* = \mu(1-\mu) + x_0^2 + y_0^2,$$
$$\alpha^2 = (x_0 - \mu)^2 + y_0^2,$$

and

$$\beta^2 = (x_0 - \mu + 1)^2 + y_0^2.$$

The function $C = C(z)$ has the following properties:

(a) $C(z) = C(-z)$;
(b) $C(0) = C_{\max} = C^* + 2(1-\mu)/|\alpha| + 2\mu/|\beta|$;
(c) $C \to C^*$ as $z \to \pm\infty$;
(d) $dC/dz \leqslant 0$, the equality sign corresponding to $z = 0$ and $z \to \pm\infty$.

Therefore to any value of C which satisfies the inequality $C^* < C < C_m$ there belong two values of z, and these are equal and of opposite sign, $z_1 = -z_2$. If the value of C is below C^* or above C_m, there will be no intersections between lines parallel to the z axis and the surfaces of zero velocity.

The intersection of the z axis with the surfaces follows from the previous considerations by writing $x_0 = y_0 = 0$. For this case we have

$$C^* = \mu(1-\mu), \qquad \alpha = \mu, \qquad \beta = (1-\mu),$$

and

$$C_m = C^* + \frac{2(2\mu^2 - 2\mu + 1)}{\mu(1-\mu)}.$$

Therefore, intersections of the z axis with the surfaces of zero velocity exist if

$$0 < \bar{C} < \frac{2(2\mu^2 - 2\mu + 1)}{\mu(1-\mu)},$$

where $\bar{C} = C - \mu(1-\mu)$.

Note that in the usual range

$$0 \leqslant \mu \leqslant \tfrac{1}{2}$$

we have, at $x_0 = y_0 = 0$,

$$0 \leqslant C^* \leqslant 1/4,$$
$$\infty \geqslant C_m \geqslant 17/4,$$

and

$$0 \geqslant \frac{dC_m}{d\mu} \geqslant -\infty.$$

Additional information on the three-dimensional picture may be obtained as follows. For large values of the Jacobian constant an inspection of Eqs. (2) and (3) shows that either $x^2 + y^2$ must be large or r_1 must be small, or, finally, r_2 must be small. In the first case, when at the same time $z = 0$, we have the outside ovals in the x, y plane with radii smaller than $\bar{C}^{1/2}$, and when $z \to \infty$ we have the asymptotic cylinders discussed before with radii $\bar{C}^{1/2}$. When r_1 or r_2 is small, approximately spherical surfaces of zero velocity appear around the primaries with radii

$$r_{10} \cong 2(1-\mu)/C \qquad \text{and} \qquad r_{20} \cong 2\mu/C,$$

as found before for the two-dimensional case [Section 4.7.2, Eqs. (96) and (99)]. So the complete picture for a large value of C—and "large" means, again, that the Jacobian constant is larger than C_2—consists of three separate surfaces: two ovoids enclosing the two primaries and a cylindrical surface outside the ovoids. This latter surface intersects the x, y plane in an oval and asymptotically approaches the shape of a circular cylinder of radius $\bar{C}^{1/2}$ as $|z| \to \infty$. The axis of this cylinder is the z coordinate axis and the cylinder is symmetric to the plane x, y.

As the value of C decreases, first the inner ovals meet at L_2 giving the familiar figure-eight curve in the x, y and in the x, z planes. The three-dimensional picture corresponding to this consists of two ovoids in contact at L_2. The intersection of the larger ovoid (located around the larger mass, $1-\mu$) with a plane through $P_1(\mu, 0, 0)$ and parallel to the z, y plane is an oval—provided $C \geqslant C_2$—which has the equation

$$C = \mu + y^2 + \frac{2(1-\mu)}{(y^2 + z^2)^{1/2}} + \frac{2\mu}{(1 + y^2 + z^2)^{1/2}}. \tag{16}$$

This oval is more elongated along the direction of the y axis than along the z axis. To show this, consider the points of intersection $(\mu, y_0, 0)$ and $(\mu, 0, z_0)$ of this oval with the x, y and with the x, z planes. From the preceding equation we have

$$C = \mu + y_0^2 + \frac{2(1-\mu)}{|y_0|} + \frac{2\mu}{(1+y_0^2)^{1/2}}$$

and

$$C = \mu + \frac{2(1-\mu)}{|z_0|} + \frac{2\mu}{(1+z_0^2)^{1/2}}.$$

Since $dC/dz_0 < 0$, we have that $y_0 > z_0$. Q.E.D.

The oval obtained by the intersection of the surface of zero velocity belonging to $C \geqslant C_2$ with the $x = \mu - 1$ plane is also more elongated along the y axis than along the z axis and has the same general shape as the intersection described previously for the section $x = \mu$.

As C further decreases the outside cylindrical surface touches the three-dimensional asymmetric dumbbell in the x, y plane at L_1. At $C = 3$ the surface touches the x, y plane at L_4 and at L_5, but when $C < 3$ intersections may occur only with the x, z and y, z planes.

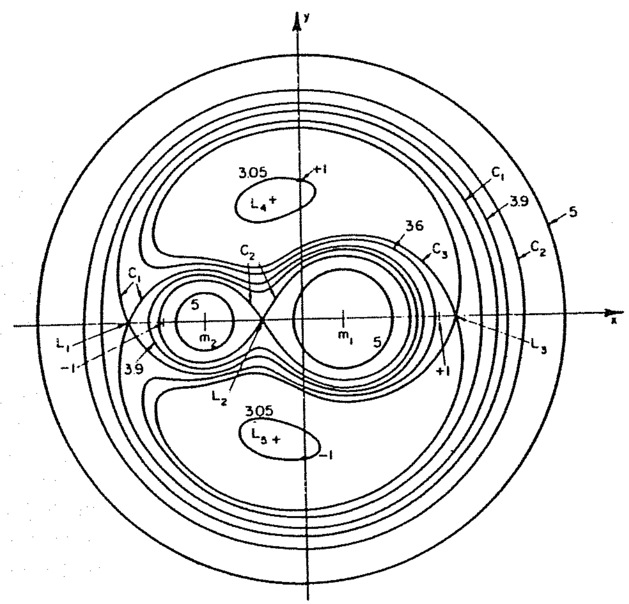


FIG. 10.4. Intersections of the surfaces of zero velocity with the x, y plane for $\mu = 0.3$, $C = 5, C_2, 3.9, C_1, 3.6, C_3, 3.05$, and 3.

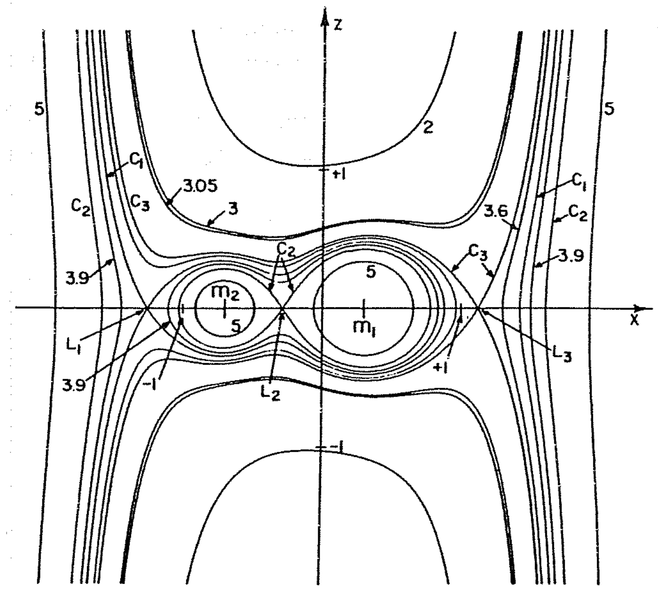


FIG. 10.5. Intersections of the surfaces of zero velocity with the x, z plane for $\mu = 0.3$, $C = 5, C_2, 3.9, C_1, 3.6, C_3, 3.05$, 3, and 2.

Note that curves of zero velocity do not exist in the x, y plane when $C < 3$ but that surfaces of zero velocity do exist for $C < 3$. The minimum value of C for the two-dimensional case is $C = 3$, as we have seen in Chapter 4. In the two-dimensional case,

$$C = (1-\mu)(r_1^2 + 2/r_1) + \mu(r_2^2 + 2/r_2),$$

with $r_1^2 = (x-\mu)^2 + y^2$ and $r_2^2 = (x - \mu + 1)^2 + y^2$, and so the lowest value of the first as well as of the second term in the equation for C occurs when $r_1 = r_2 = 1$; consequently, $C_{\min} = 3$.

For the three-dimensional case the lowest value of C is $C^* = \mu(1-\mu)$ and it corresponds to $x = y = 0$ and $z \to \pm\infty$. In this respect, therefore, it is advantageous to use the quantity $\bar{C} = C - \mu(1-\mu)$ as the Jacobian constant, the minimum value of which is zero independently on the value of the mass parameter.

The forbidden and permissible regions of the motion may be developed as follows. For large values of C the motion takes place either inside the ovoids surrounding the primaries or outside the cylindrical surface. As $C \to C_2$ the two ovoids approach each other. When $C = C_2$ the permissible volume includes both primaries with communication established between them. During this process the outside cylindrical surface shrinks but it preserves its separate identity. As $C = C_1$ the three-dimensional dumbbell touches the cylinder at L_1 and a new conic point L_1 is formed. When $C < C_1$ there are no more closed surfaces and the three-dimensional dumbbell opens up at L_1. The intersections with the x, y plane are now the familiar horseshoe-shaped curves which are the anchor figures for the surfaces symmetric to the x, y plane. Motion is possible everywhere outside the horseshoe curves (in the x, y plane) and outside the surfaces. When $3 < C < C_3$ we have two separate surfaces intersecting the x, y plane in closed curves around L_4 and L_5, which intersections disappear as $C < 3$. Motion is possible everywhere on, above, and below the x, y plane until the particle reaches the forbidden volumes surrounding the z axis. Figures 10.4, 10.5, and 10.6 show the intersections of the surfaces of zero velocity with the x, y; x, z and y, z planes for $\mu = 0.3$ and for $C = 5$, $C_2 \cong 4.13$, $C = 3.9$, $C_1 \cong 3.77$, $C = 3.6$, $C_3 \cong 3.50$, $C = 3.05$, $C = 3$, and $C = 2$.

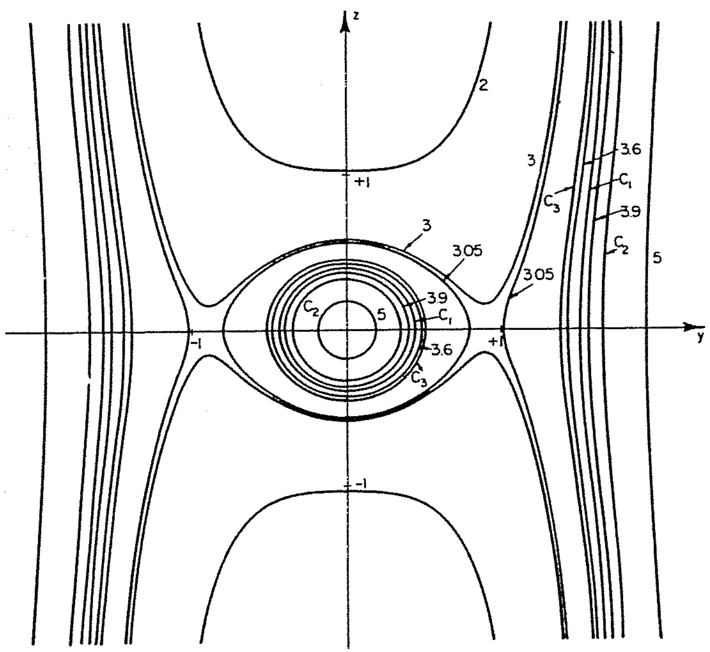


FIG. 10.6. Intersections of the surfaces of zero velocity with the y, z plane for $\mu = 0.3$, $C = 5$, C_2, 3.9, C_1, 3.6, C_3, 3.05, 3, and 2.

10.2.3 Three-dimensional motion around the equilibrium points

The variational equations in the three-dimensional case become

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \Omega^0_{xx}\xi + \Omega^0_{xy}\eta + \Omega^0_{xz}\zeta, \\ \ddot{\eta} + 2\dot{\xi} &= \Omega^0_{yx}\xi + \Omega^0_{yy}\eta + \Omega^0_{yz}\zeta, \\ \ddot{\zeta} &= \Omega^0_{zx}\xi + \Omega^0_{zy}\eta + \Omega^0_{zz}\zeta, \end{aligned} \tag{17}$$

where $x = a + \xi$, $y = b + \eta$, $z = c + \zeta$, and the superscript 0 indicates that the second derivatives are to be evaluated at the point (a, b, c). The second partial derivatives of the function Ω occurring in the last one of Eqs. (17) are

$$\begin{aligned} \Omega_{zx} &= 3z\left[\frac{(1-\mu)(x-\mu)}{r_1^5} + \frac{\mu(x-\mu+1)}{r_2^5}\right], \\ \Omega_{zy} &= 3zy\left[\frac{1-\mu}{r_1^5} + \frac{\mu}{r_2^5}\right], \\ \Omega_{zz} &= 3z^2\left(\frac{1-\mu}{r_1^5} + \frac{\mu}{r_2^5}\right) - \left(\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}\right). \end{aligned} \tag{18}$$

In the xy plane $\Omega_{zx} = \Omega_{zy} = 0$ and

$$\Omega_{zz}(x, y, 0) = -\left(\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}\right) < 0. \tag{19}$$

Recalling from Chapter 4 the forms of the second derivatives of the function Ω, we see that everywhere on the x axis, except at the primaries,

$$\Omega_{xx}(x, 0, 0) = 1 + 2\left(\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}\right) = 1 - 2\,\Omega_{zz}(x, 0, 0)$$

and

$$\Omega_{yy}(x, 0, 0) = 1 - \left(\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}\right) = 1 + \Omega_{zz}(x, 0, 0).$$

At the triangular libration points $\Omega_{zz}(L_{4,5}) = -1$ and at the collinear points of equilibrium

$$\Omega_{zz}(L_{1,2,3}) = \Omega_{yy}(L_{1,2,3}) - 1 < 0. \tag{20}$$

Consequently, and according to Section 4.6, we have

$$-8 \leqslant \Omega_{zz}(L_{1,2,3}) \leqslant -1.$$

Since the mixed partial derivatives $\Omega_{xz} = \Omega_{yz} = 0$, the third variational equation may be separated from the first two and may be treated independently at all five libration points. The motion in the xy plane therefore does not influence the motion in the z direction according to the linearized equations of motion (17). Note that the higher-order mixed derivatives of the function Ω are not zero, consequently there are nonlinear coupling effects between the first two equations and the third of Eqs. (17).

In the linear case the motion along the z axis is governed, therefore, by

$$\ddot{\zeta} = \Omega^0_{zz}\zeta. \tag{21}$$

Since at the triangular libration points $\Omega^0_{zz} = \Omega_{zz}(L_{4,5}) = -1$, the angular velocity of the oscillation along the z axis is the same as the mean motion of the primaries. The period of small oscillation perpendicularly to the plane of the ecliptic of a Trojan asteriod is the same as the orbital period of Jupiter if the assumptions of the restricted problem are accepted.

The period of the motion normal to the xy plane at the collinear points is $2\pi/[-\Omega_{zz}(L_i)]^{1/2}$. Since the motions along the z axis in the linearized case are always pure harmonic oscillations, they belong to the "critical" category and no predictions regarding nonlinear stability can be made.

Figures 10.7, 10.8, and 10.9 show the angular velocities in the xy plane (s_1, s_2, s_3) and in the direction normal to it (s_z) at the three collinear libration points for the linearized system. It is apparent that the values of s_i as obtained from Appendix I of Chapter 5, and values of s_z as computed from $s_z = (1 - \Omega_{yy})^{1/2}$, are always close. A near one-to-one commensurability ratio exists, which may be significant, especially at small and practically important values of the mass parameter.

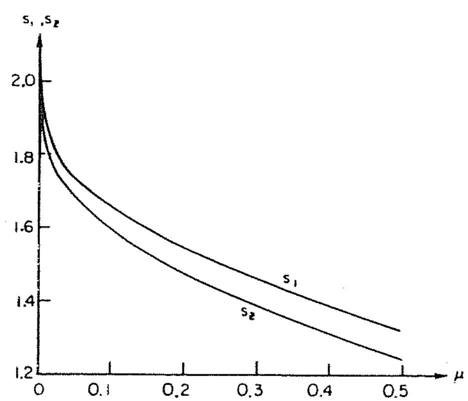


FIG. 10.7. Angular velocity in the x, y plane (s_1) and in the z-direction (s_z) at L_1.

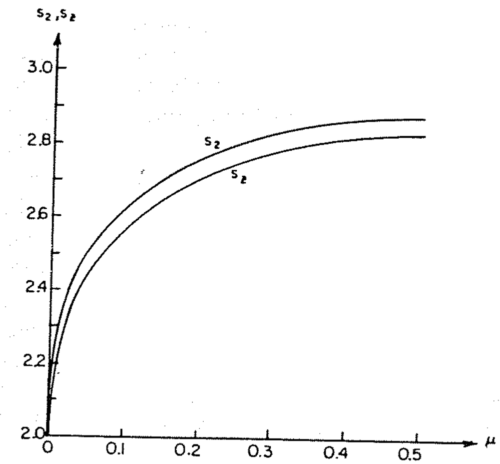


FIG. 10.8. Angular velocity in the x, y plane (s_2) and in the z-direction (s_z) at L_2.

This near-commensurability is even more pronounced at the triangular libration points, where, using again the linearized theory, the mean motion along the z direction is unity, $s_z = 1$, and the angular velocity of the short-period motion for small values of μ may be approximated by $1 - (27/8)\mu$. The nondimensional period of the resulting long-period perturbation is of the order of μ^{-1}, quite similarly to the near-resonance condition between the mean motion of the primaries and the short-period motion around $L_{4,5}$ in the x, y plane.

Returning now to Eqs. (17) of this section, or combining Eqs. (19) of Section 5.3 with Eq. (21) of the present section, we may write the

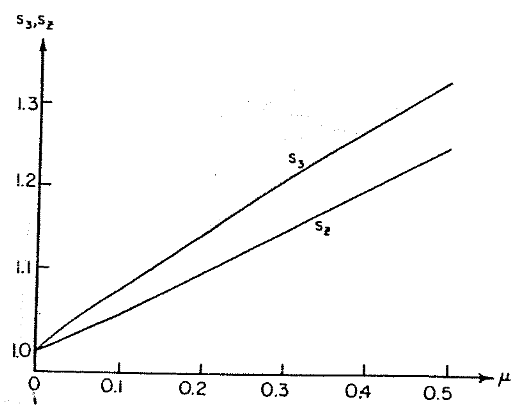


FIG. 10.9. Angular velocity in the x, y plane (s_3) and in the z-direction (s_z) at L_3.

solution of the three-dimensional linearized equations of motion around the collinear libration points as

$$\xi = \zeta_0 \cos s_i t + \frac{\eta_0}{\beta_3} \sin s_i t,$$
$$\eta = \eta_0 \cos s_i t - \beta_3 \xi_0 \sin s_i t, \qquad (22)$$
$$\zeta = C_1 \cos s_z t + C_2 \sin s_z t,$$

where s_i, with $i = 1, 2, 3$, is the angular velocity around any of the three collinear libration points and s_z is the angular velocity in the z direction.

The periodic orbits of finite size which may be generated from this solution may be classified after Moulton in three families, according to the particular solution which is continued:

(a) $\xi = \eta = 0,$
$\zeta = C_1 \cos s_z t + C_2 \sin s_z t;$

(b) $\xi = \xi_0 \cos s_i t + \dfrac{\eta_0}{\beta_3} \sin s_i t,$
$\eta = \eta_0 \cos s_i t - \beta_3 \xi_0 \sin s_i t,$
$\zeta = 0;$

(c) $\xi = \xi_0 \cos s_i t + \dfrac{\eta_0}{\beta_3} \sin s_i t,$
$\eta = \eta_0 \cos s_i t - \beta_0 \xi_0 \sin s_i t,$
$\zeta = C_1 \cos \dfrac{q}{p} s_i t + C_2 \sin \dfrac{q}{p} s_i t,$

where we denote by q/p a rational number so that

$$q/p = s_z/s_i < 1.$$

In other words, members of Class (c) are periodic if the two mean motions s_z and s_i are commensurable.

Class (a) is of special interest since, due to its simplicity, graphical presentation of its general shape is feasible without difficulty. Note first that the general solution given for ζ may be specialized to the form

$$\zeta = \frac{\epsilon}{s_z} \sin s_z t.$$

In this way the generating solution of the linearized system in Case (a) possesses the initial conditions

$$\xi_0 = \eta_0 = \zeta_0 = 0,$$
$$\dot{\xi}_0 = \dot{\eta}_0 = 0, \qquad \dot{\zeta}_0 = \epsilon.$$

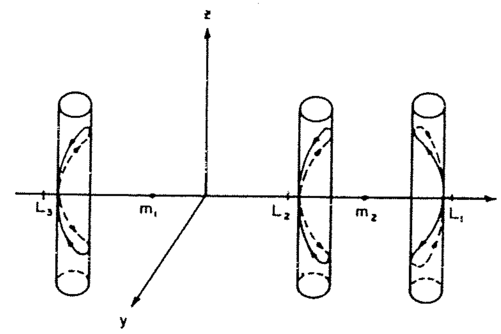


Fig. 10.10. Three-dimensional orbits in the vicinity of the collinear libration points (Moulton, 1920, Ref. 14).

The solution of the nonlinear system becomes

$$\xi = \epsilon^2(A + B \cos 2s_z t) + \mathcal{O}(\epsilon^3),$$
$$\eta = \epsilon^2 D \sin 2s_z t + \mathcal{O}(\epsilon^3), \qquad (23)$$
$$\zeta = \frac{\epsilon}{s_z} \sin s_z t + \mathcal{O}(\epsilon^3),$$

where the constants A, B, and D depend on the characteristics of the collinear libration point in question, i.e., ultimately on the value of μ.

Figure 10.10 shows the general arrangement. The ξ, η projections are approximately ellipses, the ξ, ζ projections are parabolas, and the η, ζ projections are figure-eight curves. Results of actual numerical integrations are shown and discussed in Section 10.2.4.

10.2.4 Three-dimensional orbits

Several special classes of three-dimensional periodic orbits and one group of three-dimensional nonperiodic orbits are the subject of this section.

(A) Poincaré's classification mentioned in Chapter 8 defines periodic orbits of the third kind as those which are generated from two-body orbits of $e \neq 0$ and $i \neq 0$. Consequently, the generating elliptic orbit is in a plane different from the xy plane of the motion of the primaries. A subclass of such orbits is generated from *circular* orbits, with $e = 0$, $i \neq 0$. In order to establish a family of periodic orbits consider the following initial conditions:

$$x \neq 0, \quad y = z = 0; \quad \dot{x} = 0, \quad \dot{y} \neq 0, \quad \dot{z} \neq 0, \quad \text{at} \quad t = 0.$$

Let us also require that, for the generating two-body orbit at $t = T/4$,

$$x \neq 0, \quad y = 0, \quad z \neq 0, \quad \dot{x} = 0, \quad \dot{y} \neq 0, \quad \dot{z} = 0.$$

This orbit has the period of T and it is symmetric to both the xy and xz planes. Observe that if $x(t)$, $y(t)$, $z(t)$ is a solution of the equations of motion, then so is $x(-t)$, $-y(-t)$, $\pm z(-t)$. It may be shown by analytic continuation that the doubly symmetric circular orbits defined by the previously given initial conditions may be used to generate a class of periodic orbits for sufficiently small values of the mass parameter. The proof of existence of such orbits may be given using rectangular coordinates (Goudas) or by using a combination of Delaunay's and Poincaré's variables (Jefferys) (cf. Sections 7.7.5 and 10.2.1).

Following the latter method, we have for the Hamiltonian

$$F = \tfrac{1}{2}(q_1 + q_3)^{-2} + q_1 - \tfrac{1}{2}(q_2^2 + p_2^2) + \mu F_1,$$

where the variables are defined by

$$\begin{aligned} q_1 &= L - G + H, & p_1 &= l - l', \\ q_2 &= [2(L - G)]^{1/2}\cos(l' - \varpi), & p_2 &= [2(L - G)]^{1/2}\sin(l' - \varpi), \\ q_3 &= G - H, & p_3 &= l - \Omega. \end{aligned}$$

Here l' is the longitude of Jupiter and the other symbols are explained in Section 10.2.1, Part (A).

The functional determinant introduced in Section 8.4 is

$$D = \frac{\partial(P_1, P_2, P_3)}{\partial(t, Q_1, Q_2)},$$

where Q_1, Q_2 are the initial conditions for q_1, q_2 and P_1, P_2, P_3 are the solutions of the two-body problem. Evaluation of D at $t = T/4$ gives

$$D = \frac{-3\sin T/4}{(Q_1 + Q_3)^4} \neq 0,$$

where Q_3 is the initial value of q_3.

A large number of periodic orbits of such families computed by Goudas are available. An example is shown in Fig. 10.11 for $\mu = 0.1$ with the initial conditions

$$\begin{aligned} x(0) &= 0.34508360, & y(0) &= z(0) = \dot{x}(0) = 0, \\ \dot{y}(0) &= 0.25664780, & \dot{z}(0) &= 1.2911014. \end{aligned}$$

The numbers along the curves represent the nondimensional time elapsed.

Another family of periodic orbits, shown to exist by Aksenov, has the property that the orbits are outside both primaries. These orbits may be looked at as the three-dimensional generalizations of the "external" orbits mentioned in Chapters 8 and 9. The secular motion of the node of these orbits is the characteristic feature of these solutions.

(B) An exact solution of the restricted problem—the only such solution outside the equilibrium configurations—is the one-dimensional motion along the z-axis. If we restrict the motion of the body of infinitesimal mass to the z axis, the solution may be written:

$$\begin{aligned} x(t) &\equiv 0, \\ y(t) &\equiv 0, \qquad\qquad (24) \\ z &= z(t). \end{aligned}$$

The equations of motion (1) become

$$\begin{aligned} \Omega_x(0, 0, z) &= 0, \\ \Omega_y(0, 0, z) &= 0, \qquad\qquad (25) \\ \Omega_z(0, 0, z) &= \ddot{z}. \end{aligned}$$

From the first of Eqs. (25) we have

$$r_1(0, 0, z) = r_2(0, 0, z)$$

or

$$\mu = \tfrac{1}{2}.$$

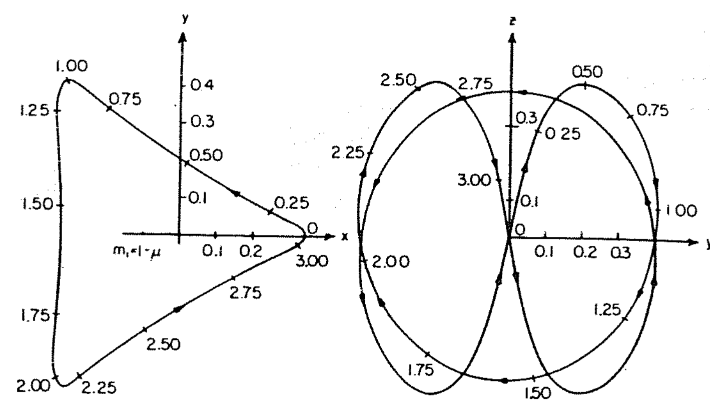


FIG. 10.11. Doubly symmetric three-dimensional periodic orbit around m_1 for $\mu = 0.1$ (Goudas, 1963, Ref. 19).

The problem, therefore, consists of two bodies of equal masses, $\frac{1}{2}$, located symmetrically at points $P_1(\frac{1}{2}, 0, 0)$ and $P_2(-\frac{1}{2}, 0, 0)$. The third body is at point $P(0, 0, z)$ and it moves on the z axis.

The third of Eqs. (25) gives

$$\ddot{z} = -\frac{z}{(z^2 + \frac{1}{4})^{3/2}}, \tag{26}$$

since

$$\Omega_z = -\frac{z}{2}\left(\frac{1}{r_1^3} + \frac{1}{r_2^3}\right)$$

and

$$r_1^2 = r_2^2 = \tfrac{1}{4} + z^2.$$

The integral of energy of this dynamical system with one degree of freedom is

$$\dot{z}^2 = 2\,\Omega(0, 0, z) - C, \tag{27}$$

where

$$\Omega(0, 0, z) = \frac{1}{8} + \frac{1}{(z^2 + \frac{1}{4})^{1/2}}. \tag{28}$$

The first-order differential equation of the problem may be obtained by the substitution of Eq. (28) into (27):

$$\dot{z}^2 = \frac{2}{(z^2 + \frac{1}{4})^{1/2}} - \bar{C}, \tag{29}$$

where $\bar{C}$ is defined as before,

$$\bar{C} = C - \mu(1 - \mu) = C - \tfrac{1}{4}.$$

The distance between either of the primaries and the third body is

$$r = (z^2 + \tfrac{1}{4})^{1/2},$$

which expression when introduced into Eq. (29) allows it to be written without the square root:

$$\dot{r}^2 = (2/r - \bar{C})(1 - 1/4r^2). \tag{30}$$

The fractional form may be eliminated by using

$$\bar{u} = 1/r$$

instead of r as the dependent variable. In this way we obtain

$$(\dot{\bar{u}})^2 = \bar{u}^4(2\bar{u} - \bar{C})(1 - \bar{u}^2/4). \tag{31}$$

To simplify, we change the scale and introduce $u = \frac{1}{2}\bar{u}$ and $c = \bar{C}/4$. The quadrature of Eq. (31) with this notation becomes

$$\int_1^u \frac{du}{4u^2[(u - c)(1 - u^2)]^{1/2}} = \int_0^t dt. \tag{32}$$

The lower limits shown on the integrals correspond to the initial conditions, according to which, at $t = 0$, $u = 1$ or $\bar{u} = 2$ or $r = 1/2$, and $\dot{z} = [4 - \bar{C}]^{1/2}$. The third particle begins its motion at the origin of the coordinate system. If $\bar{C} > 4$, or $c > 1$, then $C > 4.25 = C_2$ and motion is not possible at the origin.

The quadrature given by Eq. (32) may be reduced to Legendre's form by putting

$$v^2 = \frac{1 - u}{1 - c}$$

and

$$k^2 = \frac{1 - c}{2}.$$

In this way we obtain

$$\int_0^v \frac{dv}{(1 - 2k^2v^2)^2[(1 - v^2)(1 - k^2v^2)]^{1/2}} = 2^{3/2}\,t. \tag{33}$$

The period becomes

$$T = \frac{\pi}{2^{1/2}}\left(1 + \frac{9}{4}k^2 + \frac{345}{64}k^4 + \cdots\right).$$

The general solution is obtained, therefore, by inverting the quadrature given in Eq. (33) as an elliptic integral of the third kind. This problem has been studied in considerable detail and it was also generalized for the cases when (1) the primaries move on elliptic orbits and (2) the mass of the third body is not infinitesimal.

The linearized problem may be reduced from Eq. (33) or directly from Eq. (26) when motion around the origin along the z axis is studied. Let $z = \zeta$ be the variation, so that ζ^2 is neglected. Then we have from Eq. (26)

$$\ddot{\zeta} + 8\zeta = 0, \tag{34}$$

corresponding to a harmonic oscillator with a mean motion of $2^{3/2}$ or with a period of $T = \pi/2^{1/2}$.

(C) A systematic study of three-dimensional nonperiodic orbits connecting the neighborhood of the earth with that of the moon, using the model of the restricted problem, reveals several fascinating aspects. In what follows a short outline of the method introduced by Hoelker will be given.

Consider trajectories which satisfy the following requirements:

(1) The initial conditions are $r_1 = R_e + h$ and $\dot{r}_1 = 0$, where r_1 is the radial distance between the earth's center and the probe, R_e is the radius of the earth, and $\dot{r}_1$ is the radial velocity component. These initial conditions mean that the trajectory starts at its perigee at an altitude h above the earth.

(2) The end conditions are $r_2 = R_m + h$ and $\dot{r}_2 = 0$, where r_2 is the radial distance between the moon's center and the probe and R_m is the moon's radius. In other words the trajectory has its closest point to the moon at altitude h.

(3) The time between these two points is given and fixed at the value ΔT.

The problem is now to find the family of trajectories for which h and ΔT are given. The essential feature of the solution of this problem is that such initial conditions and requirements allow the establishment of families with important topologically invariant behavior.

To help visualization, consider $h = 185$ km and $\Delta T = 72$ hr. The search for such trajectories begins with selecting initial conditions for which $z = 0$ and $\dot{z} = 0$ at the perigee. These are two-dimensional orbits in the xy plane and are completely determined when the longitude of the starting point and the magnitude of the initial velocity are given as shown in Fig. 10.12. There are four two-dimensional trajectories which satisfy the previously given requirements, two leaving the earth in a direct sense and two in a retrograde sense. One of each of these two arrives at the moon in a direct sense, the other in a retrograde sense. These four orbits, all two-dimensional, specify two regions on the circle of radius r_1 in the xy plane. The longitudes of the two departure points giving orbits with direct departure are, for instance, both approximately 48°.

The next step is to search in the neighborhood of this region, still

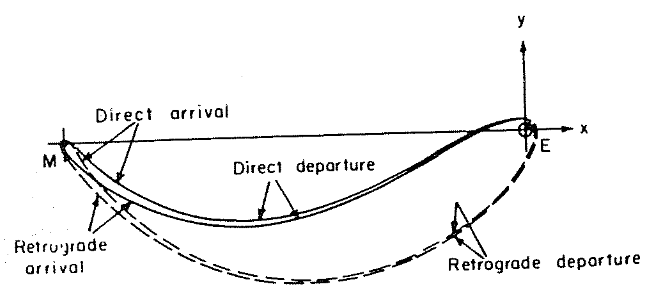


Fig. 10.12. Two-dimensional isochron earth-moon trajectories (Hoelker, 1963, Ref. 26).

in the plane x, y, for nonzero azimuths and corresponding magnitudes of the velocity vector, so that the trajectories satisfy the previously listed requirements. These are now three-dimensional orbits having their perigees in the xy plane and initial velocity vectors pointing out of the plane. The purpose of finding such a family of trajectories with direct departure is to envelop the moon in three dimensions. A similar family may be established starting from the region of the xy plane on the circle with radius r_1 and with the approximate longitude of the two-dimensional trajectories with retrograde departure. It is emphasized that the location of the point of departure up to now is varied in a one-dimensional fashion. All departure points are located on the circle with radius r_1 around the earth in the xy plane. The initial velocity vectors are varied regarding their magnitude but their direction is tangential to the sphere of radius r_1 around the earth. By proper combination of the longitude of the departure point and the magnitude and the azimuth of the initial velocity, trajectories are found which have closest approach to the moon at the distance of 185 km above the moon's surface and which have the common trip time of $\Delta T = 72$ hr. The trajectories which are described in this paragraph may be grouped into two families, one family around the departure point of the two-dimensional orbit with direct departure, the other family around the departure point of the two-dimensional orbit with retrograde departure.

These two families have several essential characteristics (the previously mentioned topologically invariant behavior), the discovery of which is the important contribution made by Hoelker:

(1) the loci of the closest points of the trajectories to the moon form approximately a circle;

(2) if the trajectories are extended beyond their closest points to the moon, they all intersect a line, at about the same point (this line connects the center of the moon with the center of the circle mentioned in 1);

(3) the departure points of the members of the family form a small segment (a few degrees) of the circle with radius r_1;

(4) the initial velocity vectors deviate from the xy plane by not more than $\pm 5°$;

(5) each family envelops the moon densely in a one-parameter fashion.

The trajectories satisfying the original requirements have these properties *independently* of the inclination of the plane of the circle on which they are generated. In other words, the previously described vo families have their departure points in the xy plane on the circle f radius r_1. The totality of departure points, of course, forms a sphere of radius r_1 around the earth. Consider now a point in the xy plane n the circle of radius r_1, which point is halfway between the regions om which the previously described two families originate. The line connecting this last-mentioned point with the center of the earth may be used as a nodal line through which pass planes of varying inclination, itersecting the sphere of radius r_1 in great circles. The inclination of ne original great circle giving the previously described two families is 0°. The fundamentally important result is now that the previously iven five properties are also valid for any other inclination.

All the results mentioned in Part (C) were obtained by numerical investigation and analytical verifications do not exist yet. Note, however, that while mentioning only trajectories from the earth to the moon, ve have established also orbits from the moon to the earth by applying he symmetry properties discussed in Chapter 8.

0.2.5 Regularization of the three-dimensional restricted problem

We have shown that the eccentric anomaly, when used as the in-lependent variable in place of the time, regularizes the problem of two odies. Consequently, such a local regularization, when the third body is in the vicinity of one of the primaries, is always possible, even when the motion of the third body is not in the plane of the motion of the primaries. On the other hand, it is also known that transformation of the ndependent variable, while necessary and sufficient for accomplishing local regularization from an analytic point of view, must be accompanied by a coordinate transformation in order to achieve increased numerical accuracy around the singularity. The question of the generalization of two-dimensional local and global regularizations to three-dimensional motion is, therefore, quite important and practical.

The generalization of the conformal mapping

$$z = w^2 \tag{35}$$

representing essentially Levi-Civita's transformation (as discussed in Chapter 3) to three-dimensional problems has been performed, via an excursion to a four-dimensional space, by Stiefel and Kustaanheimo. The basic idea is as follows:

Consider the transformation (35) in a matrix-form,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{36}$$

together with the corresponding transformation of the differentials,

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = 2 \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}. \tag{37}$$

For ease of generalization to higher-dimensional spaces let

$$x = x_1, \quad y = x_2, \quad u = u_1, \quad v = u_2,$$

and

$$A_2 = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix}. \tag{38}$$

An analogous transformation in the n-dimensional space may be obtained if a matrix A_n is found satisfying the following requirements:

(1) every element is a homogeneous linear function of u_i;
(2) the scalar product of any two rows is zero;
(3) every row has the norm $u_1^2 + u_2^2 + \cdots + u_n^2$.

These requirements are, of course, satisfied by A_2 as given by Eq. (38) but, according to Hurwitz, can be only satisfied when $n = 1, 2, 4$, or 8. Consequently, the generalizations of Levi-Civita's transformation in the sense described here and proposed by Kustaanheimo to four-dimensional space is feasible. On the other hand, generalization to three-dimensional space is not possible.

The four-dimensional form of the matrix of the transformation is

$$A_4 = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix}, \tag{39}$$

from which it follows that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} = A_4 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \tag{40}$$

and that

$$\begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \\ 0 \end{pmatrix} = 2A_4 \begin{pmatrix} du_1 \\ du_2 \\ du_3 \\ du_4 \end{pmatrix}. \tag{41}$$

Note that the product of the last line of A_4 and the column-vector u_i is identically zero according to Eq. (40) while the corresponding product from Eq. (41) is

$$u_4\, du_1 - u_3\, du_2 + u_2\, du_3 - u_1\, du_4 = 0. \tag{42}$$

This equation establishes a relation which the four differentials du_i must satisfy. The first three equations of (41) are total derivatives, quadratures of which furnish the first three equations of (40).

The equations of the transformation are now (39) and (40), which map the four-dimensional space u_1, u_2, u_3, u_4 into the three-dimensional space x_1, x_2, x_3. The inverse of this transformation may be established considering also that the four differentials du_i must satisfy Eq. (42).

The independent variable is introduced in the usual way by

$$d\tau = dt/r,$$

where

$$r = u_1^2 + u_2^2 + u_3^2 + u_4^2 = (x_1^2 + x_2^2 + x_3^2)^{1/2}.$$

The integration of the equations of motion is performed in the four-dimensional u_i space. These equations, for instance, for the perturbed problem of two bodies in a sidereal coordinate system become

$$d^2u_i/d\tau^2 + Bu_i = Q_i, \tag{43}$$

where $i = 1, 2, 3, 4$, $B = \text{const}$, and Q_i is the ith component of the disturbing function in the system u_i. These components may be obtained from

$$\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix} = 2\tilde{A}_4 \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}, \tag{44}$$

where the P_i's are the components of the disturbing function in the system x_i and $\tilde{A}_4$ is related to the transpose of the cofactor of A_4 belonging to its last element (located in the 4th row and 4th column):

$$\tilde{A}_4 = \begin{pmatrix} u_1 & u_2 & u_3 \\ -u_2 & u_1 & u_4 \\ -u_3 & -u_4 & u_1 \\ u_4 & -u_3 & u_2 \end{pmatrix}. \tag{45}$$

The corresponding three-dimensional global regularization is also available as the generalization of Birkhoff's transformation. It may be shown that the relations between the three x_i's and the four u_i's may be written as

$$x_i = a_i + \frac{c_i(u_j)}{u_1^2 + u_2^2 + u_3^2},$$

with $i = 1, 2, 3$ and $j = 1, 2, 3, 4$. The constants a_i and the third-degree polynomials c_i have been obtained by Stiefel and Waldvogel.

Three-dimensional regularization is also accomplished by the inversion of the velocity vector in conjunction with the usual transformation of the time. This approach, while regularizing singularities caused by collisions, has the disadvantage of introducing new singularities at all points of the orbit where the velocity is zero.

10.2.6 Applications

The two areas of application to be treated here are close binary systems [Part (A)] and three-dimensional nonperiodic lunar and interplanetary trajectories [Part (B)].

(A) Some of the surfaces of zero velocity are called Roche equipotentials in the field of study of binary stars. The dumbbell shape corresponding to $C = C_2$ is called Roche's limit and it represents the largest (still closed) equipotentials containing the mass of the respective components. The term "contact component" refers to a star which fills its own part of the dumbbell and the system is called a contact system when both stars are contact components. The case when $C < C_2$ is of no interest in research pertaining to binary stars since no separate components exist in this case. On the other hand, when $C > C_2$, the two ovoidal surfaces of zero velocity approximate the shapes of centrally condensed binary components. A classification of binary stars based on their shape compared to the surfaces of zero velocity is highly instructive and allows one to distinguish between detached, semidetached, and contact systems.

The identity of some of the surfaces of zero velocity with the Roche equipotentials is based on the fact that particles on these equipotentials must have zero velocities, and that the Roche equipotentials represent equilibrium surfaces which do not expand or shrink.

(B) Surfaces of zero velocity may also be used in the study of three-dimensional trajectories connecting the neighborhood of the earth with that of the moon or of the planets. These trajectories, often called lunar and interplanetary orbits, may be established by numerical integration of the three-dimensional restricted problem; however, the effects of the eccentricities of the actual orbits of the primaries and the perturbing effects of other planets (not included in the computations) might be of such magnitude that they render the assumptions of the restricted problem illusory. So-called precision orbits between planets cannot be obtained by using the model of the three-dimensional restricted problem and quick engineering estimates may often be available from simple two-body approximations. The only application of the restricted problem in this area seems to be the establishment of families of trajectories and the discovery of certain special characteristics of such orbits. This was already discussed in Section 10.2.4, Part (D), regarding trajectories from the earth to the moon, where the inclusion of the third dimension was essential in order to envelop the moon.

Interplanetary trajectories generally connect the vicinity of two planets. The orbit between the planets is mostly influenced by the sun and at the end regions by the target planets. On that part of the orbit where the influence of the target planet is negligible, other planets such as Jupiter might perturb the orbit. Such perturbation may be computed from a model of the restricted problem formed by the sun, by Jupiter, and the space probe. Another restricted problem might be set up at the beginning of the trajectory when the primaries are the earth and the sun. Finally the arrival portion of the trajectory might be computed, once again by using the restricted problem as a model with the sun and the target planet represented by the primaries. The matching of the solutions of these restricted problems would present serious difficulties even if analytical representations of the solutions of the three problems would be available, since even the matching of two-body orbits, a much simpler problem in principle, requires careful analysis, as shown by Lagerstrom and Kevorkian. Furthermore it is expected that regularization is required in the case of close approaches (at the beginning and at the end of every interplanetary orbit, and possibly at other points along the orbit, as will be discussed later) in order to achieve the necessary accuracy. The general problem of global regularization when the restricted problem is *not* a satisfactory model has not been solved yet, but local regularization methods are, of course, always available.

Computations of interplanetary orbits by numerically integrating n-body equations seems to be a rather brute-force approach leading possibly to unknown inaccuracies and few, if any, general conclusions. Nevertheless, to obtain precision trajectories, the numerical integration of the n-body equations with carefully selected ephemerides for the planets and with local regularizations seems to be the only approach available.

The highly special orbits termed "swing-by" trajectories display close approaches in addition to their terminal points. Such trajectories might, for instance, originate at the earth, "swing by" Venus, and arrive at Mars. Regularization at three parts of these orbits might be essential in order to compute efficiently and accurately these trajectories. For approximate and preliminary work, of course, arcs computed from the application of two-body dynamics are quite satisfactory, especially when Tisserand's criterion, discussed in Section 1.10, Part (C), is diligently applied to check the effect of the planet causing the "swing-by."

Note that Tisserand's criterion is not restricted to two dimensions and its general form as obtained by modifying Eq. (86) is

$$1/a + 2[a(1 - e^2)]^{1/2} \cos i = \text{const.} \tag{46}$$

This equation allows the evaluation of the orbital parameters of the probe, the semimajor axis a, the eccentricity e, and the angle of inclination i, after the "swing-by." The angle of inclination is the angle between the plane of the probe's orbit and the plane of the orbit of the planet which is responsible for the "swing-by." Since the orbital planes of the planets are close to the plane of the ecliptic, i in Eq. (46) is approximately the inclination of the probe's orbit to the ecliptic. Let, for instance, a_1 and e_1 be the orbital parameters of a probe in the ecliptic. Then the semimajor axis, after a "swing-by" which puts the probe in a plane perpendicular to the ecliptic, becomes

$$a_2 = \frac{a_1}{1 + 2[a_1^3(1 - e_1^2)]^{1/2}} < a_1 . \tag{47}$$

10.3 The elliptic restricted problem

10.3.1 Introduction

The motion of the primaries must satisfy the differential equations governing the dynamics of two gravitational bodies. Consequently the primaries might describe elliptic, parabolic, or hyperbolic orbits. The special case of circular motion of the primaries is the main subject of this work. Its simplest generalization, when the primaries describe elliptic orbits, is treated in this section.

This generalization is not trivial since its consequences are further reaching than the generalization to three-dimensional motion (discussed in Section 10.2). While the three-dimensional restricted problem still possesses the Jacobian integral, the elliptic problem does not. This property of the elliptic problem distinguishes it from other generalizations and indicates the increased degree of difficulty involved in handling it.

The terminology is not uniform. The case when the primaries move on circles is often referred to as the "circular restricted problem." The convention, as the reader will recall, suggests simply "restricted problem." The case when the primaries move on ellipses is sometimes referred to as the elliptic, or the elliptical, or the elliptically restricted problem, or the semirestricted problem, the pseudorestricted problem, the astronomical or the asteroidal restricted problem, etc. The term "elliptic restricted problem" will be accepted for our use.

The various generalizations might be combined and in this section it will be shown how the three-dimensional elliptic restricted problem might be formulated and how the two-dimensional elliptic restricted problem may be regularized.

10.3.2 Equations of motion

The differential equations of motion of the restricted problem are presented using a uniformly rotating Cartesian rectangular coordinate system. In this system the primaries are fixed and the Hamiltonian does not depend explicitly on the time. When the primaries move on elliptic orbits the introduction of a nonuniformly rotating and pulsating coordinate system results again in fixed locations for the primaries. The Hamiltonian, however, does depend explicitly on the independent variable in this case. Such a pulsating or oscillating coordinate system might be introduced by using the variable distance between the primaries as the basic length of the system by which distances are divided. This idea immediately follows realizing that the Lagrangian solution of the (general) problem of three bodies includes the case when the three bodies move on (say) elliptic orbits and the ratios of the mutual distances of the bodies are constant. In what follows dimensionless variables are introduced by using the distance between the primaries,

$$r = \frac{a(1 - e^2)}{1 + e \cos f}, \tag{48}$$

as the quantity with which distances and dimensional coordinates are divided. Here a and e are the semimajor axis and the eccentricity of the elliptic orbit of either primary around the other and f is the true anomaly of m_1.

A coordinate system which rotates with the variable angular velocity $\dot{f}$ is also introduced. This angular motion is given by

$$\frac{df}{dt^*} = \frac{k(m_1 + m_2)^{1/2}}{a^{3/2}(1 - e^2)^{3/2}} (1 + e \cos f)^2, \tag{49}$$

where t^* is the dimensional time. This equation follows from the principle of the conservation of the angular momentum. In the problem of two bodies formed by the primaries of masses m_1 and m_2 this principle is expressed by

$$\dot{f} r^2 = [a(1 - e^2)\, k^2(m_1 + m_2)]^{1/2}.$$

In the following two subsections (10.3.2.1 and 10.3.2.2) we perform the transformations of the dependent and independent variables and

show that the equations of motion of the elliptic restricted problem using the true anomaly as independent variable may be written as

$$\frac{d^2\zeta}{df^2} + 2i \frac{d\zeta}{df} = \operatorname{grad}_\zeta \omega$$

or as

$$\begin{aligned} \frac{d^2\bar{\xi}}{df^2} - 2\frac{d\bar{\eta}}{df} &= \frac{\partial \omega}{\partial \bar{\xi}}, \\ \frac{d^2\bar{\eta}}{df^2} + 2\frac{d\bar{\xi}}{df} &= \frac{\partial \omega}{\partial \bar{\eta}}, \end{aligned} \tag{50}$$

where $\bar{\xi}$, $\bar{\eta}$ are the pulsating dimensionless coordinates of the third body in the non-uniformly rotating coordinate system and

$$\omega = \frac{\Omega}{1 + e \cos f}. \tag{51}$$

The equations of motion (50), therefore, show complete formal identity with the equations describing the (circular) restricted problem. The definition (51) of the function ω, however, shows that this identity is restricted only to the form but not to the contents of the equations, since the function ω depends on the coordinates as well as on the independent variable f, while Ω depends only on the coordinates.

10.3.2.1 *Transformation to a nonuniformly rotating coordinate system.* Consider first the equations of motion in a fixed coordinate system, using dimensional quantities and variables as given by Eqs. (7) of Section 1.2 or Section 1.6:

$$\frac{d^2X}{dt^{*2}} = -m_1 k^2 \frac{X - X_1}{R_1^3} - m_2 k^2 \frac{X - X_2}{R_2^3}, \tag{52}$$

and similarly for Y. Here t^* is the dimensional time,

$$R_i^2 = (X - X_i)^2 + (Y - Y_i)^2, \tag{53}$$

$i = 1, 2$ and X, Y are the dimensional coordinates of the third body in a fixed coordinate system.

The rotating coordinate system $\bar{x}$, $\bar{y}$ may now be introduced similarly to Eq. (26) of Section 1.4 by writing

$$Z = z\, e^{if} \tag{54}$$

where

$$Z = X + iY,$$

and

$$z = \tilde{x} + i\tilde{y}.$$

The equations of motion in the fixed system using the complex vector Z become

$$\frac{d^2Z}{dt^{*2}} = -m_1k^2\frac{Z - Z_1}{R_1^{\,3}} - m_2k^2\frac{Z - Z_2}{R_2^{\,3}}. \tag{55}$$

Now, Eq. (55) is transformed by using Eq. (54). Similarly but not identically to Eq. (30) of Section 1.4 we find for the second derivative the formula

$$\frac{d^2Z}{dt^{*2}} = \left[\frac{d^2z}{dt^{*2}} + 2i\frac{df}{dt^*}\frac{dz}{dt^*} - z\left(\frac{df}{dt^*}\right)^2 + iz\frac{d^2f}{dt^{*2}}\right]e^{if}. \tag{56}$$

For circular motion of the primaries we have

$$f = nt^*$$

and Eq. (56) reduces to Eq. (30) of Section 1.4.

The equations of motion in the rotating coordinate system may be written, using the complex vector z, as

$$\frac{d^2z}{dt^{*2}} + 2i\frac{df}{dt^*}\frac{dz}{dt^*} = -k^2m_1\frac{z - z_1}{R_1^{\,3}} - k^2m_2\frac{z - z_2}{R_2^{\,3}} + z\left(\frac{df}{dt^*}\right)^2 - iz\frac{d^2f}{dt^{*2}}. \tag{57}$$

The second term on the left is the Coriolis acceleration, the first and second terms on the right-hand side are the gravitational effects, and the third and fourth terms represent the centrifugal effect and the acceleration normal to the radius vector due to the nonuniform rotation of the system.

The complex position vectors z_1 and z_2 locate the primaries. Since these are permanently on the real axis of the $\tilde{x}, \tilde{y}$ system, we have

$$z_1 = \tilde{x}_1 = \frac{p_1}{1 + e\cos f}, \qquad z_2 = \tilde{x}_2 = \frac{-p_2}{1 + e\cos f}, \tag{58}$$

where p_1 and p_2 are positive and

$$p_1/p_2 = a_1/a_2 = m_2/m_1. \tag{59}$$

Here a_1 and a_2 are the semimajor axes of the elliptic orbits of m_1 and m_2 described around their center of mass.

The distances between the primaries and the third body are

$$R_1 = |Z - Z_1| = |z - \tilde{x}_1| = [(\tilde{x} - \tilde{x}_1)^2 + \tilde{y}^2]^{1/2}$$

and

$$R_2 = |Z - Z_2| = |z - \tilde{x}_2| = [(\tilde{x} - \tilde{x}_2)^2 + \tilde{y}^2]^{1/2}.$$

Equation (57) reduces to Eq. (31) of Section 1.4 when the substitutions

$$f = nt^*, \qquad e = 0, \qquad p_1 = a_1 = b, \qquad p_2 = a_2 = a$$

are made, corresponding to circular motion of the primaries.

10.3.2.2 *Transformation to dimensionless pulsating coordinates.* The second transformation introduces dimensionless pulsating coordinates into Eq. (57) as follows. Let

$$\zeta = z/r = \tilde{\xi} + i\tilde{\eta}.$$

Then from Eq. (48) we have

$$\tilde{\xi} = \frac{\tilde{x}(1 + e\cos f)}{a(1 - e^2)}, \qquad \tilde{\eta} = \frac{\tilde{y}(1 + e\cos f)}{a(1 - e^2)}. \tag{60}$$

Prior to performing the transformation we make the following observations:

(a) The primaries are fixed in the coordinate system $\tilde{\xi}, \tilde{\eta}$ since from Eqs. (58) and (60) we have

$$\tilde{\xi}_i = \frac{\tilde{x}_i(1 + e\cos f)}{a(1 - e^2)}$$

or

$$\tilde{\xi}_i = (-1)^{i+1}\frac{p_i}{a(1 - e^2)} = (-1)^{i+1}\frac{a_i}{a}$$

and

$$\tilde{\eta}_i = 0, \qquad \text{with} \quad i = 1, 2.$$

So the fixed locations of the primaries in the $\tilde{\xi}, \tilde{\eta}$ system are $P_1(\mu, 0)$ and $P_2(\mu - 1, 0)$.

(b) Simple considerations regarding the problem of two bodies show that the following four elliptic orbits all have the same eccentricity: the orbit of m_1 relative to m_2, with semimajor axis a; the orbit of m_2

relative to m_1, also with semimajor axis a; the orbit of m_1 with respect to the center of mass, with semimajor axis $a_1 = a\mu$; and, finally, the orbit of m_2 with respect to the center of mass, with semimajor axis $a_2 = a(1-\mu)$.

(c) The true anomaly as the independent variable may be introduced by the equation

$$\frac{d}{dt^*} = \frac{df}{dt^*}\frac{d}{df}.$$

Now we are in the position to transform Eq. (57) from the variables z and t^* to the variables ζ and f. The first term on the left side of Eq. (57) becomes

$$\frac{d}{dt^*}\left(r\frac{d\zeta}{dt^*} + \frac{dr}{dt^*}\zeta\right) = r\frac{d^2\zeta}{dt^{*2}} + 2\frac{dr}{dt^*}\frac{d\zeta}{dt^*} + \zeta\frac{d^2r}{dt^{*2}}$$

or

$$\frac{d^2z}{dt^{*2}} = r\left[\frac{d\zeta}{df}\frac{d^2f}{dt^{*2}} + \frac{d^2\zeta}{df^2}\left(\frac{df}{dt^*}\right)^2\right] + 2\frac{dr}{dt^*}\frac{d\zeta}{df}\frac{df}{dt^*} + \frac{d^2r}{dt^{*2}}\zeta.$$

The second term on the left side of Eq. (57) transforms into

$$2i\frac{df}{dt^*}\frac{dz}{dt^*} = 2i\frac{df}{dt^*}\left(\zeta\frac{dr}{dt^*} + r\frac{d\zeta}{df}\frac{df}{dt^*}\right),$$

and the first term of the right side becomes

$$k^2m_1\frac{z - z_1}{R_1^3} = k^2m_1\frac{\zeta - \zeta_1}{r^2\bar{r}_1^3},$$

where

$$\bar{r}_1^2 = |\zeta - \zeta_1|^2 = (\xi - \mu)^2 + \bar{\eta}^2$$

since

$$\zeta_1 = \xi_1 = \mu.$$

After substitution into Eq. (57) and collecting terms, we have

$$\begin{aligned} r\left(\frac{df}{dt^*}\right)^2\left(\frac{d^2\zeta}{df^2} + 2i\frac{d\zeta}{df}\right) + \zeta\left[\frac{d^2r}{dt^{*2}} - r\left(\frac{df}{dt^*}\right)^2\right] \\ + \left(\frac{d\zeta}{df} + i\zeta\right)\left(r\frac{d^2f}{dt^{*2}} + 2\frac{dr}{dt^*}\frac{df}{dt^*}\right) \\ = -k^2m_1\frac{\zeta - \zeta_1}{r^2\bar{r}_1^3} - k^2m_2\frac{\zeta - \zeta_2}{r^2\bar{r}_2^3}. \end{aligned} \tag{61}$$

At this point we recall that Eq. (48) defines the function

$$r = r(f),$$

which is the solution of the two-body problem involving the two primaries. Consequently, we have the following relations involving r and f and their derivatives:

(a) the integral of angular momentum:

$$\left(r^2\frac{df}{dt^*}\right)^2 = a(1-e^2)\,k^2(m_1 + m_2);$$

(b) its derivative:

$$r\frac{d^2f}{dt^{*2}} + 2\frac{dr}{dt^*}\frac{df}{dt^*} = 0;$$

(c) the equation of motion:

$$\frac{d^2r}{dt^{*2}} - r\left(\frac{df}{dt^*}\right)^2 = -\frac{k^2(m_1+m_2)}{r^2};$$

and

(d) its modification by (a):

$$\frac{d^2r}{dt^{*2}} - r\left(\frac{df}{dt^*}\right)^2 = -\frac{r^2}{a(1-e^2)}\left(\frac{df}{dt^*}\right)^2.$$

Now, we return to Eq. (61) which we divide through by $r(df/dt^*)^2$. Utilizing the relations (a)–(d), we obtain

$$\frac{d^2\zeta}{df^2} + 2i\frac{d\zeta}{df} = \frac{r}{a(1-e^2)}\left[\zeta - \frac{m_1}{m_1+m_2}\frac{\zeta-\zeta_1}{\bar{r}_1^3} - \frac{m_2}{m_1+m_2}\frac{\zeta-\zeta_2}{\bar{r}_2^3}\right]$$

or

$$\frac{d^2\zeta}{df^2} + 2i\frac{d\zeta}{df} = \operatorname{grad}_\zeta \omega(\zeta), \tag{62}$$

where

$$\omega(\zeta) = (1 + e\cos f)^{-1}\,\Omega(\zeta) \tag{63}$$

and

$$\Omega(\zeta) = \tfrac{1}{2}(\xi^2 + \bar{\eta}^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \tfrac{1}{2}\mu(1-\mu).$$

The real and imaginary parts of Eq. (62) are the results of the derivation, giving Eqs. (50).

10.3.2.3 *Motion in the three-dimensional elliptic problem.* The equations of motion of the three-dimensional elliptic problem are

$$\begin{aligned} \bar{\xi}'' - 2\bar{\eta}' &= \omega'_{\xi}, \\ \bar{\eta}'' + 2\bar{\xi}' &= \omega'_{\eta}, \\ \bar{\zeta}'' + \bar{\zeta} &= \omega'_{\zeta}, \end{aligned} \tag{50'}$$

where

$$\omega' = (1 + e\cos f)^{-1}\Omega'$$

and

$$\Omega' = \tfrac{1}{2}(\bar{\xi}^2 + \bar{\eta}^2 + \bar{\zeta}^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \tfrac{1}{2}\mu(1-\mu)$$

or

$$\Omega' = \tfrac{1}{2}[(1-\mu)r_1^2 + \mu r_2^2] + \frac{1-\mu}{r_1} + \frac{\mu}{r_2},$$

with

$$r_1^2 = (\bar{\xi} - \mu)^2 + \bar{\eta}^2 + \bar{\zeta}^2$$

and

$$r_2^2 = (\bar{\xi} - \mu + 1)^2 + \bar{\eta}^2 + \bar{\zeta}^2.$$

The variables $\bar{\xi}$ and $\bar{\eta}$ are defined as before [see Eqs. (60)] and $\bar{\zeta}$ becomes

$$\bar{\zeta} = \bar{z}/r,$$

where $\bar{z}$ is the dimensional coordinate along the axis perpendicular to the plane x, y. This coordinate is the same in the fixed and in the rotating system. Note that the symbol ζ used in Sections 10.3.2 and 10.3.2.2 denotes the complex position vector in the $\bar{\xi}, \bar{\eta}$ plane, while $\bar{\zeta}$ is the dimensionless coordinate along the axis normal to the $\bar{\xi}, \bar{\eta}$ plane. The symbol z used in Sections 10.3.2.1 and 10.3.2.2 denotes the complex position vector in the $\tilde{x}, \tilde{y}$ plane while $\bar{z}$ is the dimensional third coordinate.

While the third coordinate does not take place in the transformation involving the rotation around the $\bar{z}$ axis, it is made dimensionless by the variable distance between the primaries. In other words $\bar{z}$ must be computed using the equation $\bar{z} = r(f)\,\bar{\zeta}$ and consequently $\bar{\zeta}$ appears in the third equation of motion of the left side.

Note that another version of the equations of motion is

$$\begin{aligned} \bar{\xi}'' - 2\bar{\eta}' &= \omega_{\xi}, \\ \bar{\eta}'' + 2\bar{\xi}' &= \omega_{\eta}, \\ \bar{\zeta}'' &= \omega_{\zeta}, \end{aligned} \tag{50''}$$

where

$$\omega = \omega' - \tfrac{1}{2}\bar{\zeta}^2 = (1 + e\cos f)^{-1}\Omega \tag{51'}$$

with

$$\Omega = \Omega' - \tfrac{1}{2}(1 + e\cos f)\,\bar{\zeta}^2. \tag{51''}$$

We see therefore that establishing the formal identity between the equations of motion of the circular problem with those of the elliptic problem requires care. Equations (50′) and (3′) of Section 10.2.1 show formal identity, but considering Eq. (50″) we note that the right-hand sides of (50″) and of Eq. (1) do not differ by a periodic factor alone.

10.3.3 An invariant relation

In Section 1.4, we saw that the equations of motion of the (circular) restricted problem

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x, \\ \ddot{y} + 2\dot{x} &= \Omega_y, \end{aligned}$$

possess the Jacobian integral. In fact when the first of these equations is multiplied by $\dot{x}$, the second by $\dot{y}$, the results added and integrated, we have

$$\dot{x}^2 + \dot{y}^2 = 2\Omega - C. \tag{64}$$

Performing the same operations with Eqs. (50), we have

$$\left(\frac{d\bar{\xi}}{df}\right)^2 + \left(\frac{d\bar{\eta}}{df}\right)^2 = 2\int (\omega_{\xi}\,d\bar{\xi} + \omega_{\eta}\,d\bar{\eta}). \tag{65}$$

The problems connected with the indicated quadrature on the right side of this equation were discussed in considerable detail in Section 1.3. Since the function $\omega = \omega(\bar{\xi}, \bar{\eta}, f)$ depends also on the independent variable, in addition to the position coordinates, the expression under the integral sign is not a total differential. In fact

$$d\omega = \omega_{\xi}\,d\bar{\xi} + \omega_{\eta}\,d\bar{\eta} + \omega_f\,df,$$

where

$$\omega_f = \frac{e\sin f}{(1 + e\cos f)^2}\,\Omega.$$

The integral in Eq. (65) may, therefore, be written as

$$2\int (d\omega - \omega_f\,df) = 2\omega - 2\int \omega_f\,df - C,$$

and Eq. (65) becomes

$$\left(\frac{d\bar{\xi}}{df}\right)^2 + \left(\frac{d\bar{\eta}}{df}\right)^2 = 2\omega - 2e\int_0^f \frac{\Omega\sin f}{(1 + e\cos f)^2}\,df - C. \tag{66}$$

We immediately may observe that when $e = 0$ the term including the integral vanishes, $\bar{\xi}$ and $\bar{\eta}$ become x and y, the true anomaly f becomes the dimensionless time t, and $\omega = \Omega$. In short, Eq. (66) becomes the Jacobian integral, Eq. (64).

Assuming now that $e \neq 0$, consider an orbit for a short time. By this we mean that we select the time to start the motion at, say, $f = 0$ and we are interested only in that part of the trajectory which takes place between $f = 0$ and $f = +\epsilon$, where ϵ is a sufficiently small positive quantity. Since f is the true anomaly, this restriction amounts to considering a sufficiently small time interval, during which the primaries describe sufficiently small arcs. The second term on the right of Eq. (66) in this case contains the product of the eccentricity and of ϵ, consequently it is smaller than the term 2ω. The curves of zero velocity, therefore, may be obtained approximately from the equation

$$2\Omega - C(1 + e\cos f) = 0$$

or

$$2\Omega - C^* = 0.$$

This relation holds as long as the integral is omitted from Eq. (66) and geometrically means that at every time (or at every value of the true anomaly) a different set of curves of zero velocity are to be constructed. At any instant the redrawn (or what is equivalent the renumbered) curves of zero velocity will govern the motion by establishing varying forbidden regions by controlling the velocity, etc. The variation of the shape of the curves of zero velocity is governed by

$$C^* = C(1 + e\cos f)$$

and therefore we might speak about pulsating curves of zero velocity.

Note that the integral occurring in Eq. (66) must be neglected to obtain the pulsating curves of zero velocity. This quasi-steady approach neglects a term in the invariant relation (66) which is of first order in the eccentricity. An expansion of Eq. (66) regarding the eccentricity shows the following:

$$\left(\frac{d\bar{\xi}}{df}\right)^2 + \left(\frac{d\bar{\eta}}{df}\right)^2 = 2\Omega - C + eF_1 + e^2F_2 + \cdots,$$

where

$$F_1 = -2\left[\Omega\cos f + \int (\Omega \sin f)\, df\right]$$

and

$$F_2 = +2\left[\Omega\cos^2 f + 2\int (\Omega \sin 2f)\, df\right].$$

The interpretation of the invariant relation (66) is simply that any solution of the differential equations of motion (50), expressed as

$$\bar{\xi} = \bar{\xi}(f),\qquad \bar{\eta} = \bar{\eta}(f), \tag{67}$$

when substituted into Eq. (66), will satisfy it. The integral occurring in Eq. (66) may be evaluated along an orbit which is given by Eqs. (67) since along an orbit

$$\Omega(\bar{\xi}, \bar{\eta}) = \Omega[\bar{\xi}(f), \bar{\eta}(f)] = \Omega(f)$$

and the integration may be performed. Without knowing the solution, the invariant relation (66) may not be evaluated since the integral occurring in it contains the independent variable explicitly.

Another way to express this is: if only total differentials occur in the invariant relation, then upon integration we obtain an integral of the motion such as in the case of the Jacobian integral of Eq. (64). If, on the other hand, the invariant relation contains terms which are not total differentials, then by integration we obtain an expression, the value of which will depend on the path of integration and not just on the beginning and end points.

10.3.4 Motion around the equilibrium points

The *formal* identity of the equations applicable to the elliptic and to the circular restricted problems allows immediate establishment of the stationary solutions. From Eqs. (50) we have, for

$$\frac{d\bar{\xi}}{df} = \frac{d\bar{\eta}}{df} = \frac{d^2\bar{\xi}}{df^2} = \frac{d^2\bar{\eta}}{df^2} = 0,$$

$$\frac{\partial\omega}{\partial\bar{\xi}} = \frac{\partial\omega}{\partial\bar{\eta}} = 0$$

or

$$\frac{\partial\Omega}{\partial\bar{\xi}} = \frac{\partial\Omega}{\partial\bar{\eta}} = 0.$$

Consequently the coordinates $\bar{\xi}$, $\bar{\eta}$ of the five libration points in the elliptic problem are the same as the coordinates x, y are in the circular problem. The triangular points of the circular problem are located at

$$z_{4,5} = \mu - \tfrac{1}{2} \pm i3^{1/2}/2.$$

The triangular points of the elliptic problem pulsate together with their own coordinate system. This pulsation appears of course only when the system ζ is projected to the system z by the equation

$$z = \frac{a(1-e^2)}{1+e\cos f}\,\zeta.$$

The coordinates of the triangular libration points of the elliptic problem expressed in the system $(\bar\xi, \bar\eta)$ are

$$\bar\xi_{4,5} = \mu - \tfrac{1}{2},$$
$$\bar\eta_{4,5} = \pm 3^{1/2}/2.$$

The reason for the invariance of the locations of the libration points of the elliptic problem in the system ζ is that the right sides of the equations of motion (50) are separable and may be written as products of two functions, one depending only on the true anomaly and the other only on the position coordinates.

The linearized equations of motion of the elliptic problem around the libration points in the system $\bar\xi, \bar\eta$ become

$$u'' - 2v' = g(e,f)[\Omega^0_{xx}u + \Omega^0_{xy}v],$$
$$v'' + 2u' = g(e,f)[\Omega^0_{xy}u + \Omega_{yy}v], \tag{68}$$

where the variations u, v are defined by the equations

$$\bar\xi = \bar\xi_i + u$$

and

$$\bar\eta = \bar\eta_i + v.$$

The subscript i indicates the number of the libration point and consequently $i = 1, 2, \ldots, 5$. The function g is defined by

$$g(e,f) = \frac{1}{1+e\cos f},$$

and so

$$g(0,f) = 1.$$

The second partial derivatives occurring in Eqs. (68) are to be computed at the libration points. Note that Eqs. (68) become the corresponding Eqs. (4) of Section 5.2 for the circular problem with $e = 0$, $u = \xi$, $v = \eta$, and $f = t$. The primes in Eqs. (68), which denote derivatives with respect to the true anomaly, become dots in Eqs. (4), denoting derivatives with respect to the time.

The system of linear differential equations (68) describing the problem has variable coefficients and its closed-form solution is not known. The result of a linearized stability analysis regarding the effect of the eccentricity on the linear stability of the triangular points has been obtained by Danby and later by Bennett and it is shown in Fig. 10.13. The shaded areas represent linear stability. The point denoted by μ^* on the μ axis corresponds to a value of the mass parameter at which any nonzero eccentricity introduces linear instability. The value of μ^* is obtained when, for one of the roots of Eq. (25), Section 5.4, $\lambda = \pm i/2$ is assumed. In fact the actual formula is given at the end of Section 5.6.2.2 as

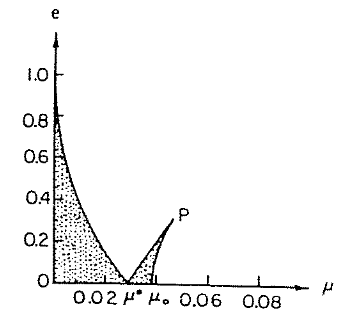


FIG. 10.13. Linear stability of the triangular points in the elliptic restricted problem (Danby, 1964, Ref. 51).

$$\mu^* = \frac{1 - [1 - 16/27s^2(1-s^2)]^{1/2}}{2},$$

which furnishes the desired value of μ^* when $s = \pm\frac{1}{2}$ is substituted:

$$\mu^* = \tfrac{1}{2} - 3^{1/2}/2 = 0.02859548.$$

The coordinates of point P, $\mu = 0.04698$ and $e = 0.3143$, may be obtained by a numerical approach.

Finally the point on the μ axis, denoted by μ_0, represents the value at which instability occurs when $e = 0$. This value is given in Section 5.4 as

$$\mu_0 = \frac{9 - (69)^{1/2}}{18} = 0.0385209.$$

The numerical analysis referred to in the preceding paragraph consists of the application of Floquet's theory and of the location of the characteristic exponents of the linear system on hand. As Fig. 10.13 shows, the eccentricity of the motion of the primaries may have a stabilizing effect according to the linear theory since if $\mu > \mu_0$, the elliptic case is stable up to $\mu = 0.04698$ with the proper value of e.

The instability of the collinear points found in the circular problem for all values of the mass parameter persists for the elliptic problem, which shows instability for any combination of the values of μ and e.

10.3.5 Regularization of the elliptic problem

The two-dimensional elliptic restricted problem will be treated in this section from the point of view of regularization. Since the transformations discussed in the previous sections do not introduce new singularities and result, formally at least, in the same equations of motion as those which describe the circular problem, it is expected that considerations employed in Chapter 3 will be transferable to the elliptic problem. Indeed this is the case, with one noteworthy exception, following from the fact that the Jacobian integral utilized in the derivation of the regularized equations of motion of the circular problem is not available; instead of this, the invariant relation discussed in Section 10.3.3 is employed. The consequence of this is that the regularized equations of motion in the elliptic case become integrodifferential equations.

The derivation begins with Eqs. (50) of Section 10.3.2, which we write in the complex form

$$\zeta'' + 2i\zeta' = \operatorname{grad}_\zeta \omega, \tag{69}$$

where primes denote derivatives with respect to the true anomaly, and the variable ζ and the function ω are defined the same way as in Section 10.3.2.

Transformations of the dependent and independent variables are introduced, similarly to the circular case, by the relations

$$\zeta = F(w) \tag{70}$$

and

$$d\tau = |\, dF/dw \,|^{-2}\, df,$$

where τ is the new independent variable. It is related to the true anomaly f in the same way the pseudotime is related to the actual nondimensional time in the circular problem. The function of the transformation is $F(w)$, which corresponds to the function $f(w)$ introduced in Section 3.3.

The velocity is transformed according to

$$\frac{d\zeta}{df} = \frac{dF}{dw}\frac{dw}{d\tau}\frac{d\tau}{df}. \tag{71}$$

Computation of the second derivative of ζ and substitution into Eq. (69) gives

$$\frac{d^2w}{d\tau^2} + 2i\frac{dw}{d\tau}\left|\frac{dF}{dw}\right|^2 = \left|\frac{dF}{dw}\right|^2 \operatorname{grad}_w V + \left|\frac{dw}{d\tau}\right|^2 \overline{\frac{d^2F}{dw^2}}\left(\overline{\frac{dF}{dw}}\right)^{-1}, \tag{72}$$

where

$$V = \omega - \tfrac{1}{2}C. \tag{73}$$

The invariant relation of the elliptic problem, corresponding to the

Jacobian integral of the restricted problem, is utilized at this point. From Eq. (66) we have

$$\left(\frac{d\zeta}{df}\right)^2 = 2(V - I), \tag{74}$$

where

$$I = e\int_0^f \frac{\Omega \sin f}{(1 + e\cos f)^2}\, df. \tag{75}$$

According to Eqs. (74) and (70) we may rewrite the formula (71) for computing the velocity in the regularized system and obtain the regularized form of the invariant relation (74) as

$$\left|\frac{dw}{d\tau}\right|^2 = 2\left|\frac{dF}{dw}\right|^2 (V - I). \tag{76}$$

The right side of Eq. (72) now becomes

$$\left|\frac{dF}{dw}\right|^2 \operatorname{grad}_w V + 2(V - I)\overline{\frac{d^2F}{dw^2}}\frac{dF}{dw} = \operatorname{grad}_w\left(V\left|\frac{dF}{dw}\right|^2\right) - 2I\overline{\frac{d^2F}{dw^2}}\frac{dF}{dw}.$$

The equation of motion (72), therefore, may be written as

$$\frac{d^2w}{d\tau^2} + 2i\frac{dw}{d\tau}|\, F' \,|^2 = \operatorname{grad}_w(V\,|\, F' \,|^2) - 2IF'\bar{F}'', \tag{77}$$

where

$$F' = dF/dw.$$

The corresponding equation for the circular problem is given in Section 3.4 as

$$\frac{d^2w}{d\tau^2} + 2i\frac{dw}{d\tau}|\, f' \,|^2 = \operatorname{grad}_w(U\,|\, f' \,|^2), \tag{78}$$

where the function $f(w)$ of Section 3.4 corresponds to $F(w)$ and $U = \Omega - C/2$ corresponds to $V = \omega - C/2$ or to

$$V = (1 + e\cos f)^{-1}\,\Omega - C/2. \tag{79}$$

Since $e < 1$, the singularities of the function V are those of the function Ω and a regularizing function $F(w)$ which eliminates the singularities of U will do the same for the function V.

Equations (77) and (78) differ by the term

$$T = 2IF'\bar{F}'' = 2eF'\bar{F}''\int_0^f \frac{\Omega \sin f}{(1 + e\cos f)^2}\, df,$$

which, by introducing the new independent variable τ, may be written as

$$T = 2eF'\bar{F}'' \int_{\tau(0)}^{\tau} \frac{\sin f(\tau)}{[1 + e \cos f(\tau)]^2} \Omega \,|\, \bar{F}' \,|^2 \, d\tau. \tag{80}$$

Note that here the true anomaly appears as a function of the new independent variable and according to Eq. (70) this relation is

$$f(\tau) = \int_{\tau(0)}^{\tau} | F' |^2 \, d\tau.$$

This shows that $f(\tau)$ is finite and defined everywhere in the finite w plane. The same is true for the integral appearing in Eq. (80) since if $F(w)$ regularizes the function Ω then $\Omega \,|\, \bar{F}' \,|^2$ is regular. At the singularities, $F' = 0$; consequently, the expression given by Eq. (80) vanishes at the collisions. This is expected since at collision the elliptic problem should be reducible to the circular problem.

10.4 Hill's problem

10.4.1 Introduction

Hill's limiting case is obtained from the model of the restricted problem with the earth and the sun as the primaries by using the following simplifications:

(1) the solar parallax is zero;
(2) the solar eccentricity is zero;
(3) the lunar inclination is zero.

Particular solutions of the differential equations of motion describing this simplified problem are searched for. These solutions are periodic in a coordinate system which rotates with uniform angular velocity equal to the sun's mean motion. The required solutions are symmetric with respect to the axes of the coordinate system. The solution corresponding to the period of the moon is called Hill's variation orbit and it represents an intermediate orbit in Hill's lunar theory. It is a distinguishing feature of Hill's theory that the intermediate orbit is not a conic section but it is the solution of the simplified restricted problem outlined here previously. Another way to emphasize the significance of Hill's idea is to state that, prior to him, in order to obtain a solution of the problem of three bodies, the problem of two bodies was solved and the solution was varied. Hill on the other hand first solved the (modified) restricted problem of three bodies and considered its variation.

The variation orbit is, of course, only a particular periodic solution of the simplified equations, the general solution of which is to be found next by varying this solution. This problem leads to Hill's equation of the form

$$\ddot{x} + \theta(t)\, x = 0, \tag{81}$$

where x is the normal deviation of the particle (the moon) from the variation orbit and θ is a periodic function of period

$$T = \frac{2\pi}{n - n'}.$$

Here n' and n are the mean motions of the sun around the earth-sun system and of the moon around the earth. The numerical values of the corresponding periods

$$T = 2\pi/n = 27^{\mathrm{d}}.321661,$$

$$T' = 2\pi/n' = 365^{\mathrm{d}}.256371,$$

give

$$n'/n = 0.0748013263.$$

The solution of Eq. (81) involves Hill's celebrated infinite determinant.

In the following sections first the equations of motion are derived. The Jacobian integral, the singularities of the manifold of the states of motion, the curves of zero velocity, and the regularized equations of motion are treated next. Following this, the family of Hill's periodic orbits is discussed, one member of which has the moon's period.

10.4.2 Formulation of the problem and equations of motion

The two derivations offered in this section use different equations as the starting point, as well as different considerations, to arrive essentially at the same results. First we show how the equations of motion in Hill's problem are derived from the equations of motion of the *general* problem of three bodies [Part (A)] and then we show how the same equations may be obtained by estimating the orders of the magnitudes of the various terms occurring in the restricted problem of three bodies [Part (B)]. In Part (C) additional remarks are made about the equations of motion.

(A) The equations of motion of the problem of three bodies were formulated in Section 1.8. Here the three bodies will be the earth, the moon, and the sun, denoted by E, M, and S, respectively. The differential equations of motion of the moon and of the earth in an inertial system are

$$\ddot{\mathbf{r}}_M = -\frac{k^2 m_E}{|\,\mathbf{r}_{EM}\,|^3}(\mathbf{r}_M - \mathbf{r}_E) - \frac{k^2 m_S}{|\,\mathbf{r}_{MS}\,|^3}(\mathbf{r}_M - \mathbf{r}_S) \tag{82}$$

and

$$\ddot{\mathbf{r}}_E = -\frac{k^2 m_S}{|\,\mathbf{r}_{ES}\,|^3}(\mathbf{r}_E - \mathbf{r}_S) - \frac{k^2 m_M}{|\,\mathbf{r}_{EM}\,|^3}(\mathbf{r}_E - \mathbf{r}_M). \tag{83}$$

Here $\mathbf{r}_E$, $\mathbf{r}_M$, and $\mathbf{r}_S$ are the position vectors of the earth, moon, and sun in an inertial system. Dots denote derivatives with respect to the time t^*. Selecting the earth as the origin of the coordinate system we have for the position vectors of the moon and of the sun

$$\boldsymbol{\rho}_M = \mathbf{r}_M - \mathbf{r}_E \quad \text{and} \quad \boldsymbol{\rho}_S = \mathbf{r}_S - \mathbf{r}_E .$$

The equation of motion of the moon in this system — relative to the earth — is obtained by subtracting Eq. (83) from Eq. (82):

$$\ddot{\boldsymbol{\rho}}_M + \frac{k^2(m_E + m_M)}{|\,\boldsymbol{\rho}_M\,|^3}\boldsymbol{\rho}_M = -k^2 m_S\left[\frac{\boldsymbol{\rho}_M - \boldsymbol{\rho}_S}{|\,\boldsymbol{\rho}_{MS}\,|^3} + \frac{\boldsymbol{\rho}_S}{|\,\boldsymbol{\rho}_S\,|^3}\right]. \tag{84}$$

If in the earth-centered system the coordinates of the moon are X, Y, and Z, and of the sun are X', Y', and Z', then the first component of Eq. (84) becomes

$$\ddot{X} + \frac{\kappa}{r^3}X = -k^2 m_S\left(\frac{X - X'}{\Delta^3} + \frac{X'}{r'^3}\right), \tag{85}$$

where

$$\kappa = k^2(m_E + m_M),$$

and the distances between the moon and the sun, the moon and the earth, and the sun and the earth are given by

$$\begin{aligned} \Delta^2 &= |\,\boldsymbol{\rho}_{MS}\,|^2 = (X - X')^2 + (Y - Y')^2 + (Z - Z')^2, \\ r^2 &= |\,\boldsymbol{\rho}_M\,|^2 = X^2 + Y^2 + Z^2, \\ r'^2 &= |\,\boldsymbol{\rho}_S\,|^2 = X'^2 + Y'^2 + Z'^2. \end{aligned} \tag{86}$$

The right side of Eq. (85) may be written as

$$\frac{\partial}{\partial X}R = \frac{\partial}{\partial X}k^2 m_S\left(\frac{1}{\Delta} - \frac{XX' + YY' + ZZ'}{r'^3}\right). \tag{87}$$

The equations of motion of the moon relative to the earth, therefore, are

$$\begin{aligned} \ddot{X} + \frac{\kappa}{r^3}X &= \frac{\partial R}{\partial X}, \\ \ddot{Y} + \frac{\kappa}{r^3}Y &= \frac{\partial R}{\partial Y}, \\ \ddot{Z} + \frac{\kappa}{r^3}Z &= \frac{\partial R}{\partial Z}. \end{aligned} \tag{88}$$

In what follows, we first find an alternate form for the disturbing function R and then apply Hill's simplifications.

From Eqs. (86) we have

$$\Delta^2 = r^2 + r'^2 - 2rr' \cos S$$

and

$$\frac{XX' + YY' + ZZ'}{r'^3} = \frac{r \cos S}{r'^2},$$

where S is the angle MES.

With these results the disturbing function becomes

$$R = \frac{k^2 m_S}{r'}\left[\frac{r'}{\Delta} - \frac{r}{r'}\cos S\right]. \tag{89}$$

The first term in the bracket is expanded in Legendre polynomials as

$$\frac{r'}{\Delta} = \left[1 - 2\frac{r}{r'}\cos S + \left(\frac{r}{r'}\right)^2\right]^{-1/2} = P_0 + \frac{r}{r'}P_1 + \left(\frac{r}{r'}\right)^2 P_2 + \cdots, \tag{90}$$

with

$$P_0 = 1, \qquad P_1 = \cos S, \qquad P_2 = -\frac{1}{2} + \frac{3}{2}\cos^2 S, \qquad \text{etc.}$$

Note that $r/r' \sim 3 \times 10^{-3} \ll 1$, so the convergence of the expansion is guaranteed.

Before substituting Eq. (90) into Eq. (89) we observe that

$$k^2 m_S\left(1 + \frac{m_E + m_M}{m_S}\right) = n'^2 a'^3$$

or

$$k^2 m_S \cong n'^2 a'^3,$$

since

$$\frac{m_E + m_M}{m_S} \sim 3 \times 10^{-6}.$$

Here n' is the sun's mean motion and a' the sun's mean distance. The new form of the disturbing function becomes

$$R = \frac{a'^3 n'^2}{r'} \left[1 + \left(\frac{r}{r'}\right)^2 P_2 + \left(\frac{r}{r'}\right)^3 P_3 + \left(\frac{r}{r'}\right)^4 P_4 + \cdots\right], \tag{91}$$

where the first term will be dropped since it does not contribute to the derivatives of the function R. Multiplying and dividing the members of Eq. (91) with the appropriate powers of the mean lunar distance a, we have

$$R = n'^2 a^2 \left[\left(\frac{r}{a}\right)^2 \left(\frac{a'}{r'}\right)^3 P_2 + \left(\frac{a}{a'}\right)\left(\frac{r}{a}\right)^3 \left(\frac{a'}{r'}\right)^4 P_3 + \left(\frac{a}{a'}\right)^2 \left(\frac{r}{a}\right)^4 \left(\frac{a'}{r'}\right)^5 P_4 + \cdots\right]. \tag{92}$$

If the eccentricity of the orbit of the sun, $e' = 0.0168$, is neglected, then $a'/r' = 1$ everywhere in the expansion of the last formula (92). Furthermore, if $a/a' = 0.0025$ is neglected then the disturbing function becomes

$$R = n'^2 r^2 \left(-\frac{1}{2} + \frac{3}{2}\cos^2 S\right). \tag{93}$$

The quantity a/a' represents the ratio between the sines of the solar and lunar parallaxes; therefore, neglecting this quantity in the expansion of the disturbing function corresponds to one of Hill's simplifying assumptions.

Furthermore, if the lunar inclination is also neglected then we have the following equations of motion of the moon in an earth-centered coordinate system:

$$\ddot{X} + \frac{\kappa}{r^3} X = \frac{\partial R}{\partial X}, \qquad \ddot{Y} + \frac{\kappa}{r^3} Y = \frac{\partial R}{\partial Y}, \tag{94}$$

where the disturbing function R is given by Eq. (93).

Now we select a coordinate system which is centered at the earth and rotates in the (fixed) plane of the ecliptic with angular velocity n'. The axis $\tilde{x}$ is directed constantly toward the sun. The transformation between the previous system XY and the new $\tilde{x}\tilde{y}$ is given by Eq. (26) of Section 1.4:

$$Z = z\, e^{in't^*}, \tag{95}$$

where $Z = X + iY$, $z = \tilde{x} + i\tilde{y}$, and t^* is the (dimensional) time.

Equations (94) in complex notation are

$$\ddot{Z} + \frac{\kappa}{|Z|^3} Z = \operatorname{grad}_Z R(Z), \tag{96}$$

where

$$\operatorname{grad}_Z R = \frac{\partial R}{\partial X} + i\frac{\partial R}{\partial Y}.$$

Applying the transformation given by Eq. (95) to Eq. (96) we obtain

$$\ddot{z} + 2in'\dot{z} = n'^2 z - \frac{\kappa z}{|z|^3} + \operatorname{grad}_z R(z). \tag{97}$$

Note that the last term follows from the equation

$$\operatorname{grad}_Z R = e^{in't^*} \operatorname{grad}_z R, \tag{98}$$

which has already been used as Eq. (76) of Section 3.3 in the form

$$\operatorname{grad}_w U = \overline{f'} \operatorname{grad}_\zeta U$$

with

$$\zeta = f(w).$$

Writing Z for w, R for U, z for ζ, and

$$z = e^{-in't^*} Z$$

for

$$\zeta = f(w)$$

we have $\overline{f'} = e^{in't^*}$, so Eq. (98) is reduced to Eq. (76) of Section 3.3 where a proof is offered.

Equation (97) may be written as

$$\ddot{z} + 2in'\dot{z} = \operatorname{grad}_z R^*, \tag{99}$$

where

$$R^* = \frac{n'^2}{2}|z|^2 + \frac{\kappa}{|z|} + R.$$

The two equations of motion in the earth-centered rotating coordinate system become

$$\ddot{\tilde{x}} - 2n'\dot{\tilde{y}} = \frac{\partial R^*}{\partial \tilde{x}}, \qquad \ddot{\tilde{y}} + 2n'\dot{\tilde{x}} = \frac{\partial R^*}{\partial \tilde{y}}. \tag{100}$$

Here

$$R^* = \frac{n'^2}{2}(\bar{x}^2 + \bar{y}^2) + \frac{\kappa}{r} + n'^2 r^2 \left(-\frac{1}{2} + \frac{3}{2}\cos^2 S\right)$$

or

$$R^* = \frac{\kappa}{r} + \frac{3}{2} n'^2 \bar{x}^2, \tag{101}$$

since

$$n'^2 r^2 \left(-\frac{1}{2} + \frac{3}{2}\cos^2 S\right) = n'^2 \left[-\frac{1}{2}(\bar{x}^2 + \bar{y}^2) + \frac{3}{2}\bar{x}^2\right].$$

Hill's problem is described by Eqs. (100) with Eq. (101) defining R^*. The variables are dimensional in these equations and dots signify derivatives with respect to the (actual) time t^*.

Equations (100), with function R^* substituted, become

$$\begin{aligned} \frac{d^2\bar{x}}{dt^{*2}} - 2n' \frac{d\bar{y}}{dt^*} &= 3n'^2 \bar{x} - \frac{\kappa \bar{x}}{r^3}, \\ \frac{d^2\bar{y}}{dt^{*2}} + 2n' \frac{d\bar{x}}{dt^*} &= -\frac{\kappa \bar{y}}{r^3}. \end{aligned} \tag{102}$$

Introducing the dimensionless variables

$$t = t^* n' \quad \text{and} \quad (\bar{\xi}, \bar{\eta}) = (\bar{x}, \bar{y})/l,$$

Eqs. (102) become

$$\ddot{\bar{\xi}} - 2\dot{\bar{\eta}} = -\frac{\bar{\xi}}{(\bar{\xi}^2 + \bar{\eta}^2)^{3/2}} + 3\bar{\xi}$$

and

$$\ddot{\bar{\eta}} + 2\dot{\bar{\xi}} = -\frac{\bar{\eta}}{(\bar{\xi}^2 + \bar{\eta}^2)^{3/2}}.$$

The scale factor l is given by

$$l = \left(\frac{\kappa}{n'^2}\right)^{1/3} = \left(\frac{k^2(M_E + M_M)}{n'^2}\right)^{1/3}.$$

The equations of motion may also be written by means of the potential Ω^* as

$$\begin{aligned} \ddot{\bar{\xi}} - 2\dot{\bar{\eta}} &= \partial \Omega^* / \partial \bar{\xi}, \\ \ddot{\bar{\eta}} + 2\dot{\bar{\xi}} &= \partial \Omega^* / \partial \bar{\eta}, \end{aligned} \tag{103}$$

where

$$\Omega^* = \tfrac{1}{2}(3\bar{\xi}^2 + 2/\bar{r}), \tag{104}$$

with

$$\bar{r} = (\bar{\xi}^2 + \bar{\eta}^2)^{1/2}.$$

(B) The second derivation of Hill's equations of motion for the simplified model of the restricted problem is now given since its simplicity and directness is a source of true joy. We start with the equations of the restricted problem as given by Eqs. (59) of Section 1.6 and subject it to a translation along the x axis. Upon obtaining in this manner a coordinate system centered at the primary with the smaller mass, we write the equations of motion in this (rotating) coordinate system. Then, the unit of distance is changed by introducing μ^α as the scale factor. The equations of motion obtained this way indicate that, if $\alpha = \frac{1}{3}$, then the gravitational force of the earth is of the same order of magnitude as the Coriolis force and as the centrifugal force. After this selection we let $\mu \to 0$ and immediately obtain the equations of motion associated with Hill's problem.

The equations of motion in the usual synodic coordinate system, using dimensionless variables, are

$$\begin{aligned} \ddot{x} - 2\dot{y} &= x - \frac{(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu(x+1-\mu)}{r_2^3}, \\ \ddot{y} + 2\dot{x} &= y\left[1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3}\right]. \end{aligned} \tag{105}$$

The equations of the transformation corresponding to a translation along the x axis are given by

$$\begin{aligned} \xi &= x + 1 - \mu, \\ \eta &= y. \end{aligned} \tag{106}$$

This transformation locates the primary of smaller mass μ at the origin of the new system, $P_2(\xi = 0, \eta = 0)$. The other primary is at $P_1(\xi = 1, \eta = 0)$. The origin of the new coordinate system ξ, η is at the earth and the new (positive) axis ξ points to the sun which latter is located at unit distance from the earth.

The equations of motion in this system become

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \xi + \mu - 1 - \frac{(1-\mu)(\xi - 1)}{r_1^3} - \frac{\mu\xi}{r_2^3}, \\ \ddot{\eta} + 2\dot{\xi} &= \eta\left[1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3}\right]. \end{aligned} \tag{107}$$

The distances r_1 and r_2 themselves do not change with this transformation, only their respective expressions become different:

$$\begin{aligned} r_1 &= [(x-\mu)^2 + y^2]^{1/2} = [(\xi - 1)^2 + \eta^2]^{1/2}, \\ r_2 &= [(x - \mu + 1)^2 + y^2]^{1/2} = (\xi^2 + \eta^2)^{1/2}. \end{aligned} \tag{108}$$

The scale is now changed according to the transformations

$$\bar\xi = \xi/\mu^\alpha \quad \text{and} \quad \bar\eta = \eta/\mu^\alpha. \tag{109}$$

When Eqs. (109) are introduced in the equations of motion (107) we obtain, after division by μ^α,

$$\begin{aligned} \ddot{\bar\xi} - 2\dot{\bar\eta} &= \bar\xi + (\mu - 1)\,\mu^{-\alpha} + \frac{(\mu-1)\,\mu^{-\alpha}(\mu^\alpha\bar\xi - 1)}{\bar r_1^{\,3}} - \frac{\mu\bar\xi}{\bar r_2^{\,3}}, \\ \ddot{\bar\eta} + 2\dot{\bar\xi} &= \bar\eta\left[1 - \frac{1-\mu}{\bar r_1^{\,3}} - \frac{\mu}{\bar r_2^{\,3}}\right], \end{aligned} \tag{110}$$

where

$$\begin{aligned} \bar r_1 &= [(\bar\xi\mu^\alpha - 1)^2 + \bar\eta^2\mu^{2\alpha}]^{1/2} = [(\bar\xi^2 + \bar\eta^2)\,\mu^{2\alpha} - 2\mu^\alpha\bar\xi + 1]^{1/2}, \\ \bar r_2 &= \mu^\alpha(\bar\xi^2 + \bar\eta^2)^{1/2}. \end{aligned} \tag{111}$$

Note that, for $\alpha > 0$, $\bar r_1 \to 1 + \frac{1}{2}\mu^{2\alpha}(\bar\xi^2 + \bar\eta^{-2}) - \mu^\alpha\bar\xi \to 1$ and $\bar r_2 \to 0$ as $\mu \to 0$. Substituting now the values of $\bar r_1$ and $\bar r_2$, as given by Eqs. (111), into Eqs. (110), we have

$$\begin{aligned} \ddot{\bar\xi} - 2\dot{\bar\eta} &= \bar\xi + (\mu-1)\,\mu^{-\alpha} + \frac{(\mu-1)\,\mu^{-\alpha}(\mu^\alpha\bar\xi - 1)}{[(\bar\xi^2 + \bar\eta^2)\,\mu^{2\alpha} - 2\mu^\alpha\bar\xi + 1]^{3/2}} - \frac{\bar\xi\mu^{1-3\alpha}}{(\bar\xi^2 + \bar\eta^2)^{3/2}}, \\ \ddot{\bar\eta} + 2\dot{\bar\xi} &= \bar\eta\left[1 - \frac{1-\mu}{[(\bar\xi^2 + \bar\eta^2)\,\mu^{2\alpha} - 2\mu^\alpha\bar\xi + 1]^{3/2}} - \frac{\mu^{1-3\alpha}}{(\bar\xi^2 + \bar\eta^2)^{3/2}}\right]. \end{aligned} \tag{112}$$

The previously mentioned selection of $\alpha = \frac{1}{3}$ results in the same orders of magnitude for the Coriolis terms $-2\dot{\bar\eta}$ and $2\dot{\bar\xi}$, for the centrifugal terms $\bar\xi$ and $\bar\eta$, and for the gravitational terms associated with the earth, $\bar\xi/(\bar\xi^2 + \bar\eta^2)^{3/2}$ and $\bar\eta/(\bar\xi^2 + \bar\eta^2)^{3/2}$.

The limit process $\mu \to 0$ now effects only the second and third terms on the right side of the first of Eqs. (112) and the second term on the right side of the second of Eqs. (112). The limit of these three terms may be evaluated as follows. First, consider the two critical terms of the first equations which may be written as

$$\frac{\mu-1}{\mu^\alpha}\left(\frac{\bar r_1^{\,3} - 1 + \mu^\alpha\bar\xi}{\bar r_1^{\,3}}\right) = (\mu - 1)\frac{\bar\xi}{\bar r_1^{\,3}} + \frac{\mu-1}{\bar r_1^{\,3}}\,\frac{\bar r_1^{\,3} - 1}{\mu^\alpha}.$$

Here the limit of the first term on the right side is $-\bar\xi$ as $\mu \to 0$. The second term becomes

$$\lim_{\mu\to 0}\frac{(\mu-1)}{\bar r_1^{\,3}}\,\frac{[1 + \frac{3}{2}(\bar\xi^2 + \bar\eta^2)\,\mu^{2\alpha} - 3\mu^\alpha\bar\xi - 1]}{\mu^\alpha} = 3\bar\xi,$$

where the previously given expansion for $\bar r_1$ is utilized. The limit of the two terms of the first equation therefore becomes $2\bar\xi$.

The limit of the second term of the second of eqs. (112) is unity since $\bar r_3 \to 1$ as $\mu = 0$.

Substituting these results into Eqs. (112) and writing $\frac{1}{3}$ for α, we obtain

$$\begin{aligned} \ddot{\bar\xi} - 2\dot{\bar\eta} &= 3\bar\xi - \frac{\bar\xi}{(\bar\xi^2 + \bar\eta^2)^{3/2}}, \\ \ddot{\bar\eta} + 2\dot{\bar\xi} &= -\frac{\bar\eta}{(\bar\xi^2 + \bar\eta^2)^{3/2}}. \end{aligned} \tag{113}$$

We may introduce again a potential function

$$\Omega^* = \tfrac{1}{2}(3\bar\xi^2 + 2/\bar r) \tag{114}$$

with

$$\bar r = (\bar\xi^2 + \bar\eta^2)^{1/2},$$

and so the equations of motion (113) may be written as

$$\begin{aligned} \ddot{\bar\xi} - 2\dot{\bar\eta} &= \partial\Omega^*/\partial\bar\xi, \\ \ddot{\bar\eta} + 2\dot{\bar\xi} &= \partial\Omega^*/\partial\bar\eta. \end{aligned} \tag{115}$$

These equations [(114) and (115)] are identical with the previously obtained results [Eqs. (103) and (104)].

(C) It is interesting to observe that the three simplifications mentioned in Section 10.4.1 enter at different places and in different forms into the two derivations. The assumption that the solar parallax is zero allows in the first derivation the writing of Eq. (92) in the form

$$R = n'^2a^2\left(\frac{r}{a}\right)^2\left(\frac{a'}{r'}\right)^3 P_2$$

and the simplification which neglects the eccentricity of the sun's orbit gives

$$R = n'a^2\left(\frac{r}{a}\right)^2 P_2.$$

The solar parallax is neglected in the second derivation, when the limit process is performed in Eqs. (112); the eccentricity of the sun's orbit is put equal to zero when the model of the (circular) restricted problem is accepted and Eqs. (105) are used.

The inclination of the moon does not enter at all into the second derivation since the equations of the (two-dimensional) restricted problem are used, while in the first derivation the plane of the ecliptic is actually chosen as the X, Y plane and it is considered fixed.

Comparison of Eqs. (114) and (115) with the equations of the problem of two bodies is of interest, through the corresponding potential functions. For Hill's problem we have

$$\Omega^* = \frac{3}{2}\bar\xi^2 + \frac{1}{\bar r}$$

and the problem of two bodies in a synodic coordinate system is described by

$$\Omega^* = \tfrac{1}{2}(\bar\xi^2 + \bar\eta^2) + 1/\bar r,$$

as given in Section 8.5.1 by Eq. (8). Consequently, the left-hand sides of the equations are identical for the two problems while the right sides are

$$3\bar\xi - \bar\xi/\bar r^3 \quad \text{and} \quad -\bar\eta/\bar r^3$$

for Hill's problem and

$$\bar\xi - \bar\xi/\bar r^3 \quad \text{and} \quad \bar\eta - \bar\eta/\bar r^3$$

for the problem of two bodies.

The Jacobian integral follows immediately from Eqs. (115):

$$\dot{\bar\xi}^2 + \dot{\bar\eta}^2 = 2\Omega^* - C. \tag{116}$$

The theorem that this is the only algebraic integral of Hill's problem follows from a similar but not identical derivation which was used by Siegel for the restricted problem.

By means of Eq. (116) Hill proposed a transformation of the equations of motion. The essential feature of this transformation is that the terms containing the only singularity, $(\bar r)^{-3}$, are eliminated. For this purpose we multiply the first of Eqs. (115) by $\bar\xi$, the second by $\bar\eta$, and add the results:

$$\bar\xi\ddot{\bar\xi} + \bar\eta\ddot{\bar\eta} + 2(\dot{\bar\xi}\bar\eta - \bar\xi\dot{\bar\eta}) = \bar\xi\Omega^*_{\bar\xi} + \bar\eta\Omega^*_{\bar\eta}. \tag{117}$$

Then multiplying the first of Eqs. (115) by $\bar\eta$, the second by $\bar\xi$, and taking the difference of the results, we obtain

$$\bar\eta\ddot{\bar\xi} - \bar\xi\ddot{\bar\eta} - 2(\bar\xi\dot{\bar\xi} + \dot{\bar\eta}\bar\eta) = \bar\eta\Omega^*_{\bar\xi} - \bar\xi\Omega^*_{\bar\eta}. \tag{118}$$

For Hill's problem the right side of Eq. (117) becomes

$$3\bar\xi^2 - 1/\bar r$$

and the right side of Eq. (118) is $3\bar\xi\bar\eta$.

For the problem of two bodies the corresponding results are

$$\bar r^2 - 1/\bar r$$

and zero.

The Jacobian integral (116) contains the term $(\bar r)^{-1}$ and so the right side of Eq. (117) may be written as

$$3\bar\xi^2 - \frac{1}{\bar r} = \frac{9}{2}\bar\xi^2 - \frac{1}{2}(\dot{\bar\xi}^2 + \dot{\bar\eta}^2) - \frac{C}{2},$$

since from Eqs. (114) and (116) we have

$$\frac{1}{\bar r} = \Omega^* - \frac{3}{2}\bar\xi^2 = \frac{1}{2}(\dot{\bar\xi}^2 + \dot{\bar\eta}^2) + \frac{C}{2} - \frac{3}{2}\bar\xi^2.$$

The corresponding result for the problem of two bodies is

$$\bar r^2 - \frac{1}{\bar r} = \frac{3}{2}\bar r^2 - \frac{1}{2}(\dot{\bar\xi}^2 + \dot{\bar\eta}^2) - \frac{C}{2}.$$

The new equations of motion for Hill's problem now may be obtained by retaining the left sides of Eqs. (117) and (118) and using the new forms of the right sides:

$$\begin{aligned} \bar\xi\ddot{\bar\xi} + \bar\eta\ddot{\bar\eta} + 2(\dot{\bar\xi}\bar\eta - \bar\xi\dot{\bar\eta}) + \frac{1}{2}(\dot{\bar\xi}^2 + \dot{\bar\eta}^2) - \frac{9}{2}\bar\xi^2 + \frac{C}{2} &= 0, \\ \bar\xi\ddot{\bar\eta} - \bar\eta\ddot{\bar\xi} + 2(\dot{\bar\xi}\bar\xi + \dot{\bar\eta}\bar\eta) + 3\bar\xi\bar\eta &= 0. \end{aligned} \tag{119}$$

The corresponding equations for the problem of two bodies are

$$\begin{aligned} \bar\xi\ddot{\bar\xi} + \bar\eta\ddot{\bar\eta} + 2(\dot{\bar\xi}\bar\eta - \bar\xi\dot{\bar\eta}) + \frac{1}{2}(\dot{\bar\xi}^2 + \dot{\bar\eta}^2) - \frac{3}{2}(\bar\xi^2 + \bar\eta^2) + \frac{C}{2} &= 0, \\ \bar\xi\ddot{\bar\eta} - \bar\eta\ddot{\bar\xi} + 2(\dot{\bar\xi}\bar\xi + \dot{\bar\eta}\bar\eta) &= 0. \end{aligned}$$

10.4.3 Libration points and curves of zero velocity

In what follows we use the notation x, y, Ω in place of the variables $\bar\xi$, $\bar\eta$, Ω^* for simplicity's sake. The Jacobian integral of Hill's problem then is written as

$$\dot x^2 + \dot y^2 = 2\Omega - C, \tag{120}$$

where

$$\Omega = \frac{3}{2}x^2 + \frac{1}{r}. \tag{121}$$

The singular points of the manifold of the state of motion are given by the equations

$$\dot{x} = 0, \quad \dot{y} = 0, \quad \Omega_x = x(3 - 1/r^3) = 0, \quad \text{and} \quad \Omega_y = -y/r^3 = 0.$$

From the last two equations we find two collinear libration points:

$$L_1(-3^{-1/3}, 0) \quad \text{and} \quad L_2(3^{-1/3}, 0).$$

The third collinear point in the restricted problem was located at the right side of the primary of the larger mass. This point does not appear in Hill's problem since m_1 moved to $x = +\infty$.

Both points are saddle points. To show this we evaluate the second partial derivatives of Ω and computing the Hessian we find the corresponding quadratic form to be indefinite at the points $x = \pm 3^{-1/3}$, $y = 0$. In fact,

$$\Omega_{xx} = 3 + \frac{2x^2 - y^2}{r^5}, \qquad \Omega_{yy} = \frac{2y^2 - x^2}{r^5}, \qquad \Omega_{xy} = \frac{3xy}{r^5}$$

and so the Hessian becomes

$$H = \frac{1}{r^{10}}[3r^5(2y^2 - x^2) - 2(x^2 + y^2)^2].$$

At both collinear points

$$\Omega_{xx} = 9, \qquad \Omega_{yy} = -3, \qquad \Omega_{xy} = 0$$

and

$$H = -27.$$

The variational equations with respect to the libration points are

$$\ddot{\xi} - 2\dot{\eta} = 9\xi,$$
$$\ddot{\eta} + 2\dot{\xi} = -3\eta,$$

and the characteristic equation becomes

$$\lambda^4 - 2\lambda^2 - 27 = 0.$$

The roots are

$$\lambda_1 = (1 + 2 \cdot 7^{1/2})^{1/2} = \lambda, \qquad \lambda_2 = -(1 + 2 \cdot 7^{1/2})^{1/2} = -\lambda,$$
$$\lambda_3 = i(2 \cdot 7^{1/2} - 1)^{1/2} = is, \qquad \lambda_4 = -i(2 \cdot 7^{1/2} - 1)^{1/2} = -is,$$

with

$$\lambda = 2.50828679$$

and

$$s = 2.07159422,$$

indicating general instability but conditional stability in the linear sense. The reader will notice the similarity between the libration points in Hill's problem and the collinear points in the restricted problem.

A set of propositions will now be given regarding the curves of zero velocity for Hill's problem.

(1) The curves of zero velocity are defined by

$$3x^2 + 2/r - C = 0, \tag{122}$$

where we observe the symmetry property with respect to the x and y axes. Consequently, the investigation may be limited to the first quadrant of the xy plane.

(2) The different roles of the coordinate axes may be seen when we evaluate the function $\Omega(x, y)$ given by Eq. (121) as $r \to \infty$. If $x \neq 0$, we have $\Omega \to \infty$ as $r \to \infty$, and when $x = 0$, $\Omega \to 0$ as $r \to \infty$. In other words, along any ray, excepting the y axis, $\Omega \to \infty$, and along the y axis, $\Omega \to 0$ as $r \to \infty$.

(3) Hill's curves intersect the x axis when $C \geqslant 3^{4/3}$. To show this we consider the function

$$Y(x) = 3x^3 - Cx + 2,$$

from which we find that

$$\frac{dY}{dx} = 0 \quad \text{at} \quad x_m = \frac{C^{1/2}}{3}.$$

At this value of x we have

$$Y(x_m) = 2[1 - (C/3^{4/3})^{3/2}],$$

which is the only minimum of the function $Y(x)$ for $x > 0$.

Depending on the value of C we have

$$Y_{\min} \begin{Bmatrix} \geq \\ = \\ < \end{Bmatrix} 0 \quad \text{if} \quad C \begin{Bmatrix} \leq \\ = \\ > \end{Bmatrix} 3^{4/3}$$

and the location of the minimum becomes

$$x_m \begin{Bmatrix} \leq \\ = \\ > \end{Bmatrix} 3^{-1/3}$$

The curve of zero velocity intersects the positive x axis if $\bar{Y}_{\min} \leqslant 0$. There are two intersections when $\bar{Y}_{\min} < 0$ and one if $\bar{Y}_{\min} = 0$. The corresponding values of the Jacobian constant are $C > 3^{4/3}$ for two and $C = 3^{4/3}$ for one intersection. The location of the intersection is

$$x_0 = x_m = 3^{-1/3}$$

when $C = 3^{4/3}$. The two intersections are located at

$$0 < x_{01} < 3^{-1/3}$$

and at

$$3^{-1/3} < x_{02} < (C/3)^{1/2}$$

when $C > 3^{4/3}$.

The upper limit of x_{02} follows from the intersection of the vertical asymptote and the x axis. From Eq. (122) this asymptote is located at

$$x_a = (C/3)^{1/2}$$

and so

$$x_a \lesseqqgtr 3^{1/6} \quad \text{if} \quad C \lesseqqgtr 3^{4/3}.$$

(4) Closed curves of zero velocity exist around the origin when $C \geqslant 3^{4/3}$. To show this, we investigate the intersections of the curves of zero velocity with the system of rays

$$x = \alpha r, \qquad y = \beta r, \qquad 1 \geqslant \alpha \geqslant 0, \qquad 1 \geqslant \beta \geqslant 0, \qquad \alpha^2 + \beta^2 = 1,$$

where $r \geqslant 0$ is a parameter measuring the radial distance along the rays from the origin. The combination $\alpha = 0$, $\beta = 1$ corresponds to the y axis, and when $\alpha = 1$, $\beta = 0$ the x axis is described. Substituting the equations of the system of rays into Eq. (122) we obtain

$$\bar{Y}(r) = 3\alpha^2 r^3 - Cr + 2 = 0. \tag{123}$$

If $\alpha = 0$ we find the intersection of the y axis with the curves of zero velocity at $r = 2/C = y$, and we conclude that the positive y axis always intersects once and only once the curve of zero velocity for any value of C.

If $\alpha = 1$, we obtain the previously investigated case of the intersection of the x axis with the curves of zero velocity.

If $0 < \alpha < 1$ the rays of the first quadrant are studied and if the equation $\bar{Y}(r) = 0$ has positive roots the rays intersect the curve of zero velocity. Depending on the value of C, we find that again

$$\bar{Y}_{\min} \gtreqqless 0 \quad \text{if} \quad C \lesseqqgtr 3^{4/3}\alpha^{2/3}$$

and the location of the minimum becomes

$$r_m \lesseqqgtr (3\alpha^2)^{-1/3}.$$

When $\bar{Y}_{\min} < 0$, there are two intersections connected with $C > 3^{4/3}\alpha^{2/3}$. If the condition for two intersections with the x axis, $C > 3^{4/3}$, is satisfied, then the condition $C > 3^{4/3}\alpha^{2/3}$ is certainly satisfied since $0 < \alpha < 1$.

Consider the case when $C > 3^{4/3}$. One intersection of the positive x axis with the curve of zero velocity occurs between $x = 0$ and $x = 3^{-1/3}$ as shown before. So when $\alpha = 1$, we have $0 < x_0 < 3^{-1/3}$, and when $\alpha = 0$, we have $y_0 = 2/C < 2/3^{4/3}$.

The intersection with the x axis is obtained at the smaller root of the equation

$$3x_0^3 - Cx_0 + 2 = 0,$$

and the intersection with the y axis is computed from the equation

$$-Cy_0 + 2 = 0.$$

Consequently, $x_0 > y_0$.

The smaller positive root of Eq. (123) for $C > 3^{1/3}$ is a monotone function of α; consequently, the curve of zero velocity around the origin is closed.

Note that

$$(y_0)_{\max} = 2/C_{\min} = 2/3^{4/3}$$

and

$$(x_0)_{\max} = 3^{-1/3}.$$

Since $(x_0)_{\max} > (y_0)_{\max}$ all closed curves of zero velocity are inside a circle of radius $3^{-1/3}$.

(5) Motion takes place inside this oval. This is the case if the oval shrinks with increasing the value of the Jacobian constant or, in other words, C is a monotone decreasing function of r. From Eq. (123)

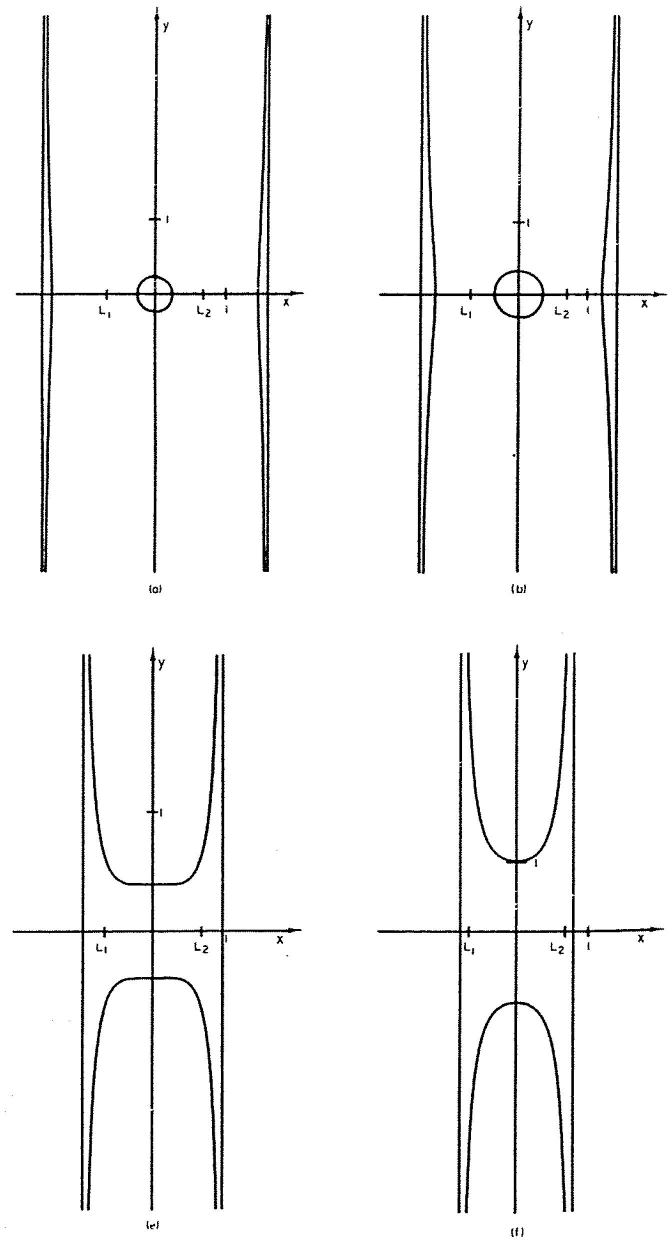
$$C = 3\alpha^2 r^2 + 2/r$$

and for a fixed $\alpha \neq 0$ we have

$$\frac{dC}{dr} = 6\left(\frac{\alpha}{r}\right)^2\left(r^3 - \frac{1}{3\alpha^2}\right).$$

So

$$\frac{dC}{dr} \gtreqqless 0 \quad \text{if} \quad r \gtreqqless (3\alpha^2)^{-1/3}$$



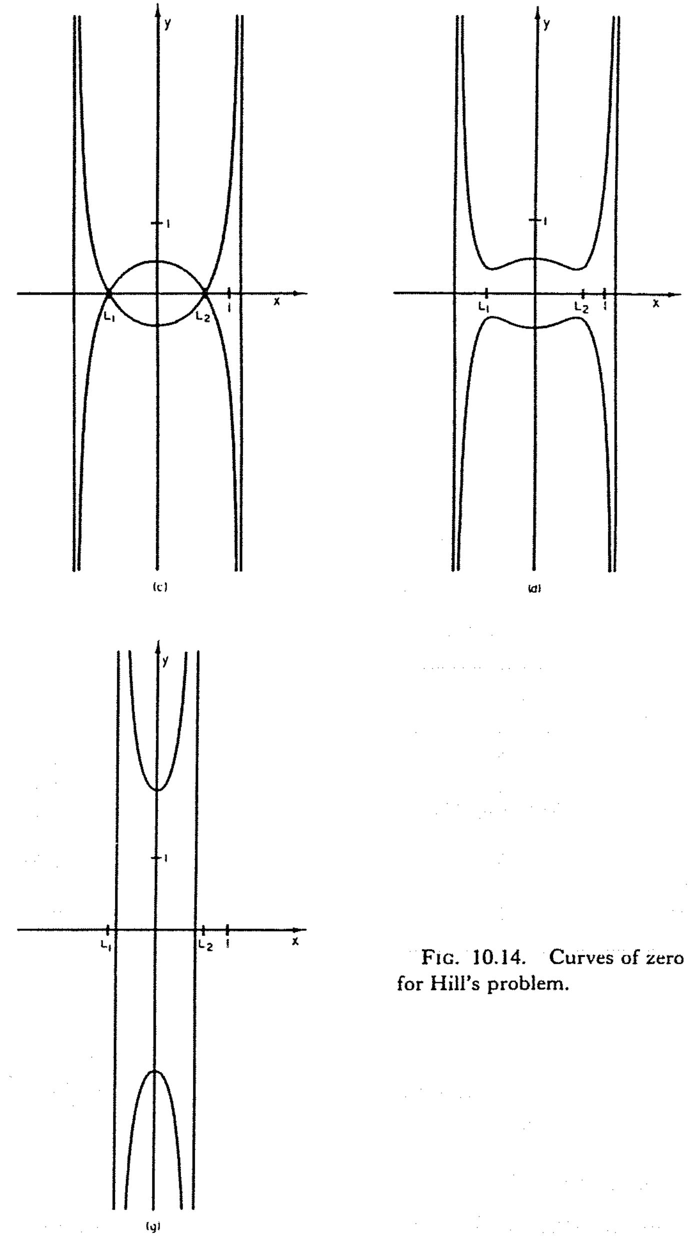


FIG. 10.14. Curves of zero velocity for Hill's problem.

and consequently the function $C(r)$ is monotone decreasing since the intersection of the ray with the curve of zero velocity occurs at a value of r which is less than the r corresponding to the minimum of C. The case of $\alpha = 0$ is simple since then $C = 2/r$ and $dC/dr < 0$.

(6) The three different types of curves of zero velocity corresponding to $C > 3^{4/3}$, $C = 3^{4/3}$, and $C < 3^{4/3}$ are shown in Fig. 10.14. When $C > 3^{4/3}$, motion may take place either inside the oval around the origin or on the concave sides of the infinite branches. When $C = 3^{4/3}$, the two branches of the curves of zero velocity join at the libration points. When $C < 3^{4/3}$, the exchange of particles between the inside of the oval around the origin and the outsides of the infinite branches becomes free, but there are still forbidden regions, given by

$$|y| > \left[\left(\frac{2}{C - 3x^2}\right)^2 - x^2\right]^{1/2}$$

and

$$|x| < (C/3)^{1/2}.$$

Table I shows the characteristics of Figs. 10.14(a)–(g).

TABLE I

CHARACTERISTICS OF CURVES SHOWN IN FIG. 10.14

Fig. No.	C	$y_0 = 2/C$	$x_a = (C/3)^{1/2}$
10.14(a)	8	1/4	1.633
10.14(b)	6	1/3	1.414
10.14(c)	$3^{4/3} = 4.327$	$2 \cdot 3^{-4/3} = 0.462$	1.201
10.14(d)	4	1/2	1.155
10.14(e)	3	2/3	1.000
10.14(f)	2	1	0.816
10.14(g)	1	2	0.577

(7) For the motion of the moon interest is focused on the case $C > 3^{4/3}$ when motion takes place inside the closed oval around the origin. Details of such a closed oval of zero velocity, together with the limiting oval at $C = 3^{4/3}$ and with the circle of radius $3^{-1/3}$ bounding the ovals, are shown in Fig. 10.15.

10.4.4 Regularization

It is shown in this section that the transformation introduced in Section 3.3 by Eq. (64) and named after Levi-Civita regularizes Hill's problem.

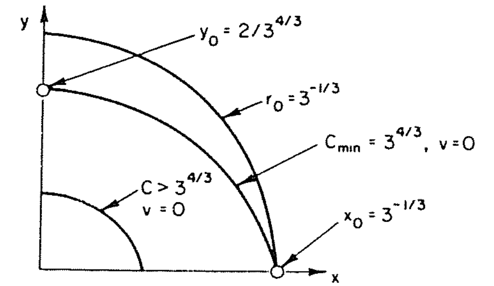


FIG. 10.15. Ovals of zero velocity in Hill's problem.

With the notations $z = x + iy$ and $w = u + iv$ we have

$$z = f(w) = w^2, \qquad x = u^2 - v^2, \qquad y = 2uv.$$

The transformation of the time becomes

$$d\tau = \frac{dt}{4(u^2 + v^2)},$$

since

$$\frac{dt}{d\tau} = |f'(w)|^2.$$

The equations of motion may be written as

$$\begin{aligned} u'' - 8(u^2 + v^2)\, v' &= \Omega_u^*(u, v), \\ v'' + 8(u^2 + v^2)\, u' &= \Omega_v^*(u, v), \end{aligned} \tag{124}$$

where primes denote derivatives with respect to the new independent variable τ, and

$$\Omega^*(u, v) = (\Omega - C/2)|f'|^2. \tag{125}$$

The function Ω is given by Eq. (121) which, when it is expressed by the new dependent variables and substituted into Eq. (125), gives

$$\Omega^*(u, v) = 6(u^2 + v^2)(u^2 - v^2)^2 + 4 - 2C(u^2 + v^2). \tag{126}$$

The right-hand sides of Eqs. (124) are

$$\begin{aligned} \Omega_u^* &= 4u[3(u^2 - v^2)(3u^2 + v^2) - C], \\ \Omega_v^* &= -4v[3(u^2 - v^2)(3v^2 + u^2) + C]. \end{aligned} \tag{127}$$

The singularity at the origin is eliminated; in fact, $\Omega^*(0, 0) = 4$,

$\Omega_u^*(0, 0) = \Omega_v^*(0, 0) = 0$, and the velocity at the origin is $2 \cdot 2^{1/2}$ which is obtained from the transformed Jacobian integral,

$$(u')^2 + (v')^2 = 2\,\Omega^*(u, v). \tag{128}$$

The motion of a particle with initial conditions $u(0) = 0$, $v(0) = 0$, and $t = \tau = 0$ corresponds to the collision of the moon with the earth. The Jacobian integral gives

$$[u'(0)]^2 + [v'(0)]^2 = 8$$

and consequently the initial conditions for the components of the velocity become

$$u'(0) = 2 \cdot 2^{1/2} \cos\alpha \qquad \text{and} \qquad v'(0) = 2 \cdot 2^{1/2} \sin\alpha,$$

where α is the angle between the initial velocity vector and the u axis.

The motion may be represented as

$$u = u(0) + u'(0)\,\tau + u''(0)\,\frac{\tau^2}{2} + \cdots,$$

and

$$v = v(0) + v'(0)\,\tau + v''(0)\,\frac{\tau^2}{2} + \cdots.$$

To find the higher derivatives in these expressions we consider Eqs. (124) from which

$$u''(0) = 8v'(0)[u^2(0) + v^2(0)] + \Omega_u^*(0, 0)$$

and

$$v''(0) = -8u'(0)[u^2(0) + v^2(0)] + \Omega_v^*(0, 0),$$

or

$$u''(0) = 0 \qquad \text{and} \qquad v''(0) = 0.$$

The third derivatives are obtained by differentiation of the equations of motion (124):

$$u''' = 8[2v'(uu' + vv') + v''(u^2 + v^2)] + \Omega_{uu}^* u' + \Omega_{uv}^* v'$$

and

$$v''' = -8[2u'(uu' + vv') + u''(u^2 + v^2)] + \Omega_{vu}^* u' + \Omega_{vv}^* v',$$

or

$$u'''(0) = -4Cu'(0) \qquad \text{and} \qquad v'''(0) = -4Cv'(0),$$

since

$$\Omega_{uu}^*(0, 0) = \Omega_{vv}^*(0, 0) = -4C \qquad \text{and} \qquad \Omega_{uv}^*(0, 0) = 0.$$

The power-series expansion therefore becomes

$$u = u_0'\tau - 4Cu_0'\,\frac{\tau^3}{3!} + \cdots$$

and

$$v = v_0'\tau - 4Cv_0'\,\frac{\tau^3}{3!} + \cdots,$$

where for simplicity's sake the notations $u_0 = u(0)$ and $v_0 = v(0)$ were introduced.

The relation between the new and the original time variables (τ and t) is

$$\int_0^t dt = 4\int_0^\tau (u_0'^2 + v_0'^2)\left(\tau^2 - \frac{8C}{3!}\,\tau^4 + \cdots\right) d\tau,$$

from which

$$t = \frac{4}{3}(u_0'^2 + v_0'^2)\,\tau^3\left(1 - \frac{4}{5}\,C\tau^2 + \cdots\right)$$

or

$$t = \frac{32}{3}\,\tau^3\left(1 - \frac{4}{5}\,C\tau^2 + \cdots\right).$$

Bringing back the original variables x and y we have

$$x = (u_0'^2 + v_0'^2)\,\tau^2 + \cdots$$

and

$$y = 2u_0'v_0'\tau^2 + \cdots,$$

or

$$x = (8\cos 2\alpha)\,\tau^2 + \cdots$$

and

$$y = (8\sin 2\alpha)\,\tau^2 + \cdots.$$

Inverting, finally, the relation connecting the two time variables, we have for small t

$$\tau = (3t/32)^{1/3} + \cdots$$

and so

$$x = [(9/2)^{1/3}\cos 2\alpha]\,t^{2/3} + \cdots$$

and

$$y = [(9/2)^{1/3}\sin 2\alpha]\,t^{2/3} + \cdots.$$

We note that a similar development may be carried out for the regularized restricted problem. The reason for delegating this derivation from Chapter 3 to Chapter 10 is that the right-hand sides of the equations of motion for Hill's problem are considerably simpler than for the restricted problem.

10.4.5 The variation orbit

We now return to the equations of motion as derived in Section 10.4.2. First consider Eq. (115) in the form

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x, \\ \ddot{y} + 2\dot{x} &= \Omega_y, \end{aligned} \tag{129}$$

where

$$\Omega = \frac{3}{2}x^2 + \frac{1}{r}.$$

The notation is simplified as before by introducing x, y, and Ω in place of $\bar{\xi}$, $\bar{\eta}$, and Ω^*.

The periodic solution of interest is symmetric to both coordinate axes. Consider the initial conditions

$$x(0) > 0, \qquad y(0) = 0, \qquad \dot{x}(0) = 0,$$

and the requirements for symmetry

$$\begin{aligned} x(t) &= x(-t) = -x(t + T/2), \\ y(t) &= -y(-t) = -y(t + T/2). \end{aligned}$$

The solution in Fourier series is

$$x = \sum_{n=0}^{\infty} A_n \cos \omega nt + B_n \sin \omega nt,$$

$$y = \sum_{n=0}^{\infty} C_n \cos \omega nt + D_n \sin \omega nt,$$

where

$$\omega = 2\pi/T.$$

This is considerably simplified when the conditions for symmetry are satisfied. Since

$$x(t) = x(-t), \qquad \text{we have} \qquad B_n = 0,$$

and similarly the equation

$$y(t) = -y(-t) \qquad \text{implies that} \qquad C_n = 0.$$

The conditions $x(t) = -x(t + T/2)$ and $y(t) = -y(t + T/2)$ are satisfied if all even terms are eliminated; consequently, we have

$$\begin{aligned} x &= A_1 \cos \omega t + A_3 \cos 3\omega t + \cdots, \\ y &= D_1 \sin \omega t + D_3 \sin 3\omega t + \cdots, \end{aligned}$$

or simply

$$\begin{aligned} x &= \sum_{n=0}^{\infty} \alpha_n \cos (2n + 1)\, \omega t, \\ y &= \sum_{n=0}^{\infty} \beta_n \sin (2n + 1)\, \omega t, \end{aligned} \tag{130}$$

where

$$\alpha_n = A_{2n+1} \qquad \text{and} \qquad \beta_n = D_{2n+1}.$$

The task, therefore, is to find the coefficients α_n, β_n for a given value of ω. Note that ω is a constant of integration and it is the *only* parameter of the solution. To a given value of $x(0) > 0$ one might be able to find a corresponding value of $\dot{y}(0) > 0$, along with $y(0) = \dot{x}(0) = 0$, which will result in a direct periodic orbit in the rotating system.

The value of ω corresponding to the moon is known with high precision from observations. Since the equations of motion as given by Eqs. (129) refer to $n' = 1$, we have in the rotating coordinate system in general

$$\omega = \frac{n - n'}{n'}.$$

Hill's parameter, defined as $m = 1/\omega$ becomes

$$m = \frac{n'/n}{1 - n'/n}.$$

Using the value of n'/n given in Section 10.4.1, we have

$$m = 0.080848933808312.$$

Table II gives Hill's results regarding four members of the family of orbits beginning with the moon's orbit.

TABLE II

CHARACTERISTICS OF HILL'S ORBITS, SHOWN IN FIG. 10.16

No.	$x(0)$	$y(T/4)$	m	C
1	0.17610	0.17864	0.0808489	6.50888
2	0.31232	0.36897	0.25	4.13277
3	0.33235	0.45973	0.3333333	3.72018
4	0.27180	0.78190	0.5609626	2.55787

The first column refers to the numbering of the curves shown in Fig. 10.16. The first orbit belongs to the earth's moon; the last one is the orbit of the moon of maximum lunation. The cusp occurs on the y axis at the curve of zero velocity. Note that the critical value of the Jacobian constant, which separates the closed ovals of zero velocity from the open curves is

$$C = 3^{4/3} = 4.3267488.$$

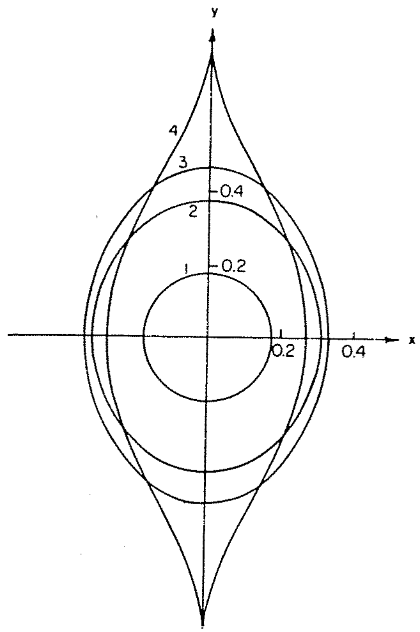


FIG. 10.16. Hill's family of orbits (Hill, 1878, Ref. 56).

Consequently, only the first orbit is inside of a closed oval of zero velocity. The family may be continued for even larger values of m, after the cusp orbit is reached. These orbits which develop loops at both ends were missed by Hill but were found by Lord Kelvin as well as by Poincaré. Their existence is discussed in the following few lines.

The cusp occurs on the y axis at $y = y_0$ where the orbit has a common point with the curve of zero velocity. Consequently for curve number 4 Table II gives

$$C = 2/y(T/4).$$

An expansion of the coordinates at this point in powers of the time gives

$$x = x_0 + \dot{x}_0 t + \ddot{x}_0 \frac{t^2}{2!} + \dddot{x}_0 \frac{t^3}{3!} + \cdots$$

and a similar expression for $y(t)$ (cf. Section 4.7.5). Note that the initial conditions now are

$$t = 0; \quad x = x_0 = 0, \quad y = y_0 = 2/C, \quad \dot{x} = \dot{y} = 0.$$

The higher-order derivatives are obtained from the equations of motion as follows:

$$\ddot{x}_0 = 2\dot{y}_0 + \Omega_x{}^0 \quad \text{and so} \quad \ddot{x}_0 = \Omega_x{}^0,$$
$$\ddot{y}_0 = -2\dot{x}_0 + \Omega_y{}^0 \quad \text{and so} \quad \ddot{y}_0 = \Omega_y{}^0.$$

On the y axis $\Omega_x{}^0 = 0$ and $\Omega_y{}^0 = -y_0^{-2}$; consequently,

$$\ddot{x}_0 = 0 \quad \text{and} \quad \ddot{y}_0 = -y_0^{-2}.$$

Computing higher-order derivatives of the equations of motion, we have

$$\dddot{x}_0 = -2y_0^{-2} \quad \text{and} \quad \dddot{y}_0 = 0.$$

So the series expansions are [cf. Eqs. (125), Section 4.7.5]

$$x = -t^3/3y_0{}^2 + \cdots,$$
$$y = y_0 - t^2/2y_0{}^2 + \cdots.$$

If there is no cusp on the y axis then the expansions in power series begin with the initial conditions

$$t = 0; \quad x = x_0 = 0, \quad y = y_0 \neq 2/C, \quad \dot{x} = \dot{x}_0 \neq 0, \quad \dot{y}_0 = 0.$$

Since the orbit has the previously discussed symmetry conditions, only odd powers of the time occur in the series for x and only even powers of t appear in the expansion for y. This follows also from the equations of motion, which give

$$\ddot{x}_0 = 0, \qquad \dddot{x}_0 = -\dot{x}_0(1 + 1/y_0^3) - 2/y_0^2,$$
$$\ddot{y}_0 = -2\dot{x}_0 - 1/y_0^2, \qquad \dddot{y}_0 = 0.$$

The expansion for x becomes simple if the value selected for y_0 is close to the cusp. In this case $\dot{x}_0$ is small and

$$\dddot{x}_0 \cong -2/y_0^2.$$

In this way we have

$$x \cong \dot{x}_0 t - t^3/3y_0^2,$$

which shows that, if $\dot{x}_0 < 0$, the equation $x(t) = 0$ has only one real root for small values of t, namely $t = 0$. On the other hand, if $\dot{x}_0 > 0$, we have three real roots,

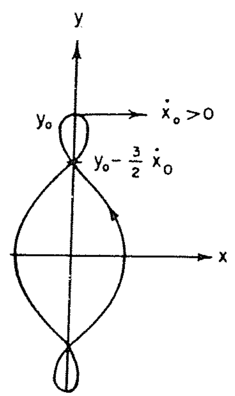


FIG. 10.17. Looped orbit for Hill's problem.

$$t_1 = 0 \qquad \text{and} \qquad t_{2,3} = \pm y_0(3\dot{x}_0)^{1/2},$$

with the corresponding values of $y = y_0$ and $y = y_0 - (3/2)\dot{x}_0$. These orbits intersect the y axis six times, having three intersections with the positive and three with the negative y axis. So there are three quadratures between two consecutive syzygies (see Fig. 10.17).

The solution of Hill's problem, as mentioned before, amounts to finding the coefficients of the Fourier expansion. This, in turn, requires the solution of an infinite system of nonlinear algebraic equations obtained by comparing the coefficients of the terms with equal frequency. The fact that the equations of motion and the variables are subjected to transformations does not change the existence of solution, it only modifies the way to find it. Hill transformed first the equations of motion from our conventional form, given by Eq. (129), to the form derived in Section 10.4.2, Part (C), and given by Eq. (119). Then he introduced new variables u, s, ζ in place of x, y, t by the transformations

$$u = x + iy, \qquad s = x - iy, \qquad \zeta = e^{it/m}.$$

The solution of the equations of motion lead to an infinite nonlinear system, as mentioned before, which Hill solved by a series approximation without investigating the convergence of the method. This was done first, by Liapunov and Happel, then by Siegel who, with a different approach, showed the existence of Hill's variational orbits, and finally by Wintner who proved the convergence of Hill's method some 48 years after Hill's publication.

As mentioned in the introduction to Hill's problem (Section 10.4.1), the most fascinating and best-known aspects of his work are connected with handling an infinite system of linear equations as the outcome of Eq. (81), which is often referred to as Hill's equation or as the generalized Gyldén-Lindstedt equation. This may be written as

$$\ddot{x} + (\theta_0 + \theta_2 \cos 2t + \theta_4 \cos 4t + \cdots)\, x = 0$$

and its solution as

$$x = \sum_{-\infty}^{\infty} a_k \cos[(c + 2k)\, t + \alpha].$$

The convergence of the infinite determinant was shown by Poincaré in several papers, references to which the reader will find in Sections 10.6 and 10.7.

The two problems of infinite systems of nonlinear and linear equations are well covered in the literature of mathematics and celestial mechanics, and inasmuch as the details of Hill's problem are outside the scope of this work we restrict the additional discussion to bibliographical remarks.

10.5 Further variations

This section mentions some additional modifications of the restricted problem. First, in Section 10.5.1, we consider two examples of the simplifications of our dynamical system. These are known as Euler's and Lagrange's problems. When the primaries are at rest in an inertial coordinate system, the centrifugal and the Coriolis terms are absent from the equations of motion. This dynamical system is referred to as the problem of two fixed centers of force and is often called Euler's problem. When the centrifugal terms are retained in the restricted problem but the Coriolis terms are ignored and in addition the value of the mass parameter is $\frac{1}{2}$, we arrive at the case which sometimes is called Lagrange's problem.

Instead of simplifying assumptions we include effects which complicate the original problem in Section 10.5.2. The complexity of the problem is increased by considering variable masses of the primaries, low-thrust forces acting on the third body in addition to the Newtonian gravitational forces, hyperbolic and parabolic orbits of the primaries, higher-order harmonics of the gravitational forces acting between the primaries and the third body, etc.

The chapter and the volume is closed with two important and powerful generalizations of the restricted problem: first, a natural modification of the Lagrangian function is introduced in Part (D) of Section 10.5.2; second, the idea of inverse problems is presented in Part (E).

10.5.1 Euler's and Lagrange's problems

A comprehensive view is offered when the Lagrangian formulation of the restricted problem is used. Let the Lagrangian be given by

$$L = \tfrac{1}{2}(\dot{x} - ny)^2 + \tfrac{1}{2}(\dot{y} + nx)^2 + \Omega^*$$

or by

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + n(x\dot{y} - y\dot{x}) + \Omega, \tag{131}$$

where the mean motion of the synodic coordinate system defined in Section 1.4 is n and

$$\Omega = \frac{n^2}{2}(x^2 + y^2) + \Omega^*,$$

with

$$\Omega^* = \tfrac{1}{2}\mu(1 - \mu) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2}.$$

If centrifugal and Coriolis forces are ignored we have Euler's problem with $n = 0$ and

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + \Omega^*.$$

The equations of motion become

$$\ddot{x} = \Omega_x^*, \qquad \ddot{y} = \Omega_y^*.$$

If the Coriolis forces are neglected, the Lagrangian is

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + \Omega,$$

and if in addition $\mu = \frac{1}{2}$, we have Lagrange's problem with

$$\Omega = \bar{\Omega} = \frac{1}{8} + \frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + \frac{n^2}{2}(x^2 + y^2).$$

Consider now the Thiele–Burrau transformation in complex form introduced in Section 3.6 as

$$q = \tfrac{1}{2}\cos w, \tag{132}$$

where the origin of the synodic coordinate system q_1, q_2 is located at the midpoint between the primaries, which themselves are on the q_1 axis at unit distance apart. The equation of motion with complex notation is given by Eq. (102) of Section 3.5 as

$$\ddot{q} + 2i\dot{q} = \operatorname{grad}_q U(q),$$

where

$$U(q) = \Omega(q) - C/2.$$

The equation of motion in terms of the variable w is

$$w'' + 2inw' \,|f'(w)|^2 = \operatorname{grad}_w |f'|^2 U$$

which, with

$$|f'(w)|^2 = \tfrac{1}{8}(\cosh 2v - \cos 2u),$$

becomes

$$u'' - \frac{nv'}{4}(\cosh 2v - \cos 2u) = (|f'|^2 U)_u \tag{132a'}$$

and

$$v'' + \frac{nu'}{4}(\cosh 2v - \cos 2u) = (|f'|^2 U)_v. \tag{132b'}$$

Here primes denote derivatives with respect to the transformed time τ, which is related to the original time t by

$$t = \frac{1}{8}\int_0^\tau (\cosh 2v - \cos 2u)\, d\tau. \tag{133}$$

The function occurring on the right side of Eqs. (132′) may be evaluated using Eq. (126) of Section 3.6. If n^2 is inserted by writing

$$\frac{n^2}{2}(x^2 + y^2) + \frac{\mu}{2}(1 - \mu)$$

in place of

$$\tfrac{1}{2}[(1 - \mu)r_1^2 + \mu r_2^2],$$

we find, with $x = q_1 - \frac{1}{2} + \mu$ and $y = q_2$, that

$$\Omega = \frac{n^2}{2}[q_1^2 + q_2^2 + 2q_1(\mu - \tfrac{1}{2}) + (\mu - \tfrac{1}{2})^2] + \frac{\mu}{2}(1 - \mu) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2}. \tag{134}$$

Substituting here the expressions of q_1, q_2, r_1, and r_2, which may be obtained from Eqs. (125) as given in Section 3.6, we have

$$U\,|f'\,|^2 = (\tfrac{1}{2}-\mu)\cos u + \tfrac{1}{2}\cosh v + \frac{C'}{2}(\cosh 2v - \cos 2u)$$
$$+ \frac{n^2}{32}(\mu - \tfrac{1}{2})(\cosh 3v \cos u - \cosh v \cos 3u)$$
$$+ \frac{n^2}{256}(\cosh 4v - \cos 4u),$$

where

$$C' = \tfrac{1}{8}[n^2(\mu - \tfrac{1}{2})^2 + \mu(1-\mu) - C]$$

is a modified Jacobian constant.

Before discussing Euler's or Lagrange's problem by writing $n = 0$ or $\mu = \frac{1}{2}$ we observe that $U\,|f'\,|^2$ is separable when either $n = 0$ or $\mu = \frac{1}{2}$, since the only term containing both variables u and v is multiplied by $n(\mu - \frac{1}{2})$. The equations of motion, on the other hand, show that if the Coriolis terms are ignored the equations reduce to

$$u'' = (U\,|f'\,|^2)_u$$

and

$$v'' = (V\,|f'\,|^2)_v.$$

Consequently, when $n = 0$ the variables are separated and Euler's problem may be written as

$$u'' = (\tfrac{1}{2}-\mu)\sin u + C'\sin 2u, \qquad v'' = \tfrac{1}{2}\sinh v + C'\sinh 2v. \tag{135}$$

When $\mu = \frac{1}{2}$ and $n \neq 0$ we have the Copenhagen problem and both variables occur in both equations:

$$u'' - \frac{nv'}{4}(\cosh 2v - \cos 2u) = C'\sin 2u + \frac{n^2}{64}\sin 4u,$$
$$v'' + \frac{nu'}{4}(\cosh 2v - \cos 2u) = \frac{1}{2}\sinh v + C'\sinh 2v + \frac{n^2}{64}\cosh 4v. \tag{136}$$

When the Coriolis terms are ignored but $n \neq 0$ and $\mu = \frac{1}{2}$ we have Lagrange's problem:

$$u'' = C'\sin 2u + \frac{n^2}{64}\sin 4u,$$
$$v'' = \frac{1}{2}\sinh v + C'\sinh 2v + \frac{n^2}{64}\cosh 4v. \tag{137}$$

So the restricted problem may be solved by quadratures if the masses of the two primaries are equal and if the Coriolis forces are ignored since Eqs. (137) may be integrated by elliptic functions.

The following few additional comments may be offered:

(1) Note that Euler's method of handling the problem of two fixed centers of force is to transform the equations of motion by Eq. (132). In this way the Lagrangian of the system becomes of the Liouville type, which may be written as

$$L = \frac{1}{2}\left[\sum_{i=1}^{n} F_i(x_i)\right]\left[\sum_{i=1}^{n} G_i(x_i)\,\dot{x}_i^2\right] - \sum_{i=1}^{n} H_i(x_i)\Big/\sum_{i=1}^{n} F_i(x_i), \tag{138}$$

where x_i is the generalized coordinate. This dynamical system may be solved by quadratures.

The same applies to the case when the Coriolis terms are ignored but the centrifugal forces are retained. In this case the Lagrangian function reduces to the Liouville type, provided the linear term in Eq. (134) is missing, since this term represents the nonseparable ($\cos u \cosh v$) combination. This requirement is satisfied if $\mu = \frac{1}{2}$.

(2) The case of neglected Coriolis forces is interesting, in spite of its artificiality, since the linear stability property of the triangular libration points changes from stable to unstable when the Coriolis effects are ignored as noted by Wintner. To show this we write the variational equations in the form given in Section 5.2.3 by Eqs. (4):

$$\ddot{\xi} - 2\alpha\dot{\eta} = \Omega^0_{xx}\xi + \Omega^0_{xy}\eta,$$
$$\ddot{\eta} + 2\alpha\dot{\xi} = \Omega^0_{xy}\xi + \Omega^0_{yy}\eta.$$

If $\alpha = 0$ the Coriolis forces are ignored, and if $\alpha = 1$ they are taken fully into consideration. For the present considerations let α be undetermined. The characteristic equation after Eq. (6) of Section 5.2.3 becomes

$$\lambda^4 - \lambda^2(\Omega^0_{xx} + \Omega^0_{yy} - 4\alpha^2) + \Omega^0_{xx}\Omega^0_{yy} - (\Omega^0_{xy})^2 = 0, \tag{139}$$

where according to Section 5.4 we have

$$\Omega^0_{xx} + \Omega^0_{yy} = 3 \qquad \text{and} \qquad \Omega^0_{xx}\Omega^0_{yy} - (\Omega^0_{xy})^2 = \frac{27}{4}\mu(1-\mu).$$

Substituting these values and $\Lambda = \lambda^2$, we obtain

$$\Lambda^2 - \Lambda(3 - 4\alpha^2) + \frac{27}{4}\mu(1-\mu) = 0,$$

corresponding to Eq. (26) in Section 5.4.

The roots of this last equation, when $\alpha = 0$, are

$$\Lambda_{12} = \tfrac{3}{2}\{1 \pm [1 - 3\mu(1 - \mu)]^{1/2}\} > 0;$$

consequently, all four roots of the characteristic equation (139) are real for any value of the mass parameter between 0 and $\frac{1}{2}$. So the equilateral libration points become unstable in general in the linear and in the nonlinear sense. More specifically the linear analysis gives conditional asymptotic stability but the motion is unstable in the nonlinear sense.

One might acquire a deeper insight into the effect of the Coriolis terms assuming other values for α, besides zero or unity. Since only small changes of the value $\alpha = 1$ are of possible physical (relativistic) interest we consider

$$\alpha = 1 + \epsilon,$$

neglect the second and higher powers of ϵ, and solve the characteristic equation. In this way we obtain

$$\Lambda^2 - \Lambda[3 - 4(1 + \epsilon)^2] + \frac{27}{4}\mu(1 - \mu) = 0.$$

The discriminant of this quadratic form is zero if

$$\mu^2 - \mu + \frac{1}{27}[4(1 + \epsilon)^2 - 3]^2 = 0.$$

The solution of this equation, when $\epsilon = 0$, for μ (in the range $0 \leqslant \mu \leqslant \frac{1}{2}$) is

$$\mu_c = \mu_0 = \tfrac{1}{2}[1 - (23/27)^{1/2}],$$

corresponding to the case when there is no deviation from the actual Coriolis forces. The critical value of the mass parameter μ_c becomes, for $\epsilon \neq 0$,

$$\mu_c = \mu_0 + \frac{16}{3(69)^{1/2}}\epsilon.$$

We conclude that, if the Coriolis terms are slightly larger than classical mechanics requires ($\epsilon > 0$), the range of values of the mass parameter giving linearly stable solutions increases, since for stability we must have

$$0 < \mu < \mu_c.$$

The corollary of this result is that the range for which periodic solutions of the linearized equations exist is increased above and beyond $\mu_0 = 0.0385$.

Note that the stability characteristics of the collinear points is not changed when the Coriolis forces are altered.

(3) Charlier's idea of using Euler's solution as reference orbit in perturbation methods of the restricted problem comes up quite naturally. For the detailed review of the contributions along this line the reader is referred to Section 10.6, and here only the latest work by Payne is mentioned, who investigates six different perturbation schemes in considerable detail. The significantly new idea in this approach is the search for systems in which the perturbations, due to the centrifugal and Coriolis terms first omitted, are minimum. Because of the elliptic functions occurring in the expressions for the reference orbit and because of the appearance of secular terms in the higher-order approximations, only a first-order theory exists.

Note that a reference orbit which might be closer to the solution may be obtained when only the centrifugal forces are maintained and the Coriolis terms are ignored; nevertheless, the analytical expression for such a reference orbit is not known unless $\mu = \frac{1}{2}$. The solution of this (Lagrange's) problem has not been considered as a reference orbit in the open literature up to now since interest is focused on a restricted problem with the earth and the moon as primaries, for which $\mu \cong 1/82$. Whether the perturbing effects of the centrifugal and Coriolis forces in Euler's problem, or the perturbing effects of the Coriolis forces and of the large difference between the assumed and actual values of the mass parameter in Lagrange's problem are more significant is still an open question. It seems to be, however, that Encke's conic sections as reference orbits might be superior to Euler's solution as reference orbits in the neighborhood of the earth and the moon.

10.5.2 Increased complexity

This chapter is concluded with some remarks regarding modifications of the restricted problem which increase the degree of complication. The discussion is restricted to a few examples.

(A) The generalization of the orbits of the primaries from circles to arbitrary orbits may be an important step within certain limitations. As long as the primaries are point masses and only Newtonian gravitational forces are acting, the orbits must be conic sections. The case of circular orbits is what we call the restricted problem proper. Elliptic orbits are discussed in Section 10.3; parabolic and hyperbolic orbits, having less significance in the solar system than elliptic orbits, are mentioned only for completeness' sake.

(B) Important generalizations of the physical nature of the primaries lead to modified versions of the restricted problem in which the primaries have either variable masses, while preserving their point-mass gravitational property, or become masses with finite size and with nonspherical density distributions. The first case is of interest in cosmological investigations and in research on binary stars. Depending on the assumption regarding the time dependence of the masses of the primaries, the existence of the Jacobian integral might be preserved. For instance, in the case of an inverse linear time dependence given by

$$m = \frac{m_0}{\alpha + \beta t},$$

properly introduced (time-dependent) new variables satisfy an equation corresponding to the Jacobian integral as shown by Razbitnaya.

Rein's example considers the dependence of the mass of the primary with the larger mass, on the distance between this primary and the third body, or $m_1 = m_1(r_1)$. Once again an invariant relation corresponding to the Jacobian integral exists.

If higher-order gravitational harmonics are present in the force field of one or both primaries, the restricted problem as a model may still be applicable, but analysis of orbits in the vicinity of the primaries must take into consideration the departure from the force law of r^{-2}. There is numerical evidence that in the computations of trajectories from the earth to the moon, neglecting the higher-order gravitational harmonics of the earth requires the altering of the initial conditions. The crucial question appears once again regarding the motion of the primaries. It may be shown that if the primaries are gravitating triaxial ellipsoids with ellipsoidal density distribution, then the system may rotate with a constant angular velocity as a rigid body and consequently some of the basic assumptions of the restricted problem remain operational. For instance, if the primaries are ellipsoids rotating on circular orbits, the collinear libration points are still unstable and the triangular points vary their linear stability according to the orientation of the ellipsoids.

(C) When the third body (of infinitesimal mass) experiences accelerations due to thrust or drag in addition to the gravitational forces, we may investigate guidance laws as embedded in the restricted problem, establish new points of libration with interesting stability characteristics, or study the motion of asteroids with retardation.

There seems to be hardly an end to the listing of ever-more complicated problems related to the restricted problem. A large variety of combinations of the cases listed in Chapter 10 is possible also, but these are outside the scope of this treatise.

(D) In this part a generalization of the restricted problem is presented

based on Birkhoff's modifications of the Lagrangian function of the problem. From Eq. (131) of Section 10.5.1 we have

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + x\dot{y} - y\dot{x} + \Omega(x, y), \tag{140}$$

where

$$\Omega(x, y) = \tfrac{1}{2}(x^2 + y^2) + \Omega^*(x, y).$$

These equations represent the restricted problem in a Cartesian rectangular synodic system which has mean motion unity.

Consider the following generalization:

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + \alpha(x, y)\,\dot{x} + \beta(x, y)\,\dot{y} + \gamma(x, y), \tag{141}$$

where α, β, and γ are real analytic functions of x and y.

Before proceeding it should be mentioned that a more general form of the first term of the Lagrangian is

$$\tfrac{1}{2}(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2),$$

with

$$a = a(x, y) > 0, \qquad b = b(x, y), \qquad c = c(x, y), \qquad ac - b^2 > 0.$$

This form, however, may be changed to (141) without difficulty.

The equations of motion of the dynamical system defined by Eq. (141) are

$$\begin{aligned} \ddot{x} + \lambda(x, y)\,\dot{y} &= \gamma_x, \\ \ddot{y} - \lambda(x, y)\,\dot{x} &= \gamma_y, \end{aligned} \tag{142}$$

where

$$\lambda(x, y) = \alpha_y - \beta_x .$$

The Jacobian integral of the Lagrangian formulation in its general form is given by

$$\sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = L + k, \tag{143}$$

where k is a constant.

This may be seen when the total derivative of the Lagrangian $L = L(q, \dot{q})$ is computed:

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i = \dot{q}_i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i}$$

or

$$\frac{dL}{dt} = \frac{d}{dt}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right)$$

from which Eq. (143) follows by integrating. Note that repeated subscripts indicate summation as usual.

Returning now to the Lagrangian as given by Eq. (141) and applying Eq. (143) we have

$$\tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) = \gamma(x, y) + k. \tag{144}$$

The fact that the Jacobian integral exists in its original form follows also directly from Eqs. (142) as one can show by the usual multiplications, addition, and integration.

The three functions $\alpha(x, y)$, $\beta(x, y)$, and $\gamma(x, y)$ are reduced to two functions, $\lambda(x, y)$ and $\gamma(x, y)$, in the equations of motion and to only one function, $\gamma(x, y)$, in the integral. Consequently, the curves of zero velocity are determined only by the function $\gamma(x, y)$. This is of considerable interest for the following reasons. The equations

$$\ddot{x} = \gamma_x, \qquad \ddot{y} = \gamma_y,$$

which are obtained from Eqs. (142) with $\lambda \equiv 0$, represent a conservative system with potential $-\gamma(x, y)$. It is also a reversible problem since when t is replaced by $-t$ the equations are not altered. Whenever $\lambda(x, y) = 0$, we speak of an irreversible dynamical system, since terms linear in the components of the velocity occur in the equations of motion (142) and these terms change sign as t is replaced by $-t$.

Both the reversible and the irreversible problems possess the same curves of zero velocity as we have seen before. The reversible problem is integrable whenever the potential is separable, that is when the $\gamma(x, y)$ may be written as the sum of a function of x and a function of y. This, of course, is not the case for the irreversible problem and this is one reason why Euler's transformation solves the problem of two fixed centers but does not solve the restricted problem. A special case of the irreversible system is when the functions λ and γ depend only on one (and the same) of the variables. This system possesses a linear integral in addition to the Jacobian integral, as was shown by Birkhoff, and the problem can be explicitly integrated.

The case of the restricted problem is inherently irreversible since, as a simple comparison of Eqs. (140) and (141) shows,

$$\alpha = -y \quad \text{and} \quad \beta = x.$$

Consequently $\lambda = -2$ and $\gamma = \Omega(x, y)$.

As mentioned in Chapter 3 the conformal transformation

$$z = f(w)$$

leaves Eqs. (142) in an unaltered form which is obtained by replacing λ by $\lambda\,|f'|^2$ and γ by $\gamma\,|f'|^2$.

We conclude that the Lagrangian of the restricted problem as given by Eq. (140) may be generalized to the form given by Eq. (141) without presenting a new problem. In other words, dynamical systems described by the Lagrangian (141) may be transformed into the restricted problem.

One may consider this conclusion to be one reason why Poincaré and Birkhoff considered the restricted problem basic not only to celestial mechanics but also to dynamics. The irreversibility and the nonintegrability of the restricted problem, in addition to its simplicity, render it the fundamental model for purposes of dynamics.

(E) This volume is closed with the idea of the inverse problem in the theory of orbits. One purpose of dynamics is to enable us to establish an orbit in a given force field with given initial conditions. The inversion of this problem is obtained when either an orbit and its initial conditions or certain properties of the orbit are given, and the force field which results in such an orbit is to be determined. Under the most general conditions this problem has no unique solution and it may be shown that, if certain realistic simplifications are introduced, it leads to the consideration of partial differential equations describing the force components. Once the force field is determined we may consider it as composed of the actual force field of the, say, restricted problem plus of an artificial field. If the latter is "small" it might be handled as a guidance force. This way a solution to a modification of the restricted problem is obtained since the initially mentioned given orbit is *the solution.* But one may go a step further. If the modification of the force field which is required to obtain a certain predetermined orbit is much smaller than the effect of other terms in the restricted problem, we may neglect it altogether and consider the given orbit either as an approximate solution of the exact equations of motion or as the exact solution of the approximate equations. Inasmuch as the restricted problem only describes a model of the actual physical situation, in principle it may happen that the approximate equations referred to in the previous sentence represent the real world better than the exact equations of the restricted problem, in which case, of course, we have obtained the exact solution of the actual problem (which solution, by the way, is only an approximate solution of the exact, but not of the actual problem).

Our ignorance regarding the precise structure of the universe, our often crude approximations of its laws, and our ever-changing opinion regarding the actual values of the constants representing it, are some of the reasons for the importance of considering models as the restricted problem with its modifications. In the fast-changing mathematical formulation of the universe, models of today may become mere perturbations of tomorrow and vice versa.

10.6 Notes

References regarding the best known and most immediate generalizations, the problem of three bodies and the problem of n bodies cannot be reviewed here since these subjects are outside our scope. Instead, attention is directed to what may be considered the major reviews of the general problem of three bodies by Gautier [1], Hill [2], Whittaker [3], Lovett [4], and Marcolongo [5]. Whittaker's [6] encyclopedia article is somewhat more general in its purpose and consequently it shows the roles played by the restricted and by the general problems of three bodies in general perturbation theories. An elegantly presented introduction to the problem of n bodies is contained in Pollard's book [7, p. 38].

The description of the three-dimensional problem in the synodic system as given in Section 10.2.1 may be found in the standard reference books. See, for instance, Moulton [8, p. 279], Brouwer and Clemence [9, p. 254], Wintner [10, p. 374], etc. The Hamiltonian formulation is also standard; the derivation of the three-dimensional Delaunay elements may be found, for instance, in Brouwer and Clemence [9, pp. 283, 539].

The question of the surfaces of zero velocity discussed in Section 10.2.2 is not covered thoroughly in the literature. The geometric considerations mentioned in Section 10.2.2.1 regarding the location of the equilibrium points for the three-dimensional case is given by Picart [11], who shows that the equilibrium locations must be either at the triangular points or on the axis where the primaries are located.

The three major references regarding the surfaces of zero velocity of the three-dimensional restricted problem (Section 10.2.2.3) are not sufficiently detailed. Kopal [12, p. 135] concentrates on the special case when $C = C_2$, giving actual coordinates for the critical points (maxima and minima) of the surface and computing its volume. Picart [11] shows the development of the surfaces for varying values of the Jacobian constant in a qualitative and highly schematic way. Moulton's [8, p. 283] treatment is also qualitative. Some of the curves representing intersections of the surfaces of zero velocity with the planes of the coordinate system display such analytic properties as would be attributed to a careless draftsman rather than to someone of Moulton's standing. His terminology regarding the cylinders around the z axis, which he calls curtains, is expressive and characteristic of his descriptive style of presentation. We mention also Hill's [13, p. 18] purely analytic description of the surfaces without pictorial details. The curves of zero velocity shown in the text were prepared by M. Standish.

Moulton's work mentioned in Section 10.2.3 on three-dimensional

orbits around the libration points is in his book on periodic orbits [14, p. 151].

Section 10.2.4 discusses three-dimensional orbits. Part (A) treats Poincaré's [15, Vol. 2, p. 144] periodic orbits of the third kind. The subclass mentioned in the text generated from two-body orbits with zero eccentricity is of special interest since Charlier [16, Vol. 2, p. 231] mentions that Poincaré "erroneously" assumed the existence of such orbits. Existence proofs by Jeffreys [17] and Goudas [18] clear up this question, indicating that Poincaré was correct. Numerical proof is offered by Goudas [19] whose orbit is shown in Fig. 10.11. Aksenov's [20] orbits, also mentioned in this part, correspond to the "outside" orbits, discussed in several previous chapters. Additional references by von Zeipel [21], regarding periodic orbits of the third kind, extending Poincaré's results, and by Huang (Huang and Wade [22]), regarding orbits normal to the plane of the orbit of the primaries, are of interest. Kobb's [22a] use of the surfaces of zero velocity as boundaries of motion is described in two of his papers.

Part (B) treats motion along the z axis. The problem has been considered by Pavanini [23], by MacMillan [24], and by Sitnikov [25].

The principal references regarding the classification of earth-to-moon trajectories [Section 10.2.4, Part (C)] are by Hoelker and Braud [26] and by Herring [27]. One of the earlier basic references by Egorov [28] is discussed in considerable detail in Chapter 9.

Section 10.2.5 introduces a general approach to the modification of the regularizations discussed in Chapter 3 when the motion is three-dimensional. The use of spinors to regularize the equations of motion in the three-dimensional case of the problem of two bodies was published in 1964 by Kustaanheimo [29]. A more detailed exposition of the same method including a perturbation theory by Kustaanheimo and Stiefel [30], shows that the necessity for the four-dimensional excursion is related to a theorem by Hurwitz [31], as mentioned in connection with Eq. (39) of Section 10.2.5. A further application demonstrating the power of the method is given by Stiefel and Waldvogel [32], who generalize Birkhoff's transformation to the three-dimensional case. Results of numerical integration of the four-dimensional regularized equations are reported by Rössler [33]. The three-dimensional regularization mentioned at the end of Section 10.2.5, which does not require the introduction of a four-vector, is described by Wintner [10, p. 330].

Part (A) of Section 10.2.6 mentions shortly the application of the zero velocity surfaces to the study of close binary systems. The principle reference is Kopal's [12] excellent treatise, especially pp. 125–139 and 476–529, where the interpretation of the zero velocity surfaces as Roche's [34] equipotentials is given.

Part (B) of Section 10.2.6 discusses the matching of conic sections as approximations to the solution of the interplanetary n-body problem. Series of papers by Lagerstrom and Kevorkian [35] and by Kevorkian [35a] offer uniformly valid asymptotic approximations and in this way legalize the matching technique. Application of the method to interplanetary trajectories is given by Perko [36].

Tisserand's criterion for the identification of comets is given in his "Mécanique Céleste" [37, Vol. 4, pp. 241, 289]. Applications of the method to comets is published by Schulhof [38].

An excellent reference on the subject of swing-by trajectories is by Gillespie and Ross [39]; application of Tisserand's cirterion to such orbits is given by Szebehely [40]. In the same paper applications of the capture problem to space dynamics are treated (see also [28]). Private communications with Hornby [40a], Ross [40b] and Lange [40c] helped to clarify Part (B) of this section. The question of priority regarding the discovery of swing-by trajectories between K. Ehricke, W. Hollister, H. Hornby, M. Hunter, S. Ross, R. Sohn, and others cannot be easily decided. The credit might belong to Tisserand [37, p. 203].

Section 10.3.2 deals with the elliptic problem. The references usually given in the literature are Nechvíle's papers of 1917 and 1926. Thus, he frequently receives erroneously the credit for introducing the transformations discussed in the text. I find that the first to suggest such transformations is Scheibner [41] who in a little known and slightly erroneous paper in 1866 gives essentially Eqs. (50′) and, therefore, treated the three-dimensional case. The next contribution is by Hill [42] who introduced the eccentric anomaly as the independent variable and used elliptic coordinates. Only 51 years after Scheibner do we see the appearance of a joint paper by Petr and Nechvíle [43] in 1917 with the English title "Two remarks on the special problem of three bodies." The first remark is by Petr who proposes the introduction of the pulsating coordinate system employing the variable distance between the primaries. The second note is by Nechvíle who continues Petr's remarks and introduces the true anomaly as the independent variable. Both remarks refer to the two-dimensional case. Nine years later Nechvíle [44] attempted to generalize the result obtained in 1917 to the three-dimensional case but unfortunately his equations are in error together with his conclusions. Nechvíle gives what corresponds to Eqs. (50″) in the text, but with the partial derivatives of ω' on the right side instead of using ω as given by Eq. (51′). Consequently, his conclusion regarding the formal equivalence of the circular and elliptic cases in three dimension is erroneous when he finds that the two cases differ only by a periodic factor on the right side of the equations.

Nechvíle's error was corrected by Rein [45] treating the three-

dimensional elliptic problem in a paper dated 1937 but published only in 1940. Note the typographical error in Rein's Eq. (2). The derivation of the pertinent equations in the text (Sections 10.3.2.1 and 10.3.2.2) follows a paper by Szebehely and Giacaglia [46].

The question of the invariant relation (Section 10.3.3) taking the place of the Jacobian integral is discussed with respect to its application to long-time predictions by Ovenden and Roy [47], by Kopal and Lyttleton [48], and by Szebehely and Giacaglia [46]. Tisserand's method of identifying comets is modified to be applicable to the elliptic case by Callandreau [49].

Motion around the libration points in the elliptic problem (Section 10.3.4) is treated by Moulton [14, p. 217] without the benefit of the special pulsating coordinate system and associated variables introduced in the text. Colombo *et al.* [50] offer numerical results; Danby [51] computes the linear stability of the orbits around L_4 in a fixed coordinate system and obtains Fig. 10.13 of the text. The same results are obtained using a rotating coordinate system by Bennett [52], first numerically, then analytically. A complete treatment of the stability in Lyapunov sense is given by Grebenikov [52a]. The question of the existence of periodic orbits around the libration points is treated in the linearized elliptic case by Szebehely [53], employing the coordinates mentioned in the text.

Application of the elliptic problem to periodic orbits of the second kind may be found in Charlier's book [16, Vol. 2, p. 226]. The problem of the Kirkwood gaps in the asteroid belt is discussed by Brouwer [54] for the elliptic case.

The equations of motion of the elliptic problem in regularized form, given in Section 10.3.5, become integrodifferential equations as shown in the text and in a previously mentioned paper by Szebehely and Giacaglia [46]. These equations have been successfully programed and numerically integrated by Bozis [55] in connection with establishing the "third integral" of the elliptic restricted problem. Contopoulos [55a] used these equations also for the same purpose in his analytic approach.

Section 10.4 offers a short outline of Hill's problem. The principal reference is, of course, Hill's basic paper [56] which is considered one of the best-written contributions to the literature of celestial mechanics. (The original version of the paper as published in the *American Journal of Mathematics*, 1887, is preferred to the version printed in his "Collected Works," 1905, since this latter has typographical errors; for instance, on p. 293 two square-root signs are missing and one sign error may also be easily detected.) This and another famous paper by Hill [2] led to E. W. Brown's work and to what is known as the Hill-Brown lunar theory. Excellent expositions of Hill's work from the astronomical point of view may be found in books by Charlier [16, p. 141], Plummer [57, p. 254], Smart [58, p. 291], Brouwer and Clemence [9, p. 335], and by Chebotarev [59, p. 211]. One of the basic books on the lunar theory is by Brown [60] who discusses Hill's theory in considerable detail [60, p. 195]. Eckert's important work with many references is described in [60a]. Along the analytical lines see Poincaré [15, p. 104], Wintner [10, p. 379], and Siegel [61, p. 104]. The first derivation of Hill's equations of motion given in Section 10.4.2, Part (A), follows Hill's treatment; the second derivation [Part. (B)] makes use of Lagerström's and Kevorkian's ideas [35, 35a]. Wintner's derivation [10, p. 381] corresponds to but is not identical with the one given in Part (B).

It is seen from the derivations that Hill's coordinate system rotates with a uniform angular velocity equal to the sun's mean motion. Note that this allows the preservation of the Jacobian integral [Section 10.4.2, Part (C)]. The same is not true in Euler's [62] second lunar theory in which he introduces a coordinate system rotating with the moon's mean angular velocity. As a consequence Euler does not mention the existence of an integral which would correspond to the Jacobian integral.

The proof that the Jacobian integral is the only *algebraic* integral for Hill's problem is given by Gravalos [63]. As far as I am able to determine, there exists no theorem regarding the nonexistence of isolating (uniform) integrals other than the Jacobian integral for Hill's problem. The reader may compare these theorems with Siegel's and Poincaré's theorems for the restricted problem (see Section 2.1) and with Bruns' and Poincaré's theorems for the general problem of three bodies.

The curves of zero velocity (Section 10.4.3) are given in most of the previously mentioned references with fewer details than in our text, where the curves were prepared by M. Standish. The problem of regularization discussed in Section 10.4.4 follows Wintner's [10, p. 384] treatment but it is not different from the regularization applied for the problem of two bodies in Chapter 3.

The simplicity of the regularized Eqs. (124) together with Eqs. (127) of Section 10.4.4 compares favorably with the original Eqs. (115) of Hill (which contain a singularity at the origin) as well as with his modified Eqs. (119). The treatment of Hill's problem via a transformation which amounts to a different regularization technique is due to Stumpff [64].

Proofs of existence for the variation orbit (Section 10.4.5) are given by Liapunov [65], Happel [66; 67, p. 332], Birkhoff [68] (for retrograde orbits), Wintner [69; 10, p. 392], Hölder [70], and by Siegel [61, p. 108].

Hill himself regretted that he was obliged to pass over the convergence problems [56, p. 8]. The existence of Hill's orbit of maximum lunation and of the looped orbits discussed in Section 10.4.5 was shown by Poincaré and by Wintner [71]. The discussion in the text follows Poincaré's [15, Vol. 1, p. 109] considerations regarding the looped orbits in spite of the fact that Wintner considers Poincaré's work "heuristic" and refers to his own work as "Präzisionmathematik" [71].

Poincaré also mentions that such loop orbits should also occur when the solar parallax is not neglected. Indeed this is the case as several figures of Chapter 9 demonstrate. Section 9.4.4 treats Class (g) orbits of the Copenhagen category ($\mu = \frac{1}{2}$). These direct orbits around either primary are generated from circular orbits of infinitesimal radii. Phase 3 of the family is shown in Fig. 9.6(c) of Section 9.4.4. The first and second drawings in this figure, with $C = 2.809$ and $C = 3.237$, show loops, but because of the presence of the other primary the orbit is not symmetric to the y axis as in Hill's case. A similar drawing is shown in Fig. 9.21(f) in Section 9.5.2 with $C = 2.9932$ for $\mu = 1/82.27$. The periodic orbit is symmetric to the x axis but not to the y axis since the "solar parallax" is not zero.

A further distinction which is attached to the orbit shown in Fig. 10.17 is that Poincaré considers it significant enough to include it as his first figure (on p. 109) in his "Méthodes Nouvelles" [15]. Charlier [16, p. 166] treats the same problem with some generalizations. Thomson [72] shows such a looped orbit to demonstrate his graphic method of integrating differential equations. The family of looped orbits was computed by Matukuma [72a].

The mathematical legalization of the infinite determinants is due to Poincaré [15, Vol. 2, 260; 73; 74; 75, Vol. 2 pt. 2, 44].

Periodic orbits, in addition to Hill's orbits, are shown to exist and are analyzed by Schubart [76] who treats the two- and three-dimensional versions of Hill's problem using orbital elements as variables.

As a general reference for Section 10.4, once again Whittaker's report is mentioned [3, p. 130].

Section 10.5.1 treats Euler's and Lagrange's problems. The original reference to the solution of the problem of two fixed centers of gravitation goes back to Euler [77] in 1760. The discussion given in Section 3.1.1 on the Thiele–Burrau method of regularization applies here also. The treatment offered in Section 10.5.1 is related to Plummer's [57, p. 252] discussion. For the classification of orbits in Euler's problem see Charlier [16, Vol. 1, p. 117], and for Lagrange's problem see Deprit's article [78]. The original reference regarding Lagrange's problem is his "Mécanique Analytique" [79, Vol. 2, p. 98]. Generalizations of Euler's problem are offered by Jacobi [80, p. 221], and by Hiltebeitel [81], the last one discussing Darboux' [82], and Liouville's [83] modifications. Darboux treats two possible applications of the problem of two fixed centers: first, when a body passes at great speed through a double-star system; second, the generalization of the problem of the moon's motion. The Liouville-type dynamical system mentioned in the text [Eq. (138)] is described by Liouville [84]. Euler's problem is also treated by Whittaker [85, p. 97] and Wintner [10, p. 145], who refers to Lagrange's problem also [10, p. 351].

Charlier's [16, Vol. 2, p. 352] idea of the solutions of Euler's problem as intermediate orbits for the restricted problem found many followers. Samter [86], Demin [87], Arenstorf and Davidson [88], Payne [89], and Langebartel [90] use the solutions of Euler's problem as reference orbits.

The second remark in Section 10.5.1 regarding the effect of the Coriolis terms generalizes Wintner's [10, p. 372] result. For the partial applicability of this generalized study of the Coriolis effects to relativity, see Pauli's book [91, p. 175]. Additional results are given by Szebehely [91a].

Part (A) of Section 10.5.2 refers to the hyperbolic and parabolic restricted problems. A detailed discussion of these cases is given by Martin [92] and a general treatment is offered by Alekseev [93]. The problem of variable masses of the primaries mentioned in Part (B) of Section 10.5.2 is discussed by Razbitnaya [94], whose work is based on Mestschersky's [95] and Lovett's [96] papers. Prior to these, however, Gyldén [97] considered the problem of two bodies with variable masses. The reference to Rein's work is [98]. The effect of ellipsoidal primaries is discussed in a paper by Lebrun and Robe [99] and a general introduction to the problem of the motion of deformable bodies is given by Finlay-Freundlich [100, p. 105].

Examples for the inclusion of extra forces referred to in Part (C) of Section 10.5.2 are treated by Danby [101] and Dusek [102]. Danby discusses the effects of forces given by $a^2r^{-2} + br^{-3}$ and describes an idea of approaching the problem of four bodies by replacing one of the bodies with a material ring (Section 5.6.2.2). Dusek generalizes the concept of libration points for the case when the third body experiences a low thrust.

The last part [Part (D)] of Section 10.5.2 discusses a group of problems reducible to the restricted problem following Birkhoff [103] and Wintner [104], and it offers the idea of inverse problems in dynamics. Aspects of this problem are also treated by Dainelli [105] and Whittaker [85, p. 95] under general conditions. A major review of and significant contributions to these problems was made by Lovett [96, p. 274]. Specific applications to the restricted problem are by Szebehely [106].

10.7 References

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