II Kinematics

Displacement:

$\mathbf{u}(\mathrm{X},t) = \chi(\mathrm{X},t) - \mathrm{X}, \quad u_i\left(X_1, X_2, X_3\right) = \chi_i\left(X_1, X_2, X_3\right) - X_i$

Velocity/Acceleration

$\dot{\mathbf{u}}(\mathbf{X},t) = \dfrac{\partial \chi(\mathbf{X},t)}{\partial t}(3.8.3)$

$\ddot{\mathbf{u}}(\mathbf{X},t) = \dfrac{\partial^2 \chi(\mathbf{X},t)}{\partial t^2}(3.8.4)$

3.2 Deformation/Displacement Gradient

$$\mathbf{F}(\mathbf{X},t) = \frac{\partial}{\partial \mathbf{X}}\chi(\mathbf{X},t), \quad F_{ij} = \frac{\partial}{\partial X_j}\chi_i\left(X_1, X_2, X_3, t\right), \quad \det \mathbf{F}(\mathbf{X},t) > 0$$

$$\mathbf{H}(\mathbf{X},t) = \frac{\partial}{\partial \mathbf{X}}\mathbf{u}(\mathbf{X},t), \quad H_{ij} = \frac{\partial}{\partial X_j}u_i\left(X_1, X_2, X_3, t\right)$$

$$\mathbf{H}(\mathbf{X},t) = \mathbf{F}(\mathbf{X},t) - 1, \quad H_{ij} = F_{ij} - \delta_{ij}$$

$$J \equiv \det\left(\frac{\partial \chi}{\partial \mathbf{X}}\right) = \det \mathbf{F} = \frac{dv}{dv_{\mathrm{R}}} \neq 0$$

3.3 Stretch & Rotation

Polar Decomposition: $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$

$$\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \quad C_{ij} = F_{ki}F_{kj} = \tfrac{\partial \chi_k}{\partial X_i}\tfrac{\partial \chi_k}{\partial X_j}$$
$$\mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^{\top}, \quad B_{ij} = F_{ik}F_{jk} = \tfrac{\partial \chi_i}{\partial X_k}\tfrac{\partial \chi_j}{\partial X_k}$$

$\lambda \overset{\text{def}}{=} \frac{ds}{dS} = |\mathbf{U}\mathbf{e}| = \sqrt{\mathbf{e}\cdot\mathbf{C}(\mathbf{X})\mathbf{e}}$

where $dS = |\mathrm{dX}|, ds = |\mathrm{dx}|, \mathrm{e} = \frac{d\mathrm{X}}{|d\mathrm{X}|}$

Engineering shear: $\gamma = \sin^{-1}\left[\frac{\mathbf{e}^{(1)}\cdot\mathbf{C}\mathbf{e}^{(2)}}{\lambda\left(\mathbf{e}^{(1)}\right)\lambda\left(\mathbf{e}^{(2)}\right)}\right]$

3.4 Strain

Green strain: $\mathrm{E} \overset{\text{def}}{=} \frac{1}{2}\left(\mathrm{F}^{\top}\mathrm{F} - 1\right) = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^{\top} + \mathbf{H}^{\top}\mathbf{H}.\right)$

Hencky's Log strain: $\ln \mathbf{U} \overset{\text{def}}{=} \sum_{i=1}^{3}\left(\ln \lambda_i\right)\mathbf{r}_i \otimes \mathbf{r}_i$

3.5.2 Infinitesimal Strain

ϵ': distortion $\epsilon_M \delta_{ij}$: dilation

$$\epsilon = \frac{1}{2}\left[\mathbf{H} + \mathbf{H}^\top\right], \qquad \epsilon = \epsilon^\top, \quad |\mathbf{H}| \ll 1$$

$$\epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right], \quad \epsilon_{ji} = \epsilon_{ij}, \left|\frac{\partial u_i}{\partial X_j}\right| \ll 1$$

3.A Linearization

$\operatorname{lin} \mathrm{Y}_o f(\mathbf{Y}) = f(\mathbf{Y}_o) + \frac{d}{d\alpha} f(\mathbf{Y}_o + \alpha(\mathbf{Y} - \mathbf{Y}_o))\big|_{\alpha=0}$ $\operatorname{lin}_0 f(\mathbf{H}) = f(0) + \frac{d}{d\alpha} f(\alpha \mathbf{H})\big|_{\alpha=0}$

3.B Compatibility