XXX

Document classes

`book`	Default is two-sided.
`report`	No `\part` divisions.
`article`	No `\part` or `\chapter` divisions.
`letter`	Letter (?).
`slides`	Large sans-serif font.

Used at the very beginning of a document: `\documentclass{`*class*`}`. Use `\begin{document}` to start contents and `\end{document}` to end the document.

Calculus

Fundamentals

$\frac{d}{dx}\int_a^x f(t)dt = f(x)$

Differentiation

In the case of a vector field, the directional derivative is also a vector each of whose components gives the rate of change of the corresponding component of v in the direction of h. The gradient in this case will be a tensor field (that when applied to h gives the directional derivative of v in the direction of h).

$$\mathbf{e}_i = \frac{\partial}{\partial x^i} = \partial_i, \quad i = 1,\, 2,\, \dots,\, n$$

define what is referred to as the local [[basis of a vector space|basis]] of the tangent space to {{math|"M"}} at each point of its domain. These can be used to define the [[metric tensor]]:

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$$

and its inverse:

$$g^{ij} = \left(g^{-1}\right)_{ij}$$

which can in turn be used to define the dual basis:

$$\mathbf{e}^i = \mathbf{e}_j g^{ji}, \quad i = 1,\, 2,\, \dots,\, n$$

Some texts write $\mathbf{g}_i$ for $\mathbf{e}_i$, so that the metric tensor takes the particularly beguiling form $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$. This convention also leaves use of the symbol e_i unambiguously for the [[vierbein]].

Gradient

$$\mathbf{e}_i \cdot \operatorname{grad}\varphi(\mathbf{x}) = [\operatorname{grad}\varphi(\mathbf{x})]_i = \tfrac{\partial \varphi(\mathbf{x})}{\partial x_i}$$
$$\mathbf{e}_i \cdot \operatorname{grad}\mathbf{v}(\mathbf{x})\mathbf{e}_j = [\operatorname{grad}\mathbf{v}(\mathbf{x})]_{ij} = \tfrac{\partial v_i(\mathbf{x})}{\partial x_j}$$

Jacobian In general, the *derivative* of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ at a point $p \in \mathbb{R}^n$, if it exists, is the unique linear transformation $Df(p) \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{h\to 0} \frac{\|f(p+h) - f(p) - Df(p)h\|}{\|h\|} = 0;$$

the matrix of $Df(p)$ with respect to the standard orthonormal bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, called the *Jacobian matrix* of f at p, therefore lies in $M_{m\times n}(\mathbb{R})$.

Now, suppose that $m = 1$, so that $f : \mathbb{R}^n \to \mathbb{R}$. Then if f is differentiable at p, $Df(p) \in L(\mathbb{R}^n, \mathbb{R}) = (\mathbb{R}^n)^*$ is a functional, and hence the Jacobian matrix, as you point out, lies in $M_{1\times n}(\mathbb{R})$, i.e., is a row vector. However, by the Riesz representation theorem, $\mathbb{R}^n \cong (\mathbb{R}^n)^*$ via the map that sends a vector $x \in \mathbb{R}^n$ to the functional $y \mapsto \langle y, x\rangle$. Hence, if f is differentiable at p, then the *gradient* of f at p is the unique (column!) vector $\nabla f(p) \in \mathbb{R}^n$ such that

$$\forall h \in \mathbb{R}^n, \quad Df(p)h = \langle \nabla f(p), h\rangle\,;$$

in particular, if you unpack definitions, you'll find that the Jacobian matrix of f at p is precisely $\nabla f(p)^T$.

The Jacobian determinant can be viewed as the ratio of an infinitesimal change in the variables of one coordinate system to another. This requires that the function $f(x)$ maps $\mathbb{R}^n$ß$\mathbb{R}^n$, which produces an nÖn square matrix for the Jacobian. For example:

$$\iiint_R f(x,y,z)dxdydz = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$$

Directional Derivative

$$\operatorname{grad}\varphi(\mathbf{x})[\mathbf{h}] = \left.\frac{d}{d\alpha}\varphi(\mathbf{x}+\alpha\mathbf{h})\right|_{\alpha=0}$$

Divergence - Curl- Laplacian

$$\operatorname{div}\mathbf{v} = \operatorname{tr}[\operatorname{grad}\mathbf{v}] = \frac{\partial v_i}{\partial x_i}$$
$$(\operatorname{div}\mathbf{T})_i = \frac{\partial T_{ij}}{\partial x_j}$$
$$(\operatorname{curl}\mathbf{v})_i = e_{ijk}\frac{\partial v_k}{\partial x_j}$$
$$(\operatorname{curl}\mathbf{T})_{ij} = e_{ipq}\frac{\partial T_{jq}}{\partial x_p}$$

$$\Delta\mathbf{v} = \operatorname{div}\operatorname{grad}\mathbf{v}, \quad \Delta v_i = \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

$$\Delta T_{ij} = \frac{\partial T_{ij}}{\partial x_k \partial x_k}$$

Integration

Integration by parts

If $u = u(x)$ and $du = u'(x)dx$, while $v = v(x)$ and $dv = v'(x)dx$, then integration by parts states that:

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx$$
$$= u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)dx$$

Divergence Theorem

$$\begin{aligned} \int \varphi n_i da &= \int_R \tfrac{\partial\varphi}{\partial x_i} dv \\ \int_{\partial R} v_i n_i da &= \int_R \tfrac{\partial v_i}{\partial x_i} dv \\ \int_{\partial R} T_{ij} n_j da &= \int_R \tfrac{\partial T_{ij}}{\partial x_j} dv \\ \partial R \quad & \quad R \end{aligned}$$

II Kinematics

Displacement:

$$\mathrm{u}(\mathrm{X},t) = \chi(\mathrm{X},t) - \mathrm{X}, \quad u_i\left(X_1,X_2,X_3\right) = \chi_i\left(X_1,X_2,X_3\right) - X_i$$

Velocity/Acceleration

$$\dot{\mathbf{u}}(\mathbf{X},t) = \frac{\partial\chi(\mathbf{X},t)}{\partial t} (3.8.3)$$

$$\ddot{\mathbf{u}}(\mathbf{X},t) = \frac{\partial^2\chi(\mathbf{X},t)}{\partial t^2} (3.8.4)$$

3.2 Deformation/Displacement Gradient

$$\mathbf{F}(\mathbf{X},t) = \frac{\partial}{\partial\mathbf{X}}\chi(\mathbf{X},t), \quad F_{ij} = \frac{\partial}{\partial X_j}\chi_i\left(X_1,X_2,X_3,t\right), \quad \det\mathbf{F}(\mathbf{X},t) > 0$$

$$\mathbf{H}(\mathbf{X},t) = \frac{\partial}{\partial\mathbf{X}}\mathbf{u}(\mathbf{X},t), \quad H_{ij} = \frac{\partial}{\partial X_j}u_i\left(X_1,X_2,X_3,t\right)$$

$$\mathbf{H}(\mathbf{X},t) = \mathbf{F}(\mathbf{X},t) - 1, \quad H_{ij} = F_{ij} - \delta_{ij}$$

$$J \equiv \det\left(\frac{\partial\chi}{\partial\mathbf{X}}\right) = \det\mathbf{F} = \frac{dv}{dv_{\mathrm{R}}} \neq 0$$

3.3 Stretch & Rotation

Polar Decomposition: $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$

$$\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \quad C_{ij} = F_{ki}F_{kj} = \tfrac{\partial\chi_k}{\partial X_i}\tfrac{\partial\chi_k}{\partial X_j}$$
$$\mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^{\top}, \quad B_{ij} = F_{ik}F_{jk} = \tfrac{\partial\chi_i}{\partial X_k}\tfrac{\partial\chi_j}{\partial X_k}$$

$$\lambda \stackrel{\text{def}}{=} \tfrac{ds}{dS} = |\mathbf{Ue}| = \sqrt{\mathbf{e}\cdot\mathbf{C}(\mathbf{X})\mathbf{e}}$$

where $dS = |\mathrm{dX}|, ds = |\mathrm{dx}|, \mathrm{e} = \frac{d\mathrm{X}}{|d\mathrm{X}|}$

Engineering shear: $\gamma = \sin^{-1}\left[\frac{\mathbf{e}^{(1)}\cdot\mathbf{Ce}^{(2)}}{\lambda\left(\mathbf{e}^{(1)}\right)\lambda\left(\mathbf{e}^{(2)}\right)}\right]$

3.4 Strain

Green strain: $\mathrm{E} \stackrel{\text{def}}{=} \frac{1}{2}\left(\mathrm{F}^{\top}\mathrm{F} - 1\right) = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^{\top} + \mathbf{H}^{\top}\mathbf{H}.\right)$

Hencky's Log strain: $\ln\mathbf{U} \stackrel{\text{def}}{=} \sum_{i=1}^{3}\left(\ln\lambda_i\right)\mathbf{r}_i\otimes\mathbf{r}_i$

3.5.2 Infinitesimal Strain

ϵ': distortion $\epsilon_M \delta_{ij}$: dilation

$$\epsilon = \frac{1}{2}\left[\mathbf{H} + \mathbf{H}^\top\right], \qquad \epsilon = \epsilon^\top, \quad |\mathbf{H}| \ll 1$$

$$\epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right], \quad \epsilon_{ji} = \epsilon_{ij}, \left|\frac{\partial u_i}{\partial X_j}\right| \ll 1$$

3.A Linearization

$$\operatorname{lin} \mathrm{Y}_o f(\mathbf{Y}) = f(\mathbf{Y}_o) + \left.\tfrac{d}{d\alpha} f(\mathbf{Y}_o + \alpha(\mathbf{Y} - \mathbf{Y}_o))\right|_{\alpha=0}$$
$$\operatorname{lin}_0 f(\mathbf{H}) = f(0) + \left.\tfrac{d}{d\alpha} f(\alpha\mathbf{H})\right|_{\alpha=0}$$

3.B Compatibility