# **IV Linear Elasticity**

#### Compatibility:

$$\operatorname{curl}(\operatorname{curl} \epsilon) = 0, \quad e_{ipq} e_{jrs} \epsilon_{qs,rp} = 0$$

# 7 Linear Constitutive Equations

#### 7.1 Free Energy: Elasticity

$$\sigma = \mathbb{C}\epsilon, \quad \sigma_{ij} = C_{ijkl}\epsilon_{kl}$$

Voight Notation:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{112} \\ C_{211} & C_{222} & C_{233} & C_{223} & C_{2213} & C_{2212} \\ C_{331} & C_{332} & C_{333} & C_{333} & C_{313} & C_{2312} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{pmatrix}$$

#### 7.5 Isotropic Relations

$$oldsymbol{\sigma} = \mathbb{C}oldsymbol{\epsilon} = 2\muoldsymbol{\epsilon} + \lambda(\operatorname{tr}oldsymbol{\epsilon})\mathbf{1}$$

$$oldsymbol{\sigma} = 2\muoldsymbol{\epsilon}' + \kappa(\mathrm{tr}\,oldsymbol{\epsilon})\mathbf{1}, \quad \sigma_{ij} = 2\muoldsymbol{\epsilon}'_{ij} + \kappa\left(oldsymbol{\epsilon}_{kk}
ight)\delta_{ij}$$

$$\mathbb{C} = 2\mu^{ ext{sym}} + \lambda \mathbf{1} \otimes \mathbf{1}, \quad C_{ijkl} = \mu \left( \delta_{ik} \delta_{il} + \delta_{il} \delta_{jk} \right) + \lambda \delta_{ij} \delta_{kl}$$

$$\epsilon = rac{1}{2\mu}\sigma' + rac{1}{9\kappa}(\mathrm{tr}\,\sigma)\mathbf{1}, \quad \epsilon_{ij} = rac{1}{2\mu}\sigma'_{ij} + rac{1}{9\kappa}\left(\sigma_{kk}
ight)\delta_{ij}$$

$$E \equiv rac{9\kappa\mu}{3\kappa+\mu}, \quad 
u \equiv rac{1}{2} \left[rac{3\kappa-2\mu}{3\kappa+\mu}
ight]$$

$$\kappa = \lambda + \frac{2}{3}\mu$$

#### 7.5.4

$$\sigma = \frac{E}{(1+\nu)} \left[ \epsilon + \frac{\nu}{(1-2\nu)} (\operatorname{tr} \epsilon) 1 \right], \quad \sigma_{ij} = \frac{E}{(1+\nu)} \left[ \epsilon_{ij} + \frac{\nu}{(1-2\nu)} (\epsilon_{kk}) \, \delta_{ij} \right]$$
$$\epsilon = \frac{1}{E} [(1+\nu)\sigma - \nu (\operatorname{tr} \sigma) 1]. \quad \epsilon_{ij} = \frac{1}{E} [(1+\nu)\sigma_{ij} - \nu (\sigma_{kk}) \, \delta_{ij}]$$

# 8 Elastostatics

Displacement Formulation (Navier)

$$C_{ijkl}u_{k,lj} + b_i = 0$$

### Isotropic

$$\mu \triangle \mathbf{u} + (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \mathbf{b} = 0$$
  
$$\mu u_{i,j} + (\lambda + \mu) u_{i,j} + b_i = 0$$

$$(\lambda + 2\mu)\nabla \operatorname{div} \mathbf{u} - \mu \operatorname{curl} \operatorname{curl} \mathbf{u} + \mathbf{b} = \mathbf{0}$$
  
 $(\lambda + 2\mu)u_{j,ji} - \mu e_{ijk}e_{klm}u_{m,lj} + b_i = 0$ 

#### Boundary:

$$\begin{array}{c} \mathbf{u} = \hat{\mathbf{u}} \; \mathrm{on} \; \mathcal{S}_1, \\ \left( \mu \left( \nabla \mathbf{u} + (\nabla \mathbf{u})^\top \right) + \lambda (\mathrm{div} \, \mathbf{u}) \mathbf{1} \right) \mathbf{n} = \hat{\mathbf{t}} \quad \mathrm{on} \; \mathcal{S}_2 \end{array} \right\}$$

Stress Formulation (Beltrami-Mitchell)

#### Compatibility:

$$\sigma_{ij,kk}+rac{1}{1+
u}\sigma_{kk,ij}=-rac{
u}{1-
u}b_{k,k}\delta_{ij}-b_{i,j}-b_{j,i} \ \Delta\sigma_{kk}=-rac{1+
u}{1-
u}b_{k,k}$$

# 8.9.1 Plane Strain

$$u_{lpha}=u_{lpha}\left(x_{1},x_{2}
ight),\quad u_{3}=0$$

$$\epsilon_{\alpha\beta} = \frac{1}{2} \left( u_{\alpha,\beta} + u_{\beta,\alpha} \right)$$

$$\epsilon_{13} = \epsilon_{23} = \epsilon_{33} = 0$$

$$\sigma_{lphaeta}=rac{E}{(1+
u)}\left(\epsilon_{lphaeta}+rac{
u}{(1-2
u)}\left(\epsilon_{\gamma\gamma}
ight)\delta_{lphaeta}
ight)$$

$$\sigma_{33} = \nu \sigma_{\alpha\alpha}$$

$$\epsilon_{lphaeta} = rac{1+
u}{F} \left(\sigma_{lphaeta} - 
u \left(\sigma_{\gamma\gamma}
ight) \delta_{lphaeta}
ight)$$

$$\sigma_{\alpha\beta,\beta} + b_{\alpha} = 0$$

#### Plane Stress

$$\sigma_{lphaeta}=\sigma_{lphaeta}\left(x_{1},x_{2}
ight),\quad\sigma_{33}=\sigma_{13}=\sigma_{23}=0$$

#### Plane Stress/Strain

$$\epsilon_{13}=\epsilon_{23}=\sigma_{13}=\sigma_{23}=0$$

	Plane Stress	Plane Strain
	$\sigma_{lphaeta}=\sigma_{lphaeta}\left(x_{1},x_{2} ight)$	$u_{lpha}=u_{lpha}\left(x_{1},x_{2} ight) \ u_{3}=0$
S	$\frac{1}{1+\nu}$	1- u
$\sigma_{33}$	0	$ u\sigma_{lphalpha}$
$\epsilon_{33}$	$-rac{ u}{1- u}\epsilon_{lphalpha}$	0

$$\text{Navier}: \left\{ \begin{array}{l} \left(\frac{E}{2(1+\nu)}\right) u_{\alpha,\beta\beta} + \left(\frac{E}{2(1+\nu)(1-2\nu)}\right) u_{\beta,\beta\alpha} + b_\alpha = 0 & \text{ for plane strain} \\ \left(\frac{E}{2(1+\nu)}\right) u_{\alpha,\beta\beta} + \frac{E}{2(1-\nu)} u_{\beta,\beta\alpha} + b_\alpha = 0 & \text{ for plane stress} \end{array} \right.$$

#### **Constitutive Relation**

$$egin{aligned} \sigma_{lphaeta} &= rac{E}{\left(1+
u
ight)} \left(\epsilon_{lphaeta} + \left(rac{1-s}{2s-1}
ight) \left(\epsilon_{\gamma\gamma}
ight) \delta_{lphaeta}
ight) \ \epsilon_{lphaeta} &= rac{\left(1+
u
ight)}{F} \left(\sigma_{lphaeta} - \left(1-s
ight) \left(\sigma_{\gamma\gamma}
ight) \delta_{lphaeta} 
ight) \end{aligned}$$

#### Equilibrium

$$\sigma_{lphaeta,eta}+b_lpha=0$$

### Compatibility

$$\Delta\left(\sigma_{lphalpha}
ight)=\left(\sigma_{11}+\sigma_{22}
ight),_{11}+\left(\sigma_{11}+\sigma_{22}
ight),_{22}=-rac{1}{s}b_{lpha,lpha}$$

# Airy Stress Function

$$\sigma_{11} = arphi_{,22} \,, \quad \sigma_{22} = arphi_{,11}, \quad \sigma_{12} = -arphi_{,12}$$

#### Compatibility

$$\Delta\Delta\varphi=\varphi,1111+2\varphi,1122+\varphi,2222=0$$

#### Displacements

$$\begin{array}{l} u_1 = \frac{(1+\nu)}{E} \left(-\varphi, 1+s\psi, 2\right) + w_1 \\ u_2 = \frac{(1+\nu)}{E} \left(-\varphi, 2+s\psi,_1\right) + w_2 \\ \text{where: } \Delta\psi = 0 \text{ and } \psi,_{12} = \Delta\varphi \end{array}$$

and w is a plane rigid displacement:

$$w_{1,1}=0, \quad w_{2,2}=0, \quad w_{1,2}+w_{2,1}=0$$

$$\begin{split} & \textbf{Polar form } (9.4.16): \\ & \sigma_{rr} = \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2} \\ & \sigma_{\theta \theta} = \frac{\partial^2 \varphi}{\partial r^2} \\ & \sigma_{r\theta} = -\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \right) \end{split}$$

$$\sigma_{ heta heta} = rac{\partial^2 arphi}{\partial r^2}$$

$$\sigma_{r heta} = -rac{\partial}{\partial r} \left(rac{1}{r}rac{\partialarphi}{\partial heta}
ight)$$

# Torsion

$$egin{aligned} u_1(\mathrm{x}) &pprox - lpha x_2 x_3 \ u_2(\mathrm{x}) &pprox lpha x_1 x_3 \ u_3(\mathbf{x}) &= lpha arphi \left( x_1, x_2 
ight) \end{aligned}$$

$$\alpha = \frac{T}{\mu J}$$

$$T=\int_{{\mathcal S}_L} \left(x_1\sigma_{23}-x_2\sigma_{13}
ight)da$$

$$ar{J} \stackrel{ ext{def}}{=} \int_{\Omega} \left( x_1^2 + x_2^2 + x_1 arphi_{,2} - x_2 arphi_{,1} 
ight) da$$

### **Displacement Formulations**

$$\begin{split} \epsilon_{11}(\mathbf{x}) &= \epsilon_{22}(\mathbf{x}) = \epsilon_{33}(\mathbf{x}) = \epsilon_{12}(\mathbf{x}) = 0 \\ \epsilon_{13}(\mathbf{x}) &= \frac{1}{2} \left( \frac{\partial \varphi}{\partial x_1} - x_2 \right) \alpha \\ \epsilon_{23}(\mathbf{x}) &= \frac{1}{2} \left( \frac{\partial \varphi}{\partial x_2} + x_1 \right) \alpha \end{split}$$

$$egin{aligned} \sigma_{11}(\mathrm{x}) &= \sigma_{22}(\mathrm{x}) = \sigma_{33}(\mathrm{x}) = \sigma_{12}(\mathrm{x}) = 0 \ \sigma_{13}(\mathrm{x}) &= \mu lpha \left( rac{\partial arphi}{\partial x_1} - x_2 
ight) \ \sigma_{23}(\mathrm{x}) &= \mu lpha \left( rac{\partial arphi}{\partial x_2} + x_1 
ight) \end{aligned}$$

#### Equilibrium:

$$\sigma_{13,1}+\sigma_{23,2}=0\quad ext{ in }\Omega$$

#### Boundary:

$$egin{aligned} \Delta arphi &= 0 \quad ext{in } \Omega \ rac{\partial arphi}{\partial n} &= x_2 n_1 - x_1 n_2 \quad ext{ on } \Gamma \end{aligned}$$

Stress Formulation

# Compatibility:

$$\begin{array}{ll} \epsilon_{13,2}-\epsilon_{23,1}=-\alpha & \text{in } \Omega \\ \Delta\Psi=\Psi_{,11}+\Psi_{,22}=-2\mu\alpha & \text{in } \Omega \text{ subject to } & \Psi=0 & \text{on } \Gamma \\ \text{for } \sigma_{13}=\frac{\partial\Psi}{\partial x_2}, & \sigma_{23}=-\frac{\partial\Psi}{\partial x_1} \\ \end{array}$$

# 9 Solutions

Crack Tip

# Mode III

$$\Delta u_z = 0 = rac{\partial^2 u_z}{\partial r^2} + rac{1}{r^2} rac{\partial^2 u_z}{\partial heta^2} + rac{1}{r} rac{\partial u_z}{\partial r}$$