$u_1(\mathrm{x}) \approx -\alpha x_2 x_3$
$u_2(\mathrm{x}) \approx \alpha x_1 x_3$
$u_3(\mathbf{x}) = \alpha\varphi(x_1, x_2)$

$\alpha = \frac{T}{\mu J}$

$T = \int_{S_L} (x_1\sigma_{23} - x_2\sigma_{13})\, da$

$\bar{J} \stackrel{\text{def}}{=} \int_\Omega \left(x_1^2 + x_2^2 + x_1\varphi_{,2} - x_2\varphi_{,1}\right) da$

For open sections:

$$\mu\bar{J} = \mu(b_1 + b_2 + b_3)\, t^3/3$$

Displacement Formulations

$\epsilon_{11}(\mathrm{x}) = \epsilon_{22}(\mathrm{x}) = \epsilon_{33}(\mathrm{x}) = \epsilon_{12}(\mathrm{x}) = 0$
$\epsilon_{13}(\mathrm{x}) = \frac{1}{2}\left(\frac{\partial\varphi}{\partial x_1} - x_2\right)\alpha$
$\epsilon_{23}(\mathrm{x}) = \frac{1}{2}\left(\frac{\partial\varphi}{\partial x_2} + x_1\right)\alpha$

$\sigma_{11}(\mathrm{x}) = \sigma_{22}(\mathrm{x}) = \sigma_{33}(\mathrm{x}) = \sigma_{12}(\mathrm{x}) = 0$
$\sigma_{13}(\mathrm{x}) = \mu\alpha\left(\frac{\partial\varphi}{\partial x_1} - x_2\right)$
$\sigma_{23}(\mathrm{x}) = \mu\alpha\left(\frac{\partial\varphi}{\partial x_2} + x_1\right)$

Equilibrium:

$\sigma_{13,1} + \sigma_{23,2} = 0 \quad \text{in } \Omega$

Boundary:

$\Delta\varphi = 0 \quad \text{in } \Omega$
$\frac{\partial\varphi}{\partial n} = x_2 n_1 - x_1 n_2 \quad \text{on } \Gamma$

Stress Formulation

Compatibility:

$\epsilon_{13,2} - \epsilon_{23,1} = -\alpha \quad \text{in } \Omega$
$\Delta\Psi = \Psi_{,11} + \Psi_{,22} = -2\mu\alpha \quad \text{in } \Omega \text{ subject to } \quad \Psi = 0 \quad \text{on } \Gamma$
for $\sigma_{13} = \frac{\partial\Psi}{\partial x_2}, \quad \sigma_{23} = -\frac{\partial\Psi}{\partial x_1}$

$T = 2\int_\Omega \Psi da$

9 Solutions

Crack Tip

Mode III: Anti-plane tearing

$\Delta u_z = 0 = \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial\theta^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}$

$\delta_3 \stackrel{\text{def}}{=} u_3(r, +\pi) - u_3(r, -\pi) = \frac{4}{\mu} K_{\mathrm{III}} \sqrt{\frac{r}{2\pi}}$

$\begin{pmatrix} \sigma_{13} \\ \sigma_{23} \end{pmatrix} = \frac{K_{\mathrm{III}}}{\sqrt{2\pi r}} \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix} + \text{ bounded terms}$

$\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{12} = 0$

$(u_3) = \frac{K_{\mathrm{M}}}{2\mu}\sqrt{\frac{r}{2\pi}}\left(4\sin\left(\frac{\theta}{2}\right)\right) +$ rigid displacenent.