# Appendix A

# Cylindrical and Spherical coordinate systems

## A.1 Introduction

The general equations of continuum mechanics are best expressed in either the direct notation, or in rectangular Cartesian coordinates. However, for the solution of particular problems it is often preferable to employ other coordinate systems. The most commonly encountered coordinate systems are

1. The **cylindrical coordinate** system which is good for solids that are symmetric around an axis.

2. The **spherical coordinate** system which is used when there is symmetry about a point.

In what follows we derive some important vector and tensor identities which will help us solve problems formulated in these coordinate systems.

## A.2 Cylindrical coordinates

Cylindrical coordinates $(r, \theta, z)$ are related to rectangular coordinates $(x_1, x_2, x_3)$ by

$$\begin{aligned} r &= [x_1^2 + x_2^2]^{\frac{1}{2}}, \quad r \geq 0, \\ \theta &= \tan^{-1}(x_2/x_1), \quad 0 \leq \theta \leq 2\pi, \\ z &= x_3, \quad -\infty < z < \infty, \end{aligned} \tag{A.2.1}$$

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad x_3 = z. \tag{A.2.2}$$

The orthonormal base vectors in the cylindrical coordinate system are directed in the radial, tangential and axial directions as illustrated in Figure A.1, and denoted by $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$.

The position vector $\mathbf{r} = \mathbf{x} - \mathbf{o}$ is given by

$$\mathbf{r} = r\cos\theta\mathbf{e}_1 + r\sin\theta\,\mathbf{e}_2 + z\,\mathbf{e}_3, \tag{A.2.3}$$

and with respect to a cylindrical coordinate system a differential element $\mathrm{d}\mathbf{x}$ at $\mathbf{x}$ is given by

$$\mathrm{d}\mathbf{x} = \mathrm{d}r\,\mathbf{e}_r + (r\mathrm{d}\theta)\,\mathbf{e}_\theta + \mathrm{d}z\,\mathbf{e}_z. \tag{A.2.4}$$

Thus, since $\mathrm{d}\mathbf{r} = \mathrm{d}\mathbf{x}$,

$$\mathbf{e}_r = \frac{\partial\mathbf{r}}{\partial r}, \quad \mathbf{e}_\theta = \frac{1}{r}\frac{\partial\mathbf{r}}{\partial\theta}, \quad \mathbf{e}_z = \frac{\partial\mathbf{r}}{\partial z}.$$

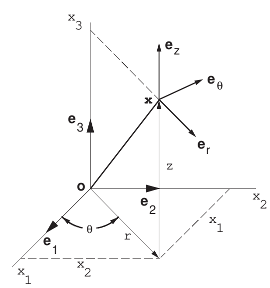


Figure A.1: Cylindrical coordinate system.

Using this and (A.2.3) we obtain that $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ are related to the orthonormal base vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in the rectangular system by

$$\begin{aligned} \mathbf{e}_r &= \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2, \\ \mathbf{e}_\theta &= -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2, \\ \mathbf{e}_z &= \mathbf{e}_3. \end{aligned} \tag{A.2.5}$$

From (A.2.5),

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta, \qquad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r, \qquad \frac{\partial \mathbf{e}_z}{\partial \theta} = \mathbf{0}. \tag{A.2.6}$$

**Components of vectors and tensors**

Let $\mathbf{v}$ be a vector with components

$$v_r = \mathbf{v}\cdot\mathbf{e}_r, \qquad v_\theta = \mathbf{v}\cdot\mathbf{e}_\theta, \qquad v_z = \mathbf{v}\cdot\mathbf{e}_z, \tag{A.2.7}$$

with respect to $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$, then

$$\mathbf{v} = v_r\,\mathbf{e}_r + v_\theta\,\mathbf{e}_\theta + v_z\,\mathbf{e}_z. \tag{A.2.8}$$

Recall that a tensor $\mathbf{S}$ is a linear map that assigns to each vector $\mathbf{u}$ a vector $\mathbf{v}$ according to

$$\mathbf{v} = \mathbf{S}\mathbf{u}.$$

Also the **tensor product** of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as the tensor $\mathbf{a}\otimes\mathbf{b}$ that assigns to each vector $\mathbf{w}$ the vector $(\mathbf{b}\cdot\mathbf{w})\,\mathbf{a}$ according to

$$(\mathbf{a}\otimes\mathbf{b})\,\mathbf{w} = (\mathbf{b}\cdot\mathbf{w})\,\mathbf{a}.$$

The components of a tensor $\mathbf{S}$ in a cylindrical coordinate system are defined by

$$\begin{aligned} S_{rr} &= \mathbf{e}_r\cdot\mathbf{S}\mathbf{e}_r, \quad & S_{r\theta} &= \mathbf{e}_r\cdot\mathbf{S}\mathbf{e}_\theta, \quad & S_{rz} &= \mathbf{e}_r\cdot\mathbf{S}\mathbf{e}_z, \\ S_{\theta r} &= \mathbf{e}_\theta\cdot\mathbf{S}\mathbf{e}_r, \quad & S_{\theta\theta} &= \mathbf{e}_\theta\cdot\mathbf{S}\mathbf{e}_\theta, \quad & S_{\theta z} &= \mathbf{e}_\theta\cdot\mathbf{S}\mathbf{e}_z, \\ S_{zr} &= \mathbf{e}_z\cdot\mathbf{S}\mathbf{e}_r, \quad & S_{z\theta} &= \mathbf{e}_z\cdot\mathbf{S}\mathbf{e}_\theta, \quad & S_{zz} &= \mathbf{e}_z\cdot\mathbf{S}\mathbf{e}_z. \end{aligned}$$

Hence

$$\begin{aligned}\mathbf{S} = &S_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + S_{r\theta}\mathbf{e}_r \otimes \mathbf{e}_\theta + S_{rz}\mathbf{e}_r \otimes \mathbf{e}_z + \\ &S_{\theta r}\mathbf{e}_\theta \otimes \mathbf{e}_r + S_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + S_{\theta z}\mathbf{e}_\theta \otimes \mathbf{e}_z + \\ &S_{zr}\mathbf{e}_z \otimes \mathbf{e}_r + S_{z\theta}\mathbf{e}_z \otimes \mathbf{e}_\theta + S_{zz}\mathbf{e}_z \otimes \mathbf{e}_z,\end{aligned} \tag{A.2.9}$$

and the matrix of the components of $\mathbf{S}$ with respect to $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is

$$[\mathbf{S}] = \begin{bmatrix} S_{rr} & S_{r\theta} & S_{rz} \\ S_{\theta r} & S_{\theta\theta} & S_{\theta z} \\ S_{zr} & S_{z\theta} & S_{zz} \end{bmatrix}.$$

Next, we consider some vector and tensor identities in cylindrical coordinates.

**Gradient of a scalar field**

We start with the gradient of a scalar field $\psi$, which is a vector quantity denoted by $\nabla\psi$. Recall that the derivative of $\psi$ in a direction $\mathbf{u}$ at a point $\mathbf{x}$ is

$$\nabla\psi(\mathbf{x}) \cdot \mathbf{u} = \lim_{\Delta s \to 0} \frac{\psi(\mathbf{x} + \Delta s\, \mathbf{u}) - \psi(\mathbf{x})}{\Delta s},$$

where $\Delta s$ is the arc length in the direction $\mathbf{u}$. That is, the scalar product of $\nabla\psi$ with $\mathbf{u}$ gives the rate of change of $\psi$ in that direction. Choosing $\mathbf{u}$ to be successively along the $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ directions, and recalling that the arc lengths in these directions are $\Delta r$, $r\Delta\theta$, and $\Delta z$, respectively, we obtain

$$\nabla\psi \cdot \mathbf{e}_r = \frac{\partial\psi}{\partial r}, \quad \nabla\psi \cdot \mathbf{e}_\theta = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \quad \nabla\psi \cdot \mathbf{e}_z = \frac{\partial\psi}{\partial z}.$$

Hence,

$$\boxed{\nabla\psi = \frac{\partial\psi}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\,\mathbf{e}_\theta + \frac{\partial\psi}{\partial z}\mathbf{e}_z.} \tag{A.2.10}$$

This can be written in operational form as

$$\nabla\psi = \left(\mathbf{e}_r\,\frac{\partial}{\partial r} + \mathbf{e}_\theta\,\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z}\right)\psi,$$

where the vector[1]

$$\nabla = \left(\mathbf{e}_r\,\frac{\partial}{\partial r} + \mathbf{e}_\theta\,\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z}\right) \tag{A.2.11}$$

denotes the **del operator** in cylindrical coordinates.

**Gradient of a vector field**

The gradient of $\mathbf{v}$, denoted by $\nabla\mathbf{v}$ is a second order tensor defined by[2]

$$\nabla\mathbf{v} = [\nabla \otimes \mathbf{v}]^\top. \tag{A.2.12}$$

Hence, using ((A.2.6), (A.2.11), and (A.2.8),

$$\begin{aligned}\nabla\mathbf{v} &= \left[\left(\mathbf{e}_r\,\frac{\partial}{\partial r} + \mathbf{e}_\theta\,\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z}\right) \otimes (v_r\,\mathbf{e}_r + v_\theta\,\mathbf{e}_\theta + v_z\,\mathbf{e}_z)\right]^\top \\ &= \left[\frac{\partial v_r}{\partial r}\mathbf{e}_r \otimes \mathbf{e}_r + \frac{\partial v_\theta}{\partial r}\mathbf{e}_r \otimes \mathbf{e}_\theta + \frac{\partial v_z}{\partial r}\mathbf{e}_r \otimes \mathbf{e}_z + \right. \\ &\quad \frac{1}{r}\frac{\partial v_r}{\partial\theta}\mathbf{e}_\theta \otimes \mathbf{e}_r + \frac{v_r}{r}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta - \frac{v_\theta}{r}\mathbf{e}_\theta \otimes \mathbf{e}_r + \frac{1}{r}\frac{\partial v_z}{\partial\theta}\mathbf{e}_\theta \otimes \mathbf{e}_z + \\ &\quad \left.\frac{\partial v_r}{\partial z}\mathbf{e}_z \otimes \mathbf{e}_r + \frac{\partial v_\theta}{\partial z}\mathbf{e}_z \otimes \mathbf{e}_\theta + \frac{\partial v_z}{\partial z}\mathbf{e}_z \otimes \mathbf{e}_z\right]^\top,\end{aligned}$$

[1]This vector has components that are *operations not numbers*! It is illegal but very useful.

[2]This is easily verified in Cartesian components: $(\nabla\mathbf{v})_{ij} = v_{i,j}$ and $((\nabla \otimes \mathbf{v})^\top)_{ij} = v_{i,j}$.

or

$$\boxed{\begin{aligned}\nabla\mathbf{v} = {} & \frac{\partial v_r}{\partial r}\mathbf{e}_r\otimes\mathbf{e}_r + \left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r}\right)\mathbf{e}_r\otimes\mathbf{e}_\theta + \frac{\partial v_r}{\partial z}\mathbf{e}_r\otimes\mathbf{e}_z + \\ & \frac{\partial v_\theta}{\partial r}\mathbf{e}_\theta\otimes\mathbf{e}_r + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r}\right)\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \frac{\partial v_\theta}{\partial z}\mathbf{e}_\theta\otimes\mathbf{e}_z + \\ & \frac{\partial v_z}{\partial r}\mathbf{e}_z\otimes\mathbf{e}_r + \frac{1}{r}\frac{\partial v_z}{\partial\theta}\mathbf{e}_z\otimes\mathbf{e}_\theta + \frac{\partial v_z}{\partial z}\mathbf{e}_z\otimes\mathbf{e}_z,\end{aligned}} \tag{A.2.13}$$

and the matrix of components of $\nabla\mathbf{v}$ with respect to $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is

$$\boxed{[\nabla\mathbf{v}] = \begin{bmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{1}{r}\dfrac{\partial v_r}{\partial\theta} - \dfrac{v_\theta}{r} & \dfrac{\partial v_r}{\partial z} \\ \dfrac{\partial v_\theta}{\partial r} & \dfrac{1}{r}\dfrac{\partial v_\theta}{\partial\theta} + \dfrac{v_r}{r} & \dfrac{\partial v_\theta}{\partial z} \\ \dfrac{\partial v_z}{\partial r} & \dfrac{1}{r}\dfrac{\partial v_z}{\partial\theta} & \dfrac{\partial v_z}{\partial z} \end{bmatrix}.} \tag{A.2.14}$$

**Divergence of a vector field**

The **divergence** of $\mathbf{v}$, denoted by $\operatorname{div}\mathbf{v}$, is the scalar field defined as

$$\operatorname{div}\mathbf{v} = \operatorname{tr}\nabla\mathbf{v} = \nabla\cdot\mathbf{v}.$$

Hence, using (A.2.6),

$$\operatorname{div}\mathbf{v} = \left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z}\right)\cdot\left(v_r\,\mathbf{e}_r + v_\theta\,\mathbf{e}_\theta + v_z\,\mathbf{e}_z\right),$$

or

$$\boxed{\operatorname{div}\mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r}.} \tag{A.2.15}$$

**Divergence of a tensor field**

The divergence of a tensor field $\mathbf{S}$, denoted by $\operatorname{div}\mathbf{S}$, is a vector field defined by

$$(\operatorname{div}\mathbf{S})\cdot\mathbf{a} = \operatorname{div}(\mathbf{S}^\top\mathbf{a}),$$

for every constant vector $\mathbf{a}$.

Using (A.2.9),

$$\mathbf{S}^\top\mathbf{a} = (\mathbf{e}_r\cdot\mathbf{a})\,\mathbf{s}_r + (\mathbf{e}_\theta\cdot\mathbf{a})\,\mathbf{s}_\theta + (\mathbf{e}_z\cdot\mathbf{a})\,\mathbf{s}_z,$$

where

$$\begin{aligned}\mathbf{s}_r &= S_{rr}\,\mathbf{e}_r + S_{r\theta}\,\mathbf{e}_\theta + S_{rz}\,\mathbf{e}_z,\\ \mathbf{s}_\theta &= S_{\theta r}\,\mathbf{e}_r + S_{\theta\theta}\,\mathbf{e}_\theta + S_{\theta z}\,\mathbf{e}_z,\\ \mathbf{s}_z &= S_{zr}\,\mathbf{e}_r + S_{z\theta}\,\mathbf{e}_\theta + S_{zz}\,\mathbf{e}_z.\end{aligned}$$

Hence,

$$
\begin{aligned}
(\mathbf{S}^{\top}\mathbf{a})_r &= S_{rr}\,(\mathbf{e}_r\cdot\mathbf{a}) + S_{\theta r}\,(\mathbf{e}_\theta\cdot\mathbf{a}) + S_{zr}\,(\mathbf{e}_z\cdot\mathbf{a}),\\
(\mathbf{S}^{\top}\mathbf{a})_\theta &= S_{r\theta}\,(\mathbf{e}_r\cdot\mathbf{a}) + S_{\theta\theta}\,(\mathbf{e}_\theta\cdot\mathbf{a}) + S_{z\theta}\,(\mathbf{e}_z\cdot\mathbf{a}),\\
(\mathbf{S}^{\top}\mathbf{a})_z &= S_{rz}\,(\mathbf{e}_r\cdot\mathbf{a}) + S_{\theta z}\,(\mathbf{e}_\theta\cdot\mathbf{a}) + S_{zz}\,(\mathbf{e}_z\cdot\mathbf{a}).
\end{aligned}
$$

Then, upon using (A.2.15)

$$
\begin{aligned}
(\operatorname{div}\mathbf{S})\cdot\mathbf{a} = {}& \frac{\partial}{\partial r}\left[S_{rr}\,(\mathbf{e}_r\cdot\mathbf{a}) + S_{\theta r}\,(\mathbf{e}_\theta\cdot\mathbf{a}) + S_{zr}\,(\mathbf{e}_z\cdot\mathbf{a})\right] +\\
& \frac{1}{r}\frac{\partial}{\partial\theta}\left[S_{r\theta}\,(\mathbf{e}_r\cdot\mathbf{a}) + S_{\theta\theta}\,(\mathbf{e}_\theta\cdot\mathbf{a}) + S_{z\theta}\,(\mathbf{e}_z\cdot\mathbf{a})\right] +\\
& \frac{\partial}{\partial z}\left[S_{rz}\,(\mathbf{e}_r\cdot\mathbf{a}) + S_{\theta z}\,(\mathbf{e}_\theta\cdot\mathbf{a}) + S_{zz}\,(\mathbf{e}_z\cdot\mathbf{a})\right] +\\
& \frac{1}{r}\left[S_{rr}\,(\mathbf{e}_r\cdot\mathbf{a}) + S_{\theta r}\,(\mathbf{e}_\theta\cdot\mathbf{a}) + S_{zr}\,(\mathbf{e}_z\cdot\mathbf{a})\right],
\end{aligned}
$$

or

$$
\begin{aligned}
(\operatorname{div}\mathbf{S})\cdot\mathbf{a} = {}& \frac{\partial S_{rr}}{\partial r}\,(\mathbf{e}_r\cdot\mathbf{a}) + \frac{\partial S_{\theta r}}{\partial r}\,(\mathbf{e}_\theta\cdot\mathbf{a}) + \frac{\partial S_{zr}}{\partial r}\,(\mathbf{e}_z\cdot\mathbf{a}) +\\
& \frac{S_{rr}}{r}\,(\mathbf{e}_r\cdot\mathbf{a}) + \frac{S_{\theta r}}{r}\,(\mathbf{e}_\theta\cdot\mathbf{a}) + \frac{S_{zr}}{r}\,(\mathbf{e}_z\cdot\mathbf{a}) +\\
& \frac{\partial S_{rz}}{\partial z}\,(\mathbf{e}_r\cdot\mathbf{a}) + \frac{\partial S_{\theta z}}{\partial z}\,(\mathbf{e}_\theta\cdot\mathbf{a}) + \frac{\partial S_{zz}}{\partial z}\,(\mathbf{e}_z\cdot\mathbf{a}) +\\
& \frac{1}{r}\left[\frac{\partial S_{r\theta}}{\partial\theta}(\mathbf{e}_r\cdot\mathbf{a}) + S_{r\theta}(\mathbf{e}_\theta\cdot\mathbf{a}) +\right.\\
& \left.\frac{\partial S_{\theta\theta}}{\partial\theta}(\mathbf{e}_\theta\cdot\mathbf{a}) - S_{\theta\theta}(\mathbf{e}_r\cdot\mathbf{a}) + \frac{\partial S_{z\theta}}{\partial\theta}(\mathbf{e}_z\cdot\mathbf{a})\right],
\end{aligned}
$$

from which

$$
\boxed{
\begin{aligned}
(\operatorname{div}\mathbf{S})_r &= \frac{\partial S_{rr}}{\partial r} + \frac{1}{r}\frac{\partial S_{r\theta}}{\partial\theta} + \frac{\partial S_{rz}}{\partial z} + \frac{1}{r}\left(S_{rr} - S_{\theta\theta}\right),\\
(\operatorname{div}\mathbf{S})_\theta &= \frac{\partial S_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta\theta}}{\partial\theta} + \frac{\partial S_{\theta z}}{\partial z} + \frac{1}{r}\left(S_{\theta r} + S_{r\theta}\right),\\
(\operatorname{div}\mathbf{S})_z &= \frac{\partial S_{zr}}{\partial r} + \frac{1}{r}\frac{\partial S_{z\theta}}{\partial\theta} + \frac{\partial S_{zz}}{\partial z} + \frac{S_{zr}}{r}.
\end{aligned}
}
\tag{A.2.16}
$$

**Curl of a vector field**

The **curl** of $\mathbf{v}$, denoted by $\operatorname{curl}\mathbf{v}$, is a vector field defined as

$$
\operatorname{curl}\mathbf{v} = \nabla\times\mathbf{v}.
$$

Hence,

$$
\operatorname{curl}\mathbf{v} = \left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z}\right)\times\left(v_r\,\mathbf{e}_r + v_\theta\,\mathbf{e}_\theta + v_z\,\mathbf{e}_z\right),
$$

which upon noting that

$$
\begin{aligned}
\mathbf{e}_r\times\mathbf{e}_r &= \mathbf{0}, \quad & \mathbf{e}_\theta\times\mathbf{e}_z &= -\mathbf{e}_z\times\mathbf{e}_\theta = \mathbf{e}_r,\\
\mathbf{e}_\theta\times\mathbf{e}_\theta &= \mathbf{0}, \quad & \mathbf{e}_z\times\mathbf{e}_r &= -\mathbf{e}_r\times\mathbf{e}_z = \mathbf{e}_\theta,\\
\mathbf{e}_z\times\mathbf{e}_z &= \mathbf{0}, \quad & \mathbf{e}_r\times\mathbf{e}_\theta &= -\mathbf{e}_\theta\times\mathbf{e}_r = \mathbf{e}_z,
\end{aligned}
$$

gives

$$\boxed{\operatorname{curl}\mathbf{v} = \left\{\frac{1}{r}\frac{\partial v_z}{\partial\theta} - \frac{\partial v_\theta}{\partial z}\right\}\mathbf{e}_r + \left\{\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right\}\mathbf{e}_\theta + \left\{\frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial\theta} + \frac{v_\theta}{r}\right\}\mathbf{e}_z.} \tag{A.2.17}$$

**Laplacian of a scalar field**

The Laplacian of a scalar $\psi$, denoted by $\triangle\,\psi$, is a scalar field defined as

$$\triangle\,\psi = \operatorname{div}\nabla\psi.$$

Thus, using (A.2.10) and (A.2.15)

$$\boxed{\triangle\,\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}.} \tag{A.2.18}$$

**Laplacian of a vector field**

The Laplacian of a vector $\mathbf{v}$, denoted by $\triangle\,\mathbf{v}$, is a vector field defined as

$$\triangle\,\mathbf{v} = \operatorname{div}\nabla\mathbf{v}.$$

From (A.2.16) and (A.2.14),

$$\boxed{\triangle\mathbf{v} = \Big(\triangle\,v_r - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{1}{r^2}v_r\Big)\mathbf{e}_r + \Big(\triangle\,v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{1}{r^2}v_\theta\Big)\mathbf{e}_\theta + (\triangle v_r)\mathbf{e}_z.} \tag{A.2.19}$$

**Infinitesimal strain tensor**

Let

$$\mathbf{u} = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + u_z\mathbf{e}_z,$$

denote the components of the displacement vector with respect to $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$. Then (A.2.14) gives the components of the displacement gradient tensor $\mathbf{H} \equiv \nabla\mathbf{u}$ in the cylindrical coordinate system, and hence the symmetric infinitesimal strain tensor $\boldsymbol{\epsilon}$ defined by

$$\boldsymbol{\epsilon} = \frac{1}{2}\left[\mathbf{H} + \mathbf{H}^\top\right],$$

has the following components in the cylindrical coordinate system

$$\boxed{\begin{aligned}
\epsilon_{rr} &= \frac{\partial u_r}{\partial r},\\
\epsilon_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r},\\
\epsilon_{zz} &= \frac{\partial u_z}{\partial z},\\
\epsilon_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = \epsilon_{\theta r},\\
\epsilon_{\theta z} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial\theta} + \frac{\partial u_\theta}{\partial z}\right) = \epsilon_{z\theta},\\
\epsilon_{zr} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) = \epsilon_{rz}.
\end{aligned}} \tag{A.2.20}$$

**Equation of motion**

The matrix of the components of the symmetric stress tensor $\boldsymbol{\sigma}$ with respect to $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{zr} & \sigma_{z\theta} & \sigma_{zz} \end{bmatrix}.$$

Let $(b_r, b_\theta, b_z)$ and $(\ddot{u}_r, \ddot{u}_\theta, \ddot{u}_z)$ denote the components of the body force $\mathbf{b}$ and the acceleration $\ddot{\mathbf{u}}$ in a cylindrical coordinate system. Then using (A.2.16) the equation of motion

$$\operatorname{div} \boldsymbol{\sigma} + \mathbf{b} = \rho \ddot{\mathbf{u}},$$

where $\rho$ is the mass density, may be written as the following three equations in the cylindrical coordinate system

$$\boxed{\begin{aligned} &\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + b_r = \rho \ddot{u}_r, \\ &\frac{\partial \sigma_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{\theta r} + b_\theta = \rho \ddot{u}_\theta, \\ &\frac{\partial \sigma_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{zr}}{r} + b_z = \rho \ddot{u}_z. \end{aligned}} \tag{A.2.21}$$