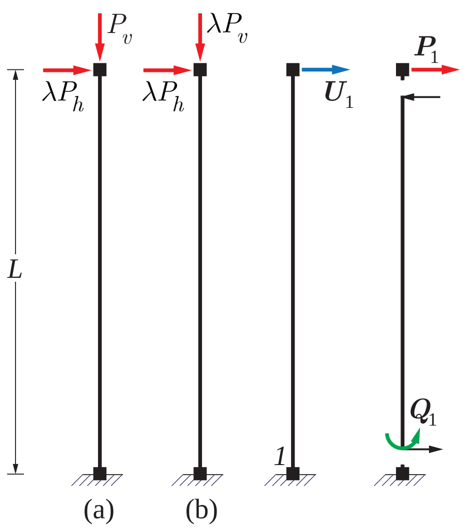
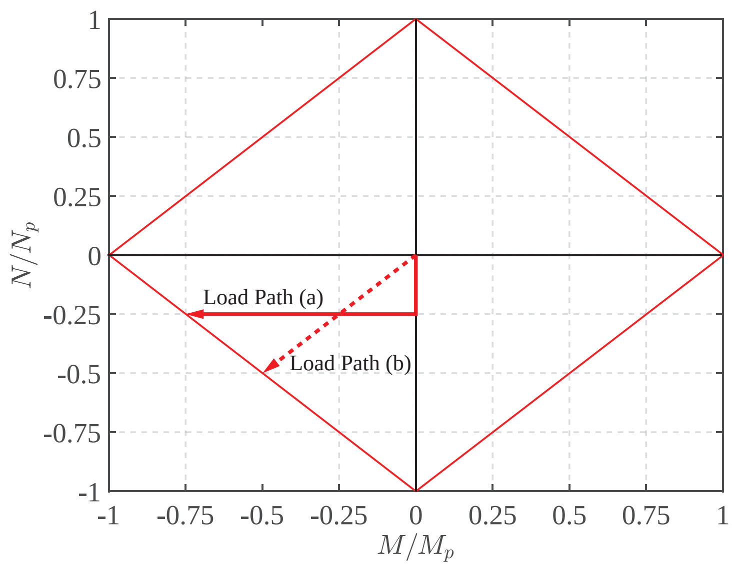
$$\frac{|N|}{N_p} + \frac{|M|}{M_p} = 1 \tag{4.9}$$



(a) Geometry and load cases



(b) Diamond shaped interaction surface

Fig. 4.6: Cantilever column under two load paths to collapse

For load case (a) $N = -P_v$ and $M = -\lambda P_h L$. Substituting these into (4.9) gives

$$\frac{P_v}{N_p} + \lambda_c \frac{P_h L}{M_p} = 1 \quad \rightarrow \quad \lambda_c = \left(\frac{1 - \frac{P_v}{N_p}}{P_h L} \right) M_p$$

Assuming $\frac{P_v}{N_p} = 0.25$ gives the load path (a) in Fig. 4.6(b) with a bending moment M at the base equal to $-0.75 M_p$ at collapse.

For load case (b) $N = -\lambda P_v$ and $M = -\lambda P_h L$. Substituting these into (4.9) gives

$$\lambda_c \frac{P_v}{N_p} + \lambda_c \frac{P_h L}{M_p} = 1 \quad \rightarrow \quad \lambda_c = \left(\frac{1}{P_v \frac{Z}{A} + P_h L} \right) M_p$$

where Z is the plastic section modulus and A is the area of the base section. The load path to collapse and the collapse load factor depends on the values of P_v and P_h as well as the ratio Z/A and the height L of the column. Fig. 4.6(b) shows the load path to collapse for load case (b) for the case that

$$P_v \frac{Z}{A} + P_h L = 2$$

4.3 Deformation State at Incipient Collapse

4.3.1 Introduction

An important application of the relations governing the response of a linear elastic, structure that is either statically determinate or statically indeterminate with a small degree of static indeterminacy, is the determination of the deformation state of a structural model at incipient collapse. *If the collapse mechanism of a structure is complete, the structure is statically determinate and stable just before the last hinge forms.* Consequently, the deformation-force relations for statically determinate structures in B-Chapter 8 can be used to determine the free dof displacements and deformations of the structure at incipient collapse, including the plastic deformations at all plastic hinges before the last.

If the collapse mechanism of a structure is partial, the structure is statically indeterminate and stable just before the last hinge forms. Because the basic forces $\boldsymbol{Q}$ are known at the plastic hinge locations, the degree of static indeterminacy of the structure is rather small relative to the large number of global degrees of freedom. Consequently, the force method of analysis is the most suitable approach for determining the basic forces at incipient collapse, especially for small structural models.

It is assumed that the number and location of the plastic hinges under the collapse load factor λ_c are known from the application of the upper bound theorem of plastic analysis.

If the number of plastic hinges is such that a unique static solution for the basic forces $\boldsymbol{Q}_c$ results, then the structure is statically determinate at incipient collapse. In this case, the strain-dependent element deformations $\boldsymbol{V}_\varepsilon$ can be established in all elements. Just before the last hinge forms, the structure is stable and the number of unknown free dof displacements matches the number of available kinematic relations. It is, therefore, possible to determine the free dof displacements and plastic hinge deformations at incipient collapse from the basic forces $\boldsymbol{Q}_c$ and the corresponding element deformations $\boldsymbol{V}_\varepsilon$ just as for any statically determinate structure.

There is one problem, however: *we need to know where the last hinge forms.* Clearly, we can use a step-by-step analysis that determines the sequence of plastic hinge formation to answer the question of which is the last plastic hinge to form, but the expense of this process is not proportional to the objective of just determining the deformation state at incipient collapse. A direct method for accomplishing this task is, therefore, required, as discussed in the next section.

4.3.2 Solution Process

The solution process for determining the last plastic hinge to form before the structure becomes a mechanism is based on the following assumptions:

1) The applied loading consists only of the reference load $\boldsymbol{P}_{ref}$, which is increased monotonically by incrementing the load factor λ until reaching the collapse load factor λ_c.
2) Plastic hinges that are "open" under a load factor λ cannot "close" under a higher load factor: this means that plastic hinge deformations increase monotonically under the monotonically increasing reference load.

Given these conditions we can determine the location of the last plastic hinge to form by making a guess and then correcting it, as described in the following process.

Before starting the process for the determination of the last hinge to form it is assumed that the following tasks are complete:

1) The upper or lower bound theorem of plastic analysis gives the collapse load factor λ_c and the plastic hinge locations for the collapse mechanism.

2) The equilibrium equations give a unique solution for the basic forces $\boldsymbol{Q}_c$ at incipient collapse, because of the complete collapse mode of the structure.
3) The element deformation-force relations give the element deformations $\boldsymbol{V}_\varepsilon$ at incipient collapse

$$\boldsymbol{V}_\varepsilon = \mathbf{F}_s \boldsymbol{Q}_c + \boldsymbol{V}_0$$

The process for the determination of the last hinge to form consists of the following steps:

1) Select any hinge of the collapse mechanism as last to form and solve the kinematic relations for the corresponding free dof displacements $\boldsymbol{U}_f^{tr}$, where the superscript tr stands for *trial result*.
2) Determine the plastic hinge deformations $\boldsymbol{V}_{hp}^{tr}$ corresponding to $\boldsymbol{U}_f^{tr}$ with

$$\boldsymbol{V} = \boldsymbol{V}_\varepsilon + \boldsymbol{V}_{hp}^{tr} = \mathbf{A}_f \boldsymbol{U}_f^{tr}$$

3) If the sign of each plastic deformation matches the sign of the corresponding basic force $\boldsymbol{Q}_c$ from the equilibrium equations, the last hinge location is correct. The trial displacements and plastic deformations from steps (1) and (2) give the free dof displacements $\boldsymbol{U}_f$ and the plastic deformations $\boldsymbol{V}_{hp}$ at incipient collapse.
4) If the sign of one or more plastic deformations does not match the sign of the corresponding basic force, *correct the free dof displacements and plastic deformations* of Step (1) and (2) in a single step as described in the following.

If the sign of one or more plastic deformations does not match the sign of the corresponding basic force, the assumption about the last plastic hinge location in Step (1) is not correct. Consequently, the free dof displacements and plastic deformations of Steps (1) and (2) need to be corrected. The correction involves the addition of the deformation state of the collapse mechanism of the structure. The independent dof of the collapse mechanism is scaled so that *the total plastic deformations match everywhere the sign of the corresponding basic force, except for one location where the plastic deformation is zero, thus, identifying the location of the last plastic hinge to form.*

The determination of the scale factor for the collapse mechanism is described next. Consider the initial assumption about the deformation state of the structure at incipient collapse. The kinematic relations give

$$\boldsymbol{V}_\varepsilon + \boldsymbol{V}_{hp}^{tr} = \mathbf{A}_f \boldsymbol{U}_f^{tr} \tag{4.10}$$

From the comparison of the sign of the plastic deformations $\boldsymbol{V}_{hp}^{tr}$ with the sign of the corresponding basic forces $\boldsymbol{Q}_c$ we identify n locations of mismatch. We need to correct these plastic deformations by adding a deformation state *without changing the element deformations* $\boldsymbol{V}_\varepsilon$. Because $\Delta\boldsymbol{V}_\varepsilon$ is zero during the collapse mechanism displacement, we conclude that we need to superimpose to the deformation state in (4.10) the deformation state of the collapse mechanism with a single independent dof $\dot{U}_p$. For this deformation state it holds that

$$\Delta\boldsymbol{V}_{hp} = \mathbf{A}_{mp}\, \Delta U_p \tag{4.11}$$

where we have converted instantaneous rates of change to increments denoted with a Δ prefix for the corresponding variable. Consequently, ΔU_p is the displacement increment of the single independent dof of the collapse mechanism and $\Delta\boldsymbol{V}_{hp}$ are the increments of the corresponding plastic deformations.