CE-222 Midterm 1

$$\int u\,dv = uv - \int v\,du \qquad \int_a^b udv = uv|_a^b - \int_a^b vdu$$

Boundary Value Problems

Weak form

1. Multiply both sides of the DE or PDE by an arbitrary function
    1. The function must be homogeneous (= 0) where displacement BC's are specified.
    2. The function must have sufficient continuity for differentiation.
2. Integrate over the domain, e.g. length of the rod
3. Integrate by parts using Green's theorem to reduce derivatives to their minimum order.
4. Replace the boundary conditions by an appropriate construction.

Note:

1. Weak form is formulated in terms of axial force (or equivalently axial stress as in general mechanics problems). This only involves equilibrium
2. Compatibility relationship is based on infinitesimal strain, .
3. We did not make statements @ stress strain relationships , so the relationship between the strong and weak forms of equilibrium are true for linear & nonlinear materials
4. The weak form (or the structural analysis interpretation as the PVD) provides a framework for finding an approximate solution
5. For the exact solution, we need to look at all possible trail functions (e.g. describing the axial force and VD), which is a formidable task.
6. Rayleigh's method (and its extension to use superposition of several functions, i.e. the Rayleigh Ritz procedure) provides a convenient way to limit the number of functions that we are examining and since the limited functions may not include the exact solution, the obtained solution will be an approximation
7. We would like to pick simple functions for easy integration & provide a set of algebraic equations that can be solved efficiently. The FEM provides a systematic way for this.

Energy

PVD

A structure is in equilibrium under a system of loads and initial strains if for any admissible virtual displacement, the internal virtual work equals the external virtual work.

$$W_{I_e} = \int_0^{L_e} (\delta\varepsilon) EA \frac{d\bar{u}}{dx} dx = \{\delta u_e\}^T \int_0^{L_e} \{B\} EA \{B\}^T dx_e \{u_e\} = \{\delta u_e\}^T [k_e] \{u_e\}$$

$$W_{E_e} = \int_0^{L_e} (\delta u) p(x) dx = \{\delta u_e\}^T \int_0^{L_s} \{N\} p(x_e)\, dx_e = -\{\delta u_e\}^T \{p_{0_e}\}$$

$$\int_\Omega \delta\varepsilon_{ij}\sigma_{ji} d\Omega = \int_\Omega \delta u_i b_i d\Omega + \int_\Gamma \delta u_i t_i d\Gamma$$

$$\sum_e \left(\int_{\Omega_e} \{\delta\varepsilon\}^T \{\sigma\} d\Omega_e - \int_{\Omega_e} \{\delta u\}^T \{b\} d\Omega_e - \int_{\Gamma_t} \{\delta u\}^T \{t\} d\Gamma_e \right) = 0$$

Minimum Potential Energy

For an elastic body, the potential energy is given by

$$\Pi(\boldsymbol{u}) = \int_\Omega W(\boldsymbol{\varepsilon}(u)) - \int_\Omega \boldsymbol{b}\cdot\boldsymbol{u} - \int_{\Gamma_t} \bar{t}\cdot\boldsymbol{u}$$

$$\frac{d}{d\alpha}\Big|_{\alpha=0} \Pi(\boldsymbol{u}+\alpha\boldsymbol{v}) = \int_\Omega \varepsilon_{ij}(v) C_{ijkl} \varepsilon_{kl}(u) - \int_\Omega b_i v_i - \int_{\Gamma_t} \bar{t}_i v_i = 0$$

Rayleigh

Brief outline

- Assume an admissible form of the solution with one unknown parameter
- Apply PVD to obtain the best estimate of the exact (but unknown) solution (e.g. displacement)
- Approximate solution satisfies governing equation (e.g. equilibrium) in an average (weak) sense.

Trial function remarks:

- The trial functions collectively satisfy the admissibility conditions:
- They satisfy the displacement BC.
- They have enough continuity that the derivatives are defined (although the derivatives are not continuous at element boundaries).
- They are linearly independent.
- They are complete as a family of piecewise linear functions over the rod.

FEM

$$\{\sigma\} = [D]\{[B][u\} - \{\alpha\}T\} + \{\sigma_0\}$$

$$\{p\} = [k]\{u\} + \{p_0\}$$

$$[k] = \int_{\Omega_e} [B]^T [D][B] d\Omega_e$$

$$\{p_0\} = \underbrace{-\int_{\Omega_e} [N]^T \{b\} d\Omega_e}_{\text{body force}} \underbrace{-\int_{\Gamma_t} [N]^T \{t\} d\Gamma_e}_{\text{boundary traction}} \underbrace{-\int_{\Omega_e} [B]^T [D] \{\varepsilon_0\} d\Omega_e}_{\text{initial strain}} \underbrace{+\int_{\Omega_e} [B]^T \{\sigma_0\} d\Omega_e}_{\text{initial stress}}$$

Isoparametric Formulation

$$\mathbf{x}(\xi,\eta) = \sum_{\alpha=1}^{4} N_\alpha(\xi,\eta)\mathbf{x}_\alpha$$

Elements

CST

$$B = \frac{1}{2A}\begin{bmatrix} x_{22}-x_{23} & 0 & x_{23}-x_{21} & 0 & x_{21}-x_{22} & 0 \\ 0 & x_{13}-x_{12} & 0 & x_{11}-x_{13} & 0 & x_{12}-x_{11} \\ x_{13}-x_{12} & x_{22}-x_{23} & x_{11}-x_{13} & x_{23}-x_{21} & x_{12}-x_{11} & x_{21}-x_{22} \end{bmatrix}$$

Q4 Isoparametric Quad

$N_i = \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)$

Govindjee

Construction

First, the solution space and the weighting function space are replaced by finite dimensional subspaces. Second, these subspaces are employed in the weak form to generate matrix equations which are then used to solve for the unknown displacements at the nodes.

The space of admissible variations is defined as

$$\mathcal{V}^h = \left\{ \boldsymbol{v}^h(\boldsymbol{x}) | v_i^h(\boldsymbol{x}) = \sum_{A\in\eta-\eta_u} N_A(\boldsymbol{x}) v_{iA}^h \right\}$$

where η is the set of all node numbers, η_u is the set of node numbers where the displacements are prescribed, and v_{iA}^h refers to the vector v_i^h at node A.

$$\begin{pmatrix} v_1(\boldsymbol{x}) \\ v_2(\boldsymbol{x}) \end{pmatrix} = \sum_{A\in\eta-\eta_u} N_A(\boldsymbol{x}) v_{iA} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \cdots & 0 & N_n \end{bmatrix} \begin{pmatrix} v_{11} \\ v_{21} \\ v_{12} \\ v_{22} \\ \vdots \\ v_{1n} \\ v_{2n} \end{pmatrix} = N\boldsymbol{v}$$

where n is the number of nodes in $\eta - \eta_u$. The N matrix is of great utility when working with the FEM equations; as given above, it is strictly for $2-$D problems but is trivially extended to 3 -D. This type of notation also extends to the symmetric gradients of functions; viz. in $2-$D we have that

$$\begin{pmatrix} v_{1,1} \\ v_{2,2} \\ v_{1,2}+v_{2,1} \end{pmatrix} = \begin{bmatrix} N_{1,1} & 0 & N_{2,1} & 0 & \dots & N_{n,1} & 0 \\ 0 & N_{1,2} & 0 & N_{2,2} & \dots & 0 & N_{n,2} \\ N_{1,2} & N_{1,1} & N_{2,2} & N_{2,1} & \dots & N_{n,2} & N_{n,1} \end{bmatrix} \begin{pmatrix} v_{11} \\ v_{21} \\ v_{12} \\ v_{22} \\ \vdots \\ v_{1n} \\ v_{2n} \end{pmatrix} = Bv$$

Fundamentals

$$\int_\Omega (\boldsymbol{B}\boldsymbol{v})^T \boldsymbol{D}\boldsymbol{B}\boldsymbol{u} = \int_\Omega (\boldsymbol{N}\boldsymbol{v})^T \boldsymbol{b} + \int_{\Gamma_t} (\boldsymbol{N}\boldsymbol{v})^T \bar{t}$$

$$k_{ab}^e = \int_{\Omega_e} \nabla N_a^e \cdot \kappa \nabla N_b^e = \int_{\Omega_e} N_{a,i}^e \kappa_{ij} N_{b,j}^e = \int_{\Omega_e} \boldsymbol{B}_a^T \boldsymbol{\kappa} \boldsymbol{B}_b \quad a,b \in \{1,2,\cdots,N_{en}\}$$

N_{en} is the number of element nodes.

The vector B_a has dimensions of N_{sd} by 1, where N_{sd} stands for number of spatial dimensions.

Integrals are usually restricted to individual elements and then the contributions are added together through an assembly operation like that of direct stiffness. In this case, the element stiffness matrix is given by

$$k^e = \int_{\Omega_e} B^T D B$$

where the dimension of B is now $3 \times 2N_{en}$ for 2 -D elasticity problems - N_{en} being the number of element nodes. In 3 -D the dimension of B becomes $6 \times 3N_{en}$. The dimension of k^e itself is either $2N_{en} \times 2N_{en}$ or $3N_{en} \times 3N_{en}$.

Nodal Forces

The element force vector is given by the following expression:

$$f^e = \int_{\Omega_e} N^T b + \int_{\Gamma_i^e} N^T \bar{t}$$

$$\boldsymbol{F} = \int_\Omega \boldsymbol{N}^T \boldsymbol{b} d\Omega + \int_{\Gamma_t} \boldsymbol{N}^T \bar{\boldsymbol{t}} d\Gamma_t$$

where N will have dimensions $2 \times 2N_{en}$ and $3 \times 3N_{en}$ for $2-$D and $3-$D problems, respectively. [Note that the last integral in (18.9) is a surface integral.]

For a 2D triangle:

$$\boldsymbol{f}^e = \begin{pmatrix} f_{11} \\ f_{21} \\ f_{12} \\ f_{22} \\ f_{13} \\ f_{23} \end{pmatrix} = \int_{\Gamma_t^e} \begin{bmatrix} N_1^e & 0 \\ 0 & N_1^e \\ N_2^e & 0 \\ 0 & N_2^e \\ N_3^e & 0 \\ 0 & N_3^e \end{bmatrix} \begin{pmatrix} 0 \\ q \end{pmatrix} d\Gamma_t^e = \int_{\Gamma_t^e} \begin{pmatrix} 0 \\ 0 \\ 0 \\ N_2^e q \\ 0 \\ N_3^e q \end{pmatrix} d\Gamma_t^e$$

An alternative and equivalent formula for computing such integrals that is often used in continuum mechanics is

$$\int_\Gamma f(\boldsymbol{x}) \boldsymbol{n} d\Gamma = \int_\gamma f(\boldsymbol{x}(\boldsymbol{X})) \det \left[\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}\right] \left[\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}}\right]^T \boldsymbol{n}_X d\gamma$$

where $\boldsymbol{x}$ and $\boldsymbol{X}$ are different parameterizations of the same edge. $\boldsymbol{x}$ are the physical coordinates along the edge Γ and X are an alternative set of coordinates along the edge $\gamma = \boldsymbol{X}(\Gamma)$; in continuum mechanics the coordinates $\boldsymbol{X}$ are usually the Lagrangian coordinates of the "material points"; in our case they will be the isoparametric coordinates. n is the normal to the edge in the physical coordinates and n_X is the normal to the edge in the X coordinates. For the case explored above this relation reduces to

$$\int_\Gamma f(\boldsymbol{x}) \boldsymbol{n} d\Gamma = \int_0^1 f\left(\boldsymbol{x}\left(t_1, 0, 1-t_1\right)\right) \det \left[\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}\left(t_1, 0, 1-t_1\right)\right] \left[\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{x}}\left(t_1, 0, 1-t_1\right)\right]^T \boldsymbol{n}_X dt_1$$

where $\boldsymbol{n}_X$ is the normal to the edge in the parent domain; in this case $\boldsymbol{n}_X = (0,-1)^T$

Isoparametric Formulations

$$\int_{\Omega_e} f(\boldsymbol{x}) d\boldsymbol{x} \longmapsto \int f(\boldsymbol{x}(\boldsymbol{\xi})) \det \left[\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}\right] d\boldsymbol{\xi} \approx \sum_{\ell=1}^{N_{int}} W_\ell f\left(\boldsymbol{x}\left(\bar{\boldsymbol{\xi}}_\ell\right)\right) \det \left[\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}\left(\bar{\boldsymbol{\xi}}_\ell\right)\right]$$

where $\bar{\xi}_\ell$ represent integration points and W_ℓ are the weights.