IV Linear Elasticity

Compatibility:

$$\operatorname{curl}(\operatorname{curl} \epsilon) = 0, \quad e_{ipq} e_{jrs} \epsilon_{qs,rp} = 0$$

7 Linear Constitutive Equations

7.1 Free Energy: Elasticity

$$\sigma = \mathbb{C}\epsilon, \quad \sigma_{ij} = C_{ijkl}\epsilon_{kl}$$

Voight Notation:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{112} \\ C_{211} & C_{222} & C_{233} & C_{223} & C_{2213} & C_{2212} \\ C_{331} & C_{332} & C_{333} & C_{333} & C_{313} & C_{2312} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{pmatrix}$$

7.5 Isotropic Relations

$$oldsymbol{\sigma} = \mathbb{C}oldsymbol{\epsilon} = 2\muoldsymbol{\epsilon} + \lambda(\operatorname{tr}oldsymbol{\epsilon})\mathbf{1}$$

$$oldsymbol{\sigma} = 2\muoldsymbol{\epsilon}' + \kappa(\operatorname{tr}oldsymbol{\epsilon})\mathbf{1}, \quad \sigma_{ij} = 2\mu\epsilon'_{ij} + \kappa\left(\epsilon_{kk}
ight)\delta_{ij}$$

$$\mathbb{C} = 2\mu^{ ext{sym}} + \lambda \mathbf{1} \otimes \mathbf{1}, \quad C_{ijkl} = \mu \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}
ight) + \lambda \delta_{ij} \delta_{kl}$$

$$\epsilon = rac{1}{2\mu}\sigma' + rac{1}{9\kappa}(\mathrm{tr}\,\sigma)1, \quad \epsilon_{ij} = rac{1}{2\mu}\sigma'_{ij} + rac{1}{9\kappa}\left(\sigma_{kk}
ight)\delta_{ij}$$

$$E \equiv rac{9\kappa\mu}{3\kappa+\mu}, \quad
u \equiv rac{1}{2} \left[rac{3\kappa-2\mu}{3\kappa+\mu}
ight]$$

$$\kappa = \lambda + \frac{2}{3}\mu$$

7.5.4

$$egin{aligned} \sigma &= rac{E}{(1+
u)} \left[\epsilon + rac{
u}{(1-2
u)} (\operatorname{tr} \epsilon) 1
ight], \quad \sigma_{ij} &= rac{E}{(1+
u)} \left[\epsilon_{ij} + rac{
u}{(1-2
u)} \left(\epsilon_{kk}
ight) \delta_{ij}
ight] \ \epsilon &= rac{1}{E} [(1+
u)\sigma -
u(\operatorname{tr} \sigma) 1]. \quad \epsilon_{ij} &= rac{1}{E} \left[(1+
u)\sigma_{ij} -
u(\sigma_{kk}) \delta_{ij}
ight] \end{aligned}$$

8 Elastostatics

Displacement Formulation (Navier)

$$C_{ijkl}u_{k,lj} + b_i = 0$$

Isotropic

$$\begin{split} \mu \triangle \mathbf{u} + (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \mathbf{b} &= 0 \\ \mu u_{i,jj} + (\lambda + \mu) u_{j,ji} + b_i &= 0 \\ (\lambda + 2\mu) \nabla \operatorname{div} \mathbf{u} - \mu \operatorname{curl} \operatorname{curl} \mathbf{u} + \mathbf{b} &= \mathbf{0} \\ (\lambda + 2\mu) u_{j,ji} - \mu e_{ijk} e_{klm} u_{m,lj} + b_i &= 0 \end{split}$$

Boundary:

$$egin{aligned} \mathbf{u} &= \hat{\mathbf{u}} ext{ on } \mathcal{S}_1, \ \left(\mu \left(
abla \mathbf{u} + (
abla \mathbf{u})^ op
ight) + \lambda (\operatorname{div} \mathbf{u}) \mathbf{1}
ight) \mathbf{n} &= \hat{\mathbf{t}} & \operatorname{on } \mathcal{S}_2 \end{aligned}
ight\}$$

Stress Formulation (Beltrami-Mitchell)

Compatibility:

$$\sigma_{ij,kk}+rac{1}{1+
u}\sigma_{kk,ij}=-rac{
u}{1-
u}b_{k,k}\delta_{ij}-b_{i,j}-b_{j,i}\ \Delta\sigma_{kk}=-rac{1+
u}{1-
u}b_{k,k}$$

8.9.1 Plane Strain

$$egin{aligned} u_{lpha} &= u_{lpha} \left(x_1, x_2
ight), \quad u_3 = 0 \ &\epsilon_{lphaeta} &= rac{1}{2} \left(u_{lpha,eta} + u_{eta,lpha}
ight) \ &\epsilon_{13} &= \epsilon_{23} = \epsilon_{33} = 0 \ &\sigma_{lphaeta} &= rac{E}{(1+
u)} \left(\epsilon_{lphaeta} + rac{
u}{(1-2
u)} \left(\epsilon_{\gamma\gamma}
ight) \delta_{lphaeta}
ight) \ &\sigma_{33} &=
u \sigma_{lphalpha} \ &\epsilon_{lphaeta} &= rac{1+
u}{E} \left(\sigma_{lphaeta} -
u \left(\sigma_{\gamma\gamma}
ight) \delta_{lphaeta}
ight) \ &\sigma_{lphaeta,eta} + b_{lpha} &= 0 \end{aligned}$$

Plane Stress

$$\sigma_{lphaeta}=\sigma_{lphaeta}\left(x_{1},x_{2}
ight),\quad\sigma_{33}=\sigma_{13}=\sigma_{23}=0$$

Plane Stress/Strain

$$\epsilon_{13} = \epsilon_{23} = \sigma_{13} = \sigma_{23} = 0$$

$$\text{Navier}: \left\{ \begin{array}{l} \left(\frac{E}{2(1+\nu)}\right) u_{\alpha,\beta\beta} + \left(\frac{E}{2(1+\nu)(1-2\nu)}\right) u_{\beta,\beta\alpha} + b_\alpha = 0 & \text{ for plane strain} \\ \left(\frac{E}{2(1+\nu)}\right) u_{\alpha,\beta\beta} + \frac{E}{2(1-\nu)} u_{\beta,\beta\alpha} + b_\alpha = 0 & \text{ for plane stress} \end{array} \right.$$

Constitutive Relation

$$egin{aligned} \sigma_{lphaeta} &= rac{E}{(1+
u)} \left(\epsilon_{lphaeta} + \left(rac{1-s}{2s-1}
ight) \left(\epsilon_{\gamma\gamma}
ight) \delta_{lphaeta}
ight) \ \epsilon_{lphaeta} &= rac{\left(1+
u
ight)}{E} \left(\sigma_{lphaeta} - \left(1-s
ight) \left(\sigma_{\gamma\gamma}
ight) \delta_{lphaeta}
ight) \end{aligned}$$

Equilibrium

$$\sigma_{\alpha\beta,\beta} + b_{\alpha} = 0$$

Compatibility

$$\Delta\left(\sigma_{lphalpha}
ight)=\left(\sigma_{11}+\sigma_{22}
ight),_{11}+\left(\sigma_{11}+\sigma_{22}
ight),_{22}=-rac{1}{s}b_{lpha,lpha}$$

Airy Stress Function

$$\sigma_{11} = \varphi_{,22} \,, \quad \sigma_{22} = \varphi_{.11}, \quad \sigma_{12} = -\varphi_{.12}$$

Compatibility

$$\Delta\Delta\varphi=\varphi,1111+2\varphi,1122+\varphi,2222=0$$

Displacements

$$u_1=rac{(1+
u)}{E}(-arphi,1+s\psi,2)+w_1 \ u_2=rac{(1+
u)}{E}\left(-arphi,2+s\psi,_1
ight)+w_2 \ ext{where:}\ \Delta\psi=0\ ext{and}\ \psi_{,12}=\Deltaarphi$$

and w is a plane rigid displacement:

$$w_{1,1}=0, \quad w_{2,2}=0, \quad w_{1,2}+w_{2,1}=0$$

Polar form (9.4.16):

$$egin{align} \sigma_{rr} &= rac{1}{r}rac{\partial arphi}{\partial r} + rac{1}{r^2}rac{\partial^2 arphi}{\partial heta^2} \ \sigma_{ heta heta} &= rac{\partial^2 arphi}{\partial r^2} \ \sigma_{r heta} &= -rac{\partial}{\partial r}\left(rac{1}{r}rac{\partial arphi}{\partial heta}
ight) \ \end{array}$$

Torsion

$$egin{aligned} u_1(\mathbf{x}) &pprox -lpha x_2 x_3 \ u_2(\mathbf{x}) &pprox lpha x_1 x_3 \ u_3(\mathbf{x}) &= lpha arphi \left(x_1, x_2
ight) \ lpha &= rac{T}{\mu J} \ T &= \int_{\mathcal{S}_L} \left(x_1 \sigma_{23} - x_2 \sigma_{13}
ight) da \ ar{J} &\stackrel{ ext{def}}{=} \int_{\Omega} \left(x_1^2 + x_2^2 + x_1 arphi_{,2} - x_2 arphi_{,1}
ight) da \end{aligned}$$

Displacement Formulations

$$egin{aligned} \epsilon_{11}(\mathbf{x}) &= \epsilon_{22}(\mathbf{x}) = \epsilon_{33}(\mathbf{x}) = \epsilon_{12}(\mathbf{x}) = 0 \ \epsilon_{13}(\mathbf{x}) &= rac{1}{2} \left(rac{\partial arphi}{\partial x_1} - x_2
ight) lpha \ \epsilon_{23}(\mathbf{x}) &= rac{1}{2} \left(rac{\partial arphi}{\partial x_2} + x_1
ight) lpha \ \sigma_{11}(\mathbf{x}) &= \sigma_{22}(\mathbf{x}) = \sigma_{33}(\mathbf{x}) = \sigma_{12}(\mathbf{x}) = 0 \ \sigma_{13}(\mathbf{x}) &= \mu lpha \left(rac{\partial arphi}{\partial x_1} - x_2
ight) \ \sigma_{23}(\mathbf{x}) &= \mu lpha \left(rac{\partial arphi}{\partial x_2} + x_1
ight) \end{aligned}$$

Equilibrium:

$$\sigma_{13.1}+\sigma_{23.2}=0\quad ext{ in }\Omega$$

Boundary:

$$egin{aligned} \Delta arphi &= 0 \quad ext{in } \Omega \ rac{\partial arphi}{\partial n} &= x_2 n_1 - x_1 n_2 \quad ext{ on } \Gamma \end{aligned}$$

Stress Formulation

Compatibility:

$$egin{aligned} \epsilon_{13,2} - \epsilon_{23,1} &= -lpha & ext{in }\Omega \ \Delta\Psi &= \Psi_{,11} + \Psi_{,22} &= -2\mulpha & ext{in }\Omega ext{ subject to } &\Psi &= 0 & ext{on }\Gamma \end{aligned}$$

for
$$\sigma_{13}=rac{\partial\Psi}{\partial x_2},\quad \sigma_{23}=-rac{\partial\Psi}{\partial x_1}$$

$$T=2\int_{\Omega}\Psi da$$

9 Solutions

Crack Tip

Mode III

$$\Delta u_z = 0 = rac{\partial^2 u_z}{\partial r^2} + rac{1}{r^2} rac{\partial^2 u_z}{\partial heta^2} + rac{1}{r} rac{\partial u_z}{\partial r}$$