UNIVERSITY OF CALIFORNIA, BERKELEY
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Dept. of Civil and Environmental Engineering
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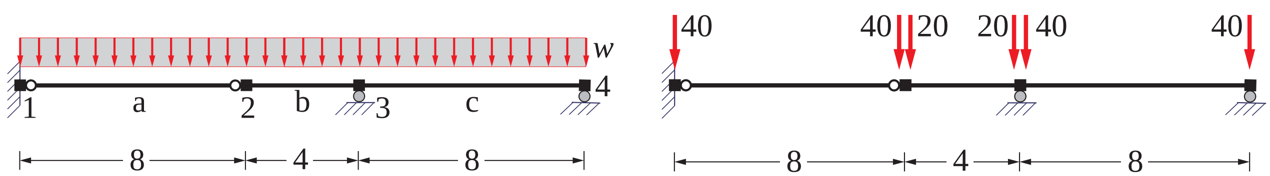
CE 220 - Structural Analysis

Solution for Homework Set #8

1. Problem

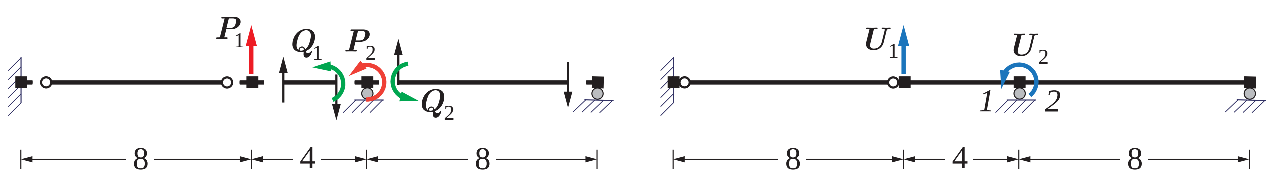
Fig. 1(a) shows a two-span girder under a uniformly distributed element load w of 10 units. The girder has flexural stiffness EI of 60,000 units. Fig. 1(b) shows the equivalent nodal forces due to the uniformly distributed element load w of 10 units.



(a) Geometry and loading (b) Equivalent nodal forces

Figure 1: Geometry, loading and equivalent nodal forces

Fig. 2(a) shows the node free bodies, the equilibrium equations at the free dofs of the structural model and the corresponding flexural basic forces $\boldsymbol{Q}$. The axial effects are uncoupled and are of no interest for the given loading. Fig. 2(b) shows the independent free dofs under consideration of the flexural effects only. The correspondence between independent free dofs and equilibrium equations is a characteristic of all structural models.



(a) Basic forces and equilibrium equations (b) Independent free dofs

Figure 2: Independent free dofs, equilibrium equations and basic forces

The equilibrium equations at the free dofs of the structural model are:

$$
\begin{aligned}
\boldsymbol{P}_1 &= \frac{\boldsymbol{Q}_1}{4} \\
\boldsymbol{P}_2 &= \boldsymbol{Q}_1 + \boldsymbol{Q}_2
\end{aligned}
\quad \rightarrow \quad
\begin{aligned}
-60 &= \frac{\boldsymbol{Q}_1}{4} \\
0 &= \boldsymbol{Q}_1 + \boldsymbol{Q}_2
\end{aligned}
\quad \rightarrow \quad
\begin{aligned}
\boldsymbol{Q}_1 &= -240 \\
\boldsymbol{Q}_2 &= 240
\end{aligned}
$$

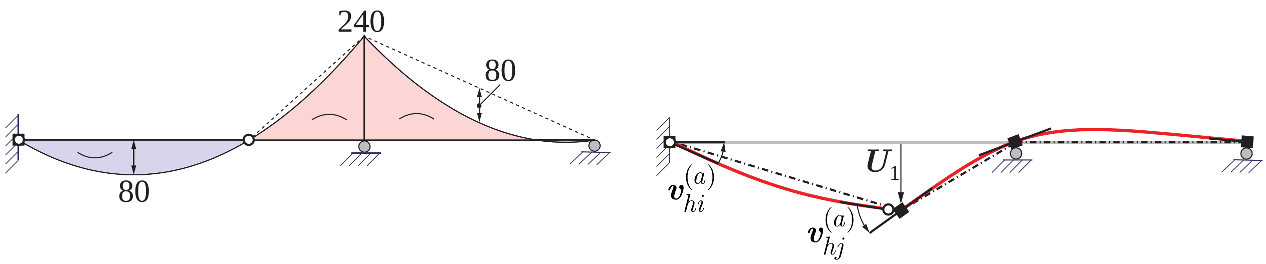
Fig. 3(a) shows the bending moment distribution consisting of the homogeneous solution depicted with a dashed line and the particular solution which is a parabola with maximum value at midspan of the corresponding element. For the elements a and c the maximum value of the particular solution is

$$\frac{w(8)^2}{8} = 80$$

as indicated in Fig. 3(a). For element b the maximum value is

$$\frac{w(4)^2}{8} = 20$$

and is too small to depict in the figure.



(a) Bending moment distribution (b) Deformed shape

Figure 3: Bending moment distribution and deformed shape

The kinematic relations of the structural model are:

$$\begin{aligned} \boldsymbol{V}_1 &= \frac{\boldsymbol{U}_1}{4} + \boldsymbol{U}_2 \\ \boldsymbol{V}_2 &= \boldsymbol{U}_2 \end{aligned}$$

Solving for the translation $\boldsymbol{U}_1$ gives

$$\boldsymbol{U}_1 = 4\left(\boldsymbol{V}_1 - \boldsymbol{V}_2\right) \tag{1}$$

We use the deformation-force relation for elements b and c to determine the deformations $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$

$$\begin{aligned} (EI)\boldsymbol{V}_1 &= \frac{L_b}{3}\boldsymbol{Q}_1 + \frac{wL_b^3}{24} \\ (EI)\boldsymbol{V}_2 &= \frac{L_c}{3}\boldsymbol{Q}_2 - \frac{wL_c^3}{24} \end{aligned} \quad \rightarrow \quad \begin{aligned} (EI)\boldsymbol{V}_1 &= \frac{4}{3}\boldsymbol{Q}_1 + \frac{(10)(4)^3}{24} \\ (EI)\boldsymbol{V}_2 &= \frac{8}{3}\boldsymbol{Q}_2 - \frac{(10)(8)^3}{24} \end{aligned} \quad \rightarrow \quad \begin{aligned} (EI)\boldsymbol{V}_1 &= -293.33 \\ (EI)\boldsymbol{V}_2 &= \quad 426.67 \end{aligned}$$

Substituting into (1) gives

$$(EI)\boldsymbol{U}_1 = 4\left(-293.33 - 426.67\right) = -2880 \quad \rightarrow \quad \boldsymbol{U}_1 = -0.048$$

The hinge rotation at end j of element a is given by the following expression

$$v_{hj}^{(a)} = \left(\beta_b + v_i^{(b)}\right) - \left(\beta_a + v_{\varepsilon j}^{(a)}\right)$$

With

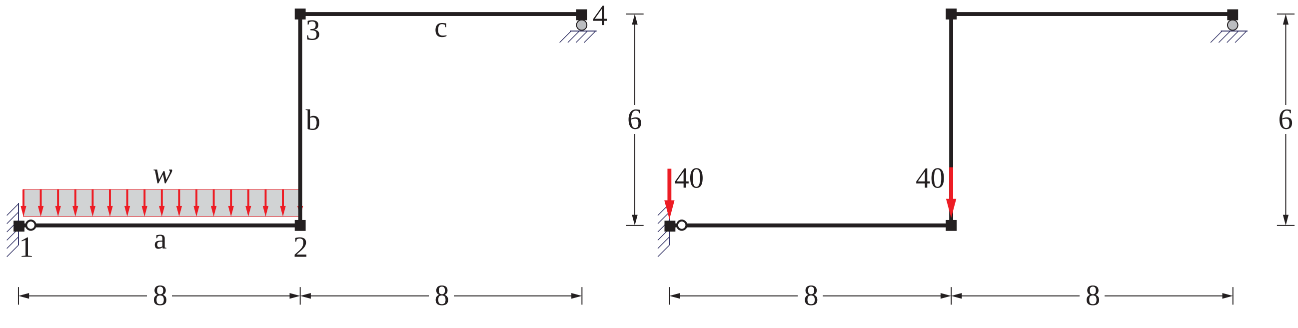
$$\beta_b = -\frac{\boldsymbol{U}_1}{4} \qquad \beta_a = \frac{\boldsymbol{U}_1}{8} \qquad (EI)\,v_i^{(b)} = -\frac{4}{6}\boldsymbol{Q}_1 - \frac{(10)(4)^3}{24} = 133.33 \qquad (EI)\,v_{\varepsilon j}^{(a)} = \frac{(10)(8)^3}{24} = 213.33$$

we get

$$(EI)\,v_{hj}^{(a)} = (720 + 133.33) - \left[(-360) + 213.33\right] = 1000 \quad \rightarrow \quad v_{hj}^{(a)} = 0.0167$$

2. Problem

The frame in Fig. 4(a) is subjected to a uniformly distributed load of w=10 units in element a. All elements have flexural stiffness EI=200,000 units. They can be considered inextensible. Fig. 4(b) shows the equivalent nodal forces for the uniformly distributed load in element a.

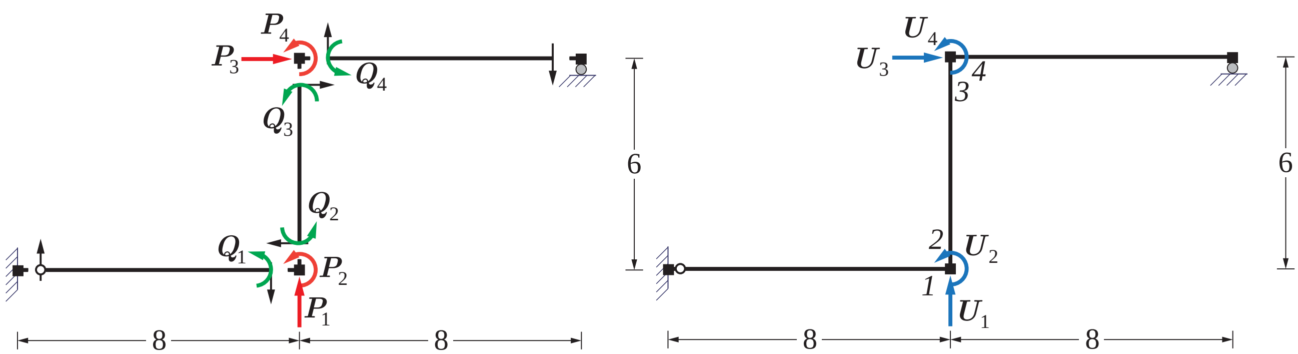


(a) Geometry and loading

(b) Equivalent nodal forces

Figure 4: Frame under uniformly distributed load w in element a

Fig. 5(a) shows the 4 equilibrium equations that do not involve the axial forces in elements a, b and c and the corresponding basic forces $\boldsymbol{Q}$ of primary interest. Fig. 5(b) shows the independent free dofs of the structural model under consideration that the elements a, b and c are inextensible. The correspondence between independent free dofs and equilibrium equations is a characteristic of all structural models.



(a) Basic forces and equilibrium equations

(b) Independent free dofs

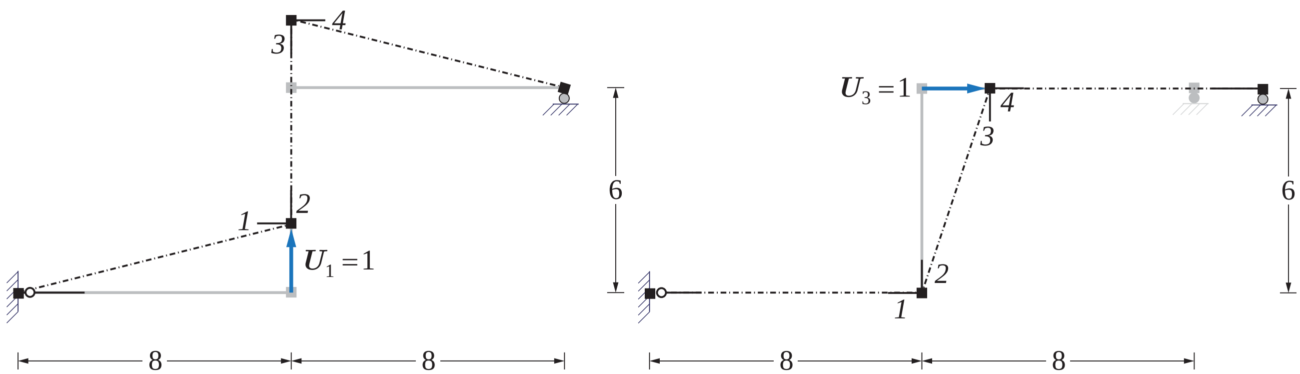
Figure 5: Independent free dofs, equilibrium equations and basic forces of primary interest

For the equilibrium equations at the free dofs of the structure we set up the kinematic matrix $\mathbf{A}_f$ with the help of Fig. 6. We have

$$\mathbf{A}_f = \begin{bmatrix} -\frac{1}{8} & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{6} & 1 \\ \frac{1}{8} & 0 & 0 & 1 \end{bmatrix}$$

The equilibrium equations for the basic forces $\boldsymbol{Q}$ with the numbering in Fig. 5(a) are

$$\boldsymbol{P}_f = \mathbf{A}_f^T \boldsymbol{Q} + \boldsymbol{P}_{wf} \quad \rightarrow \quad \begin{aligned} 0 &= -\frac{\boldsymbol{Q}_1}{8} + \frac{\boldsymbol{Q}_4}{8} + \frac{wL_a}{2} \\ 0 &= \boldsymbol{Q}_1 + \boldsymbol{Q}_2 \\ 0 &= \frac{\boldsymbol{Q}_2 + \boldsymbol{Q}_3}{6} \\ 0 &= \boldsymbol{Q}_3 + \boldsymbol{Q}_4 \end{aligned} \quad \rightarrow \quad \begin{aligned} -40 &= -\frac{\boldsymbol{Q}_1}{8} + \frac{\boldsymbol{Q}_4}{8} \\ 0 &= \boldsymbol{Q}_1 + \boldsymbol{Q}_2 \\ 0 &= \frac{\boldsymbol{Q}_2 + \boldsymbol{Q}_3}{6} \\ 0 &= \boldsymbol{Q}_3 + \boldsymbol{Q}_4 \end{aligned} \tag{2}$$



(a) Independent free dof #1

(b) Independent free dof #3

Figure 6: Kinematic relations for translation dofs

1. The structure is statically determinate. The solution of the equilibrium equations in (2) is rather straightforward. From the last three equations we get $\boldsymbol{Q}_1 = -\boldsymbol{Q}_2 = \boldsymbol{Q}_3 = -\boldsymbol{Q}_4$ and substituting into the first equation we get

$$-\boldsymbol{Q}_1 = 4(-40) \quad \rightarrow \quad \boldsymbol{Q}_1 = 160$$

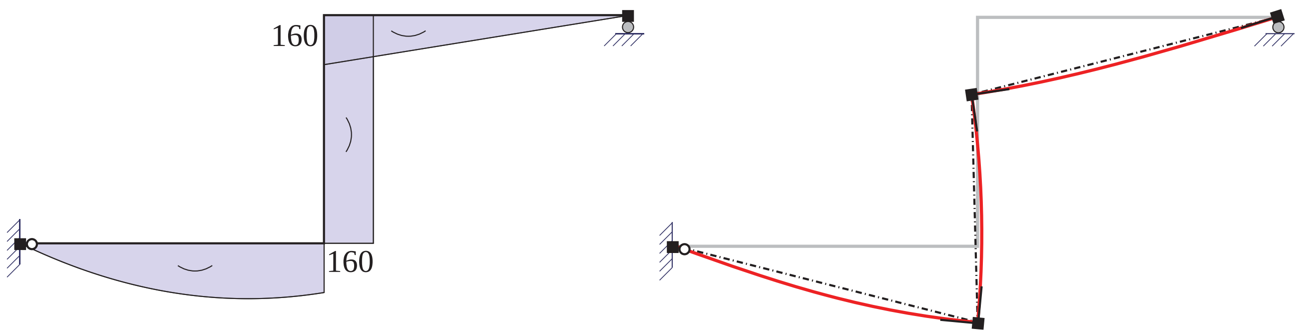
so that the basic forces $\boldsymbol{Q}$ are

$$\boldsymbol{Q} = \begin{pmatrix} 160 \\ -160 \\ 160 \\ -160 \end{pmatrix}$$

Fig. 7(a) shows the bending moment distribution under the given loading.

2. For determining the horizontal and vertical translation at node 3 we make use of the kinematic relations $\boldsymbol{V} = \mathbf{A}_f \boldsymbol{U}_f$

$$\begin{aligned} \boldsymbol{V}_1 &= -\frac{1}{8}\boldsymbol{U}_1 + \boldsymbol{U}_2 \\ \boldsymbol{V}_2 &= \boldsymbol{U}_2 + \frac{1}{6}\boldsymbol{U}_3 \\ \boldsymbol{V}_3 &= \frac{1}{6}\boldsymbol{U}_3 + \boldsymbol{U}_4 \\ \boldsymbol{V}_4 &= \frac{1}{8}\boldsymbol{U}_1 + \boldsymbol{U}_4 \end{aligned} \tag{3}$$



(a) Bending moment distribution

(b) Deformed shape

Figure 7: Bending moment distribution and deformed shape of structural model

We use the first two equations to eliminate the rotation dof $\boldsymbol{U}_2$ and the last two equations to eliminate the rotation dof $\boldsymbol{U}_4$. We get

$$\boldsymbol{V}_2 - \boldsymbol{V}_1 = \frac{1}{8}\boldsymbol{U}_1 + \frac{1}{6}\boldsymbol{U}_3$$
$$\boldsymbol{V}_4 - \boldsymbol{V}_3 = \frac{1}{8}\boldsymbol{U}_1 - \frac{1}{6}\boldsymbol{U}_3$$

From the addition of these two equations we determine the free dof translation $\boldsymbol{U}_1$

$$\boldsymbol{U}_1 = 4\,(\boldsymbol{V}_2 - \boldsymbol{V}_1 + \boldsymbol{V}_4 - \boldsymbol{V}_3)$$

Subtracting the second equation from the first gives the free dof translation $\boldsymbol{U}_3$

$$\boldsymbol{U}_3 = 3\,(\boldsymbol{V}_2 - \boldsymbol{V}_1 - \boldsymbol{V}_4 + \boldsymbol{V}_3)$$

We use the element deformation-force relations $\boldsymbol{v} = \mathbf{f}\,\boldsymbol{q} + \boldsymbol{v}_0$ to determine the element deformations $\boldsymbol{V}$. We get

$$\begin{aligned}
(EI)\boldsymbol{V}_1 &= \frac{L_a}{3}\boldsymbol{Q}_1 + \frac{wL_a^3}{24} &&= 640\\
(EI)\boldsymbol{V}_2 &= \frac{L_b}{6}(2\boldsymbol{Q}_2 - \boldsymbol{Q}_3) &&= -480\\
(EI)\boldsymbol{V}_3 &= \frac{L_b}{6}(-\boldsymbol{Q}_2 + 2\boldsymbol{Q}_3) &&= 480\\
(EI)\boldsymbol{V}_4 &= \frac{L_c}{3}\boldsymbol{Q}_4 &&= -426.67
\end{aligned}$$

With these values for the element deformations the vertical translation $\boldsymbol{U}_1$ becomes

$$\boldsymbol{U}_1 = \frac{4}{EI}\,(-480 - 640 - 426.67 - 480) = \frac{(-8106.67)}{EI} = -4.05 \cdot 10^{-2}$$

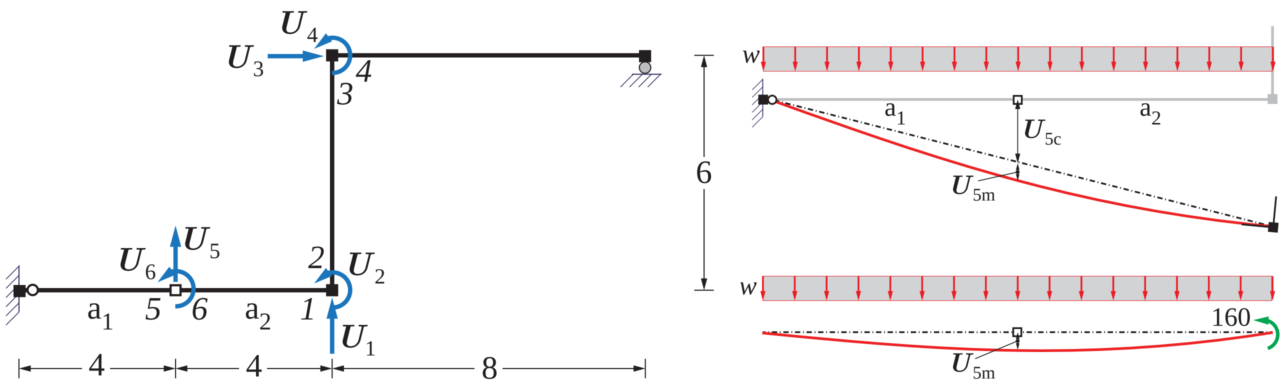
and the horizontal translation $\boldsymbol{U}_3$ becomes

$$\boldsymbol{U}_3 = \frac{3}{EI}\,(-480 - 640 + 426.67 + 480) = \frac{(-640)}{EI} = -3.2 \cdot 10^{-3}$$

Fig. 7(b) shows the deformed shape. In drawing the deformed shape it is important to note that the horizontal translation $\boldsymbol{U}_3$ is about 10 times smaller in value than the vertical translation $\boldsymbol{U}_1$. The element deformations $\boldsymbol{V}$ are measured from the element chords, which are shown with black dash-dot lines in the figure.

3. For determining the vertical translation at midspan of element a we introduce a node at the location, as shown in Fig. 8(a). Two additional dofs result with two additional element deformations on each side of the inserted node. We denote the corresponding displacements with $\boldsymbol{U}_5$ and $\boldsymbol{U}_6$ and the element deformations with $\boldsymbol{V}_5$ and $\boldsymbol{V}_6$. The additional node splits element a into two subelements a1 and a2. We use the kinematic relations for the two new dofs to determine $\boldsymbol{U}_5$. We have

$$\boldsymbol{V}_5 = \boldsymbol{v}_j^{(a1)} = -\frac{\boldsymbol{U}_5}{4} + \boldsymbol{U}_6$$
$$\boldsymbol{V}_6 = \boldsymbol{v}_i^{(a2)} = -\frac{\boldsymbol{U}_1}{4} + \frac{\boldsymbol{U}_5}{4} + \boldsymbol{U}_6$$



(a) Free dofs and corresponding element deformations

(b) Deformed shape of element a

Figure 8: Additional node for determination of translation at midspan of element a

Subtracting the first from the second equation to eliminate the rotation dof $\boldsymbol{U}_6$ gives

$$\boldsymbol{U}_5 = 2\left(\boldsymbol{v}_i^{(a2)} - \boldsymbol{v}_j^{(a1)}\right) + \frac{\boldsymbol{U}_1}{2} \tag{4}$$

We note that the translation $\boldsymbol{U}_5$ in (4) is made up of two contributions: a contribution $\boldsymbol{U}_{5c}$ in Fig. 8(b) equal to $\frac{\boldsymbol{U}_1}{2}$ due to the chord rotation of element a, and a contribution $\boldsymbol{U}_{5m}$ in Fig. 8(b) due to the deformed shape of element a *relative to the chord*. The latter depends on the element loading w and on the basic force $\boldsymbol{Q}_1$ at the end j, as reflected in the following determination of $\boldsymbol{v}_j^{(a1)}$ and $\boldsymbol{v}_i^{(a2)}$.

The deformations $\boldsymbol{v}$ of subelements a1 and a2 result from the element deformation-force relations $\boldsymbol{v} = \mathbf{f}\,\boldsymbol{q} + \boldsymbol{v}_0$

$$(EI)\,\boldsymbol{v}_j^{(a1)} = \frac{L_{a1}}{3}\boldsymbol{q}_j^{(a1)} + \frac{wL_{a1}^3}{24}$$
$$(EI)\,\boldsymbol{v}_i^{(a2)} = \frac{L_{a2}}{6}\left(2\,\boldsymbol{q}_i^{(a2)} - \boldsymbol{q}_j^{(a2)}\right) - \frac{wL_{a2}^3}{24}$$

To evaluate the above expressions we need the basic forces for subelements a1 and a2. To this end we determine the midspan moment value M_m from the superposition of the homogeneous and particular solution for element a

$$M_m = \frac{Q_1}{2} + \frac{wL_a^2}{8} = \frac{160}{2} + 80 = 160$$

so that

$$\boldsymbol{q}_j^{(a1)} = M_m = 160 \quad \text{and} \quad \boldsymbol{q}_i^{(a2)} = -M_m = -160$$

With these values and $\boldsymbol{q}_j^{(a2)} = \boldsymbol{Q}_1 = 160$ we get

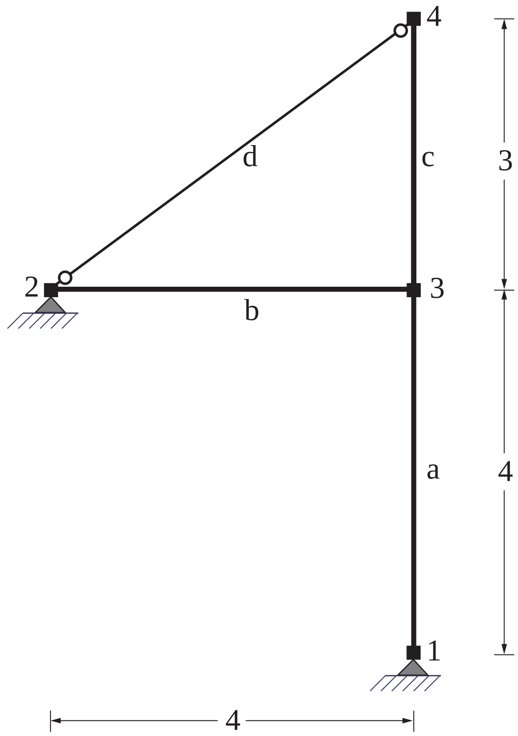
$$\begin{aligned}
(EI)\boldsymbol{v}_j^{(a1)} &= \frac{4}{3}(160) && + \frac{w(4)^3}{24} = \quad 240 \\
(EI)\boldsymbol{v}_i^{(a2)} &= \frac{4}{6}\left[2(-160) - (160)\right] && - \frac{w(4)^3}{24} = -346.67
\end{aligned}$$

The midspan translation value is

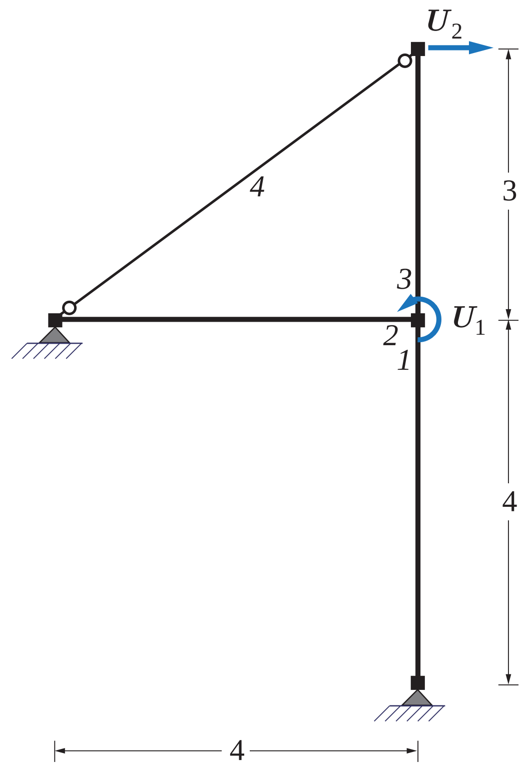
$$\boldsymbol{U}_5 = \frac{2}{EI}\left[(-346.67) - (240)\right] + \frac{1}{EI}\left(\frac{-8106.67}{2}\right) = \frac{(-5226.67)}{EI} = -2.61 \cdot 10^{-2}$$

3. Problem

The braced frame in Fig. 9(a) is subjected to initial thermal curvatures of $\kappa_0 = 3 \cdot 10^{-3}$ in element a, $\kappa_0 = -2 \cdot 10^{-3}$ in element b and $\kappa_0 = -3 \cdot 10^{-3}$ in element c. The frame elements a, b and c have flexural stiffness $EI = 30,000$ and can be assumed as inextensible. The brace element d has axial stiffness $EA = 20,000$.



(a) Geometry



(b) Independent free dofs

Figure 9: Geometry and independent free dofs

Fig. 9(b) shows the independent free dofs of the structural model under the assumption that the frame elements a, b and c are inextensible.

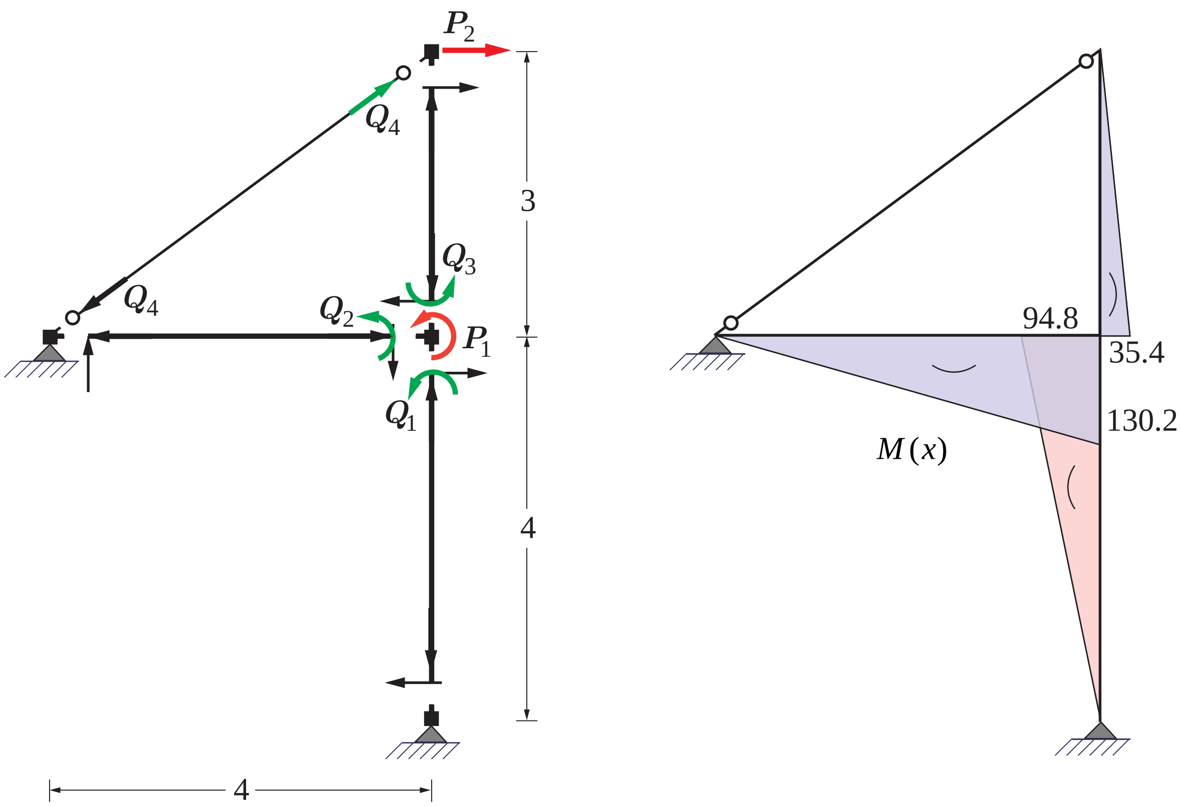
1. Fig. 10(a) shows the equilibrium equations that correspond to the independent free dofs of the structural model. The corresponding basic forces of primary interest that do not include the axial forces in the frame elements a, b and c are 4. The degree of static indeterminacy of the structural model is, therefore, 2.

The computer analysis of the braced frame under the initial thermal curvatures in elements a, b and c gives the following basic force values for these elements:

$$\boldsymbol{q}^{(a)} = \boldsymbol{Q}_1 = -94.81 \qquad \boldsymbol{q}^{(b)} = \boldsymbol{Q}_2 = 130.18 \qquad \boldsymbol{q}^{(c)} = \boldsymbol{Q}_3 = -35.37$$

2. Fig. 10(b) shows the bending moment distribution. The force $\boldsymbol{Q}_4$ in the brace is not required but can be determined from the horizontal force equilibrium at dof #2. It gives

$$\boldsymbol{P}_2 = \frac{\boldsymbol{Q}_3}{3} + 0.8\boldsymbol{Q}_4 \quad \rightarrow \quad \boldsymbol{Q}_4 = -\frac{\boldsymbol{Q}_3}{2.4} = 14.737$$



(a) Basic forces and equilibrium equations (b) Bending moment distribution

Figure 10: Basic forces, equilibrium equations and bending moment distribution

3. The kinematic relations of the structural model are

$$\begin{aligned} \boldsymbol{V}_1 &= \boldsymbol{U}_1 \\ \boldsymbol{V}_2 &= \boldsymbol{U}_1 \\ \boldsymbol{V}_3 &= \boldsymbol{U}_1 + \frac{\boldsymbol{U}_2}{3} \\ \boldsymbol{V}_4 &= 0.8\boldsymbol{U}_2 \end{aligned}$$

If we have bothered determining the brace force $\boldsymbol{Q}_4$, then the horizontal translation $\boldsymbol{U}_2$ can be readily determined from the 4th kinematic relation, namely

$$(EI)\boldsymbol{U}_2 = 1.25(EI)\boldsymbol{V}_4 = 1.25\frac{EI}{EA}\boldsymbol{Q}_4 L_d = 138.16 \quad \rightarrow \quad \boldsymbol{U}_2 = 0.0046$$

Alternatively, if we did not bother calculating the brace force, we can get the horizontal translation $\boldsymbol{U}_2$ from the second and third kinematic relation, namely

$$\boldsymbol{U}_2 = 3\left(\boldsymbol{V}_3 - \boldsymbol{V}_2\right) \tag{5}$$

We use the deformation force relation of elements b and c to determine $\boldsymbol{V}_2$ and $\boldsymbol{V}_3$, respectively.

$$\begin{aligned} (EI)\boldsymbol{V}_2 &= \frac{L_b}{3}\boldsymbol{Q}_2 + EI\frac{\kappa_0 L_b}{2} & & (EI)\boldsymbol{V}_2 = \frac{4}{3}(130.18) + \frac{(-60)4}{2} = 53.57 \\ & & \rightarrow \quad & \\ (EI)\boldsymbol{V}_3 &= \frac{L_c}{3}\boldsymbol{Q}_3 - EI\frac{\kappa_0 L_b}{2} & & (EI)\boldsymbol{V}_3 = \frac{3}{3}(-35.37) - \frac{(-90)3}{2} = 99.63 \end{aligned}$$

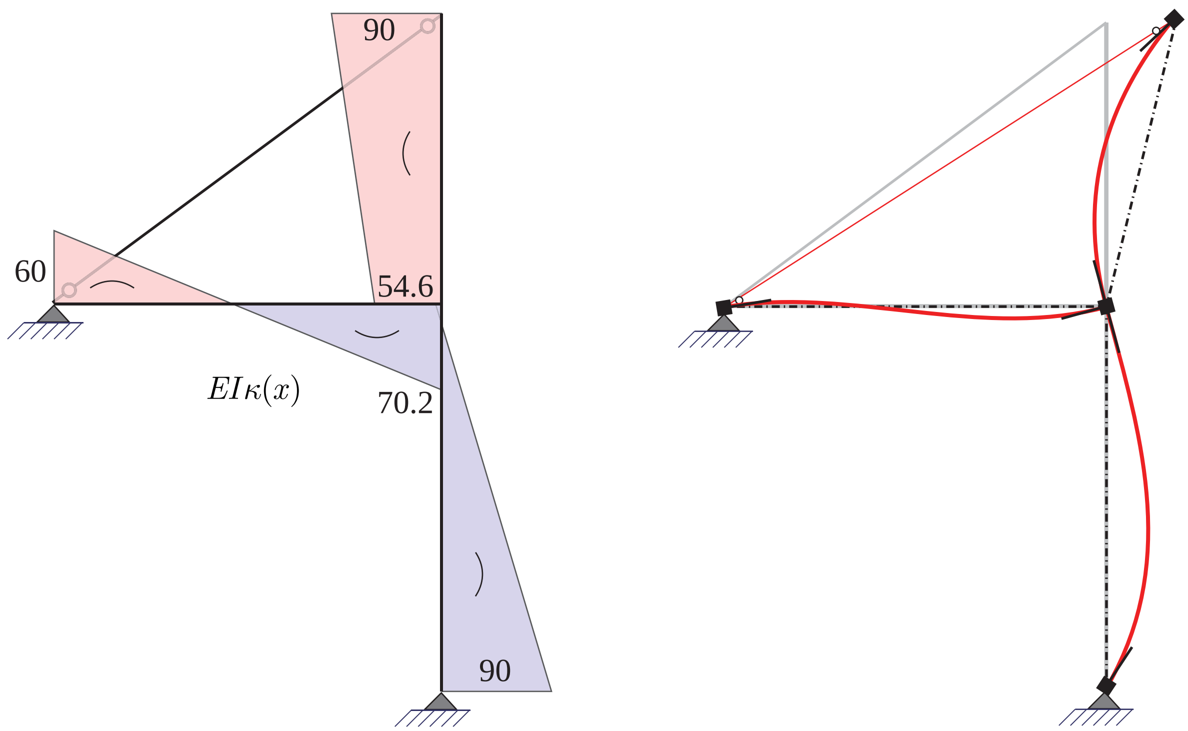
Substituting into (5) gives

$$(EI)\boldsymbol{U}_2 = 3(EI)\,(99.63 - 53.57) = 138.17 \quad \rightarrow \quad \boldsymbol{U}_2 = 0.0046$$

The rotation $\boldsymbol{U}_1$ is simply

$$(EI)\boldsymbol{U}_1 = (EI)\,\boldsymbol{V}_2 = 53.57 \quad \rightarrow \quad \boldsymbol{U}_1 = 0.0018$$

4. To draw the deformed shape we first locate the displaced node #4 with the translation $\boldsymbol{U}_2$, draw the element chords, and then use either the element deformations $\boldsymbol{V}_1$, $\boldsymbol{V}_2$ and $\boldsymbol{V}_3$ noting that these measure the angle between the element chord and the tangent at the appropriate element end, or, alternatively the curvature distribution in Fig. 11(a). The latter results from the superposition of the moment distribution with the EI-fold initial curvatures of elements a, b and c, which are 90, -60 and -90, respectively. Fig. 11(b) shows the resulting deformed shape.



(a) Curvature distribution (b) Deformed shape

Figure 11: Curvature distribution and deformed shape