UNIVERSITY OF CALIFORNIA, BERKELEY
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Dept. of Civil and Environmental Engineering
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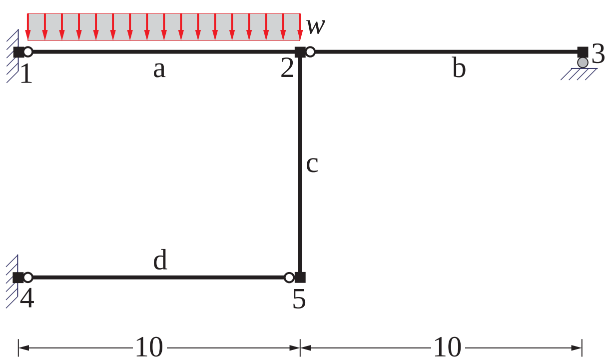
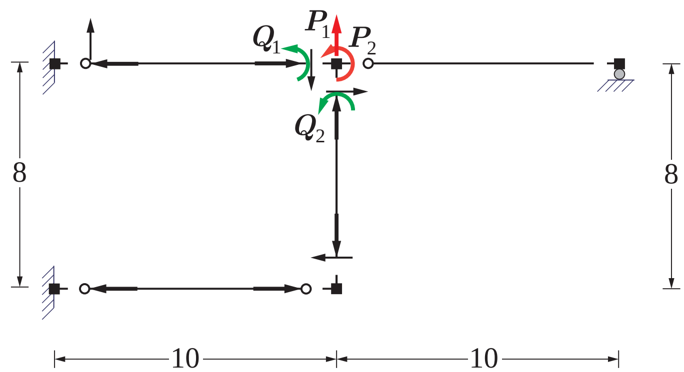
CE 220 - Structural Analysis
Solution for Homework Set #4

1. Problem

The frame in Fig. 1(a) has 8 free dofs and 8 basic forces. There are 2 trivial dofs and 4 axial forces of secondary interest. This leaves 2 free dofs and 2 basic forces of primary interest, as shown in Fig. 1(b).



(a) Model geometry



(b) Basic forces and equilibrium equations

Figure 1: Equilibrium equation and basic element force numbering w/o axial forces in frame elements

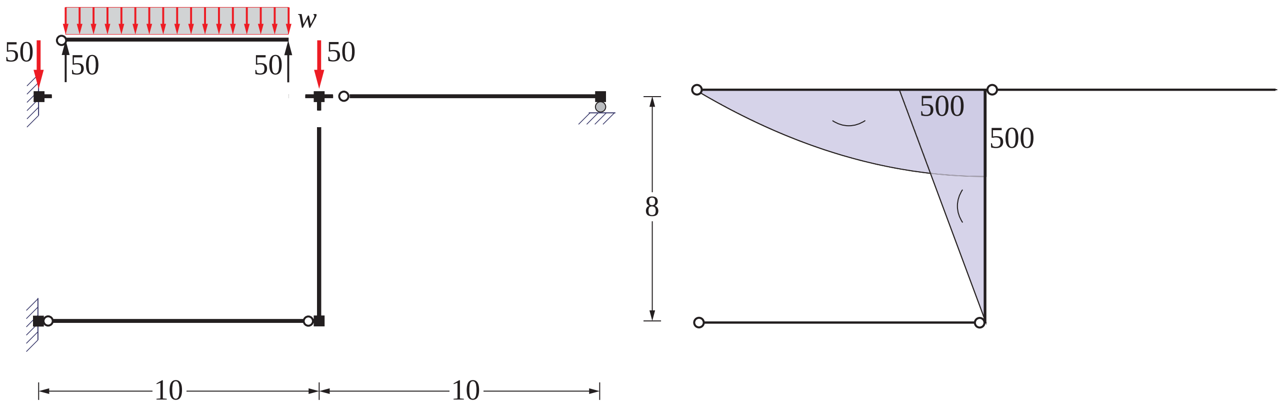
The equilibrium equations at the free dofs are

$$\begin{aligned} \boldsymbol{P}_1 &= -\frac{\boldsymbol{Q}_1}{10} \qquad + 50 \\ \boldsymbol{P}_2 &= \quad \boldsymbol{Q}_1 + \boldsymbol{Q}_2 \end{aligned} \tag{1}$$

The equivalent nodal forces due to the distributed load of element a are shown in Fig. 2(a). As transverse end forces at the element ends, they follow the sign convention for basic forces in (1). Moving the transverse end force from the element end to the node, as Fig. 2(a) shows corresponds to moving the value of 50 units to the left hand side of the equation in (1). This transforms the effect of the distributed element load to an equivalent nodal force.

$$\begin{aligned} -50 &= -\frac{\boldsymbol{Q}_1}{10} \\ 0 &= \quad \boldsymbol{Q}_1 + \boldsymbol{Q}_2 \end{aligned} \tag{2}$$

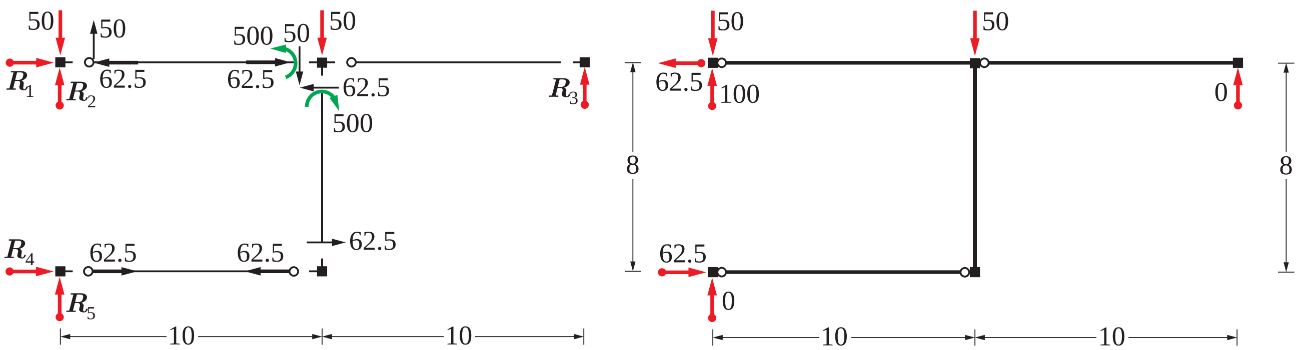
Solving for the basic forces $\boldsymbol{Q}$ gives: $\boldsymbol{Q}_1 = 500$ and $\boldsymbol{Q}_2 = -500$. The bending moment diagram is shown in Fig. 2(b). Because the shear force is zero at end j of element a *from the addition of the particular and the homogeneous solution*, the maximum moment value is equal to 500 units. *Note that the shear force is 100 units at end i of element a from the addition of the particular and the homogeneous solution.* Fig. 3(a) shows the dependent end forces of each element free body. From the node free body equilibrium we determine the axial forces in elements a through d, as Fig. 3(a) shows. A tensile axial force of 62.5



(a) Equivalent nodal forces for load of element a

(b) Bending moment diagram

Figure 2: Bending moment diagram under distributed load of element a



(a) Node free body equilibrium

(b) Structure free body equilibrium

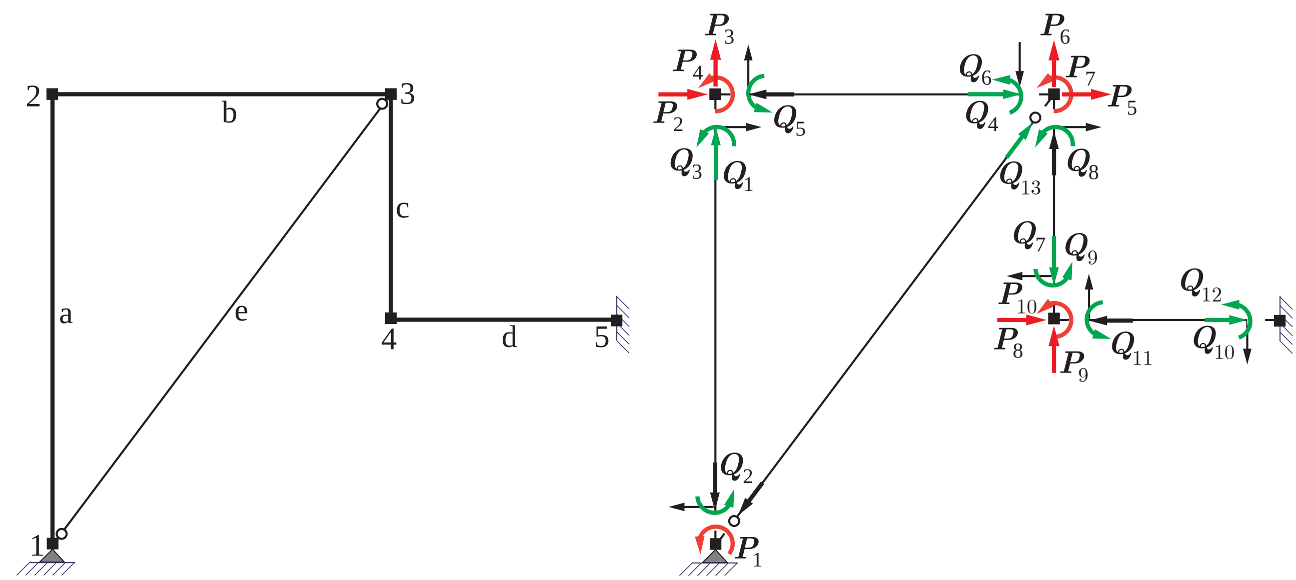
Figure 3: Node free body and structure free body equilibrium

units results in element a, and a compressive force of -62.5 units in element d. The other two elements do not have axial forces.

With the dependent element end forces and the axial forces we can readily determine the support reactions from node free body equilibrium at nodes #1, #3 and #4. Fig. 3(b) shows the structure free body with all nodal forces. It is easy to verify the horizontal and vertical force equilibrium. Also the moment equilibrium about node #1 is readily verified, if we note that 62.5 $62.5 \cdot 8 = 500$. *It is important to note that for node and structure free body equilibrium we make use of the equivalent nodal forces due to element loading.*

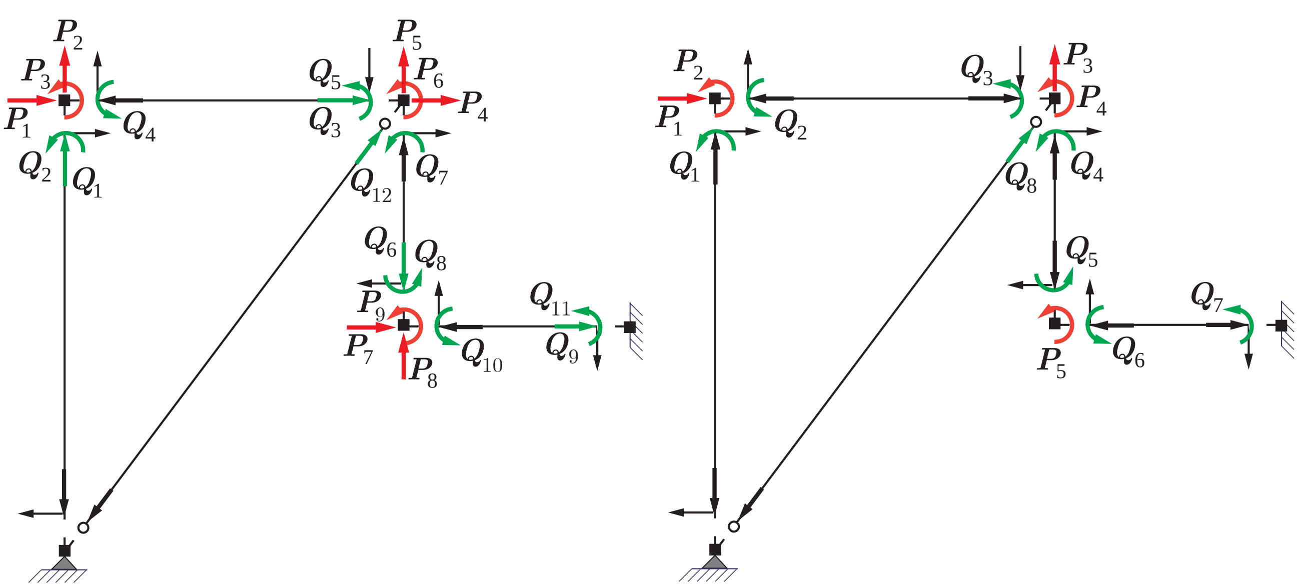
2. Problem

The degree of static indeterminacy of the structural model in Fig. 4(a) is $NOS = 3$, because there are 10 equilibrium equations for 13 basic element forces in Fig. 4(b). This number does not change when we remove one trivial equation and the corresponding basic element force in Fig. 5(a), nor does it change when we remove or combine 4 equilibrium equations involving the axial forces in elements a-d so as to reduce the basic element forces of primary interest to 8, as Fig. 5(b) shows.



(a) Structural model (b) Complete set

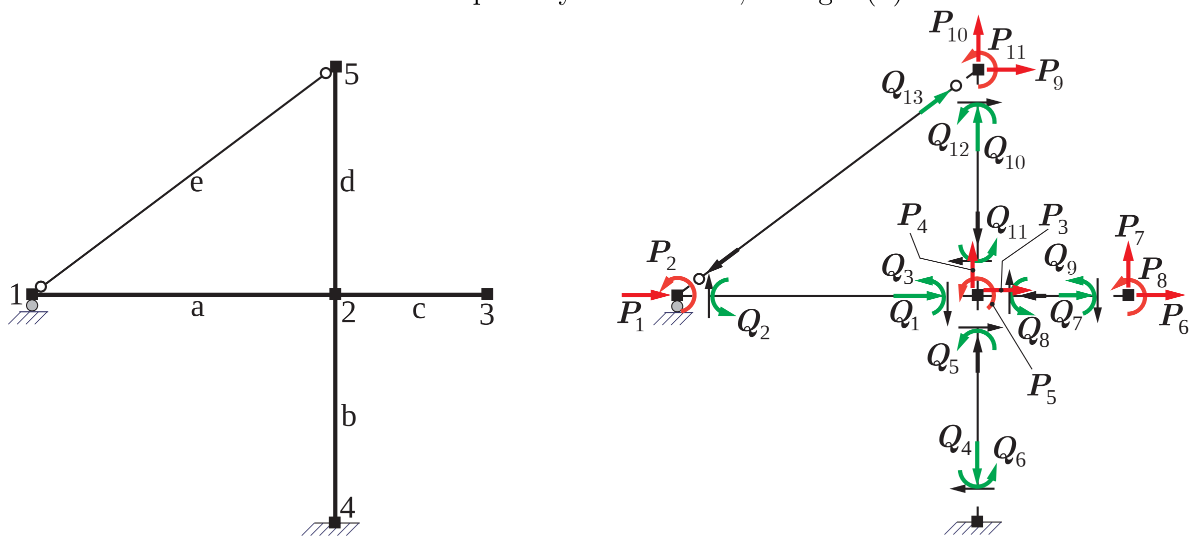
Figure 4: Complete set of free dof equilibrium equations and basic forces



(a) without trivial (b) without axial forces in a-d

Figure 5: Reduced set of free dof equilibrium equations and corresponding basic forces

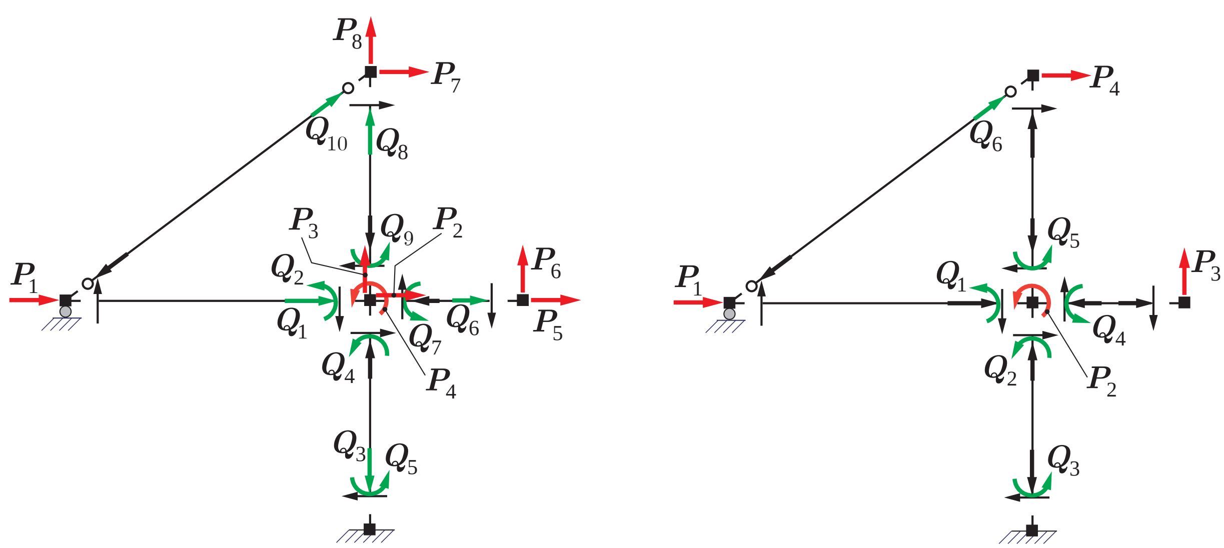
The degree of static indeterminacy of the structural model in Fig. 6(a) is $NOS = 2$, because there are 11 equilibrium equations for 13 basic element forces in Fig. 6(b). This number does not change when we remove three trivial equations and the corresponding basic element forces in Fig. 7(a), nor does it change when we remove or combine 4 equilibrium equations involving the axial forces in elements a-d so as to reduce the basic element forces of primary interest to 8, as Fig. 7(b) shows.



(a) Structural model

(b) Complete set

Figure 6: Complete set of free dof equilibrium equations and basic forces



(a) without trivial

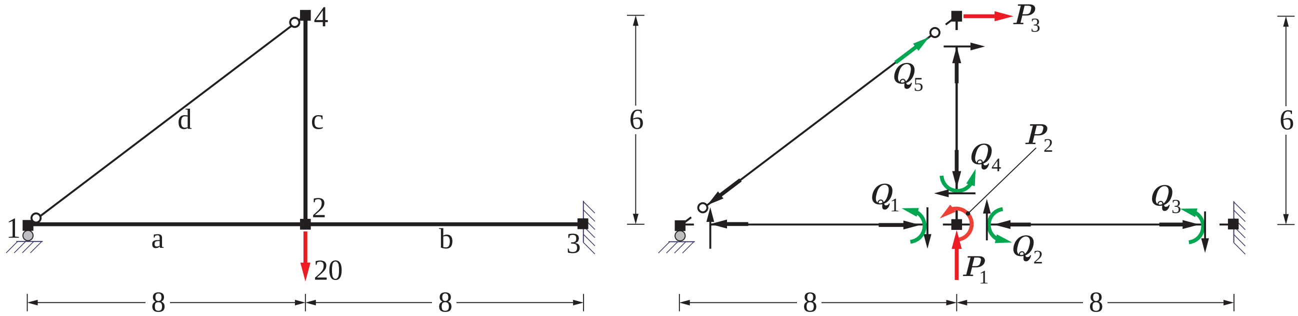
(b) without axial forces in a-d

Figure 7: Reduced set of free dof equilibrium equations and corresponding basic forces

For both structures the significant reduction in the number of relevant equilibrium equations and corresponding basic element forces of primary interest is noteworthy. While the computer analysis uses the complete set, significant insight into the structural response is gained from the reduction to the basic element forces of primary interest and the corresponding equilibrium equations.

3. Problem

The braced frame in Fig. 8(a) has 8 free dofs and 10 basic forces. There are two trivial dofs and 3 axial forces in elements a, b, and c of secondary interest. This leaves 3 free dofs and 5 basic forces of primary interest, as shown in Fig. 8(b).



(a) Structural model

(b) Numbering of equations and unknowns

Figure 8: Equilibrium equation and basic element force numbering w/o axial forces in elements a-c

The equilibrium equations at the free dofs are

$$
\begin{aligned}
\boldsymbol{P}_1 &= -\frac{\boldsymbol{Q}_1}{8} + \frac{\boldsymbol{Q}_2 + \boldsymbol{Q}_3}{8} + 0.6\boldsymbol{Q}_5 \\
\boldsymbol{P}_2 &= \boldsymbol{Q}_1 + \boldsymbol{Q}_2 + \boldsymbol{Q}_4 \\
\boldsymbol{P}_3 &= \frac{\boldsymbol{Q}_4}{6} + 0.8\boldsymbol{Q}_5
\end{aligned}
\tag{3}
$$

We select the basic forces $\boldsymbol{Q}_3$ and $\boldsymbol{Q}_4$ are redundants and solve the equilibrium equations three times: (a) the particular solution under the applied nodal force with $\boldsymbol{Q}_3 = 0$ and $\boldsymbol{Q}_4 = 0$, (b) the homogeneous solution for $\boldsymbol{Q}_3 = 1$ and $\boldsymbol{Q}_4 = 0$, and (c) the homogeneous solution for $\boldsymbol{Q}_3 = 0$ and $\boldsymbol{Q}_4 = 1$.

Particular solution: $\boldsymbol{Q}_3 = 0$ and $\boldsymbol{Q}_4 = 0$

$$
\begin{aligned}
-20 &= -\frac{\boldsymbol{Q}_1}{8} + \frac{\boldsymbol{Q}_2}{8} + 0.6\boldsymbol{Q}_5 \\
0 &= \boldsymbol{Q}_1 + \boldsymbol{Q}_2 \\
0 &= + 0.8\boldsymbol{Q}_5
\end{aligned}
\quad \rightarrow \quad \boldsymbol{Q}_5 = 0 \quad \boldsymbol{Q}_1 = 80 \quad \boldsymbol{Q}_2 = -80
\tag{4}
$$

First homogeneous solution: $\boldsymbol{Q}_3 = 1$ and $\boldsymbol{Q}_4 = 0$

$$
\begin{aligned}
0 &= -\frac{\boldsymbol{Q}_1}{8} + \frac{\boldsymbol{Q}_2 + 1}{8} + 0.6\boldsymbol{Q}_5 \\
0 &= \boldsymbol{Q}_1 + \boldsymbol{Q}_2 \\
0 &= + 0.8\boldsymbol{Q}_5
\end{aligned}
\quad \rightarrow \quad \boldsymbol{Q}_5 = 0 \quad \boldsymbol{Q}_1 = \frac{1}{2} \quad \boldsymbol{Q}_2 = -\frac{1}{2}
\tag{5}
$$

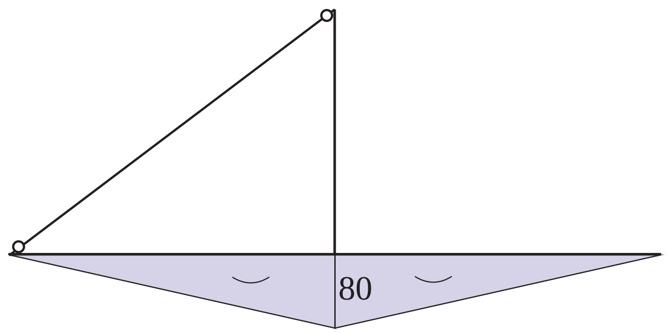
Second homogeneous solution: $\boldsymbol{Q}_3 = 0$ and $\boldsymbol{Q}_4 = 1$

$$
\begin{aligned}
0 &= -\frac{\boldsymbol{Q}_1}{8} + \frac{\boldsymbol{Q}_2}{8} + 0.6\boldsymbol{Q}_5 \\
0 &= \boldsymbol{Q}_1 + \boldsymbol{Q}_2 + 1 \\
0 &= \frac{1}{6} + 0.8\boldsymbol{Q}_5
\end{aligned}
\quad \rightarrow \quad \boldsymbol{Q}_5 = -\frac{5}{24} \quad \boldsymbol{Q}_1 = -1 \quad \boldsymbol{Q}_2 = 0
\tag{6}
$$

The complete solution of the equilibrium equations can be written in the compact form

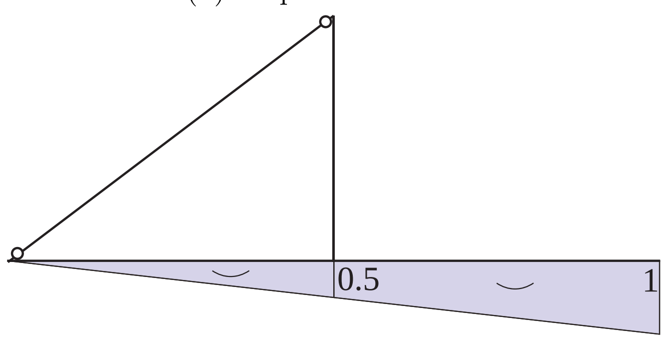
$$
\boldsymbol{Q} = \boldsymbol{Q}_p + \bar{\mathbf{B}}_x \boldsymbol{Q}_x
$$
$$
\text{with} \quad \boldsymbol{Q}_p = \begin{pmatrix} 80 \\ -80 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \bar{\mathbf{B}}_x = \begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{1}{2} & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & -\frac{5}{24} \end{bmatrix} \qquad \boldsymbol{Q}_x = \begin{pmatrix} \boldsymbol{Q}_3 \\ \boldsymbol{Q}_4 \end{pmatrix} \tag{7}
$$

With the redundant basic forces $\boldsymbol{Q}_3$ and $\boldsymbol{Q}_4$ equal to zero, there are no transverse forces in element c. Consequently, the horizontal force equilibrium at node #4 requires that the axial force in the brace be zero. The primary structure is a simply supported girder consisting of elements a and b. The nodal force acting at midspan of this girder gives rise to the bending moment distribution in the figure on the right with maximum value $\frac{P \cdot 16}{4} = 80$.



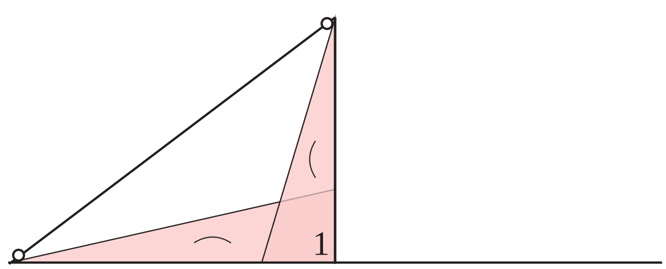
$M(x)$ for particular solution

With the redundant basic force $\boldsymbol{Q}_4$ equal to zero, there are no transverse forces in element c. Consequently, the horizontal force equilibrium at node #4 requires that the axial force in the brace be zero, as for the particular solution. The primary structure is a simply supported girder with unit moment at node #3. Because there is no transverse force acting at node #2, the moment diagram decreases linearly from the right support at node #3 to zero at the left support at node #1.



$M(x)$ for first homogeneous solution

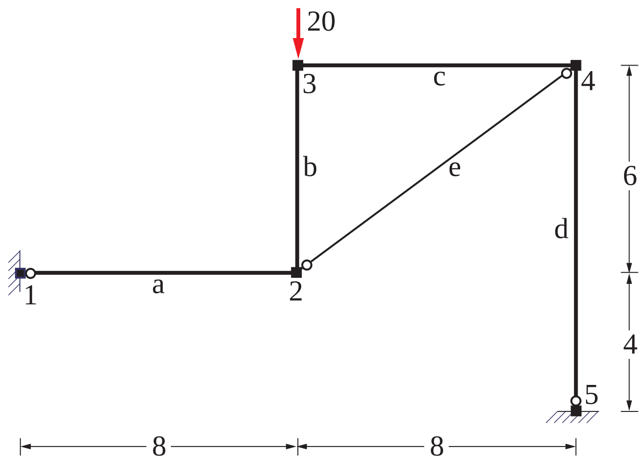
With the redundant basic force $\boldsymbol{Q}_4$ equal to 1 either $\boldsymbol{Q}_1$ or $\boldsymbol{Q}_2$ need to resist $\boldsymbol{Q}_4$ for moment equilibrium at node #2. A negative $\boldsymbol{Q}_1$ produces an upward transverse force, while a negative $\boldsymbol{Q}_2$ produces a downward transverse force. The transverse force caused by $\boldsymbol{Q}_4$ produces a compressive force in the brace and gives rise to a downward force at node #4. Thus, $\boldsymbol{Q}_1$ is the "right resistance" for vertical force equilibrium at node #2 and not $\boldsymbol{Q}_2$. An easier way to see that $\boldsymbol{Q}_2 = 0$ is to consider the global moment equilibrium about node #1. Without applied nodal forces and without moment at the right support there can be no vertical support reaction at this location and, thus, no transverse force in element b, leading to the conclusion that with $\boldsymbol{Q}_3 = 0$ $\boldsymbol{Q}_2$ must be 0.



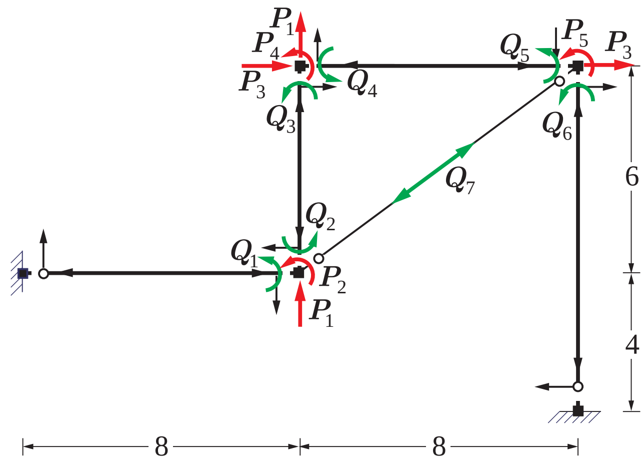
$M(x)$ for second homogeneous solution

4. Problem

The braced frame in Fig. 9(a) is subjected to a downward force at node 3. The basic forces of primary interest *exclude the axial forces in elements a through d* and are 7, as shown in Fig. 9(b). The corresponding equilibrium equations at the free dofs of the structural model that do not involve the axial forces in elements a through d are 5, as shown in Fig. 9(b).



(a) Geometry and loading



(b) Basic forces and free dofs

Figure 9: Braced frame

1. The equilibrium equations are

$$\begin{aligned} \boldsymbol{P}_1 = -20 &= -\frac{\boldsymbol{Q}_1}{8} + \frac{\boldsymbol{Q}_4 + \boldsymbol{Q}_5}{8} - 0.6\boldsymbol{Q}_7 \\ \boldsymbol{P}_2 = \quad 0 &= \boldsymbol{Q}_1 + \boldsymbol{Q}_2 \\ \boldsymbol{P}_3 = \quad 0 &= \frac{\boldsymbol{Q}_2 + \boldsymbol{Q}_3}{6} + \frac{\boldsymbol{Q}_6}{10} + 0.8\boldsymbol{Q}_7 \\ \boldsymbol{P}_4 = \quad 0 &= \boldsymbol{Q}_3 + \boldsymbol{Q}_4 \\ \boldsymbol{P}_5 = \quad 0 &= \boldsymbol{Q}_5 + \boldsymbol{Q}_6 \end{aligned} \tag{8}$$

2. There are five equations in 7 unknown basic forces of primary interest, so that the degree of static indeterminacy NOS is 2.

3. *The following solution from the CE220 homework collection determines the particular and homogeneous solutions of the problem in detail. This year's statement of the problem required only to use* FEDEASLab *to determine the particular solution with* $\bar{\mathbf{B}}_i \boldsymbol{P}_f$ *giving the result in equation* (12), *and substitute it into the equilibrium equations in* (8) *to confirm the answer, and then do the same for the homogeneous solution* $\boldsymbol{Q}_{h_1}$ *in* (14) *and* $\boldsymbol{Q}_{h_2}$ *in* (16)*; these correspond to the two columns of the* $\bar{\mathbf{B}}_x$ *matrix that the function* `BbariBbarx_matrix` *returns with the specification of* $\boldsymbol{Q}_1$ *and* $\boldsymbol{Q}_4$ *as redundants.*

Step by Step Solution. Before proceeding with the determination of the particular and the homogeneous static solutions we reduce the number of equations and unknowns with the help of the moment equations $\boldsymbol{Q}_2 = -\boldsymbol{Q}_1$, $\boldsymbol{Q}_3 = -\boldsymbol{Q}_4$, and $\boldsymbol{Q}_5 = -\boldsymbol{Q}_6$. We also change the sign of the first equation and remove some fractions in equations (1) and (3) by multiplication of both equation sides with 8 and 6, respectively. This gives

$$\begin{aligned} 160 &= \quad \boldsymbol{Q}_1 - \boldsymbol{Q}_4 + \boldsymbol{Q}_6 + 4.8\boldsymbol{Q}_7 \\ 0 &= -\left(\boldsymbol{Q}_1 + \boldsymbol{Q}_4\right) + 0.6\boldsymbol{Q}_6 + 4.8\boldsymbol{Q}_7 \end{aligned} \tag{9}$$

These are now 2 equations in 4 unknowns. Setting $\boldsymbol{Q}_1 = 0$ and $\boldsymbol{Q}_4 = 0$ for the particular solution $\boldsymbol{Q}_p$ and subtracting the second from the first equation gives

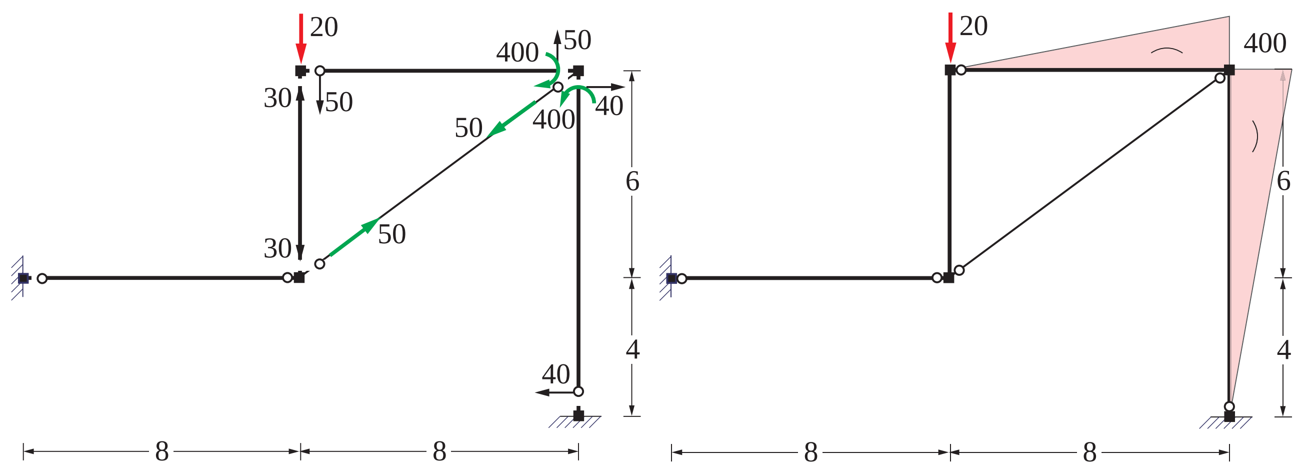
$$160 = 0.4\boldsymbol{Q}_6 \quad \rightarrow \quad \boldsymbol{Q}_6 = 400 \tag{10}$$

Substituting into the first equation gives

$$160 = 400 + 4.8\boldsymbol{Q}_7 \quad \rightarrow \quad \boldsymbol{Q}_7 = -50 \tag{11}$$

So that

$$\boldsymbol{Q}_p = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -400 \\ 400 \\ -50 \end{pmatrix} \tag{12}$$



(a) Force flow for particular solution

(b) $M(x)$ for particular solution

Figure 10: Particular solution of structural equilibrium equations

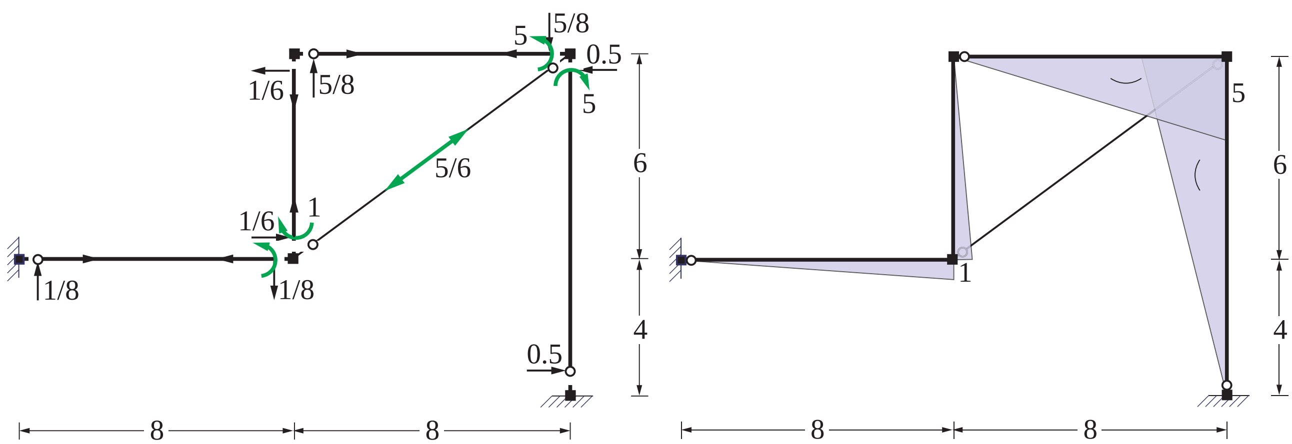
Fig. 10(a) shows the element free bodies and the relevant forces for establishing the particular solution with force flow considerations (or, allow us to check the accuracy of the numerical results): with elements a and b unable to carry transverse forces, the applied nodal force needs to be resisted by the vertical component of the brace force and the transverse end force of element c. The latter arises because of $\boldsymbol{Q}_5 = -\boldsymbol{Q}_6$ with $\boldsymbol{Q}_6$ necessary to resist the horizontal component of the brace force. The signs are such that the highest contribution for the vertical force equilibrium at dof #1 comes from element c. The node equilibrium in Fig. 10(a) is rather insightful for the force distribution resisting the applied nodal force of 20 units. Fig. 10(a) shows the corresponding moment distribution $M(x)$ for the particular solution.

For the first homogeneous solution we set $\boldsymbol{Q}_1 = 1$ and $\boldsymbol{Q}_4 = 0$ into the equilibrium equations without any nodal forces on the left hand side, namely

$$\begin{aligned} 0 &= 1 + \boldsymbol{Q}_6 + 4.8\boldsymbol{Q}_7 \\ 0 &= -1 + 0.6\boldsymbol{Q}_6 + 4.8\boldsymbol{Q}_7 \end{aligned} \tag{13}$$

Subtracting the second equation from the first gives $\boldsymbol{Q}_6 = -5$, which when substituted into the first equation gives $\boldsymbol{Q}_7 = 5/6$, so that the first homogeneous solution is

$$\boldsymbol{Q}_{h_1} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 5 \\ -5 \\ \frac{5}{6} \end{pmatrix} \tag{14}$$



(a) Force flow for first homogeneous solution (b) $M(x)$ for for first homogeneous solution

Figure 11: First homogeneous solution of structural equilibrium equations

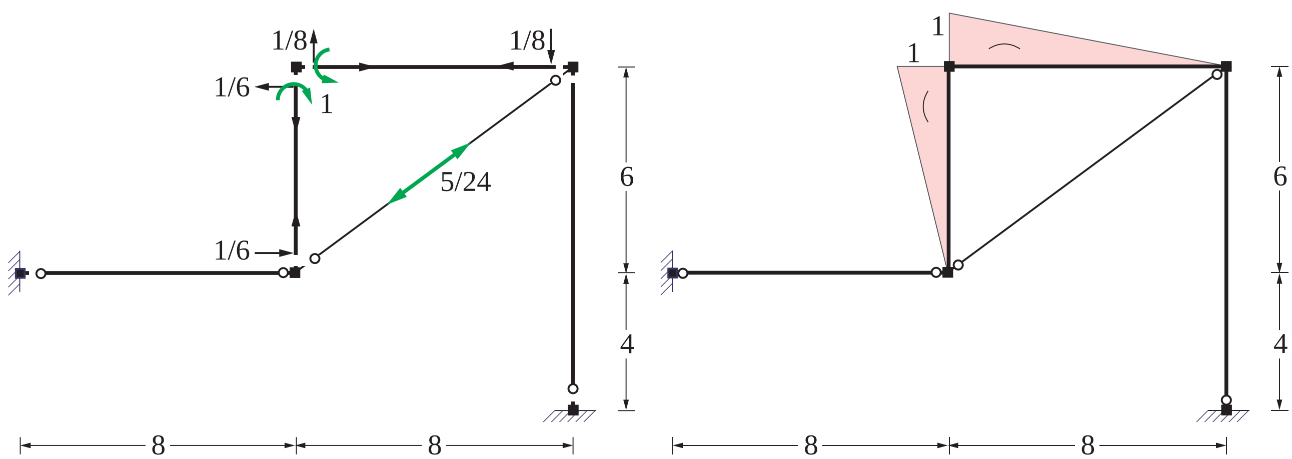
Fig. 11(a) shows the element free bodies and the relevant forces for establishing the first homogeneous solution with force flow considerations: the vertical component of the brace force needs to balance the transverse end force of element a, while its horizontal component needs to balance the transverse end force of element b. The direction of these forces is, however, such that an additional variable, the end moment at the top of element d is required to satisfy the horizontal and vertical force equilibrium. The relative contribution of the brace and the moment at the top of element d is best established by writing the two equilibrium equations and solving them. Fig. 11(a) shows the moment distribution $M(x)$ for the particular solution. Fig. 11(b) shows the moment distribution $M(x)$ for the first homogeneous solution.

For the second homogeneous solution we set $\boldsymbol{Q}_1 = 0$ and $\boldsymbol{Q}_4 = 1$ into the equilibrium equations without any nodal forces on the left hand side, namely

$$\begin{aligned} 0 &= -1 + \boldsymbol{Q}_6 + 4.8\boldsymbol{Q}_7 \\ 0 &= -1 + 0.6\boldsymbol{Q}_6 + 4.8\boldsymbol{Q}_7 \end{aligned} \tag{15}$$

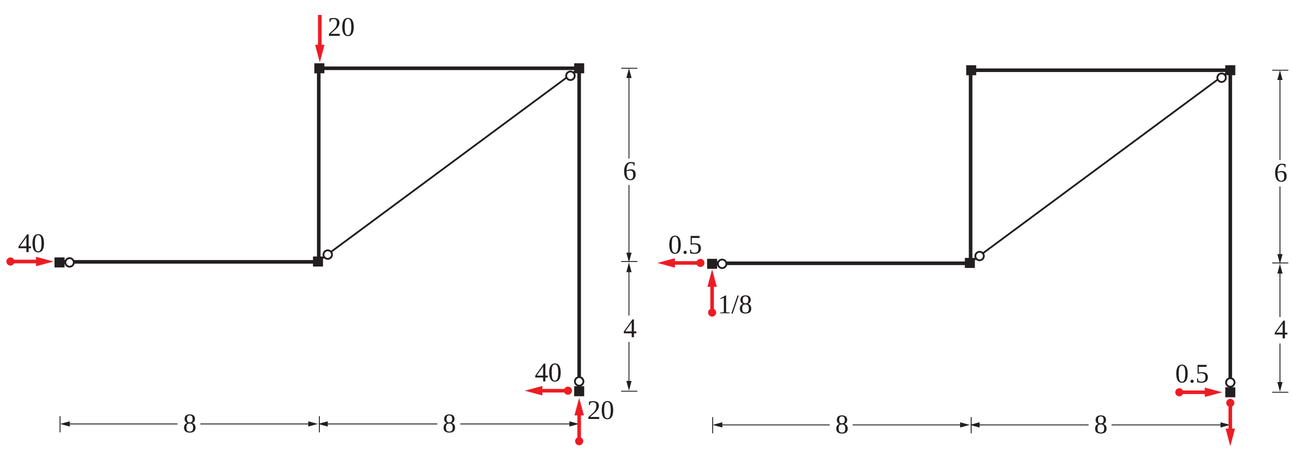
Subtracting the second equation from the first gives $\boldsymbol{Q}_6 = 0$, which when substituted into the first equation gives $\boldsymbol{Q}_7 = 5/24$, so that the second homogeneous solution is

$$\boldsymbol{Q}_{h_2} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ \frac{5}{24} \end{pmatrix} \tag{16}$$



(a) Force flow for second homogeneous solution (b) $M(x)$ for for second homogeneous solution

Figure 12: Second homogeneous solution of structural equilibrium equations



(a) Support forces for particular solution (b) Support forces for first homogeneous solution

Figure 13: Support reactions for particular and homogeneous solutions

Fig. 12(a) shows the element free bodies and the relevant forces for establishing the second homogeneous solution with force flow considerations: with element a unable to carry transverse forces the vertical component of the brace force must balance the transverse end force of element c, and the horizontal component of the brace force must balance the transverse end force of element b. Because the lengths of elements b and c relate to each other like the direction cosines of the brace and because of the direction of the transverse end forces in elements b and c, the brace force is able to satisfy *both the vertical and the horizontal force equilibrium* without assistance from element d. *A closed force loop results between elements b, c and d, so that no support reactions arise*. Fig. 12(b) shows the moment distribution $M(x)$ for the second homogeneous solution.

4. Fig. 13 shows the support reactions for the particular and for the first homogeneous solution. These follow directly from the corresponding force flow considerations in Fig. 10(a) and Fig. 11(a). We have already concluded that the second homogeneous solution does not give rise to support reactions, because it produces a closed force loop that does not involve elements a and d, which are connected to restrained dofs. By contrast, the first homogeneous solution gives rise to forces in these elements, which, in turn, give rise to support reactions.

5. Problem

The objective of this problem is to compare the response of the Howe truss with rigid connections in Fig. 14 with the response of the structural system that results after removing the diagonals from the Howe truss in Fig. 15. This system is known as Vierendeel truss, even though it is not really a truss since its load carrying capacity depends on the rigid connections between the vertical posts and the upper and the lower chord.

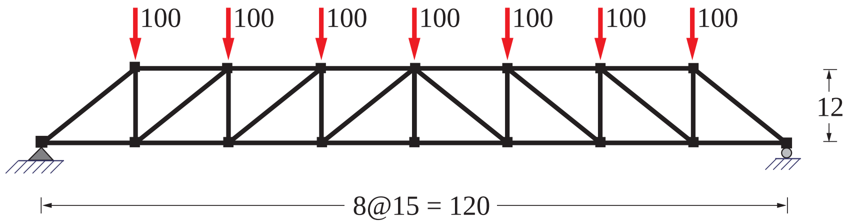


Figure 14: Howe truss with rigid connections

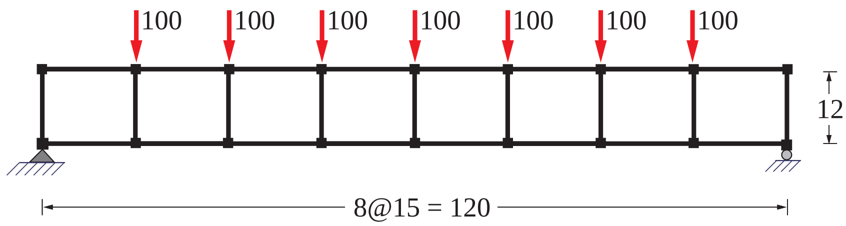


Figure 15: Vierendeel truss

The analysis of the Howe truss with pinned and rigid connections in the 3. Problem of the last homework revealed that the type of connection has little effect on the axial forces in the truss members and on the midspan deflection of the simply supported system. Doubling the flexural stiffness EI of all truss members from $EI = 2 \cdot 10^5$ to $EI = 4 \cdot 10^5$ reduced the midspan deflection from 0.5806 to 0.5667, a reduction of 2.4%, while increasing significantly the bending moments in the truss members without affecting much the axial forces.
For the following comparison the element properties are $EA = 5 \cdot 10^5$ and $EI = 4 \cdot 10^5$.
Fig. 16 shows the axial forces in the lower chord, upper chord, diagonals and vertical posts for the Howe truss with rigid connections. The midspan deflection under the given loading is -0.5667. Fig. 17 shows the bending moments that arise because of the rigid connections.
The deformed shape of the Howe truss with pinned or rigid connections is practically the same. The deformed shape for rigid connections in Fig. 18 resembles the deformed shape of a simply supported girder *under flexural deformations* with the vertical posts practically normal to the deformed shape of the upper and lower chord in good agreement with the Euler-Bernoulli assumption for beams that plane sections remain plane after deformation.
The effect of the diagonals is evident in the response of the Vierendeel truss in Fig. 15 that does not have diagonal members. The midspan deflection of the Vierendeel truss is -1.1195, *practically double the value of the Howe truss with rigid connections* clearly demonstrating the *stiffening effect* of the diagonals in the Howe truss. Equally interesting is the resemblance of the deformed shape for the Vierendeel truss in Fig. 19 with the deformed shape of a simply supported beam with significant participation of shear deformations, *a shear flexible beam*. This shear participation is evident in the

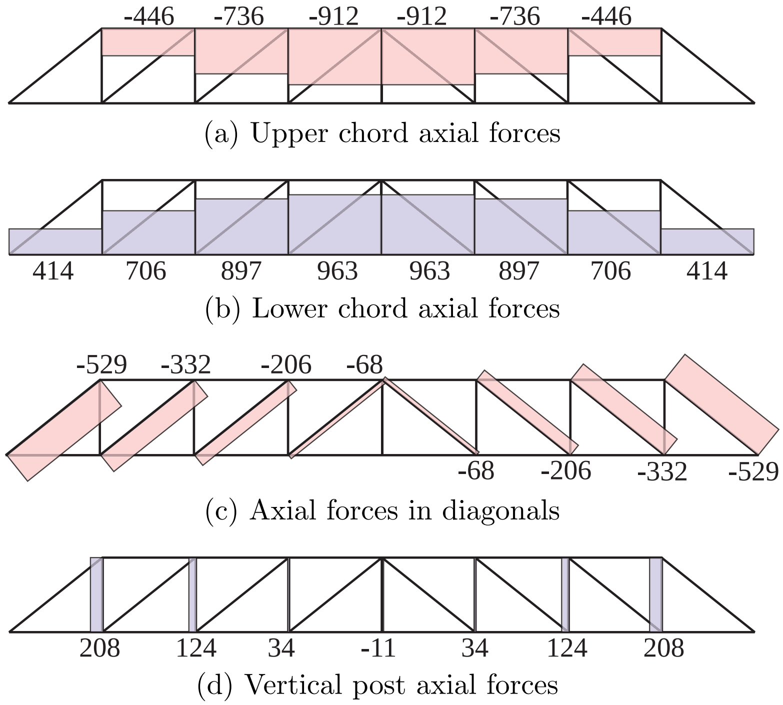


Figure 16: Axial forces in Howe truss with rigid connections, $EI = 4 \cdot 10^5$

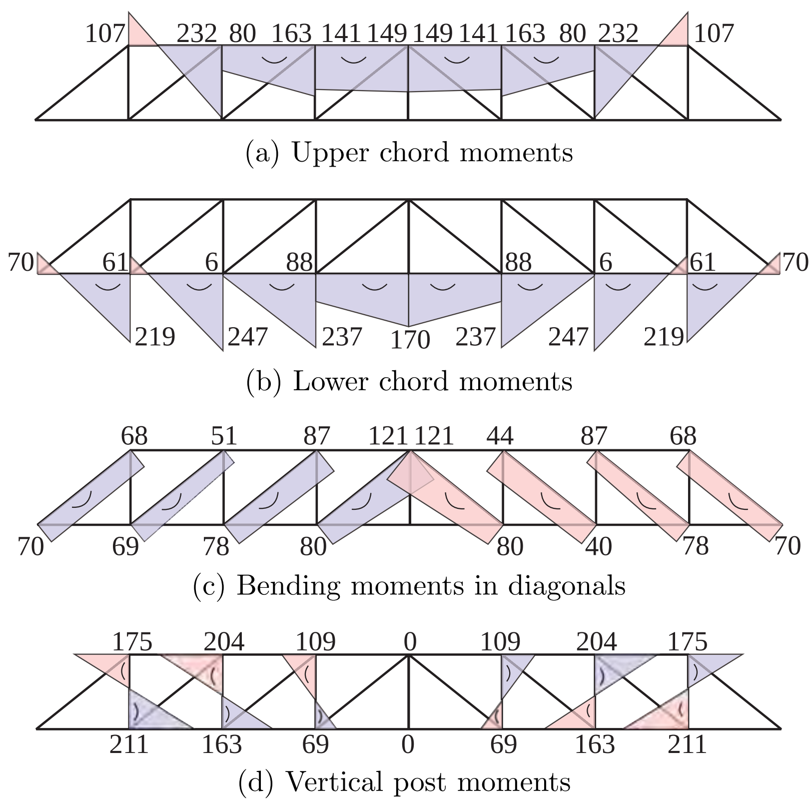


Figure 17: Bending moments in Howe truss with rigid connections, $EI = 4 \cdot 10^5$

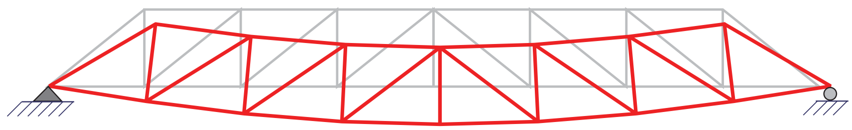


Figure 18: Deformed shape of Howe truss

deformation of each panel resulting in the vertical posts no longer being normal to the deformed shape of the upper and the lower chord in Fig. 19.

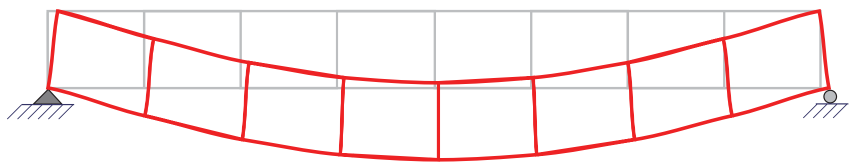
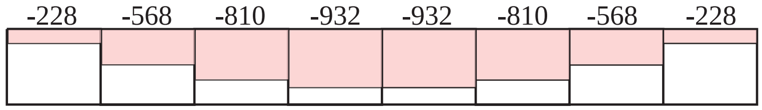
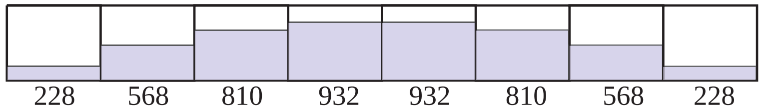


Figure 19: Deformed shape of Vierendeel truss

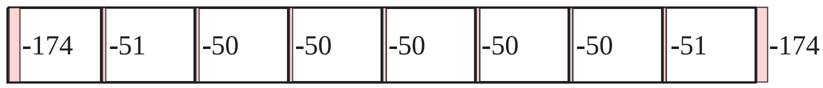
While the value of the axial force in the upper and lower chord in Fig. 20 is not affected much by the removal of the diagonals, since the pair of equal and opposite forces in the upper and lower chord provides a significant portion of the bending moment resistance at a virtual cut through a panel, the magnitude of the moments of the upper and lower chord increases dramatically in order to contribute to the *shear resistance* of the system that is weakened by the absence of the diagonals. Fig. 21 shows that most of the upper and lower chord members are in double curvature with a negative moment at the element end nearest to the closest support and a positive bending moment at the other end, so as to generate the highest possible transverse forces for shear resistance of the system. Consequently, these moments increase towards the supports and also give rise to large bending moments in the vertical posts from the sum of the moments of the upper and the lower chord elements framing into them.
The axial force in the vertical posts also contributes to the shear resistance, but the distribution of these forces over the span is quite different from the axial force distribution in the vertical posts of the Howe truss: in the Vierendeel truss the axial force in the vertical posts is practically constant, except for the vertical posts directly over each support.
Despite of its limitations the Vierendeel truss is often used as transfer girder in high-rise buildings for functional considerations and as a visible vertical or lateral force system for aesthetic considerations.



(a) Upper chord axial forces

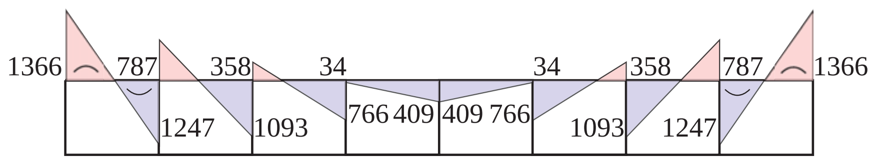


(b) Lower chord axial forces

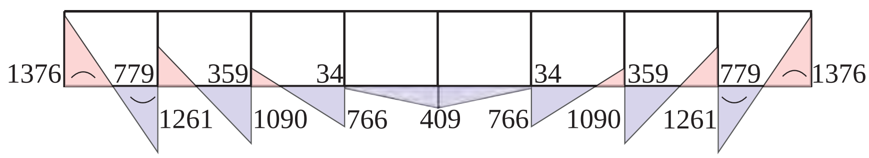


(c) Vertical post axial forces

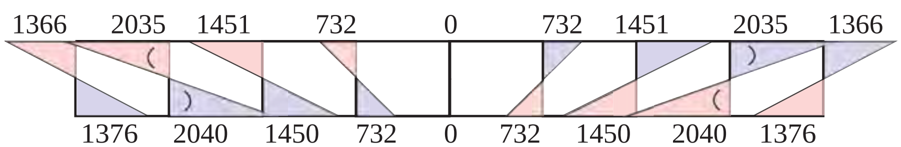
Figure 20: Axial forces in Vierendeel truss with $EI = 4 \cdot 10^5$



(a) Upper chord moments



(b) Lower chord moments



(c) Vertical post moments

Figure 21: Bending moments in Vierendeel truss with $EI = 4 \cdot 10^5$

Figure 22: Vierendeel truss for pedestrian bridge in Germany