IIX Plasticity

19 Limits & Failure Criteria

Shear flow strength: $k = Y/\sqrt{3}$

Invariants

1. Mean normal pressure.

$$\bar{p} = -\frac{1}{3}\operatorname{tr}\sigma = -\frac{1}{3}\left(\sigma_{kk}\right)$$

$$\bar{p} = -\frac{1}{3}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)$$

2. Equivalent shear/tensile stress.

$$\bar{\tau} = \frac{1}{\sqrt{2}}\left|\sigma'\right| = \sqrt{\frac{1}{2}\operatorname{tr}\left(\sigma'^2\right)} = \sqrt{\frac{1}{2}\sigma'_{ij}\sigma'_{ij}}$$

$$\bar{\tau} = \left[\frac{1}{6}\left(\left(\sigma_{11}-\sigma_{22}\right)^2 + \left(\sigma_{22}-\sigma_{33}\right)^2 + \left(\sigma_{33}-\sigma_{11}\right)^2\right) + \left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right)\right]^{1/2}$$

$$\bar{\sigma} = \sqrt{\frac{3}{2}}\left|\sigma'\right| = \sqrt{\frac{3}{2}\operatorname{tr}\left(\sigma'^2\right)} = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}}$$

$$\bar{\sigma} = \left[\frac{1}{2}\left(\left(\sigma_{11}-\sigma_{22}\right)^2 + \left(\sigma_{22}-\sigma_{33}\right)^2 + \left(\sigma_{33}-\sigma_{11}\right)^2\right) + 3\left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right)\right]^{1/2}$$

$$\bar{\sigma} = \sqrt{3}\bar{\tau}$$

3. Third stress invariant.

$$\bar{r} = \left(\frac{9}{2}\operatorname{tr}\left(\sigma'^3\right)\right)^{\frac{1}{3}} = \left(\frac{9}{2}\sigma'_{ik}\sigma'_{kj}\sigma'_{jk}\right)^{\frac{1}{3}}$$

Failure Criteria

1. **Von Mises**

$$f = \bar{\sigma} - Y \le 0$$

2. **Tresca**

$$f = \left(\sigma_1 - \sigma_3\right) \le Y$$
$$f = \tau_{\max} - \tau_{y,\,\text{Tresca}} \le 0$$

$$\tau_{\max} \stackrel{\text{def}}{=} \frac{1}{2}\left(\sigma_1 - \sigma_3\right) \ge 0$$
$$\tau_{y,\,\text{Tresca}} \stackrel{\text{def}}{=} \frac{1}{2}Y$$

3. **Mohr-Coulomb**

$$f = \frac{1}{2}\left(\sigma_1 - \sigma_3\right) + \frac{1}{2}\left(\sigma_1 + \sigma_3\right)\sin\phi - c\cos\phi \le 0$$

$$\mu = \tan\phi$$

4. **Drucker-Prager**

$$f(\bar{\tau}, \bar{p}, S) = \bar{\tau} - (S + \alpha\bar{p}) \le 0$$

20 One-Dimensional Plasticity

20.1 Elastic-Plastic Response

Isotropic hardening

Kinematic hardening

$$\sigma_b = \frac{1}{2}\left(\sigma_f + \sigma_r\right)$$
$$\left|\sigma - \sigma_b\right| \le Y_0$$

20.2 Isotropic rate-independent theory

1. Kinematic decomposition

$$\epsilon = \epsilon^e + \epsilon^p$$

2. Constitutive relation

$$\sigma = E\epsilon^e = E\left(\epsilon - \epsilon^p\right)$$

3. Yield condition

$$f = |\sigma| - Y\left(\bar{\epsilon}^{\mathrm{p}}\right) \le 0$$

4. Flow rule

$$\dot{\epsilon}^{\mathrm{p}} = \chi\beta\dot{\epsilon}$$
$$\beta = \frac{E}{E + H\left(\bar{\epsilon}^{\mathrm{p}}\right)} > 0 \quad \text{(by hypothesis)}$$

$$H\left(\bar{\epsilon}^{\mathrm{P}}\right) = \frac{dY\left(\bar{\epsilon}^{\mathrm{P}}\right)}{d\bar{\epsilon}^{\mathrm{P}}}$$

5. Kuhn-Tucker condition

$$\chi = 0 \quad \text{if } f < 0, \text{ or if } f = 0 \text{ and } n^p\dot{\epsilon} < 0, \quad \text{where} \quad n^p = \operatorname{sign}(\sigma)$$

6. Consistency condition

$$\chi = 1 \quad \text{if } f = 0 \quad \text{and} \quad n^p\dot{\epsilon} > 0$$

$\dot{\sigma} = E[1-\chi\beta]\dot{\epsilon}$

$\dot{\sigma} = E_{\tan}\dot{\epsilon}$

$E_{\tan} = \begin{cases} E & \text{if } \chi = 0 \\ \frac{EH(\bar{\epsilon}^{\mathrm{p}})}{E+H(\bar{\epsilon}^{\mathrm{p}})} & \text{if } \chi = 1 \end{cases}$

21 3D plasticity with isotropic hardening

21.1 Introduction

21.2 Basic equations

21.3 Kinematical assumptions

21.4 Separability hypothesis

21.5 Constitutive characterization of elastic response

21.6 Constitutive equations for plastic response

21.7 Summary of Mises-Hill Theory

1. Strain decomposition, (21.7.1)

$$\epsilon = \epsilon^{\mathrm{e}} + \epsilon^{\mathrm{P}}$$

2. Constitutive relation (21.7.2)

$$\sigma = 2\mu\left(\epsilon - \epsilon^{\mathrm{p}}\right) + (\kappa - (2/3)\mu)(\operatorname{tr}\epsilon)\mathbf{1}$$

   with $\mu > 0$ and $\kappa > 0$ the elastic shear and bulk moduli.

3. Yield condition (21.7.3)

$$f = \bar{\sigma} - Y\left(\bar{\epsilon}^{\mathrm{P}}\right) \leq 0$$

4. Evolution equations, (21.7.5)

$$\begin{aligned} &\dot{\epsilon}^{\mathrm{p}} = \chi\beta\left(\bar{\epsilon}^{\mathrm{p}}\right)\left(\mathbf{n}^{\mathrm{p}} : \dot{\epsilon}\right)\mathbf{n}^{\mathrm{p}}, \quad \mathbf{n}^{\mathrm{p}} = \sqrt{3/2}\,\frac{\sigma'}{\bar{\sigma}} \\ &\dot{\bar{\epsilon}}^{\mathrm{p}} = \sqrt{2/3}\,|\dot{\epsilon}^{\mathrm{p}}| \end{aligned}$$

   with
   Stiffness ratio:

$$\beta\left(\bar{\epsilon}^{\mathrm{p}}\right) = \frac{3\mu}{3\mu + H\left(\bar{\epsilon}^{\mathrm{p}}\right)} > 0$$

   Hardening modulus (21.7.6):

$$H\left(\bar{\epsilon}^{\mathrm{p}}\right) = \frac{dY\left(\bar{\epsilon}^{\mathrm{p}}\right)}{d\bar{\epsilon}^{\mathrm{p}}}$$

   Switching parameter (21.7.7)

$$\chi = \begin{cases} 0 & \text{if } f < 0, \text{ or if } f = 0 \text{ and } n^{\mathrm{p}} : \dot{\epsilon} \leq 0 \\ 1 & \text{if } f = 0 \text{ and } n^{\mathrm{p}} : \dot{\epsilon} > 0 \end{cases}$$

   and typical **initial conditions:**

$$\epsilon(\mathrm{x}, 0) = \epsilon^{\mathrm{p}}(\mathrm{x}, 0) = 0, \quad \text{and} \quad \bar{\epsilon}^{\mathrm{p}}(\mathrm{x}, 0) = 0$$

Also note:

$$d\epsilon_{ij} = \underbrace{\frac{1+\nu}{E}d\sigma_{ij} - \frac{\nu}{E}\left(d\sigma_{kk}\right)\delta_{ij}}_{d\epsilon_{ij}^{e}} + \underbrace{(3/2)d\bar{\epsilon}^{\mathrm{p}}\,\frac{\sigma'_{ij}}{\bar{\sigma}}}_{d\epsilon_{ij}^{\mathrm{p}}}$$

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Onset of yield 390

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Onset of yield 400

Partly-plastic spherical shell 400

Fully-plastic spherical shell 403

Residual stresses upon unloading

25 Rigid-perfectly-plastic materials. Two extremum principles

25.1 Mixed boundary value problem for a rigid-perfectly-plastic solid

25.3 Limit Analysis

25.4 Lower bound theorem

Statically admissible stress field, with respect to to traction field $\mathbf{t}^*$ satisfies:

1. Equilibrium, $\operatorname{div} \boldsymbol{\sigma}^* + \mathbf{b}^* = \mathbf{0} \quad$ on $\quad \mathcal{B}$
2. Traction B.Cs, $\sigma^* \mathbf{n} = \mathbf{t}^* \quad$ on $\quad \partial\mathcal{B}$
3. Yield condition: $f\left(\sigma^*, Y\right) = \sqrt{3/2}\left|\sigma^{*\prime}\right| - Y \leq 0 \quad$ in $\quad \mathcal{B}$

$Q = Q_{pl}$

25.5 Upper bound theorem

Kinematically admissible velocity field, $\mathbf{v}^*$, satisfies:

1. Stretching-velocity relation, $\dot{\epsilon}^* = \frac{1}{2}\left(\left(\nabla \mathbf{v}^*\right) + \left(\nabla \mathbf{v}^*\right)^{\top}\right)$;
2. gives no volume change, $\operatorname{tr} \dot{\epsilon}^* = 0$
3. Satisfies velocity B.Cs, $\mathbf{v}^* = \hat{\mathbf{v}} \quad$ on $\quad \mathcal{S}_1$

Upper bound:

$$\Phi\left\{\mathbf{v}^*\right\} = \mathcal{D}_{\text{int}}\left\{\mathbf{v}^*\right\} - \mathcal{W}_{\text{ext}}\left\{\mathbf{v}^*\right\} \geq \underbrace{\boldsymbol{\Phi}\{\mathbf{v}\}}_{=0}$$

$$\beta_U = \frac{\sum_{\mathcal{S}_{\tilde{d}}} A_{\mathcal{S}_d^*} k\left[v^*\right]}{\int_{\mathcal{B}} \tilde{\mathbf{b}} \cdot \mathrm{v}^* dv + \int_{\partial\mathcal{B}} \tilde{\mathrm{t}} \cdot \mathrm{v}^* da} = \frac{\sum_{\mathcal{S}_{\dot{A}}} A_{\mathcal{S}_{\dot{d}}} \cdot k\left[v^*\right]}{\int_{\mathcal{B}} \tilde{\mathbf{b}} \cdot \mathrm{v}^* dv + \int_{\partial\mathcal{B}} \tilde{\mathrm{t}} \cdot \mathrm{v}^* da}$$

25.5.1 Block-sliding velocity fields

25.6.2 Hodograph

25.6.3 Plane strain frictionless extrusion

25.7 Plane-strain indentation of a semi-infinite solid with a flat punch