$$egin{align*} u_1(\mathbf{x}) &pprox -lpha x_2 x_3 \ u_2(\mathbf{x}) &pprox lpha x_1 x_3 \ u_3(\mathbf{x}) &= lpha arphi \left(x_1, x_2
ight) \ lpha &= rac{T}{\mu J} \ T &= \int_{\mathcal{S}_L} \left(x_1 \sigma_{23} - x_2 \sigma_{13}
ight) da \ ar{J} &\stackrel{\mathrm{def}}{=} \int_{\Omega} \left(x_1^2 + x_2^2 + x_1 arphi_{,2} - x_2 arphi_{,1}
ight) da \ \end{aligned}$$

For open sections:

$$\mu ar{J} = \mu \left(b_1 + b_2 + b_3
ight) t^3 / 3$$

Displacement Formulations

$$\begin{split} \epsilon_{11}(\mathbf{x}) &= \epsilon_{22}(\mathbf{x}) = \epsilon_{33}(\mathbf{x}) = \epsilon_{12}(\mathbf{x}) = 0 \\ \epsilon_{13}(\mathbf{x}) &= \frac{1}{2} \left(\frac{\partial \varphi}{\partial x_1} - x_2 \right) \alpha \\ \epsilon_{23}(\mathbf{x}) &= \frac{1}{2} \left(\frac{\partial \varphi}{\partial x_2} + x_1 \right) \alpha \\ \sigma_{11}(\mathbf{x}) &= \sigma_{22}(\mathbf{x}) = \sigma_{33}(\mathbf{x}) = \sigma_{12}(\mathbf{x}) = 0 \\ \sigma_{13}(\mathbf{x}) &= \mu \alpha \left(\frac{\partial \varphi}{\partial x_1} - x_2 \right) \\ \sigma_{23}(\mathbf{x}) &= \mu \alpha \left(\frac{\partial \varphi}{\partial x_2} + x_1 \right) \end{split}$$

Equilibrium:

$$\sigma_{13,1}+\sigma_{23,2}=0\quad ext{ in }\Omega$$

Boundary:

$$egin{aligned} \Delta arphi &= 0 \quad ext{in } \Omega \ rac{\partial arphi}{\partial n} &= x_2 n_1 - x_1 n_2 \quad ext{ on } \Gamma \end{aligned}$$

Stress Formulation

Compatibility:

$$\begin{array}{ll} \epsilon_{13,2}-\epsilon_{23,1}=-\alpha & \text{in } \Omega \\ \Delta\Psi=\Psi_{,11}+\Psi_{,22}=-2\mu\alpha & \text{in } \Omega \text{ subject to } & \Psi=0 & \text{on } \Gamma \\ \text{for } \sigma_{13}=\frac{\partial\Psi}{\partial x_2}, & \sigma_{23}=-\frac{\partial\Psi}{\partial x_1} \\ \end{array}$$

9 Solutions

Crack Tip

Mode III: Anti-plane tearing

$$\begin{split} \Delta u_z &= 0 = \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} \\ \delta_3 &\stackrel{\text{def}}{=} u_3(r, +\pi) - u_3(r, -\pi) = \frac{4}{\mu} K_{\text{III}} \sqrt{\frac{r}{2\pi}} \\ \left(\begin{array}{c} \sigma_{13} \\ \sigma_{23} \end{array} \right) &= \frac{K_{\text{III}}}{\sqrt{2\pi r}} \left(\begin{array}{c} -\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \end{array} \right) + \text{ bounded terms} \\ \sigma_{11} &= \sigma_{22} = \sigma_{33} = \sigma_{12} = 0 \\ \left(u_3 \right) &= \frac{K_{\text{III}}}{2\mu} \sqrt{\frac{r}{2\pi}} \left(4 \sin\left(\frac{\theta}{2}\right) \right) + \text{rigid displacement.} \end{split}$$