IV Linear Elasticity

Compatibility:

$\operatorname{curl}(\operatorname{curl}\epsilon)=0, \quad e_{ipq}e_{jrs}\epsilon_{qs,rp}=0$

7 Linear Constitutive Equations

7.1 Free Energy: Elasticity

$$\sigma=\mathbb{C}\epsilon, \quad \sigma_{ij}=C_{ijkl}\epsilon_{kl}$$

Voight Notation:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{112} \\ C_{211} & C_{222} & C_{233} & C_{223} & C_{2213} & C_{2212} \\ C_{331} & C_{332} & C_{333} & C_{333} & C_{313} & C_{2312} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{pmatrix}$$

7.5 Isotropic Relations

$\boldsymbol{\sigma}=\mathbb{C}\boldsymbol{\epsilon}=2\mu\boldsymbol{\epsilon}+\lambda(\operatorname{tr}\boldsymbol{\epsilon})\mathbf{1}$

$\boldsymbol{\sigma}=2\mu\boldsymbol{\epsilon}'+\kappa(\operatorname{tr}\boldsymbol{\epsilon})\mathbf{1}, \quad \sigma_{ij}=2\mu\epsilon'_{ij}+\kappa\left(\epsilon_{kk}\right)\delta_{ij}$

$\mathbb{C}=2\mu^{\text{sym}}+\boldsymbol{\lambda}\mathbf{1}\otimes\mathbf{1}, \quad C_{ijkl}=\mu\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)+\lambda\delta_{ij}\delta_{kl}$

$\epsilon=\frac{1}{2\mu}\sigma'+\frac{1}{9\kappa}(\operatorname{tr}\sigma)1, \quad \epsilon_{ij}=\frac{1}{2\mu}\sigma'_{ij}+\frac{1}{9\kappa}\left(\sigma_{kk}\right)\delta_{ij}$

$E\equiv\frac{9\kappa\mu}{3\kappa+\mu}, \quad \nu\equiv\frac{1}{2}\left[\frac{3\kappa-2\mu}{3\kappa+\mu}\right]$

$\kappa=\lambda+\frac{2}{3}\mu$

7.5.4

$$\sigma=\frac{E}{(1+\nu)}\left[\epsilon+\frac{\nu}{(1-2\nu)}(\operatorname{tr}\epsilon)1\right], \quad \sigma_{ij}=\frac{E}{(1+\nu)}\left[\epsilon_{ij}+\frac{\nu}{(1-2\nu)}\left(\epsilon_{kk}\right)\delta_{ij}\right]$$

$$\epsilon=\frac{1}{E}[(1+\nu)\sigma-\nu(\operatorname{tr}\sigma)1]. \quad \epsilon_{ij}=\frac{1}{E}\left[(1+\nu)\sigma_{ij}-\nu\left(\sigma_{kk}\right)\delta_{ij}\right]$$

8 Elastostatics

Displacement Formulation (Navier)

$C_{ijkl}u_{k,lj} + b_i = 0$

Isotropic

$$\mu\triangle\mathbf{u} + (\lambda+\mu)\nabla\,\mathrm{div}\,\mathbf{u} + \mathbf{b} = 0$$
$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} + b_i = 0$$

$$(\lambda+2\mu)\nabla\,\mathrm{div}\,\mathbf{u} - \mu\,\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} + \mathbf{b} = \mathbf{0}$$
$$(\lambda+2\mu)u_{j,ji} - \mu e_{ijk}e_{klm}u_{m,lj} + b_i = 0$$

Boundary:

$$\left.\begin{array}{r}\mathbf{u} = \hat{\mathbf{u}} \text{ on } \mathcal{S}_1, \\ \left(\mu\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top\right) + \lambda(\mathrm{div}\,\mathbf{u})\mathbf{1}\right)\mathbf{n} = \hat{\mathbf{t}} \quad \text{on } \mathcal{S}_2\end{array}\right\}$$

Stress Formulation (Beltrami-Mitchell)

Compatibility:

$$\sigma_{ij,kk} + \tfrac{1}{1+\nu}\sigma_{kk,ij} = -\tfrac{\nu}{1-\nu}b_{k,k}\delta_{ij} - b_{i,j} - b_{j,i}$$
$$\Delta\sigma_{kk} = -\tfrac{1+\nu}{1-\nu}b_{k,k}$$

8.9.1 Plane Strain

$u_\alpha = u_\alpha(x_1, x_2), \quad u_3 = 0$

$\epsilon_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right)$

$\epsilon_{13} = \epsilon_{23} = \epsilon_{33} = 0$

$\sigma_{\alpha\beta} = \frac{E}{(1+\nu)}\left(\epsilon_{\alpha\beta} + \frac{\nu}{(1-2\nu)}\left(\epsilon_{\gamma\gamma}\right)\delta_{\alpha\beta}\right)$

$\sigma_{33} = \nu\sigma_{\alpha\alpha}$

$\epsilon_{\alpha\beta} = \frac{1+\nu}{E}\left(\sigma_{\alpha\beta} - \nu\left(\sigma_{\gamma\gamma}\right)\delta_{\alpha\beta}\right)$

$\sigma_{\alpha\beta,\beta} + b_\alpha = 0$

Plane Stress

$\sigma_{\alpha\beta} = \sigma_{\alpha\beta}(x_1, x_2), \quad \sigma_{33} = \sigma_{13} = \sigma_{23} = 0$

Plane Stress/Strain

$\epsilon_{13} = \epsilon_{23} = \sigma_{13} = \sigma_{23} = 0$

	Plane Stress	Plane Strain
	$\sigma_{\alpha\beta} = \sigma_{\alpha\beta}\left(x_1, x_2\right)$	$u_\alpha = u_\alpha\left(x_1, x_2\right)$ $u_3 = 0$
s	$\frac{1}{1+\nu}$	$1-\nu$
σ_{33}	0	$\nu\sigma_{\alpha\alpha}$
ϵ_{33}	$-\frac{\nu}{1-\nu}\epsilon_{\alpha\alpha}$	0

$$\text{Navier}: \begin{cases} \left(\frac{E}{2(1+\nu)}\right) u_{\alpha,\beta\beta} + \left(\frac{E}{2(1+\nu)(1-2\nu)}\right) u_{\beta,\beta\alpha} + b_\alpha = 0 & \text{for plane strain} \\ \left(\frac{E}{2(1+\nu)}\right) u_{\alpha,\beta\beta} + \frac{E}{2(1-\nu)} u_{\beta,\beta\alpha} + b_\alpha = 0 & \text{for plane stress} \end{cases}$$

Constitutive Relation

$$\sigma_{\alpha\beta} = \frac{E}{(1+\nu)}\left(\epsilon_{\alpha\beta} + \left(\frac{1-s}{2s-1}\right)\left(\epsilon_{\gamma\gamma}\right)\delta_{\alpha\beta}\right)$$

$$\epsilon_{\alpha\beta} = \frac{(1+\nu)}{E}\left(\sigma_{\alpha\beta} - (1-s)\left(\sigma_{\gamma\gamma}\right)\delta_{\alpha\beta}\right)$$

Equilibrium

$$\sigma_{\alpha\beta,\beta} + b_\alpha = 0$$

Compatibility

$$\Delta\left(\sigma_{\alpha\alpha}\right) = \left(\sigma_{11} + \sigma_{22}\right),_{11} + \left(\sigma_{11} + \sigma_{22}\right),_{22} = -\frac{1}{s} b_{\alpha,\alpha}$$

Airy Stress Function

$\sigma_{11} = \varphi,_{22}\,, \quad \sigma_{22} = \varphi,_{11}, \quad \sigma_{12} = -\varphi,_{12}$

Compatibility

$\Delta\Delta\varphi = \varphi, 1111 + 2\varphi, 1122 + \varphi, 2222 = 0$

Displacements

$u_1 = \frac{(1+\nu)}{E}(-\varphi, 1 + s\psi, 2) + w_1$
$u_2 = \frac{(1+\nu)}{E}\left(-\varphi, 2 + s\psi,_1\right) + w_2$
where: $\Delta\psi = 0$ and $\psi,_{12} = \Delta\varphi$

and w is a plane rigid displacement:
$w_{1,1} = 0, \quad w_{2,2} = 0, \quad w_{1,2} + w_{2,1} = 0$

Polar form (9.4.16):

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}$$
$$\sigma_{\theta\theta} = \frac{\partial^2 \varphi}{\partial r^2}$$
$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \varphi}{\partial \theta}\right)$$

Torsion

$$u_1(\mathrm{x}) \approx -\alpha x_2 x_3$$
$$u_2(\mathrm{x}) \approx \alpha x_1 x_3$$
$$u_3(\mathbf{x}) = \alpha\varphi\left(x_1, x_2\right)$$

$\alpha = \frac{T}{\mu \bar{J}}$

$T = \int_{\mathcal{S}_L} \left(x_1\sigma_{23} - x_2\sigma_{13}\right) da$

$\bar{J} \stackrel{\text{def}}{=} \int_\Omega \left(x_1^2 + x_2^2 + x_1\varphi_{,2} - x_2\varphi_{,1}\right) da$

Displacement Formulations

$$\epsilon_{11}(\mathrm{x}) = \epsilon_{22}(\mathrm{x}) = \epsilon_{33}(\mathrm{x}) = \epsilon_{12}(\mathrm{x}) = 0$$
$$\epsilon_{13}(\mathrm{x}) = \frac{1}{2}\left(\frac{\partial \varphi}{\partial x_1} - x_2\right)\alpha$$
$$\epsilon_{23}(\mathrm{x}) = \frac{1}{2}\left(\frac{\partial \varphi}{\partial x_2} + x_1\right)\alpha$$

$$\sigma_{11}(\mathrm{x}) = \sigma_{22}(\mathrm{x}) = \sigma_{33}(\mathrm{x}) = \sigma_{12}(\mathrm{x}) = 0$$
$$\sigma_{13}(\mathrm{x}) = \mu\alpha\left(\frac{\partial \varphi}{\partial x_1} - x_2\right)$$
$$\sigma_{23}(\mathrm{x}) = \mu\alpha\left(\frac{\partial \varphi}{\partial x_2} + x_1\right)$$

Equilibrium:

$$\sigma_{13,1} + \sigma_{23,2} = 0 \quad \text{in } \Omega$$

Boundary:

$$\Delta\varphi = 0 \quad \text{in } \Omega$$
$$\frac{\partial \varphi}{\partial n} = x_2 n_1 - x_1 n_2 \quad \text{on } \Gamma$$

Stress Formulation

Compatibility:

$$\epsilon_{13,2} - \epsilon_{23,1} = -\alpha \quad \text{in } \Omega$$
$$\Delta\Psi = \Psi_{,11} + \Psi_{,22} = -2\mu\alpha \quad \text{in } \Omega \ \text{ subject to } \quad \Psi = 0 \quad \text{on } \Gamma$$

for $\sigma_{13} = \frac{\partial \Psi}{\partial x_2}, \quad \sigma_{23} = -\frac{\partial \Psi}{\partial x_1}$

$T = 2 \int_\Omega \Psi da$

9 Solutions

Crack Tip

Mode III

$\Delta u_z = 0 = \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \theta^2} + \frac{1}{r} \frac{\partial u_z}{\partial r}$