1 Algebra

$S_{ij} \stackrel{\text{def}}{=} \mathbf{e}_i \cdot \mathbf{S}\mathbf{e}_j$

$C_{ijkl} \stackrel{\text{def}}{=} (\mathbf{e}_i \otimes \mathbf{e}_j) : \mathbb{C}\,(\mathbf{e}_k \otimes \mathbf{e}_l)$

$\mathbf{S} = S_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$

$v_i = S_{ij}u_j$

$\mathbb{I}^{\text{sym}} \to \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$

1. $\nabla(\mathbf{T}\cdot\mathbf{v}) = T_{ij,m}v_j + T_{ij}v_{j,m}$
2. $\nabla\cdot(\mathbf{T}\cdot\mathbf{v}) = T_{ij,i}v_j + T_{ij}v_{j,i}$
3. $\nabla^2\mathbf{T} = T_{ij,mm}$
4. $\nabla^4 f = f_{,iijj}$
5. $\nabla\cdot(\mathbf{v}\otimes\mathbf{x}) = v_{i,j}x_j + 3v_i$
6. $\mathbf{x}\cdot\mathbf{A} = x_iA_{ijk}$
7. $\boldsymbol{\sigma} : \mathbb{C} : \boldsymbol{\varepsilon} = \sigma_{ij}\mathbb{C}_{ijkl}\varepsilon_{kl}$
8. $\mathbf{T} : \mathbf{A}\cdot\mathbf{v} = T_{ij}A_{ijk}v_k$

Products

1.2.3 Outer Product

$(\mathbf{u}\otimes\mathbf{v})\mathbf{w} = (\mathbf{v}\cdot\mathbf{w})\mathbf{u}$

1.2.8 $\quad (\mathbf{ST})_{ij} = S_{ik}T_{kj}$

1.2.11 Inner / Dot Products

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u} = u_iv_j\delta_{ij} = u_iv_i$$
$$\mathbf{S}:\mathbf{T} = \mathbf{T}:\mathbf{S} = S_{ij}T_{ij}$$
$$\mathbf{S}\cdot\mathbf{T} = S_{ij}T_{jl}\mathrm{e}_i\mathrm{e}_l \neq \mathbf{T}\cdot\mathbf{S}$$

1.2.1 Cross Product

$$(\mathbf{u}\times\mathbf{v})_i = e_{ijk}u_jv_k$$

6 Symmetric - Skew

$$\mathbf{S} = \mathbf{S}^\top, \quad S_{ij} = S_{ji}$$
$$\mathbf{S} = -\mathbf{S}^\top, \quad S_{ij} = -S_{ji}$$
$$(\operatorname{sym}\mathbf{S})_{ij} = \tfrac{1}{2}(S_{ij} + S_{ji})$$
$$(\operatorname{skw}\mathbf{S})_{ij} = \tfrac{1}{2}(S_{ij} - S_{ji})$$

11 - Inner Product & Norm

$$|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{u_iu_i}$$
$$|\mathbf{S}| = \sqrt{\mathbf{S}:\mathbf{S}} = \sqrt{S_{ij}S_{ij}}$$

17 - Transformations

$$v_i^* = \mathrm{e}_i^*\cdot\mathrm{v} \quad \text{and} \quad S_{ij}^* = \mathrm{e}_i^*\cdot\mathrm{S}\mathrm{e}_j^*$$
$$\mathrm{Q} \stackrel{\text{def}}{=} \mathrm{e}_k \otimes \mathrm{e}_k^*$$
$$v_i^* = Q_{ij}v_j$$
$$S_{ij}^* = Q_{ik}Q_{jl}S_{kl}$$
$$C_{ijkl}^* = Q_{ip}Q_{jq}Q_{kr}Q_{ls}C_{pqrs}$$

18 - Eigenvalues

$$\omega^3 - I_1(\mathbf{S})\omega^2 + I_2(\mathbf{S})\omega - I_3(\mathbf{S}) = 0$$

Principal Invariants:

$$I_1(\mathbf{S}) = \operatorname{tr}\mathbf{S} = \omega_1 + \omega_2 + \omega_3$$
$$I_2(\mathbf{S}) = \tfrac{1}{2}\left[(\operatorname{tr}(\mathbf{S}))^2 - \operatorname{tr}\left(\mathbf{S}^2\right)\right] = \omega_1\omega_2 + \omega_2\omega_3 + \omega_3\omega_1$$
$$I_3(\mathbf{S}) = \det\mathbf{S} = \omega_1\omega_2\omega_3$$

A free index appears only once within each additive term in an expression.
A free index implies three distinct equations.
The same letter must be used for the free index in every additive term. You can rename this index if you rename it in every single term.
Terms in an expression may have more than one free index, which will indicate the dimension/rank of the term.

For a second order tensor $\mathbf{A}$, the first index will correspond to the row, and the second index corresponds to the column.

A dummy index appears twice within an additive term of an expression. A dummy index can only appear twice, so an individual term with 3 is not allowed.
A dummy index implies a summation over the range from 1 to 3.
A dummy index may be renamed to any letter not currently being used as a free index (or already in use as another dummy index pair in that term). The dummy index is local to an individual additive term, such that one can rename the dummy index in one term and it would not need to be renamed in other terms.

Calculus of Variations

First variation of a functional, I: $\delta I\{u;w\} \stackrel{\text{def}}{=} \frac{d}{d\zeta}I\{u(x) + \zeta w(x)\}\Big|_{\zeta=0}$

The condition: $\delta I\{u;w\} = 0$ for all admissible w is a necessary condition for $u(x)$ to be a minimizer of I.

Taking variations

$$\delta I = \tfrac{d}{d\zeta}I(\Psi + \zeta\delta\Psi)\Big|_{\zeta=0}$$
$$\delta F = \left(\tfrac{\partial F}{\partial u}\right)\delta u + \left(\tfrac{\partial F}{\partial u'}\right)\delta u'$$

$$\delta I \equiv \int_{x_0}^{x_1}\delta F(x,u,u')\,dx = \int_{x_0}^{x_1}\left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'\right)dx = \frac{\partial F}{\partial u'}\delta u\Big|_{x_0}^{x_1} + \int_{x_0}^{x_1}\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]\delta u dx$$

Euler-Lagrance Equation

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0 \quad \forall x \in (x_0, x_1)$$

Calculus $\quad \frac{d}{dx}\int_a^x f(t)dt = f(x)$

Convolution $\quad \int_{0^-}^t f(t-\tau)g(\tau)d\tau \equiv (f*g)(t)$

Laplace Transformations

$$L[f(t)] = \int_{0^-}^\infty e^{-st}f(t)dt \equiv \bar{f}(s)$$

2 Analysis

Derivatives

$$\Delta\mathbf{v} = \operatorname{div}\operatorname{grad}\mathbf{v}, \quad \Delta v_i = \frac{\partial^2 v_i}{\partial x_j\partial x_j}$$
$$\Delta T_{ij} = \frac{\partial T_{ij}}{\partial x_k\partial x_k}$$

Gradient

$$\mathbf{e}_i\cdot\operatorname{grad}\varphi(\mathbf{x}) = [\operatorname{grad}\varphi(\mathbf{x})]_i = \frac{\partial\varphi(\mathbf{x})}{\partial x_i}$$
$$\mathbf{e}_i\cdot\operatorname{grad}\mathbf{v}(\mathbf{x})\mathbf{e}_j = [\operatorname{grad}\mathbf{v}(\mathbf{x})]_{ij} = \frac{\partial v_i(\mathbf{x})}{\partial x_j}$$

Directional Derivative

$$\operatorname{grad}\varphi(\mathbf{x})[\mathbf{h}] = \frac{d}{d\alpha}\varphi(\mathbf{x}+\alpha\mathbf{h})\Big|_{\alpha=0}$$

$$(\operatorname{curl}\mathbf{v})_i = e_{ijk}\frac{\partial v_k}{\partial x_j}$$
$$(\operatorname{curl}\mathbf{T})_{ij} = e_{ipq}\frac{\partial T_{jq}}{\partial x_p}$$
$$\operatorname{div}\mathbf{v} = \operatorname{tr}[\operatorname{grad}\mathbf{v}] = \frac{\partial v_i}{\partial x_i}$$
$$(\operatorname{div}\mathbf{T})_i = \frac{\partial T_{ij}}{\partial x_j}$$

for a vector field, the directional derivative is also a vector each of whose components gives the rate of change of the corresponding component of v in the direction of h. The gradient in this case will be a tensor field (that when applied to h gives the directional derivative of v in the direction of h).

A.2 Cylindrical coordinates

$$\begin{aligned} r &= [x_1^2 + x_2^2]^{\frac{1}{2}}, \quad r \geq 0, \\ \theta &= \tan^{-1}(x_2/x_1), \quad 0 \leq \theta \leq 2\pi, \\ z &= x_3, \quad -\infty < z < \infty, \end{aligned}$$

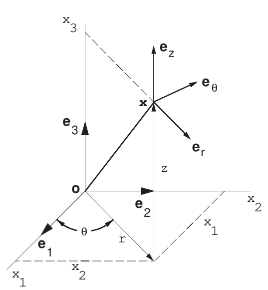
$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad x_3 = z.$$

$$\boxed{\begin{aligned} &\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + b_r = \rho\ddot{u}_r, \\ &\frac{\partial\sigma_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{\theta r} + b_\theta = \rho\ddot{u}_\theta, \\ &\frac{\partial\sigma_{zr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{z\theta}}{\partial\theta} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{zr}}{r} + b_z = \rho\ddot{u}_z. \end{aligned}}$$

$$\boxed{\triangle\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}.}$$

$$dA = rdrd\theta$$

$$\boxed{\begin{aligned} \epsilon_{rr} &= \frac{\partial u_r}{\partial r}, \\ \epsilon_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}, \\ \epsilon_{zz} &= \frac{\partial u_z}{\partial z}, \\ \epsilon_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = \epsilon_{\theta r}, \\ \epsilon_{\theta z} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial\theta} + \frac{\partial u_\theta}{\partial z}\right) = \epsilon_{z\theta}, \\ \epsilon_{zr} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) = \epsilon_{rz}. \end{aligned}}$$



$$\frac{\partial\mathbf{e}_r}{\partial\theta} = \mathbf{e}_\theta, \qquad \frac{\partial\mathbf{e}_\theta}{\partial\theta} = -\mathbf{e}_r, \qquad \frac{\partial\mathbf{e}_z}{\partial\theta} = \mathbf{0}.$$

$$\mathrm{d}\mathbf{x} = \mathrm{d}r\,\mathbf{e}_r + (r\mathrm{d}\theta)\,\mathbf{e}_\theta + \mathrm{d}z\,\mathbf{e}_z.$$

$$\mathbf{e}_r = \frac{\partial\mathbf{r}}{\partial r}, \quad \mathbf{e}_\theta = \frac{1}{r}\frac{\partial\mathbf{r}}{\partial\theta}, \quad \mathbf{e}_z = \frac{\partial\mathbf{r}}{\partial z}.$$

3 Kinematics

Displacement:

$$\mathbf{u}(\mathrm{X},t) = \chi(\mathrm{X},t) - \mathrm{X},$$

3.2 Deformation/Displacement Gradient

$$\mathbf{F}(\mathbf{X},t) = \frac{\partial}{\partial\mathbf{X}}\chi(\mathbf{X},t), \quad F_{ij} = \frac{\partial}{\partial X_j}\chi_i(X_1,X_2,X_3,t), \quad \det\mathbf{F}(\mathbf{X},t) > 0$$
$$\mathbf{H}(\mathbf{X},t) = \frac{\partial}{\partial\mathbf{X}}\mathbf{u}(\mathbf{X},t), \quad H_{ij} = \frac{\partial}{\partial X_j}u_i(X_1,X_2,X_3,t)$$
$$\mathbf{H}(\mathbf{X},t) = \mathbf{F}(\mathbf{X},t) - 1, \quad H_{ij} = F_{ij} - \delta_{ij}$$
$$J \equiv \det\left(\frac{\partial\chi}{\partial\mathbf{X}}\right) = \det\mathbf{F} = \frac{dv}{dv_{\mathrm{R}}} \neq 0$$

3.3 Stretch & Rotation

Polar Decomposition: $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$

$$\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \quad C_{ij} = F_{ki}F_{kj} = \tfrac{\partial\chi_k}{\partial X_i}\tfrac{\partial\chi_k}{\partial X_j}$$
$$\mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^\top, \quad B_{ij} = F_{ik}F_{jk} = \tfrac{\partial\chi_i}{\partial X_k}\tfrac{\partial\chi_j}{\partial X_k}$$

$$\lambda \stackrel{\text{def}}{=} \tfrac{ds}{dS} = |\mathbf{Ue}| = \sqrt{\mathbf{e}\cdot\mathbf{C}(\mathbf{X})\mathbf{e}}$$

where $dS = |\mathrm{dX}|, ds = |\mathrm{dx}|, \mathrm{e} = \frac{d\mathrm{X}}{|d\mathrm{X}|}$

Engineering shear: $\gamma = \sin^{-1}\left[\frac{\mathbf{e}^{(1)}\cdot\mathbf{C}\mathbf{e}^{(2)}}{\lambda(\mathbf{e}^{(1)})\lambda(\mathbf{e}^{(2)})}\right]$

$$\mathbf{n}_{\mathrm{R}}\mathrm{d}a_{\mathrm{R}} = \mathbf{a}\times\mathbf{b}\,.$$
$$\mathbf{n}\mathrm{d}a = \mathbf{Fa}\times\mathbf{Fb}\,.$$

$$\boxed{\mathbf{n}\mathrm{d}a = J\mathbf{F}^{-\top}\mathbf{n}_{\mathrm{R}}\mathrm{d}a_{\mathrm{R}}.}$$

area stretch at a point:

$$\lambda_{\text{area}} = da/da_{\mathrm{R}} = |J\mathbf{F}^{-\top}\mathbf{n}_{\mathrm{R}}|$$
$$e_{\text{area}} = \lambda_{\text{area}} - 1.$$

Volume change:

For finite deformations: $\quad \mathrm{d}v = \det\mathbf{F}\,\mathrm{d}v_{\mathrm{R}}.$

For small strains: $\quad \frac{\mathrm{d}v - \mathrm{d}v_{\mathrm{R}}}{\mathrm{d}v_{\mathrm{R}}} \doteq \operatorname{tr}\boldsymbol{\epsilon} = \epsilon_{kk} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}.$

3.4 Strain

Green strain: $\mathrm{E} \stackrel{\text{def}}{=} \frac{1}{2}(\mathrm{F}^\top\mathrm{F} - 1) = \frac{1}{2}(\mathbf{H} + \mathbf{H}^\top + \mathbf{H}^\top\mathbf{H}.)$

Hencky's Log strain: $\ln\mathbf{U} \stackrel{\text{def}}{=} \sum_{i=1}^3(\ln\lambda_i)\,\mathbf{r}_i\otimes\mathbf{r}_i \qquad \epsilon = \ln\lambda.$

3.5.2 Infinitesimal Strain

ϵ': distortion $\qquad \epsilon_M\delta_{ij}$: dilation

$$\epsilon = \frac{1}{2}\left[\mathbf{H} + \mathbf{H}^\top\right], \qquad \epsilon = \epsilon^\top, \quad |\mathbf{H}| \ll 1$$
$$\epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right], \quad \epsilon_{ji} = \epsilon_{ij}, \left|\frac{\partial u_i}{\partial X_j}\right| \ll 1$$

3.A Linearization

$$\operatorname{lin}\mathrm{Y}_o f(\mathbf{Y}) = f(\mathbf{Y}_o) + \tfrac{d}{d\alpha}f(\mathbf{Y}_o + \alpha(\mathbf{Y} - \mathbf{Y}_o))\Big|_{\alpha=0}$$
$$\operatorname{lin}_0 f(\mathbf{H}) = f(0) + \tfrac{d}{d\alpha}f(\alpha\mathbf{H})\Big|_{\alpha=0}$$

3.B Compatibility

$$\operatorname{curl}(\operatorname{curl}\epsilon) = 0, \quad e_{ipq}e_{jrs}\epsilon_{qs,rp} = 0$$

4 Momentum Balance

$\sigma(\mathrm{x},t)=\sigma^{\top}(\mathrm{x},t), \quad \sigma_{ij}=\sigma_{ji}$

$\mathbf{t}(\mathbf{n})=\sigma\mathbf{n}, \quad t_i=\sigma_{ij}n_j$

$\operatorname{div}\sigma+\mathbf{b}=\rho\ddot{\mathbf{u}}, \quad \sigma_{ij,j}+b_i=\rho\ddot{u}_i$

$\sigma'=\sigma-\frac{1}{3}(\operatorname{tr}\sigma)1, \quad \sigma'_{ij}\overset{\text{def}}{=}\sigma_{ij}-\frac{1}{3}(\sigma_{kk})\delta_{ij}$

5 Energy-Entropy Balance

Infinitesimal specializations boxed.

5.1 Energy

$$\rho\dot{\varepsilon}_m=\sigma:\mathbf{D}-\operatorname{div}\mathbf{q}+r \qquad \rho\dot{\varepsilon}_m=\sigma_{ij}D_{ij}-\frac{\partial q_i}{\partial x_i}+r$$

$$\mathbf{D}\overset{\text{def}}{=}\operatorname{sym}[\operatorname{grad}\dot{\mathbf{u}}], \qquad D_{ij}=\frac{1}{2}(\dot{u}_{i,j}+\dot{u}_{j,i})$$

$$\boxed{\dot{\varepsilon}=\sigma:\dot{\epsilon}-\operatorname{div}\mathbf{q}+r, \quad \dot{\varepsilon}=\sigma_{ij}\dot{\epsilon}_{ij}-\frac{\partial q_i}{\partial x_i}+r}$$

5.2 Entropy

$$\rho\dot{\eta}_m\geq-\operatorname{div}\left(\frac{\mathrm{q}}{\vartheta}\right)+\frac{r}{\vartheta}, \quad \rho\dot{\eta}_m\geq-\frac{\partial}{\partial x_i}\left(\frac{q_i}{\vartheta}\right)+\frac{r}{\vartheta}$$

$$\boxed{\dot{\eta}\geq-\operatorname{div}\left(\frac{\mathrm{q}}{\vartheta}\right)+\frac{r}{\vartheta}, \quad \dot{\eta}\geq-\frac{\partial}{\partial x_i}\left(\frac{q_i}{\vartheta}\right)+\frac{r}{\vartheta}}$$

5.3 Free Energy

$\psi=\frac{1}{2}\boldsymbol{\epsilon}:\mathbb{C}\boldsymbol{\epsilon}\,, \qquad \psi=\frac{1}{2}\epsilon_{ij}C_{ijkl}\epsilon_{kl}\,,$

Where thermal influences are negligible:

$$\mathcal{D}\overset{\text{def}}{=}\sigma:\mathbf{D}-\rho\dot{\psi}_m\geq 0$$

$$\boxed{\mathcal{D}\overset{\text{def}}{=}\sigma:\dot{\epsilon}-\dot{\psi}\geq 0}$$

$\mathcal{D}>0$ generated by plastic deformations.

$\boldsymbol{\sigma}=\frac{\partial\psi(\boldsymbol{\epsilon})}{\partial\boldsymbol{\epsilon}}=\mathbb{C}\boldsymbol{\epsilon}\,,$

- $\rho(\mathbf{x})>0$ denotes the *constant* (time independent) mass density of $\mathcal{B}$;
- $\mathbf{u}(\mathbf{x},t)$ denotes the displacement field of $\mathcal{B}$;
- $\mathbf{H}(\mathbf{x},t)=\nabla\mathbf{u}(\mathbf{x},t)$ denotes the displacement gradient in $\mathcal{B}$, with $|\mathbf{H}|\ll 1$;
- $\boldsymbol{\epsilon}=\frac{1}{2}(\nabla\mathbf{u}+(\nabla\mathbf{u})^{\top})$ denotes the symmetric small strain measure;
- $\boldsymbol{\sigma}(\mathbf{x},t)$ denotes the (symmetric) stress;
- $\mathbf{b}(\mathbf{x},t)$ denotes the (non-inertial) body force field on the body per unit volume in $\mathcal{B}$.
- $\varepsilon(\mathbf{x},t)\overset{\text{def}}{=}\rho\varepsilon_m$ denotes the *internal energy* measured *per unit volume* in $\mathcal{B}$;
- $\eta(\mathbf{x},t)\overset{\text{def}}{=}\rho\eta_m$ denotes the *internal entropy* measured *per unit volume* in $\mathcal{B}$;
- $\psi(\mathbf{x},t)\overset{\text{def}}{=}\rho\psi_m$ denotes the *Helmholtz free energy* measured *per unit volume* in $\mathcal{B}$;
- $\vartheta(\mathbf{x},t)>0$ is the absolute temperature of $\mathcal{B}$;
- $\mathbf{q}(\mathbf{x},t)$ denotes the *heat flux* measured *per unit area* in $\mathcal{B}$;
- $r(\mathbf{x},t)$ denotes the *body heat supply* measured *per unit volume* in $\mathcal{B}$.

7 Linear Constitutive Equations

7.1 Free Energy: Elasticity

$\sigma=\mathbb{C}\epsilon, \quad \sigma_{ij}=C_{ijkl}\epsilon_{kl}$

$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{pmatrix}=\begin{pmatrix}C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{112}\\ C_{211} & C_{222} & C_{233} & C_{223} & C_{2213} & C_{2212}\\ C_{331} & C_{332} & C_{333} & C_{333} & C_{313} & C_{2312}\\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312}\\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212}\end{pmatrix}\begin{pmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ 2\epsilon_{23}\\ 2\epsilon_{13}\\ 2\epsilon_{12}\end{pmatrix}$$

7.5 Isotropic Relations

$$\boldsymbol{\sigma}=\mathbb{C}\boldsymbol{\epsilon}=2\mu\boldsymbol{\epsilon}+\lambda(\operatorname{tr}\boldsymbol{\epsilon})\mathbf{1}$$

$$\boldsymbol{\sigma}=2\mu\boldsymbol{\epsilon}'+\kappa(\operatorname{tr}\boldsymbol{\epsilon})\mathbf{1}, \quad \sigma_{ij}=2\mu\epsilon'_{ij}+\kappa(\epsilon_{kk})\delta_{ij}$$

$$\mathbb{C}=2\mu^{\text{sym}}+\lambda\mathbf{1}\otimes\mathbf{1}, \quad C_{ijkl}=\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+\lambda\delta_{ij}\delta_{kl}$$

$$\epsilon=\tfrac{1}{2\mu}\sigma'+\tfrac{1}{9\kappa}(\operatorname{tr}\sigma)1, \quad \epsilon_{ij}=\tfrac{1}{2\mu}\sigma'_{ij}+\tfrac{1}{9\kappa}(\sigma_{kk})\delta_{ij}$$

$$E\equiv\tfrac{9\kappa\mu}{3\kappa+\mu}, \qquad \nu\equiv\tfrac{1}{2}\left[\tfrac{3\kappa-2\mu}{3\kappa+\mu}\right] \qquad \kappa=\lambda+\tfrac{2}{3}\mu$$

$$\sigma=\frac{E}{(1+\nu)}\left[\epsilon+\frac{\nu}{(1-2\nu)}(\operatorname{tr}\epsilon)1\right],$$

$$\sigma_{ij}=\frac{E}{(1+\nu)}\left[\epsilon_{ij}+\frac{\nu}{(1-2\nu)}(\epsilon_{kk})\delta_{ij}\right]$$

$$\epsilon=\frac{1}{E}[(1+\nu)\sigma-\nu(\operatorname{tr}\sigma)1]. \quad \epsilon_{ij}=\frac{1}{E}[(1+\nu)\sigma_{ij}-\nu(\sigma_{kk})\delta_{ij}]$$

8 Elastostatics

Displacement Formulation (Navier)

$C_{ijkl}u_{k,lj}+b_i=0$

Isotropic

$$\mu\triangle\mathbf{u}+(\lambda+\mu)\nabla\operatorname{div}\mathbf{u}+\mathbf{b}=0$$
$$\mu u_{i,jj}+(\lambda+\mu)u_{j,ji}+b_i=0$$

$$(\lambda+2\mu)\nabla\operatorname{div}\mathbf{u}-\mu\operatorname{curl}\operatorname{curl}\mathbf{u}+\mathbf{b}=\mathbf{0}$$
$$(\lambda+2\mu)u_{j,ji}-\mu e_{ijk}e_{klm}u_{m,lj}+b_i=0$$

Boundary:

$$\left.\begin{aligned}\mathbf{u}&=\hat{\mathbf{u}}\text{ on }\mathcal{S}_1,\\ \left(\mu\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^{\top}\right)+\lambda(\operatorname{div}\mathbf{u})\mathbf{1}\right)\mathbf{n}&=\hat{\mathbf{t}}\quad\text{on }\mathcal{S}_2\end{aligned}\right\}$$

Stress Formulation (Beltrami-Mitchell)

Compatibility:

$$\sigma_{ij,kk}+\tfrac{1}{1+\nu}\sigma_{kk,ij}=-\tfrac{\nu}{1-\nu}b_{k,k}\delta_{ij}-b_{i,j}-b_{j,i}$$
$$\Delta\sigma_{kk}=-\tfrac{1+\nu}{1-\nu}b_{k,k}$$

Plane Stress/Strain

$\epsilon_{13}=\epsilon_{23}=\sigma_{13}=\sigma_{23}=0$

	Plane Stress	Plane Strain
	$\sigma_{\alpha\beta}=\sigma_{\alpha\beta}(x_1,x_2)$	$u_\alpha=u_\alpha(x_1,x_2)$ $u_3=0$
s	$\frac{1}{1+\nu}$	$1-\nu$
σ_{33}	0	$\nu\sigma_{\alpha\alpha}$
ϵ_{33}	$-\frac{\nu}{1-\nu}\epsilon_{\alpha\alpha}$	0

Airy Stress Function

$\sigma_{11}=\varphi_{,22}\,, \;\; \sigma_{22}=\varphi_{,11}\,, \;\; \sigma_{12}=-\varphi_{,12=}$

Compatibility

$$\Delta\Delta\varphi c=\varphi,1111+2\varphi,1122+\varphi,2222=0$$

Displacements

$u_1=\frac{(1+\nu)}{E}(-\varphi,1+s\psi,2)+w_1$

$u_2=\frac{(1+\nu)}{E}(-\varphi,2+s\psi_{,1})+w_2$

where: $\Delta\psi=0$ and $\psi_{,12}=\Delta\varphi$

and w is a plane rigid displacement:

$w_{1,1}=0, \quad w_{2,2}=0, \quad w_{1,2}+w_{2,1}=0$

Polar form (9.4.16):

$$\sigma_{rr}=\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}$$
$$\sigma_{\theta\theta}=\frac{\partial^2\varphi}{\partial r^2}$$
$$\sigma_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right)$$

$$\text{Navier}:\begin{cases}\left(\frac{E}{2(1+\nu)}\right)u_{\alpha,\beta\beta}+\left(\frac{E}{2(1+\nu)(1-2\nu)}\right)u_{\beta,\beta\alpha}+b_\alpha=0 & \text{plane strain}\\ \left(\frac{E}{2(1+\nu)}\right)u_{\alpha,\beta\beta}+\frac{E}{2(1-\nu)}u_{\beta,\beta\alpha}+b_\alpha=0 & \text{plane stress}\end{cases}$$

Constitutive Relation

$$\sigma_{\alpha\beta}=\frac{E}{(1+\nu)}\left(\epsilon_{\alpha\beta}+\left(\frac{1-s}{2s-1}\right)(\epsilon_{\gamma\gamma})\delta_{\alpha\beta}\right)$$

$$\epsilon_{\alpha\beta}=\frac{(1+\nu)}{E}(\sigma_{\alpha\beta}-(1-s)(\sigma_{\gamma\gamma})\delta_{\alpha\beta})$$

Equilibrium

$$\sigma_{\alpha\beta,\beta}+b_\alpha=0$$

Compatibility $\quad \Delta(\sigma_{\alpha\alpha})=(\sigma_{11}+\sigma_{22})_{,11}+(\sigma_{11}+\sigma_{22})_{,22}=-\frac{1}{s}b_{\alpha,\alpha}$

Torsion

$$u_1(\mathrm{x})\approx-\alpha x_2x_3$$
$$u_2(\mathrm{x})\approx\alpha x_1x_3$$
$$u_3(\mathbf{x})=\alpha\varphi(x_1,x_2)$$

$$T=\int_{\mathcal{S}_L}(x_1\sigma_{23}-x_2\sigma_{13})\,da$$

$$\bar{J}\overset{\text{def}}{=}\int_\Omega\left(x_1^2+x_2^2+x_1\varphi_{,2}-x_2\varphi_{,1}\right)da$$

$$\alpha=\frac{T}{\mu\bar{J}} \qquad \Theta=\int_0^L\alpha\,dz=\int_0^L\frac{T}{\mu J}\,dz\,.$$

If T, μ and J are constant $\quad \Theta=\frac{TL}{\mu J}.$

Stress Formulation

Compatibility:

$$\epsilon_{13,2}-\epsilon_{23,1}=-\alpha \quad \text{in }\Omega$$
$$\Delta\Psi=\Psi_{,11}+\Psi_{,22}=-2\mu\alpha \quad \text{in }\Omega$$

subject to $\quad \Psi=0 \quad$ on Γ

$$\sigma_{13}=\frac{\partial\Psi}{\partial x_2}, \quad \sigma_{23}=-\frac{\partial\Psi}{\partial x_1}$$

$$T=2\int_\Omega\Psi da$$

Displacement Formulations

$$\epsilon_{11}(\mathrm{x})=\epsilon_{22}(\mathrm{x})=\epsilon_{33}(\mathrm{x})=\epsilon_{12}(\mathrm{x})=0$$
$$\epsilon_{13}(\mathrm{x})=\frac{1}{2}\left(\frac{\partial\varphi}{\partial x_1}-x_2\right)\alpha$$
$$\epsilon_{23}(\mathrm{x})=\frac{1}{2}\left(\frac{\partial\varphi}{\partial x_2}+x_1\right)\alpha$$

$$\sigma_{11}(\mathrm{x})=\sigma_{22}(\mathrm{x})=\sigma_{33}(\mathrm{x})=\sigma_{12}(\mathrm{x})=0$$
$$\sigma_{13}(\mathrm{x})=\mu\alpha\left(\frac{\partial\varphi}{\partial x_1}-x_2\right)$$
$$\sigma_{23}(\mathrm{x})=\mu\alpha\left(\frac{\partial\varphi}{\partial x_2}+x_1\right)$$

Equilibrium: $\sigma_{13,1}+\sigma_{23,2}=0 \quad$ in Ω

Boundary: $\quad \Delta\varphi=0 \quad$ in Ω

$\frac{\partial\varphi}{\partial n}=x_2n_1-x_1n_2 \quad$ on Γ

X Viscoelasticity

and is sketched in Fig. 29.5. The short-term value of this function is called its “glassy” value, $E_r(0_+) \equiv E_{rg}$. The long-term value of this function is called its “equilibrium” value, $E_r(\infty) \equiv E_{re}$.

$\sigma(t) = \eta\dot{\epsilon}(t)$

29.2 Stress relaxation & Creep

Strain (Creep test), given $\sigma(t) = \sigma_0 h(t)$

$$\epsilon(t) = \sum_{i=1}^{N} J_c\left(t - t_i\right)\Delta\sigma_i \quad (p497)$$

$$\epsilon(t) = \int_{0^-}^{t} J_c(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau \quad (29.2.6)$$

$$J_c(t) \stackrel{\text{def}}{=} \frac{\epsilon(t)}{\sigma_0}$$

Stress (Relaxation test), given $\epsilon(t) = \epsilon_0 h(t), \quad \dot{\epsilon}(t) = \epsilon_0\delta(t)$

$$\sigma(t) = \sum_{i=1}^{N} h\left(t - t_i\right)\Delta\sigma_i$$

$$\sigma(t) = \int_{0^-}^{t} E_r(t-\tau)\frac{d\epsilon(\tau)}{d\tau}d\tau = \left(E_r * \dot{\epsilon}\right)(t)$$

29.7 Oscillatory Response

$$J^* E^* = 1$$

Applied stress, $\sigma(t) = \sigma_0\cos(\omega t)$

$$\begin{aligned}\epsilon(t) &= \epsilon_0\cos(\omega t - \delta)\\ &= \epsilon_0\cos(\delta)\cos(\omega t) - \epsilon_0\sin(\delta)\sin(\omega t)\\ &= \sigma_0\left(J'\cos(\omega t) + J''\sin(\omega t)\right)\end{aligned}$$

Storage compliance, $J' \stackrel{\text{def}}{=} \frac{\epsilon_0}{\sigma_0}\cos(\delta)$.

Loss compliance, $J'' \stackrel{\text{def}}{=} \frac{\epsilon_0}{\sigma_0}\sin(\delta)$.

Applied strain, $\epsilon(t) = \epsilon_0\cos(\omega t)$

$$\begin{aligned}\sigma(t) &= \sigma_0\cos(\omega t + \delta)\\ &= \sigma_0\cos(\delta)\cos(\omega t) - \sigma_0\sin(\delta)\sin(\omega t)\\ &= \epsilon_0\left(E'\cos(\omega t) - E''\sin(\omega t)\right)\end{aligned}$$

Storage modulus, $E' \stackrel{\text{def}}{=} \frac{\sigma_0}{\epsilon_0}\cos(\delta)$.

Loss modulus, $E'' \stackrel{\text{def}}{=} \frac{\sigma_0}{\epsilon_0}\sin(\delta)$.

29.8 Complex formulation of oscillatory response

Applied stress, $\sigma(t) = \sigma_0 e^{i\omega t}$

$$\begin{aligned}\epsilon(t) &= \epsilon_0 e^{i(\omega t-\delta)}\\ &= \sigma_0\left[J' - iJ''\right]e^{i\omega t}\\ &= J^*\sigma(t)\end{aligned}$$

Complex compliance, $J^*(\omega) \stackrel{\text{def}}{=} \frac{\epsilon(t)}{\sigma_0 e^{i\omega t}} = J' - iJ''$

Applied strain, $\epsilon(t) = \epsilon_0 e^{i\omega t}$

$$\begin{aligned}\sigma(t) &= \sigma_0 e^{i(\omega t+\delta)}\\ &= \epsilon_0\left[E' + iE''\right]e^{i\omega t}\\ &= E^*\epsilon(t)\end{aligned}$$

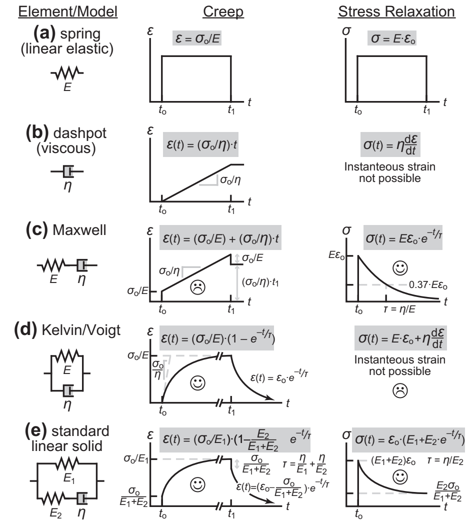
Complex modulus, $E^*(\omega) \stackrel{\text{def}}{=} \frac{\sigma(t)}{\epsilon_0 e^{i\omega t}} = E' + iE''$.

29.9 More on complex variable representation

	Kelvin-Voight	Maxwell	Std	Gen
$E'(\omega)$	E	$\frac{\tau^2\omega^2}{\tau^2\omega^2+1}E$	$\frac{E_{re}+E_{rg}\left(\tau_R^{(1)}\omega\right)^2}{1+\left(\tau_R^{(1)}\omega\right)^2}$	$E^{(0)} + \sum_\alpha \frac{E^{(\alpha)}\left(\omega\tau_R^{(\alpha)}\right)^2}{1+\left(\omega\tau_R^{(\alpha)}\right)^2}$
$E''(\omega)$	$\eta\omega = E\tau_R\omega$	$\frac{\tau\omega}{\tau^2\omega^2+1}E$	$\frac{\left(E_{rg}-E_{re}\right)\left(\tau_R^{(1)}\omega\right)}{1+\left(\tau_R^{(1)}\omega\right)^2}$	$\sum_\alpha \frac{E^{(\alpha)}\left(\omega\tau_R^{(\alpha)}\right)}{1+\left(\omega\tau_R^{(\alpha)}\right)^2}$
$\tan\delta(\omega)$			$\frac{\left(E_{rg}-E_{re}\right)\left(\tau_R^{(1)}\omega\right)}{E_{re}+E_{rg}\left(\tau_R^{(1)}\omega\right)^2}$	$\frac{E''(\omega)}{E'(\omega)}$

$$\tau_C \stackrel{\text{def}}{=} \frac{\tau_R(E_1+E_2)}{E_1}$$

$$E_r(t) = E^{(0)} + \sum_\alpha E^{(\alpha)}\exp\left(-\frac{\ }{\tau_R^{(\alpha)}}\right).$$



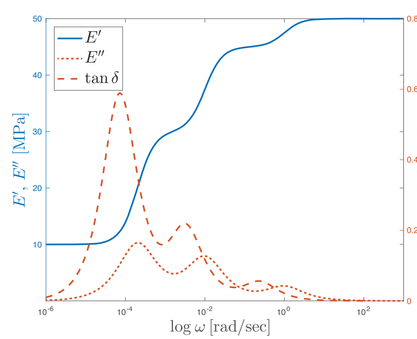


Figure 11.11: Storage and loss moduli and tan-delta versus frequency for a three element generalized standard linear solid.

29.8.1 Energy dissipation under oscillatory conditions

29.5 Correspondence Principle (1D)

$$\bar{\sigma}(s) = \bar{E}_r^*(s)\bar{\epsilon}(s)$$

where $\bar{E}_r^*(s) = s\bar{E}_r(s)$

$$\bar{J}_c(s)\bar{E}_r(s) = \frac{1}{s^2}$$

Correspondence Principle 1 for bending:[4] *If a statically determinate viscoelastic beam is subjected to forced displacement boundary conditions which are all applied at time $t = 0$ and held constant, then:*

(a) *The strain distribution is time-independent, and the same as that in an elastic beam.*

(b) *The stress distribution is time-dependent, and derived from the elastic solution by replacing the Young's modulus E with the stress-relaxation function $E_r(t)$.*

(c) *The reactions and loadings applied to the beam and the bending moment and shear force distribu-tion along the beam, being consistent with the time-dependent stress field, are also time-dependent, and can be derived from the elastic solution by replacing the Young's modulus E with the stress-relaxation function $E_r(t)$.*

Correspondence Principle 2 for bending:[5] *If a statically determinate viscoelastic beam is subjected to loads which are all applied simultaneously at time $t = 0$, and held constant, then:*

(a) *The stress distribution is time-independent, and the same as that in an elastic beam under the same load.*

(b) *The strain and displacement distributions depend on time, and are derived from the elastic solution by replacing the elastic compliance $1/E$ by the creep compliance $J_c(t)$.*

Correspondence Principle 1 for torsion: *If a statically determinate viscoelastic shaft is subjected to an angle of twist which is applied at time $t = 0$ and held constant, then:*

(a) *The shear strain distribution is time-independent, and the same as that in an elastic shaft.*

(b) *The shear stress distribution and the torque are time-dependent, and derived from the elastic solution by replacing the shear modulus G with shear-stress-relaxation function $G_r(t)$.*

Correspondence Principle 2 for torsion: *If a statically determinate viscoelastic shaft is subjected to a torque which is applied at time $t = 0$, and held constant, then:*

(a) *The shear stress distribution is time-independent, and the same as that in an elastic shaft under the same torque.*

(b) *The angle of twist depends on time, and is derived from the elastic solution by replacing the shear compliance $1/G$ by the shear creep compliance $L_c(t)$.*

IIX Elastic Limits

19 Limits & Failure Criteria

when volume is conserved:

$$\sigma = \frac{P}{A} = \frac{P}{A_0}\frac{L}{L_0} = s(1+e),$$

$$\epsilon = \ln(1+e).$$

Invariants

1. Mean normal pressure.

$$\bar{p} = -\tfrac{1}{3}\operatorname{tr}\sigma = -\tfrac{1}{3}(\sigma_{kk})$$

$$\bar{p} = -\tfrac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

2. Equivalent shear/tensile stress.

$$\bar{\tau} = \tfrac{1}{\sqrt{2}}|\sigma'| = \sqrt{\tfrac{1}{2}\operatorname{tr}(\sigma'^2)} = \sqrt{\tfrac{1}{2}\sigma'_{ij}\sigma'_{ij}}$$

$$\bar{\tau} = \left[\tfrac{1}{6}\left((\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2\right) + (\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2)\right]^{1/2}$$

$$\bar{\sigma} = \sqrt{\tfrac{3}{2}}|\sigma'| = \sqrt{\tfrac{3}{2}\operatorname{tr}(\sigma'^2)} = \sqrt{\tfrac{3}{2}\sigma'_{ij}\sigma'_{ij}}$$

$$\bar{\sigma} = \left[\tfrac{1}{2}\left((\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2\right) + 3(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2)\right]^{1/2}$$

$$\bar{\sigma} = \sqrt{3}\bar{\tau}$$

3. Third stress invariant.

$$\bar{r} = \left(\tfrac{9}{2}\operatorname{tr}(\sigma'^3)\right)^{\frac{1}{3}} = \left(\tfrac{9}{2}\sigma'_{ik}\sigma'_{kj}\sigma'_{jk}\right)^{\frac{1}{3}}$$

Failure Criteria

1. **Von Mises**

$$f = \bar{\sigma} - Y \le 0$$

2. **Tresca**

$$f = (\sigma_1 - \sigma_3) \le Y$$
$$f = \tau_{\max} - \tau_{y,\,\text{Tresca}} \le 0$$

$$\tau_{\max} \overset{\text{def}}{=} \tfrac{1}{2}(\sigma_1 - \sigma_3) \ge 0$$
$$\tau_{y,\,\text{Tresca}} \overset{\text{def}}{=} \tfrac{1}{2}Y$$

3. **Mohr-Coulomb**

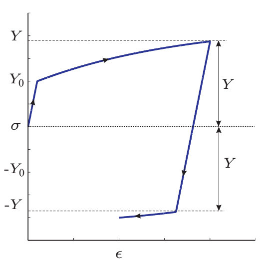
$$f = \tfrac{1}{2}(\sigma_1 - \sigma_3) + \tfrac{1}{2}(\sigma_1 + \sigma_3)\sin\phi - c\cos\phi \le 0$$

$$\mu = \tan\phi$$

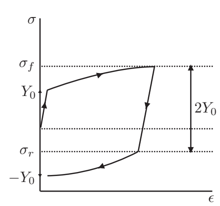
4. **Drucker-Prager**

$$f(\bar{\tau}, \bar{p}, S) = \bar{\tau} - (S + \alpha\bar{p}) \le 0$$

Isotropic hardening



Kinematic hardening



20 One-Dimensional Plasticity

20.1 Elastic-Plastic Response

Kinematic hardening

$$\sigma_b = \tfrac{1}{2}(\sigma_f + \sigma_r)$$
$$|\sigma - \sigma_b| \le Y_0$$

20.2 Isotropic rate-independent theory

1. Kinematic decomposition

$$\epsilon = \epsilon^e + \epsilon^p$$

2. Constitutive relation

$$\sigma = E\epsilon^e = E(\epsilon - \epsilon^p)$$

3. Yield condition

$$f = |\sigma| - Y(\bar{\epsilon}^{\mathrm{p}}) \le 0$$

4. Flow rule

$$\dot{\epsilon}^{\mathrm{p}} = \chi\beta\dot{\epsilon}$$

$$\beta = \frac{E}{E + H(\bar{\epsilon}^{\mathrm{p}})} > 0 \quad \text{(by hypothesis)}$$

$$H(\bar{\epsilon}^{\mathrm{p}}) = \frac{dY(\bar{\epsilon}^{\mathrm{p}})}{d\bar{\epsilon}^{\mathrm{p}}}$$

5. Kuhn-Tucker condition

$$\chi = 0 \quad \text{if } f < 0, \text{ or if } f = 0 \text{ and } n^p\dot{\epsilon} < 0, \quad \text{where} \quad n^p = \operatorname{sign}(\sigma)$$

6. Consistency condition

$$\chi = 1 \quad \text{if } f = 0 \quad \text{and} \quad n^p\dot{\epsilon} > 0$$

$$\dot{\sigma} = E[1 - \chi\beta]\dot{\epsilon}$$
$$\dot{\sigma} = E_{\tan}\dot{\epsilon}$$

$$E_{\tan} = \begin{cases} E & \text{if } \chi = 0 \\ \frac{EH(\bar{\epsilon}^{\mathrm{p}})}{E + H(\bar{\epsilon}^{\mathrm{p}})} & \text{if } \chi = 1 \end{cases}$$

$$\dot{\bar{\epsilon}}^p \overset{\text{def}}{=} |\dot{\epsilon}^p| \ge 0 \qquad \bar{\epsilon}^{\mathrm{p}}(t) = \int_0^t \dot{\bar{\epsilon}}^{\mathrm{p}}(\zeta)\, d\zeta$$

21.7 Summary of Mises-Hill Theory

1. Strain decomposition, (21.7.1)

$$\epsilon = \epsilon^{\mathrm{e}} + \epsilon^{\mathrm{P}}$$

2. Constitutive relation (21.7.2)

$$\sigma = 2\mu(\epsilon - \epsilon^{\mathrm{p}}) + (\kappa - (2/3)\mu)(\operatorname{tr}\epsilon)\mathbf{1}$$

with $\mu > 0$ and $\kappa > 0$ the elastic shear and bulk moduli.

3. Yield condition (21.7.3)

$$f = \bar{\sigma} - Y(\bar{\epsilon}^{\mathrm{P}}) \le 0$$

4. Evolution equations, (21.7.5)

$$\dot{\epsilon}^{\mathrm{p}} = \chi\beta(\bar{\epsilon}^{\mathrm{p}})(\mathbf{n}^{\mathrm{p}} : \dot{\epsilon})\,\mathbf{n}^{\mathrm{p}}, \quad \mathbf{n}^{\mathrm{p}} = \sqrt{3/2}\frac{\sigma'}{\bar{\sigma}}$$
$$\dot{\bar{\epsilon}}^{\mathrm{p}} = \sqrt{2/3}\,|\dot{\epsilon}^{\mathrm{p}}|$$

with

Stiffness ratio: $\beta(\bar{\epsilon}^{\mathrm{p}}) = \dfrac{3\mu}{3\mu + H(\bar{\epsilon}^{\mathrm{p}})} > 0$

Hardening modulus (21.7.6): $H(\bar{\epsilon}^{\mathrm{p}}) = \dfrac{dY(\bar{\epsilon}^{\mathrm{p}})}{d\bar{\epsilon}^{\mathrm{p}}}$

Switching parameter (21.7.7)

$$\chi = \begin{cases} 0 & \text{if } f < 0, \text{ or if } f = 0 \text{ and } n^{\mathrm{p}} : \dot{\epsilon} \le 0 \\ 1 & \text{if } f = 0 \text{ and } n^{\mathrm{p}} : \dot{\epsilon} > 0 \end{cases}$$

and typical **initial conditions:**

$$\epsilon(\mathrm{x}, 0) = \epsilon^{\mathrm{p}}(\mathrm{x}, 0) = 0, \quad \text{and} \quad \bar{\epsilon}^{\mathrm{p}}(\mathrm{x}, 0) = 0$$

Also note:

$$d\epsilon_{ij} = \underbrace{\frac{1+\nu}{E}d\sigma_{ij} - \frac{\nu}{E}(d\sigma_{kk})\,\delta_{ij}}_{d\epsilon^{\mathrm{e}}_{ij}} + \underbrace{(3/2)d\bar{\epsilon}^{\mathrm{p}}\frac{\sigma'_{ij}}{\bar{\sigma}}}_{d\epsilon^{\mathrm{p}}_{ij}}$$

$$\dot{\epsilon} = \underbrace{\frac{1+\nu}{E}\dot{\sigma} - \frac{\nu}{E}(\operatorname{tr}\dot{\sigma})\mathbf{1}}_{\dot{\epsilon}^{\mathrm{e}}} + \underbrace{(3/2)\dot{\bar{\epsilon}}^{\mathrm{p}}\frac{\sigma'}{\bar{\sigma}}}_{\dot{\epsilon}^{\mathrm{p}}},$$

equivalent tensile plastic strain rate

$$\dot{\bar{\epsilon}}^{\mathrm{p}} \overset{\text{def}}{=} \sqrt{(2/3)}\,|\dot{\epsilon}^{\mathrm{p}}| = \sqrt{(2/3)\dot{\epsilon}^{\mathrm{p}}_{ij}\dot{\epsilon}^{\mathrm{p}}_{ij}},$$

equivalent tensile plastic strain

$$\bar{\epsilon}^{\mathrm{p}}(t) \overset{\text{def}}{=} \int_0^t \dot{\bar{\epsilon}}^{\mathrm{p}}(\zeta)\, d\zeta.$$

25.1 Mixed BVP for a rigid-perfectly-plastic solid

Shear flow strength: $k = Y/\sqrt{3}$

25.4 Lower bound theorem

Statically admissible stress field, with respect to to traction field $\mathbf{t}^*$ satisfies:

1. Equilibrium, $\operatorname{div}\boldsymbol{\sigma}^* + \mathbf{b}^* = \mathbf{0}$ on $\mathcal{B}$
2. Traction B.Cs, $\boldsymbol{\sigma}^*\mathbf{n} = \mathbf{t}^*$ on $\partial\mathcal{B}$
3. Yield condition: $f(\sigma^*, Y) = \sqrt{3/2}\,|\sigma^{*\prime}| - Y \le 0$ in $\mathcal{B}$

25.5 Upper bound theorem

Kinematically admissible velocity field, $\mathbf{v}^*$, satisfies:

1. Stretching-velocity relation, $\dot{\epsilon}^* = \frac{1}{2}\left((\nabla\mathbf{v}^*) + (\nabla\mathbf{v}^*)^\top\right)$;
2. gives no volume change, $\operatorname{tr}\dot{\epsilon}^* = 0$
3. Satisfies velocity B.Cs, $\mathbf{v}^* = \hat{\mathbf{v}}$ on $\mathcal{S}_1$

Upper bound:

$$\Phi\{\mathbf{v}^*\} = \mathcal{D}_{\text{int}}\{\mathbf{v}^*\} - \mathcal{W}_{\text{ext}}\{\mathbf{v}^*\} \ge \underbrace{\Phi\{\mathbf{v}\}}_{=0}$$

$$\beta_U = \frac{\sum_{\mathcal{S}_{\hat{d}}} A_{\mathcal{S}_d^*} k\,[v^*]}{\int_{\mathcal{B}}\tilde{\mathbf{b}}\cdot\mathbf{v}^* dv + \int_{\partial\mathcal{B}}\tilde{\mathbf{t}}\cdot\mathbf{v}^* da} = \frac{\sum_{\mathcal{S}_{\hat{A}}} A_{\mathcal{S}_{\hat{d}}} \cdot k\,[v^*]}{\int_{\mathcal{B}}\tilde{\mathbf{b}}\cdot\mathbf{v}^* dv + \int_{\partial\mathcal{B}}\tilde{\mathbf{t}}\cdot\mathbf{v}^* da}$$

Example 29.2 (Extensional response of a viscoelastic rod)**.** Consider a viscoelastic rod of cross-sectional area A that is subjected to a time varying axial load

$$P(t) = [P_o + P_1 \sin(\omega t)]h(t)\,.$$

The creep compliance for the rod is known to be

$$J_c(t) = J_{ce} - \Delta J_c e^{-t/\tau_C} \qquad \text{with} \qquad \Delta J_c = J_{ce} - J_{cg}.$$

$$\epsilon = \frac{\sigma}{E},$$

for a viscoelastic material in the transform domain,

$$\bar{\epsilon}(s) = \frac{\bar{\sigma}(s)}{s\bar{E}_r(s)} = \frac{\bar{\sigma}(s)}{s/(s^2\bar{J}_c(s))} = s\bar{J}_c(s)\bar{\sigma}(s)\,.$$

The necessary Laplace transforms of σ and J_c are easily computed (or found in tables) as:

$$\bar{\sigma}(s) = \frac{1}{A}\left[\frac{P_o}{s} + \frac{P_1\omega}{s^2+\omega^2}\right], \qquad \text{and} \qquad \bar{J}_c(s) = \frac{J_{ce}}{s} - \frac{}{s+1/\tau_C}\,.$$

Hence the Laplace transform of the strain

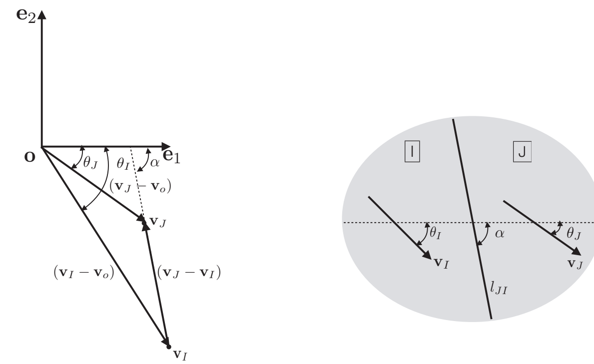
$$\bar{\epsilon}(s) = \frac{s}{A}\left[\frac{J_{ce}}{s} - \frac{}{s+1/\tau_C}\right]\left[\frac{P_o}{s} + \frac{P_1\omega}{s^2+\omega^2}\right].$$

The inverse Laplace transform:

$$\epsilon(t) = \frac{J_{ce}}{A}\Big(P_o + P_1\sin(\omega t)\Big) - \Delta J_c\left[\frac{P_1\omega\tau_C\,(\omega\tau_C\sin(\omega t)+\cos(\omega t))}{1+\omega^2\tau_C^2} + \exp\Big(- \quad\Big)\frac{\omega\tau_C\,(\omega\tau_C P_o - P_1) + P_o}{\tau_C \;\; 1+\omega^2\tau_C^2}\right].$$

Hodograph

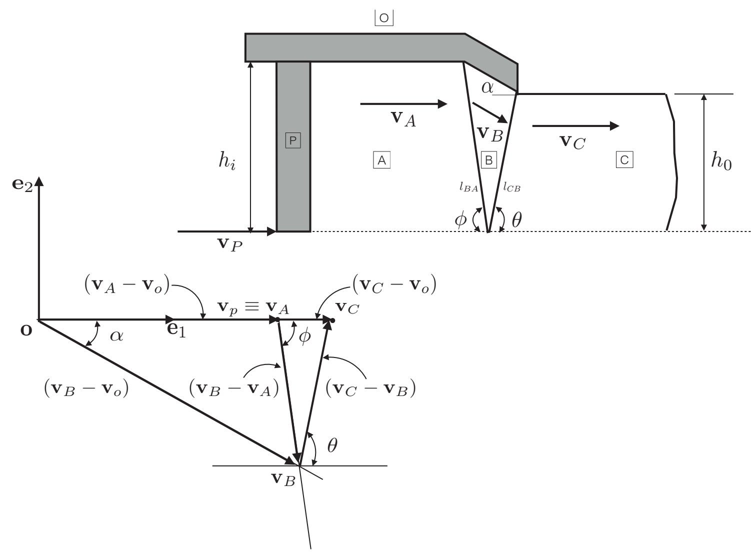
The hodograph for two rigid-blocks I and J moving at velocities $\mathbf{v}_I$ and $\mathbf{v}_J$ at angles θ_I and θ_J to the horizontal is shown in Fig. 25.7. Note that the relative velocity between the two blocks $(\mathbf{v}_J - \mathbf{v}_I)$ is necessarily tangential to l_{JI} in order for the velocity field to be kinematically admissible, in particular not give rise to volume strains.



rate of energy dissipation per unit depth along a single line of tangential relative velocity discontinuity

$$\mathcal{D}_{\text{int}} = \sum_{\mathcal{S}_d^*} k \times l_{JI} \times |\mathbf{v}_J - \mathbf{v}_I|. \qquad k \stackrel{\text{def}}{=} \frac{Y}{\sqrt{3}},$$

k is the shear flow strength of the material,
l_{JI} is the length of discontinuity line between blocks J and I,
$|\mathbf{v}_J - \mathbf{v}_I|$ is the absolute value of the relative velocity.



1. Plot a vector of magnitude $|\mathbf{v}_P - \mathbf{v}_0| \equiv |\mathbf{v}_A - \mathbf{v}_0|$ starting from **o** parallel to $\mathbf{e}_1$. This represents the velocities of the punch P and the block A relative to the die O.
2. Plot a line emanating from the origin **o** at angle $-\alpha$ to $\mathbf{e}_1$. This represents the direction of velocity $(\mathbf{v}_B - \mathbf{v}_0)$ of the block B relative to that of the die O. The magnitude of $(\mathbf{v}_B - \mathbf{v}_0)$ is not known at this point.
3. Plot a line emanating from the tip of the relative velocity vector $(\mathbf{v}_A - \mathbf{v}_0)$ at angle $-\phi$ to $\mathbf{e}_1$. For plastic incompressibility, the relative velocity vector $(\mathbf{v}_B - \mathbf{v}_A)$ must be parallel to this line. The intersection of the lines from Step 2 and this step give the relative velocities $(\mathbf{v}_B - \mathbf{v}_0)$ and $(\mathbf{v}_B - \mathbf{v}_A)$.
4. Since $\mathbf{v}_C$ is parallel to the $\mathbf{e}_1$ direction, draw a line emanating from the origin **o**. The magnitude of the relative velocity $(\mathbf{v}_C - \mathbf{v}_0)$ is not known at this point.
5. Since the relative velocity between blocks C and B is at angle θ to $\mathbf{e}_1$, plot a line emanating from the tip of $(\mathbf{v}_B - \mathbf{v}_0)$ at an angle θ to $\mathbf{e}_1$. The intersection of this line with the line from Step 4 fixes the magnitudes of the relative velocities $(\mathbf{v}_C - \mathbf{v}_0)$ and $(\mathbf{v}_C - \mathbf{v}_B)$.

Green strain	$\operatorname{Lin}_{\mathbf{0}} \mathbf{E}(\mathbf{H}) = \boldsymbol{\epsilon}$
Deformation gradient	$\operatorname{Lin}_{\mathbf{0}} \mathbf{F}(\mathbf{H}) = \mathbf{1} + \mathbf{H}$
Inverse deformation gradient	$\operatorname{Lin}_{\mathbf{0}} \mathbf{F}^{-1}(\mathbf{H}) = \mathbf{1} - \mathbf{H}$
Inverse right stretch	$\operatorname{Lin}_{\mathbf{0}} \mathbf{U}^{-1}(\mathbf{H}) = \mathbf{1} - \boldsymbol{\epsilon}$
Rotation	$\operatorname{Lin}_{\mathbf{0}} \mathbf{R}(\mathbf{H}) = \mathbf{1} + \boldsymbol{\omega}$
Jacobian	$\operatorname{Lin}_{\mathbf{0}} J(\mathbf{H}) = 1 + \operatorname{tr} \boldsymbol{\epsilon}$
Area strain	$\operatorname{Lin}_{\mathbf{0}} \lambda_{\text{area}}(\mathbf{n}_{\mathrm{R}}) - 1 = (\mathbf{1} - \mathbf{n}_{\mathrm{R}} \otimes \mathbf{n}_{\mathrm{R}}) : \boldsymbol{\epsilon}\,,$

where $\boldsymbol{\epsilon} = \operatorname{sym} \mathbf{H}$ and $\boldsymbol{\omega} = \operatorname{skw} \mathbf{H}$;

(a) In elementary beam theory the moment on the cross-section is given by

$$M = \int_A -y\sigma \, dA \,.$$

Triclinic, 21

Monoclinic 13: $\{\mathcal{C}_{11}, \mathcal{C}_{22}, \mathcal{C}_{33}, \mathcal{C}_{44}, \mathcal{C}_{55}, \mathcal{C}_{66}, \mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{16}, \mathcal{C}_{23}, \mathcal{C}_{26}, \mathcal{C}_{36}, \mathcal{C}_{45}\}$

Orthotropic, 9: $\{\mathcal{C}_{11}, \mathcal{C}_{22}, \mathcal{C}_{33}, \mathcal{C}_{44}, \mathcal{C}_{55}, \mathcal{C}_{66}, \mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}\}$

Tetragonal, 6: $\{\mathcal{C}_{11}, \mathcal{C}_{33}, \mathcal{C}_{44}, \mathcal{C}_{66}, \mathcal{C}_{12}, \mathcal{C}_{13}\}$

Transversely isotropic, 5: $\{\mathcal{C}_{11}, \mathcal{C}_{33}, \mathcal{C}_{44}, \mathcal{C}_{12}, \mathcal{C}_{13}\}$

Cubic, 3: $\{\mathcal{C}_{11}, \mathcal{C}_{12}, \mathcal{C}_{44}\}$

Thus, for a symmetric tensor, eigenvectors corresponding to distinct eigenvalues are orthogonal. When the eigenvalues are repeated, the situation is a bit more complex and is treated below. Notwithstanding, if $\mathbf{S}$ is **symmetric**, with distinct eigenvalues $\{\omega_1 > \omega_2 > \omega_3\}$, then they are **real**, and there exists an **orthonormal** set of corresponding eigenvectors $\{\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\mu}_3\}$ such that $\mathbf{S}\boldsymbol{\mu}_i = \omega_i \boldsymbol{\mu}_i$ (no sum).

SPECTRAL THEOREM *Let* $\mathbf{S}$ *be symmetric with distinct eigenvalues* ω_i. Then there is a corresponding orthonormal set $\{\boldsymbol{\mu}_i\}$ of eigenvectors of $\mathbf{S}$ and, what is most important, one uniquely has that

$$\mathbf{S} = \sum_{i=1}^{3} \omega_i \boldsymbol{\mu}_i \otimes \boldsymbol{\mu}_i \,. \tag{1.2.50}$$

V Variational Formulation

Traction Equilibrium Condition: Weak Form

Virtual power balance,

$$\int_{\mathcal{B}} (\operatorname{sym} \nabla \mathbf{w}) : \mathbb{C}(\operatorname{sym} \nabla \mathbf{u}) dv - \int_{\mathcal{S}_2} \hat{\mathbf{t}} \cdot \mathbf{w} da - \int_{\mathcal{B}} \mathbf{b} \cdot \mathbf{w} dv = 0 \quad \forall \,\text{admissible w.}$$

is satisfied if and only if $\operatorname{div} \sigma + \mathrm{b} = 0 \text{ in } \mathcal{B}$ and $\sigma \mathbf{n} = \hat{\mathbf{t}}$ on $\mathcal{S}_2$

Where $\mathbf{w}$ is a virtual velocity field such that $\mathbf{w} = \mathbf{0}$ on $\mathcal{S}_1$.

10.2

Elastostatic Displacement Problem: Weak Form

Given $\mathbb{C}$, $\mathbf{b}$, and boundary data $\hat{\mathbf{u}}$ and $\hat{\mathbf{t}}$, find a displacement field $\mathbf{u}$ equal to $\hat{\mathbf{u}}$ on $\mathcal{S}_1$ such that:

$$\int_{\mathcal{B}} (\operatorname{sym} \nabla \mathbf{w}) : \mathbb{C}(\operatorname{sym} \nabla \mathbf{u}) dv - \int_{\mathcal{S}_2} \hat{\mathbf{t}} \cdot \mathbf{w} da - \int_{\mathcal{B}} \mathbf{b} \cdot \mathbf{w} dv = 0 \quad \forall \,\text{admissible w.}$$

9 Solutions

Crack Tip

Mode III: Anti-plane tearing

$\Delta u_z = 0 = \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}$

$\delta_3 \overset{\text{def}}{=} u_3(r, +\pi) - u_3(r, -\pi) = \frac{4}{\mu} K_{\mathrm{III}} \sqrt{\frac{r}{2\pi}}$

$\begin{pmatrix} \sigma_{13} \\ \sigma_{23} \end{pmatrix} = \frac{K_{\mathrm{III}}}{\sqrt{2\pi r}} \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix} + \text{ bounded terms}$

$\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{12} = 0$

$(u_3) = \frac{K_{\mathrm{M}}}{2\mu} \sqrt{\frac{r}{2\pi}} \left(4 \sin\left(\frac{\theta}{2}\right)\right) +$ rigid displacenent.

where K_{III} is the Mode III (or anti-plane) stress intensity factor.

$$\Theta = \int_0^L \alpha \, dz = \int_0^L \frac{T}{\mu J} \, dz \, .$$

If T, μ and J are constant, we get

$$\boxed{\Theta = \frac{TL}{\mu J}.}$$

Correspondence Principle 1 for bending:[4] *If a statically determinate viscoelastic beam is subjected to forced displacement boundary conditions which are all applied at time $t = 0$ and held constant, then:*

(a) The strain distribution is time-independent, and the same as that in an elastic beam.

(b) The stress distribution is time-dependent, and derived from the elastic solution by replacing the Young's modulus E with the stress-relaxation function $E_r(t)$.

(c) The reactions and loadings applied to the beam and the bending moment and shear force distribution along the beam, being consistent with the time-dependent stress field, are also time-dependent, and can be derived from the elastic solution by replacing the Young's modulus E with the stress-relaxation function $E_r(t)$.

Correspondence Principle 2 for bending:[5] *If a statically determinate viscoelastic beam is subjected to loads which are all applied simultaneously at time $t = 0$, and held constant, then:*

(a) The stress distribution is time-independent, and the same as that in an elastic beam under the same load.

(b) The strain and displacement distributions depend on time, and are derived from the elastic solution by replacing the elastic compliance $1/E$ by the creep compliance $J_c(t)$.

Correspondence Principle 1 for torsion: *If a statically determinate viscoelastic shaft is subjected to an angle of twist which is applied at time $t = 0$ and held constant, then:*

(a) The shear strain distribution is time-independent, and the same as that in an elastic shaft.

(b) The shear stress distribution and the torque are time-dependent, and derived from the elastic solution by replacing the shear modulus G with shear-stress-relaxation function $G_r(t)$.

Correspondence Principle 2 for torsion: *If a statically determinate viscoelastic shaft is subjected to a torque which is applied at time $t = 0$, and held constant, then:*

(a) The shear stress distribution is time-independent, and the same as that in an elastic shaft under the same torque.

(b) The angle of twist depends on time, and is derived from the elastic solution by replacing the shear compliance $1/G$ by the shear creep compliance $L_c(t)$.