Dept. of Civil and Environmental Engineering
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CE 220 - Structural Analysis

Solution for Homework Set #7

1. Problem

Fig. 1 shows a braced frame.

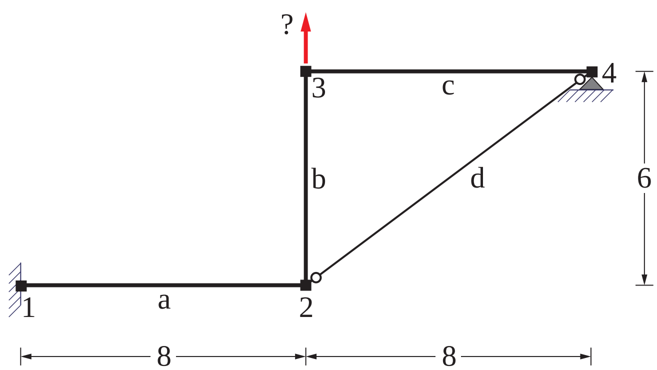


Figure 1: Braced frame under unknown vertical nodal force at node #3

A computer analysis of the frame under the assumption that elements a through c are inextensible gives the following basic element forces $\boldsymbol{Q}$ of primary interest:

$$\boldsymbol{q}^{(a)} = \begin{pmatrix} 87.947 \\ 54.824 \end{pmatrix} \qquad \boldsymbol{q}^{(b)} = \begin{pmatrix} -54.824 \\ 22.844 \end{pmatrix} \qquad \boldsymbol{q}^{(c)} = -22.844 \qquad \boldsymbol{q}^{(d)} = 15.497$$

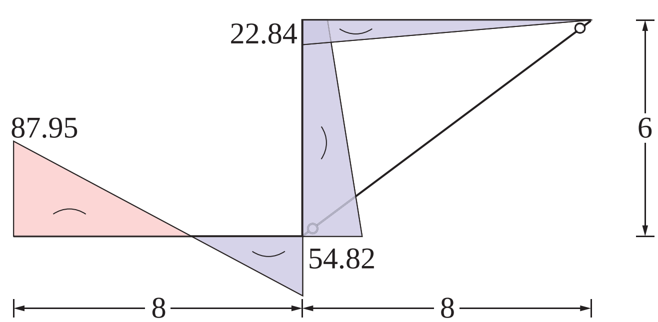
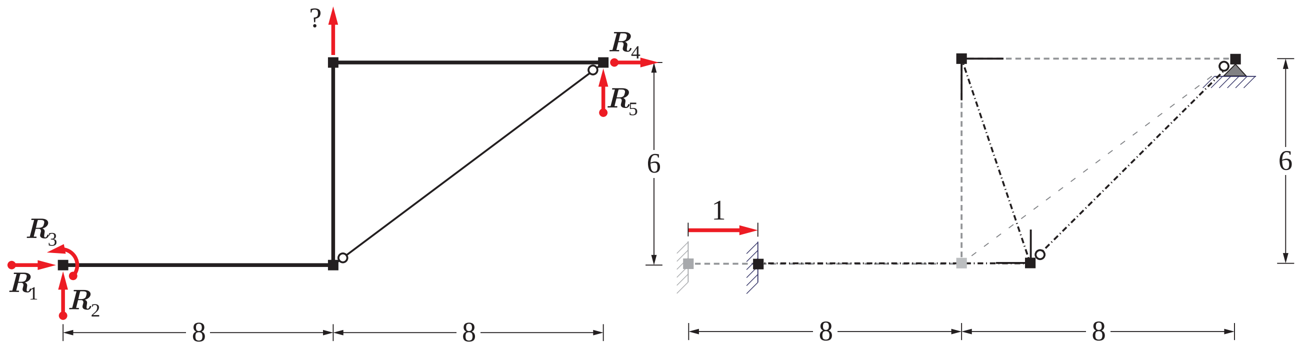


Figure 2: Bending moment distribution of braced frame

1. Fig. 2 shows the bending moment distribution of the braced frame. The bending moment diagram is drawn on the tension side of the element.
2. For determining the horizontal support reaction at node #1 we make use of the virtual displacements and deformations in Fig. 3: with a unit horizontal translation at node #1 we make sure not to displace



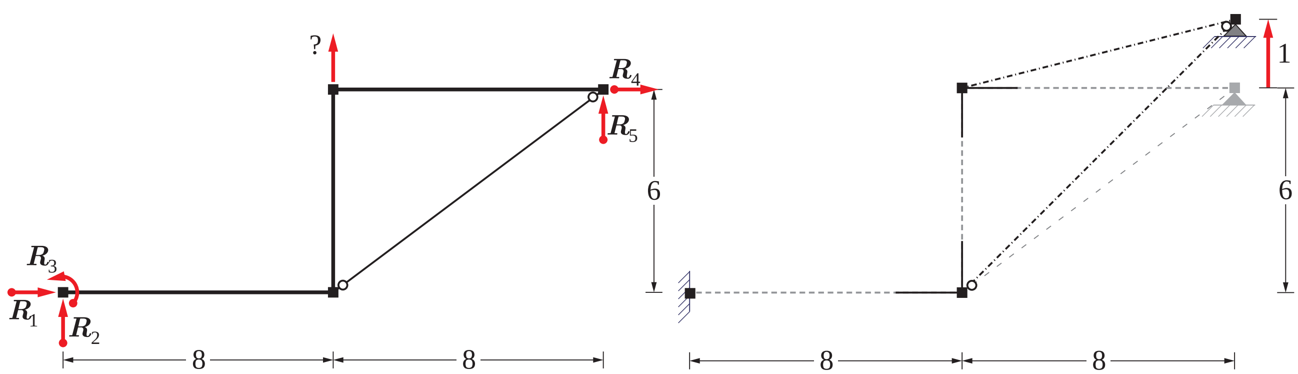
(a) Support reaction numbering (b) Virtual deformation state

Figure 3: Virtual displacements and deformations for unit horizontal translation at node #1

vertically node #3, since it carries an unknown nodal force, and not to extend axially elements a, b and c, since we do not know the axial forces.
The equality of the external and internal virtual work gives

$$(1)\boldsymbol{R}_1 = \left(-\tfrac{1}{6}\right)\boldsymbol{q}_i^{(b)} + \left(-\tfrac{1}{6}\right)\boldsymbol{q}_j^{(b)} + (-0.8)\,\boldsymbol{q}^{(d)} = -\tfrac{1}{6}\,(-54.824 + 22.844) - 0.8(15.497) = -7.068$$

3. For determining the vertical support reaction at node #4 we make use of the virtual displacements and deformations in Fig. 4: with a unit vertical translation at node #4 we make sure not to displace vertically node #3, since it carries an unknown nodal force, and not to extend axially elements a, b and c, since we do not know the axial forces.



(a) Support reaction numbering (b) Virtual deformation state

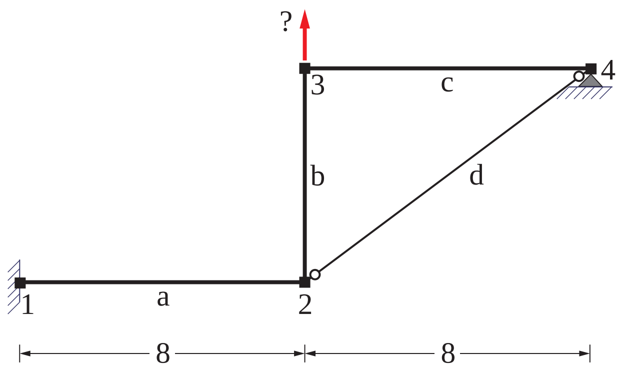
Figure 4: Virtual displacements and deformations for unit vertical translation at node #4

The equality of the external and internal virtual work gives

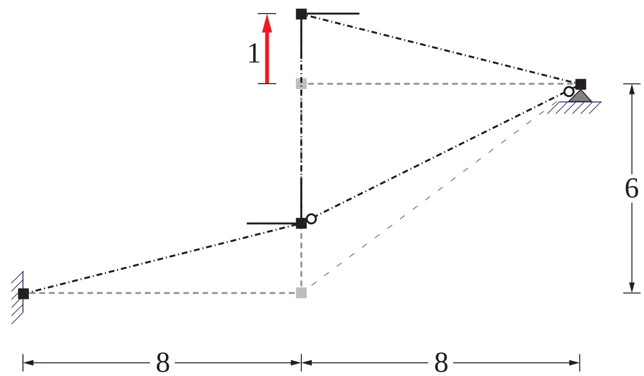
$$(1)\boldsymbol{R}_5 = \left(-\tfrac{1}{8}\right)\boldsymbol{q}^{(c)} + (0.6)\,\boldsymbol{q}^{(d)} = -\tfrac{1}{8}(-22.844) + 0.6(15.497) = 12.154$$

We conclude that this virtual work equation is identical with the equation of vertical force equilibrium of the free body at node #4.
4. For determining the magnitude of the applied nodal force at node #3 we make use of the virtual displacements and deformations in Fig. 5: with a unit vertical translation at node #3 we make sure

not to displace the supports so that we do not run the risk of accumulating an error, and not to extend axially elements a, b and c, since we do not know the axial forces.



(a) Applied nodal force numbering



(b) Virtual deformation state

Figure 5: Virtual displacements and deformations for unit vertical translation at node #3

The equality of the external and internal virtual work gives

$$
\begin{aligned}
(1)P &= \left(-\frac{1}{8}\right)\boldsymbol{q}_i^{(a)} + \left(-\frac{1}{8}\right)\boldsymbol{q}_j^{(a)} + \left(\frac{1}{8}\right)\boldsymbol{q}^{(c)} + (-0.6)\boldsymbol{q}^{(d)} \\
&= -\frac{1}{8}(87.947 + 54.824) + \frac{1}{8}(-22.844) - 0.6(15.497)(15.497) = -30
\end{aligned}
$$

We recognize that the equation corresponds to the equilibrium equation of the corresponding free dof without inclusion of the axial force in element b.

$$
P = -\frac{1}{8}\left(\boldsymbol{Q}_1 + \boldsymbol{Q}_2\right) + \frac{1}{8}\boldsymbol{Q}_5 - 0.6\boldsymbol{Q}_6
$$

2. Problem

Fig. 7(a) shows a braced frame. A computer analysis of the frame under the assumption that elements a through d are inextensible gives the following element deformations $\boldsymbol{V}$:

$$\boldsymbol{v}^{(a)} = \begin{pmatrix} -0.6937 \\ 1.3874 \end{pmatrix} 10^{-3} \qquad \boldsymbol{v}^{(b)} = \begin{pmatrix} 0.0765 \\ -2.9278 \end{pmatrix} 10^{-3} \qquad \boldsymbol{v}^{(c)} = \begin{pmatrix} -0.1310 \\ 3.1515 \end{pmatrix} 10^{-3}$$
$$\boldsymbol{v}^{(d)} = \begin{pmatrix} -3.5215 \\ -2.2149 \end{pmatrix} 10^{-3} \qquad \boldsymbol{v}^{(e)} = 6.2922 \cdot 10^{-3}$$

The corresponding basic element forces $\boldsymbol{Q}$ of primary interest are:

$$\boldsymbol{q}^{(a)} = 27.748 \quad \boldsymbol{q}^{(b)} = \begin{pmatrix} -27.748 \\ -57.791 \end{pmatrix} \quad \boldsymbol{q}^{(c)} = \begin{pmatrix} 57.791 \\ 123.440 \end{pmatrix} \quad \boldsymbol{q}^{(d)} = \begin{pmatrix} -123.440 \\ -106.019 \end{pmatrix} \quad \boldsymbol{q}^{(e)} = 12.584$$

1. Fig. 6 shows the bending moment distribution.

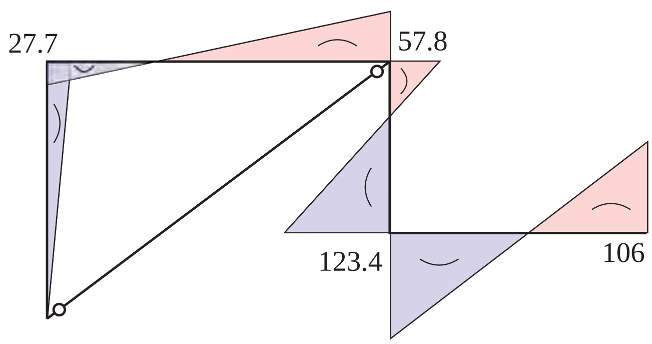
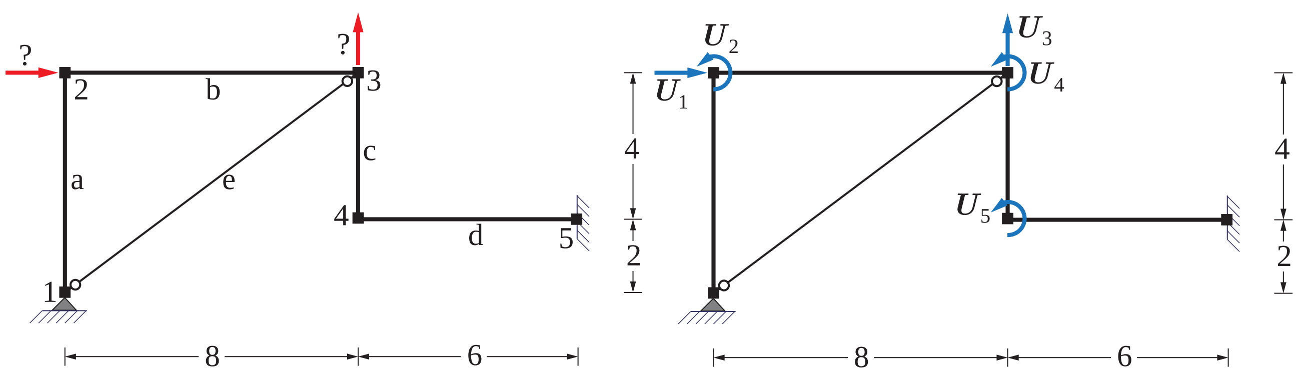


Figure 6: Bending moment distribution

With inextensible frame elements a through d the braced frame has 5 independent free dofs, as shown in Fig. 7(b). Fig. 8 shows the corresponding equilibrium equations and the basic element forces of primary interest which do not include the axial forces in elements a through d. These number 8 so that the degree of static indeterminacy is $NOS = 3$.



(a) Geometry and loading

(b) Independent free dofs

Figure 7: Braced frame under unknown nodal forces

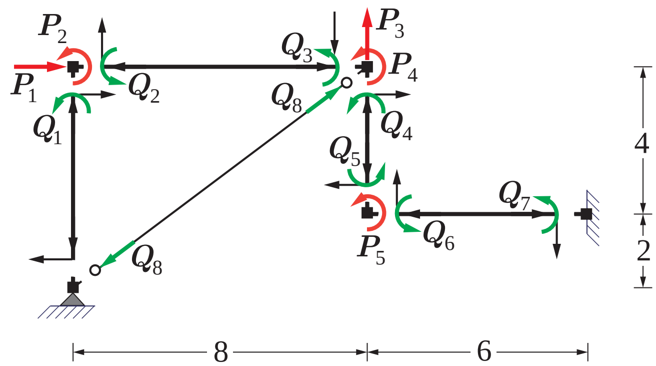


Figure 8: Equilibrium equations and basic element forces of primary interest

2. The equilibrium equations at the free dofs of the structural model are

$$
\begin{aligned}
\boldsymbol{P}_1 &= \frac{1}{6}\boldsymbol{Q}_1 + \frac{1}{4}\boldsymbol{Q}_4 + \frac{1}{4}\boldsymbol{Q}_5 + 0.8\boldsymbol{Q}_8 \\
\boldsymbol{P}_2 &= \boldsymbol{Q}_1 + \boldsymbol{Q}_2 \\
\boldsymbol{P}_3 &= -\frac{1}{8}\boldsymbol{Q}_2 - \frac{1}{8}\boldsymbol{Q}_3 + \frac{1}{6}\boldsymbol{Q}_6 + \frac{1}{6}\boldsymbol{Q}_7 + 0.6\boldsymbol{Q}_8 \\
\boldsymbol{P}_4 &= \boldsymbol{Q}_3 + \boldsymbol{Q}_4 \\
\boldsymbol{P}_5 &= \boldsymbol{Q}_5 + \boldsymbol{Q}_6
\end{aligned}
\tag{1}
$$

Upon substitution of the given basic element forces in the first and third equation of (1) we get

$$
\begin{aligned}
\boldsymbol{P}_1 &= \frac{1}{6}\boldsymbol{Q}_1 + \frac{\boldsymbol{Q}_4 + \boldsymbol{Q}_5}{4} + 0.8\boldsymbol{Q}_8 \\
&= \frac{27.748}{6} + \frac{57.791 + 123.440}{4} + 0.8(12.584) = 60
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{P}_3 &= -\frac{\boldsymbol{Q}_2 + \boldsymbol{Q}_3}{8} + \frac{\boldsymbol{Q}_6 + \boldsymbol{Q}_7}{6} + 0.6\boldsymbol{Q}_8 \\
&= -\frac{-27.748 - 57.791}{8} + \frac{-123.440 - 106.019}{6} + 0.6(12.584) = -20
\end{aligned}
$$

3. For determining the vertical support reaction at node #1 we make use of the virtual displacements and deformations in Fig. 9: with a unit vertical translation at node #1 we make sure not to displace vertically node #3 *so as to determine the support reaction independently of the applied nodal force* and also make sure not to extend axially elements a, b, c and d, since we do not know the axial forces. The equality of the external and internal virtual work gives the vertical support reaction R at node #1

$$
(1)R = \tfrac{1}{8}\left[\boldsymbol{q}_i^{(b)} + \boldsymbol{q}_j^{(b)}\right] + (-0.6)\boldsymbol{q}^{(e)} = \frac{-27.748 - 57.791}{8} - 0.6(12.584) = -18.243
$$

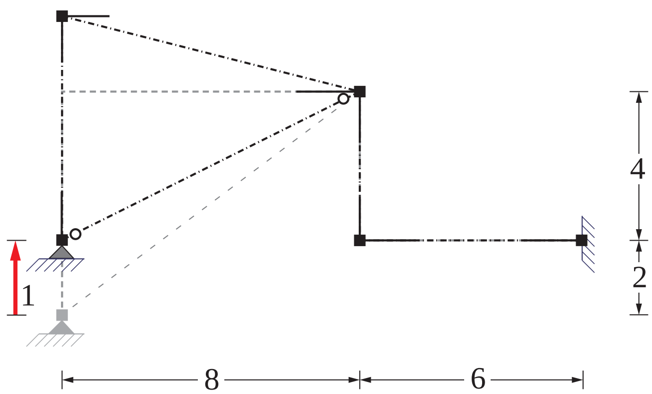
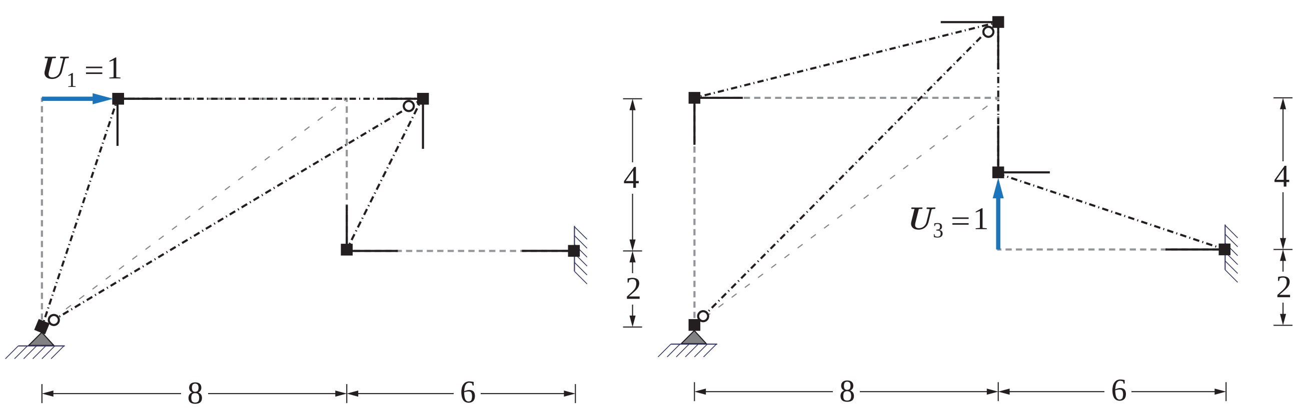


Figure 9: Virtual displacements and deformations for unit vertical translation at node #1

4. The kinematic relations $\boldsymbol{V} = \mathbf{A}_f \boldsymbol{U}_f$ of the structural model can be set up with the help of Fig. 10

$$
\begin{aligned}
\boldsymbol{v}_j^{(a)} &= \boldsymbol{V}_1 = \frac{1}{6}\boldsymbol{U}_1 + \boldsymbol{U}_2 \\
\boldsymbol{v}_i^{(b)} &= \boldsymbol{V}_2 = \boldsymbol{U}_2 - \frac{1}{8}\boldsymbol{U}_3 \\
\boldsymbol{v}_j^{(b)} &= \boldsymbol{V}_3 = -\frac{1}{8}\boldsymbol{U}_3 + \boldsymbol{U}_4 \\
\boldsymbol{v}_i^{(c)} &= \boldsymbol{V}_4 = \frac{1}{4}\boldsymbol{U}_1 + \boldsymbol{U}_4 \\
\boldsymbol{v}_j^{(c)} &= \boldsymbol{V}_5 = \frac{1}{4}\boldsymbol{U}_1 + \boldsymbol{U}_5 \\
\boldsymbol{v}_i^{(d)} &= \boldsymbol{V}_6 = \frac{1}{6}\boldsymbol{U}_3 + \boldsymbol{U}_5 \\
\boldsymbol{v}_j^{(d)} &= \boldsymbol{V}_7 = \frac{1}{6}\boldsymbol{U}_3 \\
\boldsymbol{v}^{(e)} &= \boldsymbol{V}_8 = 0.8\boldsymbol{U}_1 + 0.6\boldsymbol{U}_3
\end{aligned}
\tag{2}
$$



(a) Intra-node deformations for dof #1

(b) Intra-node deformations for dof #3

Figure 10: Intra-node deformations for translation dofs

We realize that the translation dofs $\boldsymbol{U}_1$ and $\boldsymbol{U}_3$ appear in the last two equations of (2) without other displacements. Consequently, we can use equation #7 of (2) to solve for $\boldsymbol{U}_3$

$$\boldsymbol{U}_3 = 6\boldsymbol{v}_j^{(d)} = 6(-2.2149)10^{-3} = -0.01329 \tag{3}$$

and then substitute its value into equation #8 to get $\boldsymbol{U}_1$

$$\boldsymbol{U}_1 = 1.25\left(\boldsymbol{v}^{(e)} - 0.6\boldsymbol{U}_3\right) = 1.25\,(6.2922 - 0.6(-13.29))\,10^{-3} = 0.01783$$

5. We confirm the translation values with the principle of virtual forces. For confirming the horizontal translation at dof #1 we apply a unit virtual force at dof #1. With $\delta\boldsymbol{P}_1 = 1$ and $\delta\boldsymbol{P}_2 = \delta\boldsymbol{P}_3 = \delta\boldsymbol{P}_4 = \delta\boldsymbol{P}_5 = 0$ in (1) we realize that a simple virtual force system results, if the virtual basic forces are selected such that $\delta\boldsymbol{Q}_1 = \delta\boldsymbol{Q}_2 = \delta\boldsymbol{Q}_3 = \delta\boldsymbol{Q}_4 = \delta\boldsymbol{Q}_7 = 0$. These satisfy the second and fourth equation in (1) leaving us with

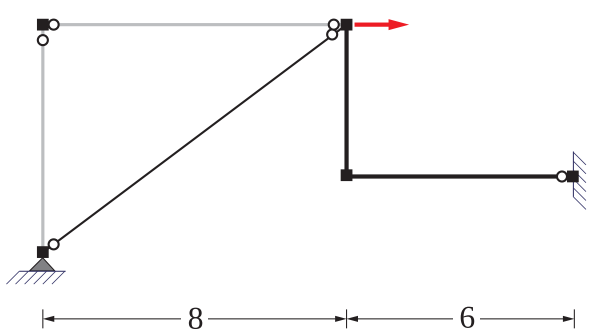
$$\begin{aligned} 1 &= \frac{1}{4}\delta\boldsymbol{Q}_5 \qquad\qquad + 0.8\delta\boldsymbol{Q}_8 \\ 0 &= \qquad\quad + \frac{1}{6}\delta\boldsymbol{Q}_6 + 0.6\delta\boldsymbol{Q}_8 \\ 0 &= \delta\boldsymbol{Q}_5 + \delta\boldsymbol{Q}_6 \end{aligned} \tag{4}$$

Multiplying the second equation by 1.5 and adding it to the first gives on account of the third equation

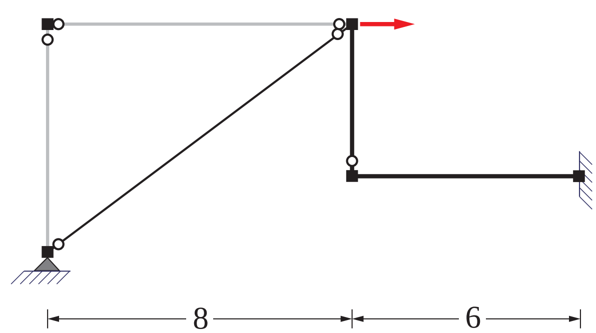
$$1 = 1.7\delta\boldsymbol{Q}_8 \quad \rightarrow \quad \delta\boldsymbol{Q}_8 = 0.5882$$

From the second equation we then get $\delta\boldsymbol{Q}_6 = -2.1176$ and from the third $\delta\boldsymbol{Q}_5 = 2.1176$. The equality of external and internal work for the principle of virtual forces gives

$$\begin{aligned} (1)\boldsymbol{U}_1 &= \delta\boldsymbol{Q}_5\boldsymbol{V}_5 + \delta\boldsymbol{Q}_6\boldsymbol{V}_6 + \delta\boldsymbol{Q}_8\boldsymbol{V}_8 \\ &= (2.1176)(3.1515)10^{-3} + (-2.1176)(-3.5215)10^{-3} + (0.5882)(6.2922)10^{-3} \\ &= 0.01783 \quad \surd \end{aligned}$$



(a) Virtual force system A



(b) Virtual force system B

Figure 11: Virtual force systems for translation dof #1

Fig. 11(a) gives the representation of the virtual force system with flexural releases for zero virtual basic forces. As a result, elements a and b do not carry any forces and are grayed in the figure. Fig. 11(b) depicts an even simpler virtual force system with the cantilever element d resisting the vertical component of the brace force. It results by setting $\delta\boldsymbol{Q}_1 = \delta\boldsymbol{Q}_2 = \delta\boldsymbol{Q}_3 = \delta\boldsymbol{Q}_4 = \delta\boldsymbol{Q}_5 = \delta\boldsymbol{Q}_6 = 0$ in (1) with $\delta\boldsymbol{P}_1 = 1$ and $\delta\boldsymbol{P}_2 = \delta\boldsymbol{P}_3 = \delta\boldsymbol{P}_4 = \delta\boldsymbol{P}_5 = 0$. This gives $\delta\boldsymbol{Q}_8 = 1.25$ from the first equation and $\delta\boldsymbol{Q}_7 = -4.50$ from the third. The other equations are satisfied automatically with the choice of virtual basic forces. The equality of external and internal work for the principle of virtual forces gives

$$(1)\boldsymbol{U}_1 = \delta\boldsymbol{Q}_7\boldsymbol{V}_7 + \delta\boldsymbol{Q}_8\boldsymbol{V}_8 = (-4.5)(-2.2149)10^{-3} + (1.25)(6.2922)10^{-3} = 0.01783 \quad \surd$$

The simplest virtual force system for a unit virtual force at dof #3 is even simpler than the last: with a unit value for $\boldsymbol{P}_3$ in the third equation of (1) and zeroes for the other nodal forces, we set all basic forces equal to zero except for $\boldsymbol{Q}_7$, which only appears in equation #3. With this choice we get $\delta\boldsymbol{Q}_7 = 6$ and the principle of virtual forces gives an equation identical with (3).

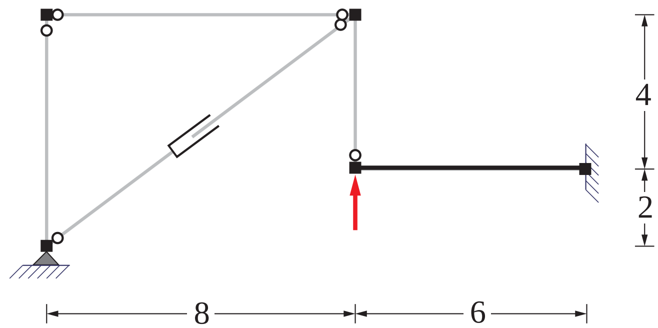


Figure 12: Virtual force system for translation dof #3

Fig. 12 gives the representation of the virtual force system with flexural releases for zero virtual basic forces and an axial release in the brace. Elements a, b, c and e do not carry any forces and are grayed in the figure.

6. With the translations $\boldsymbol{U}_1$ and $\boldsymbol{U}_3$ and all element deformations we can draw the deformed shape of the braced frame under the given element deformations. The element chords are displayed with a dashed line. The element deformations are measured from the element chord. The small value of $\boldsymbol{v}_i^{(b)}$ means that the deformed shape is essentially tangent to the chord at end i of element b.

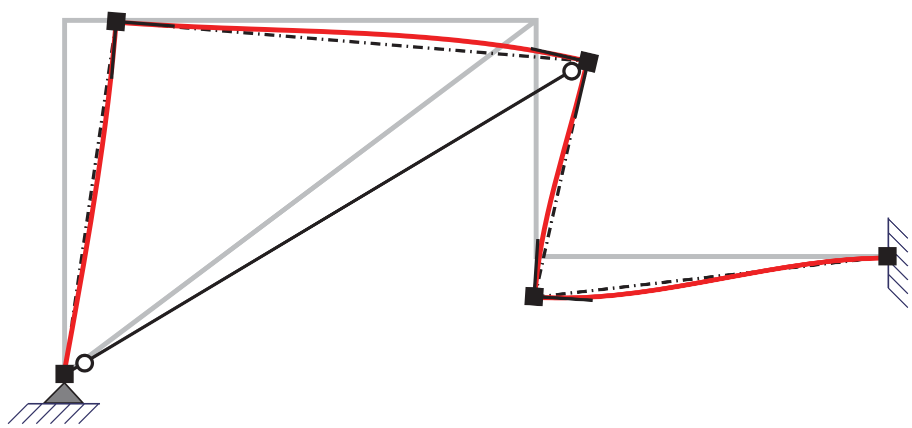
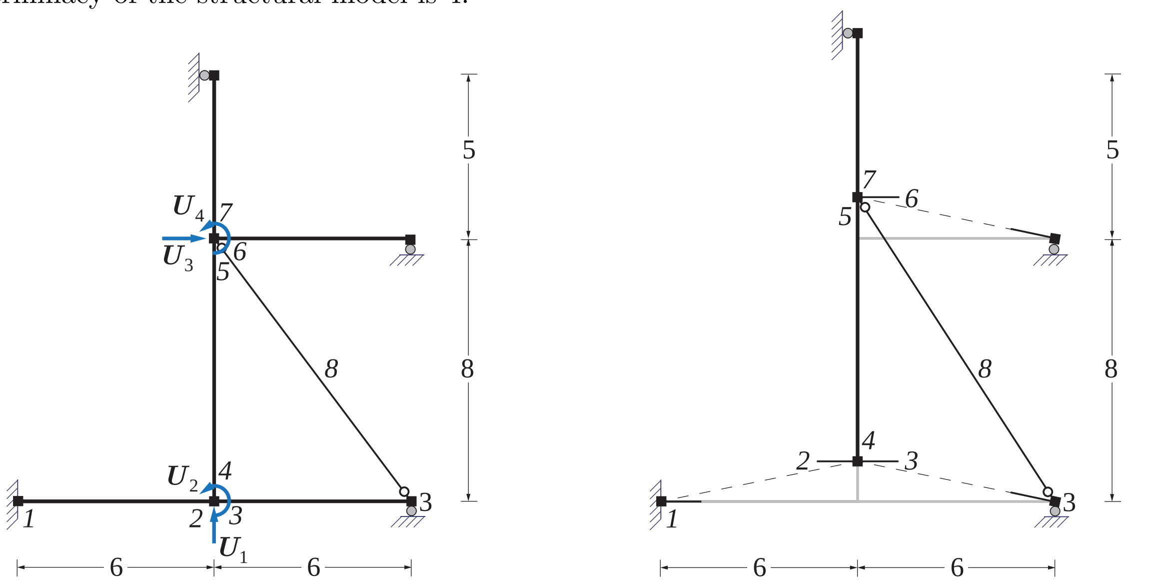


Figure 13: Deformed shape of braced frame for given element deformations

For the curvature of the deformed shape at the element ends we take into consideration that in the absence of thermal effects the curvatures are proportional to the bending moments in Fig. 6.

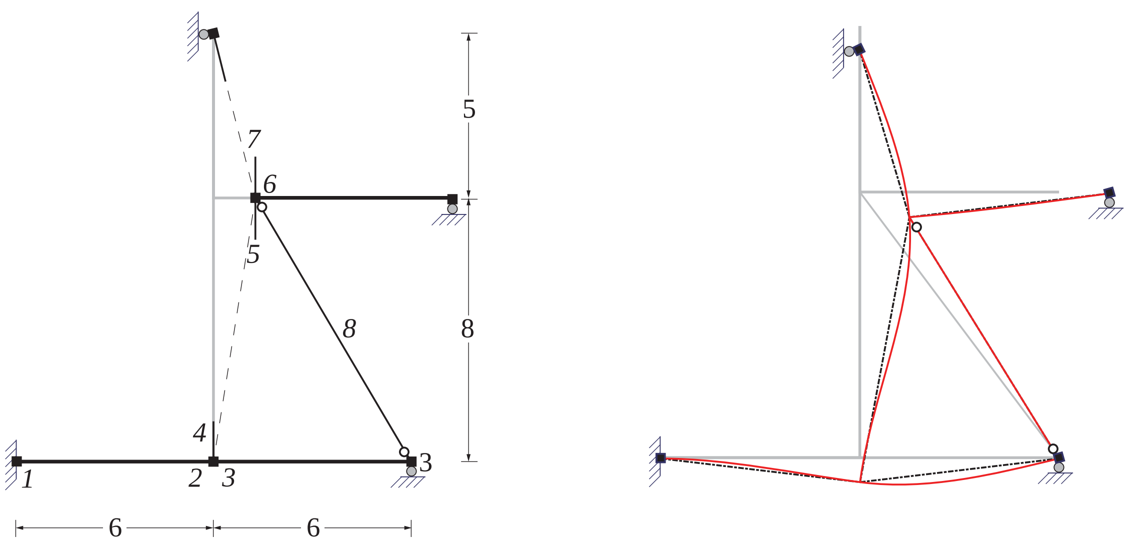
3. Problem

1. There are 4 non-trivial independent free dofs and corresponding equilibrium equations and 8 basic forces and corresponding element deformations, as shown in Fig. 14. Consequently, the degree of static indeterminacy of the structural model is 4.



(a) Free dof and deformation numbering (b) Deformation mode for independent dof $\boldsymbol{U}_1$

Figure 14: Free dof and element deformation numbering



(a) Deformation mode for independent dof $\boldsymbol{U}_3$ (b) Deformed shape

Figure 15: Deformed shape

2. For determining the horizontal translation at node #4 we apply a virtual unit horizontal load (dummy horizontal load) at node #4, as shown in Fig. 16(a). We set as many basic forces as possible equal to zero. We do this by inspection realizing that the simply supported beam consisting of elements c and e is able to equilibrate the applied horizontal load. With $\delta\boldsymbol{Q}_5 = -\delta\boldsymbol{Q}7$ on account of the moment

equilibrium at node #4, the horizontal force equilibrium gives

$$1 = \frac{\delta \boldsymbol{Q}_5}{8} - \frac{(-\delta \boldsymbol{Q}_5)}{5} \quad \rightarrow \quad \delta \boldsymbol{Q}_5 = \frac{40}{13}$$

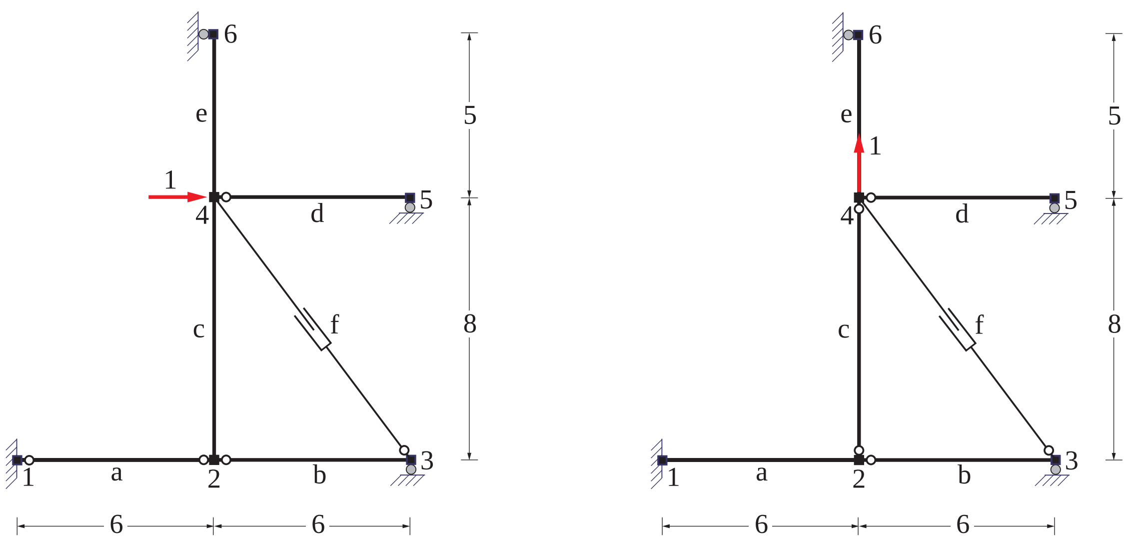
The principle of virtual forces gives

$$(1)\boldsymbol{U}_3 = \delta \boldsymbol{Q}^T \boldsymbol{V} = \frac{40}{13}\boldsymbol{V}_5 + \left(-\frac{40}{13}\right)\boldsymbol{V}_7$$

$$\rightarrow \qquad \boldsymbol{U}_3 = \frac{40}{13}(2.6814 \cdot 10^{-3}) + \left(-\frac{40}{13}\right)(-2.1568 \cdot 10^{-3}) = 14.89 \cdot 10^{-3}$$

3. The geometric interpretation of the preceding results is simply that the horizontal translation at node #4 is directly related to the element deformations $\boldsymbol{V}_5$ and $\boldsymbol{V}_7$. This can be readily concluded from the corresponding kinematic relations, which are

$$\begin{aligned} \boldsymbol{V}_5 &= \frac{\boldsymbol{U}_3}{8} + \boldsymbol{U}_4 \\ \boldsymbol{V}_7 &= -\frac{\boldsymbol{U}_3}{5} + \boldsymbol{U}_4 \end{aligned} \quad \rightarrow \quad \boldsymbol{U}_3\left(\frac{1}{8} + \frac{1}{5}\right) = \boldsymbol{V}_5 - \boldsymbol{V}_7 \quad \rightarrow \quad \boldsymbol{U}_3 = \frac{40}{13}(\boldsymbol{V}_5 - \boldsymbol{V}_7)$$



(a) Virtual force system for translation $\boldsymbol{U}_3$ (b) Virtual force system for translation $\boldsymbol{U}_1$

Figure 16: Virtual force systems for translations

4. For determining the vertical translation at node #4 we apply a virtual unit vertical load (dummy vertical load) at node #4, as shown in Fig. 16(b). We set as many basic forces as possible equal to zero. We do this by inspection realizing that the cantilever beam consisting of element a is able to equilibrate the applied vertical load. It is worth noting that the structural system in Fig. 16(b) is actually unstable, since it is not capable of resisting a horizontal force at node #4, *but this is not a requirement for the virtual force system*. The principle of virtual forces gives

$$(1)\boldsymbol{U}_1 = \delta \boldsymbol{Q}^T \boldsymbol{V} = (-6)\boldsymbol{V}_1 \quad \rightarrow \quad \boldsymbol{U}_1 = (-6)(1.1861 \cdot 10^{-3}) = -7.12 \cdot 10^{-3}$$

The negative sign means that node #4 translates *in the opposite direction of the applied dummy load*, i.e. downwards.

5. With the given element deformations and the determined translations it is possible to draw the deformed shape of the structural model under the given deformations in Fig. 15(b).