3.8 Summary of major kinematical concepts related to infinitesimal strains

1. **Motion**:
$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t), \qquad x_i = \chi_i(X_1, X_2, X_3, t). \tag{3.8.1}$$
Here $\mathbf{X}$ is a material point in the reference configuration, and $\mathbf{x}$ is the place occupied by $\mathbf{X}$ in the deformed body at time t.

2. **Displacement**:
The displacement vector $\mathbf{u}$ of each material point $\mathbf{X}$ at time t is given by
$$\mathbf{u}(\mathbf{X}, t) = \boldsymbol{\chi}(\mathbf{X}, t) - \mathbf{X}, \qquad u_i(X_1, X_2, X_3) = \chi_i(X_1, X_2, X_3) - X_i\,. \tag{3.8.2}$$

3. **Velocity and acceleration**:
The vectors
$$\dot{\mathbf{u}}(\mathbf{X}, t) = \frac{\partial \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t}\,, \qquad \dot{u}_i(X_1, X_2, X_3, t) = \frac{\partial \chi_i(X_1, X_2, X_3, t)}{\partial t}\,, \tag{3.8.3}$$
and
$$\ddot{\mathbf{u}}(\mathbf{X}, t) = \frac{\partial^2 \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t^2}, \qquad \ddot{u}_i(X_1, X_2, X_3, t) = \frac{\partial^2 \chi_i(X_1, X_2, X_3, t)}{\partial t^2}, \tag{3.8.4}$$
represent the velocity and acceleration of the material point $\mathbf{X}$ at time t.

4. **Deformation Gradient, Displacement Gradient**:
$$\mathbf{F}(\mathbf{X}, t) = \frac{\partial}{\partial \mathbf{X}} \boldsymbol{\chi}(\mathbf{X}, t), \quad F_{ij} = \frac{\partial}{\partial X_j} \chi_i(X_1, X_2, X_3, t), \quad \det \mathbf{F}(\mathbf{X}, t) > 0$$
$$\mathbf{H}(\mathbf{X}, t) = \frac{\partial}{\partial \mathbf{X}} \mathbf{u}(\mathbf{X}, t) \qquad H_{ij}(X_1, X_2, X_3, t) = \frac{\partial}{\partial X_j} u_i(X_1, X_2, X_3, t).$$
$$\mathbf{H}(\mathbf{X}, t) = \mathbf{F}(\mathbf{X}, t) - \mathbf{1}, \qquad H_{ij} = F_{ij} - \delta_{ij},$$
where
$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j, \end{cases}$$
is the **Kronecker delta**.

5. **Infinitesimal strain**:
$$\boldsymbol{\epsilon} = \frac{1}{2}\left[\mathbf{H} + \mathbf{H}^{\top}\right], \qquad \boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\top}, \qquad |\mathbf{H}| \ll 1. \tag{3.8.5}$$
$$\epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right], \qquad \epsilon_{ji} = \epsilon_{ij}, \quad \left|\frac{\partial u_i}{\partial X_j}\right| \ll 1.$$
The components $\{\epsilon_{11}, \epsilon_{22}, \epsilon_{33}\}$ are called the **normal strain** components, while the components $\{\epsilon_{12}, \epsilon_{13}, \epsilon_{23}\}$ are called the **tensorial shear strain** components. The **engineering shear strain** components are defined as **twice** the value of the tensorial shear strain components:
$$\gamma_{12} = 2\epsilon_{12}, \quad \gamma_{13} = 2\epsilon_{13}, \quad \gamma_{23} = 2\epsilon_{23}. \tag{3.8.6}$$
Volume change:
For finite deformations:
$$dv = \det \mathbf{F}\, dv_{\mathrm{R}}.$$
For small strains:
$$\frac{dv - dv_{\mathrm{R}}}{dv_{\mathrm{R}}} \doteq \operatorname{tr} \boldsymbol{\epsilon} = \epsilon_{kk} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}.$$
Strain deviator:
$$\boldsymbol{\epsilon}' = \boldsymbol{\epsilon} - \frac{1}{3}(\operatorname{tr} \boldsymbol{\epsilon})\,\mathbf{1}, \qquad \epsilon'_{ij} = \epsilon_{ij} - \frac{1}{3}(\epsilon_{kk})\,\delta_{ij}.$$