HE7, Semantic tableaux in predicate logic

The solutions must be complete and easy to follow. Correct answers will not yield full points without explanations as to how the answer was reached.

For each of the three problems, up to 5 points can be gained.

Check which of the following deductions hold by applying the method of semantic tableaux. *For any semantic tableau where not all branches close, provide a countermodel.*

1

$\{\exists x\,((P(x) \rightarrow R(x)) \rightarrow (Q(x) \rightarrow S(x)),\ \exists x\,(P(x) \wedge Q(x) \rightarrow R(x) \wedge S(x))\} \vdash$

$\exists x((P(x) \vee Q(x) \rightarrow R(x) \vee S(x))$

In addition of Tableaux from Propositional Logic, now, roughly we need to add these rules $\neg\forall xA \Leftrightarrow \neg\exists xA$ and $\exists x\neg A \Leftrightarrow \forall x\neg A$, and

$\exists xA$

with b not occurring anywhere above on the same branch.

$A[b/x]$

and

$\forall xA$

for any choice of c, only if no constant symbol has been introduced so far on the same branch.

$A[c/x]$

Using the method of semantic tableaux. The counterexample set is:

$\{\exists x\,((P(x) \rightarrow R(x)) \rightarrow (Q(x) \rightarrow S(x)),\ \exists x\,(P(x) \wedge Q(x) \rightarrow R(x) \wedge S(x)),$

$\neg\exists x((P(x) \vee Q(x) \rightarrow R(x) \vee S(x))\}$

let's do a simplification of this notation, where any P(x) means Px, and so on:
So the original set is rewrite as:

$\{\exists x\,((Px \rightarrow Rx) \rightarrow (Qx \rightarrow Sx)),\ \exists x\,(Px \wedge Qx \rightarrow Rx \wedge Sx),\ \neg\exists x(Px \vee Qx \rightarrow Rx \vee Sx)\}$

or

$\{\exists x\,((Px \rightarrow Rx) \rightarrow (Qx \rightarrow Sx)),\ \exists x\,(Px \wedge Qx \rightarrow Rx \wedge Sx),\ \forall x\neg(Px \vee Qx \rightarrow Rx \vee Sx)\}$

To build this tableaux the notation is follow of the examples and the booklet "Theory on predicate logic"

(1a) $\exists x\,((Px \rightarrow Rx) \rightarrow (Qx \rightarrow Sx))$

(1b) $\exists x\,(Px \wedge Qx \rightarrow Rx \wedge Sx)$

(1c) $\neg\exists x(Px \vee Qx \rightarrow Rx \vee Sx)$

(2) $\forall x\neg(Px \vee Qx \rightarrow Rx \vee Sx)$ {from 1c}

(3) $(Pb \wedge Qb \rightarrow Rb \wedge Sb)$ {1b}

(4a) $\neg(Pb \wedge Qb)$ {3} (4b) $(Rb \wedge Sb)$ {3}

(5a) $\neg Pb$ {4a} (5b) $\neg Qb$ {4a} (6a) Rb {4b}

(6b) Sb {4b}

(repeat this left branch here, to obtain Qb)

(derivations of 6a and 6b follow below)

....

(7) $\neg(Pb \vee Qb \rightarrow Rb \vee Sb)$ {from 2, x/b due $\forall x$ }

(8a) $(Pb \vee Qb)$ {7}

(8b) $\neg(Rb \vee Sb)$ {7}

(9a) Pb {8a} (9b) Qb {8a}

$\perp$ (5a) and (9a)

Resuming the nodes (6a) and (6b) from the tree above, we have:

(6a) Rb {4b}

(6b) Sb {4b}

(15) $\neg(Pb \vee Qb \rightarrow Rb \vee Sb)$ {from 2, x/b due $\forall x$ }

(16a) $(Pb \vee Qb)$ {15}

(16b) $\neg(Rb \vee Sb)$ {15}

(17a) $\neg Rb$ {4b}

(17b) $\neg Sb$ {4b}

$\perp$ (17a) and (6a)
and (6b) and (17b)

PS: formula (2) was re-instanced with different ‘x’ to allow the branches to close.

2

$\{\exists x((P(x) \rightarrow Q(x)) \rightarrow S(x)), \exists x((\neg P(x) \rightarrow R(x)) \rightarrow T(x)), \forall x(Q(x) \rightarrow R(x))\} \vdash$
$\exists x\ (Q(x) \land R(x) \rightarrow S(x) \land T(x))$

Rewriting:

$\{\exists x((Px \rightarrow Qx) \rightarrow Sx), \exists x((\neg Px \rightarrow Rx) \rightarrow Tx), \forall x(Qx \rightarrow Rx)\} \vdash$
$\exists x\ (Qx \land Rx \rightarrow Sx \land Tx)$

$\exists x((Px \rightarrow Qx) \rightarrow Sx) \land \exists x((\neg Px \rightarrow Rx) \rightarrow Tx) \land \forall x(Qx \rightarrow Rx) \models \exists x\ ((Qx \land Rx) \rightarrow (Sx \land Tx))$
(from generator)

Conclusion negated and included in our original set of the premisses:

H= $\{\exists x((Px \rightarrow Qx) \rightarrow Sx), \exists x((\neg Px \rightarrow Rx) \rightarrow Tx), \forall x(Qx \rightarrow Rx), \neg\exists x\ (Qx \land Rx \rightarrow Sx \land Tx)\}$

H= $\{\exists x((Px \rightarrow Qx) \rightarrow Sx), \exists x((\neg Px \rightarrow Rx) \rightarrow Tx), \forall x(Qx \rightarrow Rx), \forall x\neg\ (Qx \land Rx \rightarrow Sx \land Tx)\}$

Observing the consequence:

$\forall x\neg\ (Qx \land Rx \rightarrow Sx \land Tx)$ is equivalence at to $\forall x\ (\neg(Sx \land Tx) \land (Qx \land Rx))$ and

$\forall x\ (\neg(Sx \land Tx) \land (Qx \land Rx)) \Leftrightarrow \forall x\ \neg(Sx \land Tx) \land \forall x(Qx \land Rx)$, so rewriting H:

H= $\{\exists x((Px \rightarrow Qx) \rightarrow Sx), \exists x((\neg Px \rightarrow Rx) \rightarrow Tx), \forall x(Qx \rightarrow Rx), \forall x\ (\neg(Sx \land Tx) \land (Qx \land Rx))\}$

Building the tableaux:

(1a) $\exists x\,((Px \to Qx) \to Sx)$

(1b) $\exists x\,((\neg Px \to Rx) \to Tx)$

(1c) $\forall x\,(Qx \to Rx)$

(1d) $\forall x\,(\neg(Sx \land Tx) \land (Qx \land Rx))$

(2) $(Pa \to Qa) \to Sa$ $\{1a,\ x/a\ \exists x\}$

(3) $(\neg Pb \to Rb) \to Tb$ $\{1b,\ x/b\ \exists x\}$

(4) $\neg(Sa \land Ta) \land (Qa \land Ra)$ $\{1d,\ x/a\}$

(5a) $\neg(Sa \land Ta)$ $\{4\}$

(5b) $(Qa \land Ra)$ $\{4\}$

(6a) Qa {from 5a}

(6b) Ra {from 5b}

(7) $\neg(Sb \land Tb) \land (Qb \land Rb)$ $\{1d,\ x/b\}$

(8a) $\neg(Sb \land Tb)$ $\{7\}$

(8b) $(Qb \land Rb)$ $\{7\}$

(9a) Qb $\{8a\}$

(9b) Rb $\{8b\}$

(10a) $\neg(Pa \to Qa)$ $\{2\}$ (10b) Sa $\{2\}$

(11a) Pa $\{10a\}$

(11b) $\neg Qa$ $\{10a\}$

(12a) $\neg\neg Sa$ $\{5a\}$ (12b) $\neg Ta$ $\{5a\}$

(13) Sa $\{12a\}$

$\bot$ (6a) and (11b)

3

$\{\forall x\,((P(x) \to R(x)) \to (P(x) \lor Q(x) \to R(x))),\ \exists x(R(x) \lor S(x) \to P(x))\} \vdash$

$\forall x\,(Q(x) \to P(x))$

Rewriting:

$\{\forall x\,((Px \to Rx) \to (Px \lor Qx \to Rx)),\ \exists x(Rx \lor Sx \to Px)\} \vdash$

$\forall x\,(Qx \to Px)$

Conclusion negated and included in our original set of the premises:

$H=\{\forall x\,((Px \to Rx) \to (Px \lor Qx \to Rx)),\ \exists x(Rx \lor Sx \to Px),\ \neg\forall x\,(Qx \to Px)\}$

$H=\{\forall x\,((Px \to Rx) \to (Px \lor Qx \to Rx)),\ \exists x(Rx \lor Sx \to Px),\ \exists x\neg\,(Qx \to Px)\}$

Remember: $\exists x\neg\,(Qx \to Px) \Leftrightarrow \neg\forall x(Qx \to Px)$

Building the tableaux:

(1a) $\forall x\,((Px \to Rx) \to (Px \lor Qx \to Rx))$

(1b) $\exists x(Rx \lor Sx \to Px)$

(1c) $\neg\forall x\,(Qx \to Px)$

(2) $(Ra \lor Sa) \to Pa$ $\{1b\}$

(3a) $\neg(Ra \lor Sa)$ $\{2\}$

(4a) $\neg Ra$ $\{3a\}$

(4b) $\neg Sa$ $\{3a\}$

(7) $(Pa \to Ra) \to (Pa \lor Qa \to Ra)$ $\{1a\}$

(8a) $\neg(Pa \to Ra)$ $\{7\}$

(9a) Pa $\{8a\}$

(9b) $\neg Ra$ $\{8a\}$

(12) $\neg(Qa \to Pa)$ $\{1c, x/a\}$

(13a) Qa $\{12\}$

(13b) $\neg Pa$ $\{12\}$

$\perp$ (13b) and (9a)

(8b) $(Pa \lor Qa \to Ra)$ $\{7\}$

(10a) $\neg(Pa \lor Qa)$ $\{8b\}$

(11a) $\neg Pa$ $\{10a\}$

(11b) $\neg Qa$ $\{10a\}$

(14) $\neg(Qa \to Pa)$ $\{1c, x/a\}$

(15a) Qa $\{14\}$

(15b) $\neg Pa$ $\{14\}$

$\perp$ (15a) and (11b)

(10b) Ra $\{8b\}$

$\perp$ (4a) and (10b)

(3b) Pa $\{2\}$

(5) $\neg(Qa \to Pa)$ $\{1c, x/a\}$

(6a) Qa $\{5\}$

(6b) $\neg Pa$ $\{5\}$

$\perp$ (3b) and (6b)

This tableaux is closed, so H is tautology, or $\{\forall x\,((Px \to Rx) \to (Px \lor Qx \to Rx)),\ \exists x(Rx \lor Sx \to Px)\} \vdash \forall x\,(Qx \to Px)$