Check which of the following deductions hold by applying the method of semantic tableaux. For any semantic tableau where not all branches close, provide a countermodel.

1. $$\forall x \exists y (P(x) \leftrightarrow Q(y)) \dashv\vdash \exists y \exists z \forall x [(P(x) \to Q(y)) \land (Q(z) \to P(x))]$$

2. $$\{\forall x \exists y (P(x, y) \lor P(y, x)), \exists x (P(x, x) \to Q(x, x))\} \vdash \exists x \exists y Q(x, y)$$

3. $$\begin{aligned}\{\forall x \forall y \forall z (R(x, y) \land R(y, z) \to R(x, z)), \forall x \forall y (R(x, y) \to R(y, x))\} \vdash \\ \forall x (\exists y (R(x, y) \lor R(y, x)) \to R(x, x))\end{aligned}$$

1 Solutions

1.1 Problem 1

$\exists y \exists z \forall x[(Px \to Qy) \wedge (Qz \to Px)]$

$\neg \forall x \exists y (Px \leftrightarrow Qy)$

$\exists z \forall x[(Px \to Qa) \wedge (Qz \to Px)]$

(3) $\forall x[(Px \to Qa) \wedge (Qb \to Px)]$

$\exists x \neg \exists y (Px \leftrightarrow Qy)$

$\exists x \forall y \neg (Px \leftrightarrow Qy)$

(6) $\forall y \neg (Pc \leftrightarrow Qy)$

(7) $\neg (Pc \leftrightarrow Qa)$ $\{6\}$

$(Pc \to Qa) \wedge (Qb \to Pc)$ $\{3\}$

(9a) $(Pc \to Qa)$

(9b) $(Qb \to Pc)$

$\neg Pc$ $\{7\}$

Qa $\{7\}$

$\neg (Pc \leftrightarrow Qb)$ $\{6\}$

$\neg Pc$

Qb

$\neg Qb$ $\{9b\}$

$\perp$

Pc $\{9b\}$

$\perp$

Pc

$\neg Qb$

$\perp$

Pc $\{7\}$

$\neg Qa$ $\{7\}$

$\neg Pc$ $\{9a\}$

$\perp$

Qa $\{9a\}$

$\perp$

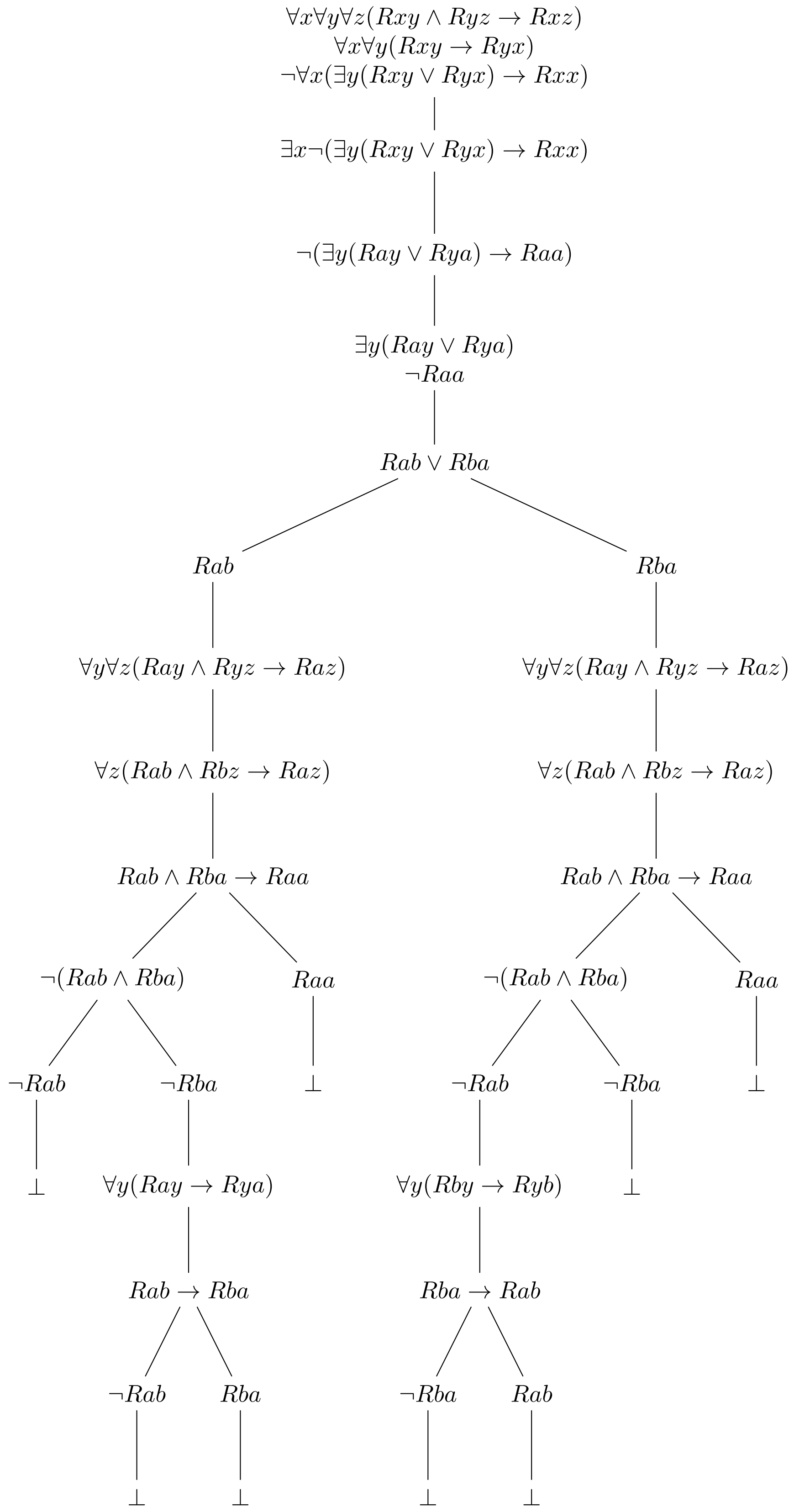
1.2 Problem 2

$$\forall x \exists y (Pxy \lor Pyx)$$
$$\exists x (Pxx \to Qxx)$$
$$\neg \exists x \exists y Qxy$$
$$\forall x \neg \exists y Qxy$$
$$\forall x \forall y \neg Qxy$$
$$Paa \to Qaa$$
$$\forall y \neg Qay$$
$$\neg Qaa$$

- $\neg Paa$
  - $\exists y (Pay \lor Pya)$
  - $Pab \lor Pba$
    - Pab
    - Pba
- Qaa
  - $\perp$

A countermodel can be got by letting the domain of discourse be $\{a, b\}$, with $P = \{(a, b)\}$ and $Q = \varnothing$.

1.3 Problem 3



$\forall x \forall y \forall z(Rxy \wedge Ryz \rightarrow Rxz)$
$\forall x \forall y(Rxy \rightarrow Ryx)$
$\neg \forall x(\exists y(Rxy \vee Ryx) \rightarrow Rxx)$
$\exists x \neg(\exists y(Rxy \vee Ryx) \rightarrow Rxx)$
$\neg(\exists y(Ray \vee Rya) \rightarrow Raa)$
$\exists y(Ray \vee Rya)$
$\neg Raa$
$Rab \vee Rba$
Rab
Rba
$\forall y \forall z(Ray \wedge Ryz \rightarrow Raz)$
$\forall y \forall z(Ray \wedge Ryz \rightarrow Raz)$
$\forall z(Rab \wedge Rbz \rightarrow Raz)$
$\forall z(Rab \wedge Rbz \rightarrow Raz)$
$Rab \wedge Rba \rightarrow Raa$
$Rab \wedge Rba \rightarrow Raa$
$\neg(Rab \wedge Rba)$
Raa
$\neg(Rab \wedge Rba)$
Raa
$\neg Rab$
$\neg Rba$
$\perp$
$\neg Rab$
$\neg Rba$
$\perp$
$\perp$
$\forall y(Ray \rightarrow Rya)$
$\forall y(Rby \rightarrow Ryb)$
$\perp$
$Rab \rightarrow Rba$
$Rba \rightarrow Rab$
$\neg Rab$
Rba
$\neg Rba$
Rab
$\perp$
$\perp$
$\perp$
$\perp$