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HE7, Semantic tableaux in predicate logic

Question 1

1. $\{\exists x ((P(x) \rightarrow R(x)) \rightarrow (Q(x) \rightarrow S(x))), \exists x (P(x) \wedge Q(x) \rightarrow R(x) \wedge S(x))\} \vdash$
 $\exists x ((P(x) \vee Q(x) \rightarrow R(x) \vee S(x))$

In addition of Tableaux from Propositional Logic, we need to add these rules $\neg \forall x A \Leftrightarrow \neg \exists x A$ and $\exists x \neg A \Leftrightarrow \forall x \neg A$, and

$\exists x A$

| with b not occurring anywhere above on the same branch.
A[b/x]

and

$\forall x A$

| for any choice of c, only if no constant symbol has been introduced so far on the same branch.

A[c/x]

Using the method of semantic tableaux, the set is given by:

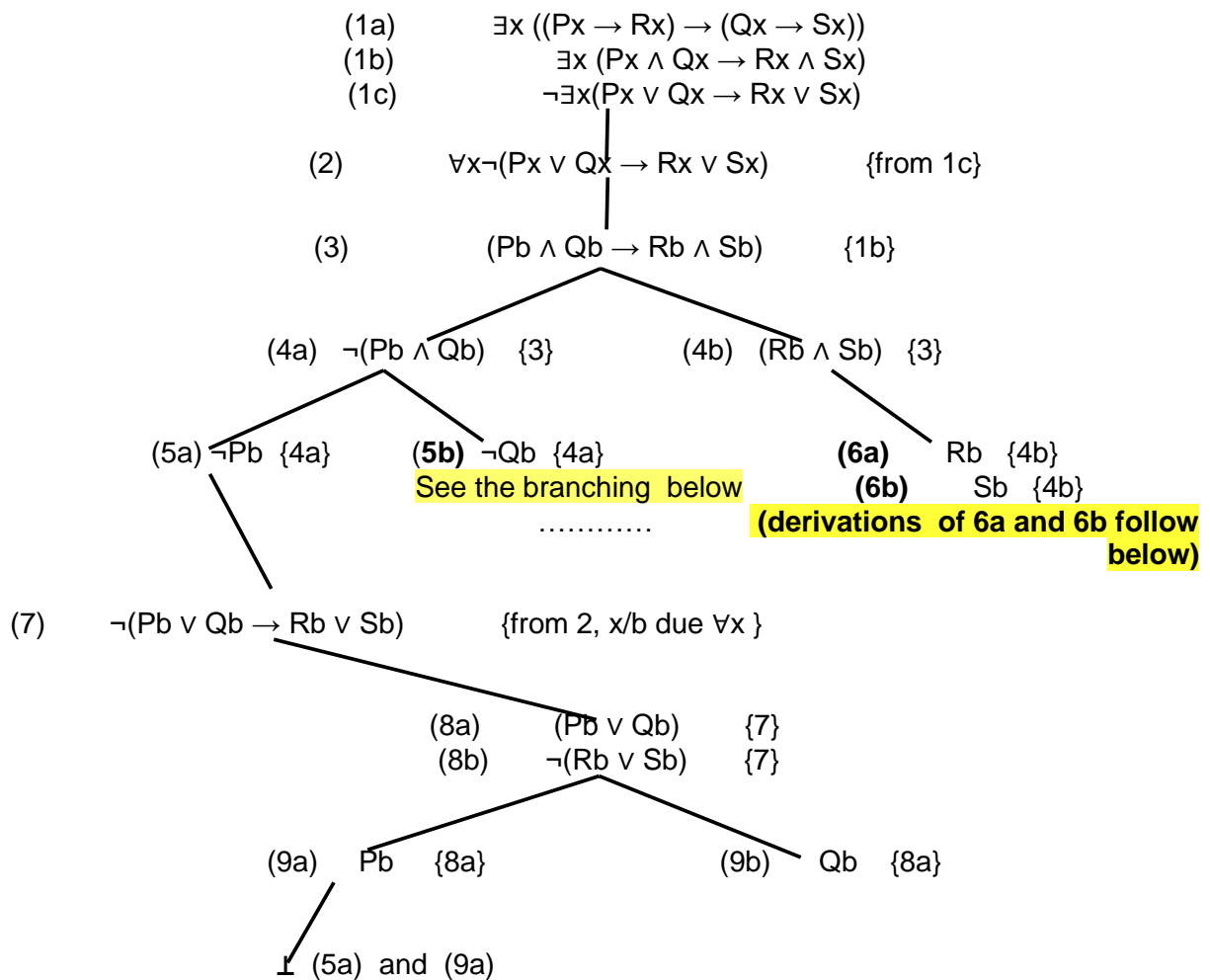
$\{\exists x ((P(x) \rightarrow R(x)) \rightarrow (Q(x) \rightarrow S(x))), \exists x (P(x) \wedge Q(x) \rightarrow R(x) \wedge S(x)),$
 $\neg \exists x ((P(x) \vee Q(x) \rightarrow R(x) \vee S(x)))\}$

Next I will do a simplification of this notation, where any $P(x)$ means Px , and so on:
So the original set is rewritten as:

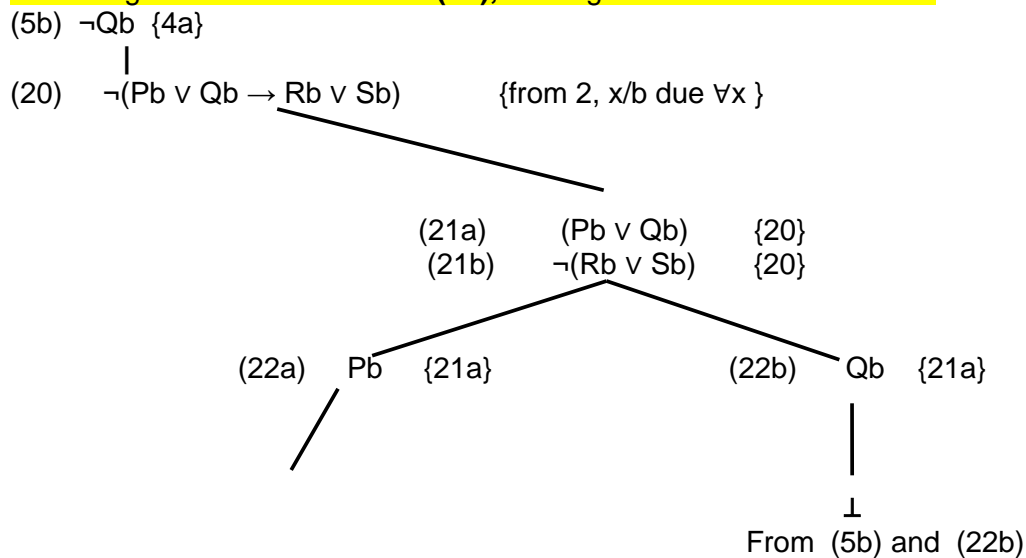
$\{\exists x ((Px \rightarrow Rx) \rightarrow (Qx \rightarrow Sx)), \exists x (Px \wedge Qx \rightarrow Rx \wedge Sx), \neg \exists x (Px \vee Qx \rightarrow Rx \vee Sx)\}$

or

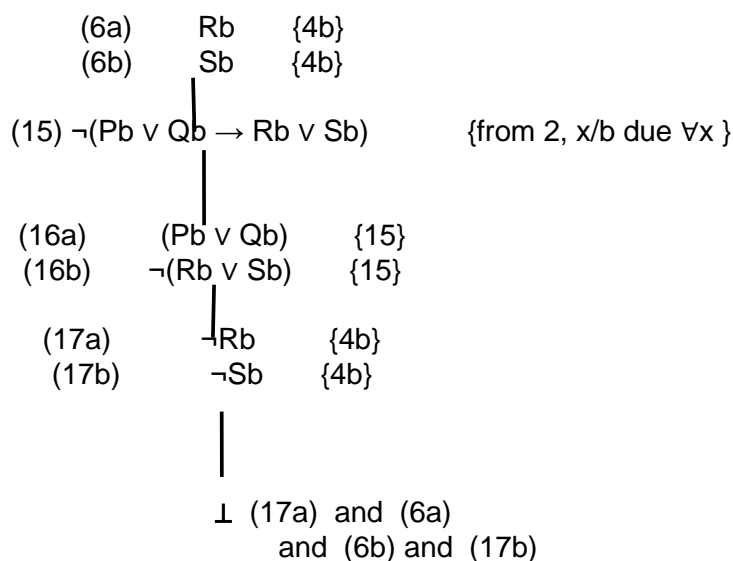
$\{\exists x ((Px \rightarrow Rx) \rightarrow (Qx \rightarrow Sx)), \exists x (Px \wedge Qx \rightarrow Rx \wedge Sx),$
 $\forall x \neg (Px \vee Qx \rightarrow Rx \vee Sx)\}$



Extending the branch from node (5b), starting a node numeration in 20:



Resuming the nodes (6a) and (6b) from the tree above, we have:



Note: formula (2) was re-instantiated with different 'x' to allow new branches to close.

Analysis:

This tableaux does not close. Although is saturated, depends of the branch is taken, it generates Pb and Qb as open nodes. However, there are branches were Pb and Qb become closed nodes. So for this domain {b} there is countermodel. A countermodel of a formula A is an interpretation that falsifies A. So we need to find a false for this domain {b} and P={b} and Q={b}.

Resuming:

- | | |
|------|---|
| (1a) | $\exists x ((Px \rightarrow Rx) \rightarrow (Qx \rightarrow Sx))$ |
| (1b) | $\exists x (Px \wedge Qx \rightarrow Rx \wedge Sx)$ |
| (1c) | $\neg \exists x (Px \vee Qx \rightarrow Rx \vee Sx)$ |

Where is a?

in addition R={} and S={}. Doing such interpretations (Int(formula):{ \perp ,T}) in these premises:

From 1a: $\text{Int}((T \rightarrow \perp) \rightarrow (T \rightarrow \perp)) = \text{Int}(\perp \rightarrow \perp) = T$

From 1b: $\text{Int}((T \wedge T) \rightarrow (\perp \wedge \perp)) = \text{Int}(T \rightarrow \perp) = \perp$ A counter model makes all premises true but the conclusion false.

From 1c: $\text{Int}(\neg((T \vee T) \rightarrow (\perp \vee \perp))) = \text{Int}(\neg T) = \perp$

The formula 1b is not a logical consequence of this set, once 1b entails a false conclusion.

3p

Question 2

2. $\{\exists x((P(x) \rightarrow Q(x)) \rightarrow S(x)), \exists x((\neg P(x) \rightarrow R(x)) \rightarrow T(x)), \forall x(Q(x) \rightarrow R(x))\} \vdash \exists x (Q(x) \wedge R(x) \rightarrow S(x) \wedge T(x))$

Rewriting:

$\{\exists x((Px \rightarrow Qx) \rightarrow Sx), \exists x((\neg Px \rightarrow Rx) \rightarrow Tx), \forall x(Qx \rightarrow Rx)\} \vdash$

$\exists x (Qx \wedge Rx \rightarrow Sx \wedge Tx)$

$$\exists x((Px \rightarrow Qx) \rightarrow Sx) \wedge \exists x((\neg Px \rightarrow Rx) \rightarrow Tx) \wedge \forall x(Qx \rightarrow Rx) \models \exists x((Qx \wedge Rx) \rightarrow (Sx \wedge Tx))$$

Conclusion negated and included in our original set of the premises:

$$H = \{\exists x((Px \rightarrow Qx) \rightarrow Sx), \exists x((\neg Px \rightarrow Rx) \rightarrow Tx), \forall x(Qx \rightarrow Rx), \neg \exists x(Qx \wedge Rx \rightarrow Sx \wedge Tx)\}$$

$$H = \{\exists x((Px \rightarrow Qx) \rightarrow Sx), \exists x((\neg Px \rightarrow Rx) \rightarrow Tx), \forall x(Qx \rightarrow Rx), \forall x \neg (Qx \wedge Rx \rightarrow Sx \wedge Tx)\}$$

Observing the consequence:

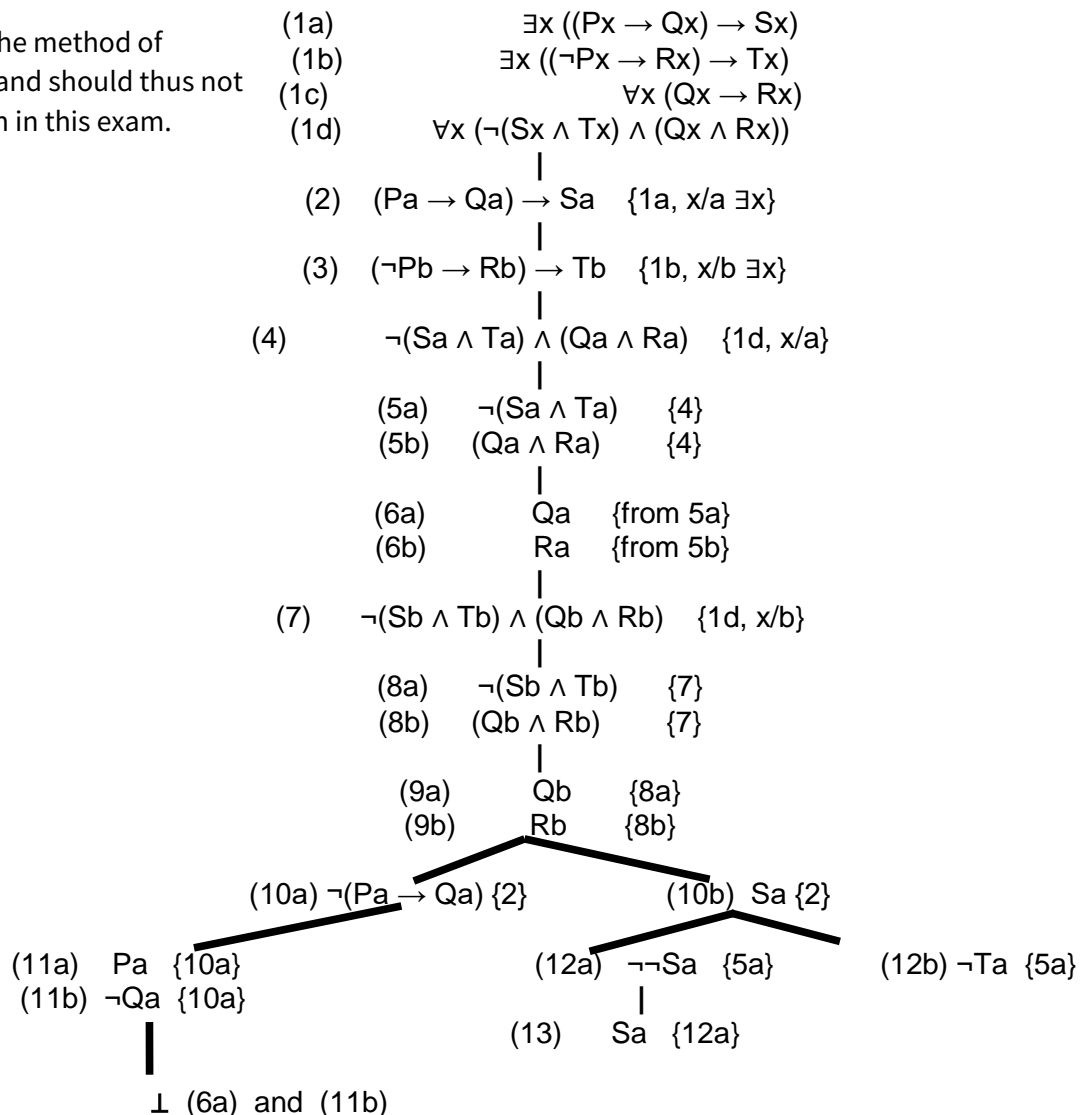
$\forall x \neg (Qx \wedge Rx \rightarrow Sx \wedge Tx)$ is equivalence at to $\forall x (\neg(Sx \wedge Tx) \wedge (Qx \wedge Rx))$ and

$\forall x (\neg(Sx \wedge Tx) \wedge (Qx \wedge Rx)) \Leftrightarrow \forall x \neg(Sx \wedge Tx) \wedge \forall x(Qx \wedge Rx)$, so rewriting H:

$$H = \{\exists x((Px \rightarrow Qx) \rightarrow Sx), \exists x((\neg Px \rightarrow Rx) \rightarrow Tx), \forall x(Qx \rightarrow Rx), \forall x (\neg(Sx \wedge Tx) \wedge (Qx \wedge Rx))\}$$

Building the tableaux:

That is not part of the method of semantic tableaux and should thus not be part of a solution in this exam.



Analysis:

This tableaux does not close. Although is saturated, depends of the branch is taken, it generates Sa and $\neg Ta$ as open nodes. So for this domain $\{a\}$ there is countermodel. A countermodel of a formula A is an interpretation that falsifies A . So we need to find a false for this domain $\{a\}$ with $S=\{a\}$ and $T=\{\}$.

Resuming our formula set, some of these formulas must falsifies under this interpretation (falsifies one of them):

Where is b?

- (1a) $\exists x ((Px \rightarrow Qx) \rightarrow Sx)$
- (1b) $\exists x ((\neg Px \rightarrow Rx) \rightarrow Tx)$
- (1c) $\forall x (Qx \rightarrow Rx)$
- (1d) $\forall x (\neg(Sx \wedge Tx) \wedge (Qx \wedge Rx))$

in addition $P=\{\}$, and $Q=\{\}$ and $R=\{\}$. Doing such interpretations $(\text{Int}(\text{formula}): \{\perp, T\})$ in these premises:

From 1a: $\text{Int}((\perp \rightarrow \perp) \rightarrow (T)) = \text{Int}(\perp \rightarrow T) = T$

From 1b: $\text{Int}((\neg \perp \rightarrow \perp) \rightarrow \perp) = \text{Int}(\perp \rightarrow \perp) = T$

From 1c: $\text{Int}(\perp \rightarrow \perp) = \text{Int}(T) = T$

From 1d: $\text{Int}((\neg(T \wedge \perp) \wedge (\perp \wedge \perp)) = \text{Int}(T \wedge \perp) = \perp$

3p

The formula 1d is not a logical consequence of this set, once 1d entails false conclusion.

Question 3

3. $\{\forall x ((P(x) \rightarrow R(x)) \rightarrow (P(x) \vee Q(x) \rightarrow R(x))), \exists x(R(x) \vee S(x) \rightarrow P(x))\} \vdash$
 $\forall x (Q(x) \rightarrow P(x))$

Rewriting:

$\{\forall x ((Px \rightarrow Rx) \rightarrow (Px \vee Qx \rightarrow Rx)), \exists x(Rx \vee Sx \rightarrow Px)\} \vdash$

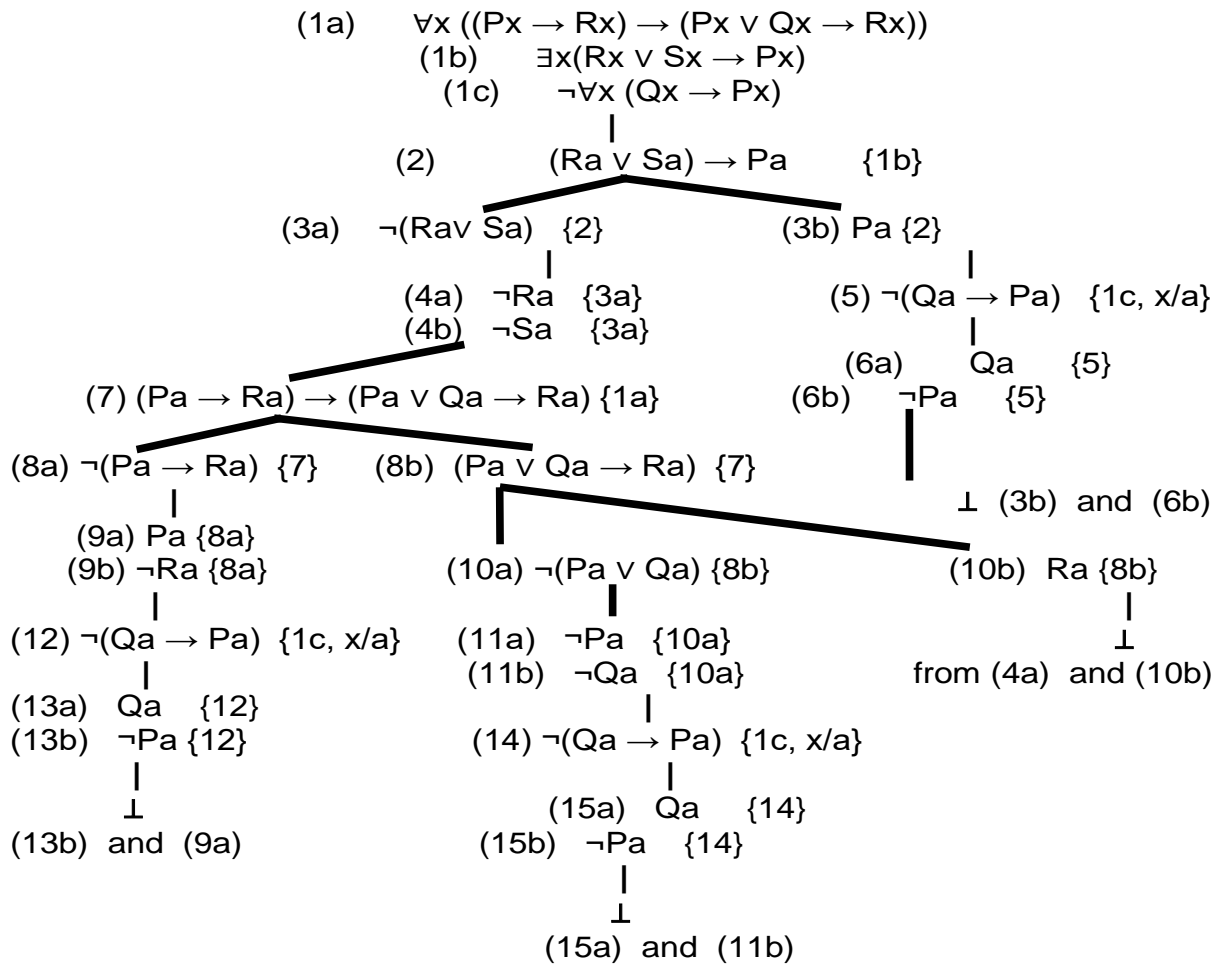
$\forall x (Qx \rightarrow Px)$

Conclusion negated and included in our original set of the premises:

$H=\{\forall x ((Px \rightarrow Rx) \rightarrow (Px \vee Qx \rightarrow Rx)), \exists x(Rx \vee Sx \rightarrow Px), \neg \forall x (Qx \rightarrow Px)\}$

$H=\{\forall x ((Px \rightarrow Rx) \rightarrow (Px \vee Qx \rightarrow Rx)), \exists x(Rx \vee Sx \rightarrow Px), \exists x \neg (Qx \rightarrow Px)\}$

Building the tableaux:



The above tableaux is closed, no countermodel to be provided. So H is tautology, or $\{\forall x ((Px \rightarrow Rx) \rightarrow (Px \vee Qx \rightarrow Rx)), \exists x (Rx \vee Sx \rightarrow Px)\} \vdash \forall x (Qx \rightarrow Px)$

2p

Total 2p