

HE7, Semantic tableaux in predicate logic

The solutions must be complete and easy to follow. Correct answers will not yield full points without explanations as to how the answer was reached.

For each of the three problems, up to 5 points can be gained.

Check which of the following deductions hold by applying the method of semantic tableaux. *For any semantic tableau where not all branches close, provide a countermodel.*

1

$$\{\exists x ((P(x) \rightarrow R(x)) \rightarrow (Q(x) \rightarrow S(x))), \exists x (P(x) \wedge Q(x) \rightarrow R(x) \wedge S(x))\} \vdash \\ \exists x((P(x) \vee Q(x) \rightarrow R(x) \vee S(x))$$

In addition of Tableaux from Propositional Logic, now, roughly we need to add these rules $\neg \forall x A \Leftrightarrow \neg \exists x \neg A$ and $\exists x \neg A \Leftrightarrow \forall x \neg A$, and

$\exists x A$

| with b not occurring anywhere above on the same branch.
 $A[b/x]$

and

$\forall x A$

| for any choice of c, only if no constant symbol has been introduced so far on the same branch.
 $A[c/x]$

Using the method of semantic tableaux. The counterexample set is:

$$\{\exists x ((P(x) \rightarrow R(x)) \rightarrow (Q(x) \rightarrow S(x))), \exists x (P(x) \wedge Q(x) \rightarrow R(x) \wedge S(x)), \\ \neg \exists x((P(x) \vee Q(x) \rightarrow R(x) \vee S(x))\}$$

let's do a simplification of this notation, where any $P(x)$ means Px , and so on:

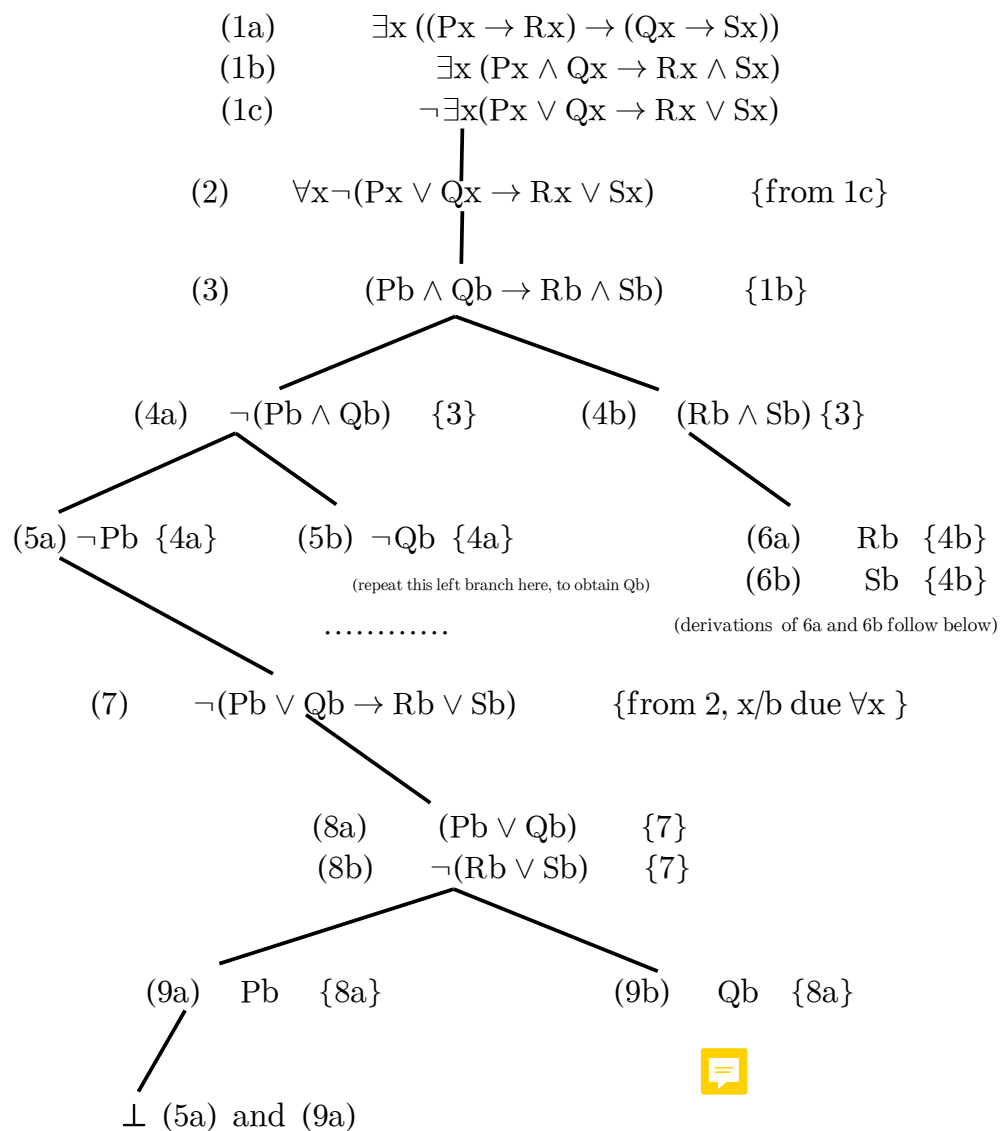
So the original set is rewrite as:

$$\{\exists x ((Px \rightarrow Rx) \rightarrow (Qx \rightarrow Sx)), \exists x (Px \wedge Qx \rightarrow Rx \wedge Sx), \neg \exists x(Px \vee Qx \rightarrow Rx \vee Sx)\}$$

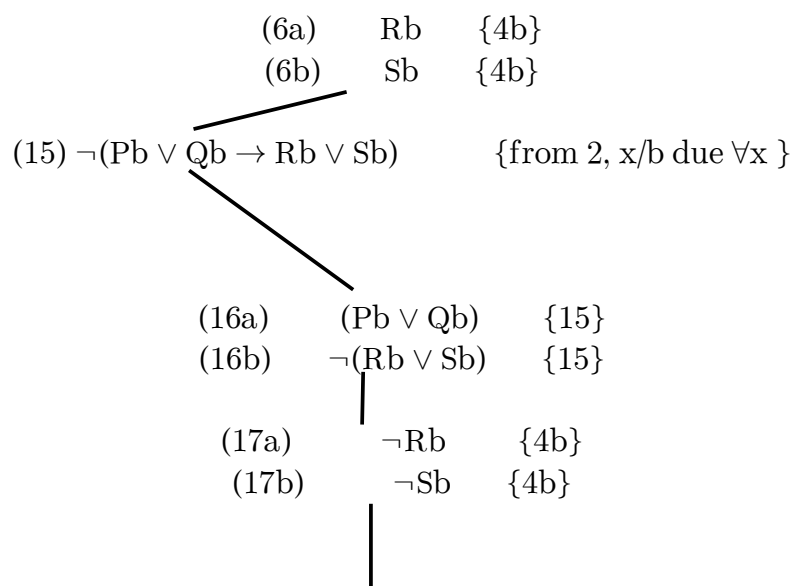
or

$$\{\exists x ((Px \rightarrow Rx) \rightarrow (Qx \rightarrow Sx)), \exists x (Px \wedge Qx \rightarrow Rx \wedge Sx), \forall x \neg(Px \vee Qx \rightarrow Rx \vee Sx)\}$$

To build this tableaux the notation is follow of the examples and the booklet “Theory on predicate logic”



Resuming the nodes (6a) and (6b) from the tree above, we have:



\perp (17a) and (6a)
and (6b) and (17b)

PS: formula (2) was re-instanced with different 'x' to allow the branches to close.

2

$\{\exists x((P(x) \rightarrow Q(x)) \rightarrow S(x)), \exists x((\neg P(x) \rightarrow R(x)) \rightarrow T(x)), \forall x(Q(x) \rightarrow R(x))\} \vdash$

$\exists x (Q(x) \wedge R(x) \rightarrow S(x) \wedge T(x))$

Rewriting:

$\{\exists x((Px \rightarrow Qx) \rightarrow Sx), \exists x((\neg Px \rightarrow Rx) \rightarrow Tx), \forall x(Qx \rightarrow Rx)\} \vdash$

$\exists x (Qx \wedge Rx \rightarrow Sx \wedge Tx)$

$\exists x((Px \rightarrow Qx) \rightarrow Sx) \wedge \exists x((\neg Px \rightarrow Rx) \rightarrow Tx) \wedge \forall x(Qx \rightarrow Rx) \models \exists x ((Qx \wedge Rx) \rightarrow (Sx \wedge Tx))$

(from generator)

Conclusion negated and included in our original set of the premisses:

$H = \{\exists x((Px \rightarrow Qx) \rightarrow Sx), \exists x((\neg Px \rightarrow Rx) \rightarrow Tx), \forall x(Qx \rightarrow Rx), \neg \exists x (Qx \wedge Rx \rightarrow Sx \wedge Tx)\}$

$H = \{\exists x((Px \rightarrow Qx) \rightarrow Sx), \exists x((\neg Px \rightarrow Rx) \rightarrow Tx), \forall x(Qx \rightarrow Rx), \forall x \neg (Qx \wedge Rx \rightarrow Sx \wedge Tx)\}$

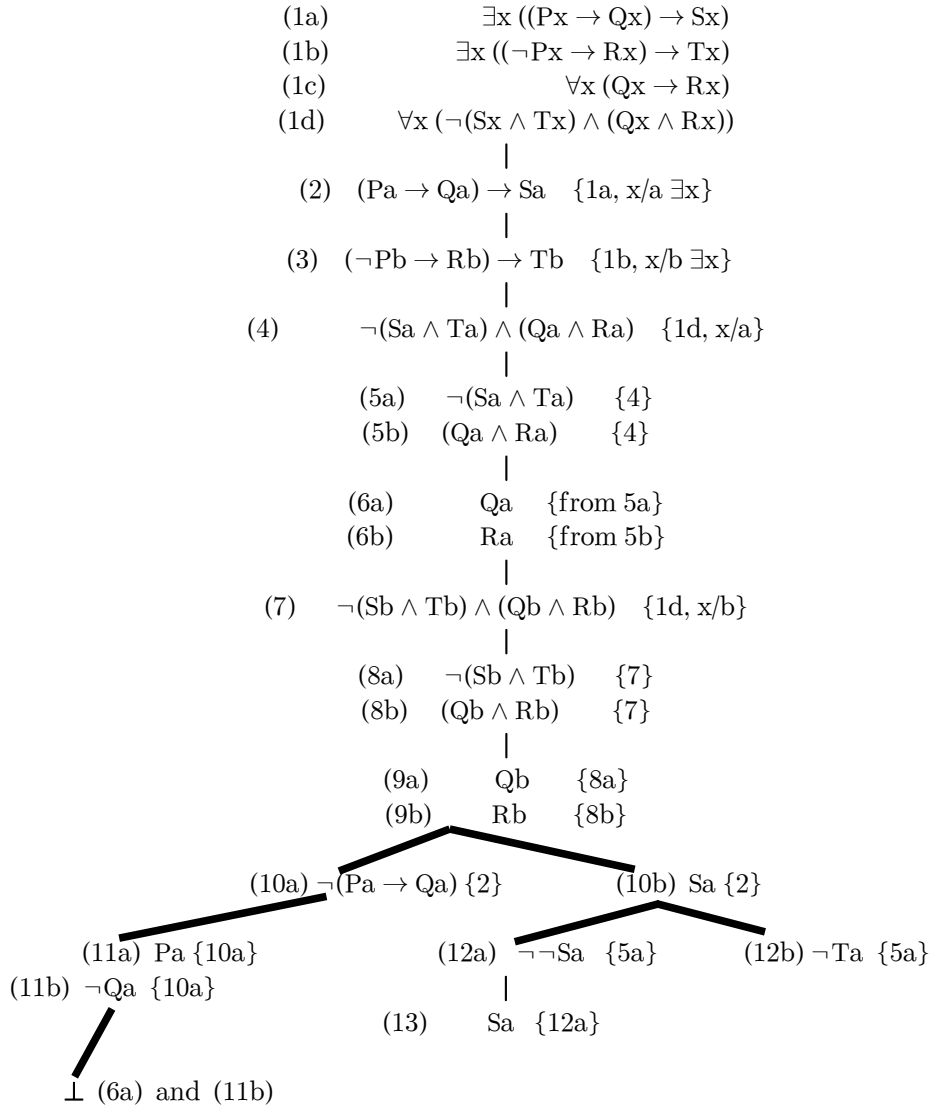
Observing the consequence:

$\forall x \neg (Qx \wedge Rx \rightarrow Sx \wedge Tx)$ is equivalence at to $\forall x (\neg(Sx \wedge Tx) \wedge (Qx \wedge Rx))$ and

$\forall x (\neg(Sx \wedge Tx) \wedge (Qx \wedge Rx)) \iff \forall x \neg(Sx \wedge Tx) \wedge \forall x(Qx \wedge Rx)$, so rewriting H:

$H = \{\exists x((Px \rightarrow Qx) \rightarrow Sx), \exists x((\neg Px \rightarrow Rx) \rightarrow Tx), \forall x(Qx \rightarrow Rx), \forall x (\neg(Sx \wedge Tx) \wedge (Qx \wedge Rx))\}$

Building the tableaux:



3

$\{\forall x ((P(x) \rightarrow R(x)) \rightarrow (P(x) \vee Q(x) \rightarrow R(x))), \exists x (R(x) \vee S(x) \rightarrow P(x))\} \vdash$
 $\forall x (Q(x) \rightarrow P(x))$

Rewriting:

$\{\forall x ((Px \rightarrow Rx) \rightarrow (Px \vee Qx \rightarrow Rx)), \exists x (Rx \vee Sx \rightarrow Px)\} \vdash$
 $\forall x (Qx \rightarrow Px)$

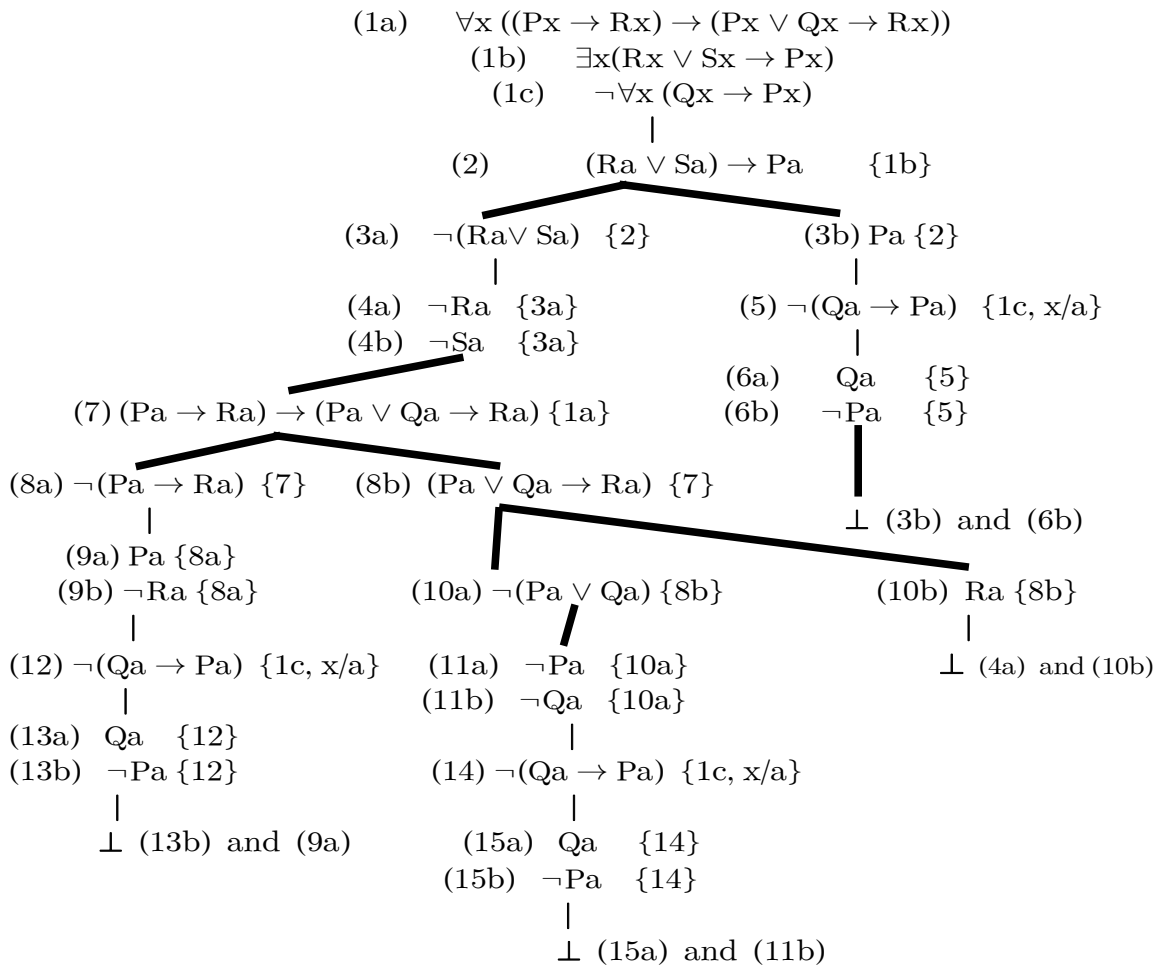
Conclusion negated and included in our original set of the premises:

$H = \{\forall x ((Px \rightarrow Rx) \rightarrow (Px \vee Qx \rightarrow Rx)), \exists x (Rx \vee Sx \rightarrow Px), \neg \forall x (Qx \rightarrow Px)\}$

$H = \{\forall x ((Px \rightarrow Rx) \rightarrow (Px \vee Qx \rightarrow Rx)), \exists x(Rx \vee Sx \rightarrow Px), \exists x \neg (Qx \rightarrow Px)\}$

Remember: $\exists x \neg (Qx \rightarrow Px) \Leftrightarrow \neg \forall x (Qx \rightarrow Px)$

Building the tableaux:



This tableaux is closed, so H is tautology, or $\{\forall x ((Px \rightarrow Rx) \rightarrow (Px \vee Qx \rightarrow Rx)), \exists x(Rx \vee Sx \rightarrow Px)\} \vdash \forall x (Qx \rightarrow Px)$