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HE7, Semantic tableaux in predicate logic

Question 1

1. $\{\exists x\,((P(x) \to R(x)) \to (Q(x) \to S(x)),\ \exists x\,(P(x) \land Q(x) \to R(x) \land S(x))\} \vdash$

$\exists x((P(x) \lor Q(x) \to R(x) \lor S(x))$

In addition of Tableaux from Propositional Logic, we need to add these rules $\neg\forall xA \Leftrightarrow \neg\exists xA$ and $\exists x\neg A \Leftrightarrow \forall x\neg A$, and

$\exists xA$

| with b not occurring anywhere above on the same branch.

$A[b/x]$

and

$\forall xA$

| for any choice of c, only if no constant symbol has been introduced so far on the same branch.

$A[c/x]$

Using the method of semantic tableaux, the set is given by:

$\{\exists x\,((P(x) \to R(x)) \to (Q(x) \to S(x)),\ \exists x\,(P(x) \land Q(x) \to R(x) \land S(x)),$

$\neg\exists x((P(x) \lor Q(x) \to R(x) \lor S(x))\}$

Next I will do a simplification of this notation, where any P(x) means Px, and so on:
So the original set is rewriten as:

$\{\exists x\,((Px \to Rx) \to (Qx \to Sx)),\ \exists x\,(Px \land Qx \to Rx \land Sx),\ \neg\exists x(Px \lor Qx \to Rx \lor Sx)\}$

or

$\{\exists x\,((Px \to Rx) \to (Qx \to Sx)),\ \exists x\,(Px \land Qx \to Rx \land Sx),$

$\forall x\neg(Px \lor Qx \to Rx \lor Sx)\}$

(1a) $\exists x\,((Px \rightarrow Rx) \rightarrow (Qx \rightarrow Sx))$

(1b) $\exists x\,(Px \wedge Qx \rightarrow Rx \wedge Sx)$

(1c) $\neg\exists x(Px \vee Qx \rightarrow Rx \vee Sx)$

(2) $\forall x\neg(Px \vee Qx \rightarrow Rx \vee Sx)$ {from 1c}

(3) $(Pb \wedge Qb \rightarrow Rb \wedge Sb)$ {1b}

(4a) $\neg(Pb \wedge Qb)$ {3} (4b) $(Rb \wedge Sb)$ {3}

(5a) $\neg Pb$ {4a} **(5b)** $\neg Qb$ {4a}

See the branching below

…………

(6a) Rb {4b}

(6b) Sb {4b}

(derivations of 6a and 6b follow below)

(7) $\neg(Pb \vee Qb \rightarrow Rb \vee Sb)$ {from 2, x/b due $\forall x$ }

(8a) $(Pb \vee Qb)$ {7}

(8b) $\neg(Rb \vee Sb)$ {7}

(9a) Pb {8a} (9b) Qb {8a}

$\perp$ (5a) and (9a)

Extending the branch from node **(5b),** starting a node numeration in 20:

(5b) $\neg Qb$ {4a}

(20) $\neg(Pb \vee Qb \rightarrow Rb \vee Sb)$ {from 2, x/b due $\forall x$ }

(21a) $(Pb \vee Qb)$ {20}

(21b) $\neg(Rb \vee Sb)$ {20}

(22a) Pb {21a} (22b) Qb {21a}

$\perp$

From (5b) and (22b)

Resuming the nodes **(6a) and (6b)** from the tree above, we have:

(6a) Rb {4b}
(6b) Sb {4b}

(15) ¬(Pb ∨ Qb → Rb ∨ Sb) {from 2, x/b due ∀x }

(16a) (Pb ∨ Qb) {15}
(16b) ¬(Rb ∨ Sb) {15}

(17a) ¬Rb {4b}
(17b) ¬Sb {4b}

⊥ (17a) and (6a) and (6b) and (17b)

Note: formula (2) was re-instanced with different 'x' to allow new branches to close.

Analysis:
This tableaux does not close. Although is saturated, depends of the branch is taken, it generates Pb and Qb as open nodes. However, there are branches were Pb and Qb become closed nodes. So for this domain {b} there is countermodel. A countermodel of a formula A is an interpretation that falsifies A. So we need to find a false for this domain {b} and P={b} and Q={b}.

Where is a?

Resuming:

(1a) ∃x ((Px → Rx) → (Qx → Sx))
(1b) ∃x (Px ∧ Qx → Rx ∧ Sx)
(1c) ¬∃x(Px ∨ Qx → Rx ∨ Sx)

in addition R={} and S={}. Doing such interpretations (Int(formula):{⊥,T}) in these premises:
From 1a: **Int((T→⊥)→(T→⊥))= Int(⊥→⊥)=T**
From 1b: **Int((T ∧ T)→(⊥ ∧ ⊥))= Int(T→⊥)=⊥**
From 1c: **Int(¬((T ∨ T)→(⊥∨ ⊥))= Int(¬T)=T**

A counter model makes all premises true but the conclusion false.

The formula 1b is not a logical consequence of this set, once 1b entails a false conclusion.

3P

Question 2

2. {∃x((P(x) → Q(x)) → S(x)), ∃x((¬P(x) → R(x)) → T(x)), ∀x(Q(x) → R(x))} ⊢ ∃x (Q(x) ∧ R(x) → S(x) ∧ T(x))

Rewriting:

{∃x((Px → Qx) → Sx), ∃x((¬Px → Rx) → Tx), ∀x(Qx → Rx)} ⊢

∃x (Qx ∧ Rx → Sx ∧ Tx)

$\exists x((Px \to Qx) \to Sx) \land \exists x((\neg Px \to Rx) \to Tx) \land \forall x(Qx \to Rx) \models \exists x ((Qx \land Rx) \to (Sx \land Tx))$

Conclusion negated and included in our original set of the premises:

$H = \{\exists x((Px \to Qx) \to Sx), \exists x((\neg Px \to Rx) \to Tx), \forall x(Qx \to Rx), \neg\exists x (Qx \land Rx \to Sx \land Tx)\}$

$H = \{\exists x((Px \to Qx) \to Sx), \exists x((\neg Px \to Rx) \to Tx), \forall x(Qx \to Rx), \forall x\neg (Qx \land Rx \to Sx \land Tx)\}$

Observing the consequence:

$\forall x\neg (Qx \land Rx \to Sx \land Tx)$ is equivalence at to $\forall x (\neg(Sx \land Tx) \land (Qx \land Rx))$ and

$\forall x (\neg(Sx \land Tx) \land (Qx \land Rx)) \Leftrightarrow \forall x \neg(Sx \land Tx) \land \forall x(Qx \land Rx)$, so rewriting H:

$H = \{\exists x((Px \to Qx) \to Sx), \exists x((\neg Px \to Rx) \to Tx), \forall x(Qx \to Rx), \forall x (\neg(Sx \land Tx) \land (Qx \land Rx))\}$

Building the tableaux:

That is not part of the method of semantic tableaux and should thus not be part of a solution in this exam.

(1a) $\exists x ((Px \to Qx) \to Sx)$

(1b) $\exists x ((\neg Px \to Rx) \to Tx)$

(1c) $\forall x (Qx \to Rx)$

(1d) $\forall x (\neg(Sx \land Tx) \land (Qx \land Rx))$

(2) $(Pa \to Qa) \to Sa$ {1a, x/a ∃x}

(3) $(\neg Pb \to Rb) \to Tb$ {1b, x/b ∃x}

(4) $\neg(Sa \land Ta) \land (Qa \land Ra)$ {1d, x/a}

(5a) $\neg(Sa \land Ta)$ {4}

(5b) $(Qa \land Ra)$ {4}

(6a) Qa {from 5a}

(6b) Ra {from 5b}

(7) $\neg(Sb \land Tb) \land (Qb \land Rb)$ {1d, x/b}

(8a) $\neg(Sb \land Tb)$ {7}

(8b) $(Qb \land Rb)$ {7}

(9a) Qb {8a}

(9b) Rb {8b}

(10a) $\neg(Pa \to Qa)$ {2} | (10b) Sa {2}

(11a) Pa {10a}

(11b) $\neg Qa$ {10a}

$\perp$ (6a) and (11b)

(12a) $\neg\neg Sa$ {5a} | (12b) $\neg Ta$ {5a}

(13) Sa {12a}

Analysis:

This tableaux does not close. Although is saturated, depends of the branch is taken, it generates Sa and ¬Ta as open nodes. So for this domain {a} there is countermodel. A countermodel of a formula A is an interpretation that falsifies A. So we need to find a false for this domain {a} with S={a} and T={}.
Resuming our formula set, some of these formulas must falsifies under this interpretation (falsifies one of them):

Where is b?

(1a) ∃x ((Px → Qx) → Sx)
(1b) ∃x ((¬Px → Rx) → Tx)
(1c) ∀x (Qx → Rx)
(1d) ∀x (¬(Sx ∧ Tx) ∧ (Qx ∧ Rx))

in addition P={}, and Q={} and R={}. Doing such interpretations (Int(formula):{⊥,T}) in these premises:
From 1a: **Int((⊥→⊥)→(T))= Int(⊥→T)=T**
From 1b: **Int((¬⊥→⊥)→ ⊥)= Int(⊥→⊥)=T**
From 1c: **Int(⊥→⊥)= Int(T)=T**
From 1d: **Int((¬(T ∧ ⊥) ∧ (⊥∧ ⊥))= Int(T ∧ ⊥)=⊥**

3p

The formula 1d is not a logical consequence of this set, once 1d entails false conclusion.

Question 3

3. {∀x ((P(x) → R(x)) → (P(x) ∨ Q(x) → R(x))), ∃x(R(x) ∨ S(x) → P(x))} ⊢ ∀x (Q(x) → P(x))

Rewriting:

{∀x ((Px → Rx) → (Px ∨ Qx → Rx)), ∃x(Rx ∨ Sx → Px)} ⊢ ∀x (Qx → Px)

Conclusion negated and included in our original set of the premises:

H={∀x ((Px → Rx) → (Px ∨ Qx → Rx)), ∃x(Rx ∨ Sx → Px), ¬∀x (Qx → Px)}

H={∀x ((Px → Rx) → (Px ∨ Qx → Rx)), ∃x(Rx ∨ Sx → Px), ∃x¬ (Qx → Px)}

Building the tableaux:

(1a) $\forall x\,((Px \rightarrow Rx) \rightarrow (Px \vee Qx \rightarrow Rx))$

(1b) $\exists x(Rx \vee Sx \rightarrow Px)$

(1c) $\neg\forall x\,(Qx \rightarrow Px)$

(2) $(Ra \vee Sa) \rightarrow Pa$ {1b}

(3a) $\neg(Ra \vee Sa)$ {2}

(4a) $\neg Ra$ {3a}

(4b) $\neg Sa$ {3a}

(7) $(Pa \rightarrow Ra) \rightarrow (Pa \vee Qa \rightarrow Ra)$ {1a}

(8a) $\neg(Pa \rightarrow Ra)$ {7}

(9a) Pa {8a}

(9b) $\neg Ra$ {8a}

(12) $\neg(Qa \rightarrow Pa)$ {1c, x/a}

(13a) Qa {12}

(13b) $\neg Pa$ {12}

$\perp$

(13b) and (9a)

(8b) $(Pa \vee Qa \rightarrow Ra)$ {7}

(10a) $\neg(Pa \vee Qa)$ {8b}

(11a) $\neg Pa$ {10a}

(11b) $\neg Qa$ {10a}

(14) $\neg(Qa \rightarrow Pa)$ {1c, x/a}

(15a) Qa {14}

(15b) $\neg Pa$ {14}

$\perp$

(15a) and (11b)

(10b) Ra {8b}

$\perp$

from (4a) and (10b)

(3b) Pa {2}

(5) $\neg(Qa \rightarrow Pa)$ {1c, x/a}

(6a) Qa {5}

(6b) $\neg Pa$ {5}

$\perp$ (3b) and (6b)

The above tableaux is closed, no countermodel to be provided. So H is tautology, or $\{\forall x\,((Px \rightarrow Rx) \rightarrow (Px \vee Qx \rightarrow Rx)), \exists x(Rx \vee Sx \rightarrow Px)\} \vdash \forall x\,(Qx \rightarrow Px)$

2p

Total 8p