HE7, Semantic tableaux in predicate logic

The solutions must be complete and easy to follow. Correct answers will not yield full points without explanations as to how the answer was reached.

For each of the three problems, up to 5 points can be gained.

Check which of the following deductions hold by applying the method of semantic tableaux. For any semantic tableau where not all branches close, provide a countermodel.

1

$$\{\exists x \ ((P(x) \to R(x)) \to (Q(x) \to S(x)), \ \exists x \ (P(x) \land Q(x) \to R(x) \land S(x))\} \vdash \\ \exists x ((P(x) \lor Q(x) \to R(x) \lor S(x))$$

In addition of Tableaux from Propositional Logic, now, roughly we need to add these rules $\neg \forall x A \Leftrightarrow \neg \exists x A \text{ and } \exists x \neg A \Leftrightarrow \forall x \neg A, \text{ and}$

 $\exists xA$

with b not occurring anywhere above on the same branch.

A[b/x]

and

 $\forall xA$

for any choice of c, only if no constant symbol has been introduced so far on the same branch.

A[c/x]

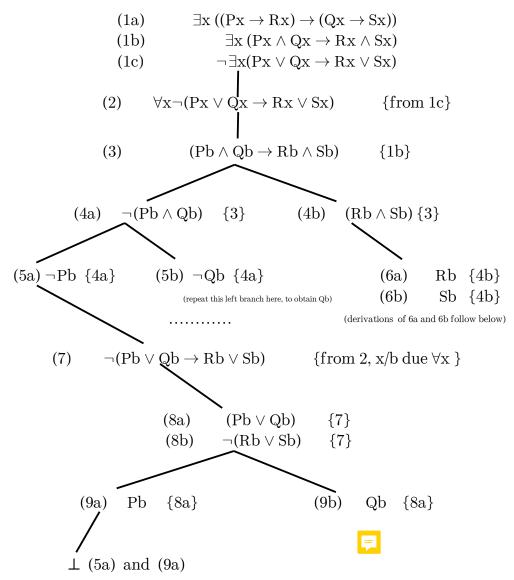
Using the method of semantic tableaux. The counterexample set is:

$$\{\exists x ((P(x) \to R(x)) \to (Q(x) \to S(x)), \exists x (P(x) \land Q(x) \to R(x) \land S(x)), \\ \neg \exists x ((P(x) \lor Q(x) \to R(x) \lor S(x))\}$$

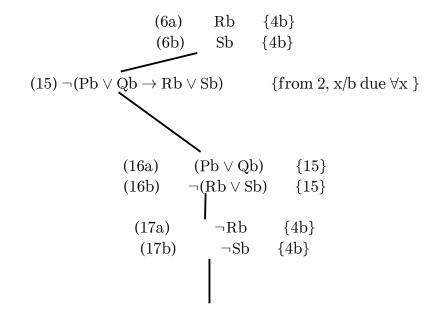
let's do a simplification of this notation, where any P(x) means Px, and so on: So the original set is rewrite as:

$$\begin{split} & \{\exists x \ ((Px \to Rx) \to (Qx \to Sx)), \ \exists x \ (Px \land Qx \to Rx \land Sx), \ \neg \exists x (Px \lor Qx \to Rx \lor Sx)\} \\ & \text{or} \\ & \{\exists x \ ((Px \to Rx) \to (Qx \to Sx)), \ \exists x \ (Px \land Qx \to Rx \land Sx), \ \forall x \neg (Px \lor Qx \to Rx \lor Sx)\} \end{split}$$

To build this tableaux the notation is follow of the examples and the booklet "Theory on predicate logic"



Resuming the nodes (6a) and (6b) from the tree above, we have:



$$\perp$$
 (17a) and (6a) and (6b) and (17b)

PS: formula (2) was re-instanced with different 'x' to allow the branches to close.

2

$$\{\exists x ((P(x) \to Q(x)) \to S(x)), \exists x ((\neg P(x) \to R(x)) \to T(x)), \forall x (Q(x) \to R(x))\} \vdash \exists x (Q(x) \land R(x) \to S(x) \land T(x))$$

Rewriting:

$$\{\exists x ((Px \to Qx) \to Sx), \exists x ((\neg Px \to Rx) \to Tx), \forall x (Qx \to Rx)\} \vdash \\ \exists x (Qx \land Rx \to Sx \land Tx)$$

$$\exists x ((Px \to Qx) \to Sx) \land \exists x ((\neg Px \to Rx) \to Tx) \land \forall x (Qx \to Rx) \mid = \exists x ((Qx \land Rx) \to (Sx \land Tx))$$

(from generator)

Conclusion negated and included in our original set of the premisses:

$$H = \{\exists x ((Px \to Qx) \to Sx), \exists x ((\neg Px \to Rx) \to Tx), \forall x (Qx \to Rx), \neg \exists x (Qx \land Rx \to Sx \land Tx)\}$$

$$H = \{\exists x ((Px \to Qx) \to Sx), \exists x ((\neg Px \to Rx) \to Tx), \forall x (Qx \to Rx), \forall x \neg (Qx \land Rx \to Sx \land Tx)\}$$

Observing the consequence:

$$\forall x \neg (Qx \land Rx \rightarrow Sx \land Tx)$$
 is equivalence at to $\forall x (\neg (Sx \land Tx) \land (Qx \land Rx))$ and

$$\forall x (\neg (Sx \land Tx) \land (Qx \land Rx)) \iff \forall x \neg (Sx \land Tx) \land \forall x (Qx \land Rx), \text{ so rewriting H:}$$

$$H = \{\exists x ((Px \to Qx) \to Sx), \exists x ((\neg Px \to Rx) \to Tx), \forall x (Qx \to Rx), \forall x (\neg (Sx \land Tx) \land (Qx \land Rx))\}$$

Building the tableaux:

$$(1a) \qquad \exists x ((Px \to Qx) \to Sx)$$

$$(1b) \qquad \exists x ((\neg Px \to Rx) \to Tx)$$

$$(1c) \qquad \forall x (Qx \to Rx)$$

$$(1d) \qquad \forall x (\neg (Sx \land Tx) \land (Qx \land Rx))$$

$$(2) \qquad (Pa \to Qa) \to Sa \qquad \{1a, x/a \exists x\}$$

$$(3) \qquad (\neg Pb \to Rb) \to Tb \qquad \{1b, x/b \exists x\}$$

$$(4) \qquad \neg (Sa \land Ta) \land (Qa \land Ra) \qquad \{1d, x/a\}$$

$$(5a) \qquad \neg (Sa \land Ta) \qquad \{4\}$$

$$(5b) \qquad (Qa \land Ra) \qquad \{4\}$$

$$(6a) \qquad Qa \qquad \{from 5a\}$$

$$(6b) \qquad Ra \qquad \{from 5b\}$$

$$(7) \qquad \neg (Sb \land Tb) \land (Qb \land Rb) \qquad \{1d, x/b\}$$

$$(8a) \qquad \neg (Sb \land Tb) \qquad \{7\}$$

$$(8b) \qquad (Qb \land Rb) \qquad \{7\}$$

$$(8a) \qquad \neg (Sb \land Tb) \qquad \{7\}$$

$$(8b) \qquad (Qb \land Rb) \qquad \{7\}$$

$$(9a) \qquad Qb \qquad \{8a\}$$

$$(9b) \qquad Rb \qquad \{8b\}$$

$$(10a) \neg (Pa \to Qa) \{2\} \qquad (10b) Sa \{2\}$$

$$(11a) \quad Pa \{10a\} \qquad (12a) \quad \neg \neg Sa \quad \{5a\} \qquad (12b) \neg Ta \quad \{5a\}$$

$$(11b) \quad \neg Qa \quad \{10a\} \qquad (13) \qquad Sa \quad \{12a\}$$

$$\begin{aligned} \mathbf{3} \\ \{ \forall x \ ((P(x) \rightarrow R(x)) \rightarrow (P(x) \lor Q(x) \rightarrow R(x))), \ \exists x (R(x) \lor S(x) \rightarrow P(x)) \} \vdash \\ \forall x \ (Q(x) \rightarrow P(x)) \end{aligned}$$

Rewriting:

$$\{ \forall x ((Px \to Rx) \to (Px \lor Qx \to Rx)), \exists x (Rx \lor Sx \to Px) \} \vdash \\ \forall x (Qx \to Px)$$

Conclusion negated and included in our original set of the premises:

$$H = \{ \forall x \: ((Px \to Rx) \to (Px \lor Qx \to Rx)), \: \exists x (Rx \lor Sx \to Px), \: \neg \forall x \: (Qx \to Px) \}$$

$$\begin{split} &H {=} \{ \forall x \: ((Px \to Rx) \to (Px \lor Qx \to Rx)), \: \exists x (Rx \lor Sx \to Px), \: \exists x \neg \: (Qx \to Px) \} \\ &Remember: \: \exists x \neg \: (Qx \to Px) \Longleftrightarrow \neg \forall x (Qx \to Px) \end{split}$$

Building the tableaux:

$$(1a) \quad \forall x ((Px \to Rx) \to (Px \lor Qx \to Rx))$$

$$(1b) \quad \exists x (Rx \lor Sx \to Px)$$

$$(1c) \quad \neg \forall x (Qx \to Px)$$

$$(1c) \quad \neg \forall x (Qx \to Px)$$

$$(1c) \quad \neg \forall x (Qx \to Px)$$

$$(2) \quad (Ra \lor Sa) \to Pa \quad \{1b\}$$

$$(3a) \quad \neg (Ra \lor Sa) \quad \{2\} \quad (3b) Pa \{2\}$$

$$(4a) \quad \neg Ra \quad \{3a\} \quad (5) \quad \neg (Qa \to Pa) \quad \{1c, x/a\}$$

$$(4b) \quad \neg Sa \quad \{3a\} \quad (6a) \quad Qa \quad \{5\}$$

$$(7) (Pa \to Ra) \to (Pa \lor Qa \to Ra) \quad \{1a\} \quad (6b) \quad \neg Pa \quad \{5\}$$

$$(8a) \quad \neg (Pa \to Ra) \quad \{7\} \quad (8b) \quad (Pa \lor Qa \to Ra) \quad \{7\}$$

$$(9a) Pa \quad \{8a\} \quad (10a) \quad \neg (Pa \lor Qa) \quad \{8b\} \quad (10b) \quad Ra \quad \{8b\}$$

$$(9b) \quad \neg Ra \quad \{8a\} \quad (10a) \quad \neg (Pa \lor Qa) \quad \{8b\} \quad (10b) \quad Ra \quad \{8b\}$$

$$(12) \quad \neg (Qa \to Pa) \quad \{1c, x/a\} \quad (11a) \quad \neg Pa \quad \{10a\} \quad \qquad \bot \quad (4a) \quad \text{and} \quad (10b)$$

$$(13a) \quad Qa \quad \{12\} \quad (1b) \quad \neg Qa \quad \{10a\} \quad \qquad \bot \quad (4a) \quad \text{and} \quad (10b)$$

$$(13a) \quad Qa \quad \{12\} \quad (14) \quad \neg (Qa \to Pa) \quad \{1c, x/a\}$$

$$\downarrow \quad \downarrow \quad (15b) \quad \neg Pa \quad \{14\}$$

$$\downarrow \quad (15a) \quad \text{and} \quad (11b)$$

This tableaux is closed, so H is tautology, or $\{\forall x ((Px \to Rx) \to (Px \lor Qx \to Rx)), \exists x (Rx \lor Sx \to Px)\}$ $\vdash \forall x (Qx \to Px)$