Seminar 6

1. Calculati derivata de ordinul $n \in \mathbb{N}$ a functiilor de mai jos si precizati multimea pe care aceste functii sunt indefinit derivabile
   a) $f(x) = \sin x$
   b) $f(x) = \ln(x+1)$
   c) $f(x) = (x^2 - x)\,\mathrm{e}^x$
   d) $f(x) = \sqrt{1-x}$

2. Pentru functiile de la exercitiul anterior, punctul $x_0 = 0$ si numarul $n \in \mathbb{N}$, determinati
   a) Polinomul lui Taylor de grad n asociat functiei f in punctul x_0
   b) Multimea de convergenta a seriei Taylor corespunzatoare.

3. Utilizand operatii cu serii de puteri, justificati egalitatile
   a)
$$\sum_{n=0}^{\infty} (-1)^n (n+1) x^n = \frac{1}{(1+x)^2}, \quad \forall x \in (-1,1)$$
   b)
$$1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!}\, x^n = \frac{1}{\sqrt{1-x}}, \quad \forall x \in [-1,1)$$

4. Determinati multimea de convergenta a seriei de puteri $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}\,(x-1)^n$

Exercitii suplimentare

1. Calculati derivata de ordinul $n \in \mathbb{N}$ a functiilor de mai jos si precizati multimea pe care aceste functii sunt indefinit derivabile
   a) $f(x) = \cos x$
   b) $f(x) = x\sqrt{x+1}$
   c) $f(x) = \ln(1-x^2)$
   d) $f(x) = \frac{1}{1-x^2}$

2. Pentru functiile de la exercitiul anterior, punctul $x_0 = 0$ si numarul $n \in \mathbb{N}$, determinati
   a) Polinomul lui Taylor de grad n asociat functiei f in punctul x_0
   b) Multimea de convergenta a seriei Taylor corespunzatoare.

3. Determinati multimea de convergenta a seriilor de puteri
   a) $\sum\limits_{n=0}^{\infty} \frac{2^n}{(n+1)^3} (x+1)^n$
   b) $\sum\limits_{n=0}^{\infty} \left(\frac{\pi}{2} - \operatorname{arctg} n\right) x^n$
   c) $\sum\limits_{n=1}^{\infty} \frac{n^n}{n!} x^n$

4. Utilizand operatii cu serii de puteri, justificati egalitatile
   a)
$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(x-1)^2}, \quad \forall x \in (-1,1)$$
   b)
$$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x}), \quad \forall x \in \mathbb{R}$$
   c)
$$x + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!\,(2n+1)} x^{2n+1} = \arcsin x, \quad \forall x \in [-1,1]$$

1. $f'(x) = \cos(x)$

$f''(x) = -\sin(x)$

$f^{(3)}(x) = -\cos(x)$

$f^{(4)}(x) = \sin(x)$

$$\left.\begin{aligned} f'(x) &= \cos(x) \\ f''(x) &= -\sin(x) \\ f^{(3)}(x) &= -\cos(x) \\ f^{(4)}(x) &= \sin(x) \end{aligned}\right\} \Rightarrow f^{(n)}(x) = \begin{cases} \sin(x), & n = 4k \\ \cos(x), & n = 4k+1 \\ -\sin(x) & n = 4k+2 \\ -\cos(x) & n = 4k+3 \end{cases}$$

$f(x) = \sin x$, cont, deriv pe $\mathbb{R}$, toate deriv. sunt cont.

$f(x) = \sin(x)$ este indefinit derivabilă pe $\mathbb{R}$ în x_0 deoarece $\exists\ f^{(n)}(x_0) \in \mathbb{R}, \forall n \in \mathbb{N}$

$$f^{(n)}(x) = \begin{cases} (-1)^k \sin x, & n = 2k \\ (-1)^k \cos x, & n = 2k+1 \end{cases}$$

$D = \mathbb{R}$

b) $f(x) = \ln(x+1)$

$x+1 > 0 \Rightarrow x > -1 \Rightarrow x \in (-1, \infty)$

$f'(x) = \frac{1}{x+1} \cdot (x+1)' = (x+1)^{-1}$

$f''(x) = (-1) \cdot (x+1)^{-2}$

$f'''(x) = (-1)(-2)(x+1)^{-3}$

$f^{(4)}(x) = (-1)(-2)(-3)(x+1)^{-4}$

$f^{(n)}(x) = (-1)^{n+1}(n-1)!\,(x+1)^{-n}, \forall n \geq 1$

(La examen trebuie demonstrat cu inducție)

$D = (-1, \infty)$

c) $f(x) = (x^2 - x) \cdot e^x$

$(g \cdot h)^{(n)} = \sum_{k=0}^{n} C_n^k g^{(k)} \cdot h^{(n-k)}$

$$f^{(n)}(x) = \sum_{k=0}^{n} C_n^k \, (x^2-x)^{(k)} \cdot e^{x(n-k)}$$

$$f^{(n)}(x) = C_n^0 \, (x^2-x)^0 \cdot e^x + C_n^1 \, (2x-1)\, e^x +$$

$$C_n^2 \; 2 \cdot e^x + \underbrace{C_n^3 \, 0 \cdot e^x}_{0} + \dots + 0$$

$$= (x^2-x) \cdot e^x + n(2x-1) e^x + 2 e^x \cdot \frac{n(n-1)}{2}$$

$$= e^x \left(x^2 - x + 2nx - n + n^2 - n \right)$$

$$= e^x \left(x^2 + n^2 + 2nx - 2n \right)$$

$$f^{(n)}(x) = e^x \cdot \left[x^2 + (2n-1) \cdot x + n^2 - 2n \right]$$

$$\forall n \in \mathbb{N}$$

$$D = \mathbb{R}$$

$n=2$

$$e^x \left(x^2 + (4-1)x + 4 - 4\right)$$

$$e^x \left(x^2 + 3x\right)$$

$$e^x ($$

$n=1$ $$e^x \left(x^2 + x - 1\right)$$

$$e^x \left(x - 1 + x^2\right)$$

$n=2$ $$3xe^x + x^2$$

d) $f'(x) = \left[(1-x)^{\frac{1}{2}}\right]' = \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)$

$= -\frac{1}{2}(1-x)^{-\frac{1}{2}}$

$f''(x) = \frac{-1}{2} \cdot \left(-\frac{1}{2}\right)(1-x)^{-\frac{3}{2}}(-1)$

$= -\frac{1}{2^2} \cdot (1-x)^{-\frac{3}{2}}$

$f''' = -\frac{1}{2^2} \cdot \left(-\frac{3}{2}\right) \cdot 1 (1-x)^{-\frac{5}{2}}(-1)$

$= -\frac{3}{2^3}(1-x)^{-\frac{5}{2}}$

$f^{(4)}(x) = -\frac{3}{2^3}\left(-\frac{5}{2}\right) \cdot (1-x)^{-\frac{7}{2}}(-1)$

$= -\frac{15}{2^4}(1-x)^{-\frac{7}{2}}$

$$f^{(n)}(x) = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)}{2^n} \cdot (1-x)^{\frac{-2n-1}{2}}$$

$$= \frac{(2n-3)!!}{2^n} (1-x)^{\frac{-(2n-1)}{2}}$$

$\forall\, n \geq 2$

$1 - x \geq 0 \Rightarrow x \leq 1$

$D = (-\infty, 1)$

2. Pentru functiile de la exercitiul anterior, punctul $x_0 = 0$ si numarul $n \in \mathbb{N}$, determinati
   a) Polinomul lui Taylor de grad n asociat functiei f in punctul x_0
   b) Multimea de convergenta a seriei Taylor corespunzatoare.

a) $f(x) = \sin(x)$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + \ldots$$

$$+ \begin{cases} 0 & , n = 2p \\ \frac{(-1)^p}{(2p+1)!} x^{2p+1} & , n = 2p+1 \end{cases}$$

$$f^{(k)}(0) = \begin{cases} 0, & k=2p \\ (-1)^p, & k=2p+1 \end{cases}$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{p=0}^{\infty} \frac{f^{(2p+1)}(0)}{(2p+1)!} x^{2p+1} =$$

$$= \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p+1)!} \cdot x^{2p+1} \quad (*)$$

$$(x_0-r, x_0+r) \subseteq I \subseteq [x_0-r, x_0+r]$$

r - raza de convergență

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

$$r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|, \text{ dacă limita există,}$$

$$r = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$$

$$(*) = \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p+1)!} x^{2p+1} = a_i a x \quad \forall x \in \mathbb{R}$$

$$a_n = \begin{cases} 0, & n = 2p \\ \frac{(-1)^p}{(2p+1)!}, & n = 2p+1 \end{cases}$$

$$\lim \left(\sqrt[n]{|a_n|} \right) = \{0\}$$

$$\sqrt[n]{|a_n|} = \begin{cases} 0, & n = 2p \\ \sqrt[n]{\frac{1}{n!}}, & n = 2p+1 \end{cases}$$

$$\lim_{n=\infty} \sup \sqrt[n]{|a_n|} = 0+ \Rightarrow R = \frac{1}{0+} = +\infty$$

$$\Rightarrow I = \mathbb{R}$$

b) $f^{(n)}(x) = (-1)^{n+1}(n-1)! \cdot (x+1)^{-n} \quad \forall n \geq 1$

$$f^{(n)}(0) = (-1)^{n+1}(n-1)!$$

$$f(0) = \ln(1) = 0$$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \cdot x^k =$$

$$= 0 + \sum_{k=1}^{n} \frac{(-1)^{k+1}\cancel{(k-1)!}}{\cancel{k!}\,k} \cdot x^k =$$

$$= \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \cdot x^k$$

$$= \sum_{n=1}^{\infty} \underbrace{\frac{(-1)^{n+1}}{n}}_{a_n} \cdot x^n$$

$$r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{n}}{\frac{(-1)^{n+2}}{n+1}} \right| =$$

$$= \lim_{n\to\infty} \frac{n+1}{n} = 1$$

$$(-1,1) \subseteq I \subseteq [-1,1]$$

pt $x=-1$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot (-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n}$

$\displaystyle = -\sum_{n=1}^{\infty} \frac{1}{n}$ – divergentă

$\Rightarrow -1 \notin I$

pt $x=1 \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \longrightarrow$ convergentă

$\Rightarrow 1 \in I$

$$\Rightarrow I = (-1,1]$$

d) $f(x)=\sqrt{1-x}$, $f^{(n)}(x) = -\dfrac{(2n-3)!!}{2^n} \cdot (1-x)$ $\forall n \geq 2$

$$f^{(n)}(0) = -\frac{(2n-3)!!}{2^n}$$

$f(0) = \sqrt{1-0} = 1$

$f'(0) = -\dfrac{1}{2} \cdot \dfrac{1}{\sqrt{1-0}} = -\dfrac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} x^n = 1 - \frac{1}{2}x - \sum_{n=2}^{\infty} \frac{(2n-3)!!}{2^n n!} x^n =$$

$$= \sqrt{1-x}, \; x \in [-1,1]$$

$$r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(2n-3)!!}{2^n n!} \cdot \frac{2^{n+1}(n+1)!}{(2n-1)!!} \right|$$

$$= \lim_{n\to\infty} \frac{2n+2}{2n-1} = 1$$

$$(-1,1) \subseteq J \subseteq [-1,1]$$

pt $x = 1$

$$\sum_{n=2}^{\infty} \frac{(2n-3)!!}{2^n \cdot n!}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1 \quad \text{nu decide}$$

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{2n+2}{2n-1} - 1\right)$$

$$\lim_{n\to\infty} n\left(\frac{\cancel{2n} + 2 - \cancel{2n} + 1}{2n - 1}\right) =$$

$$\lim_{n \to \infty} \frac{3n}{2n-1} = \frac{3}{2} > 1 \Rightarrow \text{conv} \Rightarrow 1 \in J$$

pt $x = -1$

$$\sum_{n=0}^{\infty} \frac{(2n-3)!!}{2^n n!} (-1)^n = \text{absolut convergenta}$$

$\Rightarrow$ conv.

$$\Rightarrow -1 \in J \Rightarrow J = [-1, 1]$$

3. Utilizand operatii cu serii de puteri, justificati egalitatile

a)

$$\sum_{n=0}^{\infty} (-1)^n (n+1) x^n = \frac{1}{(1+x)^2}, \quad \forall x \in (-1, 1)$$

b)

$$1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} x^n = \frac{1}{\sqrt{1-x}}, \quad \forall x \in [-1, 1)$$

a) $$\sum_{n=0}^{\infty} \frac{(2n-1)!!}{2n!!} x^n = \frac{1}{\sqrt{1-x}}, \forall x \in [-1, 1)$$

folosim $$1 - \frac{x}{2} - \sum_{n=2}^{\infty} \frac{(2n-3)}{2^n \cdot n!} x^n$$

$$= \sqrt{1-x} \Big|' \ \forall x \in [-1,1]$$

$$-\frac{1}{2} - \sum_{n=2}^{\infty} n \cdot \frac{(2n-3)!!}{2^n \cdot n!} \cdot x^{n-1} = \frac{-1}{2\sqrt{1-x}} \Big| \cdot (-2)$$

$$1 + \sum_{n=2}^{\infty} \frac{(2n-3)!!}{2^{n-1} \cdot (n-1)!} \cdot x^{n-1} = \frac{1}{\sqrt{1-x}}$$

$$(n-1) = m$$

$$1 + \sum_{m=1}^{\infty} \frac{(2m-1)!!}{2^m \cdot m!} \cdot x^m = \frac{1}{\sqrt{1-x}}, \ r = 1$$

$$2^m \cdot m! = (2m)!!$$

$$\Rightarrow (-1,1] \subseteq I \subseteq [-1,1]$$

pt $x = 1 \Rightarrow \sum_{m=1}^{\infty} \frac{(2m-1)!!}{(2m)!!}$ s.t.p. diverg. $\Rightarrow 1 \notin I$

pt $x = -1 \Rightarrow \sum_{m=1}^{\infty} \underbrace{\frac{(2m-1)!!}{(2m)!!}}_{a_m} (-1)^m$ $a_m \searrow 0$ (descresc și are lim 0)

Reamintim tabel variatie ...