④ (Funcția Gama)

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \cdot e^{-x}\, dx, \quad \alpha \in \mathbb{R}$$

a) $\Gamma(\alpha)$ conv. $\forall\, \alpha > 0$

b) $\Gamma(n+1) = n!\ \forall\, n \in \mathbb{N}$

c) $\Gamma(\alpha+1) = \alpha\Gamma(\alpha),\ \forall\, \alpha > 0$

d) $\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n} \cdot \Gamma\left(\frac{1}{2}\right),\ \forall\, n \in \mathbb{N}^*$

a) $\Gamma(\alpha) = \underbrace{\int_{0+0}^1 x^{\alpha-1} \cdot e^{-x}\, dx}_{I_1} + \underbrace{\int_1^\infty x^{\alpha-1} \cdot e^{-x}\, dx}_{I_2}$

st I_1 $\quad x^{\alpha-1} \cdot e^{-x} \le x^{\alpha-1},\ \forall x \in (0,1]$

$$\int_{0+0}^1 x^{\alpha-1}\, dx = \left.\frac{x^\alpha}{\alpha}\right|_{0+0}^1 = \frac{1}{\alpha} < +\infty$$

$\overset{C.C.}{\Longrightarrow} I_1$ conv

st I_2 : $f : [1,\infty) \to [0,\infty)$, $f(x) = x^{\alpha-1} \cdot e^{-x}$

$$\lambda = \lim_{x\to\infty} x^p f(x) = \lim_{x\to\infty} x^{p+\alpha-1} \cdot e^{-x} = \lim_{x\to\infty} \frac{x^{p+\alpha-1}}{e^x} = 0$$

alegem $p=2>1$, $\lambda<+\infty \Rightarrow I_2$ conv

$\Rightarrow \Gamma(\alpha)$ conv

b) $\Gamma(n+1)=\int_0^\infty x^n \cdot e^{-x}\,dx = n!$, $\forall n\in\mathbb{N}$

$n! = n(n-1)! \Rightarrow \Gamma(n+1) = n\cdot\Gamma(n)$

c) temă (integrarea prin părți)

d) inducție

$\Gamma\left(\frac{3}{2}\right)=\Gamma\left(1+\frac{1}{2}\right)=\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)$

o. a. m. d.

$\Gamma\left(\frac{1}{2}\right)=\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx=\sqrt{\pi}$

(5) Exprimați cu ajutorul lui Γ

a) $\int_0^\infty e^{-x^2}\,dx$

b) $\int_0^1 (\ln x)^{\frac{1}{3}}\,dx$

a) $x^2=t$

$x=\sqrt{t}$

$dx=\frac{1}{2\sqrt{t}}\,dt$

$\int_0^\infty e^{-t}\cdot\frac{1}{2\sqrt{t}}\,dt$

$\frac{1}{2}\int_0^\infty t^{-\frac{1}{2}}\cdot e^{-t}\,dt =$

$$= \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right)$$

e) $\int_{0+0}^{1} (\ln x)^{\frac{1}{3}} dx = \int_{-\infty}^{0} (\ln e^t)^{\frac{1}{3}} \cdot e^t dt$

$x = e^t$

$dx = e^t dt$

$x \searrow 0 \Rightarrow t \to -\infty$

$x \to 1 \Rightarrow t = 0$

$$= \int_{-\infty}^{0} t^{\frac{1}{3}} \cdot e^t dt$$

$t = -u$

$dt = -du$

$$= -\int_{\infty}^{0} (-u)^{\frac{1}{3}} \cdot e^{-u} du =$$

$$= \int_{0}^{\infty} (-u)^{\frac{1}{3}} \cdot e^{-u} du$$

$$= -\int_{0}^{\infty} u^{\frac{1}{3}} \cdot e^{-u} du$$

$$= -\Gamma\left(\frac{4}{3}\right)$$

$$= -\frac{1}{3} \Gamma\left(\frac{1}{3}\right)$$

Spațiul $\mathbb{R}^n$

① Fie $x = (1, 0, -1)$, $y = (3, -1, 1) \in \mathbb{R}^3$ calc.

$x+y,\ x\cdot y,\ \|x\|,\ \|-2y\|,\ \|x-y\|$

$x\cdot y \overset{not}{=} \langle x,y\rangle$

$x+y=(1,0,-1)+(3,-1,1)=(4,-1,0)$

$x\cdot y= 1\cdot 3+0\cdot(-1)+(-1)\cdot 1=2$

$\|x\|=\sqrt{x\cdot x}=\sqrt{1\cdot 1+0\cdot 0+(-1)(-1)}=\sqrt{2}$

$\|-2y\|=|-2|\,\|y\|=2\sqrt{3\cdot 3+(-1)(-1)+1\cdot 1}=2\sqrt{11}$

$\|x-y\|=\sqrt{4+1+4}=\sqrt{9}=3$

$x-y=(-2,1,-2)$

(2) Fie $x,y\in\mathbb{R}^n$, notam $a=x\cdot y$

$b=\|x\|$

$c=\|y\|$

Exprimati cu a,b,c

$(x+y)y=xy+y\cdot y=a+c^2$

$x(2x-y)=x(2x)-x\cdot y=2b^2-a$

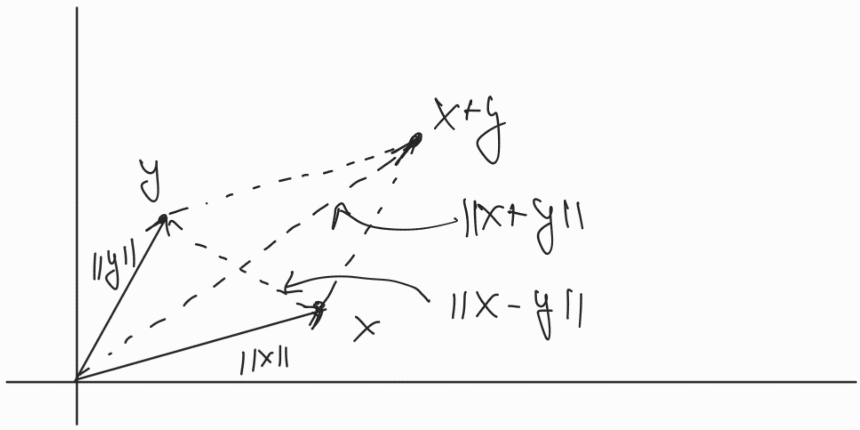
$\|x-y\|=\sqrt{(x-y)\cdot(x-y)}=\sqrt{x\cdot x-x\cdot y-x\cdot y+y\cdot y}=$

$=\sqrt{b^2-2a+c^2}$

③ Identitatea paralelogramului

$\forall x, y \in \mathbb{R}^m: \|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

$m = 2$



$x = (x_1, x_2, \dots, x_m)$

$y = (y_1, y_2, \dots y_m)$

$\|x\|^2 = x_1^2 + x_2^2 + \dots + x_m^2$

$\|y\|^2 = y_1^2 + y_2^2 + \dots + y_m^2$

$\|x+y\|^2 = (x_1+y_1)^2 + (x_2+y_2)^2 + \dots + (x_m+y_m)^2$

$\|x-y\|^2 = (x_1-y_1)^2 + (x_2-y_2)^2 + \dots + (x_m-y_m)^2$ (+)

$\|x+y\|^2 + \|x-y\|^2 = 2x_1^2 + 2y_1^2 + 2x_2^2 + \dots$

$= 2(x_1^2 + y_1^2 + x_2^2 + y_2^2 + \dots)$

$= 2(\|x\|^2 + \|y\|^2)$

$x \in \mathbb{R}^m, r > 0$

$B(x,r) = \{ y \in \mathbb{R}^m \mid \|x-y\| < r \}$

$A \subseteq \mathbb{R}^m$

$\text{int } A = \{ x \in \mathbb{R}^m \mid \exists\ r>0 \text{ a.î. } B(x,r) \subseteq A \}$

$\text{fr } A = \{ x \in \mathbb{R}^m \mid \forall\ r>0 : B(x,r) \cap A \neq \emptyset$ şi $B(x,r) \cap (\mathbb{R}^m \setminus A) \neq \emptyset \}$

↓ pct. frontieră

A deschisă $\Leftrightarrow A = \text{int } A \Leftrightarrow A \cap \text{fr } A = \emptyset$

A închisă $\Leftrightarrow \mathbb{R}^n \setminus A$ deschisă

$\Leftrightarrow \text{fr } A \subseteq A$

(4) Det int A, fr A, şi dacă A e deschisă sau închisă

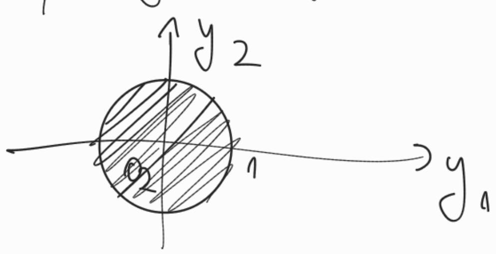
a) $A = B(O_2, 1) \subseteq \mathbb{R}^2$

b) $A = [2,+\infty) \times (2,+\infty) \subseteq \mathbb{R}$

c) $A = \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$

a) $B(O_2,1) = \{ y \in \mathbb{R}^2 \mid \|O_2 - y\| < 1 \}$

$= \{ (y_1, y_2) \in \mathbb{R}^2 \mid (y_1)^2 + (y_2)^2 < 1 \}$

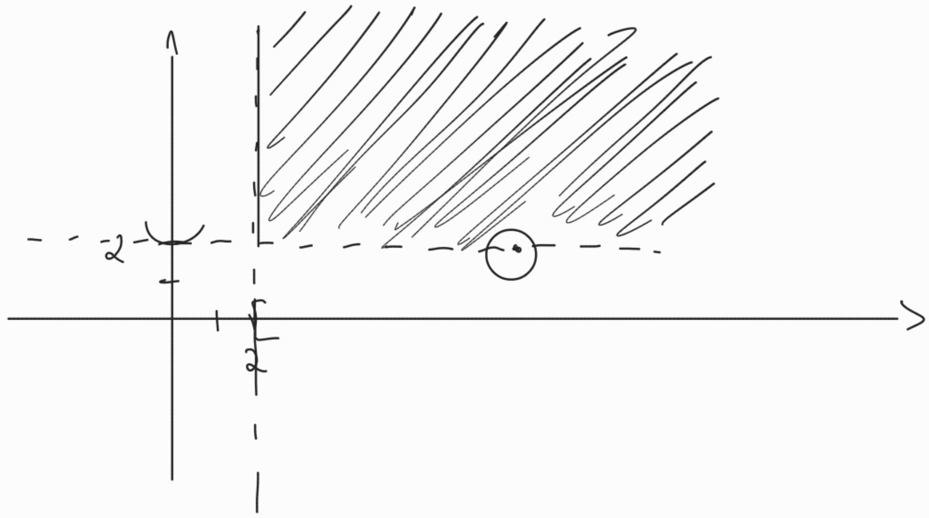


$\text{int} A = B(0_2, 1) = A$

$\text{fr} A = \{(y_1, y_2) \in \mathbb{R}^2 \mid (y_1)^2 + (y_2)^2 = 1\}$

$A = \text{int} A \Rightarrow A$ deschisă

b)



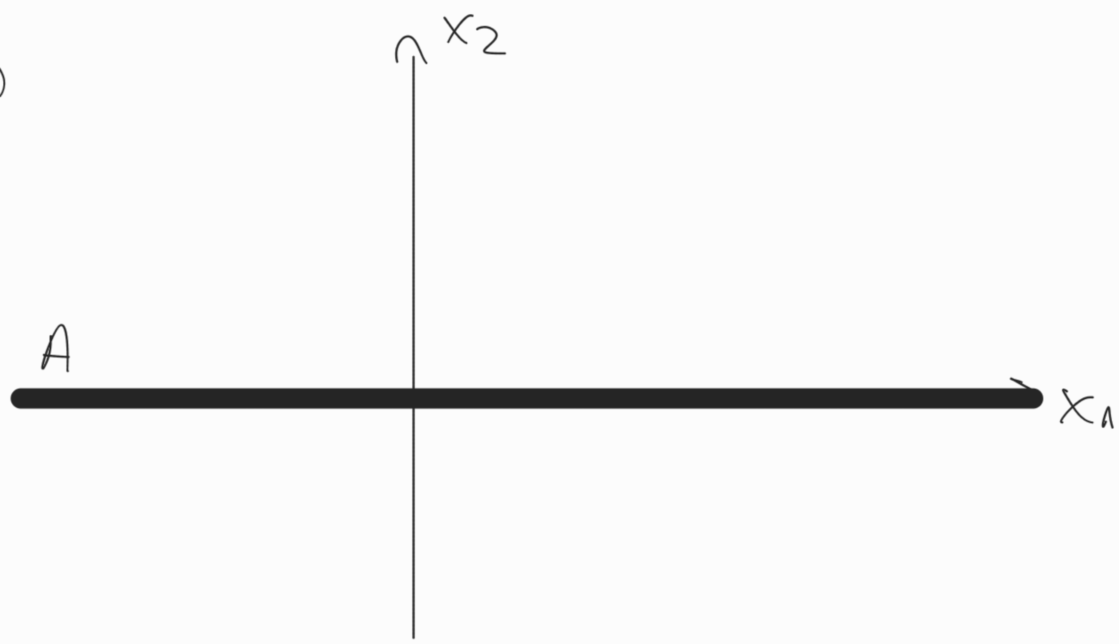
$\text{int} A = (2, +\infty) \times (2, +\infty)$

$\text{fr}(A) = \{(2, a), (a, 2) \mid a \geq 2\}$

$\text{fr}(A) \subseteq A \Rightarrow A$ nu e închisă

$A = \text{int} A \Rightarrow A$ nu e deschisă

c)



$\text{int} A = \emptyset$

$fr A \subseteq A \Rightarrow A$ închisă

⑤ $\forall A \subseteq \mathbb{R}^m$ nevidă

a) $\text{int} A \subseteq A$

b) $\text{int} A \cap fr A = \emptyset$

c) $A \subseteq \text{int} A \cup fr A$ (cu egalitate dacă A închisă)

d) $\text{int} A \cup fr A \cup \text{int}(\mathbb{R}^m \setminus A) = \mathbb{R}^m$

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a) Fie $x \in \text{int} A \Rightarrow \exists r > 0$ a.î. $B(x,r) \subseteq A$

$x \in B(x,r) \Leftrightarrow \|x-x\| < r \Leftrightarrow 0 < r$ ✓

$\Rightarrow x \in A \Rightarrow \text{int} A \subseteq A$

c) Fie $x \in A$ vom presupune că $x \notin \text{int} A \Rightarrow$

$\Rightarrow \forall r > 0 : B(x,r) \not\subseteq A$

$\Rightarrow \forall r > 0 : B(x,r) \cap (\mathbb{R}^m \setminus A) \neq \emptyset$ (1)

$x \in A \Rightarrow x \in B(x,r) \cap A \neq \emptyset \quad \forall r > 0$ (2)

$\left.\right\} \Rightarrow$

$x \in fr A$

Dacă A închisă $\Rightarrow fr A \subseteq A$
$\text{int} A \subseteq A$
$\left.\right\} \Rightarrow \text{int} A \cup fr(A) \subseteq A$

$\Rightarrow A = \text{int} A \cup fr A$