Nr reale

(1) $x, y \in \mathbb{R}$

$$\max\{x,y\} = \frac{|x-y| + (x+y)}{2}$$

$\min\{x,y\} = ?$

Cazul 1: $x \geq y$

$$\max\{x,y\} = \frac{x-\cancel{y}+x+\cancel{y}}{2} = \frac{2x}{2} = x \quad (A)$$

Cazul 2: $x < y$

$$\max\{x,y\} = \frac{y-\cancel{x}+\cancel{x}+y}{2} = \frac{2y}{2} = y \quad (A)$$

$$\min\{x,y\} + \max\{x,y\} = x+y$$

$$\min\{x,y\} = x+y - \max\{x,y\}$$

$$\min\{x,y\} = {}^{2)}(x+y) - \frac{|x-y|+(x+y)}{2} = \frac{(x+y) - |x-y|}{2}$$

(2) $x, y \in \mathbb{R}$

a) $|x+y| \leq |x| + |y| \;|^2 \iff (x+y)^2 \leq x^2+y^2+2|x|\cdot|y|$

$$\cancel{x^2}+2xy+\cancel{y^2} \leq \cancel{x^2}+\cancel{y^2}+2|x||y|$$

$$2xy \leq 2|x||y|$$

$$xy \leq |x||y|$$

b) $|x-y| \geq |x| - |y|$

$\iff |x-y| + |y| \geq |x|$

$|x| = |x-y+y| \leq |x-y| + |y|$

(3) Determinați $\inf A$, $\sup A$, $\min A$, $\max A$

a) $A = [0,7) \cup [8,+\infty)$

$MIN(A) = (-\infty, 0]$

$\inf(A) = \max(MIN(A)) = 0$

$MAJ(A) = \emptyset$, $\sup A = +\infty$

$\min A = 0$

$\max A = \nexists$

$$A = [-1,2] \setminus \mathbb{Q}$$

d) $MIN(A) = (-\infty, -1]$ $\quad$ $\sup A = 2$

$\inf A = -1$ $\quad$ $\min A = \nexists$

$MAJ(A) = [2, +\infty)$ $\quad$ $\max A = \nexists$

c) $A = \{ \frac{1}{n} \mid n \in \mathbb{N}^* \}$

$MIN(A) = (-\infty, 0]$

$\inf(A) = \max((-\infty, 0]) = 0$

$MAJ(A) = [1, +\infty)$

$\sup(A) = 1$

$\min(A) = \nexists$

$\max(A) = 1$

d) $A = \{ \frac{1}{x - [x]} \mid x \in \mathbb{R} \setminus \mathbb{Z} \}$

$[0,$

(4) $A \subseteq \mathbb{R}$ atunci

a) Dacă $\exists \min A \Rightarrow \min A = \inf A$

Dacă $\exists \max A \Rightarrow \max A = \sup A$

$m = \inf A \Leftrightarrow \begin{cases} i) & m \in MIN(A) \\ ii) & \forall m' \in MIN(A): m' \leq m \end{cases}$

$m = \min A \Leftrightarrow \begin{cases} m \in A \\ m \leq a, \forall a \in A \Leftrightarrow \underline{m \in MIN(A)} \end{cases}$

a) $m = \min A \Rightarrow i)$ ✓

Fie $m' \in MIN(A) \Rightarrow \left. \begin{array}{l} m' \leq a, \forall a \in A \\ m \in A \end{array} \right\} \Rightarrow m' \leq m \Rightarrow ii)$

analog b) Temă

(5) $A, B \subseteq \mathbb{R}$ nevide, mărginite și $A \subseteq B$

$\Rightarrow \inf B \le \inf A \le \sup A \le \sup B$

↳ Temă

$\inf A \le a \le \sup A$, $\forall a \in A$

fie $m = \inf B \Rightarrow m \in MIN(B) \Rightarrow m \le b \ \forall b \in B$

$\Rightarrow m \le a, \forall a \in A \Rightarrow m \in MIN(A) \Rightarrow m \le \inf A$

(6) Demonstrați că între oricare 2 nr reale distincte ∃ cel puțin un nr rațional (respectiv irațional)

$a < b$, $a, b \in \mathbb{R}$

$\frac{1}{b-a} > 0 \Rightarrow \exists n \in \mathbb{N}^* : n > \frac{1}{b-a}$

$\forall x \in \mathbb{R}$ $\quad x-1 < [x] \le x$

$[x] = \max\{k \in \mathbb{Z} \mid k \le x\}$

fie $m = [n \cdot a] + 1 \in \mathbb{Z}$

$\frac{m}{n} \in \mathbb{Q}$

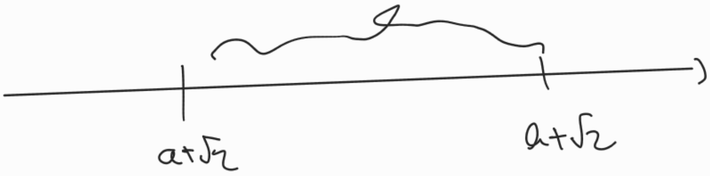
$n \cdot a - 1 < [n \cdot a] \le n \cdot a \mid +1$

$n \cdot a < m \le n \cdot a + 1 \mid : n \ne 0$

$a < \frac{m}{n} \le a + \frac{1}{n} \le a + (b-a) = b$

$\Rightarrow \frac{m}{n} \in (a, b) \cap \mathbb{Q}$

$\sqrt{2} \notin \mathbb{Q}$

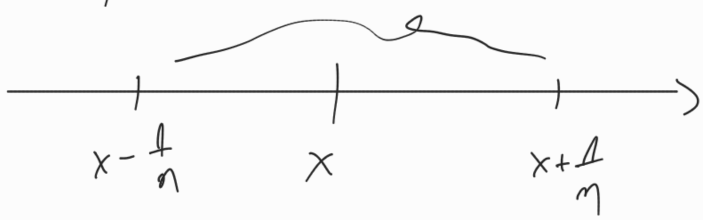


$\Rightarrow \exists\, r \in (a+\sqrt{2},\, b+\sqrt{2}) \cap \mathbb{Q}$

$\Rightarrow r - \sqrt{2} \in (a,b) \cap (\mathbb{R} \setminus \mathbb{Q})$

(7) Orice nr real este limita unui șir de nr raționale (și iraționale). Exemple concrete pt $x = 2$, $y = \sqrt{2}$

Fie $x \in \mathbb{R}$



$\forall\, n \in \mathbb{N}^*$

$\overset{6}{\Rightarrow} \exists\, r_n \in \left(x - \frac{1}{n},\, x + \frac{1}{n}\right) \cap \mathbb{Q}$

$(r_n) \subseteq \mathbb{Q}$ șir, $\underbrace{x - \frac{1}{n}}_{\to x} < r_n < \underbrace{x + \frac{1}{n}}_{\to x}$, $n \to \infty$

$\Rightarrow \lim\limits_{n \to \infty} r_n = x$

$x = 2$

$r_n = 2 + \frac{1}{n}$, $n \in \mathbb{N}^*$

$i_n = 2 + \frac{\sqrt{2}}{n}$, $n \in \mathbb{N}^*$

$y = \sqrt{2}$

$i_n = \sqrt{2} + \frac{1}{n}$, $n \in \mathbb{N}$

$r_n = 1 + \underbrace{\frac{1}{2+\frac{1}{2+\dots\frac{1}{2+}}}}_{n\ ori} \in \mathbb{Q}$

$r_n \to \sqrt{2}$

$r_{n+1} = 1 + \frac{1}{2 + r_n - 1} = 1 + \frac{1}{r_n + 1} = \frac{r_n + 2}{r_n + 1}$

$r_0 = 1$

Ex: $r_{n+1} = \frac{r_n}{2} + \frac{1}{r_n}$, $\forall n \in \mathbb{N}$, $r_0 \in \mathbb{Q}$, $r_0 > 0$

$(r_n) \subseteq \mathbb{Q}$

$r_{n+1} = \frac{r_n}{2} + \frac{1}{r_n} \geq 2\sqrt{\frac{r_n}{2} \cdot \frac{1}{r_n}} = \sqrt{2}$

$r_n \geq \sqrt{2}$, $\forall n \geq 1$ $\Rightarrow (r_n)$ mărg inf (1)

$r_{n+1} - r_n = \frac{r_n}{2} + \frac{1}{r_n} - r_n = \frac{1}{r_n} - \frac{r_n}{2} = \frac{2 - r_n^2}{2 \cdot r_n} \leq 0$

$\Rightarrow (r_n)$ desc. (2)

(1), (2) $\Rightarrow (r_n)$ conv, $\exists\, l = \lim_{n\to\infty} r_n \in \mathbb{R}$

$l = \frac{l}{2} + \frac{1}{l} \Rightarrow \frac{l}{2} = \frac{1}{l} \Rightarrow l^2 = 2 \Rightarrow l = \sqrt{2}$

generalizare

$r_{n+1} = \frac{r_n}{2} + \frac{a}{2 \cdot r_n}$, $\forall n \in \mathbb{N}$, $r_0 > 0$ $\Rightarrow \lim_{n\to\infty} r_n = \sqrt{a}$

$a > 0$ param

(8) justificați că $\alpha = \sqrt{2} + \sqrt[3]{3} \notin \mathbb{Q}$

$\alpha - \sqrt{2} = \sqrt[3]{3}$ $\uparrow^3$

$(\alpha - \sqrt{2})^2 (\alpha - \sqrt{2}) = 3$

$(\alpha^2 - 2\alpha\sqrt{2} + 2)(\alpha - \sqrt{2}) = 3$

$\alpha^3 - \alpha^2\sqrt{2} - 2\alpha^2\sqrt{2} + 4\alpha + 2\alpha - 2\sqrt{2} = 3$

$\alpha^3 - 3\sqrt{2}\alpha^2 + 6\alpha + 2\sqrt{2} - 3 = 0$

$\alpha^3 + 6\alpha - 3 = \sqrt{2}(3\alpha^2 + 2)$ $\uparrow^2$

$(\alpha^3 + 6\alpha - 3)^2 = 2(3\alpha^2 + 2)^2 \Rightarrow$

$\Rightarrow \alpha^6 + \ldots + 1 = 0$

P.p. $\alpha = \frac{p}{q}$, $p, q \in \mathbb{Z}$, $q \neq 0$

$\overset{T}{\Longrightarrow} \left.\begin{array}{l} p \mid 1 \\ q \mid 1 \end{array}\right\} \frac{p}{q} \in \{\pm 1\} \Rightarrow$ absurd $\cancel{\alpha}$

$\alpha \notin \mathbb{Q}$