① $\lim\limits_{n\to\infty} x_n$

a) $x_n = \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)$

b) $x_n = \dfrac{n+\cos n}{n+\sin n}$

c) $x_n = \dfrac{(\sqrt{2}+1)}{(\sqrt{2})^n+1}$

a) $\lim\limits_{n\to\infty} \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right) = \lim\limits_{n\to\infty} \sqrt{n}\ \dfrac{n+1-n}{\sqrt{n+1}+\sqrt{n}} =$

$= \lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{\sqrt{n\left(1+\frac{1}{n}\right)}+\sqrt{n}} =$

$= \lim\limits_{n\to\infty} \dfrac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}\left(\sqrt{1+\underset{\downarrow 0}{\frac{1}{n}}}+1\right)} = \dfrac{1}{2}$

b) $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \dfrac{n+\cos n}{n+\sin n} = \lim\limits_{n\to\infty} \dfrac{\cancel{n}\left(1+\overset{\to 0}{\frac{\cos n}{n}}\right)}{\cancel{n}\left(1+\underset{\downarrow 0}{\frac{\sin n}{n}}\right)} = 1$

$\begin{cases} -1 \le \sin n \le 1 \\ -1 \le \cos n \le 1 \end{cases} \Big| :n \iff \begin{cases} -\frac{1}{n} \le \sin n \le \frac{1}{n} \\ -\frac{1}{n} \le \cos n \le \frac{1}{n} \end{cases}$

0

c) $\lim\limits_{n\to\infty} \dfrac{(\sqrt{2}+1)^n}{(\sqrt{2})^n+1} = \lim\limits_{n\to\infty} \dfrac{\cancel{(\sqrt{2})^n}\left(1+\frac{1}{\sqrt{2}}\right)^n}{\cancel{(\sqrt{2})^n}\left(1+\left(\frac{1}{\sqrt{2}}\right)^n\right)} = \dfrac{\infty}{1+0} = \infty$

② Justificați cu definiția val limitelor

a) $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}} = 0$

b) $\lim\limits_{n\to\infty} \dfrac{n^2}{n+1} = +\infty$

a) $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0 \iff \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ a.î. $\forall n \geq n_0, \left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon$

$$\frac{1}{\sqrt{n}} < \varepsilon \Rightarrow \sqrt{n} > \frac{1}{\varepsilon} \Rightarrow \underbrace{n > \frac{1}{\varepsilon^2}}_{n \geq n_0 > \frac{1}{\varepsilon^2}}$$

alegem $n_0 = \left[\frac{1}{\varepsilon^2}\right] + 1$

b) $\lim\limits_{n\to\infty} \frac{n^2}{n+1} = +\infty$

$\iff \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ a.î. $\forall n \geq n_0$

$$\frac{n^2}{n+1} > \varepsilon$$

$$\frac{n^2 - 1 + 1}{n+1} = \frac{(n+1)(n-1)}{n+1} + \frac{1}{n+1} =$$

$$= (n-1) + \frac{1}{n+1} > n-1 > \varepsilon$$

$$n > \varepsilon + 1$$

alegem $n_0 = [\varepsilon] + 2$

$$n \geq n_0 > [\varepsilon] + 2 > \varepsilon + 1$$

③ Indicați convergența și limita acolo unde este posibil

a) $x_n = a^n, a \in \mathbb{R}$

b) $x_n = \frac{2^n}{n!}$

c) $x_n = \sqrt[n]{n}$

d) $x_n = \left(1 + \frac{1}{n}\right)^n$

e) $x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}$

f) $x_n = \frac{\sin(1!)}{1\cdot 2} + \frac{\sin(2!)}{2\cdot 3} + \dots + \frac{\sin(n!)}{n(n+1)}$

a) $\lim_{n\to\infty} a^n = \begin{cases} 0, & a\in(-1,1) \\ 1, & a=1 \\ \infty, & a>1 \\ \nexists, & a\le -1 \end{cases}$

$a=-2$

$x_n=(-2)^n$

$x_{2k}=2^{2k}\to\infty$

$x_{2k+1}=-2^{2k+1}\to-\infty$

$\Rightarrow \nexists \lim_{n\to\infty} x_n$

b) $x_n=\frac{2^n}{n!}$

$\lim_{n\to\infty}\frac{2^n}{n!}=$

$\frac{x_{n+1}}{x_n}=\frac{\frac{\cancel{2^{n+1}}}{(n+1)!}}{\frac{\cancel{2^n}}{\cancel{n!}}}=\frac{2}{n+1}$

$\frac{2}{n+1}\le 1, \forall n\ge 1$

$\Rightarrow (x_n)$ șir descrescător (1)

$x_n>0, \forall n\in\mathbb{N} \Rightarrow (x_n)$ mărginit inf (2)

din (1), (2) $\Rightarrow (x_n)$ convergent

$\Rightarrow \exists \lim_{n\to\infty} x_n=l\in\mathbb{R}$

$\frac{x_{n+1}}{x_n}=\frac{2}{n+1} \quad \Big|\, x_n$

$x_{n+1}=\frac{2}{n+1}\cdot x_n \quad \Big|\, n\to\infty$

$\lim_{n\to\infty} x_{n+1}=\lim_{n\to\infty}\frac{2}{n+1}\cdot\lim_{n\to\infty} x_n \Rightarrow l=0\cdot l \Rightarrow l=0$

c) $x_n=\sqrt[n]{n}$

$y_n=\sqrt[n]{n}-1\ge 0, \forall n\ge 1$

$\sqrt[n]{n}=1+y_n \;\; |^n$

$n=(1+y_n)^n=1+C_n^1 y_n+\underbrace{C_n^2 y_n^2}+\dots+C_n^n\cdot y_n^n > C_n^2\cdot y_n^2$

$\Rightarrow n>\frac{n(n-1)}{2}\cdot y_n^2 \Rightarrow y_n^2<\frac{2}{n-1} \Rightarrow 0\le y_n<\sqrt{\frac{2}{n-1}}, \forall n\ge 2$

$\to 0$

$\Rightarrow \lim_{n\to\infty} y_n=0$

d) $X_n = \left(1+\frac{1}{n}\right)^n$

$$X_n = 1 + C_n^1 \cdot \frac{1}{n} + C_n^2 \frac{1}{n^2} + \ldots + C_n^n \frac{1}{n^n} =$$

$$= 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \ldots + \frac{n(n-1)}{n!} \cdot \frac{1}{n^n} =$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right)$$

$$X_{n+1} = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \ldots + \frac{1}{n!}\left(1-\frac{1}{n+1}\right)\cdot\ldots\cdot\left(1-\frac{n-1}{n+1}\right)$$

$$\underbrace{+\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)}_{>0}$$

$\Rightarrow X_{n+1} > X_n, \forall n \geq 1 \Rightarrow (X_n)$ s. cresc.

$$X_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} < 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}\right) = 1 + \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} =$$

$$= 1 + 2\underbrace{\left(1-\frac{1}{2^n}\right)}_{<1} < 3$$

$$\frac{1}{n!} = \frac{1}{1\cdot 2\cdot 3\cdot\ldots\cdot n} < \frac{1}{1\cdot \underbrace{2\cdot\ldots\cdot 2}_{n-1}} = \frac{1}{2^{n-1}} \quad \forall n \geq 3 \Rightarrow$$

$\Rightarrow (X_n)$ mărg sup $\Rightarrow (X_n)$ conv $\Rightarrow \exists \lim\limits_{n\to\infty} x_n \overset{not}{=} e \simeq 2{,}71$

e) (X_n) s crescător
$X_n < 3, \forall n \in \mathbb{N} \Rightarrow (X_n)$ mărg sup $\Big| \Rightarrow (X_n)$ conv $\Rightarrow \exists\, L = \lim\limits_{n\to\infty} X_n$

notam $Y_n = \left(1+\frac{1}{n}\right)^n$

$\overset{d)}{\Rightarrow} Y_n < X_n, \forall n \geq 1 \Rightarrow \lim\limits_{n\to\infty} Y_n \leq \lim\limits_{n\to\infty} X_n \Rightarrow e \leq L$

Fie $p \in \mathbb{N}^*$ fixat și $n > p$

$\overset{d}{\Longrightarrow} Y_n \quad 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \ldots + \frac{1}{p!}\left(1-\frac{1}{n}\right)\cdot\ldots\cdot\left(1-\frac{p-1}{n}\right), \forall n \geq 1$

$n\to\infty \Rightarrow \lim\limits_{n\to\infty} Y_n \geq 1 + 1 + \frac{1}{2!} + \ldots + \frac{1}{p!} = X_p \Rightarrow e \geq X_p, \forall p \geq 1, p \to \infty$

$\Rightarrow \boxed{e \geq \lim\limits_{p\to\infty} X_p = L} \Rightarrow L = e$

f) $x_n = \frac{\sin(1!)}{1\cdot 2} + \frac{\sin(2!)}{2\cdot 3} + \dots + \frac{\sin(n!)}{n(n+1)}$

x_n fundamental $\overset{\text{dacă}}{\Longleftrightarrow}$

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ a.î. $\forall n \geq n_0 \ \forall p \in \mathbb{N}$

$|x_{n+p} - x_n| < \varepsilon$

$|x_{n+p} - x_n| =$

$$\left| \cancel{\frac{\sin(1!)}{1\cdot 2}} + \cancel{\frac{\sin(2!)}{2\cdot 3}} + \dots + \cancel{\frac{\sin(n!)}{n(n+1)}} + \dots + \frac{\sin((n+p)!)}{(n+p)(n+p+1)} - \left(\cancel{\frac{\sin(1!)}{1\cdot 2}} + \cancel{\frac{\sin(2!)}{2\cdot 3}} + \dots + \cancel{\frac{\sin(n!)}{n(n+1)}} \right) \right|$$

$$\left| \frac{\sin((n+1)!)}{(n+1)(n+2)} + \dots + \frac{\sin((n+p)!)}{(n+p)(n+p+1)} \right| \leq \frac{|\sin((n+1)!)|}{(n+1)(n+2)} + \dots + \frac{|\sin((n+p)!)|}{(n+p)(n+p+1)}$$

$$\leq \frac{1}{(n+1)(n+2)} + \dots + \frac{1}{(n+p)(n+p+1)} \leq \frac{1}{n+1} - \cancel{\frac{1}{n+2}} + \cancel{\frac{1}{n+2}} - \cancel{\frac{1}{n+3}} + \dots + \cancel{\frac{1}{n+p}} - \frac{1}{n+p+1}$$

$$= \frac{1}{n+1} - \frac{1}{n+p+1} < \frac{1}{n+1} < \varepsilon$$

$\frac{1}{n+1} < \varepsilon \iff n+1 > \frac{1}{\varepsilon} \iff n > \frac{1}{\varepsilon} - 1$

Aleg $n_0 = \left[\frac{1}{\varepsilon}\right] \Rightarrow x_n$ șir fundamental $\Rightarrow$

$\Rightarrow x_n$ șir convergent.