① Fie $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=x^3-3xy^2-15x-12y$ și $a=(-2,-1)$. Precizați:

a) $\nabla f(a)$, $H(f)(a)$, $d^2f(a)$

b) natura punctului a

a) $$\frac{\partial f}{\partial x}(x,y)=3x^2+3y^2-15$$

$$\frac{\partial f}{\partial y}(x,y)=6xy-12$$

$$\nabla f(a)=(12+3-15,\ 12-12)=(0,0)$$

$$H(f)(x,y)=\begin{pmatrix}\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y\partial x}\\ \frac{\partial^2 f}{\partial x\partial y} & \frac{\partial^2 f}{\partial y^2}\end{pmatrix}=\begin{pmatrix}6x & 6y\\ 6y & 6x\end{pmatrix}$$

$$H(f)(a)=\begin{pmatrix}-12 & -6\\ -6 & -12\end{pmatrix}$$

$$d^2f(a)(u_1,u_2)=-12u_1^2-6u_1u_2-6u_2u_1-12u_2^2=$$

$$=-12u_1^2-12u_1u_2-12u_2^2=$$

$$=-12(u_1^2+u_1u_2+u_2^2)$$

b) $\nabla f(a)=0_2 \Rightarrow$ a punct critic

pct minim local
pct maxim local
pct şa.

$\Delta_1 = -12$

$\Delta_2 = \begin{vmatrix} -12 & -6 \\ -6 & -12 \end{vmatrix} = 144 - 36 > 0$ | $\xrightarrow{\text{Sylvester}}$

$d^2 f(a)$ neg definită

$\Rightarrow a$ max local

② Det pt critice şi natura lor pt

a) $f: \mathbb{R}^3 \to \mathbb{R}, \; f(x,y,z) = 2x^2 - xy + 2xz - y + y^3 + z^2$

a) $f: \mathbb{R}^2 \to \mathbb{R}, \; f(x,y) = x^4 + y^4 - 2x^2$

a) $\frac{df}{dx}(x,y,z) = 4x - y + 2z$

$\frac{df}{dy}(x,y,z) = -x - 1 + 3y^2$

$\frac{df}{dz}(x,y,z) = 2x + 2z$

$\nabla f(x,y,z) = \begin{cases} 4x - y + 2z = 0 \\ -x - 1 + 3y^2 = 0 \\ 2x + 2z = 0 \end{cases} \Leftrightarrow \Big\{$

$$\begin{cases} 2x = -2z \\ -4z - y + 2z = 0 \\ -z - 1 + 3y^2 = 0 \end{cases} \Leftrightarrow \begin{cases} x = -z \\ -y = -2z \Rightarrow y = 2z \\ 12z^2 - z - 1 = 0 \end{cases}$$

$\Delta = 1 + 48$

$z_{1,2} = \frac{1 \pm 7}{24}$

$a = \left(-\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}\right)$

$b = \left(\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right)$

$$H(f) = \begin{pmatrix} 4 & -1 & 2 \\ -1 & 6y & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$H(f)(a) = \begin{pmatrix} 4 & -1 & 2 \\ -1 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$\Delta_1 = 4 > 0$

$\Delta_2 = \begin{vmatrix} 4 & -1 \\ -1 & 4 \end{vmatrix} = 15 > 0$

$\Delta_3 = 14 > 0$

$\Rightarrow d^2(f)(a)$ poz def.

pct de min.

$$H(f)(a) = \begin{pmatrix} 4 & -1 & 2 \\ -1 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$\Delta_1 = 4 > 0$ $\Delta_2 = \begin{vmatrix} 4 & -1 \\ -1 & -3 \end{vmatrix} = -12 - 1 = -13 < 0$

$\delta_3 = -14 < 0$

$d^2(f)(a)(u_1, u_2, u_3) = 4u_1^2 - 3u_2^2 + 2u_3^2 - 2u_1u_2 + 4u_1u_3$

$d^2(f)(a)(1, 0, 0) = 4 > 0$

$\Rightarrow$ indef.

$d^2(f)(a)(0, 1, 0) = -3 < 0$

$\Rightarrow a$ - pct ș.a.

b) $f(x, y) = x^4 + y^4 - 2x^2$

$\frac{\partial f}{\partial x} = 4x^3 - 4x$

$\frac{\partial f}{\partial y} = 4y^3$

$\begin{cases} 4y^3 = 0 \Rightarrow y = 0 \\ 4x^3 - 4x = 0 \Rightarrow x^3 - x = 0 \end{cases}$

$x(x^2 - 1) = 0 \nearrow x = 0$

$x = \pm 1$

$a = (0, 0)$
$b = (1, 0)$
$c = (-1, 0)$

$H(f) = \begin{pmatrix} 12x^2 - 4 & 0 \\ 0 & 12y^2 \end{pmatrix}$

$$H(f)(0,0) = \begin{pmatrix} -4 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} \Delta_1 = -4 \\ \Delta_2 = 0 \end{matrix}$$

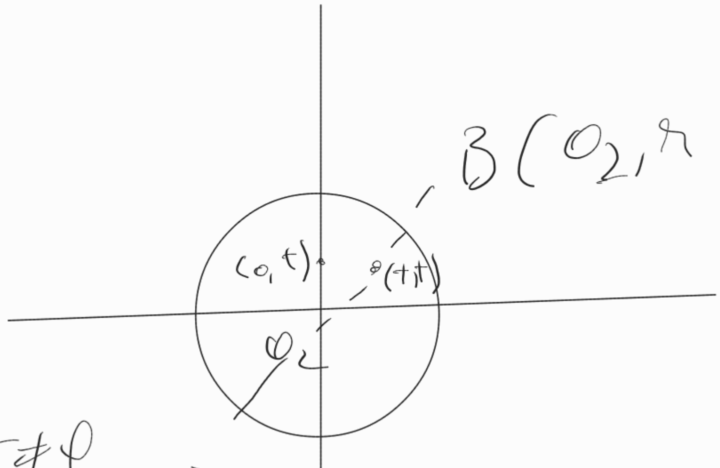
$$d^2 f(0,0)(u_1, u_2) = -4u_1^2 \le 0$$

negative
semi definită

$$H(f)(1,0) = \begin{pmatrix} 8 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} \Delta_1 = 8 \\ \Delta_2 = 0 \end{matrix}$$

$$d^2 f(1,0)(u_1, u_2) = 8u_1^2 \ge 0$$

analol $(-1,0)$



$a = (0,0)$

$f(0,0) = 0$

$\forall\, r > 0$

$f(0,t) = t^4 > 0, \; t \neq 0$

$f(t,t) = 2t^4 - 2t^2 = 2t^2(t^2-1)$

$= 2t^2(t-1)(t+1) \quad \forall\, t \in (-1,1)$

conc: $(0,0)$ punct şa

$f(1,0) = f(-1,0) = -1$

$f(x,y) = \underbrace{(x^2-1)^2 + y^4}_{\geq 0} - 1 \geq -1 \quad \forall (x,y) \in \mathbb{R}^2$

$\Rightarrow$ pct minim global

③ Det pct de extrem condiționat și val extreme ale lui f relativ la S

a) $f: \mathbb{R}^2 \to \mathbb{R}, \ f(x,y) = (1-x)(1-y)$

$S = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$

S compactă, f cont $\xRightarrow{t.W}$ f își atinge val extreme $\Rightarrow$ f are pct de minim și maxim cond relativ la S

$F(x,y) = x^2+y^2-1$; $L(x,y,\lambda) = f(x,y) + \lambda F(x,y)$

$L(x,y,\lambda) = (1-x)(1-y) + \lambda(x^2+y^2-1)$

$= 1 - x - y + xy + \lambda x^2 + \lambda y^2 - \lambda$

$\frac{\partial L}{\partial x} = -1 + y + 2\lambda x$

$$\frac{dL}{dy} = -1 + x + 2\lambda y$$

$$\begin{cases} -1 + y + 2\lambda x = 0 \\ -1 + x + 2\lambda y = 0 \\ x^2 + y^2 - 1 = 0 \end{cases}$$

$$\frac{dL}{d\lambda} = x^2 + y^2 - 1$$

$$\Rightarrow \left\{ 2\lambda x - 2\lambda y + y - x \Leftrightarrow (y - x)(1 - 2\lambda) = 0 \right.$$

$$y = x$$

$$2y^2 = 1$$

$$y^2 = \frac{1}{2}$$

$$y = \pm \frac{\sqrt{2}}{2}$$

$$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \lambda_1\right)$$

$$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \lambda_2\right)$$

— pct critice

CII $\lambda = \frac{1}{2}$

$$-1 + x + y = 0$$

$$x + y = 1$$

$$x = 1 - y$$

$$(1 - y)^2 + y^2 - 1 = 0$$

$1-2y+2y^2-1=0$

$y_1=0 \quad x_1=1 \quad (0,1,\frac{1}{2})$

$y_2=0 \quad x_2=0 \quad (1,0,\frac{1}{2})$ pct crit

$f(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = (1-\frac{\sqrt{2}}{2})^2 > 0$

$f(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) = (1+\frac{\sqrt{2}}{2})^2 > 0$ - maximul lui f|S

$f(0,1)=0$

$f(1,0)=0$

$(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ pct de maxim condit

$(0,1)(1,0)$ - pct de min. condit.

b) $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z)=xyz$

$S=\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; \begin{array}{l} x+y+z=0 \\ x^2+y^2+z^2=1 \end{array}\right\}$

S compactă
f cont $\overset{\text{TW.}}{\Rightarrow}$ f are pct de minim și maxim cond relativ. la S

$$F_1(x,y,z) = x+y+z$$

$$F_2(x,y,z) = x^2+y^2+z^2-1$$

$$L(x,y,z,\lambda_1,\lambda_2) = f(x,y,z) + \lambda_1 F_1(x,y,z) + \lambda_2 F_2(x,y,z)$$

$$= xyz + \lambda_1(x+y+z) + \lambda_2(x^2+y^2+z^2-1)$$

$$\nabla L = 0_5 \Rightarrow \begin{cases} yz + \lambda_1 + 2\lambda_2 x = 0 \\ xz + \lambda_1 + 2\lambda_2 y = 0 \\ xy + \lambda_1 + 2\lambda_2 z = 0 \\ x+y+z = 0 \\ x^2+y^2+z^2 = 0 \end{cases}$$

$$\overset{(+)}{\Rightarrow} yz+xz+xy+3\lambda_1 = 0$$

$$?^2 = x^2+y^2+z^2+2\underbrace{(xy+xz+yz)}_{=0}$$

$$= 1+2(-3\lambda_1) = 0$$