4. Det $\lim (x_n)$ ∄

a) $x_n = (-1)^n \cdot n \cdot \sin \frac{n\pi}{2}$

b) $x_n = \left(1 + \frac{\cos(n\pi)}{n}\right)^n$

a) $x_{2k} = (-1)^{2k} \cdot 2k \cdot \sin \frac{2k\pi}{2} = 2k \cdot \sin k\pi = 0, \forall k \in \mathbb{N}$

$x_{2k+1} = (-1)^{2k+1} \cdot (2k+1) \cdot \sin \frac{(2k+1)\pi}{2} = (-1)(2k+1) \cdot \sin \frac{(2k+1)\pi}{2}$

$= (-1)(2k+1) \cdot \sin\left(k\pi + \frac{\pi}{2}\right)$

$= (-1)^{k+1}(2k+1)$

$k = 2p, p \in \mathbb{N}$

$x_{4p+1} = (-1)^{2p+1}(4p+1) = (-1)(4p+1) \to -\infty$

$k = 2p+1, p \in \mathbb{N}$

$x_{4p+3} = (-1)^{2p+2}(4p+3) \longrightarrow \infty$

$\mathrm{Lim}(x_n) = \{-\infty, 0, +\infty\}$

Criteriul Stolz - Cesaro

(a_n) șir oarecare

(b_n) șir strict monoton și divergent,

Dacă există următoarea limită $\exists \lim\limits_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = l$

$\Rightarrow \exists \lim\limits_{n\to\infty} \frac{a_n}{b_n} = l$

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1) (x_n) șir termeni strict pozitivi

$a_n = \ln x_n$

$b_n = n \nearrow +\infty$

$\lim\limits_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim\limits_{n\to\infty} \frac{\ln x_{n+1} - \ln x_n}{(n+1) - n} = \lim\limits_{n\to\infty} \ln \frac{x_{n+1}}{x_n} = \ln\left(\underbrace{\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n}}_{L \in [0, +\infty)}\right) =$

$= \ln L$, cu convenția că $\ln 0 = -\infty$
$\ln(+\infty) = +\infty$

$\overset{S.C.}{\Longrightarrow} \ln L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln x_n}{n} = \lim_{n\to\infty} \frac{1}{n} \cdot \ln x_n$

$\lim_{n\to\infty} \ln \sqrt[n]{x_n} = \ln\left(\lim_{n\to\infty} \sqrt[n]{x_n}\right)$

$\boxed{\Rightarrow \lim_{n\to\infty} \sqrt[n]{x_n} = L = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}}$ Consecința lui. S.C.

(2) Calculați $\lim_{n\to\infty} y_n$

a) $y_n = \dfrac{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}}{\ln n}$

b) $y_n = \sqrt[n]{n!}$

c) $y_n = \dfrac{\sqrt[n]{n!}}{n}$

a) $a_n = 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}$

$b_n = \ln n \nearrow +\infty$

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{1+\frac{1}{2}+\dots+\frac{1}{n+1} - 1 - \frac{1}{2} - \dots - \frac{1}{n} - \frac{1}{n}}{\ln(n+1) - \ln n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\ln \frac{n+1}{n}} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{1}{\ln \frac{n+1}{n}} = \lim_{n\to\infty} \frac{1}{(n+1)\ln \frac{n+1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\ln\left(\frac{n+1}{n}\right)^{n+1}} = \lim_{n\to\infty} \frac{1}{\ln\left[\left(1+\frac{1}{n}\right)^n\right]^{\frac{n+1}{n}}} = \frac{1}{\ln e} = 1$$

b) $y_n = \sqrt[n]{n!}$ $\quad x_n = n!$ $\quad \lim_{n\to+\infty} \sqrt[n]{n!} = \lim_{n\to\infty} \frac{(n+1)!}{n!} = +\infty$

c) $y_n = \frac{\sqrt[n]{n!}}{n} = \sqrt[n]{\frac{n!}{n^n}}$, $x_n = \frac{n!}{n^n}$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\frac{n!\,(n+1)}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{\cancel{(n+1)}\cdot n^n}{(n+1)^{n+\cancel{1}}}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(1+ \frac{n-n-1}{n+1}\right)^n =$$

$$= \lim_{n\to\infty} \left[\left(1+\frac{-1}{n+1}\right)^{-(n+1)}\right]^{-\left(\frac{n}{n+1}\right)} = e^{\lim_{n\to\infty} \frac{-n}{n+1}} = e^{-1} = \frac{1}{e}$$

$$\Rightarrow \lim_{n\to\infty} y_n = \frac{1}{e}$$

$a_n \to 0$

$$\lim_{n\to\infty} (1+a_n)^{\frac{1}{a_n}} = e$$

③ Calculati lim :

$a_n = \sum_{k=1}^{n} \frac{1+(-1)^k}{2}$, $b_n = n$, $\forall n \in \mathbb{N}$

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{\frac{1+(-1)^{n+1}}{2}}{(n+1)-n} = \lim_{n\to\infty} \frac{1+(-1)^{n+1}}{2} \ \nexists$$

$$a_n = 0+1+0+1+\ldots+\frac{1+(-1)^n}{2} = \begin{cases} k, & n=2k \Rightarrow k=\frac{n}{2} \\ k, & n=2k+1 \Rightarrow k=\frac{n-1}{2} \end{cases}$$

$$\left.\begin{aligned} &\lim_{k\to\infty} \frac{a_{2k}}{b_{2k}} = \lim_{k\to\infty} = \frac{k}{2k} = \frac{1}{2} \\ &\lim_{k\to\infty} \frac{a_{2k+1}}{b_{2k+1}} = \lim_{k\to\infty} \frac{k}{2k+1} = \frac{1}{2} \end{aligned}\right\} \Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{2}$$

(4) a) $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\dots = \sum_{n=1}^{\infty} \frac{1}{2n-1} = \sum_{n=0}^{\infty} \frac{1}{2n+1}$

b) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots = \sum_{n=0}^{\infty} \frac{1}{2^n}$

c) $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}\dots = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

(5) a) $\sum_{n=0}^{\infty} \frac{1}{n!}$ f) $\sum_{n=0}^{\infty} \frac{n \cdot 2^n}{(n+2)!}$

b) $\sum_{n=1}^{\infty} \frac{1}{5^n}$ g) $\sum_{n=1}^{\infty} \operatorname{arctg} \frac{1}{n^2+n+1}$

c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n-1}}$

d) $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

e) $\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right)$

a) $S = \lim_{n\to\infty} \left(1+\frac{1}{1!}+\dots\frac{1}{n!}\right) = e$

b) $\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n - 1 = \frac{1}{1-\frac{1}{5}} - 1 = \frac{5}{4} - 1 = \frac{1}{4}$

$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \forall a \in (-1,1)$

c) $S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}+\sqrt{k-1}} = \sum_{k=1}^{n} \frac{\sqrt{k}-\sqrt{k-1}}{k-(k-1)} = \sum_{k=1}^{n} \sqrt{k}-\sqrt{k-1} =$

$= \cancel{\sqrt{1}} - \sqrt{0} + \cancel{\sqrt{2}} - \cancel{\sqrt{1}} + \dots + \sqrt{n} - \cancel{\sqrt{n-1}} = \sqrt{n}$

$\Rightarrow S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sqrt{n} = \infty$ divergentă

d) $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

$$S_n = \sum_{k=1}^{n} \frac{1}{4k^2-1}$$

$$S_n = \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) =$$

$$= \frac{1}{2}\left(\left(1 - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \cancel{\ldots} + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\right)$$

$$= \frac{1}{2}\left(\frac{2n+1-1}{2n+1}\right) =$$

$$= \frac{n}{2n+1} \Rightarrow S = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$$

e) $\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right)$

$$S_n = \sum_{k=2}^{n} \ln\left(1 - \frac{1}{k^2}\right) = \sum_{k=2}^{n} \ln\left(\frac{(k+1)(k-1)}{k^2}\right) = \sum_{k=2}^{n} \ln\left(\frac{k+1}{k} \cdot \frac{k-1}{k}\right)$$

$$= \sum_{k=2}^{n} \ln\left(\frac{k+1}{k}\right) + \ln\left(\frac{k-1}{k}\right) = \sum_{k=2}^{n} \ln\left(\frac{k+1}{k}\right) - \ln\left(\frac{k}{k-1}\right)$$

$$= \cancel{\ln\frac{3}{2}} - \ln 2 + \cancel{\ln\frac{4}{3}} - \cancel{\ln\frac{3}{2}} + \ldots + \ln\frac{n+1}{n} - \cancel{\ln\frac{n}{n-1}} =$$

$$= \ln\frac{n+1}{n} - \ln 2$$

$$S = \lim_{n\to\infty} \ln\frac{n+1}{n} - \ln = -\ln 2$$

f) $\sum_{n=0}^{\infty} \frac{n \cdot 2^n}{(n+2)!}$

$$\frac{n \cdot 2^n}{(n+2)!} = X + \frac{1}{(n+2)!}$$

$$S_n = \sum_{k=0}^{n} \frac{k \cdot 2^k}{(k+2)!} = \sum_{k=0}^{n} \frac{(k+2-2)\cdot 2^k}{(k+2)!} =$$

$$= \sum_{k=0}^{n} \frac{(k+2)\cdot 2^k}{(k+2)!} - \frac{2^{k+1}}{(k+2)!}$$

$$= \sum_{k=0}^{n} \frac{2^k}{(k+1)!} - \frac{2^{k+1}}{(k+2)!}$$

$$= \frac{2^0}{1!} - \frac{2^1}{2!} + \frac{2^1}{2!} - \frac{2^2}{3!} + \dots + \frac{2^n}{(n+1)!} - \frac{2^{n+1}}{(n+2)!}$$

$$= 1 - \frac{2^{n+1}}{(n+2)!}$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{2^{n+1}}{(n+2)!}\right) = 1$$

$\operatorname{arctg} \alpha - \operatorname{arctg} \beta = ?$

formula