① Calculați derivatele parțiale, ∇f și df pentru

gradientul

a) $f:\mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^2y^3 + y\sin x - 2z$

$$\frac{df}{dx}(x,y,z) = 2xy^3 + y\cos x$$

$$\frac{df}{dy} = 3x^2y^2 + \sin x$$

$$\frac{df}{dz} = -2$$

$$\nabla f = \left(\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz}\right)$$

$$\left(\frac{f}{g}\right)'$$

$df(x,y,z): \mathbb{R}^3 \to \mathbb{R}$

$$df(x,y,z)(u_1,u_2,u_3) = \nabla f(x,y,z) \cdot u =$$

$$= (2xy^3 + y\cos x)u_1 + (3x^2y^2 + \sin x)u_2 - 2u_3$$

b) $f:(0,\infty)^2 \to \mathbb{R}$, $f(x,y) = \operatorname{arctg}\frac{x-y}{x+y}$

$$\frac{df}{dx}(x,y) = \frac{1}{1+\left(\frac{x-y}{x+y}\right)^2} \cdot \left(\frac{x-y}{x+y}\right)'_x =$$

$$= \frac{1}{1+\left(\frac{x-y}{x+y}\right)^2} \cdot \frac{(x+y)-(x-y)}{(x+y)^2}$$

$$= \frac{(x+y)^2}{(x+y)^2+(x-y)^2} \cdot \frac{2y}{(x+y)^2} =$$

$$= \frac{2y}{2x^2+2y^2} = \frac{y}{x^2+y^2}$$

$$\frac{df}{dy}(x,y) = \frac{1}{1+\left(\frac{x-y}{x+y}\right)^2} \left(\frac{x-y}{x+y}\right)'_y = \frac{(x+y)^2}{(x+y)^2+(x-y)^2} \cdot \frac{(-1)(x+y)-(x-y)}{(x+y)^2}$$

$$= \frac{-2x}{2x^2+2y^2} = \frac{-x}{x^2+y^2}$$

c) $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x\sqrt{x^2+y^2}$

$$\frac{df}{dx} = (x)'\sqrt{x^2+y^2} + x\left(\sqrt{x^2+y^2}\right)'_x$$

$$= \sqrt{x^2+y^2} + x\frac{x}{\sqrt{x^2+y^2}}$$

$$= \frac{x^2+y^2+x^2}{\sqrt{x^2+y^2}} = \frac{2x^2+y^2}{\sqrt{x^2+y^2}}$$

$$\frac{df}{dy} = \left(x\sqrt{x^2+y^2}\right)'_y = 0 + x \cdot \frac{2y}{2\sqrt{x^2+y^2}} = \frac{xy}{\sqrt{x^2+y^2}}$$

$$(x,y) \neq 0_2$$

$$\frac{df}{dx}(0,0) = \frac{f(x,0) - f(0,0)}{x-0} = \frac{x \cdot \sqrt{x^2} - 0}{x} = \frac{x \cdot |x|}{x} = |x|$$

$$\lim_{x \to 0} |x| = 0$$

$$\frac{df}{dy}(0,0) = \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y-0} = \frac{0-0}{y-0} = \lim_{y \to 0} \frac{0}{y} = 0$$

② functia $f(x,y) = y \cdot \ln(x^2 - y^2)$ verifică

$$\frac{1}{x} \cdot \frac{df}{dx} + \frac{1}{y}\frac{df}{dy} = \frac{f(x,y)}{y^2} \qquad (f-g)' = f'$$

$$\frac{df}{dx} = y \cdot \ln(x^2 - y^2) = y \cdot \frac{1}{x^2 - y^2} \cdot 2x$$

$$= \frac{2xy}{x^2 - y^2} \cdot \frac{1}{x} = \frac{2y}{x^2 - y^2}$$

$$\frac{df}{dy} = (y)'_y \cdot \ln(x^2 - y^2 + y\,(\ln(x^2 - y^2))'_y$$

$$= \ln(x^2 - y^2) + y \cdot \frac{1}{x^2 - y^2}(x^2 - y^2)'_{y^2}$$

$$= \ln(x^2 - y^2) - \frac{2y^2}{x^2 - y^2}$$

$$\frac{2y}{x^2-y^2} + \frac{1}{y}\left(\ln(x^2-y^2) - \frac{2y^2}{x^2-y^2}\right) =$$

$$= \cancel{\frac{2y}{x^2-y^2}} + \frac{\ln(x^2-y^2)}{y} - \cancel{\frac{2y}{x^2-y^2}} =$$

$$= \frac{\ln(x^2-y^2)}{y} = \frac{y \cdot \ln(x^2-y^2)}{y^2} = \frac{f(x,y)}{y^2}$$

(3) $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \frac{x^2 \cdot y}{x^4+y^2}, & (x,y) \neq O_2 \\ 0, & (x,y) = O_2 \end{cases}$$

Calc. deriv. partiale în $(0,0)$ și deriv după direcție în $(0,0)$

$v = (v_1, v_2) \in \mathbb{R}$

$$f'_v(0,0) = \lim_{t\to 0} \frac{1}{t}\left[f((0,0)+t\cdot v) - f(0,0)\right] =$$

$$= \lim_{t\to 0} \frac{1}{t}\left[f(tv_1, tv_2) - f(0,0)\right]$$

$$\frac{df}{dx}(0,0) = 0$$

$$\frac{df}{dy} = 0$$

$$v = (v_1, v_2); \quad f'_v(0,0) = \lim_{t \to 0} \frac{1}{t}\left[\frac{(tv_1)^2 \cdot tv_2}{(tv_1)^4 + (tv_2)^2} - 0\right]$$

$$= \lim_{t \to 0} \frac{1}{\cancel{t}}\left(\frac{t^{\cancel{3}} \cdot v_1^2 \cdot v_2}{\cancel{t^2}(t^2 v_1^4 + v_2^2)}\right) =$$

$$= \lim_{t \to 0} \frac{v_1^2 \cdot v^2}{t^2 v_1^4 + v_2^2} = \frac{v_1^2 v_2}{v_2^2} = \frac{v_1^2}{v_2}, \quad v_2 \neq 0$$

Dacă $v_2 = 0$
$f'_v(0,0) = 0$ $\Big\} \Rightarrow$ funcția este derivabilă după orice direcție

(4) Se consideră funcțiile

$f: \mathbb{R}^2 \to \mathbb{R}^3$

$$f(x,y) = (\underbrace{x^2 - y}_{f_1}, \underbrace{3x - 2y}_{f_2}, \underbrace{2xy + y^2}_{f_3})$$

și $g = g(u, v, w): \mathbb{R}^3 \to \mathbb{R}$ de clasă C^1

a) $J(f)(1,1)$

a) D.p ale $g \circ f$

$$J(f)(x,y) = \begin{pmatrix} \frac{df_1}{dx}, \frac{df_1}{dy} \\ \frac{df_2}{dx}, \frac{df_2}{dy} \\ \frac{df_3}{dx}, \frac{df_3}{dy} \end{pmatrix} = \begin{pmatrix} 2x & -1 \\ 3 & -2 \\ 2y & 2x+2y \end{pmatrix}$$

$$J(f)(1,1) = \begin{pmatrix} 2 & -1 \\ 3 & -2 \\ 2 & 4 \end{pmatrix}$$

e)

$g \circ f : \mathbb{R}^2 \to \mathbb{R}$

$$(g \circ f)(x,y) = g(f(x,y)) = g(x^2 - y, 3x - 2y, 2xy + y^2)$$

$$\nabla (g \circ f)(x,y) = \nabla g(f(x,y)) \cdot J(f)(x,y)$$

$$\Rightarrow \left(\frac{d(g \circ f)}{dx}(x,y), \frac{d(g \circ f)}{dy}(x,y) \right) =$$

$$= \left(\frac{dg}{du}(f(x,y)), \frac{dg}{dv}(f(x,y)), \frac{dg}{dw}(f(x,y)) \right) \cdot J(f)(x,y)$$

$$= \frac{d(g\circ f)}{dx}(x,y) = 2x\frac{dg}{du}(f(x,y)) + 3\frac{dg}{dv}(f(x,y)) +$$
$$2y\frac{dg}{dw}(f(x,y))$$

⑤ Deriv part. de ord 2. M

$f: \mathbb{R}\times(0,+\infty)\to\mathbb{R}$,

$$f(x,y) = x\cdot y\cdot e^{\frac{x}{y}}$$

$$\frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2f}{dx^2}$$

$$\frac{d}{dx}\left(\frac{df}{dy}\right) = \frac{d^2f}{dx\,dy}$$

$$\frac{d}{dy}\left(\frac{df}{dx}\right) = \frac{d^2f}{dy\,dx}$$

$$\frac{d}{dy}\left(\frac{df}{dy}\right) = \frac{d^2f}{dy^2}$$

$$\frac{df}{dx} = e^{\frac{x}{y}}(x+y)$$

$$\frac{df}{dy} = x \cdot e^{\frac{x}{y}} + yx \cdot e^{\frac{x}{y}} \cdot \left(-\frac{x}{y^2}\right)$$

$$\frac{df}{dy} = e^{\frac{x}{y}}\left(x + \frac{-x^2}{y}\right)$$

$$\frac{d^2 f}{dx^2} = \left(e^{\frac{x}{y}}(x+y)\right)'_x = \left(e^{\frac{x}{y}}\right)'_x \cdot (x+y) + e^{\frac{x}{y}} \cdot (x+y)'_x$$

$$= e^{\frac{x}{y}}\left(\frac{x}{y} + 2\right)$$

$$\frac{d^2 f}{d(yx)} = \left(e^{\frac{x}{y}}(x+y)\right)'_y = e^{\frac{x}{y}}\left(-\frac{x}{y^2}\right)(x+y) + e^{\frac{x}{y}} =$$

$$= e^{\frac{x}{y}}\left(\frac{x^2}{y^2} - \frac{x}{y} + 1\right)$$