1. Natura seriilor cu termeni poz crit seriile indicate

i) comparației

a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{4n^2-1}}$

b) $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$

ii) cons crit Kummer

a) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

b) $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{\sqrt{n}}$

c) $\sum_{n=1}^{\infty} \left[\frac{(2n)!!}{(2n+1)!!}\right]^2$

i)

a) $\frac{1}{\sqrt{4n^2-1}} > \frac{1}{\sqrt{4n^2}} = \frac{1}{2n}, \forall n \geq 1$

$\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{1}{n}$ div

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{4n^2-1}}$ div

b) Fie $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty} n^2 \ln\left(1+\frac{1}{n^2}\right) =$$

$$= \lim_{n\to\infty} \ln\left(1+\frac{1}{n^2}\right)^{n^2} = 1$$

$1 \in (0,+\infty) \Rightarrow \sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right) \sim \sum_{n=1}^{\infty} \frac{1}{n^2}$ convergentă

ii) a) $\sum_{n=0}^{\infty} \qquad x_n = \frac{2^n}{n!}$

$$\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{2^n}{n!} \cdot \frac{(n+1)!}{2^{n+1}} =$$

$$= \lim_{n\to\infty} \frac{n+1}{2} = \infty > 1 \Rightarrow \sum_{n=1}^{\infty} x_n \text{ conv}$$

b) $x_n = \left(\frac{1}{2}\right)^{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{\left(\frac{1}{2}\right)^{\sqrt{n}}}{\left(\frac{1}{2}\right)^{\sqrt{n+1}}} = \lim_{n\to\infty} 2^{\sqrt{n+1}-\sqrt{n}}$$

$$\lim_{n\to\infty} 2^{\frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}} = \lim_{n\to\infty} 2^{\frac{1}{\sqrt{n+1}+\sqrt{n}}} \to 0$$

$= 1$ nu decide

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) =$$

$$= \lim_{n\to\infty} n\left(2^{\frac{1}{\sqrt{n+1}-\sqrt{n}}} - 1\right)$$

$$= \lim_{x\to 0} \frac{a^x - 1}{x} = \ln a \ \forall a > 0$$

$$= \lim_{n\to\infty} \frac{\left(2^{\frac{1}{\sqrt{n+1}-\sqrt{n}}} - 1\right)}{\frac{1}{\sqrt{n+1}-\sqrt{n}}} \cdot \lim_{n\to\infty} \frac{n}{\sqrt{n+1}-\sqrt{n}}$$

$$= \ln 2 \cdot \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{1+\frac{1}{n}}+1} = +\infty > 1$$

$$\Rightarrow \sum_{n=0}^{\infty} x_n \text{ convergentă}$$

$$x_n = \left(\frac{(2n)!!}{(2n+1)!!}\right)^2$$

$$\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \frac{\left(\frac{(2n)!!}{(2n+1)!!}\right)^2}{\left(\frac{(2n+2)!!}{(2n+3)!!}\right)^2} =$$

$$= \lim_{n\to\infty}\left(\frac{2n+3}{2n+2}\right)^2 = 1$$ Nu decide

$$\lim_{n\to\infty} n\left(\frac{4n^2+12n+9}{4n^2+8n+4} - 1\right)$$

$$\lim_{n\to\infty} n\left(\frac{\cancel{4n^2}+12n+9-\cancel{4n^2}-4-8n}{4n^2+8n+4}\right)$$

$$\lim \frac{4n^2+5n}{4n^2+4+8n} = 1$$ nu decide

$$\lim_{n\to\infty} \ln(n)\left(\frac{4n^2+5n}{4n^2+8n+4} - 1\right)$$

$$\lim_{n\to\infty} \ln(n)\left(\frac{-3n-4}{4n^2+8n+4}\right)$$

$$\lim_{n\to\infty} \frac{\ln(n)}{n} \cdot \frac{-3n^2-4n}{4n^2+8n+4}$$

$$-\frac{3}{4} \cdot \lim_{n\to\infty} \frac{\ln(n)}{n}$$

$$= -\frac{3}{4} \cdot \lim_{n\to\infty} \ln\left(\underset{\underset{1}{\parallel}}{\sqrt[n]{n}}\right) =$$

$$= -\frac{3}{4} \cdot \lim_{n\to\infty} \underset{\underset{0}{\parallel}}{\ln(1)} = 0 < 1$$

divers.

iii) radicalului

$$\sum_{n=1}^{\infty} \frac{n^2}{\left(2+\frac{1}{n}\right)^n}$$

$$x_n = \frac{n^2}{\left(2+\frac{1}{n}\right)^n}$$

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n^2}{\left(2+\frac{1}{n}\right)^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n^2}}{\sqrt[n]{\left(2+\frac{1}{n^2}\right)^n}}$$

$$\lim_{n\to\infty} \frac{\left(\sqrt[n]{n}\right)^2}{2+\frac{1}{n}} = \frac{1}{2} < 1 \Rightarrow \text{convergentă}$$

iv) condensării

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

$p > 0$

$$x_n = \frac{1}{n(\ln n)^p}, \quad p > 0$$

$$x_n = \frac{1}{n(\ln n)^p} \text{ descrescator}$$

$$\overset{CC}{\Rightarrow} \sum_{n=2}^{\infty} x_n \sim \sum_{n=1}^{\infty} 2^n \cdot x_{2^n} =$$

$$= \sum_{n=1}^{\infty} \cancel{2^n} \cdot \frac{1}{\cancel{2^n}(\ln 2^n)^p}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n \cdot \ln 2)^p} =$$

$$= \sum_{n=1}^{\infty} = \frac{1}{(\ln 2)^p} \sum_{n=1}^{\infty} \frac{1}{n^p}$$

este convergentă d.a.n.d $p > 1$

Caz particular

$p = 1 \Rightarrow \sum_{n=2}^{\infty} \frac{1}{n \cdot \ln n}$ divergentă

Studiați convergența și absolut conv urm serii

a) $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{2n+1}{3^n}$

$$a_n = \frac{2n+1}{3^n}$$

$$\frac{a_n}{a_{n+1}} = \frac{2n+1}{\cancel{3^n}} \cdot \frac{\overset{3}{\cancel{3^{n+1}}}}{2n+3} =$$

$$= \frac{2n+1}{2n+3} \cdot 3 > 1 \Rightarrow a_n \text{ desc.}$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{2n+1}{3^n} = 0$$

$$\overset{C.L}{\Longrightarrow} \sum_{n=0}^{\infty} (-1)^n a_n - \text{conv}$$

absolut conv

$\sum_{n=0}^{\infty} |x_n|$ - conv

$$\sum_{n=0}^{\infty} \left|(-1)^n \cdot \frac{2n+1}{3^n}\right| = \sum_{n=0}^{\infty} \frac{2n+1}{3^n}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{6n+3}{2n+3} = 3 > 1 \text{ conv}$$

$\Rightarrow$ abs. conv

b) $\sum_{n=1}^{\infty} \frac{\sin n}{2^n}$

$$\sum_{n=1}^{\infty} \left|\frac{\sin n}{2^n}\right| = Y$$

$\frac{|\sin n|}{2^n} \le \frac{1}{2^n} \quad \forall n\in\mathbb{N}$ | crit comp $\Rightarrow$ Y - conv.

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ = convergentă

$\Rightarrow$ seria init e abs. conv deci și conv

(3) criteriul raportului pt șiruri

(x_n) șir cu termeni strict pozitivi

și $\exists \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = l$ au loc

i) dacă $l > 1 \Rightarrow \lim_{n\to\infty} x_n = 0$

ii) dacă $l < 1 \Rightarrow \lim_{n\to\infty} x_n = +\infty$

Ex: 1) $x_n = \frac{2n+1}{3^n}$

$$\frac{x_n}{x_{n+1}} = \frac{2n+1}{3^n} \cdot \frac{3^{n+1}}{2n+3} \xrightarrow{n\to\infty} 3 > 1 \Rightarrow \lim_{n\to\infty} x_n = 0$$

d) (temă) $x_n = \frac{3^n \cdot n!}{n^2}$, $n \geq 1$

Dem

i) fie s.t.p. $\sum_{n=1}^{\infty} x_n$, $\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = l > 1 \Rightarrow \sum x_n$ conv

$$\Rightarrow \lim_{n\to\infty} x_n = 0$$

ii) fie stp $\sum_{n=1}^{\infty} \frac{1}{x_n}$, $\lim_{n\to\infty} \frac{\frac{1}{x_n}}{\frac{1}{x_{n+1}}} = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{1}{l} > 1 \Rightarrow$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{x_n} \text{ conv} \rightarrow \lim_{n\to\infty} \frac{1}{x_n} = 0_+ \Rightarrow \lim_{n\to\infty} x_n = +\infty$$