① Evaluați:

a) $\displaystyle\int_0^3 \frac{1}{\sqrt{x^2+16}}\,dx = \underbrace{\ln\left(x+\sqrt{x^2+16}\right)}_{F(x)}\Big|_0^3 = F(3)-F(0)$

b) $\displaystyle\int_1^{\sqrt3} \frac{\operatorname{arctg} x}{x^2}\,dx$

c) $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx$

d) $\displaystyle\int_2^4 \frac{\sqrt{x^2-4}}{x}\,dx$

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b) $\displaystyle = \int_1^{\sqrt3} \operatorname{arctg} x \left(-\frac{1}{x}\right)' dx = \operatorname{arctg} x\left(-\frac{1}{3}\right)\Big|_1^{\sqrt3} + \int_1^{\sqrt3} \frac{1}{x^2+1}\cdot\frac{1}{x}\,dx = \frac{\pi}{3}\left(-\frac{1}{\sqrt3}\right)+\frac{\pi}{4}+\int_1^{\sqrt3}\frac{-x}{x^2+1}\,dx+\int_1^{\sqrt3}\frac{1}{x}\,dx =$

$$\frac{1}{x^2+1}\cdot\frac{1}{x} = \frac{Ax+B}{x^2+1}+\frac{C}{x} = \frac{Ax^2+Bx+Cx^2+C}{x(x^2+1)} =$$

$$= \frac{x^2(A+C)+Bx+C}{x(x^2+1)}$$

$$\left.\begin{aligned} &A+C=0 \Rightarrow A=-1\\ &B=0\\ &C=1 \end{aligned}\right| \quad \frac{1}{(x^2+1)x} = \frac{-x}{x^2+1}+\frac{1}{x}$$

$$= \frac{\pi}{2}-\frac{\pi}{3\sqrt3}-\frac{1}{2}\ln(x^2+1)\Big|_1^{\sqrt3}+\ln x\Big|_1^{\sqrt3}$$

Substitutiile trigonometrice pt int algebrice

$R(u,v)$ - funcție rațională, $a>0$

$$\int R(x,\sqrt{a^2-x^2})\,dx, \quad x=a\sin t \text{ sau } x=a\cdot\cos t$$

$$\int R(x,\sqrt{a^2+x^2})\,dx \quad x=a\cdot tg\,t \text{ sau } x=a\,ctg\,t$$

$$\int R(x,\sqrt{x^2-a^2})\,dx \quad x=\frac{a}{\sin t},\ x=\frac{a}{\cos t}$$

c) $$\int_{-1}^{1}\sqrt{1-x^2}\,dx = \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}}\sqrt{1-\sin^2 t}\cdot\cos t\,dt = \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}}|\cos t|\cdot\cos t\,dt$$

$x=\sin t$

$dx=\cos t\,dt$

$x=-1 \Rightarrow \sin t=-1$

$\sin\frac{3\pi}{2}=-1$

$x=1 \rightarrow \sin t=1$

$\sin\frac{\pi}{2}=1$

$$= \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\,dt =$$

$$\cos 2\alpha = 2\cos^2\alpha-1 \Rightarrow \cos^2\alpha=\frac{\cos 2\alpha+1}{2}$$

$$= \left(\frac{1}{2}\left(\int_{\frac{3\pi}{2}}^{\frac{\pi}{2}}\cos 2t\,dt + \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}}1\,dt\right)\right)$$

$$= \frac{1}{2}\left(\frac{\sin 2t}{2} \Big|_{\frac{3\pi}{2}}^{\frac{\pi}{2}} + t \Big|_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \right)$$

d) $x = \frac{2}{\sin t}$, $dx = \left(\frac{2}{\sin t}\right)' dt = \frac{-2\cos t}{\sin^2 t} dt$

$$x = 2 \Rightarrow \frac{2}{\sin t} = 2 \Rightarrow \sin t = 1 \Rightarrow t = \frac{\pi}{2}$$

$$x = 4 \Rightarrow \frac{2}{\sin t} = 4 \Rightarrow \sin t = \frac{1}{2} \Rightarrow t = \frac{\pi}{6}$$

$$\frac{\sqrt{x^2-4}}{x} = \frac{\sqrt{\frac{4}{\sin^2 t} - 4}}{\frac{2}{\sin t}} = \sqrt{\frac{4(1-\sin^2 t)}{\sin^2 t}} \cdot \frac{\sin t}{2} = \left|\frac{\cos t}{\sin t}\right| \cdot \sin t$$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} \left|\frac{\cos t}{\sin t}\right| \cdot \sin t \, \frac{-2\cos t}{\sin^2 t} dt = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos t}{\sin t} \cdot \sin t \cdot \frac{2\cos t}{\sin^2 t} dt$$

$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos^2 t}{\sin^2 t} dt = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1-\sin^2 t}{\sin^2 t} dt = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{1}{\sin^2 t} - 1\right) dt$$

$$= 2\left(-\mathrm{ctg} \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} - t \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} \right)$$