① Justificați că

a) $\frac{1}{n+1} < \ln(n+1) - \ln(n) < \frac{1}{n}, \forall n \in \mathbb{N}^*$

b) Șirul $c_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n$ e convergent

c) Seria $\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^{1+\frac{1}{2}+\dots\frac{1}{n}}$ este divergentă

d) Șirul $S_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{(-1)^{n+1}}{n}$ are limita $\ln 2$.

a) $f(x) = \ln(x)$ $f : [n, n+1] \to \mathbb{R}$

f derivabilă pe $(0, +\infty) \Rightarrow$

$\exists x_0 \in (n, n+1)$ a.î. $\frac{f(n+1) - f(n)}{n+1-n} = f'(x_0)$

$\ln(n+1) - \ln(n) = \frac{1}{x_0}$ (1)

$n < x_0 < n+1$

$$\frac{1}{n} > \frac{1}{x_0} > \frac{1}{n+1} \quad (2)$$

Din (1) și (2) => a)

b) c_n - convergent

monotonie:

$$c_{n+1} - c_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n+1} - \ln(n+1) - 1 - \frac{1}{2} -$$

$$- \dots - \frac{1}{n} + \ln(n)$$

$$= \frac{1}{n+1} + \ln n - \ln n+1$$

$$= \frac{1}{n+1} + \ln\left(\frac{n}{n+1}\right) < 0$$

=> șir (n - monot. desc)

$n = 1 \Rightarrow \ln 2 - \ln 1 < 1$

$n = 2 \Rightarrow \ln 3 - \ln 2 < \frac{1}{2}$

$\vdots$

$n = n \Rightarrow \ln(n+1) - \ln(n) < \frac{1}{n}$ (+)

$$-\underbrace{\ln(1)}_{=0} + \ln(n+1) < 1 + \frac{1}{2} + \dots \frac{1}{n} \Big| - \ln(n)$$

$$0 < \ln(n+1) - \ln(n) < c_n$$

$\Rightarrow c_n > 0 \ \forall n \in \mathbb{N}^*$

$\Rightarrow c_n$ marg inf

$\Rightarrow c_n$ convergent

$\Rightarrow \exists \lim_{n \to \infty} c_n \overset{not}{=} \gamma \simeq 0{,}57$ (constanta lui Euler)

$$\lim_{n \to \infty} \frac{\left(\frac{1}{e}\right)^{1 + \frac{1}{2} + \dots + \frac{1}{n}}}{\left(\frac{1}{e}\right)^{\ln n}} = \lim_{n \to \infty} \left(\frac{1}{e}\right)^{c_n} = \left(\frac{1}{e}\right)^{\gamma} \in (0, \infty)$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^{1 + \frac{1}{2} + \dots \frac{1}{n}} \sim \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^{\ln n} =$$

$= \sum_{n=1}^{\infty} \frac{1}{n}$ divergentă

d)

$$S_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots - \frac{1}{2n} =$$

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2n} - 2\left(\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2n}\right) =$$

$= 1+\frac{1}{2}+\ldots+\frac{1}{2n}-\ln(2n)-\left[1+\frac{1}{2}+\ldots+\frac{1}{n}-\ln n\right]+$

$+\ln(2n)-\ln(n) = C_{2n}-C_n+\ln 2 \Rightarrow$

$\Rightarrow \lim\limits_{n\to\infty} S_{2n} = \cancel{\gamma}-\cancel{\gamma}+\ln 2 = \ln 2$

$\lim\limits_{n\to\infty} S_{2n+1} = \lim\limits_{n\to\infty}\left(S_{2n}+\frac{1}{2n+1}\right) = \ln 2$

$\Rightarrow \exists \lim\limits_{n\to\infty} S_n = \ln 2$

(2) Determinați mulțimea pct de acumulare A' n†

a) $A = \left\{ \frac{1}{2^n} \middle| n \in \mathbb{N} \right\}$

b) $A = \mathbb{Q}$

a) $\lim\limits_{n\to\infty} \frac{1}{2^n} = 0 \Rightarrow 0 \in A' \Rightarrow A' = \{0\}$

b) $\forall x \in \mathbb{R}, \exists (q_n) \subseteq \mathbb{Q} \setminus \{x\}$ a.î. $\lim\limits_{n\to\infty} q_n = x \Rightarrow x \in A' \Rightarrow A' = \overline{\mathbb{R}}$

③ Determinați val extreme și verif dacă acestea se ating pt funcțiile

a) $f:(-1,1)\to\mathbb{R}$, $f(x)=\ln\frac{1-x}{1+x}$

b) $f:[0,1]\to\mathbb{R}$, $f(x)=\begin{cases}\frac{1}{2}, & x=0\\ x, & x\in(0,1]\end{cases}$

c) $f:[-1,1]\to\mathbb{R}$, $f(x)=x\cdot\sqrt{1-x^2}$

a) $A=(-1,1)$

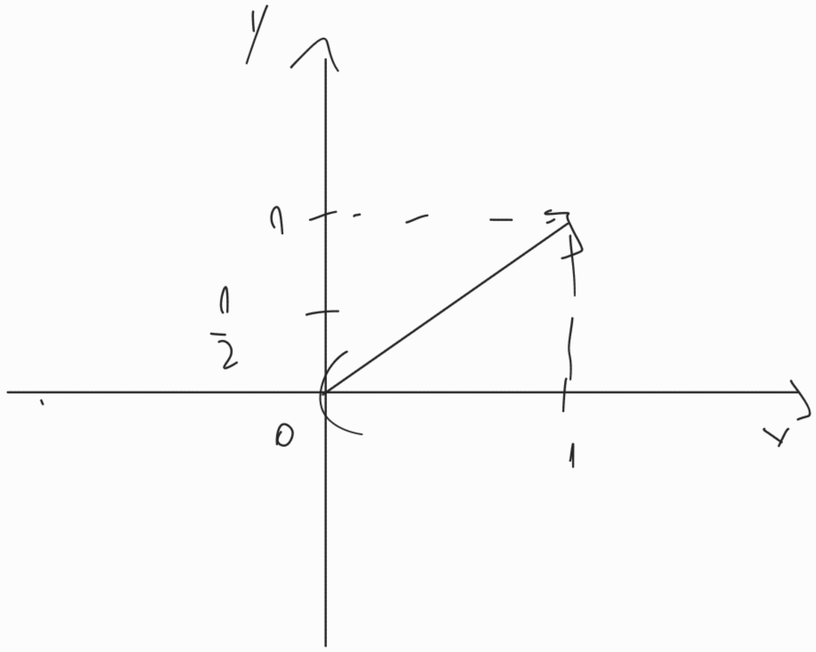
$$\left.\begin{aligned}&\lim_{x\searrow -1}\ln\frac{1-x}{1+x}=\ln\frac{2}{0_+}=+\infty\\ &\lim_{x\nearrow 1}\ln\frac{1-x}{1+x}=\ln\frac{0_+}{2}=\ln 0_+=-\infty\end{aligned}\right\}\Rightarrow$$

$\Rightarrow f(A)=\mathbb{R}$

$\inf f(A)=-\infty$

$\sup f(A)=+\infty$

b)



$A = [0,1]$ $\inf f(A) = 0$ nu se atinge

$f(A) = (0,1]$ $\sup f(A) = 1 = f(1)$

c) f cont pe $[-1,1] \overset{TW}{\Rightarrow} f$ atinge val extreme

$$f'(x) = \left(x \cdot \sqrt{1-x^2}\right)' = \sqrt{1-x^2} + x\left(\frac{-2x}{2\sqrt{1-x^2}}\right) = \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} =$$

$$= \frac{1-x^2-x^2}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}, \forall x \in (-1,1) \quad f'(x) = 0 \Rightarrow$$

$$\Rightarrow \frac{1-2x^2}{\sqrt{1-x^2}} = 0$$

$\sqrt{1-x^2} \neq 0 \Rightarrow 1-2x^2 = 0$ $\quad 2x^2 = 1$

$$x = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$

x	-1		$-\frac{\sqrt{2}}{2}$		$\frac{\sqrt{2}}{2}$		1
$f'(x)$		$- - -$	0	$+ +$	0	$- - -$	
$f(x)$	0	$\searrow$	$-\frac{1}{2}$	$\nearrow$	$\frac{1}{2}$	$\searrow$	0

$$f\left(\pm\frac{\sqrt{2}}{2}\right) = \pm\frac{\sqrt{2}}{2}\sqrt{1-\frac{2}{4}} = \frac{\sqrt{2}}{2} \cdot \sqrt{\frac{1}{2}} = \pm\frac{1}{2}$$

$$A = [-1, 1] \rightarrow f(A) = \left[-\frac{1}{2}, \frac{1}{2}\right]$$

$\inf f(A) = -\frac{1}{2}$ min se atinge

$\sup f(A) = \frac{1}{2}$

(4) Det pct. de extrem locale functiei ant

c) $x = -\frac{\sqrt{2}}{2}$ pct min local

$x = \frac{\sqrt{2}}{2}$ pct max local

$x=-1$ pct de max loc

$x=1$ pct de min loc

a) f - derivabilă pe $(-1,1)$

$f(x)=\ln(1-x)-\ln(1+x)$

$f'(x)=\frac{1}{1-x}(1-x)'-\frac{1}{1+x}(1+x)'$

$f'(x)=-\frac{1}{1-x}-\frac{1}{1+x}$

$f'(x)=\frac{-2}{1-x^2}\neq 0,\ \forall x\in(-1,1)$

$\Rightarrow f$ nu are puncte de extrem local

b) $x=1$ pct maximal

$1=f(1)\geq f(x),\ \forall x\in[0,1)$

$x=0$ pt de max local $\overset{def}{\Leftrightarrow}\exists\,\delta>0$ a.î.

$\forall x\in(-\delta,\delta)\cap[0,1]:\ \frac{1}{2}f(0)\geq f(x)$

pt $\delta = \frac{1}{4} \Rightarrow \forall x \in [0, \frac{1}{4}] : \frac{1}{2} \geq f(x)$

$\Rightarrow x = 0$ pt de maxim local

Calculul unor limite

a) $\lim\limits_{x \to} \frac{e - (1+x)^{\frac{1}{x}}}{x}$; b) $\lim\limits_{x \to \infty} \frac{x^{\alpha}}{e^{x}}$, $\forall \alpha \in \mathbb{R}$

$u = u(x)$

$v = v(x)$

$(u^{v})' = \left(e^{\ln u^{v}}\right)' = \left(e^{v \cdot \ln u}\right)' = e^{v \cdot \ln u} \cdot (v \cdot \ln u)' =$

$= u^{v} \cdot \left[v' \cdot \ln u + v \cdot \frac{u'}{u}\right]$

a) $\lim\limits_{x \to 0} \frac{e - (1+x)^{\frac{1}{x}}}{x} \underset{l'H}{\overset{\frac{0}{0}}{=}} \lim\limits_{x \to 0} \frac{(1+x)^{\frac{1}{x}} \left[-\frac{1}{x^2} \cdot \ln(1+x) + \frac{1}{x} \cdot \frac{1}{1+x}\right]}{1}$

$= -e \cdot \lim\limits_{x \to 0} \frac{(-1-x) \cdot \ln(1+x)}{x^2(1+x)} + \frac{x}{x^2(1+x)} \overset{\left[\frac{0}{0}\right]}{=}$

$= -e \cdot \lim\limits_{x \to 0} \frac{(-1-x)' \cdot \ln(1+x) + \ln(1+x)' \cdot (-1-x) + 1}{2x(1+x) + x^2}$

$= -e \cdot \lim\limits_{x \to 0} \frac{-\ln(1+x) + \frac{1}{1+x-1} \cdot (-1-x) + 1}{2x + 2x^2 + x^2}$

$$= -e \cdot \lim_{x \to 0} \frac{-\ln(1+x)}{3x^2+2x} \overset{\frac{0}{0}}{=} -e \cdot \lim_{x \to 0} \frac{\frac{-1}{1+x}}{6x+2}$$

$$= -e \cdot \lim_{x \to 0} -\frac{1}{(6x+2)(1+x)} = \frac{e}{2}$$

a) I) $\alpha \leq 0 \Rightarrow L = 0$

II) $\alpha > 0$

$$f(x) = \frac{x^\alpha}{e^x} \Rightarrow \ln f(x) = \ln \frac{x^\alpha}{e^x} = 2 \cdot \ln x - x =$$

$$= x\Big(\alpha \cdot \underbrace{\frac{\ln x}{x}}_{\to 0} - 1\Big) \xrightarrow[x \to \infty]{}$$

$$\lim_{x \to \infty} \ln f(x) = -\infty \Rightarrow \lim_{x \to \infty} f(x) = 0$$