1. Calculați integralele

a) $\int_0^{\infty} \frac{\operatorname{arctg} x}{1+x^2}\,dx = \lim_{v\to\infty} \int_0^v \frac{\operatorname{arctg} x}{1+x^2}\,dx = \cancel{*}$

b) $\int_{-1}^{1} \frac{x+1}{\sqrt{1-x^2}}\,dx$

c) $\int_0^{\infty} x^n \cdot e^{-x}\,dx, \; n\in\mathbb{N}$

d) $\int_1^2 \frac{1}{\sqrt{x(2-x)}}\,dx$

$* = \lim_{v\to\infty} \frac{1}{2}\operatorname{arctg}^2 x \Big|_0^v = \lim_{v\to\infty} \frac{1}{2}\operatorname{arctg}^2 v = \frac{1}{2}\left(\frac{\pi}{2}\right)^2$

b) $\int_{-1}^{0} \frac{x+1}{\sqrt{1-x^2}} + \int_0^1 \frac{x+1}{\sqrt{1-x}} = \lim_{u\to -1} \int_u^0 \frac{x+1}{\sqrt{1-x^2}}\,dx + \lim_{v\to 1} \int_0^v \frac{x+1}{\sqrt{1-x^2}}\,dx$

$\int \frac{x+1}{\sqrt{1-x^2}}\,dx = -\int \frac{-x}{\sqrt{1-x^2}}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx =$

$= -\sqrt{1-x^2} + \arcsin x + C$

$= \lim_{u\to -1} \left(-\sqrt{1-x^2} + \arcsin x\right)\Big|_u^0 + \lim_{v\to 1} \left(-\sqrt{1-x^2} + \arcsin x\right)\Big|_0^v$

$$= \lim_{u \searrow -1} \left(-1 + 0 + \sqrt{1-u^2} - \arcsin u \right) +$$

$$+ \lim_{v \nearrow 1} \left(-\sqrt{1-v^2} + \arcsin u - (-1) \right)$$

$$= -1 + \frac{\pi}{2} + (+1) + \frac{\pi}{2} = \pi$$

c) $$J_n = \int_0^\infty x^n \cdot e^{-x} dx = \int_0^\infty x^n \cdot (-e^{-x})' dx$$

$$= -\frac{x^n}{e^x} \Big|_0^\infty - n \int_0^\infty x^{n-1} \cdot e^{-x} dx =$$

$$= \lim_{v \to \infty} \frac{-x^n}{e^x} \Big|_0^v + n J_{n-1} =$$

$$= \underbrace{\lim_{v \to \infty} \left(-\frac{v^n}{e^v} + \frac{0}{e^0} \right)}_{= 0} + n \cdot J_{n-1} =$$

$$= n \, J_{n+1}$$

$J_n = n \cdot J_{n-1}, \forall n \geq 1$

$$J_0 = \int_0^{\infty} e^{-x}\,dx = \int_0^{\infty} (-e^{-x})'\,dx = \lim_{v\to\infty} -\frac{1}{e^x}\Big|_0^v = 1$$

$$J_1 = 1 \cdot J_0 = 1$$

$$J_2 = 2 \cdot 1 = 2$$

$$J_3 = 3 \cdot 2 = 6$$

$$J_4 = 4 \cdot 3 \cdot 2 \cdot 1 = 4!$$

$$J_n = n!$$

d) $$\int_1^{2-0} \frac{1}{\sqrt{x(2-x)}}\,dx = \int_1^{\sqrt{2}-0} \frac{1}{\sqrt{t^2(2-t^2)}} \cdot 2t\,dt =$$

$x = t^2$

$t = \sqrt{x}$

$dx = 2t\,dt$

$x = 1 \Rightarrow t = 1$

$x \nearrow 2 \Rightarrow t \nearrow \sqrt{2}$

$$= \int_{1}^{\sqrt{2}-0} \frac{2\cancel{t}}{\cancel{t}\sqrt{2-t^2}}\, dt = 2\int_{1}^{\sqrt{2}-0} \frac{1}{\sqrt{2-t^2}}\, dt$$

$$= 2 \arcsin \frac{t}{\sqrt{2}} \Big|_{1}^{\sqrt{2}} =$$

$$= 2 \arcsin 1 - 2 \arcsin \frac{\sqrt{2}}{2} =$$

$$= \cancel{2} \cdot \frac{\pi}{\cancel{2}} - \cancel{2} \cdot \frac{\pi}{\cancel{4}2} = \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

② Studiați convergența integralelor

a) $\displaystyle\int_{0}^{3} \frac{x^3+1}{\sqrt{9-x^2}}\, dx$

b) $\displaystyle\int_{0}^{\infty} \frac{\operatorname{arctg} x}{x}\, dx$

c) $\displaystyle\int_{0}^{\pi} x \cdot \ln(\sin x)\, dx$

a) $\displaystyle\int_{0}^{3-0} \frac{x^3+1}{\sqrt{9-x^2}}\, dx$

$f:[0,3) \to [0,+\infty), \ f(x) = \frac{x^3+1}{\sqrt{9-x^2}}$

Prop 1

$\lambda = \lim_{x \to 3} (3-x)^p \cdot f(x) = \lim_{x \to} (3-x)^p \cdot \frac{x^3+1}{\sqrt{(3-x)(3+x)}}$

$= \lim_{x \to 3} (3-x)^{p-\frac{1}{2}} \cdot \underbrace{\frac{x^3+1}{\sqrt{3+x}}}_{\frac{28}{\sqrt{6}}} \overset{p=\frac{1}{2}}{=} \frac{28}{\sqrt{6}} < +\infty \Big\} \Rightarrow$ int conv

b) $\int_{0+0}^{\infty} \frac{\operatorname{arctg} x}{x} dx = \underbrace{\int_{0+0}^{1} \frac{\operatorname{arctg} x}{x} dx}_{J_1} + \underbrace{\int_{1}^{\infty} \frac{\operatorname{arctg} x}{x} dx}_{J_2}$

Pt J_1:

$f:(0,1] \to (0,+\infty) \ f(x) = \frac{\operatorname{arctg} x}{x}$

$\lambda = \lim_{x \searrow 0} (x-0)^p \cdot f(x) = \lim_{x \searrow 0} x^p \cdot \underbrace{\frac{\operatorname{arctg} x}{x}}_{1} =$

$= \lim_{x \searrow 0} x^p \overset{p=0}{=} 1 \Rightarrow$ conv

Pt J_2

$f : [1, +\infty) \to [0, +\infty)$

$$\lambda = \lim_{x \to \infty} x^p \cdot f(x) = \lim_{x \to \infty} x^p \cdot \frac{\operatorname{arctg} x}{x} =$$

Dacă $p > 1$ și $\lambda < +\infty \Rightarrow$ conv

Dacă $p \leq 1$ și $\lambda > 0 \Rightarrow$ div

$$= \lim_{x \to \infty} x^{p-1} \cdot \operatorname{arctg} x = \frac{\pi}{2} \cdot \lim_{x \to \infty} x^{p-1} \overset{p=1}{=}$$

$$= \frac{\pi}{2} \Rightarrow \text{div}$$

$\Rightarrow J_1 + J_2 =$ divergentă

astfel, f se poate prelungi prin continuitate la $[0,1]$

$$f^*(x) = \begin{cases} f(x), & x \in [0,1] \\ 1, & x = 0 \end{cases} \Rightarrow I_1 \text{ conv}$$

d) $\displaystyle\int_{0+0}^{\pi-0} x \cdot \ln(\sin x)\,dx =$

$$= -\underbrace{\int_{0+0}^{\frac{\pi}{2}} x \cdot \ln(\sin x)\,dx}_{J_1} + \underbrace{\int_{\frac{\pi}{2}}^{\pi-0} x \cdot \ln(\sin x)\,dx}_{J_2}$$

Pt J_1

$$f:\left(0,\frac{\pi}{2}\right] \to (0,+\infty),\ f(x) = -x\ln(\sin x)\,dx$$

$$\lambda = \lim_{x\searrow 0} x^{p+1}\cdot(-\ln(\sin x)) \xrightarrow{p=0}$$

$$\Rightarrow \lambda = -\lim_{x\searrow 0} x\cdot\ln(\sin x) =$$

$$= -\lim_{x\searrow 0}\frac{\ln(\sin x)}{\frac{1}{x}} \overset{\frac{\infty}{\infty}}{=}$$

$$= -\lim_{x\searrow 0}\frac{\frac{1}{\sin x}\cdot\cos x}{-\frac{1}{x^2}}$$

$$= \lim_{x \to 0} \frac{x^2 \cos x}{\sin x} = \lim_{x\to 0} \frac{x}{\sin x} \cdot x\cos x =$$

$$= 0$$

$\Rightarrow J_1 =$ convergentă

Pt J_2

$f : [\frac{\pi}{2}, \pi) \to [0, +\infty)$ $f(x) = -x\ln(\sin x)$

$$\lambda = \lim_{x \to \pi} (\pi - x)^p (-x) \cdot \ln(\sin x)$$

Not: $\pi - x = y \Rightarrow \lambda = \lim_{y \searrow 0} y^p (-\pi + y) \ln(\sin(\pi - y))$

$$\lambda = (-\pi) \lim_{y \searrow 0} y^p \cdot \ln(\sin y) \Rightarrow p = \frac{1}{2}$$

$\Rightarrow \lambda = 0$

$\Rightarrow$ integrala conv

$-J_1 - J_2 =$ conv

③ Studiați conv integralei

$$J(\alpha) = \int_0^1 \left(\frac{x}{(1-x)}\right)^\alpha dx \text{ , } \forall \alpha \in \mathbb{R}$$

I) $\alpha > 0$, $J(\alpha) = \int_0^{1-0} \left(\frac{x}{1-x}\right)^\alpha dx$

$f : [0,1) \to [0,+\infty)$, $f(x) = \frac{x^\alpha}{(1-x)^\alpha}$

$$\lambda = \lim_{x \nearrow 1} (1-x)^p \cdot \frac{x^\alpha}{(1-x)^\alpha}$$

$$\lambda = \lim_{x \nearrow 1} (1-x)^{p-\alpha} \cdot x^\alpha =$$

$$\lambda = \lim_{x \to 1} (1-x)^{p-\alpha} \overset{p=\alpha}{=} 1$$

Dacă $p<1$ $(\alpha<1) \Rightarrow J(\alpha)$ conv

Dacă $p \geq 1$ $(p \geq 1) \Rightarrow J(\alpha)$ diverg.

II $\alpha < 0$, $I(\alpha) = \int_{0+0}^{1} \left(\frac{x}{1-x}\right)^{\alpha} dx$

$f : (0,1] \to [0,+\infty)$, $f(x) = \frac{x^{\alpha}}{(1-x)^{\alpha}}$,

$\lambda = \lim_{x \searrow 0} (x-0)^{p} \cdot f(x)$

$\lambda = \lim_{x \searrow 0} \frac{x^{p+\alpha}}{(1-x)^{\alpha}} = \lim_{x \searrow 0} x^{p+\alpha} \stackrel{p=-\alpha}{=} 1$

Dacă $p < 1$ $(\alpha > -1) \Rightarrow I(\alpha)$ conv

Dacă $p \geq 1$ $(\alpha \leq -1) \Rightarrow I(\alpha)$ div

III $\alpha = 0 \Rightarrow I(\alpha)$ conv

$I(\alpha)$ conv $\Leftrightarrow \alpha \in (-1,1)$

în rest div

Temă $I\left(\frac{1}{2}\right) = \int_{0}^{1-0} \sqrt{\frac{x}{1-x}}\, dx$