① Studiați existența limitelor

a) $\lim\limits_{(x,y)\to(0,2)} \dfrac{x\cdot y}{\sqrt{1+xy}-1} \overset{x\cdot y=t}{=\!=} \lim\limits_{t\to 0} \dfrac{t}{\sqrt{1+t}-1} = 2$

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$

c) $\lim\limits_{(x,y)\to(\infty,\infty)} \dfrac{x^2+y^2}{x^4+y^4}$

d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x\cdot \sin(x^2-y^2)}{x^2+y^2}$

e) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x\cdot y}$

f) $\lim\limits_{(x,y)\to(1,1)} \dfrac{(x-1)(y-1)}{xy-1}$

g) $\lim\limits_{(x,y,z)\to 0_3} \dfrac{(x+y+z)^2}{x^2+y^2+z^2}$

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b) $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} \dfrac{x^2-y^2}{x^2+y^2}\right) = \lim\limits_{x\to 0} \dfrac{x^2}{x^2} = 1$

$\neq \Rightarrow \nexists \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$

$\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0} \dfrac{x^2-y^2}{x^2+y^2}\right) = -1$

$$f(x,y) = \frac{x^2-y^2}{x^2+y^2}, \quad f: \mathbb{R}^2 \setminus \{0_2\} \to \mathbb{R}$$

$a^n, b^n \to (0,0), \; n \to \infty$

$$\lim_{n\to\infty} f(a_n) \neq \lim_{n\to\infty} f(b_n) \Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y)$$

$$f(a^n) = \frac{\frac{1}{n^2} - 0}{\frac{1}{n^2} + 0} = 1$$

$$b^n = \left(0, \frac{1}{n}\right) \to (0,0)$$

$$f(b^n) = \frac{-\frac{1}{n^2}}{\frac{1}{n^2}} = -1$$

$$\lim_{n\to\infty} f(a^n) \neq \lim_{n\to\infty} f(b^n) \Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y)$$

c) $f(x,y) = \dfrac{x^2+y^2}{x^4+y^4}$

$$0 \leq |f(x,y) - 0| = \frac{x^2+y^2}{x^4+y^4} \overset{\left[\frac{\infty}{\infty}\right]}{=\!=} \frac{x^2}{x^4+y^4} + \frac{y^2}{x^4 y^4} \leq$$

$$\leq \frac{x^2}{x^4} + \frac{y^2}{y^4} = \frac{1}{x^2} + \frac{1}{y^2} \to 0, \; (x,y) \to (\infty, \infty)$$

$$\exists \lim_{(x,y)\to(0,0)} = 0$$

d) $$\lim_{(x,y)\to(0,0)} \frac{x\cdot \sin(x^2-y^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{\sin(x^2-y^2)}{x^2-y^2} \cdot \frac{x(x^2-y^2)}{(x^2+y^2)}$$

not $t=x^2-y^2$

$$\underbrace{\lim_{t\to0} \frac{\sin t}{t}}_{1} \cdot \lim_{(x,y)\to(0,0)} \underbrace{\frac{x(x^2-y^2)}{x^2+y^2}}_{f(x,y)} =$$

$$f(x,y) = \frac{x(x^2-y^2)}{x^2+y^2}$$

$$0\le |f(x,y)-0| = \frac{|x(x^2-y^2)|}{x^2+y^2} = \frac{|x^3-xy^2|}{x^2+y^2} \le \frac{|x^3|}{x^2+y^2} + \frac{|-xy^2|}{x^2+y^2}$$

$$= \frac{|x|\cdot x^2}{\underbrace{x^2+y^2}_{\le 1}} + \frac{|x|\cdot y^2}{\underbrace{x^2+y^2}_{\le 1}} \le 2\underbrace{|x|}_{\to 0} \Rightarrow \exists \lim_{(x,y)\to(0,0)} f(x,y) = 0$$

e) $$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x\cdot y}$$

$$f(x,y) = \frac{x^3+y^3}{x\cdot y}$$

$$a^n = \left(\frac{1}{n}, \frac{1}{n}\right) \to (0,0) \qquad b^n = \left(\frac{1}{n}, \frac{1}{n^2}\right)$$

$$f(a^n) = \frac{\frac{1}{n^3} + \frac{1}{n^3}}{\frac{1}{n^2}} = \frac{2}{n} \to 0$$

$$f(b^n) = \frac{\frac{1}{n^3} + \frac{1}{n^6}}{\frac{1}{n^3}} = 1 + \frac{n^3}{n^6} = 1 + \frac{1}{n^3} \to 1$$

$\Rightarrow$

$$f(a^n) \neq f(b^n) \Rightarrow \nexists \lim.$$

f) $$\lim_{(x,y)\to(1,1)} \frac{(x-1)(y-1)}{xy-1} = \lim_{(u,v)\to(0,0)} \frac{u \cdot v}{(u+1)(v+1)-1} =$$

Not $x-1=u$
$y-1=v$

$$= \lim_{(u,v)\to(0,0)} \frac{u \cdot v}{uv+u+v}$$

$$a_n = \left(\frac{1}{n}, 0\right)$$

$$f(a^n) = \frac{\frac{1}{n} \cdot 0}{\frac{1}{n} \cdot 0 + \frac{1}{n} + 0} = \frac{0}{\frac{1}{n}} = 0$$

$$b^n = \left(-\frac{1}{n}, \frac{1}{n}\right)$$

$$f(b^n) = \frac{\left(-\frac{1}{n}\right) \cdot \frac{1}{n}}{\left(-\frac{1}{n}\right) \cdot \frac{1}{n} - \frac{1}{n} + \frac{1}{n}} = \frac{-\frac{1}{n^2}}{-\frac{1}{n^2}} = 1$$

$$\Rightarrow \nexists \lim.$$

$$\lim_{(x,y,z)\to 0_3} \frac{(x+y+z)^2}{x^2+y^2+z^2} = \frac{x^2+y^2+z^2+2xy+2xz+2yz}{x^2+y^2+z^2} <$$

② Fie $f:\mathbb{R}^2\to\mathbb{R}$

$$f(x,y) = \begin{cases} x\cdot\cos\frac{1}{y^2}+y\cdot\cos\frac{1}{x^2}, & x\cdot y\neq 0 \\ 0, & x\cdot y=0 \end{cases}$$

Este f cont in $(0,0)$? Dar in $(1,0)$?

f cont in $(0,0) \Leftrightarrow \exists \lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)=0$

$$|f(x,y)-0| = \left|x\cdot\cos\frac{1}{y^2}+y\cdot\cos\frac{1}{x^2}\right| \leq$$

$$\leq |x|\cdot\left|\cos\frac{1}{y^2}\right| + |y|\left|\cos\frac{1}{x^2}\right| \leq |x|+|y|\to 0$$

$\Rightarrow f$ cont in $(0,0)$

$$\lim_{(x,y)\to(1,0)}\left(x\cdot\cos\frac{1}{y^2}+y\cdot\cos\frac{1}{x}\right) =$$

$$= \lim_{(x,y)\to(1,0)} x\cdot\cos\frac{1}{y^2} + \underbrace{\lim_{(x,y)\to(1,0)} y\cdot\cos\frac{1}{x^2}}_{0}$$

$\lim\limits_{x\to 0}\cos\frac{1}{x^2} \nexists$

$a^n = \left(1, \frac{1}{\sqrt{2n\pi}}\right) \to (1,0)$
$n \to \infty$

$b^n = \left(1, \frac{1}{\sqrt{(2n+1)\pi}}\right) \to (1,0)$

$\lim\limits_{n\to\infty} g(a_n) \neq \lim\limits_{n\to\infty} g(b_n) \Rightarrow f$ nu e cont în $(1,0)$

③ Det. valorile extreme și verificați dacă se ating

a) $f:(0,+\infty)^2 \to \mathbb{R}$, $f(x,y) = \frac{x}{y} + \frac{y}{x}$

b) $f: B(0_2,1) \to \mathbb{R}$, $f(x,y) = \frac{1}{1+x^2+y^2}$

c) $f: A \to \mathbb{R}$, $f(x,y) = x \cdot y \cdot (1-x-y)$

$A = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x+y \leq 1\}$

$\inf f(A)$, $\sup f(A)$

a) $A = (0,+\infty)^2$, $x,y > 0$

$f(x,y) = \frac{x}{y} + \frac{y}{x}$

$\frac{x}{y} + \frac{y}{x} \geq 2 \quad \forall x,y > 0$

$\Rightarrow \inf f(A) = 2 = f(1,1)$

$f(x,1) = x + \frac{1}{x}$

$\lim\limits_{x\to\infty} f(x,1) = +\infty$

$\Rightarrow \sup f(A) = +\infty$

a) $A = B(O_2, 1) = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$

$f(x,y) = \dfrac{1}{1+x^2+y^2}$

A nu este închisă $\Rightarrow$
A nu e compactă

$0 \le x^2+y^2 < 1$

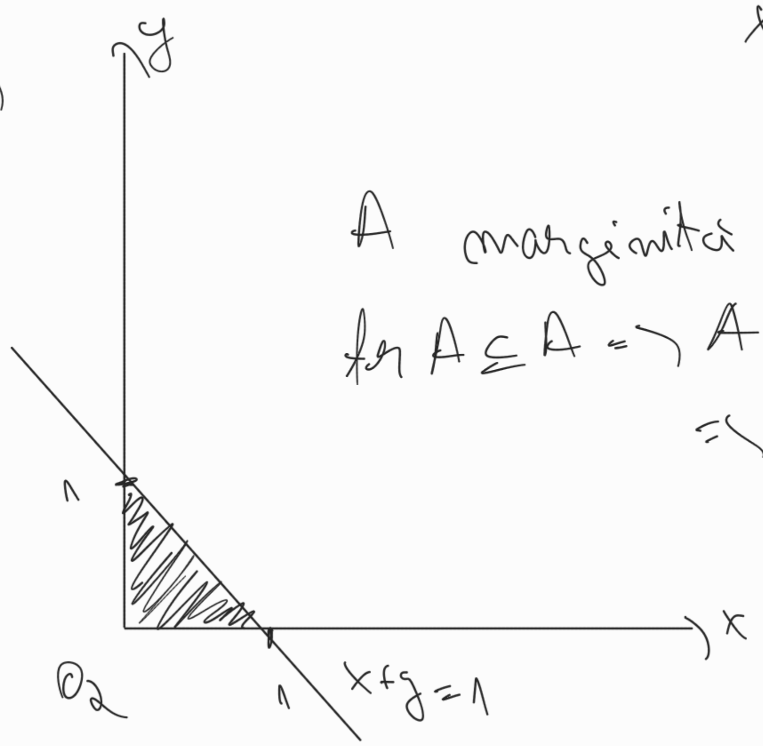
$1 \le 1+x^2+y^2 < 2$

$\dfrac{1}{2} < \dfrac{1}{1+x^2+y^2} \le 1$ $\Rightarrow$ $\inf f(A) = \dfrac{1}{2}$ nu se atinge
$\sup f(A) = 1$ se atinge; $f(0,0) = 1$

a) A nemărginită $\Rightarrow$ A nu e compactă

c)

$x+y=1$



A mărginită

$Fr\,A \subseteq A \Rightarrow A$ închisă

$\Rightarrow A$ compactă

$\xrightarrow{TW}$ f își atinge extremele

$f(x,y) = x \cdot y\,(1-x-y)$

$x \geq 0$

$y \geq 0$

$1-x-y \geq 0$

$f(x,y) \geq 0 = f(0,0) \Rightarrow \min f(A) = 0$

$\frac{u+v+w}{3} \geq \sqrt[3]{u \cdot v \cdot w} \Rightarrow \frac{1}{3} \geq \sqrt[3]{f(x,y)} \Rightarrow f(x,y) \leq \frac{1}{3}$

$\max f(A) = \frac{1}{27} = f\left(\frac{1}{3}, \frac{1}{3}\right)^3$

$u = v = w \iff x = y = 1-x-y \iff (x,y) = \left(\frac{1}{3}, \frac{1}{3}\right)$

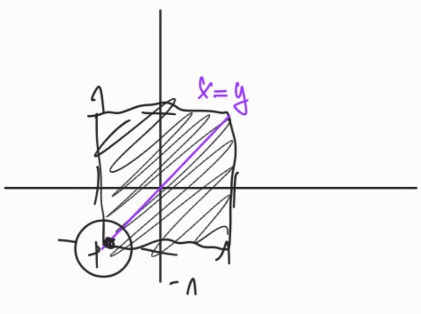
(4) $A = \{(x,y) \in [-1,1]^2 \mid x \neq y\}$ și $f : A \to \mathbb{R}$

$f(x,y) = \frac{x^2+y^2}{(x-y)^2}$

a) $\exists \lim\limits_{(x,y)\to(0,0)} f(x,y)$? Nu există (cu șiruri)

b) Este A compactă?

c) Det $\inf f(A)$ și $\sup f(A)$. Se ating?

b) 

A mărginită

$\left.\begin{array}{l}(-1,1) \notin A \\ (-1,-1) \in \text{fr} A\end{array}\right\} \Rightarrow \text{fr} A \not\subset A \Rightarrow A$ nu este închisă

$\Rightarrow A$ nu e compactă

c) $\inf f(A) = \frac{1}{2}$

$\sup f(A) = +\infty$