THE GEOMETRICAL LEMMA FOR SMOOTH REPRESENTATIONS IN NATURAL CHARACTERISTIC

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Abstract. The Geometrical Lemma is a classical result in the theory of (complex) smooth representations of p-adic reductive groups, which helps to analyze the parabolic restriction of a parabolically induced representation by providing a filtration whose graded pieces are (smaller) parabolic inductions of parabolic restrictions. In this article, we establish the Geometrical Lemma for the derived category of smooth mod p representations of a p-adic reductive group.

As an important application we compute higher extension groups between parabolically induced representations, which in a slightly different context had been achieved by Hauseux assuming a conjecture of Emerton concerning the higher ordinary parts functor. We also compute the (cohomology functors of the) left adjoint of derived parabolic induction on principal series and generalized Steinberg representations.

Contents

1. Introduction

1.1. History and motivation. Fix a finite extension $\mathfrak{F}/\mathbb{Q}_p$ and let G be (the group of $\mathfrak{F}$-points of) a connected reductive $\mathfrak{F}$-group. In the theory of complex smooth representations of G, the Geometrical Lemma, independently due to Bernstein–Zelevinsky [BZ77] and Casselman [Cas95], is one of the main tools in the classification of irreducible smooth representations in terms of parabolically induced representations. To state it, we fix a parabolic subgroup $P = M \ltimes U$ of G

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with Levi quotient M and unipotent radical U. Denote $\boldsymbol{i}_P^G\colon \operatorname{Rep}_{\mathbb{C}}(M) \to \operatorname{Rep}_{\mathbb{C}}(G)$ the normalized parabolic induction functor on the categories of smooth representations. Its left adjoint is the normalized parabolic restriction functor $\boldsymbol{r}_P^G$. If $Q = L \ltimes N$ is another parabolic subgroup with Levi quotient L and unipotent radical N, then the Geometrical Lemma states that the functor $\boldsymbol{r}_Q^G\,\boldsymbol{i}_P^G$ admits a filtration by subfunctors such that the graded pieces are given by functors of the form

$$\boldsymbol{i}^L_{g^{-1}Pg\cap L} \circ g_*^{-1} \circ \boldsymbol{r}^M_{M\cap gQg^{-1}},$$

where $g_*^{-1}\colon \operatorname{Rep}_{\mathbb{C}}\big(M\cap gLg^{-1}\big) \xrightarrow{\cong} \operatorname{Rep}_{\mathbb{C}}\big(g^{-1}Mg\cap L\big)$ is the equivalence of categories induced by conjugation with g^{-1} and g runs through a certain set of double coset representatives of $P\backslash G/Q$. All known proofs rely on the use of Haar measures and the exactness of $\boldsymbol{r}_P^G$. They are easily adapted to prove a Geometrical Lemma for smooth representations over a field k of characteristic $\neq p$.

However, if k is a field of characterestic p, then the proofs break down completely. Since there is no modulus character, one cannot talk about normalization. We denote $i_P^G\colon \operatorname{Rep}_k(M) \to \operatorname{Rep}_k(G)$ the unnormalized parabolic induction functor and $\mathrm{L}^0(U,\_)$ its left adjoint. In this context, Haar measures do not exist and $\mathrm{L}^0(U,\_)$ is not exact; the Geometrical Lemma takes on a much simpler but also less satisfactory form: the functor $\mathrm{L}^0(N,\_)\,i_P^G$ is isomorphic to $i^L_{P\cap L}\,\mathrm{L}^0(M\cap N,\_)$, see [AHV19, Theorem 5.5]; in other words, there is a filtration with only one graded piece. The reason for this pathological behaviour comes down to the fact that there exists no non-trivial Haar measure on U.

As it turns out, there is a full Geometrical Lemma once we pass to the derived categories.

1.2. Main results. From now on, let k be a field of characteristic p. For any p-adic Lie group H we denote by $\mathrm{D}(H)$ the unbounded derived category of the category $\operatorname{Rep}_k(H)$ of smooth k-linear representations of G. The parabolic induction extends to a derived functor $\mathrm{R}\,i_P^G\colon \mathrm{D}(M) \to \mathrm{D}(G)$. By the main result of [Hey22], there exists a left adjoint $\mathrm{L}(U,\_)$. To state the Geometrical Lemma, we fix a set $\mathcal{N}_{P,Q}$ of double coset representatives of $P\backslash G/Q$ which normalizes a maximal $\mathfrak{F}$-split torus (of G) that is contained in $P\cap Q$. For the notion of filtration on a triangulated functor, we refer to Definition 3.2.7.

A. Theorem (Corollary 3.3.6). *The functor* $\mathrm{L}(N,\_)\circ i_P^G\colon \mathrm{D}(M)\to\mathrm{D}(L)$ *admits a filtration of length* $|\mathcal{N}_{P,Q}|$ *with graded pieces of the form*

$$i^L_{n^{-1}Pn\cap L}\circ(\omega_n\otimes_k \_)\circ n_*^{-1}\circ \mathrm{L}(M\cap nNn^{-1},\_),$$

for $n\in\mathcal{N}_{P,Q}$*, where* $\omega_n\in\mathrm{D}(n^{-1}Mn\cap L)$ *is a character in cohomological degree* $-\dim(n^{-1}\overline{U}n\cap N)$ *and* $\overline{U}$ *is the unipotent radical of the parabolic opposite* P*.*

To fix ideas, we note that, if one chooses $\mathcal{N}_{P,Q}$ carefully and if G is $\mathfrak{F}$-split, then ω_n is concentrated in degree $-[\mathfrak{F}:\mathbb{Q}_p]\ell(w)$, where $\ell(w)$ denotes the length of the image w of n in the (finite) Weyl group. Thus, ω_n contributes a cohomological shift which is not detected on the abelian categories; this gives a conceptual explanation of why there is no proper Geometrical Lemma for (underived) smooth mod p representations. The twist by the character ω_n is not surprising as it occurs also in the classical context although it is hidden in the normalization of the parabolic induction and restriction functors.

The proof of Theorem A follows the general strategy employed by Bernstein–Zelevinsky and Casselman in that the problem is reduced to checking certain compatibilities between compact induction and (derived) coinvariants. Since we do not have Haar measures at our disposal (which are the primary tool in the classical setting), this is the point where our proof departs from the classical approach. Instead, we show that the Geometrical Lemma holds almost entirely for purely formal reasons. The only non-formal input seems to be that the derived inflation functor $\mathrm{R}\operatorname{Inf}_P^M\colon \mathrm{D}(M)\to\mathrm{D}(P)$ is fully faithful, which follows from the fact that U is a unipotent group, see [Hey22, Example 3.4.24]. Hence, the same proof will apply in other contexts as well.

A general method to determine explicitly the characters ω_n, for $n\in\mathcal{N}_{P,Q}$, is presented in Propositions 2.4.3 and 2.4.8. This is applied in Remark 4.1.5 to deduce a concrete description of the characters ω_n.

As an application, we will compute several Ext-groups between parabolically induced representations. These are virtually identical with the main results of [Hau16, Hau18]; the important differences are that Hauseux computes higher extensions in the category of locally admissible representations and relies for the strongest of these results on an open conjecture of Emerton [Eme10, Conjecture 3.7.2], which to my knowledge has only been resolved for GL_2. In contrast, the Ext-groups in this paper are computed in the category of all smooth representations and do not rely on conjectural statements.

To state the results concerning Ext-groups, we introduce some notation. We fix a maximal $\mathfrak{F}$-split torus contained in a minimal parabolic subgroup B. These choices come with a (relative) root system together with a set of simple roots. Fix standard parabolic subgroups $P = M \ltimes U$ and $Q = L \ltimes N$. We choose a distinguished set $\mathcal{N}_{P,Q}$ of double coset representatives of $P\backslash G/Q$, see §4.1.2 for more details. For each $n \in \mathcal{N}_{P,Q}$ we consider the smooth character δ_n of $n^{-1}Mn \cap L$ given by $\omega_n = \delta_n[\dim(n^{-1}\overline{U}n \cap N)]$.

B. Theorem (Theorems 4.2.8 and 4.2.11)**.**

(a) Assume $Q = B$. Let $\chi\colon M \to k^\times$ be a smooth character, let $r \in \mathbb{Z}_{\geq 0}$, and denote $Z(L)$ the center of L.

(i) Let $\chi'\colon L \to k^\times$ be a smooth character. If

$$\mathrm{Ext}^r_G\big(i_P^G\,\chi, i_B^G\,\chi'\big) \neq 0,$$

then there exists $n \in \mathcal{N}_{P,B}$ such that $\dim(n^{-1}\overline{U}n \cap N) \leq r$ and $\chi' \cong \delta_n \otimes_k n_*^{-1}\chi$ after restriction to $Z(L)$.

(ii) Assume $\delta_n \otimes_k n_*^{-1}\chi \ncong \chi$ after restriction to $Z(L)$, for all $n \in \mathcal{N}_{P,B}$. For each $n \in \mathcal{N}_{P,B}$ with $\dim(n^{-1}\overline{U}n \cap N) \leq r$ one has k-linear isomorphisms

$$\mathrm{Ext}^r_G\big(i_P^G\,\chi, i_B^G(\delta_n \otimes_k n_*^{-1}\chi)\big) \cong \mathrm{Ext}_L^{r-\dim(n^{-1}\overline{U}n\cap N)}(\mathbf{1},\mathbf{1}) \cong \mathrm{H}^{r-\dim(n^{-1}\overline{U}n\cap N)}(L,k),$$

where $\mathrm{H}^*(L,k)$ denotes continuous group cohomology.

(b) Let $V \in \mathrm{Rep}_k(M)$ and $W \in \mathrm{Rep}_k(L)$.

(i) Assume $P \nsubseteq Q$ and $P \nsupseteq Q$, that V is left cuspidal and that W is right cuspidal (see Definition 4.2.9). Then

$$\mathrm{Ext}^1_G\big(i_P^G\,V, i_Q^G\,W\big) = 0.$$

(ii) Assume $P = Q$. For each $0 \leq i < [\mathfrak{F} : \mathbb{Q}_p]$ the functor i_P^G induces a k-linear isomorphism

$$\mathrm{Ext}^i_M(V,W) \xrightarrow{\cong} \mathrm{Ext}^i_G\big(i_P^G\,V, i_P^G\,W\big).$$

If moreover V is left cuspidal or W is right cuspidal, and V and W admit distinct central characters, then

$$\mathrm{Ext}_G^{[\mathfrak{F}:\mathbb{Q}_p]}\big(i_P^G\,V, i_P^G\,W\big) \cong \bigoplus_{\alpha\in\Delta_M^{\perp,1}} \mathrm{Hom}_M\big(\delta_{n_\alpha} \otimes_k n_{\alpha*}^{-1}V, W\big),$$

where $\Delta_M^{\perp,1}$ denotes the set of simple roots α of G which are orthogonal to all simple roots of M and such that the associated root space has dimension $[\mathfrak{F} : \mathbb{Q}_p]$ as a p-adic manifold. Here, $n_\alpha \in \mathcal{N}_{P,P}$ denotes the lift of the simple reflection corresponding to α.

(iii) Assume $P \supsetneqq Q$ and that V is left cuspidal. For all $0 \leq i \leq [\mathfrak{F} : \mathbb{Q}_p]$, the functor i_P^G induces a k-linear isomorphism

$$\mathrm{Ext}^i_M\big(V, i_{M\cap Q}^M\,W\big) \xrightarrow{\cong} \mathrm{Ext}^i_G\big(i_P^G\,V, i_Q^G\,W\big).$$

(iv) Dually, assume $P \subsetneqq Q$ and that W is right cuspidal. For all $0 \leq i \leq [\mathfrak{F} : \mathbb{Q}_p]$, the functor i_Q^G induces a k-linear isomorphism

$$\mathrm{Ext}^i_L\big(i_{P\cap L}^L\,V, W\big) \xrightarrow{\cong} \mathrm{Ext}^i_G\big(i_P^G\,V, i_Q^G\,W\big).$$

We further compute the representations $\mathrm{L}^{-j}(N, \mathrm{Sp}_P^G)$ for all $j \geq 0$, where Sp_P^G denotes the generalized Steinberg representation attached to P, *i.e.*, the unique simple quotient of $i_P^G(\mathbf{1})$; see Theorem 4.3.9 and Corollary 4.3.10. To my knowledge, this is the first computation of this kind. This raises the question whether one can compute $\mathrm{L}^{-j}(N, V)$ for all irreducible smooth representations V. However, for supersingular V the answer seems to be out of reach with the current methods available. A naive hope would be that $\mathrm{L}(N, V) = 0$ for all supersingular V, but [HW22, Theorem 10.37] shows that already for $G = \mathrm{GL}_2(\mathfrak{F})$, where $\mathfrak{F} \supsetneq \mathbb{Q}_p$ is unramified, there exist a supersingular representation V and a principal series $i_P^G W$ such that $\mathrm{Ext}_G^1(V, i_P^G W) \neq 0$, and this implies $\mathrm{L}^{-1}(U, V) \neq 0$. In view of this it is unclear what one should expect.

1.3. Organization of the paper. The Geometrical Lemma is the content of Corollary 3.3.6. The preparatory lemmas concerning compact induction and derived coinvariants are proved in §2.3; these in turn rely on the abstract results about functors in monoidal categories which are presented in §2.2. The various characters that implicitly appear in the Geometrical Lemma are completely determined in §2.4.

Regarding applications, we compute $\mathrm{L}^{-j}(N, V)$ whenever V is a principal series representation (Example 4.2.1) or a generalized Steinberg representation (Theorem 4.3.9 and Corollary 4.3.10). Finally, we use the Geometrical Lemma to compute many Ext-groups between parabolically induced representations in Theorems 4.2.8 and 4.2.11.

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2. Preliminaries

2.1. Notation and conventions. If $\mathbf{H}$ is an algebraic group defined over a field $\mathfrak{F}$, we denote its group of $\mathfrak{F}$-points by the corresponding lightface letter, that is, $H = \mathbf{H}(\mathfrak{F})$.

We fix a field k of characteristic $p > 0$.

For a p-adic Lie group G, we denote $\mathrm{Rep}_k(G)$ the Grothendieck abelian category of smooth k-linear G-representations.

The (unbounded) derived category of $\mathrm{Rep}_k(G)$ is denoted $\mathrm{D}(G)$; it is a *tensor triangulated category*, that is, a triangulated category which is symmetric monoidal and such that the functors $\_ \otimes_k X$ are triangulated for any X in $\mathrm{D}(G)$. The tensor unit is $\mathbf{1} = k[0]$. We denote $\mathrm{hom}_G(X, \_)\colon \mathrm{D}(G) \to \mathrm{D}(G)$ the right adjoint of $\_ \otimes_k X$. The *smooth dual* of X is denoted $X^\vee := \mathrm{hom}_G(X, \mathbf{1})$.

The category $\mathrm{D}(G)$ is naturally enriched over $\mathrm{D}(k) := \mathrm{D}(\{1\})$, the unbounded derived category of the category of k-vector spaces; we denote $\mathrm{RHom}_G(X, Y) \in \mathrm{D}(k)$ the derived Hom-complex, for each $X, Y \in \mathrm{D}(G)$; it defines a triangulated functor in each variable.

The derived category $\mathrm{D}(G)$ comes with a natural t-structure. For any integer $n \in \mathbb{Z}$ we denote $\mathrm{D}^{\leq n}(G)$ (resp. $\mathrm{D}^{\geq n}(G)$) the full subcategory of objects X in $\mathrm{D}(G)$ satisfying $\mathrm{H}^i(X) = 0$ for all $i > n$ (resp. $i < n$). Denote $\mathrm{H}^i\colon \mathrm{D}(G) \to \mathrm{Rep}_k(G)$ the i-th cohomology functor. Note that $\mathrm{RHom}_G(X, Y) \in \mathrm{D}^{\geq 0}(k)$ provided $X \in \mathrm{D}^{\leq 0}(G)$ and $Y \in \mathrm{D}^{\geq 0}(G)$.

2.2. General abstract nonsense.

2.2.1. The purpose of this section is to formulate some formal statements about adjunctions in monoidal categories. These will only be applied in Lemmas 2.3.6, 2.3.8, and 2.3.14 and hence this section should be skipped on first reading. For the relevant notions concerning monoidal categories we refer to [Lur23, Section 00BL].

2.2.2. Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$ be monoidal categories and consider a diagram

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\overline{g}^*"] \arrow[d, "\overline{f}^*"'] \arrow[dr, phantom, "\overset{\alpha}{\nearrow}"] & \mathscr{B} \arrow[d, "f^*"] \\ \mathscr{C} \arrow[r, "g^*"'] & \mathscr{D} \end{tikzcd}$$

of (strongly) monoidal functors, where $\alpha\colon g^*\overline{f}^* \overset{\cong}{\Longrightarrow} f^*\overline{g}^*$ is a monoidal natural isomorphism. We make two assumptions:

(**A1**) We have the following adjunctions (of the underlying ordinary categories)

$$f_{(1)} \dashv f^* \dashv f_* \dashv f^{(1)}, \qquad g^* \dashv g_*$$
$$\overline{f}_{(1)} \dashv \overline{f}^* \dashv \overline{f}_* \dashv \overline{f}^{(1)}, \qquad \overline{g}^* \dashv \overline{g}_*,$$

where the notation "$F \dashv G$" means "F is left adjoint to G".

As f^* is monoidal, we have a natural isomorphism

$$\mathsf{mon}_f\colon f^*(b') \otimes f^*(b) \xrightarrow{\cong} f^*(b' \otimes b).$$

The right mate of $f^* \circ (\_ \otimes b) \Rightarrow (\_ \otimes f^*(b)) \circ f^*$ yields a natural map

$$\mathsf{rpf}_f\colon f_*(d) \otimes b \longrightarrow f_*\big(d \otimes f^*(b)\big).$$

Similarly, $\mathsf{rpf}_{\overline{f}}$ and mon_h, for $h \in \{\overline{f}, g, \overline{g}\}$, are defined. Define $\mathsf{rpf}_g\colon c \otimes g_*(d) \to g_*(g^*(c) \otimes d)$ as the right mate of $g^* \circ (c\otimes \_) \Rightarrow (g^*(c)\otimes \_) \circ g^*$ and similarly for $\mathsf{rpf}_{\overline{g}}$. The second assumption we make is:

(**A2**) rpf_f and $\mathsf{rpf}_{\overline{f}}$ are isomorphisms. In other words: the adjunctions $f^* \dashv f_*$ and $\overline{f}^* \dashv \overline{f}_*$ satisfy (right) projection formulas.

Finally, we consider the natural maps (which we do not require to be isomorphisms)

$$\begin{aligned} \mathsf{groth}_f\colon \quad & f^{(1)}(b') \otimes f^*(b) \quad \longrightarrow \quad f^{(1)}(b' \otimes b) \\ \mathsf{wirth}_f\colon \quad & f_*\big(f^{(1)}(b) \otimes d\big) \quad \longrightarrow \quad b \otimes f_{(1)}(d), \end{aligned}$$

and similarly for f replaced by $\overline{f}$. Here, the morphism groth_f arises as the right mate of the map $\mathsf{rpf}_f^{-1}\colon f_* \circ (\_ \otimes f^*(b)) \Rightarrow (\_ \otimes b) \circ f_*$ and the map wirth_f arises as the left mate of the natural transformation $\mathsf{groth}_f\colon (f^{(1)}(b)\otimes \_) \circ f^* \Rightarrow f^{(1)} \circ (b\otimes \_)$.

Denoting $r(\,\cdot\,)$ and $l(\,\cdot\,)$ the passage to right and left mates,[1] respectively, we consider the following natural transformations:

$$\begin{aligned} \alpha\colon \quad & g^*\overline{f}^* & \overset{\cong}{\Longrightarrow} \quad & f^*\overline{g}^* \\ r^2(\alpha^{-1})\colon \quad & \overline{f}_*g_* & \overset{\cong}{\Longrightarrow} \quad & \overline{g}_*f_* \\ \beta \coloneqq r(r(\alpha^{-1})^{-1})\colon \quad & g^*\overline{f}^{(1)} & \Longrightarrow \quad & f^{(1)}\overline{g}^* \\ \gamma \coloneqq l(r(\alpha)^{-1})\colon \quad & \overline{f}_{(1)}g_* & \Longrightarrow \quad & \overline{g}_*f_{(1)}. \end{aligned}$$

Note also that $l(\gamma) = l^2(r(\alpha)^{-1}) = l^2(r(\alpha))^{-1} = l(\alpha)^{-1}$ and therefore $\gamma = r(l(\alpha)^{-1})$.

2.2.3. Lemma. *For all $a', a \in \mathscr{A}$ and $d \in \mathscr{D}$, the following diagrams commute:*

[1] These operations are not well-defined, because there are usually several ways to pass to right/left mates. However, the context provides enough information to make the intended meaning unambiguous.

$$
\text{(a)}\qquad
\begin{tikzcd}
g^*\overline{f}^{(1)}(a') \otimes g^*\overline{f}^*(a) \arrow[r, "\mathsf{mon}_g"] \arrow[d, "\beta\otimes\alpha"'] & g^*\big(\overline{f}^{(1)}(a') \otimes \overline{f}^*(a)\big) \arrow[r, "g^*\mathsf{groth}_{\overline{f}}"] & g^*\overline{f}^{(1)}(a' \otimes a) \arrow[d, "\beta"] \\
f^{(1)}\overline{g}^*(a') \otimes f^*\overline{g}^*(a) \arrow[r, "\mathsf{groth}_f"'] & f^{(1)}\big(\overline{g}^*(a') \otimes \overline{g}^*(a)\big) \arrow[r, "f^{(1)}\mathsf{mon}_{\overline{g}}"'] & f^{(1)}\overline{g}^*(a' \otimes a);
\end{tikzcd}
$$

$$
\text{(b)}\qquad
\begin{tikzcd}
\overline{f}_*g_*\big(g^*\overline{f}^{(1)}(a) \otimes d\big) \arrow[r, "r^2(\alpha^{-1})"] & \overline{g}_*f_*\big(g^*\overline{f}^{(1)}(a) \otimes d\big) \arrow[r, "\overline{g}_*f_*(\beta\otimes\mathrm{id})"] & \overline{g}_*f_*\big(f^{(1)}\overline{g}^*(a) \otimes d\big) \arrow[d, "\overline{g}_*\mathsf{wirth}_f"] \\
\overline{f}_*\big(\overline{f}^{(1)}(a) \otimes g_*(d)\big) \arrow[u, "\overline{f}_*\mathsf{rpf}_g"] \arrow[d, "\mathsf{wirth}_{\overline{f}}"'] & & \overline{g}_*\big(\overline{g}^*(a) \otimes f_{(1)}(d)\big) \\
a \otimes \overline{f}_{(1)}g_*(d) \arrow[rr, "\mathrm{id}\otimes\gamma"'] & & a \otimes \overline{g}_*f_{(1)}(d). \arrow[u, "\mathsf{rpf}_{\overline{g}}"']
\end{tikzcd}
$$

$\square$

2.2.4. Consider a diagram

$$
\begin{tikzcd}
\mathscr{A} \arrow[rrr, "\alpha^*"] \arrow[dr, "f^*"] \arrow[dd, bend right, "h^*"'] & & & \overline{\mathscr{A}} \arrow[dl, "\overline{f}^*"'] \arrow[dd, bend left, "\overline{h}^*"] \\
& \mathscr{B} \arrow[r, "\beta^*"] \arrow[dl, "g^*"'] \arrow[ur, phantom, "\overset{\sigma}{\Rightarrow}"] & \overline{\mathscr{B}} \arrow[dr, "\overline{g}^*"] & \\
\mathscr{C} \arrow[rrr, "\gamma^*"'] \arrow[ur, dashed, "g_!"'] \arrow[urr, phantom, "\overset{\tau}{\Rightarrow}"] & & & \overline{\mathscr{C}}, \arrow[ul, dashed, "\overline{g}_!"]
\end{tikzcd}
$$

where the solid diagram commutes and consists of (strongly) monoidal functors between monoidal categories, and where $\sigma\colon \beta^*f^* \overset{\cong}{\Longrightarrow} \overline{f}^*\alpha^*$ and $\tau\colon \gamma^*g^* \overset{\cong}{\Longrightarrow} \overline{g}^*\beta^*$ are monoidal natural isomorphisms. Identify $h^* = g^*f^*$ and $\overline{h}^* = \overline{g}^*\overline{f}^*$ and put $\rho \coloneqq \overline{g}^*\sigma \circ \tau f^*\colon \gamma^*h^* \overset{\cong}{\Longrightarrow} \overline{h}^*\alpha^*$. Let further

$$\phi\colon \overline{g}_!\gamma^* \overset{\cong}{\Longrightarrow} \beta^*g_!$$

be a natural isomorphism. We make the following assumptions:

(**A3**) The functors $f^*, \overline{f}^*, h^*, \overline{h}^*$ admit left adjoints:

$$f_{(1)} \dashv f^*, \qquad \overline{f}_{(1)} \dashv \overline{f}^*, \qquad h_{(1)} \dashv h^*, \qquad \overline{h}_{(1)} \dashv \overline{h}^*.$$

(**A4**) There exist natural isomorphisms

$$
\begin{aligned}
\mathsf{pf}_g\colon \quad & g_!\big(c \otimes g^*(b)\big) \xrightarrow{\cong} g_!(c) \otimes b \\
\mathsf{pf}_{\overline{g}}\colon \quad & \overline{g}_!\big(\bar{c} \otimes \overline{g}^*(\bar{b})\big) \xrightarrow{\cong} \overline{g}_!(\bar{c}) \otimes \bar{b}
\end{aligned}
$$

such that the following diagram commutes:

$$\begin{tikzcd}
\overline{g}_!\big(\gamma^*(c)\otimes\overline{g}^*\beta^*(b)\big) \arrow[r, "\mathsf{pf}_{\overline{g}}"] \arrow[d, "\overline{g}_!(\mathrm{id}\otimes\tau^{-1})"'] & \overline{g}_!\gamma^*(c)\otimes\beta^*(b) \arrow[d, "\phi\otimes\mathrm{id}"] \\
\overline{g}_!\big(\gamma^*(c)\otimes\gamma^*g^*(b)\big) \arrow[d, "\overline{g}_!\mathsf{mon}_\gamma"'] & \beta^*g_!(c)\otimes\beta^*(b) \arrow[dd, "\mathsf{mon}_\beta"] \\
\overline{g}_!\gamma^*\big(c\otimes g^*(b)\big) \arrow[d, "\phi"'] & \\
\beta^*g_!\big(c\otimes g^*(b)\big) \arrow[r, "\mathsf{pf}_g"'] & \beta^*\big(g_!(c)\otimes b\big).
\end{tikzcd}$$

By (**A3**), the left mate of $\mathsf{mon}_f\colon (\_\otimes f^*(a))\circ f^*\xRightarrow{\cong} f^*\circ(\_\otimes a)$ yields a natural map

$$\mathsf{lpf}_f\colon f_{(1)}\big(b\otimes f^*(a)\big)\longrightarrow f_{(1)}(b)\otimes a,$$

and similarly for $\mathsf{lpf}_{\overline{f}}$. Passing to the left mate of the composite

$$f_{(1)}g_!\circ(c\otimes\_)\circ g^*f^*\xrightarrow{f_{(1)}\mathsf{pf}_g} f_{(1)}\circ\big(g_!(c)\otimes\_\big)\circ f^*\xrightarrow{\mathsf{lpf}_f} f_{(1)}g_!(c)\otimes\_,$$

and using $h^*=g^*f^*$, we obtain a natural map

$$\mathsf{ltm}_{f,g}\colon f_{(1)}g_!(c'\otimes c)\longrightarrow f_{(1)}g_!(c')\otimes h_{(1)}(c).$$

Similarly, the map $\mathsf{ltm}_{\overline{f},\overline{g}}\colon \overline{f}_{(1)}\overline{g}_!(\overline{c}'\otimes\overline{c})\to\overline{f}_{(1)}\overline{g}_!(\overline{c}')\otimes\overline{h}_{(1)}(\overline{c})$ is defined.

2.2.5. Lemma. *For all $c',c\in\mathscr{C}$ the following diagram commutes:*

$$\begin{tikzcd}
\overline{f}_{(1)}\overline{g}_!\big(\gamma^*(c')\otimes\gamma^*(c)\big) \arrow[r, "\mathsf{ltm}_{\overline{f},\overline{g}}"] \arrow[d, "\overline{f}_{(1)}\overline{g}_!\mathsf{mon}_\gamma"'] & \overline{f}_{(1)}\overline{g}_!\gamma^*(c')\otimes\overline{h}_{(1)}\gamma^*(c) \arrow[d, "\overline{f}_{(1)}\phi\otimes\mathrm{id}"] \\
\overline{f}_{(1)}\overline{g}_!\gamma^*(c'\otimes c) \arrow[d, "\overline{f}_{(1)}\phi"'] & \overline{f}_{(1)}\beta^*g_!(c')\otimes\overline{h}_{(1)}\gamma^*(c) \arrow[d, "l(\sigma)\otimes l(\rho)"] \\
\overline{f}_{(1)}\beta^*g_!(c'\otimes c) \arrow[d, "l(\sigma)"'] & \alpha^*f_{(1)}g_!(c')\otimes\alpha^*h_{(1)}(c) \arrow[d, "\mathsf{mon}_\alpha"] \\
\alpha^*f_{(1)}g_!(c'\otimes c) \arrow[r, "\alpha^*\mathsf{ltm}_{f,g}"'] & \alpha^*\big(f_{(1)}g_!(c')\otimes h_{(1)}(c)\big).
\end{tikzcd}$$

□

2.2.6. We stay in the context of §2.2.4 but restrict to the subdiagram

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "f^*"] \arrow[dr, bend right, "h^*"'] & \mathscr{B} \arrow[r, "\beta^*"] \arrow[d, shift right, "g^*"'] & \overline{\mathscr{B}} \arrow[d, shift left, "\overline{g}^*"] \\
& \mathscr{C} \arrow[r, "\gamma^*"'] \arrow[u, dashed, shift right, "g_!"'] & \overline{\mathscr{C}}. \arrow[u, dashed, shift left, "\overline{g}_!"]
\end{tikzcd}$$

Besides (**A3**) and (**A4**) we additionally assume:

(**A5**) The functors β^* and γ^* admit left adjoints:

$$\beta_{(1)} \dashv \beta^*, \qquad\qquad \gamma_{(1)} \dashv \gamma^*.$$

We denote $\varepsilon_\beta\colon \beta_{(1)}\beta^* \Rightarrow \mathrm{id}_{\mathscr{B}}$ the counit. Observe that $f_{(1)}\beta_{(1)}$ is a left adjoint of β^*f^*. Passing to the left mate of the composite $f_{(1)}\beta_{(1)}\overline{g}_! \circ (\overline{c}'\otimes -) \circ \overline{g}^*\beta^*f^* \xRightarrow{\mathsf{pf}_{\overline{g}}} f_{(1)}\beta_{(1)} \circ (\overline{g}_!(\overline{c}')\otimes -) \circ \beta^*f^* \xRightarrow{\mathsf{lpf}_\beta} f_{(1)} \circ (\beta_{(1)}\overline{g}_!(\overline{c}')\otimes -) \circ f^* \xRightarrow{\mathsf{lpf}_f} f_{(1)}\beta_{(1)}\overline{g}_!(\overline{c}')\otimes -$, and using $\overline{g}^*\beta^*f^* = \gamma^*h^*$ we obtain a natural map

$$\mathsf{ltm}_{f\beta,\overline{g}}\colon f_{(1)}\beta_{(1)}\overline{g}_!(\overline{c}'\otimes\overline{c}) \longrightarrow f_{(1)}\beta_{(1)}\overline{g}_!(\overline{c}')\otimes h_{(1)}\gamma_{(1)}(\overline{c}).$$

Let $\mathsf{lpf}_\gamma\colon \gamma_{(1)}(\gamma^*(c)\otimes\overline{c}) \to c\otimes\gamma_{(1)}(\overline{c})$ be the left mate of $\mathsf{mon}_\gamma\colon (\gamma^*(c)\otimes -)\circ\gamma^* \xRightarrow{\cong} \gamma^*\circ(c\otimes -)$.

2.2.7. Lemma. *For all $c \in \mathscr{C}$ and $\overline{c} \in \overline{\mathscr{C}}$ the following diagram commutes:*

$$\begin{tikzcd}
f_{(1)}\beta_{(1)}\overline{g}_!\big(\gamma^*(c)\otimes\overline{c}\big) \arrow[r, "\mathsf{ltm}_{f\beta,\overline{g}}"] \arrow[d, "f_{(1)}l(\phi)"'] & f_{(1)}\beta_{(1)}\overline{g}_!\gamma^*(c)\otimes h_{(1)}\gamma_{(1)}(\overline{c}) \arrow[d, "f_{(1)}\beta_{(1)}\phi\otimes\mathrm{id}"] \\
f_{(1)}g_!\gamma_{(1)}\big(\gamma^*(c)\otimes\overline{c}\big) \arrow[d, "f_{(1)}g_!\mathsf{lpf}_\gamma"'] & f_{(1)}\beta_{(1)}\beta^*g_!(c)\otimes h_{(1)}\gamma_{(1)}(\overline{c}) \arrow[d, "f_{(1)}\varepsilon_\beta\otimes\mathrm{id}"] \\
f_{(1)}g_!\big(c\otimes\gamma_{(1)}(\overline{c})\big) \arrow[r, "\mathsf{ltm}_{f,g}"'] & f_{(1)}g_!(c)\otimes h_{(1)}\gamma_{(1)}(\overline{c}).
\end{tikzcd}$$

□

2.2.8. Consider a diagram

$$\begin{tikzcd}
 & & \mathscr{A} \arrow[dl, "f^*"'] \arrow[dd, bend left, "h^*"] \\
\mathscr{D} \arrow[urr, bend left, "\alpha^*"] \arrow[r, "\beta^*"] \arrow[drr, bend right, "\gamma^*"'] & \mathscr{B} \arrow[ur, dashed, "f_!"'] \arrow[dr, "g^*"'] & \\
 & & \mathscr{C} \arrow[ul, dashed, "g_!"'] \arrow[uu, dashed, bend left, "h_!"]
\end{tikzcd}$$

where the solid diagram commutes and consists of (strongly) symmetric monoidal functors between symmetric monoidal categories, and where the dashed diagram commutes. The commutativity of the right triangles is witnessed by natural isomorphisms

$$\lambda\colon h_! \xRightarrow{\cong} f_!g_! \qquad \text{and} \qquad \mu\colon h^* \xRightarrow{\cong} g^*f^*,$$

where μ is monoidal. We make the following assumption:

(**A6**) The functors α^*, β^*, and γ^* admit left adjoints:

$$\alpha_{(1)} \dashv \alpha^*, \qquad \beta_{(1)} \dashv \beta^*, \qquad \gamma_{(1)} \dashv \gamma.$$

Assume moreover that there are natural isomorphisms

$$\begin{aligned}
\mathsf{pf}_f\colon &\quad f_!\big(b\otimes f^*(a)\big) &\xrightarrow{\cong}&\quad f_!(b)\otimes a,\\
\mathsf{pf}_g\colon &\quad g_!\big(c\otimes g^*(b)\big) &\xrightarrow{\cong}&\quad g_!(c)\otimes b,\\
\mathsf{pf}_h\colon &\quad h_!\big(c\otimes h^*(a)\big) &\xrightarrow{\cong}&\quad h_!(c)\otimes a,
\end{aligned}$$

which satisfy the following conditions:

(**A7**) For all $a \in \mathscr{A}$, $c \in \mathscr{C}$ the diagram

$$\begin{tikzcd}
h_!\big(c \otimes h^*(a)\big) \arrow[rr, "\mathsf{pf}_h"] \arrow[d, "\lambda"'] & & h_!(c) \otimes a \arrow[dd, "\lambda \otimes \mathrm{id}"] \\
f_! g_!\big(c \otimes h^*(a)\big) \arrow[d, "f_! g_!(\mathrm{id} \otimes \mu)"'] & & \\
f_! g_!\big(c \otimes g^* f^*(a)\big) \arrow[r, "f_! \mathsf{pf}_g"'] & f_!\big(g_!(c) \otimes f^*(a)\big) \arrow[r, "\mathsf{pf}_f"'] & f_! g_!(c) \otimes a
\end{tikzcd}$$

is commutative.

(**A8**) For all $a', a \in \mathscr{A}$, $b \in \mathscr{B}$ the diagram

$$\begin{tikzcd}
f_!\big((b \otimes f^*(a')) \otimes f^*(a)\big) \arrow[r, "\mathsf{pf}_f"] \arrow[d, "f_! \mathsf{assoc}"', "\cong"] & f_!\big(b \otimes f^*(a')\big) \otimes a \arrow[r, "\mathsf{pf}_f \otimes \mathrm{id}"] & \big(f_!(b) \otimes a'\big) \otimes a \arrow[d, "\cong"', "\mathsf{assoc}"] \\
f_!\big(b \otimes (f^*(a') \otimes f^*(a))\big) \arrow[r, "f_!(\mathrm{id} \otimes \mathsf{mon}_f)"'] & f_!\big(b \otimes f^*(a' \otimes a)\big) \arrow[r, "\mathsf{pf}_f"'] & f_!(b) \otimes (a' \otimes a)
\end{tikzcd}$$

is commutative, and similarly with f replaced by g. Here, assoc denotes the associativity constraint in the respective monoidal categories.

Finally, let $\widetilde{\mathsf{pf}}_g$ be the unique natural isomorphism making the diagram

$$\begin{tikzcd}
g_!\big(g^*(b) \otimes c\big) \arrow[r, "\widetilde{\mathsf{pf}}_g"] \arrow[d, "\mathsf{sym}"', "\cong"] & b \otimes g_!(c) \arrow[d, "\cong"', "\mathsf{sym}"] \\
g_!\big(c \otimes g^*(b)\big) \arrow[r, "\mathsf{pf}_g"'] & g_!(c) \otimes b
\end{tikzcd}$$

commutative, where sym denotes the symmetry constraints in the respective symmetric monoidal categories.

2.2.9. Lemma. *For all $b \in \mathscr{B}$ and $c', c \in \mathscr{C}$ the following diagram commutes:*

$$\begin{tikzcd}
\alpha_{(1)} h_!\big((g^*(b) \otimes c') \otimes c\big) \arrow[r, "\mathsf{ltm}_{\alpha,h}"] \arrow[d, "\alpha_{(1)}\lambda"'] & \alpha_{(1)} h_!\big(g^*(b) \otimes c'\big) \otimes \gamma_{(1)}(c) \arrow[r, "\alpha_{(1)}\lambda \otimes \mathrm{id}"] & \alpha_{(1)} f_! g_!\big(g^*(b) \otimes c'\big) \otimes \gamma_{(1)}(c) \arrow[d, "\alpha_{(1)} f_! \widetilde{\mathsf{pf}}_g \otimes \mathrm{id}"] \\
\alpha_{(1)} f_! g_!\big((g^*(b) \otimes c') \otimes c\big) \arrow[d, "\alpha_{(1)} f_! g_! \mathsf{assoc}"'] & & \alpha_{(1)} f_!\big(b \otimes g_!(c')\big) \otimes \gamma_{(1)}(c) \arrow[d, "\mathsf{ltm}_{\alpha,f} \otimes \mathrm{id}"] \\
\alpha_{(1)} f_! g_!\big(g^*(b) \otimes (c' \otimes c)\big) \arrow[d, "\alpha_{(1)} f_! \widetilde{\mathsf{pf}}_g"'] & & \big(\alpha_{(1)} f_!(b) \otimes \beta_{(1)} g_!(c')\big) \otimes \gamma_{(1)}(c) \arrow[d, "\mathsf{assoc}"] \\
\alpha_{(1)} f_!\big(b \otimes g_!(c' \otimes c)\big) \arrow[r, "\mathsf{ltm}_{\alpha,f}"'] & \alpha_{(1)} f_!(b) \otimes \beta_{(1)} g_!(c' \otimes c) \arrow[r, "\mathrm{id} \otimes \mathsf{ltm}_{\beta,g}"'] & \alpha_{(1)} f_!(b) \otimes \big(\beta_{(1)} g_!(c') \otimes \gamma_{(1)}(c)\big),
\end{tikzcd}$$

where $\mathsf{ltm}_{\alpha,h}$, $\mathsf{ltm}_{\alpha,f}$, and $\mathsf{ltm}_{\beta,g}$ are defined as in §2.2.4.

□

2.3. Compact induction and derived coinvariants. Let k be a field of characteristic p.

2.3.1. Let G be a p-adic Lie group and $H \leq G$ a closed subgroup. Given a smooth H-representation V, we denote by $\operatorname{c-Ind}_H^G V \in \operatorname{Rep}_k(G)$ the space of all locally constant functions $f\colon G \to V$ which satisfy $f(hg) = hf(g)$ for all $h \in H$, $g \in G$, and have compact support in $H\backslash G$; note that f is fixed by an open subgroup of G under the right translation action. The functor $V \mapsto \operatorname{c-Ind}_H^G V$ is exact and hence extends to a triangulated functor on the (unbounded) derived categories:

$$\operatorname{c-Ind}_H^G \colon \mathrm{D}(H) \longrightarrow \mathrm{D}(G).$$

As $\operatorname{c-Ind}_H^G$ clearly commutes with direct sums, Brown representability shows that it admits a right adjoint $\mathcal{R}_H^G$, cf. [Hey22, Corollary 2.3.10(a)]. The functors $\operatorname{c-Ind}_H^G$ and $\mathcal{R}_H^G$ are transitive, *i.e.*, if $H \leq H' \leq G$ is a closed intermediate group, then $\operatorname{c-Ind}_H^G \cong \operatorname{c-Ind}_{H'}^G \operatorname{c-Ind}_H^{H'}$ and $\mathcal{R}_H^G \cong \mathcal{R}_H^{H'} \mathcal{R}_{H'}^G$. Observe that $\mathcal{R}_K^G \cong \operatorname{Res}_K^G$ provided $K \leq G$ is an open subgroup; in this case we prefer to write ind_K^G instead of $\operatorname{c-Ind}_K^G$.

2.3.2. With $H \leq G$ as above, given a smooth H-representation V, the group G acts by right translation on the space of all functions $f\colon G \to V$ which satisfy $f(hg) = hf(g)$ for all $h \in H$, $g \in G$; we denote by $\operatorname{Ind}_H^G V \in \operatorname{Rep}_k(G)$ the subspace of functions which are fixed by an open subgroup of G. The functor $V \mapsto \operatorname{Ind}_H^G V$ is left exact; if $H\backslash G$ is compact, then $\operatorname{c-Ind}_H^G \xRightarrow{\cong} \operatorname{Ind}_H^G$ is even exact. Taking the right derived functor, we obtain a triangulated functor

$$\operatorname{RInd}_H^G \colon \mathrm{D}(H) \longrightarrow \mathrm{D}(G),$$

which is right adjoint to restriction $\operatorname{Res}_H^G \colon \mathrm{D}(G) \to \mathrm{D}(H)$. By a slight abuse of notation we write Ind_H^G for RInd_H^G in case $H\backslash G$ is compact.

The restriction functor $\operatorname{Res}_H^G \colon \mathrm{D}(G) \to \mathrm{D}(H)$ satisfies the following compatibility with compact induction.

2.3.3. Lemma (Projection formula)**.** *Let $H \leq G$ be a closed subgroup. There exists an isomorphism*

$$\operatorname{c-Ind}_H^G\bigl(X \otimes_k \operatorname{Res}_H^G Y\bigr) \xrightarrow{\cong} \operatorname{c-Ind}_H^G(X) \otimes_k Y \qquad \textit{in } \mathrm{D}(G)$$

which is natural in $X \in \mathrm{D}(H)$ and $Y \in \mathrm{D}(G)$. Moreover, the natural map

$$\hom_G\bigl(\operatorname{c-Ind}_H^G X, Y\bigr) \xrightarrow{\cong} \operatorname{RInd}_H^G \hom_H\bigl(X, \mathcal{R}_H^G Y\bigr)$$

is an isomorphism in $\mathrm{D}(G)$, for all $X \in \mathrm{D}(H)$ and $Y \in \mathrm{D}(G)$.

Proof. We describe the inverse map. For any $V \in \operatorname{Rep}_k(H)$ and $W \in \operatorname{Rep}_k(G)$, the map $\operatorname{c-Ind}_H^G(V) \otimes_k W \to \operatorname{c-Ind}_H^G(V \otimes_k \operatorname{Res}_H^G W)$ given by $f \otimes w \mapsto [g \mapsto f(g) \otimes w]$ is a natural isomorphism, cf. [AHV19, Lemma 2.5]. Since $\operatorname{c-Ind}_H^G$, Res_H^G, and $\_ \otimes_k \_$ are exact functors, this isomorphism readily extends to the derived categories.

The isomorphism $\operatorname{c-Ind}_H^G(X \otimes_k \_) \operatorname{Res}_H^G \xRightarrow{\cong} \operatorname{c-Ind}_H^G X \otimes_k \_$ yields, by passing to the right adjoints, an isomorphism $\hom_G(\operatorname{c-Ind}_H^G X, \_) \xRightarrow{\cong} \operatorname{RInd}_H^G \hom_G(X, \_)\mathcal{R}_H^G$, which proves the last assertion. □

2.3.4. Given a closed normal subgroup $N \trianglelefteq G$, the inflation $\operatorname{Inf}_G^{G/N} \colon \mathrm{D}(G/N) \to \mathrm{D}(G)$ along the projection $G \to G/N$ admits a right adjoint $\mathrm{RH}^0(N, \_)$ as well as a left adjoint, see [Hey22, Theorem 3.2.3], which we call the functor of *derived coinvariants* and denote

$$\mathrm{L}_N \colon \mathrm{D}(G) \longrightarrow \mathrm{D}(G/N).$$

If G is compact and torsion-free, then $\mathrm{RH}^0(N, \_)$ admits a right adjoint denoted $F_G^{G/N}$, [Hey22, Lemma 3.1.3]. In this case, we call $\omega_G \coloneqq F_G^{G/N}(\mathbf{1}) \in \mathrm{D}(G)$ the *dualizing complex*; we remark that $\omega_G \cong k[\dim N]$, [Hey22, Proposition 3.1.10].

2.3.5. Lemma. *Let G be compact and torsion-free, and let $N \leq H \leq G$ be closed subgroups such that $N \trianglelefteq G$. Then one has a commutative diagram*

$$\begin{tikzcd} \mathrm{D}(G/N) \arrow[r, "\mathrm{Res}^{G/N}_{H/N}"] \arrow[d, "F^{G/N}_G"'] & \mathrm{D}(H/N) \arrow[d, "F^{H/N}_H"] \\ \mathrm{D}(G) \arrow[r, "\mathrm{Res}^G_H"'] & \mathrm{D}(H). \end{tikzcd}$$

Proof. We apply Lemma 2.2.3(a) to the isomorphism $\alpha\colon \mathrm{Res}^G_H \mathrm{Inf}^{G/N}_G \overset{\cong}{\Longrightarrow} \mathrm{Inf}^{H/N}_H \mathrm{Res}^{G/N}_{H/N}$. By §2.3.4 and the projection formula for $\mathrm{RH}^0(N, \_)$, [Hey22, Lemma 3.1.2], the assumptions (**A1**) and (**A2**) of §2.2.2 are satisfied. Thus, we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Res}^G_H \omega^{G/N}_G \otimes_k \mathrm{Res}^G_H \mathrm{Inf}^{G/N}_G(X) \arrow[r, "\cong"] \arrow[d, "\beta_{\mathbf{1}}\otimes\alpha"'] & \mathrm{Res}^G_H F^{G/N}_G(X) \arrow[d, "\beta"] \\ \omega^{H/N}_H \otimes_k \mathrm{Inf}^{H/N}_H \mathrm{Res}^{G/N}_{H/N}(X) \arrow[r, "\cong"'] & F^{H/N}_H \mathrm{Res}^{G/N}_{H/N}(X), \end{tikzcd}$$

where $\beta \coloneqq r(r(\alpha^{-1})^{-1})$ in the notation of §2.2.2. The top and bottom horizontal maps are isomorphisms by [Hey22, Corollary 3.1.7]. The assertion is that β is an isomorphism. But note that β is non-zero (as the right mate of a non-zero map) and hence $\beta_{\mathbf{1}} \neq 0$ by the commutativity of the diagram. But since $\omega^{G/N}_G$ and $\omega^{H/N}_H$ are characters, $\beta_{\mathbf{1}}$ is necessarily an isomorphism. Since also α is invertible, we deduce that the left vertical map in the diagram is an isomorphism. Therefore, β is an isomorphism. $\square$

2.3.6. Lemma. *Let $H, N \trianglelefteq G$ be closed normal subgroups such that HN is closed. Assume $\mathrm{L}_{H\cap N}(\mathbf{1}) \cong \mathbf{1}$. Then one has a commutative diagram*

$$\begin{tikzcd} \mathrm{D}(G/H) \arrow[r, "\mathrm{Inf}^{G/H}_G"] \arrow[d, "\mathrm{L}_{HN/H}"'] & \mathrm{D}(G) \arrow[d, "\mathrm{L}_N"] \\ \mathrm{D}(G/HN) \arrow[r, "\mathrm{Inf}^{G/HN}_{G/N}"'] & \mathrm{D}(G/N). \end{tikzcd}$$

Proof. The natural isomorphism $\mathrm{Inf}^{G/H}_G \mathrm{Inf}^{G/HN}_{G/H} \overset{\cong}{\Longrightarrow} \mathrm{Inf}^{G/N}_G \mathrm{Inf}^{G/HN}_{G/N}$ induces, by passing to the left mates, a natural transformation $\mathrm{L}_N \mathrm{Inf}^{G/H}_G \Rightarrow \mathrm{Inf}^{G/HN}_{G/N} \mathrm{L}_{HN/H}$ which we claim is an isomorphism of functors $\mathrm{D}(G/H) \to \mathrm{D}(G/N)$. This can be checked after applying the conservative functor $\mathrm{Res}^{G/N}_1$. By the compatibility of restriction with derived coinvariants, [Hey22, Proposition 3.2.19], and inflation we reduce to the case $G = N$. Hence, we have to show that the natural map

$$\mathrm{L}_G \mathrm{Inf}^{G/H}_G \Longrightarrow \mathrm{L}_{G/H} \tag{2.3.7}$$

which arises as the left mate of $\varphi\colon \mathrm{Inf}^{G/H}_G \mathrm{Inf}^1_{G/H} \overset{\cong}{\Longrightarrow} \mathrm{Inf}^1_G$ is an isomorphism. Let us denote $\psi\colon \mathrm{L}_G \overset{\cong}{\Longrightarrow} \mathrm{L}_{G/H}\mathrm{L}_H$ the isomorphism obtained from φ by passing to the left adjoints. By [Hey22, Corollary 3.4.23], the hypothesis $\mathrm{L}_H(\mathbf{1}) \cong \mathbf{1}$ means that the counit $\varepsilon\colon \mathrm{L}_H \mathrm{Inf}^{G/H}_G \overset{\cong}{\Longrightarrow} \mathrm{id}_{\mathrm{D}(G/H)}$ is an isomorphism. Consider the following commutative diagram

$$\begin{tikzcd}
& \mathrm{Nat}\big(L_{G/H}, \mathrm{L}_{G/H}\big) \arrow[r, leftrightarrow] \arrow[d, "(\mathrm{L}_{G/H}\varepsilon)^*"'] & \mathrm{Nat}\big(\mathrm{Inf}^1_{G/H}, \mathrm{Inf}^1_{G/H}\big) \ni \mathrm{id}_{\mathrm{Inf}^1_{G/H}} \arrow[d, "\mathrm{Inf}^{G/H}_G"] \\
& \mathrm{Nat}\big(\mathrm{L}_{G/H}\mathrm{L}_H \mathrm{Inf}^{G/H}_G, \mathrm{L}_{G/H}\big) \arrow[r, leftrightarrow] \arrow[d, "(\psi \mathrm{Inf}^{G/H}_G)^*"'] & \mathrm{Nat}\big(\mathrm{Inf}^{G/H}_G \mathrm{Inf}^1_{G/H}, \mathrm{Inf}^{G/H}_G \mathrm{Inf}^1_{G/H}\big) \arrow[d, "\varphi_*"] \\
(2.3.7) \in & \mathrm{Nat}\big(\mathrm{L}_G \mathrm{Inf}^{G/H}_G, \mathrm{L}_{G/H}\big) \arrow[r, leftrightarrow] & \mathrm{Nat}\big(\mathrm{Inf}^{G/H}_G \mathrm{Inf}^1_{G/H}, \mathrm{Inf}^1_G\big),
\end{tikzcd}$$

where the horizontal maps are given by passing to the right/left mates. Now, the map (2.3.7) is the image of $\mathrm{id}_{\mathrm{Inf}^1_{G/H}}$ under the lower-right circuit. By the commutativity of the diagram we deduce that (2.3.7) coincides with the composition

$$\mathrm{L}_G \mathrm{Inf}^{G/H}_G \xRightarrow{\psi \mathrm{Inf}^{G/H}_G} \mathrm{L}_{G/H}\mathrm{L}_H \mathrm{Inf}^{G/H}_G \xRightarrow{\mathrm{L}_{G/H}\varepsilon} \mathrm{L}_{G/H}$$

of two isomorphisms and is thus itself an isomorphism. □

2.3.8. Lemma. *Let $N \leq H \leq G$ be closed subgroups such that $N \trianglelefteq G$ is normal. One has commutative diagrams*

$$\begin{tikzcd}
\mathrm{D}(H) \arrow[r, "\text{c-Ind}^G_H"] \arrow[d, "\mathrm{L}_N"'] & \mathrm{D}(G) \arrow[d, "\mathrm{L}_N"] \\
\mathrm{D}(H/N) \arrow[r, "\text{c-Ind}^{G/N}_{H/N}"'] & \mathrm{D}(G/N)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{D}(G/N) \arrow[r, "\mathcal{R}^{G/N}_{H/N}"] \arrow[d, "\mathrm{Inf}^{G/N}_G"'] & \mathrm{D}(H/N) \arrow[d, "\mathrm{Inf}^{H/N}_H"] \\
\mathrm{D}(G) \arrow[r, "\mathcal{R}^G_H"'] & \mathrm{D}(H).
\end{tikzcd}$$

Proof. Denoting $\mathrm{pr}\colon G \to G/N$ the projection map, we consider the natural isomorphism

$$\text{c-Ind}^G_H \mathrm{Inf}^{H/N}_H \overset{\cong}{\Longrightarrow} \mathrm{Inf}^{G/N}_G \text{c-Ind}^{G/N}_{H/N} \tag{2.3.9}$$

of functors $\mathrm{D}(H/N) \to \mathrm{D}(G)$, whose inverse is given by $f \mapsto f \circ \mathrm{pr}$ on the level of underived categories. Passing to the left and right mates, respectively, we obtain natural transformations

$$\mathrm{L}_N \text{c-Ind}^G_H \Longrightarrow \text{c-Ind}^{G/N}_{H/N} \mathrm{L}_N \tag{2.3.10}$$

$$\mathrm{Inf}^{H/N}_H \mathcal{R}^{G/N}_{H/N} \Longrightarrow \mathcal{R}^G_H \mathrm{Inf}^{G/N}_G . \tag{2.3.11}$$

Note that (2.3.11) is obtained from (2.3.10) by passing to the right adjoints, and hence one is an isomorphism if and only if the other is. Thus, it suffices to prove that (2.3.11) is an isomorphism.

Let $K \leq G$ be any open subgroup; we write $K_H = K \cap H$ and $K_N = K \cap N$. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Res}^H_{K_H} \mathrm{Inf}^{H/N}_H \mathcal{R}^{G/N}_{H/N} \arrow[r, Rightarrow, "\cong"] \arrow[d, Rightarrow, "\mathrm{Res}^H_{K_H}(2.3.11)"'] & \mathrm{Inf}^{K_H/K_N}_{K_H} \mathrm{Res}^{H/N}_{K_H/K_N} \mathcal{R}^{G/N}_{H/N} \arrow[r, Rightarrow, "\cong"] & \mathrm{Inf}^{K_H/K_N}_{K_H} \mathcal{R}^{K/K_N}_{K_H/K_N} \mathrm{Res}^{G/N}_{K/K_N} \arrow[d, Rightarrow, "(2.3.11)\mathrm{Res}^{G/N}_{K/K_N}"] \\
\mathrm{Res}^H_{K_H} \mathcal{R}^G_H \mathrm{Inf}^{G/N}_G \arrow[r, Rightarrow, "\cong"'] & \mathcal{R}^K_{K_H} \mathrm{Res}^G_K \mathrm{Inf}^{G/N}_G \arrow[r, Rightarrow, "\cong"'] & \mathcal{R}^K_{K_H} \mathrm{Inf}^{K/K_N}_K \mathrm{Res}^{G/N}_{K/K_N} .
\end{tikzcd}$$

Here, the upper right and lower left horizontal maps are isomorphisms, because compact induction, hence also its right adjoint, is transitive, [Vig96, I.5.3]. As $\mathrm{Res}^H_{K_H}$ is conservative, (2.3.11) is an isomorphism if and only if the left vertical map is an isomorphism, if and only if the right vertical map is an isomorphism.

Thus, replacing G by K, we may assume from the beginning that G is compact and torsion-free. In this setting, the map (2.3.10) reads

$$\mathrm{L}_N \operatorname{Ind}_H^G \Longrightarrow \operatorname{Ind}_{H/N}^{G/N} \mathrm{L}_N. \tag{2.3.12}$$

We finish by proving that (2.3.12) is an isomorphism. In the proof of Lemma 2.3.5 we verified that the isomorphism $\alpha\colon \operatorname{Res}_H^G \operatorname{Inf}_G^{G/N} \xRightarrow{\cong} \operatorname{Inf}_H^{H/N} \operatorname{Res}_{H/N}^{G/N}$ satisfies the assumptions (**A1**) and (**A2**) of §2.2.2 in the notation of which the map (2.3.12) is just $\gamma \coloneqq l(r(\alpha)^{-1})$. Lemma 2.2.3(b) shows that the diagram

$$\begin{tikzcd}
\mathrm{RH}^0(N, \omega_G^{G/N} \otimes_k -) \operatorname{Ind}_H^G \arrow[r, Rightarrow, "\cong"] \arrow[d, Rightarrow, "\cong"'] & \mathrm{RH}^0(N, -) \operatorname{Ind}_H^G\big(\operatorname{Res}_H^G \omega_G^{G/N} \otimes_k -\big) \arrow[r, Rightarrow, "\cong"] & \operatorname{Ind}_{H/N}^{G/N} \mathrm{RH}^0(N, \omega_H^{H/N} \otimes_k -) \arrow[d, Rightarrow, "\cong"] \\
\mathrm{L}_N \operatorname{Ind}_H^G \arrow[rr, Rightarrow, "\gamma"'] & & \operatorname{Ind}_{H/N}^{G/N} \mathrm{L}_N
\end{tikzcd}$$

commutes. The left and right vertical maps are isomorphisms by [Hey22, Corollary 3.1.8]. Hence, γ is an isomorphism. □

2.3.13. Remark. Let $N \trianglelefteq G$ and $H \le G$ be closed subgroups such that $G = HN$. Then one has a natural isomorphism

$$\mathrm{RH}^0(N, -) \operatorname{RInd}_H^G \xRightarrow{\cong} \mathrm{RH}^0(H \cap N, -)$$

of functors $\mathrm{D}(H) \to \mathrm{D}(G/N)$ arising from the isomorphism $\operatorname{Inf}_H^{H/H\cap N} \xRightarrow{\cong} \operatorname{Res}_H^G \operatorname{Inf}_G^{G/N}$ by passing to the right adjoints.

We will need the following dual statement.

2.3.14. Lemma. *Let $N \trianglelefteq G$ and $H \le G$ be closed subgroups such that $G = HN$. Assume there exists a subnormal series $H \cap N = N_r \trianglelefteq N_{r-1} \trianglelefteq \cdots \trianglelefteq N_1 \trianglelefteq N_0 = N$ by closed subgroups such that $\mathrm{L}_{N_i}(\mathbf{1}) = \mathbf{1}$ for all $i = 1, \dots, r$. Then*

$$\omega_{H\backslash G} \coloneqq \mathrm{L}_N \operatorname{c-Ind}_H^G(\mathbf{1}) \quad \in \mathrm{D}(G/N) \tag{2.3.15}$$

is a character sitting in degree $-\dim H\backslash G$, and the following diagram commutes:

$$\begin{tikzcd}
\mathrm{D}(H) \arrow[r, "\operatorname{c-Ind}_H^G"] \arrow[d, "\mathrm{L}_{H\cap N}"'] & \mathrm{D}(G) \arrow[d, "\mathrm{L}_N"] \\
\mathrm{D}(H/H \cap N) \arrow[r, "\omega_{H\backslash G} \otimes -"'] & \mathrm{D}(G/N).
\end{tikzcd}$$

Proof. We will implicitly make the identification $H/H \cap N \cong G/N$ throughout the proof. Let $Y \in \mathrm{D}(k)$ be arbitrary and put $X \coloneqq \operatorname{Inf}_H^1 Y$. Consider the natural isomorphism

$$\mathrm{L}_N \operatorname{c-Ind}_H^G(X \otimes_k -) \operatorname{Res}_H^G \operatorname{Inf}_G^{G/N} \xRightarrow[\cong]{\mathrm{L}_N \mathsf{pf}} \mathrm{L}_N(\operatorname{c-Ind}_H^G X \otimes_k -) \operatorname{Inf}_G^{G/N} \xRightarrow[\cong]{\mathsf{lpf}} \mathrm{L}_N \operatorname{c-Ind}_H^G X \otimes_k -,$$

where pf is the isomorphism in Lemma 2.3.3 and lpf is the map lpf_f from §2.2.4 for the functor $f^* = \operatorname{Inf}_G^{G/N}$; note that lpf is an isomorphism by [Hey22, Corollary 3.3.5]. Using $\operatorname{Res}_H^G \operatorname{Inf}_G^{G/N} = \operatorname{Inf}_H^{G/N}$ and passing to the left mate, we obtain a natural map

$$\rho_{N,H}\colon \mathrm{L}_N \operatorname{c-Ind}_H^G(X \otimes X') \Longrightarrow \mathrm{L}_N \operatorname{c-Ind}_H^G X \otimes_k \mathrm{L}_{H\cap N} X'$$

for any $X' \in \mathrm{D}(H)$. We claim that $\rho_{N,H}$ is an isomorphism, which then for $X = \mathbf{1}$ witnesses the commutativity of the asserted diagram.

We reduce to the case $N = G$ as follows: apply the discussion of §2.2.4 to the diagram

$$\begin{tikzcd}
\mathrm{D}(G/N) \arrow[rrr, "\mathrm{Res}_1^{G/N}"] \arrow[dr, "\mathrm{Inf}_G^{G/N}"] \arrow[dd, bend right, "\mathrm{Inf}_H^{G/N}"'] & & & \mathrm{D}(k) \arrow[dl, "\mathrm{Inf}_N^1"'] \arrow[dd, bend left, "\mathrm{Inf}_{H\cap N}^1"] \\
& \mathrm{D}(G) \arrow[r, "\mathrm{Res}_N^G"] \arrow[dl, "\mathrm{Res}_H^G"'] & \mathrm{D}(N) \arrow[dr, "\mathrm{Res}_{H\cap N}^N"] & \\
\mathrm{D}(H) \arrow[rrr, "\mathrm{Res}_{H\cap N}^H"'] \arrow[ur, dashed, "\text{c-Ind}_H^G"'] & & & \mathrm{D}(H\cap N). \arrow[ul, dashed, "\text{c-Ind}_{H\cap N}^N"]
\end{tikzcd}$$

Note that, since $G = HN$, restriction of functions induces a natural isomorphism whose inverse

$$\text{c-Ind}_{H\cap N}^N \mathrm{Res}_{H\cap N}^H \overset{\cong}{\Longrightarrow} \mathrm{Res}_N^G \text{c-Ind}_H^G$$

plays the role of ϕ in §2.2.4. The condition (**A3**) is satisfied by [Hey22, Theorem 3.2.3]. The isomorphisms in (**A4**) are supplied by Lemma 2.3.3 for which (**A4**) is easily verified. Now, Lemma 2.2.5 provides a commutative diagram

$$\begin{tikzcd}
\mathrm{L}_N \,\text{c-Ind}_{H\cap N}^N\big(\mathrm{Res}_{H\cap N}^H X \otimes_k \mathrm{Res}_{H\cap N}^H X'\big) \arrow[r, "\rho_{N,H\cap N}"] \arrow[d, "\cong"'] & \mathrm{L}_N \,\text{c-Ind}_{H\cap N}^N \mathrm{Res}_{H\cap N}^H X \otimes_k \mathrm{L}_{H\cap N} \mathrm{Res}_{H\cap N}^H X' \arrow[d, "\cong"] \\
\mathrm{Res}_1^{G/N} \mathrm{L}_N \,\text{c-Ind}_H^G(X \otimes_k X') \arrow[r, "\mathrm{Res}_1^{G/N} \rho_{N,H}"'] & \mathrm{Res}_1^{G/N}\big(\mathrm{L}_N \,\text{c-Ind}_H^G X \otimes_k \mathrm{L}_{H\cap N} X'\big).
\end{tikzcd}$$

The maps $l(\sigma)$ and $l(\rho)$ in Lemma 2.2.5 correspond to the natural transformations

$$l(\sigma)\colon \mathrm{L}_N \mathrm{Res}_N^G \overset{\cong}{\Longrightarrow} \mathrm{Res}_1^{G/N} \mathrm{L}_N \quad \text{and} \quad l(\rho)\colon \mathrm{L}_{H\cap N} \mathrm{Res}_{H\cap N}^H \overset{\cong}{\Longrightarrow} \mathrm{Res}_1^{G/N} \mathrm{L}_{H\cap N},$$

which are isomorphisms by [Hey22, Proposition 3.2.19]. Therefore, the vertical maps are isomorphisms. Since $\mathrm{Res}_1^{G/N}$ is conservative, it suffices to show that the top map is an isomorphism. In other words, we may assume from the beginning that $N = G$ and have to show that the map

$$\rho_{N,H}\colon \mathrm{L}_N \,\text{c-Ind}_H^N(X \otimes_k X') \longrightarrow \mathrm{L}_N \,\text{c-Ind}_H^N X \otimes_k \mathrm{L}_{H\cap N} X' \tag{2.3.16}$$

is a natural isomorphism. We induct on the length r of the subnormal series. Let $r = 1$ so that $H \trianglelefteq N$.[2] We apply the discussion in §2.2.6 to the diagram

$$\begin{tikzcd}
\mathrm{D}(k) \arrow[r, "\mathrm{Inf}_{N/H}^1"] \arrow[dr, equal, bend right] & \mathrm{D}(N/H) \arrow[r, "\mathrm{Inf}_N^{N/H}"] \arrow[d, shift right, "\mathrm{Res}_1^{N/H}"'] & \mathrm{D}(N) \arrow[d, shift left, "\mathrm{Res}_H^N"] \\
& \mathrm{D}(k) \arrow[r, "\mathrm{Inf}_H^1"'] \arrow[u, dashed, shift right, "\text{c-Ind}_1^{N/H}"'] & \mathrm{D}(H). \arrow[u, dashed, shift left, "\text{c-Ind}_H^N"]
\end{tikzcd}$$

The map $l(\phi)$ in Lemma 2.2.7 is now given by $\mathrm{L}_H \,\text{c-Ind}_H^N \overset{\cong}{\Longrightarrow} \text{c-Ind}_1^{N/H} \mathrm{L}_H$, see (2.3.10). The conditions (**A3**)–(**A5**) are clearly satisfied. By Lemma 2.2.7, and since $\mathrm{L}_N \cong \mathrm{L}_{N/H}\mathrm{L}_H$, we obtain

[2]Note that, if $r = 0$, then $N = H = G$ and the assertion of the lemma is tautological. It is true, although not trivial, that $\rho_{N,N}$ is an isomorphism and will be left as an easy exercise.

a commutative diagram

$$
\begin{tikzcd}
\mathrm{L}_N\,\text{c-Ind}_H^N\big(\mathrm{Inf}_H^1 Y \otimes_k X'\big) \arrow[r, "\rho_{N,H}"] \arrow[d, "\cong"'] & \mathrm{L}_N\,\text{c-Ind}_H^N \mathrm{Inf}_H^1 Y \otimes_k \mathrm{L}_H X' \arrow[d, "\cong"] \\
\mathrm{L}_{N/H}\,\text{c-Ind}_1^{N/H}\,\mathrm{L}_H\big(\mathrm{Inf}_H^1 Y \otimes_k X'\big) \arrow[d, "\cong"'] & \mathrm{L}_N\,\mathrm{Inf}_N^{N/H}\,\text{c-Ind}_1^{N/H} Y \otimes_k \mathrm{L}_H X' \arrow[d, "\cong"', "\varepsilon\otimes\mathrm{id}"] \\
\mathrm{L}_{N/H}\,\text{c-Ind}_1^{N/H}(Y \otimes_k \mathrm{L}_H X') \arrow[r, "\cong", "\rho_{N/H,1}"'] & \mathrm{L}_{N/H}\,\text{c-Ind}_1^{N/H} Y \otimes \mathrm{L}_H X'.
\end{tikzcd}
$$

The lower left vertical map is an isomorphism by [Hey22, Corollary 3.3.5], and the lower right vertical map is an isomorphism by [Hey22, Corollary 3.4.23] and the hypothesis $\mathrm{L}_H(\mathbf{1}) \cong \mathbf{1}$. It is obvious from the construction that $\rho_{N/H,1}$ is an isomorphism. Hence, $\rho_{N,H}$ is an isomorphism, which settles the case $r = 1$.

Let now $r \geq 2$. We apply the discussion in §2.2.8 to the diagram

$$
\begin{tikzcd}
 & & \mathrm{D}(N) \arrow[dl, "\mathrm{Res}^N_{N_1}"'] \arrow[dd, bend left, "\mathrm{Res}^N_H"] \\
\mathrm{D}(k) \arrow[urr, bend left, "\mathrm{Inf}^1_N"] \arrow[r, "\mathrm{Inf}^1_{N_1}"] \arrow[drr, bend right, "\mathrm{Inf}^1_H"'] & \mathrm{D}(N_1) \arrow[ur, dashed, "\text{c-Ind}^N_{N_1}"] \arrow[dr, "\mathrm{Res}^{N_1}_H"'] & \\
 & & \mathrm{D}(H) \arrow[ul, dashed, "\text{c-Ind}^{N_1}_H"] \arrow[uu, dashed, "\text{c-Ind}^N_H"]
\end{tikzcd}
$$

Note that there is a natural isomorphism $\text{c-Ind}_H^N \overset{\cong}{\Longrightarrow} \text{c-Ind}_{N_1}^N\,\text{c-Ind}_H^{N_1}$, [Vig96, I.5.3]. Again, the conditions (**A6**)–(**A8**) are easily verified. Let $Y' \in \mathrm{D}(k)$ be arbitrary and put $Z \coloneqq \mathrm{Inf}^1_{N_1} Y'$. By Lemma 2.2.9 we obtain a commutative diagram

$$
\begin{tikzcd}
\mathrm{L}_N\,\text{c-Ind}_H^N\big(\mathrm{Res}_H^{N_1} Z \otimes_k X \otimes_k X'\big) \arrow[r, "\cong"] \arrow[d, "\rho_{N,H}"'] & \mathrm{L}_N\,\text{c-Ind}_{N_1}^N\big(Z \otimes_k \text{c-Ind}_H^{N_1}(X \otimes X')\big) \arrow[d, "\cong"', "\rho_{N,N_1}"] \\
\mathrm{L}_N\,\text{c-Ind}_H^N\big(\mathrm{Res}_H^{N_1} Z \otimes_k X\big) \otimes_k \mathrm{L}_H X' \arrow[d, "\cong"'] & \mathrm{L}_N\,\text{c-Ind}_{N_1}^N Z \otimes_k \mathrm{L}_{N_1}\,\text{c-Ind}_H^{N_1}\big(X \otimes X'\big) \arrow[d, "\cong"', "\mathrm{id}\otimes\rho_{N_1,H}"] \\
\mathrm{L}_N\,\text{c-Ind}_{N_1}^N\big(Z \otimes_k \text{c-Ind}_H^{N_1} X\big) \otimes \mathrm{L}_H X' \arrow[r, "\cong", "\rho_{N,N_1}\otimes\mathrm{id}"'] & \mathrm{L}_N\,\text{c-Ind}_{N_1}^N Z \otimes_k \mathrm{L}_{N_1}\,\text{c-Ind}_H^{N_1} X \otimes_k \mathrm{L}_H X'.
\end{tikzcd}
$$

Here, the top horizontal and lower left vertical map are isomorphisms by Lemma 2.3.3. The maps ρ_{N,N_1} and $\rho_{N_1,H}$ are isomorphisms by the induction hypothesis. It follows that the top left vertical map is an isomorphism. In particular, for $Y' = \mathbf{1}$, this proves that (2.3.16) is an isomorphism. This finishes the induction step.

It remains to prove that $\omega_{H\backslash G} = \mathrm{L}_N\,\text{c-Ind}_H^G\,\mathbf{1}$ is a character sitting in degree $-\dim H\backslash G$. Note that $\mathrm{Res}_1^{G/N}\,\omega_{H\backslash G} \cong \omega_{H\cap N\backslash N}$, so that we may assume $N = G$ from the beginning. We first treat the case $H = \{1\}$ so that the claim becomes $\mathrm{L}_N\,\text{c-Ind}_1^N(\mathbf{1}) \cong k[\dim N]$. Fix a torsion-free open compact subgroup $K_N \leq N$; note that $\dim K_N = \dim N$. We compute

$$\begin{aligned}
\mathrm{L}_N \,\text{c-Ind}_1^N(\mathbf{1}) &\cong \mathrm{L}_N \operatorname{ind}_{K_N}^N \operatorname{Ind}_1^{K_N}(\mathbf{1}) \\
&\cong \mathrm{L}_{K_N} \operatorname{Ind}_1^{K_N}(\mathbf{1}) && \text{(by [Hey22, Proposition 3.2.11])} \\
&\cong \mathrm{RH}^0\big(K_N, \omega_{K_N} \otimes_k \operatorname{Ind}_1^{K_N}(\mathbf{1})\big) && \text{(by [Hey22, Proposition 3.1.8])} \\
&\cong \mathrm{RH}^0\big(K_N, \operatorname{Ind}_1^{K_N} k[\dim K_N]\big) && \text{(by [Hey22, Proposition 3.1.10])} \\
&\cong k[\dim K_N],
\end{aligned}$$

where the last isomorphism is Shapiro's lemma (or Remark 2.3.13). This settles the case $H = \{1\}$.

If $r = 1$, we compute

$$\mathrm{L}_N \,\text{c-Ind}_H^N(\mathbf{1}) \cong \mathrm{L}_N \operatorname{Inf}_N^{N/H} \text{c-Ind}_1^{N/H}(\mathbf{1}) \cong \mathrm{L}_{N/H} \,\text{c-Ind}_1^{N/H}(\mathbf{1}) \cong k[\dim N/H],$$

where the first isomorphism is (2.3.9), and the second follows from the assumption $\mathrm{L}_H(\mathbf{1}) \cong \mathbf{1}$. Now, for $r \geq 2$, we have canonical isomorphisms

$$\begin{aligned}
\omega_{H\backslash N} &= \mathrm{L}_N \,\text{c-Ind}_H^N(\mathbf{1}) \\
&\cong \mathrm{L}_{N/N_1} \mathrm{L}_{N_1} \,\text{c-Ind}_{N_1}^N \,\text{c-Ind}_H^{N_1}(\mathbf{1}) \\
&\cong \mathrm{L}_{N/N_1} \,\text{c-Ind}_1^{N/N_1} \mathrm{L}_{N_1} \,\text{c-Ind}_H^{N_1}(\mathbf{1}) && \text{(by Lemma 2.3.8)} \\
&\cong \mathrm{L}_{N/N_1} \,\text{c-Ind}_1^{N/N_1}(\mathbf{1}) \otimes_k \mathrm{L}_{N_1} \,\text{c-Ind}_H^{N_1}(\mathbf{1}) && \text{(induced by } \rho_{N/N_1,1}) \\
&= \omega_{N/N_1} \otimes_k \omega_{H\backslash N_1}.
\end{aligned}$$

Since $\dim(N/N_1) + \dim(H\backslash N_1) = \dim(H\backslash N)$, it follows that $\omega_{H\backslash N} \cong k[\dim H\backslash N]$. This finishes the proof. □

2.4. The duality character.

2.4.1. The purpose of this section is to give an explicit description of the character (2.3.15). Let G be a p-adic Lie group of dimension d and denote $\mathfrak{g}$ its $\mathbb{Q}_p$-Lie algebra (cf. [Sch11, p. 100]). The determinant of the adjoint action of G on $\mathfrak{g}$ provides a character $\bigwedge^d \mathfrak{g} \colon G \to \mathbb{Q}_p^\times$. Denote $\delta_G \colon G \to \mathbb{Q}_p^\times$ the modulus character of G (cf. [Vig96, Definition I.2.6]). By [Koh17, Theorem 5.1] the character $\bigwedge^d \mathfrak{g} \otimes_{\mathbb{Q}_p} \delta_G$ takes values in $\mathbb{Z}_p^\times$. Denote $\mathfrak{d}_G$ the composition

$$\mathfrak{d}_G \colon G \xrightarrow{\bigwedge^d \mathfrak{g} \otimes_{\mathbb{Q}_p} \delta_G} \mathbb{Z}_p^\times \twoheadrightarrow \mathbb{F}_p^\times \subseteq k^\times.$$

2.4.2. Definition. Let $\Delta \colon G \to G \times G$, $g \mapsto (g, g)$ be the diagonal and write $G_1 \coloneqq G \times \{1\}$ and $G_2 \coloneqq \{1\} \times G$ as subgroups of $G \times G$. The *dualizing character* of G is defined as

$$\omega_G \coloneqq \mathrm{L}_{G_2} \,\text{c-Ind}_{\Delta(G)}^{G\times G}(\mathbf{1}) \in \mathrm{D}(G).$$

2.4.3. Proposition. *One has* $\omega_G \cong \mathfrak{d}_G[d]$ *and* $\omega_G^\vee \cong \mathcal{R}_{\Delta(G)}^{G\times G}(\mathbf{1})$.

Proof. We compute

$$\begin{aligned}
\hom_G(\omega_G, \mathbf{1}) &\cong \mathrm{RH}^0\big(G_2, \hom_{G\times G}(\text{c-Ind}_{\Delta(G)}^{G\times G}(\mathbf{1}), \mathbf{1})\big) && \text{([Hey22, Corollary 3.3.7])} \\
&\cong \mathrm{RH}^0\big(G_2, \operatorname{RInd}_{\Delta(G)}^{G\times G} \hom_G\big(\mathbf{1}, \mathcal{R}_{\Delta(G)}^{G\times G}(\mathbf{1})\big)\big) && \text{(Lemma 2.3.3)} \\
&\cong \mathcal{R}_{\Delta(G)}^{G\times G}(\mathbf{1}),
\end{aligned}$$

where the last isomorphism uses Remark 2.3.13 and that canonically $\hom_G(\mathbf{1}, X) \cong X$ for each $X \in \mathrm{D}(G)$.

We prove $\omega_G \cong \mathfrak{d}_G[d]$ by showing $\omega_G^\vee \cong \chi_G[-d]$, where χ_G is Kohlhaase's duality character. The claim then follows from [Koh17, Theorem 5.1]. Observe that restriction of functions induces a k-linear isomorphism $\text{c-Ind}_{\Delta(G)}^{G\times G}(\mathbf{1}) \xrightarrow{\cong} \text{c-Ind}_1^{G_2}(\mathbf{1})$. Under this identification the $G \times G$ action on $\text{c-Ind}_1^{G_2}(\mathbf{1})$ is given by $\big((g_1, g_2)f\big)(g) = f(g_1^{-1} g g_2)$. In this way, G acts on the 1-dimensional

k-vector space $\mathrm{Ext}^d_G(\text{c-Ind}_1^{G_2}(k), k)$ through the opposite G_1-action on $\text{c-Ind}_1^{G_2}(k)$, which is denoted χ_G in [Koh17, Definition 3.12]. We finish by computing

$$\begin{aligned}
\mathrm{Res}_1^G \omega_G^\vee &\cong \mathrm{RHom}_{\mathrm{D}(k)}\big(\mathrm{Res}_1^G \mathrm{L}_{G_2} \text{c-Ind}_{\Delta(G)}^{G\times G}(\mathbf{1}), \mathbf{1}\big) \\
&\cong \mathrm{RHom}_{\mathrm{D}(k)}\big(\mathrm{L}_{G_2} \mathrm{Res}_{G_2}^{G\times G} \text{c-Ind}_{\Delta(G)}^{G\times G}(\mathbf{1}), \mathbf{1}\big) && \text{([Hey22, Proposition 3.2.19])} \\
&\cong \mathrm{RHom}_{\mathrm{D}(G)}\big(\mathrm{Res}_{G_2}^{G\times G} \text{c-Ind}_{\Delta(G)}^{G\times G}(\mathbf{1}), \mathbf{1}\big) && \text{(adjunction and } \mathrm{Inf}_G^1(\mathbf{1}) = \mathbf{1}) \\
&\cong \chi_G[-d]. && \square
\end{aligned}$$

Remark. The idea for defining the dualizing character as ω_G comes from the theory of six functor formalisms. The computation in [Sch19, Proof of Theorem 11.6] (which goes back to an idea of Verdier [Ver69, Proof of Thorem 3]) shows that $\mathcal{R}_{\Delta(G)}^{G\times G}(\mathbf{1})$ has to be the inverse dualizing object if derived smooth mod p representations were part of a six functor formalism.

2.4.4. Lemma. *Let $H \leq G$ be a closed subgroup. Then $\mathcal{R}_H^G(\mathbf{1})$ is a character sitting in degree* $\dim(G/H)$.

Proof. Let $K \leq G$ be a torsion-free compact open subgroup. By our observations in §2.3.1, we compute $\mathrm{Res}_{H\cap K}^H \mathcal{R}_H^G(\mathbf{1}) \cong \mathcal{R}_{H\cap K}^K \mathrm{Res}_K^G(\mathbf{1})$. Therefore, we may assume from the beginning that G and H are compact and torsion-free. The isomorphism $\mathrm{Res}_H^G \mathrm{Inf}_G^1 \overset{\cong}{\Longrightarrow} \mathrm{Inf}_H^1$ yields an isomorphism $\mathcal{R}_H^G F_G^1 \overset{\cong}{\Longrightarrow} F_H^1$ by passing to the right adjoints twice. Since $F_G^1(\mathbf{1}) = \mathbf{1}[\dim G]$ and $F_H^1(\mathbf{1}) = \mathbf{1}[\dim H]$ (see §2.3.4), we deduce $\mathcal{R}_H^G(\mathbf{1}[\dim G]) \cong \mathbf{1}[\dim H]$. The assertion follows. $\square$

2.4.5. Lemma. *Let $H \leq G$ be a closed subgroup and let $X, Y \in \mathrm{D}(G)$. The natural map*

$$\mathcal{R}_H^G(X) \otimes_k \mathrm{Res}_H^G(Y) \longrightarrow \mathcal{R}_H^G(X \otimes_k Y)$$

is an isomorphism provided Y is dualizable.

Proof. Recall that Y is called *dualizable* if $Y^\vee \otimes_k -$ is right adjoint to $Y \otimes_k -$. Observe that we always have the evaluation map $\mathrm{ev}\colon Y \otimes Y^\vee \to \mathbf{1}$. By [Lur23, Tag 02E3], being dualizable is equivalent to the existence of a coevaluation map $\mathrm{coev}\colon \mathbf{1} \to Y^\vee \otimes Y$ such that the compositions

$$Y \xrightarrow{\mathrm{id}_Y \otimes \mathrm{coev}} Y \otimes_k Y^\vee \otimes_k Y \xrightarrow{\mathrm{ev} \otimes \mathrm{id}_Y} Y$$

and

$$Y^\vee \xrightarrow{\mathrm{coev} \otimes \mathrm{id}_{Y^\vee}} Y^\vee \otimes_k Y \otimes_k Y^\vee \xrightarrow{Y^\vee \otimes \mathrm{ev}} Y^\vee$$

are the identity morphisms on Y and $Y^\vee$, respectively. As $\mathrm{D}(G)$ is symmetric monoidal, it follows that $Y^\vee \otimes_k -$ is also left adjoint to $Y \otimes_k -$. Since Res_H^G is monoidal, we deduce that $\mathrm{Res}_H^G Y$ is dualizable and that $\mathrm{Res}_H^G(Y^\vee) \xrightarrow{\cong} (\mathrm{Res}_H^G Y)^\vee$. Recall from Lemma 2.3.3 the natural isomorphism $\text{c-Ind}_H^G(- \otimes_k \mathrm{Res}_H^G Y) \overset{\cong}{\Longrightarrow} (- \otimes_k Y)\,\text{c-Ind}_H^G$ of functors $\mathrm{D}(H) \to \mathrm{D}(G)$. Passing to the left resp. right mates yields natural transformations

$$(- \otimes_k Y^\vee)\,\text{c-Ind}_H^G \Longrightarrow \text{c-Ind}_H^G(- \otimes_k \mathrm{Res}_H^G Y^\vee), \tag{2.4.6}$$

$$(- \otimes_k \mathrm{Res}_H^G Y)\mathcal{R}_H^G \Longrightarrow \mathcal{R}_H^G(- \otimes_k Y). \tag{2.4.7}$$

Note that (2.4.7) arises from (2.4.6) by passing to the right adjoints. It thus suffices to show that (2.4.6) is an isomorphism. We contemplate the commutative diagram

$$\begin{tikzcd}
\text{c-Ind}_H^G(X) \otimes_k Y^\vee \arrow[r, equal] \arrow[d, "\mathrm{coev}"'] & \text{c-Ind}_H^G(X) \otimes_k Y^\vee \arrow[d, "\mathrm{coev}"] \\
\text{c-Ind}_H^G(X \otimes_k \operatorname{Res}_H^G Y^\vee \otimes_k \operatorname{Res}_H^G Y) \otimes_k Y^\vee \arrow[r, "\cong"] \arrow[d, "\cong"] & \text{c-Ind}_H^G(X) \otimes_k Y^\vee \otimes_k Y \otimes_k Y^\vee \arrow[d, equal] \\
\text{c-Ind}_H^G(X \otimes_k \operatorname{Res}_H^G Y^\vee) \otimes_k Y \otimes_k Y^\vee \arrow[r, "\cong"] \arrow[d, "\mathrm{ev}"'] & \text{c-Ind}_H^G(X) \otimes_k Y^\vee \otimes_k Y \otimes_k Y^\vee \arrow[d, "\mathrm{ev}"] \\
\text{c-Ind}_H^G(X \otimes_k \operatorname{Res}_H^G Y^\vee) \arrow[r, "\cong"'] & \text{c-Ind}_H^G(X) \otimes_k Y^\vee,
\end{tikzcd}$$

where the isomorphisms are given by the projection formula. The composite of the left vertical arrows is (2.4.6), and the composite of the right vertical arrows is the identity. We deduce that (2.4.6), and hence also (2.4.7), is an isomorphism. □

2.4.8. Proposition. *Let $H \leq G$ be a closed subgroup. There is an isomorphism*

$$\big(\mathcal{R}_H^G(\mathbf{1})\big)^\vee \cong \omega_H^\vee \otimes_k \operatorname{Res}_H^G \omega_G.$$

In particular, if H is open, one has $\operatorname{Res}_H^G \omega_G \cong \omega_H$.

Proof. We first compute

$$\begin{aligned}
\operatorname{Res}_{\Delta(H)}^{H\times H} \mathcal{R}_{H\times H}^{G\times H}(\mathbf{1}) &= \operatorname{Res}_{\Delta(H)}^{H\times H} \mathcal{R}_{H\times H}^{G\times H}\big(\operatorname{Inf}_{G\times H}^{G}(\mathbf{1})\big) \\
&\cong \operatorname{Res}_{\Delta(H)}^{H\times H} \operatorname{Inf}_{H\times H}^{H} \mathcal{R}_H^G(\mathbf{1}) && \text{(Lemma 2.3.8)} \\
&\cong \mathcal{R}_H^G(\mathbf{1}),
\end{aligned}$$

where the inflations are taken along the first projection maps. A similar computation shows $\operatorname{Res}_{\Delta(H)}^{G\times H} \mathcal{R}_{G\times H}^{G\times G}(\mathbf{1}) \cong \mathcal{R}_H^G(\mathbf{1})$. Hence, we compute

$$\begin{aligned}
\mathcal{R}_H^G(\mathbf{1}) \otimes_k \operatorname{Res}_H^G \mathcal{R}_{\Delta(G)}^{G\times G}(\mathbf{1}) &\cong \mathcal{R}_H^G \mathcal{R}_{\Delta(G)}^{G\times G}(\mathbf{1}) && \text{(Lemma 2.4.5)} \\
&\cong \mathcal{R}_{\Delta(H)}^{H\times H} \mathcal{R}_{H\times H}^{G\times H} \mathcal{R}_{G\times H}^{G\times G}(\mathbf{1}) \\
&\cong \mathcal{R}_{\Delta(H)}^{H\times H}(\mathbf{1}) \otimes_k \operatorname{Res}_{\Delta(H)}^{H\times H} \mathcal{R}_{H\times H}^{G\times H}(\mathbf{1}) \otimes_k \operatorname{Res}_{\Delta(H)}^{G\times H} \mathcal{R}_{G\times H}^{G\times G}(\mathbf{1}) \\
&\cong \mathcal{R}_{\Delta(H)}^{H\times H}(\mathbf{1}) \otimes_k \mathcal{R}_H^G(\mathbf{1}) \otimes_k \mathcal{R}_H^G(\mathbf{1}).
\end{aligned}$$

Since $\mathcal{R}_H^G(\mathbf{1})$ is invertible by Lemma 2.4.4, we can cancel it on both sides. By Proposition 2.4.3 we have $\operatorname{Res}_H^G \omega_G^\vee \cong \omega_H^\vee \otimes_k \mathcal{R}_H^G(\mathbf{1})$, which is equivalent to the first assertion. The last assertion follows from the fact that $\mathcal{R}_H^G = \operatorname{Res}_H^G$ if H is open. □

2.4.9. Example. With the notation and assumptions of Lemma 2.3.14, we consider the character $\omega_{H\backslash G} = \mathrm{L}_N \operatorname{c-Ind}_H^G(\mathbf{1})$ in $\mathrm{D}(G/N)$. The same computation as in the proof of Proposition 2.4.3 shows

$$\omega_{H\backslash G}^\vee \cong \mathrm{RH}^0\big(N, \operatorname{RInd}_H^G \mathcal{R}_H^G(\mathbf{1})\big) \cong \mathrm{RH}^0\big(H \cap N, \mathcal{R}_H^G(\mathbf{1})\big),$$

where the second isomorphism uses Remark 2.3.13. Since $\mathcal{R}_H^G(\mathbf{1})$ is invertible, we deduce an isomorphism $\omega_{H\backslash G} \cong \mathrm{L}_{H\cap N}\big(\mathcal{R}_H^G(\mathbf{1})^\vee\big)$. In the specific cases appearing in §3, the group $H \cap N$ has no non-trivial characters; in this case, we deduce

$$\operatorname{Inf}_H^{G/N} \omega_{H\backslash G} \cong \mathcal{R}_H^G(\mathbf{1})^\vee \cong \omega_H^\vee \otimes_k \operatorname{Res}_H^G \omega_G.$$

3. The Geometrical Lemma

3.1. Setup.

3.1.1. Let k be a field of characteristic $p > 0$. We fix a finite field extension $\mathfrak{F}/\mathbb{Q}_p$. Let $\mathbf{G}$ be a connected reductive $\mathfrak{F}$-group and $\mathbf{T}$ a maximal $\mathfrak{F}$-split torus of $\mathbf{G}$. Let $\mathbf{P} = \mathbf{M} \ltimes \mathbf{U}$ and $\mathbf{Q} = \mathbf{L} \ltimes \mathbf{N}$ be semistandard parabolic $\mathfrak{F}$-subgroups of $\mathbf{G}$, that is, the Levi factors $\mathbf{M}$ and $\mathbf{L}$ both contain $\mathbf{Z_G}(\mathbf{T})$. Denote $\overline{\mathbf{P}} = \mathbf{M} \ltimes \overline{\mathbf{U}}$ the parabolic opposite $\mathbf{P}$. We note the following:

- The semistandard condition is not a restriction, as any two parabolic $\mathfrak{F}$-subgroups contain a common minimal Levi, [BT65, 4.18. Corollaire].
- For any $n \in \mathbf{N_G}(\mathbf{T})(\mathfrak{F})$ also $n\mathbf{P}n^{-1}$ is semistandard.
- One has a decomposition $\mathbf{P} \cap \mathbf{Q} = (\mathbf{M} \cap \mathbf{L}) \ltimes \big((\mathbf{U} \cap \mathbf{L})(\mathbf{M} \cap \mathbf{N})(\mathbf{U} \cap \mathbf{N})\big)$, cf. [Car85, Proposition 2.8.2 and Theorem 2.8.7].

Recall our convention that for an algebraic $\mathfrak{F}$-group $\mathbf{H}$ the corresponding lightface letter $H := \mathbf{H}(\mathfrak{F})$ denotes its associated group of $\mathfrak{F}$-points. Then H is a p-adic Lie group, and we denote $\mathrm{Rep}_k(H)$ the abelian category of smooth k-linear representations of H. We write $\mathrm{C}(H)$ (resp. $\mathrm{K}(H)$, resp. $\mathrm{D}(H)$) for the category of unbounded complexes (resp. the homotopy category of unbounded complexes, resp. the unbounded derived category) of $\mathrm{Rep}_k(H)$.

3.1.2. We write $i_P^G := \mathrm{Ind}_P^G \circ \mathrm{Inf}_P^M : \mathrm{D}(M) \to \mathrm{D}(G)$ for the functor of *parabolic induction*. It has a left adjoint $\mathrm{L}(U, \_) := \mathrm{L}_U \circ \mathrm{Res}_P^G$ and a right adjoint $\mathrm{R}(U, \_) := \mathrm{RH}^0(U, \_) \circ \mathcal{R}_P^G$. For any integer i we denote $\mathrm{L}^i(U, \_), \mathrm{R}^i(U, \_)\colon \mathrm{Rep}_k(G) \to \mathrm{Rep}_k(M)$ the corresponding cohomology functors given by composing $\mathrm{L}(U, \_)$, resp. $\mathrm{R}(U, \_)$, with H^i. By the proof of [Hey22, Theorem 4.1.1] we have natural isomorphisms

$$\mathrm{RHom}_M\big(\mathrm{L}(U, X), Y\big) \cong \mathrm{RHom}_G\big(X, i_P^G Y\big) \quad \text{and} \quad \mathrm{RHom}_M\big(Y, \mathrm{R}(U, X)\big) \cong \mathrm{RHom}_G\big(i_P^G Y, X\big)$$

for all $X \in \mathrm{D}(G)$ and $Y \in \mathrm{D}(M)$.

In this section we will study the composite functor

$$\mathrm{L}(N, \_) \circ i_P^G \colon \mathrm{D}(M) \longrightarrow \mathrm{D}(L)$$

by constructing certain filtrations in §3.2 whose graded pieces will be described explicitly in §3.3

3.2. Filtrations.

3.2.1. The group $P \times Q$ acts continuously on G via $(x, y) \cdot g := xgy^{-1}$. By the Bruhat decomposition, the coset space $P \backslash G / Q$ admits a finite representing system $\mathcal{N}_{P,Q} \subseteq \mathbf{N_G}(\mathbf{T})(\mathfrak{F})$. It follows from [BZ76, 1.5. Proposition] that the orbits PnQ are locally closed in G. We define a partial order on $\mathcal{N}_{P,Q}$ via

$$n \leq n' \overset{\text{def}}{\iff} Pn'Q \subseteq \overline{PnQ},$$

where the overline means topological closure. The following lemma is well-known.

3.2.2. Lemma. *For each $n \in \mathcal{N}_{P,Q}$ the subset $X_n := \bigcup_{n' \leq n} Pn'Q$ is the smallest open $P \times Q$-invariant subset in G containing n. In particular, PnQ is open if n is minimal and is closed if n is maximal.*

Proof. Note that $G \smallsetminus X_n = \bigcup_{n' \not\leq n} Pn'Q$ is closed, because $n' \not\leq n$ and $n' \leq n''$ implies $n'' \not\leq n$. If Y is any open $P \times Q$-invariant subset containing n and if $n' \leq n$, then $Y \cap \overline{Pn'Q}$ contains n. From the definition of topological closure and the $P \times Q$-invariance of Y it follows that $Pn'Q \subseteq Y$. Hence $X_n \subseteq Y$. The last assertions are immediate. □

Remark. (a) Every open $P \times Q$-invariant subset of G is a union of X_n's.
(b) The poset $\mathcal{N}_{P,Q}$ has a smallest and a largest element but is in general not totally ordered.

3.2.3. We extend the notation introduced in §2.3.1. Let $Z \subseteq G$ be a $P \times Q$-invariant subset. For any $V \in \mathrm{Rep}_k(P)$ we denote $\text{c-Ind}_P^Z V$ the k-vector space of locally constant functions $f\colon Z \to V$ which satisfy $f(xz) = xf(z)$, for all $x \in P$, $z \in Z$, and have compact support in $P\backslash Z$. The group Q acts smoothly by right translation on $\text{c-Ind}_P^Z V$. Observe that $\text{c-Ind}_P^Z V \subseteq \mathrm{Ind}_P^G V$ provided Z is open in G, that is, every element of $\text{c-Ind}_P^Z V$ is fixed by an open compact subgroup of G. The functor $\text{c-Ind}_P^Z\colon \mathrm{Rep}_k(P) \to \mathrm{Rep}_k(Q)$ is exact and hence extends to a triangulated functor on the derived categories:

$$\text{c-Ind}_P^Z\colon \mathrm{D}(P) \longrightarrow \mathrm{D}(Q).$$

It is clear that c-Ind_P^Z commutes with small direct sums and hence admits a right adjoint $\mathcal{R}_P^Z$.

The next lemma is well-known, cf. [Cas95, Lemma 6.1.1].

3.2.4. Lemma. *Let $Z \subseteq Z' \subseteq G$ be $P \times Q$-invariant subsets such that Z is open in Z'. For every smooth P-representation V one has an exact sequence*

$$0 \longrightarrow \text{c-Ind}_P^Z V \longrightarrow \text{c-Ind}_P^{Z'} V \longrightarrow \text{c-Ind}_P^{Z'\smallsetminus Z} V \longrightarrow 0.$$

Here, the first map is extension by zero and the second map is restriction of functions.

Proof. Choose a continuous section of the projection map $G \to P\backslash G$, cf. [AHV19, Lemma 2.3]; for any $P \times Q$-invariant subset $Y \subseteq G$ it restricts to a continuous section $\sigma\colon P\backslash Y \to Y$. For any k-vector space W we denote $\mathcal{C}_c^\infty(P\backslash Y, W)$ the space of locally constant functions $P\backslash Y \to W$ with compact support. We have k-linear isomorphisms $\text{c-Ind}_P^Y V \xrightarrow{\cong} \mathcal{C}_c^\infty(P\backslash Y, V) \xleftarrow{\cong} \mathcal{C}_c^\infty(P\backslash Y, k) \otimes_k V$, where the first isomorphism is given by $f \mapsto f \circ \sigma$ and the second by $f \otimes v \mapsto [Py \mapsto f(Py)v]$. Put $Z'' \coloneqq Z' \smallsetminus Z$. Under all these identifications the sequence in the lemma arises by applying $\_ \otimes_k V$ to the sequence

$$0 \longrightarrow \mathcal{C}_c^\infty(P\backslash Z, k) \longrightarrow \mathcal{C}_c^\infty(P\backslash Z', k) \longrightarrow \mathcal{C}_c^\infty(P\backslash Z'', k) \longrightarrow 0. \tag{3.2.5}$$

It therefore suffices to show that (3.2.5) is exact. Injectivity of the first map and exactness in the middle are clear. Let $f\colon P\backslash Z'' \to k$ be a locally constant function with compact support. We then find an open compact subgroup $K \leq G$ such that f takes a constant value, say, $c_{(Pz)K}$ on each $(Pz)K \cap P\backslash Z''$. We put $c_{(Pz)K} \coloneqq 0$ whenever $(Pz)K \cap P\backslash Z'' = \varnothing$. As the cosets $(Pz)K$ are pairwise disjoint, the function $P\backslash Z' \to k$ that is constantly $c_{(Pz)K}$ on each $(Pz)K$ gives a well-defined element of $\mathcal{C}_c^\infty(P\backslash Z', k)$, which extends f. □

3.2.6. Remark. If $Z' \smallsetminus Z$ is open in Z', then the sequence in Lemma 3.2.4 splits, so that we obtain a natural isomorphism $\text{c-Ind}_P^{Z'} \cong \text{c-Ind}_P^Z \oplus \text{c-Ind}_P^{Z'\smallsetminus Z}$.

3.2.7. Definition. Let $F\colon \mathscr{C} \to \mathscr{D}$ be a triangulated functor. A *(ascending) filtration* of length n on F is a sequence of natural transformations

$$F_i \overset{\lambda_i}{\Longrightarrow} F_{i+1} \overset{\mu_{i+1}}{\Longrightarrow} F_{i+1,i} \overset{\nu_i}{\Longrightarrow} F_i[1] \tag{3.2.8}$$

of triangulated functors $\mathscr{C} \to \mathscr{D}$, for $i = 0, 1, \dots, n-1$, such that:

- $F_0 = 0$ and $F_n = F$;
- evaluating (3.2.8) at any object of $\mathscr{C}$ gives a distinguished triangle in $\mathscr{D}$.

The functor $F_{i,i-1}$ is called the *i-th graded piece* of the filtration; note that $\mu_1\colon F_1 \overset{\cong}{\Longrightarrow} F_{1,0}$ is an isomorphism. TODO: Find a better definition of filtration.

3.2.9. Lemma. *Let $Z \subseteq Z' \subseteq G$ be $P \times Q$-invariant subsets such that Z is open in Z'. There exists a natural transformation $\nu\colon \text{c-Ind}_P^{Z'\smallsetminus Z} \Rightarrow \text{c-Ind}_P^Z[1]$ of triangulated functors $\mathrm{D}(P) \to \mathrm{D}(Q)$ such that for all $X \in \mathrm{D}(P)$ one has a distinguished triangle*

$$\text{c-Ind}_P^Z X \longrightarrow \text{c-Ind}_P^{Z'} X \longrightarrow \text{c-Ind}_P^{Z'\smallsetminus Z} X \overset{\nu}{\longrightarrow} \text{c-Ind}_P^Z X[1].$$

Proof. Denote $\lambda\colon \operatorname{c-Ind}_P^Z \Rightarrow \operatorname{c-Ind}_P^{Z'}$ and $\mu\colon \operatorname{c-Ind}_P^{Z'} \Rightarrow \operatorname{c-Ind}_P^{Z'\smallsetminus Z}$ the obvious maps. In order to construct ν let $F\colon \mathrm{C}(P) \to \mathrm{C}(Q)$ be the cone of λ. More precisely, given a complex (X, d) in $\mathrm{C}(P)$, define $F(X)^n := \operatorname{c-Ind}_P^Z X^{n+1} \oplus \operatorname{c-Ind}_P^{Z'} X^n$ with differential $F(X)^n \to F(X)^{n+1}$ given by the matrix $\begin{pmatrix} -\operatorname{c-Ind}_P^Z d^{n+1} & 0 \\ \lambda & \operatorname{c-Ind}_P^{Z'} d^n \end{pmatrix}$. It is clear that $F\colon \mathrm{C}(P) \to \mathrm{C}(Q)$ is indeed a functor which comes with natural transformations $\nu_1\colon F \Rightarrow \operatorname{c-Ind}_P^Z[1]$ and $\nu_2\colon F \Rightarrow \operatorname{c-Ind}_P^{Z'\smallsetminus Z}$ given by $(\mathrm{id}, 0)$ and $(0, \mu)$, respectively. In combination with Lemma 3.2.4, the usual long exact sequence argument and the five lemma show that $\nu_2(X)$ is a quasi-isomorphism for all $X \in \mathrm{C}(P)$. Since $\operatorname{c-Ind}_P^Z$ and $\operatorname{c-Ind}_P^{Z'}$ are exact and mapping cones are functorial in $\mathrm{C}(Q)$, a similar argument shows that F preserves quasi-isomorphisms. Hence F descends to a functor $\mathrm{D}(P) \to \mathrm{D}(Q)$, and then $\nu := \nu_1 \circ \nu_2^{-1}$ is the desired natural transformation. □

3.2.10. For any $P \times Q$-invariant subset $Z \subseteq G$ we define

$$\Phi_Z := \mathrm{L}(N, \_) \circ \operatorname{c-Ind}_P^Z \circ \operatorname{Inf}_P^M \colon \mathrm{D}(M) \to \mathrm{D}(L).$$

Observe that, if $Z = \bigsqcup_i Pn_iQ$ is a union of open orbits in Z, then $\Phi_Z \cong \bigoplus_i \Phi_{Pn_iQ}$.

3.2.11. Proposition. *Let $\varnothing = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \cdots \subseteq Z_r$ be a chain of open $P \times Q$-invariant subsets of Z_r. There exists a filtration*

$$0 \Longrightarrow \Phi_{Z_1} \Longrightarrow \Phi_{Z_2} \Longrightarrow \cdots \Longrightarrow \Phi_{Z_r}$$

of Φ_{Z_r}, whose i-th graded piece is $\Phi_{Z_i \smallsetminus Z_{i-1}}$.

Proof. Use Lemma 3.2.9. □

3.2.12. Definition. Let $\varnothing = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \cdots \subseteq Z_r = G$ be the unique chain of open $P \times Q$-invariant subsets of G for which the $P \times Q$-orbits in $Z_i \smallsetminus Z_{i-1}$ are the open orbits in $G \smallsetminus Z_{i-1}$ of maximal dimension. We call $\mathrm{ht}(\mathcal{N}_{P,Q}) := r$ the *height* of $\mathcal{N}_{P,Q}$. The *height* $\mathrm{ht}(n)$ of $n \in \mathcal{N}_{P,Q}$ is defined as the smallest integer i with $n \in Z_i$.

3.3. The Geometrical Lemma.

We keep the notations of the previous subsections.

3.3.1. Proposition. *The diagram*

$$\begin{tikzcd}
\mathrm{D}(M\cap Q) \arrow[r, "\operatorname{Inf}_{P\cap Q}^{M\cap Q}"] \arrow[d, "\mathrm{L}_{M\cap N}"'] & \mathrm{D}(P\cap Q) \arrow[r, "\operatorname{c-Ind}_{P\cap Q}^{Q}"] & \mathrm{D}(Q) \arrow[d, "\mathrm{L}_N"] \\
\mathrm{D}(M\cap L) \arrow[r, "\omega\otimes-"'] & \mathrm{D}(M\cap L) \arrow[r, "i_{P\cap L}^L"'] & \mathrm{D}(L)
\end{tikzcd}$$

is commutative, where $\omega := \operatorname{Res}_{M\cap L}^{P\cap L} \mathrm{L}_N \operatorname{c-Ind}_{P\cap Q}^{(P\cap L)N}(\mathbf{1})$ is a character sitting in degree $-\dim(\overline{U}\cap N)$.

Proof. We first verify the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{D}(M\cap Q) \arrow[r, "\operatorname{Inf}_{P\cap Q}^{M\cap Q}"] \arrow[d, "\mathrm{L}_{M\cap N}"'] & \mathrm{D}(P\cap Q) \arrow[r, "\operatorname{c-Ind}_{P\cap Q}^{(P\cap L)N}"] \arrow[d, "\mathrm{L}_{P\cap N}"'] & \mathrm{D}\big((P\cap L)N\big) \arrow[r, "\operatorname{c-Ind}_{(P\cap L)N}^{Q}"] \arrow[d, "\mathrm{L}_N"] & \mathrm{D}(Q) \arrow[d, "\mathrm{L}_N"] \\
\mathrm{D}(M\cap L) \arrow[r, "\operatorname{Inf}_{P\cap L}^{M\cap L}"'] & \mathrm{D}(P\cap L) \arrow[r, "\omega'\otimes-"'] & \mathrm{D}(P\cap L) \arrow[r, "\operatorname{c-Ind}_{P\cap L}^{L}"'] & \mathrm{D}(L),
\end{tikzcd}$$

where $\omega' := \omega_{(P\cap Q)\backslash(P\cap L)N} \in \mathrm{D}(P\cap L)$ is defined as in (2.3.15). The left diagram commutes by Lemma 2.3.6 applied to $(G, H, N) = (P\cap Q, U\cap Q, P\cap N)$; note that $\mathrm{L}_{U\cap N}(\mathbf{1}) \cong \mathbf{1}$ by [Hey22, Example 3.4.24]. The diagram on the right commutes by Lemma 2.3.8. Since $\mathbf{N}$ is a unipotent algebraic group, the hypotheses of Lemma 2.3.14 are easily seen to be satisfied. Hence, also the middle diagram commutes and ω' is a character of $P\cap L$ sitting in degree $-\dim(\overline{U}\cap N)$; here, the computation of the degree uses the decomposition $(P\cap L)N = (P\cap Q) \times (\overline{U}\cap N)$ as p-adic manifolds. As the only character of $U \cap L$ is the trivial one, we have $\omega' = \operatorname{Inf}_{P\cap L}^{M\cap L} \omega$. Hence, since $(\omega' \otimes_k -) \operatorname{Inf}_{P\cap L}^{M\cap L} = \operatorname{Inf}_{P\cap L}^{M\cap L}(\omega \otimes_k -)$ and compact induction is transitive, this proves the assertion. □

3.3.2. Notation. Given a closed subgroup $H \le G$ and $g \in G$, we write $g(H) \coloneqq gHg^{-1}$ and denote $g_*\colon \mathrm{D}(H) \xrightarrow{\cong} \mathrm{D}(g(H))$ the "inflation" along the conjugation map $g(H) \xrightarrow{\cong} H$.

3.3.3. Lemma. *There is a natural isomorphism*

$$\text{c-Ind}_P^{PnQ} \overset{\cong}{\Longrightarrow} \text{c-Ind}_{n^{-1}(P)\cap Q}^{Q} \operatorname{Res}_{n^{-1}(P)\cap Q}^{n^{-1}(P)} n_*^{-1}$$

of functors $\mathrm{D}(P) \to \mathrm{D}(Q)$.

Proof. Note that, by [BZ76, 1.6. Corollary][3], the inclusion $Q \hookrightarrow n^{-1}(P)Q$ and $n^{-1}(P)Q \xrightarrow{n\cdot} PnQ$ induce homeomorphisms $n^{-1}(P)\cap Q\backslash Q \xrightarrow{\cong} n^{-1}(P)\backslash n^{-1}(P)Q \xrightarrow{\cong} P\backslash PnQ$. It follows that for any $V \in \operatorname{Rep}_k(P)$ the natural map

$$\begin{aligned} \text{c-Ind}_P^{PnQ} V &\longrightarrow \text{c-Ind}_{n^{-1}(P)\cap Q}^{Q} \operatorname{Res}_{n^{-1}(P)\cap Q}^{n^{-1}(P)} n_*^{-1} V, \\ f &\longmapsto [q \mapsto f(nq)] \end{aligned}$$

is an isomorphism in $\operatorname{Rep}_k(Q)$. As all functors involved are exact, the assertion follows. □

3.3.4. For any $n \in \mathcal{N}_{P,Q}$ we consider the functors $\mathrm{D}(M) \to \mathrm{D}(L)$ given by

$$\begin{aligned} \Phi_{PnQ} &= \mathrm{L}_N \circ \text{c-Ind}_P^{PnQ} \circ \operatorname{Inf}_P^M, \\ \Psi_n &\coloneqq i_{n^{-1}(P)\cap L}^{L} \circ (\omega_n \otimes_k -) \circ n_*^{-1} \circ \mathrm{L}(M \cap n(N), -), \end{aligned}$$

where $\omega_n \coloneqq \operatorname{Res}_{n^{-1}(M)\cap L}^{n^{-1}(P)\cap L} \mathrm{L}_N \,\text{c-Ind}_{n^{-1}(P)\cap Q}^{(n^{-1}(P)\cap L)N}(\mathbf{1})$ is by Lemma 2.3.14 a character of $n^{-1}(M)\cap L$ sitting in degree $-\dim(n^{-1}(\overline{U}) \cap N)$.

3.3.5. Theorem. *For every* $n \in \mathcal{N}_{P,Q}$ *there is a natural isomorphism* $\Phi_{PnQ} \overset{\cong}{\Longrightarrow} \Psi_n$.

Proof. Applying Lemma 3.3.3 and Proposition 3.3.1 to $(P, Q) = (n^{-1}(P), Q)$, we compute

$$\begin{aligned} \Phi_{PnQ} &\overset{\cong}{\Longrightarrow} \mathrm{L}_N \,\text{c-Ind}_{n^{-1}(P)\cap Q}^{Q} \operatorname{Res}_{n^{-1}(P)\cap Q}^{n^{-1}(P)} n_*^{-1} \operatorname{Inf}_P^M \\ &= \mathrm{L}_N \,\text{c-Ind}_{n^{-1}(P)\cap Q}^{Q} \operatorname{Inf}_{n^{-1}(P)\cap Q}^{n^{-1}(M)\cap Q} n_*^{-1} \operatorname{Res}_{M\cap n(Q)}^{M} \\ &\overset{\cong}{\Longrightarrow} i_{n^{-1}(P)\cap L}^{L}(\omega_n \otimes_k -) \mathrm{L}_{n^{-1}(M)\cap N} n_*^{-1} \operatorname{Res}_{M\cap n(Q)}^{M} \\ &\cong \Psi_n. \end{aligned}$$

□

TODO: Does this theorem have any implications for [Hau18, Conjecture 3.3.4]?

3.3.6. Corollary (Geometrical Lemma)**.** *Let* $\varnothing = Z_0 \subsetneq Z_1 \subsetneq Z_2 \subsetneq \cdots \subsetneq Z_r = G$ *be a chain of open* $P \times Q$*-invariant subsets of* G *such that* $Z_i \smallsetminus Z_{i-1} = Pn_iQ$ *for all* $1 \le i \le r$*, where* $n_i \in \mathcal{N}_{P,Q}$*. It induces a filtration on* $\Phi_G = \mathrm{L}(N, -)\, i_P^G$ *whose i-th graded piece is* Ψ_{n_i}.

Proof. Combine Proposition 3.2.11 and Theorem 3.3.5. □

3.3.7. Corollary. *The functor* $\mathrm{L}(N, -) \circ i_P^G$ *admits a filtration of length* $\operatorname{ht}(\mathcal{N}_{P,Q})$ *whose i-th graded piece is* $\bigoplus_{n,\operatorname{ht}(n)=i} \Psi_n$.

Proof. Combine Proposition 3.2.11, §3.2.10, and Theorem 3.3.5. □

[3]The condition "countable at infinity" is automatic for $\mathfrak{F}$-points of linear algebraic groups.

3.3.8. Remark. There is a dual version of Corollary 3.3.6 which states that there exists a descending filtration on $\mathrm{R}(U, -)\, i_Q^G$ with graded pieces (in some order) $i^M_{M\cap n(Q)}(\Omega_n \otimes_k -) n_* \mathrm{R}\big(n^{-1}(U)\cap L, -\big)$, where $\Omega_n := \operatorname{Res}^{P\cap n(Q)}_{M\cap n(L)} \mathcal{R}^{(P\cap n(L))n(N)}_{P\cap n(Q)}(\mathbf{1})$ is a character sitting in degree $\dim(\overline{U}\cap n(N))$ and $n \in \mathcal{N}_{P,Q}$. However, for the proof it seems we need to employ the theory of ∞-categories. Granted this more general framework, we could deduce this version by passing everywhere in the filtration to the right adjoints. TODO: explain in more detail in an appendix?

4. Applications

4.1. Setup and notation.

4.1.1. Fix a field k of characteristic $p > 0$. Let $\mathfrak{F}/\mathbb{Q}_p$ be a finite extension. Let $\mathbf{G}$ be a connected reductive $\mathfrak{F}$-group. Fix a maximal $\mathfrak{F}$-split torus $\mathbf{T} \leq \mathbf{G}$. We choose a set of simple roots Δ_G inside the (relative) reduced root system $\Sigma := \Phi(\mathbf{G}, \mathbf{T})^{\mathrm{red}}$ together with the associated set Σ^+ of positive roots, which corresponds to some minimal parabolic subgroup $\mathbf{B}$ containing $\mathbf{Z} := \mathbf{Z}_{\mathbf{G}}(\mathbf{T})$. Put

$$\Delta_G^1 := \{\alpha \in \Delta_G \,|\, \dim_{\mathfrak{F}} \mathbf{U}_\alpha = 1\},$$

where $\mathbf{U}_\alpha$ denotes the root $\mathfrak{F}$-group associated with $\alpha \in \Sigma$. We denote W the finite Weyl group associated with $(\mathbf{G}, \mathbf{T})$. We fix once and for all a set $\mathcal{N} \subseteq \mathbf{N}_{\mathbf{G}}(\mathbf{T})(\mathfrak{F})$ of representatives of W and denote n_w the element of $\mathcal{N}$ corresponding to $w \in \mathrm{W}$. We denote $n_\alpha \in \mathcal{N}$ the element lifting the simple reflection s_α attached to $\alpha \in \Delta_G$. For each $w \in \mathrm{W}$ we put

$$d_w := d_{n_w} := \sum_{\alpha \in \Sigma^+ \cap w^{-1}(-\Sigma^+)} \dim_{\mathfrak{F}} \mathbf{U}_\alpha.$$

We also abbreviate $d_\alpha := d_{s_\alpha}$ for $\alpha \in \Sigma$.

To each standard parabolic subgroup $\mathbf{P} = \mathbf{M} \ltimes \mathbf{U}$ corresponds a subset $\Delta_M \subseteq \Delta_G$. Set

$$\Delta_M^\perp := \{\alpha \in \Delta_G \,|\, \langle \alpha, \beta^\vee \rangle = 0 \text{ for all } \beta \in \Delta_M\} \quad \text{and} \quad \Delta_M^{\perp,1} := \Delta_M^\perp \cap \Delta_G^1.$$

We denote $\Sigma_M = \Sigma(\mathbf{M}, \mathbf{T}) \subseteq \Sigma$ the root system of $\mathbf{M}$; the positive roots corresponding to Δ_M are $\Sigma_M^+ = \Sigma_M \cap \Sigma^+$. We denote $\mathrm{W}_M \leq \mathrm{W}$ the finite Weyl group associated with $(\mathbf{M}, \mathbf{T})$.

4.1.2. Fix standard parabolic subgroups $\mathbf{P} = \mathbf{M} \ltimes \mathbf{U}$ and $\mathbf{Q} = \mathbf{L} \ltimes \mathbf{N}$ of $\mathbf{G}$. We will choose a distinguished set $\mathcal{N}_{P,Q}$ of double coset representatives of $P\backslash G/Q$ as follows: Put

$$D_M := \big\{ w \in \mathrm{W} \,\big|\, w(\Delta_M) \subseteq \Sigma^+ \big\}$$

and define D_L similarly. For the various properties of the set

$$D_{M,L} := (D_M)^{-1} \cap D_L = \big\{ w \in \mathrm{W} \,\big|\, w \in D_L \text{ and } w^{-1} \in D_M \big\}$$

we refer to [Car85, §2.7]. Let us only mention that $D_{M,L}$ is a set of double coset representatives of $\mathrm{W}_M\backslash \mathrm{W}/\mathrm{W}_L$ and that each $w \in D_{M,L}$ is the unique element of minimal length in $\mathrm{W}_M w \mathrm{W}_L$. Finally, let $\mathcal{N}_{P,Q} \subseteq \mathcal{N}$ be the subset corresponding to $D_{M,L}$.

4.1.3. Lemma. *For each $w \in D_{M,L}$ one has*

$$[\mathfrak{F} : \mathbb{Q}_p] \cdot d_w = \dim\big(n_w^{-1}(\overline{U}) \cap N\big).$$

Proof. Note that $\dim_{\mathfrak{F}}(\cdot) = [\mathfrak{F} : \mathbb{Q}_p] \cdot \dim(\cdot)$. Put $\Sigma_N = \Sigma^+ \smallsetminus \Sigma_L^+$ and $\Sigma_{\overline{U}} = -(\Sigma^+ \smallsetminus \Sigma_M^+)$. Then

$$n_w^{-1}(\overline{\mathbf{U}}) \cap \mathbf{N} = \prod_{\alpha \in \Sigma_N \cap w^{-1}(\Sigma_{\overline{U}})} \mathbf{U}_\alpha.$$

The claim thus reduces to $\Sigma_N \cap w^{-1}(\Sigma_{\overline{U}}) = \Sigma^+ \cap w^{-1}(-\Sigma^+)$. By definition of $D_{M,L}$ we have $w^{-1}(\Sigma_M^+) \subseteq \Sigma^+$ and $\Sigma_L^+ \subseteq w^{-1}(\Sigma^+)$. Hence,

$$\begin{aligned} \Sigma_N \cap w^{-1}(\Sigma_{\overline{U}}) &= \Sigma^+ \smallsetminus \Sigma_L^+ \cap -w^{-1}(\Sigma^+ \smallsetminus \Sigma_M^+) \\ &= \Sigma^+ \smallsetminus w^{-1}(-\Sigma_M^+) \cap w^{-1}(-\Sigma^+) \smallsetminus \Sigma_L^+ = \Sigma^+ \cap w^{-1}(-\Sigma^+). \end{aligned}$$

□

4.1.4. Notation. For each $w \in D_{M,L}$, let $\delta_w \in \mathrm{Rep}_k(n_w^{-1}(M) \cap L)$ be the character defined by

$$\omega_w \coloneqq \omega_{n_w} = \delta_w\big[[\mathfrak{F} : \mathbb{Q}_p] d_w\big],$$

see §3.3.4 and Lemma 4.1.3. If w is the simple reflection corresponding to $\alpha \in \Delta_G$, we also write δ_α for δ_w and ω_α for ω_w.

4.1.5. Remark. A concrete description of δ_w can be extracted from Example 2.4.9. The adjoint action on $\bigwedge^{[\mathfrak{F}:\mathbb{Q}_p]d_w} \mathrm{Lie}(n_w^{-1}(\overline{\mathbf{U}}) \cap \mathbf{N})$ yields a character $\delta'_w \colon n_w^{-1}(M) \cap L \to \mathbb{Q}_p^\times$. Recall the mod p cyclotomic character $\varepsilon \colon \mathbb{Q}_p^\times \to \mathbb{Z}_p^\times \to \mathbb{F}_p^\times \subseteq k^\times$, where the first map is given by $x \mapsto xp^{-\mathrm{val}_p(x)}$. The character δ_w is given as the composite $\delta_w = \varepsilon \circ \delta'_w$.

4.2. Computation of Ext-groups. Recall the setup in §4.1.1

4.2.1. Example. Let $\chi \colon M \to k^\times$ be a smooth character. We then have

$$\mathrm{L}^{-j}(N, i_P^G \chi) = \bigoplus_{\substack{w \in D_{M,L} \\ [\mathfrak{F}:\mathbb{Q}_p]d_w = j}} i^L_{n_w^{-1}(P)\cap L}\big(\delta_w \otimes_k n_{w*}^{-1}\chi\big). \tag{4.2.2}$$

Indeed, fix $j \geq 0$. Consider the open $P \times Q$-invariant subsets $Z \subseteq Z' \subseteq G$ defined by $Z \coloneqq \bigsqcup_{\substack{w \in D_{M,L} \\ [\mathfrak{F}:\mathbb{Q}_p]d_w > j}} P n_w Q$ and $Z' \coloneqq \bigsqcup_{\substack{w \in D_{M,L} \\ [\mathfrak{F}:\mathbb{Q}_p]d_w \geq j}} P n_w Q$. By Proposition 3.2.11 we obtain a filtration

$$\Phi_Z(\chi) \longrightarrow \Phi_{Z'}(\chi) \longrightarrow \Phi_G(\chi) = \mathrm{L}(N, i_P^G \chi).$$

Applying $\mathrm{H}^{-j}(\_)$ to the triangle $\Phi_{Z'}(\chi) \to \Phi_G(\chi) \to \Phi_{G\smallsetminus Z'}(\chi) \xrightarrow{+}$ yields an exact sequence

$$\mathrm{H}^{-j-1}\big(\Phi_{G\smallsetminus Z'}(\chi)\big) \longrightarrow \mathrm{H}^{-j}\big(\Phi_{Z'}(\chi)\big) \longrightarrow \mathrm{H}^{-j}\big(\Phi_G(\chi)\big) \longrightarrow \mathrm{H}^{-j}\big(\Phi_{G\smallsetminus Z'}(\chi)\big). \tag{4.2.3}$$

By Proposition 3.2.11 and Theorem 3.3.5 there exists a filtration on $\Phi_{G\smallsetminus Z'}(\chi)$ with graded pieces of the form $i^L_{n_w(P)\cap L}\big(\delta_w\big[[\mathfrak{F} : \mathbb{Q}_p]d_w\big] \otimes_k n_{w*}^{-1}\chi\big) \in \mathrm{D}^{\geq 1-j}(L)$. Since $\mathrm{D}^{\geq 1-j}(L)$ is closed under extensions, we deduce $\Phi_{G\smallsetminus Z'}(\chi) \in \mathrm{D}^{\geq 1-j}(L)$. Hence, we have $\mathrm{H}^{-j-1}(\Phi_{G\smallsetminus Z'}(\chi)) = \mathrm{H}^{-j}(\Phi_{G\smallsetminus Z'}) = 0$, and then (4.2.3) shows $\mathrm{H}^{-j}(\Phi_{Z'}(\chi)) \xrightarrow{\cong} \mathrm{H}^{-j}(\Phi_G(\chi))$. A similar argument applied to the triangle $\Phi_Z(\chi) \to \Phi_{Z'}(\chi) \to \Phi_{Z'\smallsetminus Z}(\chi) \xrightarrow{+}$ implies

$$0 = \mathrm{H}^{-j}\big(\Phi_Z(\chi)\big) \longrightarrow \mathrm{H}^{-j}\big(\Phi_{Z'}(\chi)\big) \xrightarrow{\cong} \mathrm{H}^{-j}\big(\Phi_{Z'\smallsetminus Z}(\chi)\big) \longrightarrow \mathrm{H}^{-j+1}\big(\Phi_Z(\chi)\big) = 0.$$

The discussion shows $\mathrm{L}^{-j}(N, i_P^G \chi) = \mathrm{H}^{-j}(\Phi_G(\chi)) \cong \mathrm{H}^{-j}(\Phi_{Z'\smallsetminus Z}(\chi))$. Moreover, by Remark 3.2.6 and Theorem 3.3.5, we have

$$\begin{aligned}
\mathrm{H}^{-j}\big(\Phi_{Z'\smallsetminus Z}(\chi)\big) &\cong \bigoplus_{\substack{w \in D_{M,L} \\ [\mathfrak{F}:\mathbb{Q}_p]d_w = j}} \mathrm{H}^{-j}\big(\Phi_{P n_w Q}(\chi)\big) \\
&\cong \bigoplus_{\substack{w \in D_{M,L} \\ [\mathfrak{F}:\mathbb{Q}_p]d_w = j}} \mathrm{H}^{-j}\big(\Psi_{n_w}(\chi)\big) \cong \bigoplus_{\substack{w \in D_{M,L} \\ [\mathfrak{F}:\mathbb{Q}_p]d_w = j}} i^L_{n_w^{-1}(P)\cap L}(\delta_w \otimes_k n_{w*}^{-1}\chi),
\end{aligned}$$

where the last isomorphism follows from the fact that, since $M \cap n_w(N)$ acts trivially on χ, we have $\mathrm{L}(M \cap n_w(N), \chi) \cong \chi$ viewed as a character of $M \cap n_w(L)$. This proves (4.2.2).

4.2.4. Lemma. *Let $V, W \in \mathrm{Rep}_k(G)$. Assume there exists a central element $z \in G$ such that the action of z on V and W is given by multiplication with distinct scalars. Then*

$$\mathrm{Ext}^i_G(V, W) = 0, \qquad \textit{for all } i \geq 0.$$

Proof. This is well-known, see [BW00, I, §4]. □

4.2.5. Remark. In [Hau19, Conjecture 3.17] Hauseux states a conjecture regarding the computation of Ext^1-groups of representations which are parabolically induced from a supersingular representation. This conjecture is conditionally resolved in [Hau18, Corollary 5.2.9]; parts of the argument rely, however, on an open conjecture of Emerton, [Eme10, Conjecture 3.7.2], which states that the higher ordinary parts functor $H^\bullet \mathrm{Ord}_P$ is the right derived functor of Ord_P on the category of locally admissible representations. Granting this conjecture, Hauseux is able to prove even more general forms of his conjecture, see particularly [Hau18, Remarks 5.2.6 and 5.2.8]. Further, Hauseux computes higher Ext-groups for principal series representations in [Hau16, Théorème 5.3.1], which again relies on the conjecture of Emerton.

In Theorems 4.2.8 and 4.2.11 below I give unconditional proofs of the computation, resp. the generalized conjecture, of Hauseux in a slightly different context: where Hauseux is computing (higher) extensions in the category of locally admissible representations, our computations naturally work in the category of all smooth representations.

4.2.6. Observe that restriction of characters induces an inclusion $\mathrm{X}^*(Z) \subseteq \mathrm{X}^*(T)$ with finite index, which is W-equivariant for the natural actions of W on $\mathrm{X}^*(Z)$ and $\mathrm{X}^*(T)$, respectively. (A priori, only $\mathbf{N_G}(\mathbf{T})(\mathfrak{F})$ acts on $\mathrm{X}^*(Z)$, but the inclusion shows that the induced action of Z is trivial.) For each $\alpha \in \Sigma$, the composite $\mathbf{T} \subseteq \mathbf{Z} \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathrm{Lie}\,\mathbf{U}_\alpha) \to \mathrm{Aut}(\bigwedge^{d_\alpha} \mathrm{Lie}\,\mathbf{U}_\alpha) \cong \mathbb{G}_{\mathrm{m},\mathfrak{F}}$ coincides with $d_\alpha \alpha$. Hence, we have $d_\alpha \alpha \in \mathrm{X}^*(Z)$ under the above inclusion.

For any $w \in \mathrm{W}$, we put $\alpha_w := \sum_{\beta \in \Sigma^+ \cap w^{-1}(-\Sigma^+)} d_\beta \beta \in \mathrm{X}^*(Z)$. The character δ_w from Notation 4.1.4 is then given by $\delta_w = \varepsilon_{\mathfrak{F}} \circ \alpha_w$, where $\varepsilon_{\mathfrak{F}} = \varepsilon \circ \mathrm{Nm}_{\mathfrak{F}/\mathbb{Q}_p}$ (here, $\mathrm{Nm}_{\mathfrak{F}/\mathbb{Q}_p} \colon \mathfrak{F}^\times \to \mathbb{Q}_p^\times$ is the norm map) and where we view α_w as a character $Z \to \mathfrak{F}^\times$.

Inspired by the definition in [BG10, Definition 5.2.1], we call a character $\theta \in \mathrm{X}^*(Z)$ a *twisting element* if $\langle \theta, \alpha^\vee \rangle = d_\alpha$ for all $\alpha \in \Delta_G$. If $\rho := \frac{1}{2} \sum_{\alpha \in \Sigma^+} d_\alpha \alpha$ lies in $\mathrm{X}^*(Z)$ (*e.g.*, if $\mathbf{G}$ is $\mathfrak{F}$-split, semisimple, and simply connected), then we show below that ρ is a twisting element.

4.2.7. Lemma.

(a) *One has $\alpha_w = \rho - w^{-1}(\rho)$ in $\mathrm{X}^*(Z) \otimes_{\mathbb{Z}} \mathbb{Q}$, for all $w \in \mathrm{W}$.*
(b) *The function $\mathrm{W} \to \mathrm{X}^*(Z)$, $w \mapsto \alpha_w$ is injective and satisfies the cocycle condition $\alpha_{vw} = \alpha_w + w^{-1}(\alpha_v)$ for all $v, w \in \mathrm{W}$.*
(c) *The assignment $\chi \star w := \delta_w \otimes_k w_*^{-1} \chi$ defines a right action of W on the group of smooth characters of Z.*
(d) *If $\rho \in \mathrm{X}^*(Z)$, then ρ is a twisting element.*
(e) *Assume there exists a twisting element θ. Then $\chi \star w = (\varepsilon_{\mathfrak{F}} \circ \theta) \otimes_k w_*^{-1}\big(\chi \otimes_k \varepsilon_{\mathfrak{F}}^{-1} \circ \theta\big)$ for all $w \in \mathrm{W}$ and all smooth characters $\chi \colon Z \to k^\times$.*

Remark. The action $(\chi, w) \mapsto \chi \star w$ is analogous to the "dot action" in the representation theory of semisimple Lie algebras.

Proof of Lemma 4.2.7. Note that $d_\alpha = d_{w(\alpha)}$ for all $\alpha \in \Sigma$, $w \in \mathrm{W}$. For (a) we compute

$$\begin{aligned} 2w^{-1}(\rho) &= \sum_{\alpha \in \Sigma^+} d_\alpha \cdot w^{-1}(\alpha) = \sum_{\alpha \in \Sigma^+ \cap w(\Sigma^+)} d_{w^{-1}(\alpha)} w^{-1}(\alpha) + \sum_{\alpha \in \Sigma^+ \cap w(-\Sigma^+)} d_{w^{-1}(\alpha)} w^{-1}(\alpha) \\ &= \sum_{\alpha \in w^{-1}(\Sigma^+) \cap \Sigma^+} d_\alpha \alpha - \sum_{\alpha \in w^{-1}(-\Sigma^+) \cap \Sigma^+} d_\alpha \alpha = 2\rho - 2\alpha_w. \end{aligned}$$

The cocycle condition in (b) can be verified in $\mathrm{X}^*(Z) \otimes_{\mathbb{Z}} \mathbb{Q}$, in which case it is obvious from (a). If $v, w \in \mathrm{W}$ are such that $\alpha_v = \alpha_w$, then $\alpha_v = \alpha_{vw^{-1}w} = \alpha_w + w^{-1}(\alpha_{vw^{-1}})$, and hence $\alpha_{vw^{-1}} = 0$; but this necessitates $v = w$. Part (c) follows from (b) noting that also $w \mapsto \delta_w = \varepsilon_{\mathfrak{F}} \circ \alpha_w$ is a cocycle. Let us prove (d). For each $\alpha \in \Delta_G$ we compute, using (a),

$$\langle \rho, \alpha^\vee \rangle = \langle s_\alpha(\rho), s_\alpha(\alpha^\vee) \rangle = \langle \rho - d_\alpha \alpha, -\alpha^\vee \rangle = -\langle \rho, \alpha^\vee \rangle + d_\alpha \langle \alpha, \alpha^\vee \rangle.$$

In combination with $\langle \alpha, \alpha^\vee \rangle = 2$, this shows $\langle \rho, \alpha^\vee \rangle = d_\alpha$. Finally, we prove (e). Note that $\langle \theta - \rho, \alpha^\vee \rangle = 0$ for all $\alpha \in \Delta_G$, by (d). Hence, W fixes $\theta - \rho$, and therefore $\theta - w^{-1}(\theta) = \rho - w^{-1}(\rho) = \alpha_w$ in $\mathrm{X}^*(Z)$ (the computation is carried out in $\mathrm{X}^*(Z) \otimes_{\mathbb{Z}} \mathbb{Q}$). We deduce $\delta_w = (\varepsilon_{\mathfrak{F}} \circ \theta) \otimes_k w_*^{-1}\big(\varepsilon_{\mathfrak{F}}^{-1} \circ \theta\big)$, from which the assertion follows. $\square$

4.2.8. Theorem. *Let $\mathbf{P} = \mathbf{M} \ltimes \mathbf{U}$ a standard parabolic subgroup of $\mathbf{G}$ and write $\mathbf{B} = \mathbf{Z} \ltimes \mathbf{N}$. Denote $C(Z)$ the center of Z. Let $\chi \colon M \to k^\times$ be a smooth character and let $r \in \mathbb{Z}_{\geq 0}$.*

(a) *Let $\chi'\colon Z \to k^\times$ be another smooth character. If*
$$\mathrm{Ext}^r_G\big(i_P^G\,\chi, i_B^G\,\chi'\big) \neq 0,$$
then there exists $w \in D_{M,Z}$ such that $\chi' \cong \delta_w \otimes_k n_{w*}^{-1}\chi$ as characters of $C(Z)$ and with $[\mathfrak{F} : \mathbb{Q}_p]d_w \le r$.

(b) *Assume $\delta_w \otimes_k n_{w*}^{-1}\chi \not\cong \chi$ as characters of $C(Z)$, for all $w \in D_{M,Z}$. For each $w \in D_{M,Z}$ with $[\mathfrak{F} : \mathbb{Q}_p]d_w \le r$ one has k-linear isomorphisms*
$$\mathrm{Ext}^r_G\big(i_P^G\,\chi, i_B^G(\delta_w \otimes_k n_{w*}^{-1}\chi)\big) \cong \mathrm{Ext}_Z^{r-[\mathfrak{F}:\mathbb{Q}_p]d_w}(\mathbf{1},\mathbf{1}) \cong \mathrm{H}^{r-[\mathfrak{F}:\mathbb{Q}_p]d_w}(Z,k),$$
where $\mathrm{H}^*(Z,k)$ denotes the continuous group cohomology of Z with coefficients in k.

Proof. By [Hey22, Corollary 4.1.3], there is a spectral sequence
$$E_2^{i,j} = \mathrm{Ext}^i_Z\big(\mathrm{L}^{-j}(N, i_P^G\,\chi), \chi'\big) \implies \mathrm{Ext}^{i+j}_G\big(i_P^G\,\chi, i_B^G\,\chi'\big).$$
Moreover, (4.2.2) reads $\mathrm{L}^{-j}(N, i_P^G\,\chi) \cong \bigoplus_{\substack{w\in D_{M,Z}\\ [\mathfrak{F}:\mathbb{Q}_p]d_w=j}} \delta_w \otimes_k n_{w*}^{-1}\chi$. We prove (a) by showing the contrapositive. If after restriction to $C(Z)$ we have $\chi' \not\cong \delta_w \otimes_k n_{w*}^{-1}\chi$ for all $w \in D_{M,Z}$ with $[\mathfrak{F} : \mathbb{Q}_p]d_w \le r$, then Lemma 4.2.4 shows that $E_2^{i,j} = 0$ for all $j \le r$ and all i. But this implies $\mathrm{Ext}^r_G(i_P^G\,\chi, i_B^G\,\chi') = 0$. We now prove (b). Take $w \in D_{M,Z}$ with $[\mathfrak{F} : \mathbb{Q}_p]d_w \le r$ and put $\chi' := \delta_w \otimes_k n_{w*}^{-1}\chi$. By the assumption and Lemma 4.2.7.(c) we have $\chi' \not\cong \delta_v \otimes_k n_{v*}^{-1}\chi$ for all $v \neq w$ (as characters of $C(Z)$). Together with Lemma 4.2.4 we deduce $E_2^{i,j} = 0$ whenever $j \neq [\mathfrak{F} : \mathbb{Q}_p]d_w$, and
$$E_2^{i,[\mathfrak{F}:\mathbb{Q}_p]d_w} \cong \mathrm{Ext}^i_Z\big(\chi',\chi'\big) \cong \mathrm{Ext}^i_Z(\mathbf{1},\mathbf{1}) \cong \mathrm{H}^i(Z,k),$$
where the second isomorphism follows from [Fus22, Theorem 1.1]. Hence, the spectral sequence collapses on the second page and gives an isomorphism $\mathrm{Ext}_Z^{r-[\mathfrak{F}:\mathbb{Q}_p]d_w}\big(\chi',\chi'\big) \cong \mathrm{Ext}^r_G\big(i_P^G\,\chi, i_B^G\,\chi'\big)$. □

TODO: Does the theorem generalize to a computation of Ext-groups between generalized Steinberg representations?

4.2.9. Definition. A smooth representation $V \in \mathrm{Rep}_k(G)$ is called *left cuspidal* (resp. *right cuspidal*) if for all proper parabolic subgroups $\mathbf{P} = \mathbf{M} \ltimes \mathbf{U}$ of $\mathbf{G}$ it holds that $\mathrm{L}^0(U,V) = 0$ (resp. $\mathrm{R}^0(U,V) = 0$).

4.2.10. Remark. By [AHV19, Corollary 6.9], an irreducible admissible G-representation is left and right cuspidal if and only if it is supercuspidal.

4.2.11. Theorem. *Let $\mathbf{P},\mathbf{Q}$ and $\mathcal{N}_{P,Q}$ be as in §4.1.2. Let $V \in \mathrm{Rep}_k(M)$ and $W \in \mathrm{Rep}_k(L)$.*

(a) *Assume $\mathbf{P} \not\subseteq \mathbf{Q}$ and $\mathbf{P} \not\supseteq \mathbf{Q}$, that V is left cuspidal and that W is right cuspidal. Then*
$$\mathrm{Ext}^1_G\big(i_P^G\,V, i_Q^G\,W\big) = 0.$$

(b) *Assume $\mathbf{P} = \mathbf{Q}$. For each $0 \le i < [\mathfrak{F} : \mathbb{Q}_p]$, the functor i_P^G induces an isomorphism*
$$\mathrm{Ext}^i_M(V,W) \xrightarrow{\cong} \mathrm{Ext}^i_G\big(i_P^G\,V, i_P^G\,W\big),$$
and there is an exact sequence
$$0 \longrightarrow \mathrm{Ext}_M^{[\mathfrak{F}:\mathbb{Q}_p]}(V,W) \longrightarrow \mathrm{Ext}_G^{[\mathfrak{F}:\mathbb{Q}_p]}\big(i_P^G\,V, i_P^G\,W\big) \longrightarrow X \longrightarrow \mathrm{Ext}_M^{[\mathfrak{F}:\mathbb{Q}_p]+1}(V,W),$$
where
$$X := \bigoplus_{\alpha\in\Delta_G^1\smallsetminus\Delta_M} \mathrm{Hom}_{n_\alpha^{-1}(M)\cap M}\big(\delta_\alpha \otimes_k n_{\alpha*}^{-1}\mathrm{L}^0(M\cap n_\alpha(U), V), \mathrm{R}^0(n_\alpha^{-1}(U)\cap M, W)\big).$$

(c) *Assume $\mathbf{P} = \mathbf{Q}$, that V is left cuspidal or W is right cuspidal, and that V and W admit distinct central characters. Then*
$$\mathrm{Ext}_G^{[\mathfrak{F}:\mathbb{Q}_p]}\big(i_P^G\,V, i_P^G\,W\big) \cong \bigoplus_{\alpha\in\Delta_M^{\perp,1}} \mathrm{Hom}_M\big(\delta_\alpha \otimes_k n_{\alpha*}^{-1}V, W\big).$$

(d) *Assume* $\mathbf{P} \supsetneqq \mathbf{Q}$ *and that* V *is left cuspidal. For all* $0 \le i \le [\mathfrak{F} : \mathbb{Q}_p]$*, the functor* i_P^G *induces an isomorphism*

$$\operatorname{Ext}^i_M\big(V, i^M_{M\cap Q} W\big) \xrightarrow{\cong} \operatorname{Ext}^i_G\big(i^G_P V, i^G_Q W\big).$$

(e) *Assume* $\mathbf{P} \subsetneqq \mathbf{Q}$ *and that* W *is right cuspidal. For all* $0 \le i \le [\mathfrak{F} : \mathbb{Q}_p]$*, the functor* i_Q^G *induces an isomorphism*

$$\operatorname{Ext}^i_L\big(i^L_{P\cap L} V, W\big) \xrightarrow{\cong} \operatorname{Ext}^i_G\big(i^G_P V, i^G_Q W\big).$$

Proof. The $P \times Q$-invariant open subsets $\big(G \smallsetminus \bigcup_{\substack{w\in D_{M,L}\\ d_w\le 1}} P n_w Q\big) \subseteq \big(G \smallsetminus PQ\big) \subseteq G$ induce by Proposition 3.2.11 and Theorem 3.3.5 two distinguished triangles

$$Y \longrightarrow \mathrm{L}\big(N, i^G_P V\big) \longrightarrow Z \xrightarrow{+} \tag{4.2.12}$$

and

$$\bigoplus_{\substack{\alpha\in\Delta^1_G\\ n_\alpha\in\mathcal{N}_{P,Q}}} i^L_{n_\alpha^{-1}(P)\cap L}\, \omega_\alpha \otimes_k n_{\alpha*}^{-1}\mathrm{L}\big(M\cap n_\alpha(N), V\big) \longrightarrow Z \longrightarrow i^L_{P\cap L}\,\mathrm{L}(M\cap N, V) \xrightarrow{+} \tag{4.2.13}$$

in $\mathrm{D}(L)$ such that $\mathrm{H}^i(Y) = 0$ for each $i > -2[\mathfrak{F} : \mathbb{Q}_p]$. Applying $\operatorname{RHom}_L(\_, W)$ to (4.2.12) and using $\operatorname{RHom}_G\big(i^G_P V, i^G_Q W\big) \cong \operatorname{RHom}_L\big(\mathrm{L}(N, i^G_P V), W\big)$, we obtain a distinguished triangle

$$\operatorname{RHom}_L(Z, W) \longrightarrow \operatorname{RHom}_G\big(i^Q_P V, i^G_Q W\big) \longrightarrow \operatorname{RHom}_L(Y, W) \xrightarrow{+},$$

where $\mathrm{H}^i \operatorname{RHom}_L(Y, W) = 0$ for all $i < 2[\mathfrak{F} : \mathbb{Q}_p]$. Thus, for all $i < 2[\mathfrak{F} : \mathbb{Q}_p]$ we have an isomorphism

$$\mathrm{H}^i \operatorname{RHom}_L(Z, W) \xrightarrow{\cong} \operatorname{Ext}^i_G\big(i^G_P V, i^G_Q W\big). \tag{4.2.14}$$

We now turn to the proofs of the statements.

We prove (a). From $\mathbf{P} \not\subseteq \mathbf{Q}$ we deduce that $\mathbf{M}\cap\mathbf{Q}$ is a proper parabolic of $\mathbf{M}$, and similarly from $\mathbf{P} \not\supseteq \mathbf{Q}$ it follows that $\mathbf{P}\cap\mathbf{L}$ is a proper parabolic of $\mathbf{L}$. As V is left cuspidal and W is right cuspidal, we obtain

$$\mathrm{L}^0(M\cap N, V) = 0 = \mathrm{R}^0(U\cap L, W).$$

Consequently, it follows that

$$\operatorname{RHom}_L\big(i^L_{P\cap L}\,\mathrm{L}(M\cap N, V), W\big) \cong \operatorname{RHom}_{M\cap L}\big(\mathrm{L}(M\cap N, V), \mathrm{R}(U\cap L, W)\big)$$

lies in $\mathrm{D}^{\ge 2}(k)$.

Claim. Let $\alpha \in \Delta_G$. Then $\mathbf{M}\cap n_\alpha(\mathbf{N}) = \{1\} = n_\alpha^{-1}(\mathbf{U})\cap\mathbf{L}$ implies $\mathbf{P} = \mathbf{Q}$.

Proof of the claim. The condition $\mathbf{M}\cap n_\alpha(\mathbf{N}) = \{1\}$ implies $\Sigma_M \cap s_\alpha(\Sigma^+ \smallsetminus \Sigma_L) = \varnothing$, where s_α denotes the simple reflection attached to α. Since $-\Sigma_M = \Sigma_M$, we deduce $\Sigma_M \subseteq s_\alpha(\Sigma_L)$. Similarly, the condition $n_\alpha^{-1}(\mathbf{U})\cap\mathbf{L} = \{1\}$ implies $\Sigma_L \subseteq s_\alpha^{-1}(\Sigma_M)$. Hence, we have $\Sigma_M = s_\alpha(\Sigma_L)$. If $s_\alpha \in D_{M,L}$, then $s_\alpha(\Sigma_L^+) \subseteq \Sigma^+ \cap \Sigma_M = \Sigma_M^+$ and in fact $s_\alpha(\Sigma_L^+) = \Sigma_M^+$. But then $s_\alpha(\Delta_L) = \Delta_M$ by the uniqueness of root bases. Since for $\beta \in \Delta_L$ the root $s_\alpha(\beta) = \beta - \langle\beta, \alpha^\vee\rangle\alpha$ is simple only if $\langle\beta, \alpha^\vee\rangle = 0$, it follows that $\alpha \in \Delta_L^\perp$ and hence that $\Delta_L = s_\alpha(\Delta_L) = \Delta_M$. This entails $\Sigma_L = \Sigma_M$. If $s_\alpha \notin D_{M,L}$, then $s_\alpha \in \mathrm{W}_M$ or $s_\alpha \in \mathrm{W}_L$ in which case it is clear that $\Sigma_M = \Sigma_L$. In any case, we obtain $\mathbf{M} = \mathbf{L}$ and hence also $\mathbf{P} = \mathbf{Q}$. □

The (contrapositive of the) claim and the assumptions $\mathbf{P} \not\subseteq \mathbf{Q}$ and $\mathbf{P} \not\supseteq \mathbf{Q}$ imply that $\mathbf{M}\cap n_\alpha(\mathbf{Q})$ is a proper parabolic subgroup of $\mathbf{M}$ or $n_\alpha^{-1}(\mathbf{P})\cap\mathbf{L}$ is a proper parabolic subgroup of $\mathbf{L}$. As V is left cuspidal and W is right cuspidal, it follows that $\mathrm{L}^0(M\cap n_\alpha(N), V) = 0$ or $\mathrm{R}^0(n_\alpha^{-1}(U)\cap L, W) = 0$. Since also $\omega_\alpha = \delta_\alpha\big[[\mathfrak{F} : \mathbb{Q}_p]d_\alpha\big]$, we deduce that

$$\begin{aligned}&\operatorname{RHom}_L\big(i^L_{n_\alpha^{-1}(P)\cap L}\omega_\alpha \otimes_k n_{\alpha*}^{-1}\mathrm{L}\big(M\cap n_\alpha(N), V\big), W\big)\\ &\qquad\cong \operatorname{RHom}_{n_\alpha^{-1}(M)\cap L}\big(\omega_\alpha \otimes_k n_{\alpha*}^{-1}\mathrm{L}\big(M\cap n_\alpha(N), V\big), \mathrm{R}\big(n_\alpha^{-1}(U)\cap L, W\big)\big)\end{aligned}$$

lies in $\mathrm{D}^{\geq[\mathfrak{F}:\mathbb{Q}_p]d_\alpha+1}(k)$. Hence, applying $\mathrm{H}^1\,\mathrm{RHom}_L(-,W)$ to (4.2.13) and using (4.2.14), we deduce

$$\mathrm{Ext}^1_G\big(i_P^G\,V, i_Q^G\,W\big) \cong \mathrm{H}^1\,\mathrm{RHom}_L(Z,W) \cong \mathrm{H}^1\,\mathrm{RHom}_L\big(i_{P\cap L}^L\,\mathrm{L}(M\cap N,V),W\big) = 0.$$

We now prove (b), so assume $\mathbf{P}=\mathbf{Q}$. For $\alpha\in\Delta_G^1$ we have $\omega_\alpha = \delta_\alpha\big[[\mathfrak{F}:\mathbb{Q}_p]\big]$, and hence the complex

$$\bigoplus_{\substack{\alpha\in\Delta_G^1\\ n_\alpha\in\mathcal{N}_{P,P}}} \mathrm{RHom}_M\big(i^M_{n_\alpha^{-1}(P)\cap M}\,\omega_\alpha\otimes_k n_{\alpha*}^{-1}\mathrm{L}\big(M\cap n_\alpha(U),V\big),W\big) \tag{4.2.15}$$

lies in $\mathrm{D}^{\geq[\mathfrak{F}:\mathbb{Q}_p]}(k)$. Since $\mathrm{H}^{[\mathfrak{F}:\mathbb{Q}_p]}\big((4.2.15)\big)\cong X$, the assertion follows by applying $\mathrm{H}^i\,\mathrm{RHom}_M(-,W)$ to (4.2.13) and using (4.2.14).

For (c) we observe that $\mathrm{Ext}^i_M(V,W)=0$ for all $i\geq 0$ by Lemma 4.2.4 and the assumption that V and W admit distinct central characters. Note that $\mathbf{M}\cap n_\alpha(\mathbf{P})=\mathbf{M}$ if and only if $\mathbf{M}=n_\alpha(\mathbf{M})$ if and only if $\alpha\in(\Delta_M\cup\Delta_M^\perp)$. As V is left cuspidal or W is right cuspidal, we deduce $\mathrm{L}^0\big(M\cap n_\alpha(U),V\big)=0$ or $\mathrm{R}^0\big(n_\alpha^{-1}(U)\cap M,W\big)=0$, for each $\alpha\in\Delta_G\smallsetminus(\Delta_M\cup\Delta_M^\perp)$. Now, the assertion follows from (b).

The proofs of (d) and (e) are symmetric, so we only prove (d). Since V is left cuspidal, we have $\mathrm{L}^0\big(M\cap n_\alpha(U),V\big)=0$ for all $\alpha\in\Delta_G\smallsetminus(\Delta_M\cup\Delta_M^\perp)$. For all $\alpha\in\Delta_M^\perp$ we have

$$\mathrm{Hom}_M\big(\delta_\alpha\otimes_k n_{\alpha*}^{-1}V, i^M_{M\cap Q}\,W\big)=0,$$

because with V also $\delta_\alpha\otimes_k n_{\alpha*}^{-1}V$ is left cuspidal and $\mathbf{M}\cap\mathbf{Q}$ is a proper parabolic subgroup of $\mathbf{M}$. Hence, the assertion follows from (b). ☐

4.3. Generalized Steinberg representations. We will need the following well-known result, which we will prove for the convenience of the reader.

4.3.1. Lemma. *Let R be a unital associative ring and M an R-module of finite length whose constituents occur with multiplicity one. Denote $\mathrm{JH}(M)$ the set of Jordan–Hölder factors of M.*

- *(a) For each $V\in\mathrm{JH}(M)$ there exists a unique minimal submodule $M_V\leq M$ which has V as a quotient. The cosocle of M is V.*
- *(b) One has $\mathrm{JH}(N_1\cap N_2)=\mathrm{JH}(N_1)\cap\mathrm{JH}(N_2)$ and $\mathrm{JH}(N_1+N_2)=\mathrm{JH}(N_1)\cup\mathrm{JH}(N_2)$, for all submodules $N_1,N_2\leq M$.*

Proof. We first show (a). Let $V\in\mathrm{JH}(M)$ and take any submodule $N\leq M$ minimal with the property that V is a quotient of N. If we had $\mathrm{cosoc}(N)\neq V$, then the preimage of V under the surjection $N\twoheadrightarrow\mathrm{cosoc}(N)$ is strictly contained in N and has V as a quotient, contradicting the minimality assumption on N.

Let now $N,N'\leq M$ be two minimal submodules having V as a quotient. Consider the short exact sequence $0\to N'\longrightarrow N+N'\to N/(N\cap N')\to 0$. From the multiplicity one assumption, we deduce $V\notin\mathrm{JH}(N/(N\cap N'))$. Since $V\in\mathrm{JH}(N)$, it follows that $V\in\mathrm{JH}(N\cap N')$. Hence, there exists $N''\leq N\cap N'$ having V as a quotient. By the minimality of N and N' we conclude $N=N''=N'$. This proves the uniqueness claim.

For (b) it is generally true that $\mathrm{JH}(N_1\cap N_2)\subseteq\mathrm{JH}(N_1)\cap\mathrm{JH}(N_2)$ and $\mathrm{JH}(N_1+N_2)=\mathrm{JH}(N_1)\cup\mathrm{JH}(N_2)$. The remaining assertion needs the multiplicity one condition and follows from (a), since $V\in\mathrm{JH}(N_1)\cap\mathrm{JH}(N_2)$ implies $M_V\subseteq N_1\cap N_2$ and hence $V\in\mathrm{JH}(N_1\cap N_2)$. ☐

4.3.2. Recall the setup in §4.1.1. For any $I\subseteq\Delta_G$ we denote $\mathbf{P}_I=\mathbf{M}_I\ltimes\mathbf{U}_I$ the corresponding standard parabolic subgroup of $\mathbf{G}$. Note that $\mathbf{B}=\mathbf{P}_\varnothing$.

In the following, we will abbreviate $i^G_{P_I}$ for $i^G_{P_I}(\mathbf{1})$. By [GK14, Theorem D] (for split $\mathbf{G}$ with classical root system) and [Ly15, Corollary 3.2] (for general $\mathbf{G}$) the Jordan–Hölder constituents of $i^G_{P_I}$ are the *generalized Steinberg representations*

$$\mathrm{Sp}^G_{P_J}\coloneqq i^G_{P_J}\Big/\sum_{J\subsetneq J'\subseteq\Delta_G} i^G_{P_{J'}},$$

where $I \subseteq J \subseteq \Delta_G$; they are pairwise non-isomorphic and occur with multiplicity one.

4.3.3. Lemma. *Let $I, J, J_1, \dots, J_r \subseteq \Delta_G$ be arbitrary subsets. Inside i_B^G we have:*

(a) $i_{P_I}^G \cap i_{P_J}^G = i_{P_{I\cup J}}^G$*;*

(b) $i_{P_I}^G \cap \sum_{i=1}^r i_{P_{J_i}}^G = \sum_{i=1}^r \big(i_{P_I}^G \cap i_{P_{J_i}}^G\big)$.

Proof. (a) is clear as $P_{I\cup J}$ is generated as a group by P_I and P_J.

We now prove (b). Since the constituents of i_B^G occur with multiplicity one, we may prove the equality by showing that the sets of Jordan–Hölder factors are the same. Using Lemma 4.3.1 we compute

$$\begin{aligned} \mathrm{JH}\Big(i_{P_I}^G \cap \sum_{i=1}^r i_{P_{J_i}}^G\Big) &= \mathrm{JH}(i_{P_I}^G) \cap \bigcup_{i=1}^r \mathrm{JH}(i_{P_{J_i}}^G) \\ &= \bigcup_{i=1}^r \big(\mathrm{JH}(i_{P_I}^G) \cap \mathrm{JH}(i_{P_{J_i}}^G)\big) = \mathrm{JH}\Big(\sum_{i=1}^r (i_{P_I}^G \cap i_{P_{J_i}}^G)\Big). \end{aligned}$$ □

4.3.4. Choose a total ordering $\preccurlyeq$ on Δ_G. Fix subsets $I_0, I_1 \subseteq \Delta_G$ and $I, K \subseteq I_1$, and consider the power set $\mathcal{P}(I_1 \smallsetminus I_0)$ as a partially ordered set with respect to reverse inclusion. Define a functor

$$\begin{aligned} \mathcal{V}_{I,K}^{I_0,I_1} \colon \mathcal{P}(I_1 \smallsetminus I_0) &\longrightarrow \mathrm{Rep}_k(M_K), \\ J &\longmapsto i_{P_{(I\cup J)\cap K}}^{M_K}, \end{aligned}$$

where $\mathcal{V}_{I,K}^{I_0,I_1}(J \subseteq J')$ is the inclusion $i_{P_{(I\cup J')\cap K}}^{M_K} \hookrightarrow i_{P_{(I\cup J)\cap K}}^{M_K}$, and where for any $I' \subseteq \Delta_G$ we write $i_{P_{I'\cap K}}^{M_K}$ instead of $i_{P_{I'\cap K}\cap M_K}^{M_K}$ for the sake of readability. We define a complex $C_{I_1,K}^\bullet(I, I_0)$ as follows: for each integer $n \le 0$ we put

$$C_{I_1,K}^n(I, I_0) \coloneqq \bigoplus_{\substack{J \subseteq I_1 \smallsetminus I_0 \\ |J| = -n}} \mathcal{V}_{I,K}^{I_0,I_1}(J);$$

by convention, $C_{I_1,K}^n(I, I_0) = 0$ for $n > 0$. The differential $d^n \colon C_{I_1,K}^n(I, I_0) \to C_{I_1,K}^{n+1}(I, I_0)$ is defined on the J-th summand as

$$(d^n)_J = \sum_{j_0 \in J} \varepsilon_{I_0}(J \smallsetminus \{j_0\}, j_0) \cdot \mathcal{V}_{I,K}^{I_0,I_1}\big(J \smallsetminus \{j_0\} \subseteq J\big),$$

where $\varepsilon_{I_0}(J \smallsetminus \{j_0\}, j_0) \coloneqq (-1)^{|\{j \in I_0 \sqcup J \,|\, j \prec j_0\}|}$. It is an easy exercise to show that $C_{I_1,K}^\bullet(I, I_0)$ is indeed a complex.

4.3.5. Example. The complex $C_{\Delta_G,\Delta_G}^\bullet(I, I)$ can be depicted as

$$0 \longrightarrow \mathbf{1} \longrightarrow \bigoplus_{\substack{I \subseteq J \subseteq \Delta_G \\ |\Delta_G \smallsetminus J| = 1}} i_{P_J}^G \longrightarrow \cdots \longrightarrow \bigoplus_{j \in \Delta_G \smallsetminus I} i_{P_{I\cup\{j\}}}^G \longrightarrow i_{P_I}^G \longrightarrow 0. \tag{4.3.6}$$

Observe that $\mathrm{H}^0\big(C_{\Delta_G,\Delta_G}^\bullet(I, I)\big) = \mathrm{Sp}_{P_I}^G$.

4.3.7. Proposition. *The complex $C_{I_1,K}^\bullet(I, I_0)$ is a resolution of $\mathrm{H}^0\big(C_{I_1,K}^\bullet(I, I_0)\big)$.*

Proof. In view of Lemma 4.3.3, this follows from the general [KS06, Corollary 12.4.11]. □

4.3.8. Notation. For $w \in \mathrm{W}$, we put

$$I(w) \coloneqq \{\alpha \in \Delta_G \,|\, \ell(s_\alpha w) = \ell(w) + 1\},$$

where $\ell(\cdot)$ denotes the length function on W. Observe that $I(w) \subseteq \Delta_G$ is the maximal subset such that $w^{-1} \in D_{I(w)}$.

4.3.9. Theorem. *Let $I, K \subseteq \Delta_G$ and $j \in \mathbb{Z}$. Then*

$$\mathrm{L}^{-j}\big(U_K, \mathrm{Sp}_{P_I}^G\big) = \bigoplus_{\substack{w \in D_{I,K},\\ [\mathfrak{F}:\mathbb{Q}_p]d_w = j,\\ I(w)\smallsetminus I \subseteq w(K)}} i_{P_{w^{-1}(I(w))\cap K}}^{M_K}\big(\delta_w \otimes_k n_{w*}^{-1}\,\mathrm{Sp}_{P_{I\cap w(K)}}^{M_{I(w)\cap w(K)}}\big).$$

Proof. We compute $\mathrm{L}^{-j}(U_K, \mathrm{Sp}_{P_I}^G)$ via a spectral sequence coming from the resolution (4.3.6). For any complex $X^\bullet \in \mathrm{C}(G)$ and any integer r, we denote $\sigma^{\geq r}X$ the stupid truncation of $X^\bullet$ [KS06, Definition 11.3.11]; these fit into a distinguished triangle

$$\sigma^{\geq r+1}X \longrightarrow \sigma^{\geq r}X \longrightarrow X^r[-r] \xrightarrow{+}$$

in $\mathrm{D}(G)$. As, *e.g.*, described in [Hey22, Lemma 2.3.20] (applied to $X(r) \coloneqq \sigma^{\geq r}X$), we obtain for any triangulated functor $F\colon \mathrm{D}(G) \to \mathrm{D}(M_K)$ a spectral sequence

$$E_1^{r,s} = \mathrm{H}^s(F)(X^r) \implies \mathrm{H}^{r+s}(FX),$$

which converges provided $X^\bullet$ is bounded. We apply this to the bounded complex $X^\bullet = C^\bullet_{\Delta_G,\Delta_G}(I,I)$ and the functor $F = \mathrm{L}(U_K, \_)$ and obtain a third quadrant spectral sequence

$$E_1^{r,s} = \bigoplus_{\substack{I\subseteq J\subseteq \Delta_G\\ |J\smallsetminus I| = -r}} \mathrm{L}^s\big(U_K, i_{P_J}^G\big) \implies \mathrm{L}^{r+s}\big(U_K, \mathrm{Sp}_{P_I}^G\big).$$

We will show that this spectral sequence collapses on the second page. Fix any s. The differential $d^r\colon E_1^{r,s} \to E_1^{r+1,s}$ is induced by the differential of $C^\bullet_{\Delta_G,\Delta_G}(I,I)$. In order to analyse $E_1^{\bullet,s}$, we fix $J \subseteq J' \subseteq \Delta_G$ for the moment. The inclusion $i_{P_{J'}}^G \hookrightarrow i_{P_J}^G$ coincides with $i_{P_{J'}}^G(\eta)$, where $\eta\colon \mathbf{1} \to i_{P_J\cap M_{J'}}^{M_{J'}}$ is the unit map. From the Geometrical Lemma (Corollary 3.3.6) we deduce

$$\mathrm{L}^s\big(U_K, i_{P_J}^G\big) = \bigoplus_{\substack{w\in D_{J',K}\\ [\mathfrak{F}:\mathbb{Q}_p]d_w \leq -s}} i_{w^{-1}(P_{J'})\cap M_K}^{M_K}\Big(\delta_w \otimes n_{w*}^{-1}\mathrm{L}^{s+[\mathfrak{F}:\mathbb{Q}_p]d_w}\big(M_{J'}\cap n_w(U_K), i_{P_J\cap M_{J'}}^{M_{J'}}\big)\Big).$$

Hence, the map $\mathrm{L}^s(U_K, i_{P_{J'}}^G) \to \mathrm{L}^s(U_K, i_{P_J}^G)$ is induced on the w-th summand (where w satisfies $[\mathfrak{F} : \mathbb{Q}_p]d_w = -s$) by applying the functor $i_{w^{-1}(P_{J'})\cap M_K}^{M_K}\big(\delta_w \otimes n_{w*}^{-1}\mathrm{L}^0\big(M_{J'}\cap n_w(U_K), \_\big)\big)$ to the unit $\eta\colon \mathbf{1} \to i_{P_J\cap M_{J'}}^{M_{J'}}$. By Kilmoyer's Theorem [Car85, Theorem 2.7.4], we have $M_{J'} \cap n_w(M_K) = M_{J'\cap w(K)}$. Hence, $P_J \cap M_{J'} \cap w(M_K)$ is the standard parabolic subgroup of $M_{J'\cap w(K)}$ attached to $J \cap w(K)$. Observe that the diagram

$$\begin{tikzcd}
\mathrm{L}^0\big(M_{J'}\cap n_w(U_K), \mathbf{1}\big) \arrow[r] \arrow[d, "\cong"'] & \mathrm{L}^0\big(M_{J'}\cap n_w(U_K), i_{P_J\cap M_{J'}}^{M_{J'}}\big) \arrow[d, "\cong"] \\
\mathbf{1} \arrow[r] & i_{P_{J\cap w(K)}}^{M_{J'\cap w(K)}},
\end{tikzcd}$$

commutes, where the bottom map is the unit map.

To summarize, the map $\mathrm{L}^s(U_K, i_{P_{J'}}^G) \to \mathrm{L}^s(U_K, i_{P_J}^G)$ is on the w-th summand given by

$$i_{w^{-1}(P_{J'})\cap M_K}^{M_K}\,\delta_w \longrightarrow i_{w^{-1}(P_{J'})\cap M_K}^{M_K}\big(\delta_w \otimes_k n_{w*}^{-1}\, i_{P_{J\cap w(K)}}^{M_{J'\cap w(K)}}\big).$$

For any $w \in D_{I,K}$, we consider the complex $C^\bullet(w) \coloneqq C^\bullet_{I(w),I(w)\cap w(K)}(I,I)$ attached to the functor

$$\begin{aligned}
\mathcal{V}_{I,I(w)\cap w(K)}^{I,I(w)}\colon \mathcal{P}\big(I(w)\smallsetminus I\big) &\longrightarrow \mathrm{Rep}_k(M_{I(w)\cap w(K)}),\\
J &\longmapsto i_{P_{(I\cup J)\cap w(K)}}^{M_{I(w)\cap w(K)}},
\end{aligned}$$

cf. §4.3.4. Our analysis shows that

$$E_1^{\bullet,s} = \bigoplus_{\substack{w\in D_{I,K}\\ [\mathfrak{F}:\mathbb{Q}_p]d_w=-s}} i^{M_K}_{P_{w^{-1}(I(w))\cap K}}\big(\delta_w \otimes_k n_{w*}^{-1} C^\bullet(w)\big).$$

From Proposition 4.3.7 it follows that $C^\bullet(w)$ is a resolution of

$$\mathrm{H}^0\big(C^\bullet(w)\big) = \begin{cases} \mathrm{Sp}^{M_{I(w)\cap w(K)}}_{P_{I\cap w(K)}}, & \text{if } I(w)\smallsetminus I \subseteq w(K),\\ 0, & \text{otherwise.}\end{cases}$$

We conclude that $E_2^{\bullet,\bullet}$ is supported on the 0-th column, *i.e.*, the spectral sequence collapses on the second page. Since

$$E_2^{0,-j} = \bigoplus_{\substack{w\in D_{I,K},\\ [\mathfrak{F}:\mathbb{Q}_p]d_w=j,\\ I(w)\smallsetminus I\subseteq w(K)}} i^{M_K}_{P_{w^{-1}(I(w))\cap K}}\big(\delta_w \otimes_k n_{w*}^{-1} \mathrm{Sp}^{M_{I(w)\cap w(K)}}_{P_{I\cap w(K)}}\big),$$

the assertion follows. □

4.3.10. Corollary. *For every $I \subseteq \Delta_G$ and $j \in \mathbb{Z}$, one has*

$$\mathrm{L}^{-j}\big(U_\varnothing, \mathrm{Sp}^G_{P_I}\big) = \bigoplus_{\substack{w\in D_{I,\varnothing},\\ [\mathfrak{F}:\mathbb{Q}_p]d_w=j,\\ I=I(w)}} \delta_w.$$

(The condition $I = I(w)$ indicates $w \in D_{I,\varnothing} \smallsetminus \bigcup_{J\supsetneq I} D_{J,\varnothing}$.) □

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