THE LEFT ADJOINT OF DERIVED PARABOLIC INDUCTION

CLAUDIUS HEYER

Abstract. We prove that the derived parabolic induction functor, defined on the unbounded derived category of smooth mod p representations of a p-adic reductive group, admits a left adjoint $\mathrm{L}(U, -)$. We study the cohomology functors $\mathrm{H}^i \circ \mathrm{L}(U, -)$ in some detail and deduce that $\mathrm{L}(U, -)$ preserves bounded complexes and global admissibility in the sense of Schneider–Sorensen. Using $\mathrm{L}(U, -)$ we define a derived Satake homomorphism und prove that it encodes the mod p Satake homomorphisms defined explicitly by Herzig.

Contents

1. Introduction

1.1. **The setting.** This article is devoted to the investigation of the left adjoint of the derived parabolic induction functor in natural characteristic. Let G be a p-adic reductive group, that is, the group of $\mathfrak{F}$-points of a connected reductive group defined over a finite field extension $\mathfrak{F}/\mathbb{Q}_p$. Let k be

2020 *Mathematics Subject Classification.* 11C08, 20C08, 20G25.

a coefficient field of characteristic p and denote $\operatorname{Rep}_k(G)$ the category of smooth G-representations on k-vector spaces. The derived category

$$\mathrm{D}(G) \coloneqq \mathrm{D}(\operatorname{Rep}_k(G))$$

of unbounded cochain complexes was introduced in [Sch15]. The necessity to work with the derived category stems from the fact that taking K-invariants is never exact when $K \subseteq G$ is a compact open subgroup. The main result of *op. cit.* proves that, given a torsion-free open pro-p subgroup $K \subseteq G$, taking derived K-invariants induces a derived equivalence of $\mathrm{D}(G)$ with the derived category $\mathrm{D}(\mathcal{H}_K^\bullet)$ of dg-modules over a certain differential graded algebra $\mathcal{H}_K^\bullet$. This strongly suggests that the study of smooth mod p representations is best done on the derived level.

Let $P = UM$ be a parabolic subgroup of G with Levi subgroup M and unipotent radical U. The parabolic induction functor $i_M^G\colon \operatorname{Rep}_k(M) \longrightarrow \operatorname{Rep}_k(G)$ is defined as the composition

$$\operatorname{Rep}_k(M) \xrightarrow{\operatorname{Inf}_P^M} \operatorname{Rep}_k(P) \xrightarrow{\operatorname{Ind}_P^G} \operatorname{Rep}_k(G),$$

where Inf_P^M is the inflation along the canonical projection $P \twoheadrightarrow M$ and Ind_P^G is the smooth induction functor. The functors Inf_P^M and Ind_P^G are exact. Hence also i_M^G is exact and extends to a functor

$$\mathrm{R}i_M^G\colon \mathrm{D}(M) \longrightarrow \mathrm{D}(G)$$

on the derived categories: Here, $\mathrm{R}i_M^G$ is given by applying i_M^G termwise to a complex. (The different notation serves the only purpose of being able to distinguish between underived and derived parabolic induction.) The compatibility of $\mathrm{R}i_M^G$ with an analogous functor on $\mathrm{D}(\mathcal{H}_K^\bullet)$ is explored in [SS22a], which also discusses the existence of left and right adjoints of $\mathrm{R}i_M^G$, referring for the former to my unpublished notes.

Let me briefly discuss why it is not immediate that $\mathrm{R}i_M^G$ admits a left adjoint. It is well-known that the Jacquet-functor $\operatorname{Rep}_k(G) \longrightarrow \operatorname{Rep}_k(M)$, $V \longmapsto V_U$, is left adjoint to i_M^G. The left derived functor of $(-)_U$ would be a promising candidate for the left adjoint of derived parabolic induction. Unfortunately, the category $\operatorname{Rep}_k(G)$ does not have enough projectives, since $\operatorname{char}(k) = p$; in fact, cf. [DK21, Rmk. 2.20], it has no non-zero projectives at all! Therefore, one cannot use projective resolutions to prove that the left derived functor exists, and it remains unclear whether the Jacquet-functor admits a left derived functor. A second approach is to show that $\mathrm{R}i_M^G$ commutes with arbitrary small products and to then use a variant of Brown representability to deduce the existence of a left adjoint. It is easy to show that i_M^G preserves arbitrary products, but since products in $\mathrm{D}(M)$ and $\mathrm{D}(G)$ are not computed by taking products of representing complexes, it is not obvious that $\mathrm{R}i_M^G$ should have the same property.

In view of these concerns it was surprising that $\mathrm{R}i_M^G$ does in fact commute with small products. The initial proof was rather technical and relied on an analysis of Hochschild–Serre spectral sequences associated with $K_P \cap U \trianglelefteq K_P$, where K_P is a compact open subgroup of P, and analyzing how these behave with respect to restriction to smaller subgroups. The more conceptual proof presented here is by appealing to one of the main results of [BDS16]. The key ingredient is the observation that the derived functor $\mathrm{RH}^0(U \cap K_P, -)\colon \mathrm{D}(K_P) \longrightarrow \mathrm{D}(K_P/U \cap K_P)$ preserves compact objects provided K_P and $K_P/U \cap K_P$ are torsion-free.

1.2. **Main results.** Note that $\mathrm{R}i_M^G = \mathrm{R}\operatorname{Ind}_P^G \circ \mathrm{R}\operatorname{Inf}_P^M$ so that it suffices to determine the left adjoints of $\mathrm{R}\operatorname{Ind}_P^G$ and $\mathrm{R}\operatorname{Inf}_P^M$ separately. The left adjoint of $\mathrm{R}\operatorname{Ind}_P^G$ is easily seen to be the restriction functor $\mathrm{R}\operatorname{Res}_P^G\colon \mathrm{D}(G) \longrightarrow \mathrm{D}(P)$, and hence it remains to show that $\mathrm{R}\operatorname{Inf}_P^M$ admits a left adjoint.

Granted the existence, the next task would then be to find an explicit description of said left adjoint. In view of this it is convenient to study the derived inflation functor $\mathrm{R}\operatorname{Inf}_S^T\colon \mathrm{D}(T) \longrightarrow \mathrm{D}(S)$ along a surjective open monoid homomorphism $f\colon S \twoheadrightarrow T$ with kernel U, where S and T are open submonoids of some p-adic Lie groups. The most relevant example is the following: Fix a compact open subgroup K_U of U and consider the positive monoid $T = M^+$ consisting of those elements $m \in M$ with $mK_Um^{-1} \subseteq K_U$ as well as the open submonoid $S = K_UM^+$ of P (in which case the kernel of $S \longrightarrow T$ is K_U). This added flexibility is useful to deduce precise statements about the left adjoint of $\mathrm{R}\,\mathrm{Inf}_P^M$.

In this general setting, the first main result is:

Theorem A (Theorem 3.2.3)**.** *The functor* $\mathrm{R}\,\mathrm{Inf}_S^T \colon \mathrm{D}(T) \longrightarrow \mathrm{D}(S)$ *admits a left adjoint* L_U*. In particular,* $\mathrm{R}i_M^G$ *admits a left adjoint* $\mathrm{L}(U, -)$*.*

We deduce the theorem from a variant of Brown representability by proving that $\mathrm{R}\,\mathrm{Inf}_S^T$ commutes with small products. Observing that restriction to compact open subgroups is conservative and product preserving, we reduce to the case where S and T are torsion-free compact p-adic Lie groups. We then verify that the categories $\mathrm{D}(T)$ and $\mathrm{D}(S)$ are rigidly-compactly generated (in the sense of [BDS16, 1.2. Hyp.]) and that the right adjoint $\mathrm{RH}^0(U, -)$ of $\mathrm{R}\,\mathrm{Inf}_S^T$ preserves compact objects. With these hypotheses in place, the theorem follows from [BDS16, 3.3. Thm.]. In fact, under these restrictions on S and T, the shifted invariants functor $\mathrm{RH}^0(U, -)[\dim U]$ is a left adjoint of $\mathrm{R}\,\mathrm{Inf}_S^T$.

As a special case of Theorem A we obtain that $\mathrm{R}\,\mathrm{Inf}_P^M$ admits a left adjoint, which we denote by L_U. The next task then is to try to explicitly compute L_U, or at least its cohomology functors L_U^i. A formal argument shows $\mathrm{L}_U^i \cong 0$, for $i > 0$, and that L_U^0 is the usual functor of U-coinvariants (*i.e.*, the left adjoint of Inf_P^M). In order to analyze L_U^i for $i < 0$, we employ p-adic monoids. To motivate the idea, we recall a different description of the (underived) coinvariants functor L_U^0. The unipotent radical U is the union of compact open subgroups, say $U = \bigcup_{r \in \Lambda} H_r$. It is a well-known and easy fact that $\mathrm{L}_U^0 \cong \mathrm{colim}_{r \in \Lambda} \mathrm{L}_{H_r}^0$. Although this isomorphism describes L_U^0 in terms of functors of coinvariants with respect to compact open subgroups (which should be easier to understand), it is in this formulation difficult to extend to the derived setting. This is where the positive monoid enters the stage: Fixing a torsion-free compact open subgroup K_U of U, the positive monoid M^+ of elements $m \in M$ with $mK_U m^{-1} \subseteq K_U$ acts naturally on $\mathrm{L}_{K_U}^0(V)$, for $V \in \mathrm{Rep}_k(P)$. It is well-known that there exists a central positive element $z \in M$ such that $U = \bigcup_{i \geq 0} z^{-i} K_U z^i$. There is a natural isomorphism $\mathrm{colim}_i \mathrm{L}_{z^{-i}K_U z^i}^0(V) \cong \mathrm{colim}_z \mathrm{L}_{K_U}^0(V)$, where the transition maps in the latter colimit are given by multiplication with z. In fact, colim_z is the (exact) functor $\mathrm{Rep}_k(M^+) \longrightarrow \mathrm{Rep}_k(M)$ given by inverting the action of z. It is left adjoint to the restriction functor $\mathrm{Res}_{M^+}^M$ and also denoted $\mathrm{ind}_{M^+}^M$. Thus, there is a natural isomorphism $\mathrm{L}_U^0(V) \cong \mathrm{ind}_{M^+}^M \mathrm{L}_{K_U}^0(V)$, which can easily be extended to the derived categories. Concretely, writing $P^+ := K_U M^+$ (it is an open submonoid in P), the derived inflation $\mathrm{R}\,\mathrm{Inf}_{P^+}^{M^+}$ admits a left adjoint $\mathrm{L}_{K_U} \colon \mathrm{D}(P^+) \longrightarrow \mathrm{D}(M^+)$, and we find a natural isomorphism

$$\mathrm{R}\,\mathrm{ind}_{M^+}^M \mathrm{L}_{K_U} \mathrm{R}\,\mathrm{Res}_{P^+}^P \overset{\cong}{\Longrightarrow} \mathrm{L}_U$$

of functors $\mathrm{D}(P) \longrightarrow \mathrm{D}(M)$ (Proposition 3.4.6). The group K_U is a compact, torsion-free p-adic Lie group and hence has finite cohomological dimension by a result of Lazard and Serre (see [Ser65, Cor. (1)]). We deduce from this fact in Proposition 3.4.23 that the functors $\mathrm{L}_{K_U}^i$ are naturally isomorphic to the cohomology functors $\mathrm{H}^{\dim K_U + i}(K_U, -)$, where the M^+-action coincides with the "Hecke action" from [Eme10b] twisted by a certain character. To summarize:

Theorem B (Corollary 3.4.26)**.** *One has natural isomorphisms* $\mathrm{L}_U^i \cong \mathrm{ind}_{M^+}^M \mathrm{H}^{\dim K_U + i}(K_U, -)$ *as functors* $\mathrm{Rep}_k(P) \longrightarrow \mathrm{Rep}_k(M)$*, for all* $i \in \mathbb{Z}$*.*

An immediate consequence of Theorem B is that L_U restricts to the bounded derived categories.

Similar results trivially also follow for the left adjoint $\mathrm{L}(U, -) := \mathrm{L}_U \circ \mathrm{R}\,\mathrm{Res}_P^G$ of derived parabolic induction $\mathrm{R}i_M^G$. It is well-known that i_M^G and the Jacquet-functor $\mathrm{L}^0(U, -)$ preserve admissible representations, and it is a natural question whether the same still holds on the derived level. The correct notion of admissibility for $\mathrm{D}(G)$ was introduced by Schneider–Sorensen in [SS22b]: A complex X in $\mathrm{D}(G)$ is called *globally admissible* if for some torsion-free compact open pro-p subgroup $K \subseteq G$ and all $i \in \mathbb{Z}$ the cohomology groups $\mathrm{H}^i(K, X)$ are finite-dimensional k-vector spaces. They then proceed to show that globally admissible complexes are precisely the reflexive objects for a natural duality functor on $\mathrm{D}(G)$. In this direction, we prove:

Theorem C (Theorem 4.2.1)**.** *The functors* $\mathrm{R}i_M^G$ *and* $\mathrm{L}(U,-)$ *preserve globally admissible complexes.*

We finally compute $\mathrm{L}^i(U,V)$ for the irreducible smooth $\overline{\mathbb{F}}_p[\mathrm{GL}_2(\mathbb{Q}_p)]$-representations V and all $i \in \mathbb{Z}$, where U consists of the upper triangular unipotent matrices. For $i \in \{-1,0\}$ the representations $\mathrm{L}^i(U,V)$ are listed in Table 1 on page 53, while for $i \notin \{-1,0\}$ these representations are zero.

The proof uses a derived variant of the Satake homomorphism, which we will now describe. Let K be an compact open subgroup of G satisfying $G = PK$. Given $V \in \mathrm{D}(K)$, there is a natural isomorphism $\mathrm{L}(U, \mathrm{ind}_K^G V) \cong \mathrm{ind}_{K\cap M}^M \mathrm{L}(K \cap U, V)$, where we write $\mathrm{L}(K\cap U,-) := \mathrm{L}_{K\cap U}\mathrm{R}\,\mathrm{Res}_{K\cap P}^K$. Hence, the functor $\mathrm{L}(U,-)$ induces a k-algebra homomorphism

$$\mathcal{S}_V\colon \operatorname{End}_{\mathrm{D}(G)}(\mathrm{ind}_K^G V) \longrightarrow \operatorname{End}_{\mathrm{D}(M)}\big(\mathrm{ind}_{K\cap M}^M \mathrm{L}(K\cap U, V)\big)$$

which we call the derived Satake homomorphism.[1] In a different context, a version of the derived Satake homomorphism was defined and studied by Ronchetti, [Ron19]. The relation between these homomorphisms is still unclear: The fact that $\mathrm{L}(K \cap U, V)$ is a proper complex (even when $V = \mathbf{1}$ is the trivial representation) makes a comparison difficult. However, it is possible to relate $\mathcal{S}_V$ with the (underived) mod p Satake homomorphisms introduced by Herzig, [Her11b, Thm. 2.6 and §2.3]. Composing $\mathcal{S}_V$ with the $(-n)$-th cohomology functor H^{-n} yields a k-algebra homomorphism

$$\mathcal{S}_V^n\colon \operatorname{End}_{\mathrm{Rep}_k(G)}(\mathrm{ind}_K^G V) \longrightarrow \operatorname{End}_{\mathrm{Rep}_k(M)}\big(\mathrm{ind}_{K\cap M}^M \mathrm{L}^{-n}(K\cap U, V)\big),$$

and we prove:

Theorem D (Theorem 4.3.2). *The mod p Satake homomorphisms constructed in* [Her11b] *coincide with* $\mathcal{S}_V^0$ *and* $\mathcal{S}_V^{\dim U}$.

The takeaway of Theorem D is that the mod p Satake homomorphisms appearing in the literature are induced by a functor. For $\mathcal{S}_V^0$ this was surely well-known although I could not find a place in the literature where this fact was explicitly mentioned.

1.3. **Structure of the paper.** In §2 we recall several fundamental concepts starting with a recap of adjoint functors and the calculus of mates in §2.1. In §2.2 we introduce p-adic monoids and give a quick overview of their smooth representation theory. The subject of §2.3 is the derived category $\mathrm{D}(S)$ of smooth representations of a p-adic monoid S. We recall Brown representatibility, the tensor triangulated structure on $\mathrm{D}(S)$, and how to obtain spectral sequences using truncation functors.

In §3 we prove that the derived inflation functor $\mathrm{R}\,\mathrm{Inf}_S^T$ admits a left adjoint. We first prove this in the case where T and S are torsion-free compact p-adic Lie groups, see §3.1. The general case is then deduced from the compact case in §3.2. In §3.3 and §3.4 we specialize to p-adic Lie groups and show that L_U satisfies a projection formula and compute more explicit descriptions of the L_U^i.

In §4 we investigate the left adjoint of derived parabolic induction. In §4.2 we show that $\mathrm{R}i_M^G$ and $\mathrm{L}(U,-)$ preserve global admissibility, and §4.3 is concerned with the derived Satake homomorphism.

The explicit computations of $\mathrm{L}^i(U,V)$, for irreducible smooth $\overline{\mathbb{F}}_p[\mathrm{GL}_2(\mathbb{Q}_p)]$-representations V, are the subject of §5.

1.4. **Acknowledgments.** First and foremost I thank Peter Schneider for his interest and support, for some enlightening discussions, and for sharing with me the preprints [SS22a] and [SS22b]. I thank Jakob Scholbach for some helpful suggestions and for pointing me to the article [BDS16]. During the work on this project I have further profited from discussions with Eugen Hellmann, Emanuele Bodon and Yifei Zhao. I am most grateful to Tobias Schmidt, for inviting me to speak at the Colloque Tournant 2021 du GDR TLAG, and to Claus Sorensen, for inviting me to present some of these results at the UCSD Number Theory Seminar. The project was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 427320536

[1] In fact, $\mathcal{S}_V$ is the 0-th cohomology of the morphism induced by $\mathrm{L}(U,-)$ on RHom-complexes, but we will not make use of this generality.

– SFB 1442, as well as under Germany's Excellence Strategy EXC 2044 390685587, Mathematics Münster: Dynamics–Geometry–Structure.

1.5. **Notation and conventions.** Throughout this article, k will denote a coefficient field, which will be assumed to be of characteristic $p > 0$ from §3 onwards. The category of k-vector spaces is denoted Vect_k, and we write $\mathrm{D}(k)$ for the unbounded derived category of Vect_k.

If S is a locally profinite monoid, we denote $\mathrm{Rep}_k(S)$ the abelian category of smooth S-representations on k-vector spaces and by $\mathrm{D}(S)$ the unbounded derived category of $\mathrm{Rep}_k(S)$. Given $V \in \mathrm{Rep}_k(S)$, we denote $V[0]$ (or just V if no confusion is possible) the representation V viewed as a complex concentrated in degree 0.

We make the convention that a diagram of functors, say,

$$\begin{tikzcd} \mathscr{A} \arrow[r, "a"] \arrow[d, "f"'] & \mathscr{A}' \arrow[d, "f'"] \\ \mathscr{B} \arrow[r, "b"'] & \mathscr{B}' \end{tikzcd}$$

commutes if there exists a natural isomorphism $b \circ f \overset{\cong}{\Longrightarrow} f' \circ a$, which will usually be specified in the proof if it is unclear which natural isomorphism is meant.

If G is a group, $K \subseteq G$ is a subgroup and $g \in G$, we write ${}^gK := gKg^{-1}$ and $K^g := g^{-1}Kg$.

2. Foundations

This section is devoted to collecting facts about adjoint functors (§2.1), smooth representations of locally profinite monoids (§2.2), and the derived category of smooth representations (§2.3).

2.1. **About adjoint functors.** In this section we recall well-known facts about adjoint functors. For references and more details we refer to [KS06] and [ML13].

Let $\mathscr{C}, \mathscr{D}$ be categories. We fix two functors $L\colon \mathscr{C} \longrightarrow \mathscr{D}$ and $R\colon \mathscr{D} \longrightarrow \mathscr{C}$.

Definition. We say L is *left adjoint* to R, or R is *right adjoint* to L, if there is a bijection

$$\mathrm{Hom}_{\mathscr{D}}\big(L(C), D\big) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{C}}\big(C, R(D)\big) \tag{2.1.1}$$

which is natural in both $C \in \mathscr{C}$ and $D \in \mathscr{D}$.

Remark 2.1.2. If $L\colon \mathscr{C} \longrightarrow \mathscr{D}$ is an additive functor between additive categories, then a right adjoint R (if it exists) is additive as well and the map (2.1.1) is an isomorphism of abelian groups, see [ML13, IV.1, Thm. 3].

It follows immediately from the definition that one can compose adjunctions:

Lemma 2.1.3 ([KS06, Prop. 1.5.5]). *Consider the functors*

$$\mathscr{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{D} \underset{R'}{\overset{L'}{\rightleftarrows}} \mathscr{E}.$$

Assume that L and L' are left adjoint to R and R', respectively. Then $L' \circ L$ is left adjoint to $R \circ R'$.

Lemma 2.1.4. *The following assertions are equivalent:*

(a) L is left adjoint to R.

(b) There exist natural transformations $\eta\colon \mathrm{id}_{\mathscr{C}} \Longrightarrow RL$ and $\varepsilon\colon LR \Longrightarrow \mathrm{id}_{\mathscr{D}}$, called the unit *and the* counit *of the adjunction, respectively, satisfying the "triangle identities" $R\varepsilon \circ \eta R = \mathrm{id}_R$ and $\varepsilon L \circ L\eta = \mathrm{id}_L$.*

Proof. For the details we refer to [ML13, IV.1]. If L is left adjoint to R, the map η_C is given by the image of $\mathrm{id}_{L(C)}$ under (2.1.1) and ε_D is given by the preimage of $\mathrm{id}_{R(D)}$ under (2.1.1). It is then easily checked that η and ε are natural transformations and satisfy the triangle identities.

Conversely, if η and ε are as in (b) one checks that the compositions

$$\mathrm{Hom}_{\mathscr{D}}\big(L(C), D\big) \xrightarrow{R} \mathrm{Hom}_{\mathscr{C}}\big(RL(C), R(D)\big) \xrightarrow{(\eta_C)^*} \mathrm{Hom}_{\mathscr{C}}\big(C, R(D)\big)$$

and

$$\mathrm{Hom}_{\mathscr{C}}\big(C, R(D)\big) \xrightarrow{L} \mathrm{Hom}_{\mathscr{D}}\big(L(C), LR(D)\big) \xrightarrow{(\varepsilon_D)_*} \mathrm{Hom}_{\mathscr{D}}\big(L(C), D\big)$$

are inverse to each other. □

Lemma 2.1.5 ([KS06, Prop. 1.5.6])**.** *Assume that L is left adjoint to R.*

(a) The functor R is fully faithful if and only if the counit $\varepsilon\colon LR \Longrightarrow \mathrm{id}_{\mathscr{D}}$ is an isomorphism.
(b) The functor L is fully faithful if and only if the unit $\eta\colon \mathrm{id}_{\mathscr{C}} \Longrightarrow RL$ is an isomorphism.

The following result shows that adjoints are uniquely determined.

Theorem 2.1.6 ([KS06, Thm. 1.5.3])**.** *The right adjoint of L (if it exists) is unique up to unique isomorphism. Moreover, L admits a right adjoint if and only if the functor $\mathrm{Hom}_{\mathscr{D}}\big(L(-), D\big)\colon \mathscr{D} \longrightarrow (\mathrm{Sets})$ is representable for every $D \in \mathscr{D}$.*

Dually, the left adjoint of R (if it exists) is unique up to unique isomorphism. Moreover, R admits a left adjoint if and only if the functor $\mathrm{Hom}_{\mathscr{C}}\big(C, R(-)\big)\colon \mathscr{C} \longrightarrow (\mathrm{Sets})$ is representable for every $C \in \mathscr{C}$.

Proposition 2.1.7 ([KS06, Prop. 2.1.10])**.** *If $L\colon \mathscr{C} \longrightarrow \mathscr{D}$ admits a right adjoint, then L commutes with all colimits that exist in $\mathscr{C}$.*

Dually, if $R\colon \mathscr{D} \longrightarrow \mathscr{C}$ admits a left adjoint, then R commutes with all limits that exist in $\mathscr{D}$.

For a stronger (and dual) version of the next extremely useful result, see [HS71, II, Prop. 10.2].

Lemma 2.1.8. *Assume that $\mathscr{C}$ and $\mathscr{D}$ are abelian categories and $R\colon \mathscr{D} \longrightarrow \mathscr{C}$ is an additive functor admitting an exact left adjoint. Then R preserves injective objects.*

Proof. Let $I \in \mathscr{D}$ be an injective object. The functor $\mathrm{Hom}_{\mathscr{C}}\big(-, R(I)\big) \cong \mathrm{Hom}_{\mathscr{D}}\big(L(-), I\big)$ is exact, which means that $R(I) \in \mathscr{C}$ is injective. □

Finally, we discuss the formalism of "mates" of natural transformations involving adjoint functors. As usual, given two functors $F, G\colon \mathscr{C} \longrightarrow \mathscr{D}$, we denote by $\mathrm{Nat}(F, G)$ the set of natural transformations from F to G.

Proposition 2.1.9 ([KS74, Prop. 2.1])**.** *Consider a diagram of functors*

(2.1.10)
$$\begin{tikzcd}
\mathscr{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] \arrow[d, "F"'] & \mathscr{D} \arrow[l, bend left, "R"] \arrow[d, "G"] \\
\mathscr{C}' \arrow[r, bend left, "L'"] \arrow[r, phantom, "\perp"] & \mathscr{D}' \arrow[l, bend left, "R'"]
\end{tikzcd}$$

and denote $\eta\colon LR \Longrightarrow \mathrm{id}_{\mathscr{D}}$, $\varepsilon\colon \mathrm{id}_{\mathscr{C}} \Longrightarrow RL$ and $\eta'\colon L'R' \Longrightarrow \mathrm{id}_{\mathscr{D}'}$, $\varepsilon'\colon \mathrm{id}_{\mathscr{C}'} \Longrightarrow R'L'$ the units and counits. There is a natural bijection

(2.1.11)
$$\begin{aligned}
\mathrm{Nat}(L'F, GL) &\xleftrightarrow{\ \cong\ } \mathrm{Nat}(FR, R'G), \\
\alpha &\longmapsto (R'G\varepsilon) \circ (R'\alpha_R) \circ (\eta'_{FR}), \\
(\varepsilon'_{GL}) \circ (L'\beta_L) \circ (L'F\eta) &\longleftarrow\!\shortmid \beta.
\end{aligned}$$

If $\alpha\colon L'F \Longrightarrow GL$ corresponds to $\beta\colon FR \Longrightarrow R'G$ under (2.1.11), we say that α is the left mate of β and β is the right mate of α.

The "naturality" above means that the bijection (2.1.11) is compatible with horizontal and vertical composition of squares of the form (2.1.10), see [KS74, Prop. 2.2] for a precise statement.

Example 2.1.12. Let us make explicit two special cases of vertical composition. Suppose we are in the setting of Proposition 2.1.9.

(i) Let $L_1\colon \mathscr{C} \longrightarrow \mathscr{D}$ be a functor with right adjoint R_1, and let $\alpha\colon L \Longrightarrow L_1$ be a natural transformation with right mate $\beta\colon R_1 \Longrightarrow R$. Then the diagram

$$\begin{tikzcd} \mathrm{Nat}(L'F, GL) \arrow[r, leftrightarrow, "\cong"] \arrow[d, "(G\alpha)_*"'] & \mathrm{Nat}(FR, R'G) \arrow[d, "(F\beta)^*"] \\ \mathrm{Nat}(L'F, GL_1) \arrow[r, leftrightarrow, "\cong"'] & \mathrm{Nat}(FR_1, R'G) \end{tikzcd}$$

commutes.

(ii) Let $L_1'\colon \mathscr{C}' \longrightarrow \mathscr{D}'$ be a functor with right adjoint R_1', and let $\alpha'\colon L_1' \Longrightarrow L'$ be a natural transformation with right mate $\beta'\colon R' \Longrightarrow R_1'$. Then the diagram

$$\begin{tikzcd} \mathrm{Nat}(L'F, GL) \arrow[r, leftrightarrow, "\cong"] \arrow[d, "(\alpha'_F)^*"'] & \mathrm{Nat}(FR, R'G) \arrow[d, "(\beta'_G)_*"] \\ \mathrm{Nat}(L_1'F, GL) \arrow[r, leftrightarrow, "\cong"'] & \mathrm{Nat}(FR, R_1'G) \end{tikzcd}$$

commutes.

It is not true in general that (2.1.11) preserves isomorphisms. However, in very special cases this does hold.

Example 2.1.13. Let $L_i\colon \mathscr{C} \longrightarrow \mathscr{D}$ be a functor admitting a right adjoint R_i, for $i = 1, 2, 3$. Let $\alpha\colon L_3 \Longrightarrow L_2$ be a natural transformation with mate $\beta\colon R_2 \Longrightarrow R_3$. It follows from Example 2.1.12 that the diagram

$$\begin{tikzcd} \mathrm{Nat}(L_2, L_1) \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\alpha^*"'] & \mathrm{Nat}(R_1, R_2) \arrow[d, "\beta_*"] \\ \mathrm{Nat}(L_3, L_1) \arrow[r, leftrightarrow, "\cong"'] & \mathrm{Nat}(R_1, R_3) \end{tikzcd}$$

commutes. Hence, if $\alpha'\colon L_2 \Longrightarrow L_1$ has mate $\beta'\colon R_1 \Longrightarrow R_2$, then the mate of $\alpha' \circ \alpha$ is $\beta \circ \beta'$. Note also that under $\mathrm{Nat}(L_3, L_3) \overset{\cong}{\longleftrightarrow} \mathrm{Nat}(R_3, R_3)$ the mate of id_{L_3} is id_{R_3}; this is a reformulation of the fact that the unit and counit of the adjunction $L_3 \dashv R_3$ satisfy the triangle identities.

It follows from this discussion that $\alpha\colon L_3 \Longrightarrow L_2$ is an isomorphism if and only if its mate $\beta\colon R_2 \Longrightarrow R_3$ is an isomorphism. This is a strengthening of the uniqueness statement in Theorem 2.1.6.

2.1.1. *An extended example.* Beginning in §3 we will make extensive use of the results in [BDS16]. In this paragraph we record some statements pertaining to the context of *op. cit.* whose proofs will be left as an exercise in applying Proposition 2.1.9. The sole purpose of this paragraph is to elucidate the proof of Proposition 3.4.23.

Let $\mathscr{C}$, $\mathscr{D}$ (and later $\mathscr{C}'$, $\mathscr{D}'$) be symmetric monoidal categories which are rigidly-compactly generated; cf. [BDS16, 1.2 Hyp.]. We call a functor $f^*\colon \mathscr{C} \longrightarrow \mathscr{D}$ *good* if it is strongly monoidal and if it is sitting in a chain of adjunctions,

$$f_{(1)} \dashv f^* \dashv f_* \dashv f^{(1)}.$$

Recall that *strongly monoidal* means that there are natural isomorphisms $\mathbf{1} \xrightarrow{\cong} f^*(\mathbf{1})$ and $f^*(x) \otimes f^*(y) \xrightarrow{\cong} f^*(x \otimes y)$ satisfying the usual unit, associativity, and commutativity constraints. The natural isomorphism

$$f^* \circ (- \otimes y) \xrightarrow{\cong} \big(- \otimes f^*(y)\big) \circ f^*$$

induces by Proposition 2.1.9 a natural map

$$(- \otimes y) \circ f_* \xrightarrow{\cong} f_* \circ \big(- \otimes f^*(y)\big), \tag{2.1.14}$$

which is an isomorphism by [BDS16, 2.15. Prop.]. The inverse of (2.1.14) induces by [BDS16, 3.2. Prop.] a natural isomorphism

$$\big(- \otimes f^*(y)\big) \circ f^{(1)} \xrightarrow{\cong} f^{(1)} \circ (- \otimes y). \tag{2.1.15}$$

Since (2.1.15) is natural in y, we also have a natural isomorphism

$$\big(f^{(1)}(x) \otimes -\big) \circ f^* \xrightarrow{\cong} f^{(1)} \circ (x \otimes -),$$

which in turn induces a natural isomorphism

$$f_* \circ \big(f^{(1)}(x) \otimes -\big) \xrightarrow{\cong} (x \otimes -) \circ f_{(1)}. \tag{2.1.16}$$

Lemma 2.1.17. *Let $g^*, h^* \colon \mathscr{C} \longrightarrow \mathscr{D}$ be good functors, and let $\alpha^* \colon g^* \Longrightarrow h^*$ be a natural map of monoidal functors. Consider the induced maps*

$$\begin{aligned} \alpha_{(1)} \colon h_{(1)} \Longrightarrow g_{(1)}, \qquad \alpha^* \colon g^* \Longrightarrow h^*, \\ \alpha_* \colon h_* \Longrightarrow g_*, \qquad \alpha^{(1)} \colon g^{(1)} \Longrightarrow h^{(1)}. \end{aligned}$$

The diagrams

$$\begin{tikzcd}
h_*(x) \otimes y \arrow[r, "(2.1.14)", "\cong"'] \arrow[dd, "\alpha_* \otimes \mathrm{id}"'] & h_*\big(x \otimes h^*(y)\big) \arrow[d, "\alpha_*"] \\
 & g_*\big(x \otimes h^*(y)\big) \\
g_*(x) \otimes y \arrow[r, "\cong", "(2.1.14)"'] & g_*\big(x \otimes g^*(y)\big) \arrow[u, "g_*(\mathrm{id} \otimes \alpha^*)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
g^{(1)}(x) \otimes g^*(y) \arrow[r, "(2.1.15)", "\cong"'] \arrow[d, "\alpha^{(1)} \otimes \alpha^*"'] & g^{(1)}(x \otimes y) \arrow[d, "\alpha^{(1)}"] \\
h^{(1)}(x) \otimes h^*(y) \arrow[r, "\cong", "(2.1.15)"'] & h^{(1)}(x \otimes y)
\end{tikzcd}$$

$$\begin{tikzcd}
h_*\big(h^{(1)}(x) \otimes y\big) \arrow[r, "(2.1.16)", "\cong"'] & x \otimes h_{(1)}(y) \arrow[dd, "\mathrm{id} \otimes \alpha_{(1)}"] \\
h_*\big(g^{(1)}(x) \otimes y\big) \arrow[u, "h_*(\alpha^{(1)} \otimes \mathrm{id})"] \arrow[d, "\alpha_*"'] & \\
g_*\big(g^{(1)}(x) \otimes y\big) \arrow[r, "\cong", "(2.1.16)"'] & x \otimes g_{(1)}(y)
\end{tikzcd}$$

are commutative.

Lemma 2.1.18. *Let $g^* \colon \mathscr{C} \longrightarrow \mathscr{D}$ and $h^* \colon \mathscr{D} \longrightarrow \mathscr{D}'$ be good functors. Then $(hg)^* := h^* \circ g^*$ is good and we have*

$$(hg)_{(1)} = g_{(1)} \circ h_{(1)}, \quad (hg)_* = g_* \circ h_*, \quad (hg)^{(1)} = h^{(1)} \circ g^{(1)}.$$

Moreover, the diagrams

$$\begin{tikzcd}
(hg)_*(x) \otimes y \arrow[rr, "(2.1.14)"] \arrow[d, equal] & & (hg)_*\big(x \otimes (hg)^*(y)\big) \arrow[d, equal] \\
g_*\big(h_*(x)\big) \otimes y \arrow[r, "(2.1.14)"'] & g_*\big(h_*(x) \otimes g^*(y)\big) \arrow[r, "g_*(2.1.14)"'] & g_* h_*\big(x \otimes h^* g^*(y)\big)
\end{tikzcd}$$

$$\begin{tikzcd}
(hg)^{(1)}(x) \otimes (hg)^*(y) \arrow[rr, "(2.1.15)"] \arrow[d, equal] & & (hg)^{(1)}(x \otimes y) \arrow[d, equal] \\
h^{(1)} g^{(1)}(x) \otimes h^* g^*(y) \arrow[r, "(2.1.15)"'] & h^{(1)}\big(g^{(1)}(x) \otimes g^*(y)\big) \arrow[r, "h^{(1)}(2.1.15)"'] & h^{(1)} g^{(1)}(x \otimes y)
\end{tikzcd}$$

$$\begin{tikzcd}
(hg)_*\big((hg)^{(1)}(x) \otimes y\big) \arrow[rr, "(2.1.16)"] \arrow[d, equal] & & x \otimes (hg)_{(1)}(y) \arrow[d, equal] \\
g_* h_*\big(h^{(1)} g^{(1)}(x) \otimes y\big) \arrow[r, "g_*(2.1.16)"'] & g_*\big(g^{(1)}(x) \otimes h_{(1)}(y)\big) \arrow[r, "(2.1.16)"'] & x \otimes g_{(1)} h_{(1)}(y)
\end{tikzcd}$$

are commutative.

Proposition 2.1.19. *Consider a diagram*

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "c^*"] \arrow[d, "g^*"'] & \mathscr{C}' \arrow[d, "f^*"] \\
\mathscr{D} \arrow[r, "d^*"'] & \mathscr{D}'
\end{tikzcd}$$

of good functors together with a natural map $\alpha^* \colon d^* g^* \Longrightarrow f^* c^*$*. There are natural maps*

$$\alpha_{(1)} \colon c_{(1)} f_{(1)} \Longrightarrow g_{(1)} d_{(1)}, \qquad \alpha_* \colon c_* f_* \Longrightarrow g_* d_*, \qquad \alpha^{(1)} \colon d^{(1)} g^{(1)} \Longrightarrow f^{(1)} c^{(1)}$$

and the diagram

$$\begin{tikzcd}
& c_*\big(c^{(1)}(x) \otimes f_{(1)}(y)\big) \arrow[dr, "(2.1.16)"] & \\
c_* f_*\big(f^{(1)} c^{(1)}(x) \otimes y\big) \arrow[ur, "c_*(2.1.16)"] \arrow[rr, "(2.1.16)"'] & & x \otimes c_{(1)} f_{(1)}(y) \arrow[dd, "\mathrm{id} \otimes \alpha_{(1)}"] \\
c_* f_*\big(d^{(1)} g^{(1)}(x) \otimes y\big) \arrow[u, "c_* f_*(\alpha^{(1)} \otimes \mathrm{id})"] \arrow[d, "\alpha_*"'] & & \\
g_* d_*\big(d^{(1)} g^{(1)}(x) \otimes y\big) \arrow[rr, "(2.1.16)"] \arrow[dr, "g_*(2.1.16)"'] & & x \otimes g_{(1)} d_{(1)}(y) \\
& g_*\big(g^{(1)}(x) \otimes d_{(1)}(y)\big) \arrow[ur, "(2.1.16)"'] &
\end{tikzcd}$$

is commutative.

Proof. Combine Lemmas 2.1.17 and 2.1.18. □

2.2. **Smooth representations.** For our purposes it is useful to consider smooth representations over locally profinite monoids. Everything in this section is well-known for locally profinite groups, see, *e.g.*, [BH06] or [Vig96], and so we will restrain ourselves to a very brief treatment. We fix a coefficient field k.

2.2.1. *General properties of smooth representations.*

Definition. Let S be a topological monoid.

(i) We call S a *locally profinite monoid* if it satisfies the following conditions:
   - S contains an open profinite subgroup.
   - For each $s \in S$ and every open subset $U \subseteq S$ the subsets sU and Us are open in S.

(ii) We call S a *p-adic monoid* if it is a locally profinite monoid containing an open subgroup which is a p-adic Lie group.

A morphism between locally profinite monoids, resp. p-adic monoids, is a continuous monoid homomorphism.

Example 2.2.1. (i) Every open submonoid of a locally profinite group is a locally profinite monoid. Every open submonoid of a p-adic Lie group is a p-adic monoid.

(ii) We give a non-trivial example of a p-adic monoid. Let $\mathfrak{F}$ be a finite extension of $\mathbb{Q}_p$ and consider a connected reductive group $\mathbf{G}$ defined over $\mathfrak{F}$. Then every closed subgroup of $\mathbf{G}(\mathfrak{F})$ is a p-adic Lie group [DDMS99, 9.6 Thm.]. Let $\mathbf{P} = \mathbf{MU}$ be a parabolic subgroup of $\mathbf{G}$. Fix a compact open subgroup $K_P \subseteq \mathbf{P}(\mathfrak{F})$ such that $K_P = K_M K_U$, where $K_M \coloneqq K_P \cap \mathbf{M}(\mathfrak{F})$ and $K_U \coloneqq K_P \cap \mathbf{U}(\mathfrak{F})$. Then

$$M^+ \coloneqq \{m \in \mathbf{M}(\mathfrak{F}) \,|\, m K_U m^{-1} \subseteq K_U\}$$

contains K_M and hence is a p-adic monoid.

The topological monoid $P^+ \coloneqq K_U M^+$ contains K and is therefore a p-adic monoid.[2] Note that there is a surjective monoid homomorphism $P^+ \twoheadrightarrow M^+$ with kernel K_U.

Every p-adic monoid in this article will be as in one of the two examples above.

Lemma 2.2.2. *Let S be a locally profinite monoid and $S' \subseteq S$ a submonoid. Then S' is open if and only if S' contains an open subgroup of S.*

Proof. Let $S_0 \subseteq S$ be a fixed open profinite subgroup. If S' is open, then $S' \cap S_0$ is an open neighborhood of 1 in S_0 and hence contains an open subgroup, say, S_1. But then S_1 is an open subgroup contained in S'. Conversely, let S_1 be an open subgroup contained in S'. For every $s \in S'$ the subsets sS_1 are then open and contained in S'. Therefore, $S' = \bigcup_{s\in S'} sS_1$ is open. □

Definition. Let S be a locally profinite monoid. Let V be a representation of S on a k-vector space, *i.e.*, a (left) module over the monoid algebra $k[S]$.

(i) A vector $v \in V$ is called *smooth* if the stabilizer $\{s \in S \,|\, sv = v\}$ of v is open in S. We denote by $V^{\mathrm{sm}} \subseteq V$ the subset of smooth vectors in V.

(ii) V is called *smooth* if $V = V^{\mathrm{sm}}$.

We denote by $\mathrm{Rep}_k(S)$ the full subcategory of the category $\mathrm{Mod}(k[S])$ of $k[S]$-modules consisting of smooth representations of S.

Lemma 2.2.3. *Let S be a locally profinite monoid and let V be a $k[S]$-module.*

(a) Let $v \in V$. The following assertions are equivalent:
   (i) The vector $v \in V$ is smooth.
   (ii) There exists an open subgroup $S_0 \subseteq S$ fixing v.

(b) V^{sm} is a smooth S-subrepresentation of V.

Proof. The equivalence in (a) follows immediately from Lemma 2.2.2. We now prove (b). Let $v, w \in V^{\mathrm{sm}}$ and $s \in S$. It suffices to show $v + w \in V^{\mathrm{sm}}$ and $sv \in V^{\mathrm{sm}}$. By (a) there exist open subgroups S_v and S_w fixing v and w, respectively. Then $S_v \cap S_w$ fixes $v + w$, and then (a) implies $v + w \in V^{\mathrm{sm}}$. Since multiplication in S is continuous and sS_v is open in S, there exists an open subgroup $S'_v \subseteq S$ such that $S'_v s \subseteq sS_v$. But then S'_v fixes sv which, by (a) again, shows $sv \in V^{\mathrm{sm}}$. □

[2] Observe, however, that M^+K_U is *not* a monoid.

We list some immediate consequences of Lemma 2.2.3.

Corollary 2.2.4. *Let S be a locally profinite monoid.*

(a) A $k[S]$-module V is smooth if and only if V is smooth when considered as a $k[S_0]$-module for some open profinite subgroup $S_0 \subseteq S$.

(b) $\mathrm{Rep}_k(S)$ is an abelian subcategory of $\mathrm{Mod}(k[S])$.

Let $S' \subseteq S$ be a closed submonoid of a locally profinite monoid S. The restriction functor

$$\mathrm{Res}^S_{S'}\colon \mathrm{Rep}_k(S) \longrightarrow \mathrm{Rep}_k(S')$$

is exact. It admits a right adjoint which we will now describe. Given a smooth $k[S']$-module V, we obtain a smooth $k[S]$-module

$$\mathrm{Ind}^S_{S'} V := \mathrm{Hom}_{k[S']}\big(k[S], V\big)^{\mathrm{sm}},$$

called the *smooth induction* of V. Here, S acts on $\mathrm{Ind}^S_{S'} V$ via right multiplication on $k[S]$. One easily verifies that

$$\mathrm{Ind}^S_{S'} V \cong \left\{ f\colon S \longrightarrow V \;\middle|\; \begin{array}{l} f(s's) = s'f(s) \text{ for all } s \in S,\ s' \in S', \\ \exists K \subseteq S \text{ open with } f(sx) = f(s) \text{ for all } s \in S,\ x \in K \end{array} \right\},$$

where S acts on $f\colon S \longrightarrow V$ via $(sf)(t) = f(ts)$, for all $s, t \in S$. We will often implicitly make this identification.

Lemma 2.2.5 (Frobenius reciprocity I)**.** *The smooth induction $\mathrm{Ind}^S_{S'}\colon \mathrm{Rep}_k(S') \longrightarrow \mathrm{Rep}_k(S)$ is a right adjoint of $\mathrm{Res}^S_{S'}$, that is, there is a natural bijection*

$$\mathrm{Hom}_{\mathrm{Rep}_k(S')}\big(\mathrm{Res}^S_{S'} V, W\big) \cong \mathrm{Hom}_{\mathrm{Rep}_k(S)}\big(V, \mathrm{Ind}^S_{S'} W\big)$$

for all $V \in \mathrm{Rep}_k(S)$, $W \in \mathrm{Rep}_k(S')$. In particular, $\mathrm{Ind}^S_{S'}$ is left exact and preserves injective objects.

Proof. The last assertion is a formal consequence of the fact that $\mathrm{Ind}^S_{S'}$ admits an exact left adjoint, see Lemma 2.1.8. □

Remark 2.2.6. If S and S' are profinite groups, the projection $S \longrightarrow S/S'$ admits a continuous section by [Ser13, Ch. I, Prop. 1]. Hence, $\mathrm{Ind}^S_{S'}\colon \mathrm{Rep}_k(S') \longrightarrow \mathrm{Rep}_k(S)$ is exact by [Her12, Prop. 5].

Corollary 2.2.7. *The category $\mathrm{Rep}_k(S)$ has enough injectives.*

Proof. Let $V \in \mathrm{Rep}_k(S)$. The map $\varphi\colon V \longrightarrow \mathrm{Ind}^S_1 \mathrm{Res}^S_1 V$, $\varphi(v)(x) := xv$, is clearly injective. Since $\mathrm{Res}^S_1 V$ is injective in the category of k-vector spaces, it follows from Lemma 2.2.5 that $\mathrm{Ind}^S_1 \mathrm{Res}^S_1 V$ is injective in $\mathrm{Rep}_k(S)$. □

If $S' \subseteq S$ is open, then $\mathrm{Res}^S_{S'}$ also admits an exact left adjoint. Given a smooth $k[S']$-module V, we define

$$\mathrm{ind}^S_{S'} V := k[S] \otimes_{k[S']} V,$$

called the *compact induction* of V. The smoothness of $\mathrm{ind}^S_{S'} V$ follows from Lemma 2.2.3.(b) and the fact that $\mathrm{ind}^S_{S'} V$ is generated as a $k[S]$-module by V, which consists of smooth vectors.

Remark 2.2.8. Assume that S is contained in a locally profinite group and that $S' \subseteq S$ is an open subgroup. Given $s \in S$ and $v \in V$, we define $[s, v] \in \mathrm{Ind}^S_{S'} V$ by

$$[s, v](t) := \begin{cases} ts \cdot v, & \text{if } ts \in S', \\ 0, & \text{otherwise.} \end{cases}$$

In other words, $[s, v]$ is the S'-equivariant function $S \longrightarrow V$ supported on $S's^{-1}$ with value v at s^{-1}. In this case we have a canonical injective S-equivariant map

$$\begin{aligned} \mathrm{ind}^S_{S'} V &\longrightarrow \mathrm{Ind}^S_{S'} V, \\ s \otimes v &\longmapsto [s, v]. \end{aligned}$$

It is customary to identify $\mathrm{ind}^S_{S'} V$ with its image in $\mathrm{Ind}^S_{S'} V$.

Lemma 2.2.9 (Frobenius reciprocity II)**.** *Assume that $S' \subseteq S$ is an open submonoid. Then compact induction $\mathrm{ind}_{S'}^S \colon \mathrm{Rep}_k(S') \longrightarrow \mathrm{Rep}_k(S)$ is left adjoint to $\mathrm{Res}_{S'}^S$, that is, there is a natural bijection*

$$\mathrm{Hom}_{\mathrm{Rep}_k(S)}\big(\mathrm{ind}_{S'}^S V, W\big) \cong \mathrm{Hom}_{\mathrm{Rep}_k(S')}\big(V, \mathrm{Res}_{S'}^S W\big),$$

for all $V \in \mathrm{Rep}_k(S')$, $W \in \mathrm{Rep}_k(S)$. In particular, $\mathrm{ind}_{S'}^S$ is right exact.

Proof. This follows from standard properties of the tensor product. □

In later applications the extension $k[S'] \subseteq k[S]$ will be flat so that $\mathrm{ind}_{S'}^S$ is actually exact.

Corollary 2.2.10. *Let $S' \subseteq S$ be an open submonoid. For all $V \in \mathrm{Rep}_k(S)$ the map*

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Rep}_k(S)}\big(\mathrm{ind}_{S'}^S \mathbf{1}, V\big) &\xrightarrow{\cong} \mathrm{H}^0(S', V) := \{v \in V \mid sv = v \text{ for all } s \in S'\}\,,\\ \varphi &\longmapsto \varphi(1 \otimes 1)\end{aligned}$$

is a natural bijection. In particular, $\left\{\mathrm{ind}_{S'}^S \mathbf{1} \,\middle|\, S' \subseteq S \text{ open}\right\}$ is a set of generators for $\mathrm{Rep}_k(S)$.

Proof. For the definition of a set of generators we refer to [KS06, Def. 5.2.1]. The first assertion is an easy consequence of Lemma 2.2.9. Therefore, in order to prove the last statement, it suffices to prove that a morphism $V \longrightarrow W$ in $\mathrm{Rep}_k(S)$ is an isomorphism if the induced maps $V^{S'} \longrightarrow W^{S'}$ are bijective, for all open submonoids S'. But this follows from the fact that V and W are smooth. □

Proposition 2.2.11. *The category $\mathrm{Rep}_k(S)$ is Grothendieck abelian.*

Proof. By Corollaries 2.2.4 and 2.2.10 we know that $\mathrm{Rep}_k(S)$ is abelian and admits a set of generators. Given a family $\{V_i\}_{i\in I}$ of smooth representations, the direct sum $\bigoplus_{i\in I} V_i$ is again smooth. It follows that $\mathrm{Rep}_k(S)$ has all colimits and that they can be computed in $\mathrm{Mod}(k[S])$. Therefore, all filtered colimits are exact, *i.e.*, $\mathrm{Rep}_k(S)$ satisfies the Grothendieck axiom AB5. Moreover, the representation $\big(\prod_{i\in I} V_i\big)^{\mathrm{sm}}$ is a categorical product in $\mathrm{Rep}_k(S)$, so that $\mathrm{Rep}_k(S)$ satisfies the axiom AB3*. □

Remark 2.2.12. If $\mathrm{char}\, k$ divides the pro-order of every open profinite subgroup of S, the formation of infinite direct products is not exact in $\mathrm{Rep}_k(S)$. As an example, consider $S = \mathbb{Z}_p$ and $k = \mathbb{F}_p$. For each $n \geq 1$ the map $f_n \colon k[\mathbb{Z}_p/p^n\mathbb{Z}_p] \longrightarrow k$, $x + (p^n) \mapsto 1$, is a surjection between smooth $\mathbb{Z}_p$-representations, where $\mathbb{Z}_p$ acts trivially on k. Note that, for $m < n$, the induced map $\mathrm{H}^0(p^m\mathbb{Z}_p, k[\mathbb{Z}_p/p^n\mathbb{Z}_p]) \longrightarrow k$ on $p^m\mathbb{Z}_p$-invariants is zero (this uses $\mathrm{char}\, k = p$). Therefore, the map

$$\Big(\prod_{n\geq 1} k[\mathbb{Z}_p/p^n\mathbb{Z}_p]\Big)^{\mathrm{sm}} \xrightarrow{(f_n)_n} \Big(\prod_{n\geq 1} k\Big)^{\mathrm{sm}} = \prod_{n\geq 1} k$$

takes values in $\bigoplus_{n\geq 1} k$ and hence is not surjective.

The failure of having exact direct products means that $\mathrm{Rep}_k(S)$ does not have enough projective objects (see [Roo06, Thm. 1.3]). In fact, [DK21, Rmk. 2.20] shows that $\mathrm{Rep}_k(S)$ contains no non-zero projectives (the proof of *loc. cit.* works for general S). This is the source of many difficulties in the theory of smooth mod p representations of p-adic Lie groups.

2.2.2. *Inflation.* Let $f \colon S \twoheadrightarrow T$ be a surjective open morphism of locally profinite monoids with kernel U. Inflation along f defines an exact functor $\mathrm{Inf}_S^T \colon \mathrm{Rep}_k(T) \longrightarrow \mathrm{Rep}_k(S)$.

Proposition 2.2.13. Inf_S^T *admits a left and a right adjoint. More concretely:*

(a) The functor

$$\begin{aligned}\mathrm{L}_U^0 \colon \mathrm{Rep}_k(S) &\longrightarrow \mathrm{Rep}_k(T),\\ V &\longmapsto \mathrm{L}_U^0(V) := k[T] \otimes_{k[S]} V\end{aligned}$$

is a left adjoint of Inf_S^T and, in particular, right exact.

(b) The functor

$$\begin{aligned}\Pi_U\colon \operatorname{Rep}_k(S) &\longrightarrow \operatorname{Rep}_k(T),\\ V &\longmapsto \Pi_U(V) \coloneqq \operatorname{Hom}_{k[S]}(k[T],V)^{\mathrm{sm}}\end{aligned}$$

is a right adjoint of Inf_S^T *and, in particular, left exact.*

Proof. (a) We write $\operatorname{Inf}_S^T W = \operatorname{Hom}_{k[T]}(k[T],W)$, where S acts on the right on $k[T]$. The assertion follows from the $\otimes$-Hom adjunction, [Ben91a, Lem. 2.8.2]:

$$\operatorname{Hom}_{k[S]}\big(V,\operatorname{Hom}_{k[T]}(k[T],W)\big)\cong \operatorname{Hom}_{k[T]}\big(k[T]\otimes_{k[S]}V,W\big).$$

(b) We write $\operatorname{Inf}_S^T W = k[T]\otimes_{k[T]} W$, where S acts on the left on $k[T]$. By *loc. cit.* we have

$$\begin{aligned}\operatorname{Hom}_{k[S]}\big(k[T]\otimes_{k[T]}W,V\big) &\cong \operatorname{Hom}_{k[T]}\big(W,\operatorname{Hom}_{k[S]}(k[T],V)\big)\\ &\cong \operatorname{Hom}_{k[T]}\big(W,\operatorname{Hom}_{k[S]}(k[T],V)^{\mathrm{sm}}\big),\end{aligned}$$

where the second bijection comes from the fact that every homomorphic image of a smooth representation is smooth.

□

Remark 2.2.14. The description of the left and right adjoints of Inf_S^T is quite different from the more familiar description in the case where S (and hence T, U) is a group. It is instructive to make precise the distinction between both descriptions.

(i) Given $V\in\operatorname{Rep}_k(S)$, put $V(U)\coloneqq\langle uv-v\mid u\in U, v\in V\rangle_k$, where $\langle\ldots\rangle_k$ denotes the k-linear span. There is a canonical k-linear surjection

$$V_U\coloneqq V/V(U)\twoheadrightarrow k[T]\otimes_{k[S]}V, \tag{2.2.15}$$

given by mapping the coset of v to $1\otimes v$. Note that, if $sU\subseteq Us$ for all $s\in S$, then there is a natural $k[S]$-action on V_U. If even $f^{-1}(f(s))=Us$ for all $s\in S$, then this action induces a $k[T]$-action on V_U. In this case, the kernel of $k[S]\twoheadrightarrow k[T]$ is generated as a right ideal by $\{u-1\mid u\in U\}$, and hence (2.2.15) is an isomorphism of smooth T-representations.

(ii) Evaluation at $1\in k[T]$ induces an injective k-linear map

$$\mathrm{ev}_1\colon \Pi_U(V)\hookrightarrow \mathrm{H}^0(U,V).$$

We observe that, if $Us\subseteq sU$ for all $s\in S$, then there is a natural $k[S]$-action on $\mathrm{H}^0(U,V)$. If even $f^{-1}(f(s))=sU$ for all $s\in S$, then this action induces a $k[T]$-action on $\mathrm{H}^0(U,V)$. In this case, the kernel of $k[S]\twoheadrightarrow k[T]$ is generated as a left ideal by $\{u-1\mid u\in U\}$, and hence ev_1 is an isomorphism of smooth T-representations.

Example 2.2.16. In order to see where some of the above conditions may fail, let us consider the positive monoid, cf. Example 2.2.1.(ii). More concretely, let $P\subseteq\mathrm{GL}_2(\mathbb{Q}_p)$ be the subgroup of upper triangular matrices, M the subgroup of diagonal matrices, and U the subgroup of upper triangular unipotent matrices. Put $K_P=P\cap\mathrm{GL}_2(\mathbb{Z}_p)$, $K_M=M\cap K_P$, and $K_U=U\cap K_P$. Consider now the positive monoid $M^+=\big\{m\in M\,\big|\,mK_Um^{-1}\subseteq K_U\big\}$ and put $P^+=K_UM^+$. Let $f\colon P^+\longrightarrow M^+$ be the canonical projection with kernel K_U.

It is obvious that $f^{-1}(m)=K_Um$, for all $m\in M^+$, and hence $V_{K_U}\cong \mathrm{L}^0_{K_U}(V)$ as smooth $\operatorname{Rep}_k(M^+)$-representations, for all $V\in\operatorname{Rep}_k(P^+)$.

However, for $m\coloneqq\operatorname{diag}(p,1)\in M^+$, we have $K_Um\nsubseteq mK_U$. Therefore $\mathrm{H}^0(K_U,V)$ is not stable under the M^+-action and $\mathrm{ev}_1\colon\Pi_{K_U}(V)\hookrightarrow\mathrm{H}^0(K_U,V)$ is not surjective. We remark that $\mathrm{H}^0(K_U,V)$ can be endowed with a "Hecke action", cf. [Eme10a, Lem. 3.1.4], making it a smooth M^+-representation.

2.2.3. *Symmetric monoidal structure.* Given V, W in $\mathrm{Rep}_k(S)$, we let S act diagonally on the tensor product $V \otimes_k W$; this is again a smooth representation. Therefore, $\mathrm{Rep}_k(S)$ carries the structure of a symmetric monoidal category, where the $\otimes$-unit is given by the trivial representation k.

Proposition 2.2.17. *The symmetric monoidal category* $\mathrm{Rep}_k(S)$ *is closed, that is, for each* V *in* $\mathrm{Rep}_k(S)$ *the functor*

$$- \otimes_k V \colon \mathrm{Rep}_k(S) \longrightarrow \mathrm{Rep}_k(S)$$

admits a right adjoint $\mathrm{hom}_{\mathrm{Rep}_k(S)}(V, -) \colon \mathrm{Rep}_k(S) \longrightarrow \mathrm{Rep}_k(S)$. *Moreover, there is a natural* S*-equivariant isomorphism*

$$\mathrm{hom}_{\mathrm{Rep}_k(S)}\big(U \otimes_k V, W\big) \cong \mathrm{hom}_{\mathrm{Rep}_k(S)}\big(U, \mathrm{hom}_{\mathrm{Rep}_k(S)}(V, W)\big).$$

Proof. Let $V, W \in \mathrm{Rep}_k(S)$. We let S act diagonally on $k[S] \otimes_k V$ and then on the k-vector space $\mathrm{Hom}_{k[S]}\big(k[S] \otimes_k V, W\big)$ via $(sf)(s' \otimes v) := f(s's \otimes v)$. One easily verifies that the maps

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Rep}_k(S)}\big(U \otimes_k V, W\big) &\rightleftarrows \mathrm{Hom}_{\mathrm{Rep}_k(S)}\big(U, \mathrm{Hom}_{k[S]}(k[S] \otimes_k V, W)^{\mathrm{sm}}\big),\\
\varphi &\longmapsto [u \mapsto [s \otimes v \mapsto \varphi(su \otimes v)]]\\
[u \otimes v \mapsto \psi(u)(1 \otimes v)] &\longleftarrow\!\shortmid \psi
\end{aligned}$$

are natural and inverse to each other. Hence, $\mathrm{hom}_{\mathrm{Rep}_k(S)}(V, -) := \mathrm{Hom}_{k[S]}(k[S] \otimes_k V, -)^{\mathrm{sm}}$ is right adjoint to $- \otimes_k V$. Let $X, U, V, W \in \mathrm{Rep}_k(S)$. Now, there are natural bijections

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Rep}_k(S)}\big(X, \mathrm{hom}_{\mathrm{Rep}_k(S)}(U \otimes_k V, W)\big) &\cong \mathrm{Hom}_{\mathrm{Rep}_k(S)}\big(X \otimes_k U \otimes_k V, W\big)\\
&\cong \mathrm{Hom}_{\mathrm{Rep}_k(S)}\big(X \otimes_k U, \mathrm{hom}_{\mathrm{Rep}_k(S)}(V, W)\big)\\
&\cong \mathrm{Hom}_{\mathrm{Rep}_k(S)}\big(X, \mathrm{hom}_{\mathrm{Rep}_k(S)}(U, \mathrm{hom}_{\mathrm{Rep}_k(S)}(V, W))\big).
\end{aligned}$$

The last claim follows from this together with the Yoneda lemma. ☐

Remark 2.2.18. If S is a group, the right adjoint of $- \otimes_k V$ has a more familiar description: The map $\sigma \colon k[S] \otimes_k V \longrightarrow k[S] \otimes_k V$, $s \otimes v \longmapsto s \otimes s^{-1}v$ is an isomorphism of $k[S]$-modules if we let S act diagonally on the left hand side and on the first factor on the right hand side. There is a natural $k[S]$-linear isomorphism

$$\mathrm{Hom}_k(V, W) \cong \mathrm{Hom}_{k[S]}\big(k[S] \otimes_k V, W\big) \xrightarrow[\sigma^*]{\cong} \mathrm{Hom}_{k[S]}\big(k[S] \otimes_k V, W\big)$$

if we let S act on the left hand side via $(sf)(v) = s \cdot f(s^{-1}v)$. Therefore, the right adjoint of $- \otimes_k V$ is in this case given by $\mathrm{Hom}_k(V, -)^{\mathrm{sm}}$.

2.3. **The derived category.**

2.3.1. *Derived functors.* Let S be a locally profinite monoid. We denote by $\mathrm{C}(S)$ the category of unbounded cochain complexes of smooth S-representations and by $\mathrm{K}(S)$ its homotopy category. Then $\mathrm{K}(S)$ carries the structure of a triangulated category [KS06, Prop. 11.2.8]. We denote by

$$\mathrm{D}(S) := \mathrm{D}\big(\mathrm{Rep}_k(S)\big)$$

the derived category of unbounded complexes in $\mathrm{Rep}_k(S)$. It arises from $\mathrm{K}(S)$ by localization at the class of quasi-isomorphisms [KS06, §13.1]. Note that $\mathrm{D}(S)$ inherits the structure of a triangulated category from $\mathrm{K}(S)$ such that the localization functor

$$\mathbf{q} \colon \mathrm{K}(S) \longrightarrow \mathrm{D}(S)$$

is triangulated. The shift of a complex C (in $\mathrm{K}(S)$ or $\mathrm{D}(S)$) will be denoted $C[1]$, defined by $(C[1])^n := C^{n+1}$ and with differential $d_{C[1]} = -d_C$. We recall the most important properties of $\mathrm{D}(S)$, which are summarized in [KS06, Thm. 14.3.1]. The derived category $\mathrm{D}(S)$ admits infinite direct sums, and they can be computed on complexes. By the main result of [Ser03, Thm. 3.13] or by [KS06, Cor. 14.1.8] every complex X in $\mathrm{K}(S)$ admits a quasi-isomorphism $X \xrightarrow{\mathrm{qis}} I$ into a K-injective complex I. (Recall that a complex J in $\mathrm{K}(S)$ is called *K-injective* if $\mathrm{Hom}_{\mathrm{K}(S)}(Y, J) = 0$

for all exact complexes Y.) This amounts to saying that $\mathbf{q}$ admits a fully faithful triangulated right adjoint

$$\mathbf{i}\colon \mathrm{D}(S) \longrightarrow \mathrm{K}(S)$$

which identifies $\mathrm{D}(S)$ with the full subcategory of $\mathrm{K}(S)$ consisting of K-injective complexes. In particular, $\mathrm{D}(S)$ is locally small. It also follows that right derived functors always exist:

Proposition 2.3.1. *Let $\mathscr{A}$ be an abelian category and $F\colon \mathrm{Rep}_k(S) \longrightarrow \mathscr{A}$ a left exact functor. A right derived functor $\mathrm{R}F\colon \mathrm{D}(S) \longrightarrow \mathrm{D}(\mathscr{A})$ exists and is given by the composition*

$$\mathrm{R}F\colon \mathrm{D}(S) \xrightarrow{\mathbf{i}} \mathrm{K}(S) \xrightarrow{\mathrm{K}(F)} \mathrm{K}(\mathscr{A}) \xrightarrow{\mathbf{q}_{\mathscr{A}}} \mathrm{D}(\mathscr{A}),$$

where $\mathbf{q}_{\mathscr{A}}$ denotes the localization functor and $\mathrm{K}(F)$ the natural extension of F to the homotopy categories.

Notation 2.3.2. If $F\colon \mathrm{Rep}_k(S) \longrightarrow \mathscr{A}$ happens to be exact, we simply write $F\colon \mathrm{D}(S) \longrightarrow \mathrm{D}(\mathscr{A})$ for the right derived functor of F. This is justified by the fact that in this case the right adjoint can be computed by applying F termwise to the complex.

Here is a useful and well-known lemma:

Lemma 2.3.3. *Let $L\colon \mathscr{A} \longrightarrow \mathscr{B}$ be an exact functor between abelian categories admitting a right adjoint R.*

(a) $\mathrm{K}(R)\colon \mathrm{K}(\mathscr{B}) \longrightarrow \mathrm{K}(\mathscr{A})$ preserves K-injective complexes.
(b) Assume moreover that the localization functors $\mathbf{q}_{\mathscr{A}}\colon \mathrm{K}(\mathscr{A}) \longrightarrow \mathrm{D}(\mathscr{A})$ and $\mathbf{q}_{\mathscr{B}}\colon \mathrm{K}(\mathscr{B}) \longrightarrow \mathrm{D}(\mathscr{B})$ admit right adjoints $\mathbf{i}_{\mathscr{A}}$ and $\mathbf{i}_{\mathscr{B}}$, respectively. Then there is a natural isomorphism $\mathrm{K}(R)\,\mathbf{i}_{\mathscr{B}} \overset{\cong}{\Longrightarrow} \mathbf{i}_{\mathscr{A}}\,\mathrm{R}R$ of functors $\mathrm{D}(\mathscr{B}) \longrightarrow \mathrm{K}(\mathscr{A})$.
(c) Under the hypotheses of (b), the functor $\mathrm{R}R$ is a right adjoint of $L\colon \mathrm{D}(\mathscr{A}) \longrightarrow \mathrm{D}(\mathscr{B})$.

Proof. Note that $L \dashv R$ extends to an adjunction $\mathrm{K}(L) \dashv \mathrm{K}(R)$. Let $I \in \mathrm{K}(\mathscr{B})$ be K-injective and $Y \in \mathrm{K}(\mathscr{A})$ an exact complex. Then $\mathrm{K}(L)(Y)$ is exact and hence $\mathrm{Hom}_{\mathrm{K}(\mathscr{A})}(Y, \mathrm{K}(R)(I)) \cong \mathrm{Hom}_{\mathrm{K}(\mathscr{B})}(\mathrm{K}(L)(Y), I) = 0$. This proves (a).

We now prove (b). Note that the unit $\mathrm{id}_{\mathrm{K}(\mathscr{A})} \Longrightarrow \mathbf{i}_{\mathscr{A}}\,\mathbf{q}_{\mathscr{A}}$ is an isomorphism on the K-injective complexes. Since $\mathrm{K}(R)$ preserves K-injective complexes, it follows that the map $\mathrm{K}(R)\,\mathbf{i}_{\mathscr{B}} \Longrightarrow \mathbf{i}_{\mathscr{A}}\,\mathbf{q}_{\mathscr{A}}\,\mathrm{K}(R)\,\mathbf{i}_{\mathscr{B}} = \mathbf{i}_{\mathscr{A}}\,\mathrm{R}R$ is an isomorphism.

For (c), note that $\mathrm{K}(R)$ is right adjoint to $\mathrm{K}(L)\colon \mathrm{K}(\mathscr{A}) \longrightarrow \mathrm{K}(\mathscr{B})$. There are natural bijections

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{D}(\mathscr{A})}\big(\mathbf{q}_{\mathscr{A}}\,X, \mathrm{R}RY\big) &\cong \mathrm{Hom}_{\mathrm{K}(\mathscr{A})}\big(X, \mathbf{i}_{\mathscr{A}}\,\mathrm{R}RY\big)\\
&\cong \mathrm{Hom}_{\mathrm{K}(\mathscr{A})}\big(X, \mathrm{K}(R)\,\mathbf{i}_{\mathscr{B}}\,Y\big) && \text{(by (b))}\\
&\cong \mathrm{Hom}_{\mathrm{K}(\mathscr{B})}\big(\mathrm{K}(L)X, \mathbf{i}_{\mathscr{B}}\,Y\big)\\
&\cong \mathrm{Hom}_{\mathrm{D}(\mathscr{B})}\big(\mathbf{q}_{\mathscr{B}}\,\mathrm{K}(L)X, Y\big)\\
&\cong \mathrm{Hom}_{\mathrm{D}(\mathscr{B})}\big(L\,\mathbf{q}_{\mathscr{A}}\,X, Y\big)
\end{aligned}$$

for all $X \in \mathrm{K}(\mathscr{A})$ and $Y \in \mathrm{D}(\mathscr{B})$. This implies (c). □

Example 2.3.4. We present an example of the same derived functor in two different ways. Let $K \subseteq S$ be an open subgroup.

(a) The functor $\mathrm{H}^0(K,-)\,\mathrm{Res}^S_K\colon \mathrm{Rep}_k(S) \longrightarrow \mathrm{Vect}_k$ of K-invariants is left exact and hence admits a right derived functor $\mathrm{R}\big(\mathrm{H}^0(K,-)\,\mathrm{Res}^S_K\big)\colon \mathrm{D}(S) \longrightarrow \mathrm{D}(k)$. We observe that he diagram

$$\begin{tikzcd}
\mathrm{D}(S) \arrow[r, "\mathrm{Res}^S_K"] \arrow[dr, "\mathrm{R}(\mathrm{H}^0(K,-)\,\mathrm{Res}^S_K)"'] & \mathrm{D}(K) \arrow[d, "\mathrm{RH}^0(K,-)"] \\
& \mathrm{D}(k)
\end{tikzcd}$$

commutes. Indeed, since $K \subseteq S$ is a subgroup, Res_K^S admits an *exact* left adjoint ind_K^S (Lemma 2.2.9). By Lemma 2.3.3.(b) we have

$$\begin{aligned}\mathrm{RH}^0(K,-)\operatorname{Res}_K^S &= \mathbf{q}\,\mathrm{H}^0(K,-)\,\mathbf{i}\operatorname{Res}_K^S \cong \mathbf{q}\,\mathrm{H}^0(K,-)\operatorname{Res}_K^S\,\mathbf{i}\\ &= \mathrm{R}\big(\mathrm{H}^0(K,-)\operatorname{Res}_K^S\big).\end{aligned}$$

For this reason, we often allow ourselves to omit Res_K^S when it does not lead to confusion.

(b) The functor $\operatorname{Hom}_{\operatorname{Rep}_k(K)}(\mathbf{1},-)\colon \operatorname{Rep}_k(K) \longrightarrow \operatorname{Vect}_k$ is canonically isomorphic to $\mathrm{H}^0(K,-)$. This extends to a natural isomorphism $\operatorname{Hom}^\bullet_{\operatorname{Rep}_k(K)}(\mathbf{1},-) \overset{\cong}{\Longrightarrow} \mathrm{K}(\mathrm{H}^0(K,-))$ of functors $\mathrm{K}(K) \longrightarrow \mathrm{K}(\operatorname{Vect}_k)$, which induces a natural isomorphism

$$\operatorname{RHom}_{\operatorname{Rep}_k(K)}(\mathbf{1},-) \overset{\cong}{\Longrightarrow} \mathrm{RH}^0(K,-)$$

of functors $\mathrm{D}(K) \longrightarrow \mathrm{D}(k)$.

Recall that, by definition, every smooth S-representation V satisfies $V = \bigcup_K \operatorname{Hom}_{\operatorname{Rep}_k(K)}(\mathbf{1}, V)$, where K runs through the compact open subgroups of S. This property continues to hold on the level of derived categories as well.

Proposition 2.3.5. *Let $X \in \mathrm{D}(S)$. There exists a natural isomorphism*

$$\varinjlim_{\substack{K\subseteq S\\ \text{compact open}}} \operatorname{RHom}_{\operatorname{Rep}_k(K)}\big(\mathbf{1}, \operatorname{Res}_K^S X\big) \overset{\cong}{\Longrightarrow} \operatorname{Res}_1^S X \qquad \text{in } \mathrm{D}(k). \tag{2.3.6}$$

Proof. Since taking filtered colimits is exact in Vect_k, the maps

$$\begin{aligned}\varinjlim_{\substack{K\subseteq S\\ \text{compact open}}} \operatorname{RHom}_{\operatorname{Rep}_k(K)}\big(\mathbf{1}, \operatorname{Res}_K^S X\big) &\xrightarrow{\cong} \varinjlim_{\substack{K\subseteq S\\ \text{compact open}}} \mathrm{RH}^0(K,X)\\ &= \varinjlim_{\substack{K\subseteq S\\ \text{compact open}}} \mathbf{q}\,\mathrm{H}^0(K,\mathbf{i}\,X)\\ &\xrightarrow{\cong} \mathbf{q} \varinjlim_{\substack{K\subseteq S\\ \text{compact open}}} \mathrm{H}^0(K,\mathbf{i}\,X)\\ &\xrightarrow{\cong} \mathbf{q}\operatorname{Res}_1^S \mathbf{i}\,X\\ &\cong \operatorname{Res}_1^S X\end{aligned}$$

are natural isomorphisms. □

Remark 2.3.7. If S is a locally profinite group, one can make sense of the left hand side of (2.3.6) even as an object of $\mathrm{D}(S)$ und show that there is a natural isomorphism

$$\varinjlim_{\substack{K\subseteq S\\ \text{compact open}}} \operatorname{RHom}_{\operatorname{Rep}_k(K)}(\mathbf{1}, \operatorname{Res}_K^S X) \overset{\cong}{\Longrightarrow} X \qquad \text{in } \mathrm{D}(S).$$

2.3.2. *Brown representability.* We will now formulate the Brown representability theorems. We make the assumption that S is a p-adic monoid and k has characteristic p. Then S admits a compact open subgroup K which carries the structure of a p-adic Lie group of dimension, say, d. Replacing K by an open subgroup, we may even assume that K is torsion-free. Then K is a Poincaré group by [Ser65, Cor. (1)] and [Laz65, V.2.5.8]. In particular, the cohomology groups $\mathrm{H}^i(K,k)$ are finite-dimensional, and $\mathrm{H}^i(K,-) = 0$ provided $i > d$.

Proposition 2.3.8. *Let S be a p-adic monoid and let $K \subseteq S$ be a torsion-free compact open subgroup. Then $\mathrm{D}(S)$ is compactly generated by $\operatorname{ind}_K^S \mathbf{1}$.*

Proof. This is precisely [Sch15, Prop. 6] in the case where S is a p-adic Lie group, and the proof is general. We only have to observe that ind_K^S is an exact functor, since K is a group, and hence the adjunction $\operatorname{ind}_K^S \dashv \operatorname{Res}_K^S$ (Lemma 2.2.9) extends to the derived categories. $\square$

Theorem 2.3.9 (Brown representability)**.** *Denote by* (Ab) *the category of abelian groups.*

(a) Let $H\colon \mathrm{D}(S)^{\mathrm{op}} \longrightarrow (\mathrm{Ab})$ be a cohomological functor commuting with small products. Then H is representable, that is, there exists an object $X \in \mathrm{D}(S)$ and a natural isomorphism $\mathrm{Hom}_{\mathrm{D}(S)}(-, X) \cong H$. In particular, $\mathrm{D}(S)$ admits small products.

(b) Let $H\colon \mathrm{D}(S) \longrightarrow (\mathrm{Ab})$ be a cohomological functor commuting with small direct sums. Then H is corepresentable, that is, there exists an object $X \in \mathrm{D}(S)$ and a natural isomorphism $\mathrm{Hom}_{\mathrm{D}(S)}(X, -) \cong H$.

Proof. See [Nee01, Thm. 8.3.3] or [Kra02, Thm. A] for (a) and [Kra02, Thm. B] for (b). $\square$

The following corollary is a standard application of Brown representability.

Corollary 2.3.10. *Let $\mathscr{C}$ be a triangulated category and $F\colon \mathrm{D}(S) \longrightarrow \mathscr{C}$ a triangulated functor.*

(a) F admits a triangulated right adjoint if and only if F commutes with small direct sums.

(b) F admits a triangulated left adjoint if and only if F commutes with small products.

Proof. We only prove (b) and leave (a) as an exercise. It is clear that F commutes with small products if F admits a left adjoint (see Proposition 2.1.7).

Conversely, if F commutes with small products, the functor $\mathrm{Hom}_{\mathscr{C}}\big(X, F(-)\big)\colon \mathrm{D}(S) \longrightarrow (\mathrm{Ab})$ is cohomological and commutes with small products. By Theorem 2.3.9 this functor is representable. From Theorem 2.1.6 it follows that F admits a left adjoint, which, by [Nee01, Lem. 5.3.6], is even triangulated. $\square$

2.3.3. *Tensor triangulated structure.* The category $\mathrm{D}(S)$ carries the structure of a closed symmetric monoidal category, which was introduced and investigated in [SS22b]. The tensor product of cochain complexes endows the triangulated category $\mathrm{K}(S)$ with a symmetric monoidal structure. The $\otimes$-unit is given by $\mathbf{1} := k[0]$, that is, k viewed as a complex concentrated in degree 0. Since, for any $X, Y \in \mathrm{K}(S)$, the complex $X \otimes_k Y$ is exact whenever either X or Y is exact, the induced symmetric monoidal structure on $\mathrm{D}(S)$ is again given by $- \otimes_k -$.

Proposition 2.3.11. *The symmetric monoidal category $\mathrm{D}(S)$ is closed.*

Proof. Let $X \in \mathrm{D}(S)$. Then $- \otimes_k X$ is a triangulated functor commuting with small direct sums. By Corollary 2.3.10.(a), it admits a right adjoint. $\square$

Remark 2.3.12. The internal Hom objects in $\mathrm{D}(S)$ are explicitly given as follows: Let $X, Y \in \mathrm{C}(S)$ be complexes and define a new complex $\underline{\mathrm{Hom}}_S(X, Y)$ in $\mathrm{C}(S)$ by

$$\underline{\mathrm{Hom}}_S^n(X, Y) := \Big(\prod_{i \in \mathbb{Z}} \mathrm{Hom}_{k[S]}\big(k[S] \otimes_k X^i, Y^{i+n}\big)\Big)^{\mathrm{sm}},$$

with differential $(df)^i = d_Y^{n+i} \circ f^i - (-1)^n f^{i+1} \circ (1 \otimes d_X^i)$, for $f \in \underline{\mathrm{Hom}}_S^n(X, Y)$. This yields a k-linear bifunctor

$$\underline{\mathrm{Hom}}_S\colon \mathrm{C}(S)^{\mathrm{op}} \times \mathrm{C}(S) \longrightarrow \mathrm{C}(S). \tag{2.3.13}$$

A direct computation shows

$$\mathrm{H}^0\big(\underline{\mathrm{Hom}}_S(X, Y)\big) \cong \varinjlim_{\substack{S' \subseteq S \\ \text{open subgroup}}} \mathrm{Hom}_{\mathrm{K}(S)}\big(k[S/S'] \otimes_k X, Y\big). \tag{2.3.14}$$

As in the proof of Proposition 2.2.17 one constructs a natural isomorphism

$$\underline{\mathrm{Hom}}_S\big(X \otimes_k Y, Z\big) \cong \underline{\mathrm{Hom}}_S\big(X, \underline{\mathrm{Hom}}_S(Y, Z)\big).$$

Taking cohomology and then S-invariants yields a natural isomorphism

$$\mathrm{Hom}_{\mathrm{K}(S)}\big(X \otimes_k Y, Z\big) \cong \mathrm{Hom}_{\mathrm{K}(S)}\big(X, \underline{\mathrm{Hom}}_S(Y, Z)\big).$$

This has the following immediate consequences:

(i) The bifunctor (2.3.13) descends to a bifunctor
$$\underline{\mathrm{Hom}}_S\colon \mathrm{K}(S)^{\mathrm{op}} \times \mathrm{K}(S) \longrightarrow \mathrm{K}(S)$$
making $\mathrm{K}(S)$ a closed symmetric monoidal category.

(ii) As a right adjoint of $-\otimes_k Y$, the functor $\underline{\mathrm{Hom}}_S(Y,-)$ is triangulated, by [Nee01, Lem. 5.3.6]. It is also true that $\underline{\mathrm{Hom}}_S(-,Y)$ preserves triangles, cf. [KS06, Prop. 11.6.4.(ii)].

(iii) If Z is K-injective, then the complex $\underline{\mathrm{Hom}}_S(Y,Z)$ again is K-injective.

(iv) If Z is K-injective and Y is exact, then $\underline{\mathrm{Hom}}_S(Y,Z)$ is contractible and, in particular, exact. Consequently, $\underline{\mathrm{Hom}}_S(-,Z)$ preserves quasi-isomorphisms.

The last property shows that the right derived functor
$$\mathrm{R}\,\underline{\mathrm{Hom}}_S\colon \mathrm{D}(S)^{\mathrm{op}} \times \mathrm{D}(S) \longrightarrow \mathrm{D}(S)$$
exists and is computed by $\mathrm{R}\,\underline{\mathrm{Hom}}_S(X,Y) \cong \underline{\mathrm{Hom}}_S(X,\mathbf{i}\,Y)$. Using (iii) above, one shows that there is a natural isomorphism
$$\mathrm{Hom}_{\mathrm{D}(S)}\big(X\otimes_k Y, Z\big) \cong \mathrm{Hom}_{\mathrm{D}(S)}\big(X, \mathrm{R}\,\underline{\mathrm{Hom}}_S(Y,Z)\big).$$
Therefore, $\mathrm{R}\,\underline{\mathrm{Hom}}_S(Y,-)$ is the right adjoint of $-\otimes_k Y$. Moreover, (2.3.14), applied to $\mathbf{i}\,Y$, shows that there is a natural isomorphism
$$\mathrm{H}^0\big(\mathrm{R}\,\underline{\mathrm{Hom}}_S(X,Y)\big) \cong \varinjlim_{\substack{S'\subseteq S\\ \text{open subgroup}}} \mathrm{Hom}_{\mathrm{D}(S)}\big(k[S/S']\otimes_k X, Y\big).$$
If, in addition, S is a group, then Remark 2.2.18 implies that there is a natural isomorphism
$$\mathrm{H}^0\big(\mathrm{R}\,\underline{\mathrm{Hom}}_S(X,Y)\big) \cong \varinjlim_{\substack{S'\subseteq S\\ \text{open subgroup}}} \mathrm{Hom}_{\mathrm{D}(S')}(X,Y). \tag{2.3.15}$$

Notation 2.3.16. Given $X\in\mathrm{D}(S)$, we denote the right adjoint of $-\otimes_k X$ by $\mathrm{hom}_{\mathrm{D}(S)}(X,-)$ or just $\mathrm{hom}(X,-)$ if the category is clear from the context.

We call $X^\vee \coloneqq \mathrm{hom}(X,\mathbf{1})$ the *dual* of X in $\mathrm{D}(S)$.

Warning 2.3.17. The internal Hom complex $\mathrm{hom}_{\mathrm{D}(S)}(X,Y)\in\mathrm{D}(S)$ should not be confused with the RHom complex $\mathrm{RHom}_{\mathrm{Rep}_k(S)}(X,Y)$, which is only an object of $\mathrm{D}(k)$. In general, there is the relation
$$\mathrm{RH}^0(S,-)\circ \mathrm{hom}_{\mathrm{D}(S)}(X,Y) \cong \mathrm{RHom}_{\mathrm{Rep}_k(S)}(X,Y), \tag{2.3.18}$$
which follows from (iii) above, *i.e.*, the fact that $\underline{\mathrm{Hom}}_S(X,-)$ preserves K-injective complexes.

2.3.4. *Rigidly-compact generation.* Let K be a compact torsion-free p-adic Lie group and k a field of characteristic p. The next crucial observation allows us to appeal to the results of [BDS16].

Proposition 2.3.19. *The category* $\mathrm{D}(K)$ *is rigidly-compactly generated.*

Proof. Ye know from Proposition 2.3.8 that $\mathrm{D}(K)$ is compactly generated by $\mathbf{1}$. It remains to show that the set of compact objects coincides with the set of rigid objects. Recall that $X\in\mathrm{D}(K)$ is called *compact* if for every family $\{Y_i\}_{i\in I}$ in $\mathrm{D}(K)$ the natural map
$$\bigoplus_{i\in I}\mathrm{Hom}_{\mathrm{D}(K)}(X,Y_i) \longrightarrow \mathrm{Hom}_{\mathrm{D}(K)}\Big(X,\bigoplus_{i\in I}Y_i\Big)$$
is bijective. Recall also that X is called *rigid* if for all $Y\in\mathrm{D}(K)$ the natural map
$$X^\vee\otimes_k Y\longrightarrow \mathrm{hom}(X,Y)$$
is an isomorphism in $\mathrm{D}(K)$. Since $\mathbf{1}$ is compact and $\mathrm{Hom}_{\mathrm{D}(K)}\big(\mathbf{1},\mathrm{hom}(X,Y)\big)=\mathrm{Hom}_{\mathrm{D}(K)}(X,Y)$, it is immediate that rigid objects are compact. For the converse, note that $\mathbf{1}$ is rigid and that the strictly full subcategory of rigid objects is triangulated and closed under direct summands. Therefore, the strictly full, thick, triangulated subcategory $\langle\mathbf{1}\rangle$ generated by $\mathbf{1}$ consists of rigid objects. Now, [Nee92, Lem. 2.2] shows that $\langle\mathbf{1}\rangle$ consists of all the compact objects in $\mathrm{D}(K)$. $\square$

2.3.5. *Spectral sequences.* Lacking a good reference, we explain how to construct spectral sequences using the canonical t-structure on $\mathrm{D}(\mathscr{A})$, for $\mathscr{A}$ an abelian category. Everything in this paragraph is well-known and holds in much greater generality, cf. [Lur12, 1.2.2].

Our main reference is [Ben91b, 3.2], which constructs spectral sequences from filtered complexes. Our treatment is very similar, yet not the same, and so we refer to *loc. cit.* for the proofs, which apply verbatim in our case if the notation is reinterpreted accordingly.

First, let us recall the canonical t-structure on $\mathrm{D}(\mathscr{A})$, for an abelian category $\mathscr{A}$. Given a cochain complex $(C, \{d^i\}_i) \in \mathrm{C}(\mathscr{A})$, the truncated complexes $\tau^{\leq n}C$ and $\tau^{\geq n}C$ in $\mathrm{C}(\mathscr{A})$ are defined by

$$\tau^{\leq n}C\colon \qquad \cdots \xrightarrow{d^{n-3}} C^{n-2} \xrightarrow{d^{n-2}} C^{n-1} \xrightarrow{d^{n-1}} \mathrm{Ker}(d^n) \longrightarrow 0 \longrightarrow \cdots$$

and

$$\tau^{\geq n}C\colon \qquad \cdots \longrightarrow 0 \longrightarrow \mathrm{Coker}(d^{n-1}) \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} C^{n+2} \xrightarrow{d^{n+2}} \cdots$$

The truncation functors $\tau^{\leq n}$ and $\tau^{\geq n}$ preserve quasi-isomorphisms and hence give rise to truncation functors on $\mathrm{D}(\mathscr{A})$. For each complex C there is a distinguished triangle

$$\tau^{\leq n}C \longrightarrow C \longrightarrow \tau^{\geq n+1}C \xrightarrow{+}$$

in $\mathrm{D}(\mathscr{A})$, and we have $\mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(\tau^{\leq n}C, \tau^{\geq n+1}D) = 0$, for all $C, D \in \mathrm{D}(\mathscr{A})$, $n \in \mathbb{Z}$. This defines the canonical t-structure on $\mathrm{D}(\mathscr{A})$. Note also that

$$\mathrm{H}^n(C)[-n] \cong \tau^{\leq n}\tau^{\geq n}C \cong \tau^{\geq n}\tau^{\leq n}C, \qquad \text{for each } C \in \mathrm{D}(\mathscr{A}).$$

The following version of the hypercohomology spectral sequence will be useful later. For the notion of way-out functors, we refer to [Har66, §I.7]

Lemma 2.3.20. *Let $\mathscr{A}, \mathscr{B}$ be abelian categories and let $X \in \mathrm{D}(\mathscr{A})$.*

(a) Let $F\colon \mathrm{D}(\mathscr{A}) \longrightarrow \mathrm{D}(\mathscr{B})$ be a triangulated functor. Assume that one of the following properties is satisfied:
- *F is way-out right (e.g., left t-exact) and X is left bounded;*
- *F is way-out left (e.g., right t-exact) and X is right bounded;*
- *F is way-out in both directions (e.g., t-exact);*
- *X is bounded.*

Then there exists a convergent spectral sequence

$$E_2^{i,j} = (\mathrm{H}^i F)\big(\mathrm{H}^j(X)\big) \Longrightarrow \mathrm{H}^{i+j}(FX).$$

(b) Let $F\colon \mathrm{D}(\mathscr{A})^{\mathrm{op}} \longrightarrow \mathrm{D}(\mathscr{B})$ be a triangulated functor. Assume that one of the following properties is satisfied:
- *F is way-out right and X is right bounded;*
- *F is way-out left and X is left bounded;*
- *F is way-out in both directions;*
- *X is bounded.*

Then there exists a convergent spectral sequence

$$E_2^{i,j} = (\mathrm{H}^i F)\big(\mathrm{H}^{-j}(X)\big) \Longrightarrow \mathrm{H}^{i+j}(FX).$$

Proof. The following construction is adapted from [Ben91b, §3.2]. We refer to *loc. cit.* for details about the proofs.

(a) Put $X(r) := \tau^{\leq -r}X$ for each $r \in \mathbb{Z}$. Given $s \geq r$, we have a distinguished triangle $X(s) \longrightarrow X(r) \longrightarrow X(r,s) \xrightarrow{+}$ in $\mathrm{D}(\mathscr{A})$; note that $X(r, r+1) \cong \mathrm{H}^{-r}(X)[r]$. Applying F, we obtain a distinguished triangle

$$FX(s) \longrightarrow FX(r) \longrightarrow FX(r,s) \xrightarrow{+} \quad \text{in } \mathrm{D}(\mathscr{B}).$$

For $s = r+1$ we contemplate the induced long exact sequence in cohomology

$$\cdots \longrightarrow \mathrm{H}^n\big(FX(r+1)\big) \xrightarrow{i_1} \mathrm{H}^n\big(FX(r)\big) \xrightarrow{j_1} \mathrm{H}^n\big(FX(r,r+1)\big) \xrightarrow{k_1} \mathrm{H}^{n+1}\big(FX(r+1)\big) \longrightarrow \cdots.$$

Define $D_1^{r,s} \coloneqq \mathrm{H}^{r+s}\big(FX(r)\big)$ and $E_1^{r,s} \coloneqq \mathrm{H}^{r+s}\big(FX(r,r+1)\big)$, for all $r,s \in \mathbb{Z}$. We obtain an exact couple

$$\begin{tikzcd} D_1^{*,*} \arrow[rr, "i_1"] & & D_1^{*,*} \arrow[dl, "j_1"] \\ & E_1^{*,*} \arrow[ul, "k_1"] & \end{tikzcd}$$

that is, we have $\mathrm{Im}(i_1) = \mathrm{Ker}(j_1)$, $\mathrm{Im}(j_1) = \mathrm{Ker}(k_1)$, and $\mathrm{Im}(k_1) = \mathrm{Ker}(i_1)$. We denote by $d_1 = j_1 \circ k_1 \colon E_1^{*,*} \longrightarrow E_1^{*+1,*}$ the associated differential. Taking derived couples yields a spectral sequence $\{(E_n, d_n)\}_{n \geq 1}$.

For each $n \geq 1$ we put

$$Z_n^{r,s} \coloneqq \mathrm{Im}\Big(\mathrm{H}^{r+s}\big(FX(r,r+n)\big) \longrightarrow \mathrm{H}^{r+s}\big(FX(r,r+1)\big)\Big),$$

and

$$B_n^{r,s} \coloneqq \mathrm{Ker}\Big(\mathrm{H}^{r+s}\big(FX(r,r+1)\big) \longrightarrow \mathrm{H}^{r+s}\big(FX(r-n+1,r+1)\big)\Big)$$

and then $Z_\infty^{r,s} \coloneqq \bigcap_n Z_n^{r,s}$, $B_\infty^{r,s} \coloneqq \bigcup_n B_n^{r,s}$. Now, [Ben91b, Prop. 3.2.4] shows $E_n^{r,s} = Z_n^{r,s}/B_n^{r,s}$. Put $E_\infty^{r,s} \coloneqq Z_\infty^{r,s}/B_\infty^{r,s}$. We define a descending filtration on $\mathrm{H}^{r+s}(FX)$ by

$$\mathcal{F}il^r \mathrm{H}^{r+s}(FX) \coloneqq \mathrm{Im}\big(\mathrm{H}^{r+s}(FX(r)) \longrightarrow \mathrm{H}^{r+s}(FX)\big).$$

We now verify that $\{(E_n^{*,*}, d_n)\}_n$ is *biregular*, that is:

(i) For all $r,s \in \mathbb{Z}$ there exists $n \gg 0$ such that $Z_\infty^{r,s} = Z_n^{r,s}$ and $B_\infty^{r,s} = B_n^{r,s}$.
(ii) For all $m \in \mathbb{Z}$, we have $\mathcal{F}il^r \mathrm{H}^m(FX) = 0$, for $r \gg 0$, and $\mathcal{F}il^r \mathrm{H}^m(FX) = \mathrm{H}^m(FX)$, for $r \ll 0$.

It then follows that the spectral sequence converges to $\mathrm{H}^*(FX)$.

- If X is left bounded, then we have $Z_n^{r,s} = Z_\infty^{r,s}$, for $n \gg 0$, and $\mathcal{F}il^r \mathrm{H}^m(FX) = 0$, for fixed $m \in \mathbb{Z}$ and all $r \gg 0$. Indeed, there exists $n \gg 0$ such that $X(r+n) = 0$. From the distinguished triangle $X(r+n) \longrightarrow X(r) \longrightarrow X(r,r+n) \xrightarrow{+}$ we deduce $X(r) \cong X(r,r+n)$. This implies $\mathrm{H}^{r+s}(FX(r,r+n)) \cong \mathrm{H}^{r+s}(FX(r))$ and hence $Z_n^{r,s} = Z_\infty^{r,s}$, for $n \gg 0$. Moreover, for all $r \gg 0$, we have $X(r) = 0$ and hence $\mathrm{H}^m(FX(r)) = 0$, that is, $\mathcal{F}il^r \mathrm{H}^m(FX(r)) = 0$, for $r \gg 0$.
- If X is right bounded, then we have $B_n^{r,s} = B_\infty^{r,s}$, for $n \gg 0$, and $\mathcal{F}il^r \mathrm{H}^m(FX) = \mathrm{H}^m(FX)$, for fixed $m \in \mathbb{Z}$ and all $r \ll 0$. Indeed, there exists $n \gg 0$ such that $X(r-n+1) \cong X(r-n'+1)$, for all $n' \geq n$. This implies $\mathrm{H}^{r+s}(FX(r-n+1,r+1)) \cong \mathrm{H}^{r+s}(FX(r-n'+1,r+1))$ and hence $B_n^{r,s} = B_\infty^{r,s}$. Moreover, for all $r \gg 0$ we have $X(r) = X$ and hence $\mathcal{F}il^r \mathrm{H}^m(FX) = \mathrm{H}^m(FX)$.
- If F is way-out right, then we have $B_n^{r,s} = B_\infty^{r,s}$, for $n \gg r+s$, and $\mathcal{F}il^r \mathrm{H}^m(FX) = \mathrm{H}^m(FX)$, for all $r \ll -m$. Indeed, we have $\mathrm{H}^{r+s}(FX(r-n'+1,r-n+1)) = 0$ whenever $n' \geq n \gg r+s$. Hence, from the distinguished triangle

$$FX(r-n'+1,r+1) \longrightarrow FX(r-n+1,r+1) \longrightarrow FX(r-n+1,r-n'+1) \xrightarrow{+},$$

  we deduce an isomorphism

$$\mathrm{H}^{r+s}\big(FX(r-n+1,r+1)\big) \cong \mathrm{H}^{r+s}\big(FX(r-n'+1,r+1)\big),$$

  which shows $B_n^{r,s} = B_\infty^{r,s}$. Similarly, we have $\mathrm{H}^m(FX(r)) \cong \mathrm{H}^m(FX)$, whenever $r \ll -m$, and hence $\mathcal{F}il^r \mathrm{H}^m(FX) = \mathrm{H}^m(FX)$.
- If F is way-out left, then we have $Z_n^{r,s} = Z_\infty^{r,s}$, for all $n \gg -2r-s$, and $\mathcal{F}il^r \mathrm{H}^m(FX) = 0$, for $r \gg -m$.
  Indeed, we have $\mathrm{H}^{r+s}(FX(r+n,r+n')) = 0$ whenever $n' \geq n \gg -2r-s$. From the distinguished triangle

$$FX(r+n,r+n') \longrightarrow FX(r,r+n') \longrightarrow FX(r,r+n) \xrightarrow{+}$$

we deduce an isomorphism $\mathrm{H}^{r+s}(FX(r,r+n')) \cong \mathrm{H}^{r+s}(FX(r,r+n))$, which shows $Z_n^{r,s} = Z_\infty^{r,s}$. Moreover, for $r \gg -m$ we have $\mathrm{H}^m(FX(r)) = 0$, so that $\mathscr{F}\!il^r \mathrm{H}^m(FX) = 0$.

By [Ben91b, Thm. 3.2.9] we have

$$\mathscr{F}\!il^r \mathrm{H}^{r+s}(FX)/\mathscr{F}\!il^{r+1}\mathrm{H}^{r+s}(FX) \cong E_\infty^{r,s}.$$

Therefore, the spectral sequence indeed computes $\mathrm{H}^*(FX)$. Note that

$$E_1^{r,s} = \mathrm{H}^{r+s}\big(FX(r,r+1)\big) = (\mathrm{H}^{r+s}F)\big(\mathrm{H}^{-r}(X)[r]\big) = (\mathrm{H}^{2r+s}F)\big(\mathrm{H}^{-r}(X)\big).$$

Hence, reindexing $(i,j) := (2r+s,-r)$ yields the desired spectral sequence $E_2^{i,j} = E_1^{r,s} = (\mathrm{H}^iF)\big(\mathrm{H}^j(X)\big) \Longrightarrow \mathrm{H}^{i+j}(FX)$.

(b) Of course, this follows by duality from (a), but let us nevertheless give the construction in this case. Put $X(r) := \tau^{\geq r}X$ for each $r \in \mathbb{Z}$. Given $r \leq s$, we have a distinguished triangle $X(r,s) \longrightarrow X(r) \longrightarrow X(s) \xrightarrow{+}$ in $\mathrm{D}(\mathscr{A})$; note that $X(r,r+1) \cong \mathrm{H}^r(X)[-r]$. Applying F, we obtain a distinguished triangle

$$FX(s) \longrightarrow FX(r) \longrightarrow FX(r,s) \xrightarrow{+} \quad \text{in } \mathrm{D}(\mathscr{B}).$$

Put $D_1^{r,s} := \mathrm{H}^{r+s}(FX(r))$ and $E_1^{r,s} := \mathrm{H}^{r+s}(FX(r,r+1))$. As in (a) this gives rise to a spectral sequence $\{(E_n,d_n)\}_n$ which converges to $\mathrm{H}^*(FX)$. We compute

$$E_1^{r,s} = \mathrm{H}^{r+s}\big(FX(r,r+1)\big) = (\mathrm{H}^{r+s}F)\big(\mathrm{H}^r(X)[-r]\big) = (\mathrm{H}^{2r+s}F)\big(\mathrm{H}^r(X)\big).$$

Again, reindexing $(i,j) := (2r+s,-r)$ yields the desired spectral sequence $E_2^{i,j} = E_1^{r,s} = (\mathrm{H}^iF)\big(\mathrm{H}^{-j}(X)\big) \Longrightarrow \mathrm{H}^{i+j}(FX)$.

□

3. The left adjoint of derived inflation

From now on, k denotes a field of characteristic $p > 0$.

3.1. **The compact case.** Let $f\colon K_P \twoheadrightarrow K_M$ be a surjective homomorphism of torsion-free, compact p-adic Lie groups with kernel K_U. Note that f is automatically open as a surjection between profinite groups. Being torsion-free p-adic Lie groups, these groups are pro-p, and [Ser65, Cor. (1)] combined with [Laz65, V.2.5.8] shows that the groups K_P, K_M, and K_U are Poincaré, cf. [Ser13, Ch. I, §4.5]. Recall that a pro-p group K is called *Poincaré* of dimension n if $\mathrm{H}^i(K,\mathbb{F}_p)$ is finite for all i, $\dim_{\mathbb{F}_p}\mathrm{H}^n(K,\mathbb{F}_p) = 1$, and the cup-product $\mathrm{H}^i(K,\mathbb{F}_p) \times \mathrm{H}^{n-i}(K,\mathbb{F}_p) \longrightarrow \mathrm{H}^n(K,\mathbb{F}_p)$ is a non-degenerate bilinear form.

Notation 3.1.1. If $K_P' \subseteq K_P$ is an open subgroup, we write $K_M' := f(K_P')$ and $K_U' := K_U \cap K_P'$.

By Proposition 2.3.19 the categories $\mathrm{D}(K_P)$ and $\mathrm{D}(K_M)$ are rigidly-compactly generated. Note that inflation along f yields an exact functor $\mathrm{Inf}_{K_P}^{K_M}\colon \mathrm{Rep}_k(K_M) \longrightarrow \mathrm{Rep}_k(K_P)$ that is strictly monoidal, which means $\mathrm{Inf}_{K_P}^{K_M}(\mathbf{1}) = \mathbf{1}$ and $\mathrm{Inf}_{K_P}^{K_M}(V \otimes_k W) \cong \mathrm{Inf}_{K_P}^{K_M}(V) \otimes_k \mathrm{Inf}_{K_P}^{K_M}(W)$, for all $V, W \in \mathrm{Rep}_k(K_M)$. The right adjoint is given by the functor of invariants $\mathrm{H}^0(K_U,-)$.

Passing to the derived categories, we obtain a strictly monoidal functor

$$\mathrm{Inf}_{K_P}^{K_M}\colon \mathrm{D}(K_M) \longrightarrow \mathrm{D}(K_P)$$

with right adjoint $\mathrm{RH}^0(K_U,-)$. For later use we record the following result:

Lemma 3.1.2 (Projection formula)**.** *The natural map*

$$X \otimes_k \mathrm{RH}^0(K_U,Y) \xrightarrow{\cong} \mathrm{RH}^0\big(K_U, \mathrm{Inf}_{K_P}^{K_M}(X) \otimes_k Y\big)$$

is an isomorphism, for all $X \in \mathrm{D}(K_M)$ and $Y \in \mathrm{D}(K_P)$. In particular, $X \otimes_k \mathrm{RH}^0(K_U,\mathbf{1}) \xrightarrow{\cong} \mathrm{RH}^0\big(K_U, \mathrm{Inf}_{K_P}^{K_M} X\big)$, for all $X \in \mathrm{D}(K_M)$.

Proof. This follows from [BDS16, 2.15. Prop.] together with Proposition 2.3.19. □

Our goal in this section is to prove the existence of the left adjoint of $\mathrm{Inf}_{K_P}^{K_M}$ and to determine it explicitly. To start, we observe that $\mathrm{RH}^0(K_U,-)$ admits a right adjoint.

Lemma 3.1.3. *The functor* $\mathrm{RH}^0(K_U,-)\colon \mathrm{D}(K_P)\longrightarrow \mathrm{D}(K_M)$ *admits a right adjoint, denoted* $F_{K_P}^{K_M}$*. More precisely, there is a natural isomorphism*

$$\mathrm{RHom}_{\mathrm{Rep}_k(K_P)}\big(X, F_{K_P}^{K_M}(Y)\big)\cong \mathrm{RHom}_{\mathrm{Rep}_k(K_M)}\big(\mathrm{RH}^0(K_U,X),Y\big) \tag{3.1.4}$$

in $\mathrm{D}(k)$*, for all* $X\in\mathrm{D}(K_P)$ *and* $Y\in\mathrm{D}(K_M)$*.*

Further, for each open subgroup $K_P'\subseteq K_P$*, the following diagram of functors commutes:*

$$\begin{tikzcd} \mathrm{D}(K_M) \arrow[r, "F_{K_P}^{K_M}"] \arrow[d, "\mathrm{Res}_{K_M'}^{K_M}"'] & \mathrm{D}(K_P) \arrow[d, "\mathrm{Res}_{K_P'}^{K_P}"] \\ \mathrm{D}(K_M') \arrow[r, "F_{K_P'}^{K_M'}"'] & \mathrm{D}(K_P'). \end{tikzcd}$$

Proof. The functor $\mathrm{Inf}_{K_M}^{K_M}$ satisfies [BDS16, Hyp. 1.2] and hence [BDS16, Cor. 2.14] shows that the right adjoint $F_{K_P}^{K_M}$ of $\mathrm{RH}^0(K_U,-)$ exists. Then [BDS16, (2.18)] shows that there is a natural isomorphism

$$\mathrm{RH}^0\big(K_U,\mathrm{hom}_{\mathrm{D}(K_P)}(X,F_{K_P}^{K_M}(Y))\big)\xrightarrow{\cong}\mathrm{hom}_{\mathrm{D}(K_M)}\big(\mathrm{RH}^0(K_U,X),Y\big)$$

in $\mathrm{D}(K_M)$. Applying $\mathrm{RH}^0(K_M,-)$, and using $\mathrm{RH}^0(K_M,-)\circ\mathrm{RH}^0(K_U,-)\cong\mathrm{RH}^0(K_P,-)$ and (2.3.18), yields the isomorphism (3.1.4).

Let now $K_P'\subseteq K_P$ be an open subgroup. It is obvious that there is a natural isomorphism $\mathrm{Inf}_{K_P'}^{K_M'}\mathrm{Res}_{K_M'}^{K_M}\overset{\cong}{\Longrightarrow}\mathrm{Res}_{K_P'}^{K_P}\mathrm{Inf}_{K_P}^{K_M}$. Passing to the right adjoints, see Example 2.1.13, yields a natural isomorphism $\mathrm{RH}^0(K_U,-)\,\mathrm{Ind}_{K_P'}^{K_P}\overset{\cong}{\Longrightarrow}\mathrm{Ind}_{K_M'}^{K_M}\mathrm{RH}^0(K_U',-)$. Since $K_P'\subseteq K_P$ is open, the functor $\mathrm{Res}_{K_P'}^{K_P}$ is also right adjoint to $\mathrm{Ind}_{K_P'}^{K_P}$. Similarly, $\mathrm{Res}_{K_M'}^{K_M}$ is the right adjoint of $\mathrm{Ind}_{K_M'}^{K_M}$. Hence, passing to the right adjoints again yields a natural isomorphism $F_{K_P'}^{K_M'}\mathrm{Res}_{K_M'}^{K_M}\overset{\cong}{\Longrightarrow}\mathrm{Res}_{K_P'}^{K_P}F_{K_P}^{K_M}$. □

Lemma 3.1.5. *Let* $f'\colon K_M\twoheadrightarrow K_L$ *be another surjective homomorphism of torsion-free* p*-adic Lie groups. Then the following diagram of functors commutes:*

$$\begin{tikzcd} \mathrm{D}(K_L) \arrow[r, "F_{K_M}^{K_L}"] \arrow[dr, "F_{K_P}^{K_L}"'] & \mathrm{D}(K_M) \arrow[d, "F_{K_P}^{K_M}"] \\ & \mathrm{D}(K_P) \end{tikzcd}$$

Proof. Clearly, we have a natural isomorphism $\mathrm{Inf}_{K_P}^{K_M}\mathrm{Inf}_{K_M}^{K_L}\overset{\cong}{\Longrightarrow}\mathrm{Inf}_{K_P}^{K_L}$. Passing to the right adjoints, cf. Example 2.1.13, yields an isomorphism $\mathrm{RH}^0(\mathrm{Ker}(f'\circ f),-)\overset{\cong}{\Longrightarrow}\mathrm{RH}^0(\mathrm{Ker}(f'),-)\circ\mathrm{RH}^0(\mathrm{Ker}(f),-)$. The statement follows by passing to the right adjoints again. □

Proposition 3.1.6. *The functor* $\mathrm{RH}^0(K_U,-)\colon\mathrm{D}(K_P)\longrightarrow\mathrm{D}(K_M)$ *preserves compact objects.*

Proof. Since K_U a Poincaré group, the object $\mathrm{RH}^0(K_U,\mathbf{1})\in\mathrm{D}(K_M)$ is represented by a bounded complex with finite-dimensional cohomologies. Since $\mathbf{1}\in\mathrm{D}(K_M)$ is compact by Proposition 2.3.8, and because the subcategory of compact objects is triangulated, it suffices to show that $\mathrm{RH}^0(K_U,\mathbf{1})$ is contained in the strictly full triangulated subcategory $\langle\mathbf{1}\rangle$ of $\mathrm{D}(K_M)$ generated by $\mathbf{1}$. This follows from a standard argument which we recall for the benefit of the reader. Let $X\in\mathrm{D}(K_M)$ be bounded with finite-dimensional cohomology groups. There exist integers $a\le b$ with $\tau^{\le i}X\cong 0$ for $i<a$ and $\tau^{\ge i}X\cong 0$ for $i>b$. Then $\tau^{\ge b}X\cong\mathrm{H}^b(X)[-b]$ is finite-dimensional and concentrated in degree b. We obtain a distinguished triangle $\tau^{\leq b-1}X \longrightarrow X \longrightarrow \mathrm{H}^b(X)[-b] \xrightarrow{+}$. Hence, by induction on $b-a$ we are reduced to the case where X is a finite-dimensional smooth K_M-representation concentrated in one degree, say $X \cong V[i]$. Then $\mathrm{H}^0(K_M, V)$ is non-zero (since K_M is a pro-p group) and contained in $\langle \mathbf{1} \rangle$, because $\langle \mathbf{1} \rangle$ is closed under finite direct sums. By induction on $\dim_k V$ also the quotient $V/\mathrm{H}^0(K_M, V)$ is contained in $\langle \mathbf{1} \rangle$. As $\langle \mathbf{1} \rangle$ is triangulated and $\mathrm{H}^0(K_M, V) \longrightarrow V \longrightarrow V/\mathrm{H}^0(K_M, V) \xrightarrow{+}$ is a distinguished triangle in $\mathrm{D}(K_M)$, it follows that V, hence also X, is contained in $\langle \mathbf{1} \rangle$. □

We draw two important consequences of Proposition 3.1.6, but first we make a definition.

Definition. The object $\omega_{K_P} := F_{K_P}^{K_M}(\mathbf{1})$ in $\mathrm{D}(K_P)$ is called the *dualizing object*.

Corollary 3.1.7 (Grothendieck duality)**.** *For each $X \in \mathrm{D}(K_M)$ there is a natural isomorphism*

$$\omega_{K_P} \otimes_k \mathrm{Inf}_{K_P}^{K_M}(X) \xrightarrow{\cong} F_{K_P}^{K_M}(X).$$

Proof. This follows from Proposition 3.1.6 and [BDS16, 3.2. Prop.]. □

Corollary 3.1.8. *The functor $\mathrm{Inf}_{K_P}^{K_M} : \mathrm{D}(K_M) \longrightarrow \mathrm{D}(K_P)$ preserves small products. In particular, it admits a left adjoint $\mathrm{L}_{K_U} : \mathrm{D}(K_P) \longrightarrow \mathrm{D}(K_M)$. Moreover, there is a natural isomorphism*

$$\mathrm{RH}^0(K_U, \omega_{K_P} \otimes_k -) \xRightarrow{\cong} \mathrm{L}_{K_U}. \tag{3.1.9}$$

Proof. By Proposition 3.1.6 the adjunction $\mathrm{Inf}_{K_P}^{K_M} \dashv \mathrm{RH}^0(K_U, -)$ restricts to an adjunction on the full subcategories of compact objects $\mathrm{D}(K_M)^{\mathrm{c}} \subseteq \mathrm{D}(K_M)$ and $\mathrm{D}(K_P)^{\mathrm{c}} \subseteq \mathrm{D}(K_P)$. The argument in [BDS16, Lem. 3.1] shows that the inflation functor $\mathrm{D}(K_M)^{\mathrm{c}} \longrightarrow \mathrm{D}(K_P)^{\mathrm{c}}$ admits a left adjoint, and then [BDS16, Lem. 2.6(b)] shows that $\mathrm{Inf}_{K_P}^{K_M}$ preserves small products. By the Corollary 2.3.10.(b) to Brown representability it follows that the left adjoint L_{K_U} exists. The isomorphism (3.1.9) is the ur-Wirthmüller isomorphism [BDS16, (3.10)]. □

To finish our discussion of L_{K_U} we determine the dualizing complex ω_{K_P} explicitly.

Proposition 3.1.10. *One has $\omega_{K_P} \cong k[\dim K_U]$ in $\mathrm{D}(K_P)$. In particular, ω_{K_P} is $\otimes$-invertible.*

Proof. We apply Lemma 3.1.5 to the surjection $f' : K_M \longrightarrow \{1\}$, and together with Corollary 3.1.7 we obtain isomorphisms

$$F_{K_P}^1(\mathbf{1}) \cong F_{K_P}^{K_M}\big(F_{K_M}^1(\mathbf{1})\big) \cong \omega_{K_P} \otimes_k \mathrm{Inf}_{K_P}^{K_M}\big(F_{K_M}^1(\mathbf{1})\big).$$

Hence, if we can show $F_{K_P}^1(\mathbf{1}) \cong k[\dim K_P]$ and $F_{K_M}^1(\mathbf{1}) \cong k[\dim K_M]$, the assertion follows from $\dim K_P = \dim K_M + \dim K_U$, cf. [DDMS99, 4.8 Thm.].

It remains to prove that $F_K^1(\mathbf{1}) \cong k[\dim K]$ provided K is a torsion-free p-adic Lie group.

We have the following isomorphisms in $\mathrm{D}(k)$:

$$\begin{aligned}
\mathrm{Res}_1^K F_K^1(\mathbf{1}) &\cong \varinjlim_{K' \subseteq K} \mathrm{RHom}_{\mathrm{Rep}_k(K')}\big(\mathbf{1}, \mathrm{Res}_{K'}^K F_K^1(\mathbf{1})\big) && \text{(Prop. 2.3.5)}\\
&\cong \varinjlim_{K' \subseteq K} \mathrm{RHom}_{\mathrm{Rep}_k(K')}\big(\mathbf{1}, F_{K'}^1(\mathbf{1})\big) && \text{(Lem. 3.1.3)}\\
&\cong \varinjlim_{K' \subseteq K} \mathrm{RHom}_{\mathrm{Vect}_k}\big(\mathrm{RH}^0(K', \mathbf{1}), \mathbf{1}\big) && \text{(by (3.1.4)).}
\end{aligned}$$

Passing to the i-th cohomology yields isomorphisms

$$\begin{aligned}
\mathrm{H}^i\big(F_K^1(\mathbf{1})\big) &\cong \varinjlim_{K' \subseteq K} \mathrm{Hom}_{\mathrm{D}(k)}\big(\mathrm{RH}^0(K', \mathbf{1}), \mathbf{1}[i]\big)\\
&\cong \varinjlim_{K' \subseteq K} \mathrm{Hom}_k\big(\mathrm{H}^{-i}(K', \mathbf{1}), k\big).
\end{aligned}$$

Here, the transition maps are induced by the corestriction

$$\mathrm{cores} : \mathrm{H}^{-i}(K'', \mathbf{1}) \longrightarrow \mathrm{H}^{-i}(K', \mathbf{1}),$$

for $K'' \subseteq K'$. Now, (5) in the proof of [Ser13, Ch. I, Prop. 30] implies $\mathrm{H}^i\big(F_K^1(\mathbf{1})\big) = 0$ provided $i \neq -\dim K$. By (4) in the proof of *loc. cit.* the above corestriction maps are isomorphisms for $i = -\dim K$. Therefore, we have $F_K^1(\mathbf{1}) \cong \mathrm{H}^{\dim K}(K, \mathbf{1})^*[\dim K]$. Since $\mathrm{H}^{\dim K}(K, \mathbf{1})$ is one-dimensional, it follows that $F_K^1(\mathbf{1}) \cong \delta[\dim K]$, for some smooth character $\delta\colon K \longrightarrow k^\times$. Since K is a pro-p group and $\operatorname{char} k = p$, it follows that δ is the trivial character, which proves $F_K^1(\mathbf{1}) \cong k[\dim K]$. □

Proposition 3.1.10 verifies condition (W4) in [BDS16, 1.9. Thm.]. Precisely, this means:

Corollary 3.1.11. *Put $f^{(n)} \coloneqq \omega_{K_P}^{\otimes n} \otimes_k \operatorname{Inf}_{K_P}^{K_M}$ and $f_{(n)} \coloneqq \mathrm{RH}^0\big(K_U, \omega_{K_P}^{\otimes n} \otimes_k -\big)$, for $n \in \mathbb{Z}$. There is an infinite chain of adjunctions*

$$\cdots \dashv f^{(-1)} \dashv f_{(1)} \dashv f^{(0)} \dashv f_{(0)} \dashv f^{(1)} \dashv f_{(-1)} \dashv \cdots$$

3.2. **The general case.** In this section we fix an open surjective morphism $f\colon S \twoheadrightarrow T$ of p-adic monoids with kernel U. The functor $\operatorname{Inf}_S^T\colon \mathrm{D}(T) \longrightarrow \mathrm{D}(S)$ is strictly monoidal and admits the right adjoint $\mathrm{R}\Pi_U$, cf. Proposition 2.2.13 and Lemma 2.3.3.(c). The goal of this section is to show that Inf_S^T also admits a left adjoint and study its properties. We make the following observation:

Proposition 3.2.1. *For all $X \in \mathrm{D}(S)$ and $Y \in \mathrm{D}(T)$ the natural map*

$$\hom_{\mathrm{D}(T)}\big(Y, \mathrm{R}\Pi_U(X)\big) \xrightarrow{\cong} \mathrm{R}\Pi_U\big(\hom_{\mathrm{D}(S)}(\operatorname{Inf}_S^T Y, X)\big)$$

is an isomorphism in $\mathrm{D}(T)$.

Proof. Let $W \in \mathrm{D}(T)$. Since Inf_S^T is strictly monoidal, the natural transformation

$$(\operatorname{Inf}_S^T Y \otimes_k -) \circ \operatorname{Inf}_S^T \overset{\cong}{\Longrightarrow} \operatorname{Inf}_S^T \circ (Y \otimes_k -)$$

is an isomorphism of functors $\mathrm{D}(T) \longrightarrow \mathrm{D}(S)$. Passing to the right adjoints, see Example 2.1.13, shows that the natural map

$$\hom_{\mathrm{D}(T)}(Y, -) \circ \mathrm{R}\Pi_U \overset{\cong}{\Longrightarrow} \mathrm{R}\Pi_U \circ \hom_{\mathrm{D}(S)}(\operatorname{Inf}_S^T Y, -)$$

is an isomorphism of functors $\mathrm{D}(S) \longrightarrow \mathrm{D}(T)$. □

Lemma 3.2.2. *Let $K_S \subseteq S$ be an open subgroup. The restriction functor $\operatorname{Res}_{K_S}^S\colon \mathrm{D}(S) \longrightarrow \mathrm{D}(K_S)$ is conservative and preserves small products and small coproducts.*

Proof. Note that if φ is a morphism in $\mathrm{C}(S)$, then $\operatorname{Res}_{K_S}^S(\varphi)$ is a quasi-isomorphism if and only if φ is a quasi-isomorphism. This implies that $\operatorname{Res}_{K_S}^S\colon \mathrm{D}(S) \longrightarrow \mathrm{D}(K_S)$ is conservative.

Since $K_S \subseteq S$ is an open subgroup, the functor $\operatorname{Res}_{K_S}^S\colon \operatorname{Rep}_k(S) \longrightarrow \operatorname{Rep}_k(K_S)$ admits an exact left adjoint $\operatorname{ind}_{K_S}^S$ and a left exact right adjoint $\operatorname{Ind}_{K_S}^S$, by Lemmas 2.2.9 and 2.2.5, respectively. But then the induced functors $\operatorname{ind}_{K_S}^S, \operatorname{RInd}_{K_S}^S\colon \mathrm{D}(K_S) \longrightarrow \mathrm{D}(S)$ are left adjoint, resp. right adjoint, to $\operatorname{Res}_{K_S}^S$. By Proposition 2.1.7, $\operatorname{Res}_{K_S}^S$ preserves small products and small coproducts. □

Theorem 3.2.3. *The functor $\operatorname{Inf}_S^T\colon \mathrm{D}(T) \longrightarrow \mathrm{D}(S)$ preserves small products. In particular, it admits a left adjoint, denoted L_U.*

Proof. Let $K_S \subseteq S$ be an open, compact, torsion-free p-adic Lie group such that $K_T \coloneqq f(K_S) \subseteq T$ is open and torsion-free.

Let I be a set, and take $Y_i \in \mathrm{D}(T)$, for $i \in I$. We show that the natural map

$$\alpha_S\colon \operatorname{Inf}_S^T \prod_{i\in I} Y_i \longrightarrow \prod_{i \in I} \operatorname{Inf}_S^T Y_i.$$

is an isomorphism in $\mathrm{D}(S)$. Note that $\operatorname{Res}^S_{K_S} \operatorname{Inf}^T_S \cong \operatorname{Inf}^{K_T}_{K_S} \operatorname{Res}^T_{K_T}$ as functors $\mathrm{D}(T) \longrightarrow \mathrm{D}(K_S)$. By Lemma 3.2.2 the functors $\operatorname{Res}^S_{K_S}$ and $\operatorname{Res}^T_{K_T}$ preserve small products. Now, the diagram

$$\begin{tikzcd}
\operatorname{Res}^S_{K_S} \operatorname{Inf}^T_S \prod_{i\in I} Y_i \arrow[r, "\operatorname{Res}^S_{K_S}(\alpha_S)"] \arrow[d, "\cong"'] & \operatorname{Res}^S_{K_S} \prod_{i\in I} \operatorname{Inf}^T_S Y_i \arrow[d, "\cong"] \\
\operatorname{Inf}^{K_T}_{K_S} \prod_{i\in I} \operatorname{Res}^T_{K_T} Y_i \arrow[r, "\alpha_{K_S}"'] & \prod_{i\in I} \operatorname{Inf}^{K_T}_{K_S} \operatorname{Res}^S_{K_S} Y_i
\end{tikzcd}$$

is commutative. By Corollary 3.1.8 the bottom map is an isomorphism in $\mathrm{D}(K_S)$. Therefore, the top map $\operatorname{Res}^S_{K_S}(\alpha_S)$ is an isomorphism in $\mathrm{D}(K_S)$. By Lemma 3.2.2 again, $\operatorname{Res}^S_{K_S}$ is conservative, which shows that α_S is an isomorphism in $\mathrm{D}(S)$. Hence, Inf^T_S preserves small products. By the Corollary 2.3.10.(b) of Brown representability, it follows that Inf^T_S admits a left adjoint. $\square$

In fact, the adjunction $\mathrm{L}_U \dashv \operatorname{Inf}^T_S$ holds on the level of RHom complexes:

Corollary 3.2.4. *There is a natural isomorphism*

$$\mathrm{RHom}_{\mathrm{Rep}_k(T)}\big(\mathrm{L}_U(X), Y\big) \xrightarrow{\cong} \mathrm{RHom}_{\mathrm{Rep}_k(S)}\big(X, \operatorname{Inf}^T_S Y\big) \qquad \text{in } \mathrm{D}(k). \tag{3.2.5}$$

Proof. Since Inf^T_S is an exact functor, it induces a natural morphism

$$\mathrm{RHom}_{\mathrm{Rep}_k(T)}(Y_1, Y_2) \longrightarrow \mathrm{RHom}_{\mathrm{Rep}_k(S)}\big(\operatorname{Inf}^T_S Y_1, \operatorname{Inf}^T_S Y_2\big), \tag{3.2.6}$$

for all $Y_1, Y_2 \in \mathrm{D}(T)$. Let now $X \in \mathrm{D}(S)$ and $Y \in \mathrm{D}(T)$, and denote $\eta_X \colon X \longrightarrow \operatorname{Inf}^T_S \mathrm{L}_U(X)$ the unit of the adjunction. Then (3.2.5) is given as the composite

$$\begin{aligned}
\mathrm{RHom}_{\mathrm{Rep}_k(T)}\big(\mathrm{L}_U(X), Y\big) &\xrightarrow{(3.2.6)} \mathrm{RHom}_{\mathrm{Rep}_k(S)}\big(\operatorname{Inf}^T_S \mathrm{L}_U(X), \operatorname{Inf}^T_S Y\big) \\
&\xrightarrow{(\eta_V)^*} \mathrm{RHom}_{\mathrm{Rep}_k(S)}\big(X, \operatorname{Inf}^T_S Y\big).
\end{aligned}$$

On the n-th cohomology it is given by the adjunction isomorphism

$$\mathrm{Hom}_{\mathrm{D}(T)}\big(\mathrm{L}_U(X), Y[n]\big) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{D}(S)}\big(X, \operatorname{Inf}^T_S Y[n]\big).$$

Therefore, (3.2.5) is an isomorphism in $\mathrm{D}(k)$. $\square$

Notation 3.2.7. Given $n \in \mathbb{Z}$, we denote by $\mathrm{L}^n_U \colon \mathrm{Rep}_k(S) \longrightarrow \mathrm{Rep}_k(T)$ the n-th cohomology functor of L_U, *i.e.*,

$$\mathrm{L}^n_U(V) := \mathrm{H}^n(\mathrm{L}_U V[0]), \qquad \text{for all } V \in \mathrm{Rep}_k(S).$$

Proposition 3.2.8. *The functor $\mathrm{L}_U \colon \mathrm{D}(S) \longrightarrow \mathrm{D}(T)$ is right t-exact. More precisely, we have $\mathrm{L}^n_U = 0$ for all $n > 0$, and L^0_U is the left adjoint of $\operatorname{Inf}^T_S \colon \mathrm{Rep}_k(T) \longrightarrow \mathrm{Rep}_k(S)$.*

Proof. The right t-exactness is a formal consequence of the (left) t-exactness of Inf^T_S. Let us show that L^0_U is the left adjoint of Inf^T_S. Let $\mathrm{D}^{\geq 0}(T)$ be the full subcategory of $\mathrm{D}(T)$ consisting of complexes Y with $\mathrm{H}^n(Y) = 0$ for all $n < 0$. The truncation functor $\tau^{\geq 0} \colon \mathrm{D}(T) \longrightarrow \mathrm{D}^{\geq 0}(T)$ is left adjoint to the inclusion. Hence, given $V \in \mathrm{Rep}_k(S)$ and $W \in \mathrm{Rep}_k(T)$, we compute

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Rep}_k(T)}\big(\mathrm{L}^0_U(V), W\big) &\cong \mathrm{Hom}_{\mathrm{D}^{\geq 0}(T)}\big(\tau^{\geq 0}\mathrm{L}_U(V), W\big) \\
&= \mathrm{Hom}_{\mathrm{D}(T)}\big(\mathrm{L}_U(V), W\big) \\
&\cong \mathrm{Hom}_{\mathrm{D}(S)}\big(V, \operatorname{Inf}^T_S W\big) \\
&\cong \mathrm{Hom}_{\mathrm{Rep}_k(S)}(V, \operatorname{Inf}^T_S W).
\end{aligned}$$

$\square$

Remark. Proposition 3.2.8 implies that there is a natural transformation

$$\mathrm{L}_U \circ \mathbf{q}_S \Longrightarrow \mathbf{q}_T \circ \mathrm{L}^0_U$$

of functors $\mathrm{K}(S) \longrightarrow \mathrm{D}(T)$. I do not know whether the left derived functor of L^0_U exists and, if it does exist, whether it coincides with L_U.

Corollary 3.2.9. *Given $V \in \mathrm{Rep}_k(S)$ and $W \in \mathrm{Rep}_k(T)$, there is a convergent first-quadrant spectral sequence*

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathrm{Rep}_k(T)}\big(\mathrm{L}_U^{-j}(V), W\big) \Longrightarrow \mathrm{Ext}^{i+j}_{\mathrm{Rep}_k(S)}\big(V, \mathrm{Inf}_S^T W\big).$$

In particular, there is a five-term exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}^1_{\mathrm{Rep}_k(T)}\big(\mathrm{L}_U^0(V), W\big) \arrow[r] & \mathrm{Ext}^1_{\mathrm{Rep}_k(S)}\big(V, \mathrm{Inf}_S^T W\big) \arrow[r] & \mathrm{Hom}_{\mathrm{Rep}_k(T)}\big(\mathrm{L}_U^{-1}(V), W\big) \arrow[dll, "d_2^{0,1}"] \\ & \mathrm{Ext}^2_{\mathrm{Rep}_k(T)}\big(\mathrm{L}_U^0(V), W\big) \arrow[r] & \mathrm{Ext}^2_{\mathrm{Rep}_k(S)}\big(V, \mathrm{Inf}_S^T W\big). \end{tikzcd}$$

Proof. The functor $\mathrm{RHom}_{\mathrm{Rep}_k(T)}(-, W)$ is left t-exact and $\mathrm{L}_U(V)$ is right bounded by Proposition 3.2.8. Now, Lemma 2.3.20.(b) together with Corollary 3.2.4 yields the desired spectral sequence. □

Proposition 3.2.10. *Let $f' \colon T \twoheadrightarrow T'$ be another open surjective morphism of p-adic monoids. Write $U' = \mathrm{Ker}(f')$ and $U'' = \mathrm{Ker}(f' \circ f)$.*

(a) The diagram

$$\begin{tikzcd} \mathrm{D}(S) \arrow[r, "\mathrm{L}_U"] \arrow[dr, "\mathrm{L}_{U''}"'] & \mathrm{D}(T) \arrow[d, "\mathrm{L}_{U'}"] \\ & \mathrm{D}(T') \end{tikzcd}$$

is commutative.

(b) Given $V \in \mathrm{Rep}_k(S)$, there is a convergent third-quadrant spectral sequence

$$E_2^{i,j} = \mathrm{L}^i_{U'}\big(\mathrm{L}^j_U(V)\big) \Longrightarrow \mathrm{L}^{i+j}_{U''}(V).$$

In particular, there is a five-term exact sequence

$$\mathrm{L}^{-2}_{U''}(V) \longrightarrow \mathrm{L}^{-2}_{U'}\big(\mathrm{L}^0_U(V)\big) \xrightarrow{d^{-2,0}} \mathrm{L}^0_{U'}\big(\mathrm{L}^{-1}_U(V)\big) \longrightarrow \mathrm{L}^{-1}_{U''}(V) \longrightarrow \mathrm{L}^{-1}_{U'}\big(\mathrm{L}^0_U(V)\big) \longrightarrow 0.$$

Proof. (a) follows from the natural isomorphism $\mathrm{Inf}_S^T \mathrm{Inf}_T^{T'} \overset{\cong}{\Longrightarrow} \mathrm{Inf}_S^{T'}$ by passing to the left adjoints, see Example 2.1.13. Now, (b) follows from (a) by applying Lemma 2.3.20.(a) and observing that the functors $\mathrm{L}_{U'}$ and L_U are right t-exact by Proposition 3.2.8. □

Proposition 3.2.11. *Let $S' \subseteq S$ be an open submonoid and put $T' := f(S') \subseteq T$. Put $U' := U \cap S'$. Assume further that $k[S]$ is flat over $k[S']$ and $k[T]$ is flat over $k[T']$. The following diagram is commutative:*

$$\begin{tikzcd} \mathrm{D}(S') \arrow[r, "\mathrm{L}_{U'}"] \arrow[d, "\mathrm{ind}_{S'}^S"'] & \mathrm{D}(T') \arrow[d, "\mathrm{ind}_{T'}^T"] \\ \mathrm{D}(S) \arrow[r, "\mathrm{L}_U"'] & \mathrm{D}(T). \end{tikzcd}$$

In particular, there is, for each $n \in \mathbb{Z}$, a natural isomorphism $\mathrm{ind}_{T'}^T \mathrm{L}^n_{U'} \overset{\cong}{\Longrightarrow} \mathrm{L}^n_U \mathrm{ind}_{S'}^S$ of functors $\mathrm{Rep}_k(S') \longrightarrow \mathrm{Rep}_k(T)$.

Proof. Clearly there is a natural isomorphism $\mathrm{Res}^S_{S'} \mathrm{Inf}_S^T \overset{\cong}{\Longrightarrow} \mathrm{Inf}_{S'}^{T'} \mathrm{Res}^T_{T'}$. Passing to the left adjoints, see Example 2.1.13, yields the first assertion. The second assertion follows from the first by passing to the n-th cohomology and using that $\mathrm{ind}_{T'}^T$ and $\mathrm{ind}_{S'}^S$ are exact. □

Proposition 3.2.12. *Let $\chi \colon T \longrightarrow k^\times$ be a smooth character viewed as a character of S via inflation along f. There is a natural isomorphism $\mathrm{L}_U(\chi \otimes_k V) \overset{\cong}{\longrightarrow} \chi \otimes_k \mathrm{L}_U(V)$, for each $V \in \mathrm{D}(S)$.*

Proof. The functor $\chi \otimes_k -$ is an equivalence of triangulated categories with quasi-inverse $\chi^{-1} \otimes_k -$. In particular, $\chi \otimes_k -$ is left adjoint to $\chi^{-1} \otimes_k -$. Moreover, there is a natural isomorphism $\operatorname{Inf}_S^T(\chi^{-1} \otimes_k W) \xrightarrow{\cong} \chi^{-1} \otimes_k \operatorname{Inf}_S^T(W)$, for $W \in \mathrm{D}(T)$. Passing to the left adjoints, see Example 2.1.13, yields the result. $\square$

Remark 3.2.13. If S and T are p-adic Lie groups, we will obtain in Corollary 3.3.5 a stronger version of the above proposition.

Next, we will study how L_U behaves under restriction. This requires some preparatory lemmas. We will employ a version of the Mackey decomposition for derived categories.

Notation 3.2.14. Let G be locally profinite group and $S \subseteq G$ a closed submonoid. Given $g \in G$ and $V \in \operatorname{Rep}_k(S)$, we define a smooth representation of gSg^{-1} on V via $(gsg^{-1}) \star v := sv$, for all $s \in S$, $v \in V$. We denote this modified representation by g_*V. In this way, we obtain a functor

$$g_*\colon \operatorname{Rep}_k(S) \longrightarrow \operatorname{Rep}_k(gSg^{-1})$$

which is easily seen to be an equivalence of categories. The induced equivalence on the derived categories is again denoted g_*.

Lemma 3.2.15 (derived Mackey decomposition)**.** *Assume that S is an open submonoid of a locally profinite group. Let $K, H \subseteq S$ be two subgroups with K open and H closed.*

(a) There is a natural isomorphism of functors $\mathrm{D}(H) \longrightarrow \mathrm{D}(K)$,

$$\operatorname{Res}_K^S \operatorname{RInd}_H^S \xRightarrow{\cong} \prod_{s \in H\backslash S/K} \operatorname{Ind}_{s^{-1}Hs\cap K}^K s_*^{-1} \operatorname{Res}_{H\cap sKs^{-1}}^H .$$

(b) There is a natural isomorphism of functors $\mathrm{D}(K) \longrightarrow \mathrm{D}(H)$,

$$\operatorname{Res}_H^S \operatorname{ind}_K^S \xLeftarrow{\cong} \bigoplus_{s \in H\backslash S/K} \operatorname{ind}_{H\cap sKs^{-1}}^H s_* \operatorname{Res}_{s^{-1}Hs\cap K}^K .$$

Proof. Note that $\operatorname{Ind}_{s^{-1}Hs\cap K}^K$ is exact by Remark 2.2.6. Both statements follow from the Mackey decomposition, [Vig96, I.5.5], by passing to the derived categories. For (a) we observe that the functors $\operatorname{Res}_{H\cap sKs^{-1}}^H$, s_*^{-1}, and $\operatorname{Ind}_{s^{-1}Hs\cap K}^K$ preserve K-injective complexes by Lemma 2.3.3.(b), since they all admit exact left adjoints. $\square$

Recall that we denoted by U the kernel of $f\colon S \twoheadrightarrow T$.

Lemma 3.2.16. *Let $H_S \subseteq S$ be a subgroup and put $H_T := f(H_S)$. Assume that f induces a bijection $H_S\backslash S \xrightarrow{\cong} H_T\backslash T$.*

(a) One has $sU \subseteq H_S s$, for all $s \in S$. In particular, $U \subseteq H_S$ is a group.

(b) Let $K_S \subseteq S$ be a subgroup and put $K_T := f(K_S)$. The map f induces a bijection $H_S\backslash S/K_S \xrightarrow{\cong} H_T\backslash T/K_T$.

Proof. The first assertion in (a) is immediate from the hypothesis, because $f(sU) = \{f(s)\} \subseteq H_T f(s)$ is contained in a single coset. For $s = 1$ this shows $U \subseteq H_S$. Thus, $U = U \cap H_S$ is the kernel of the group homomorphism $f\big|_{H_S}$, hence a group. For (b), note that K_S acts on $H_S\backslash S$ by right multiplication. By (a), this action factors through K_T, and this implies the assertion. $\square$

Lemma 3.2.17. *Let K_P be a profinite group and $K_U \subseteq K_P' \subseteq K_P$ closed subgroups with K_U normal in K_P. Write $K_M := K_P/K_U$ and $K_M' := K_P'/K_U$.*

The natural transformation

$$\operatorname{Inf}_{K_P}^{K_M} \operatorname{Ind}_{K_M'}^{K_M} \xRightarrow{\cong} \operatorname{Ind}_{K_P'}^{K_P} \operatorname{Inf}_{K_P'}^{K_M'} \tag{3.2.18}$$

is an isomorphism of functors $\mathrm{D}(K_M') \longrightarrow \mathrm{D}(K_P)$.

Proof. It suffices to show that (3.2.18) is an isomorphism of functors $\mathrm{Rep}_k(K'_M) \longrightarrow \mathrm{Rep}_k(K_P)$, since by Remark 2.2.6, the underived functors $\mathrm{Ind}_{K'_M}^{K_M}$ and $\mathrm{Ind}_{K'_P}^{K_P}$ are exact. But this can be checked directly or by observing that the natural map $\mathrm{L}^0_{K_U} \mathrm{Res}^{K_P}_{K'_P} \xRightarrow{\cong} \mathrm{Res}^{K_M}_{K'_M} \mathrm{L}^0_{K_U}$ is obviously an isomorphism of functors $\mathrm{Rep}_k(K_P) \longrightarrow \mathrm{Rep}_k(K'_M)$, and then passing to the right adjoints, see Example 2.1.13. □

For the rest of this section we assume that S and T are open submonoids of p-adic Lie groups P and M, respectively, and that f arises from a continuous surjection $P \twoheadrightarrow M$. This condition will be satisfied for the p-adic monoids we consider. Note that, since M with the quotient topology is a p-adic Lie group, [DDMS99, Thm. 9.6 (ii)], and the p-adic analytic structure on M is unique, [DDMS99, Cor. 9.5], it follows that $P \twoheadrightarrow M$ is a quotient map of topological groups, hence is open. Therefore, f is automatically open.

Proposition 3.2.19. *Let $H_S \subseteq S$ be a closed subgroup and put $H_T := f(H_S) \subseteq T$. Assume that f induces a bijection $H_S\backslash S \xrightarrow{\cong} H_T\backslash T$. The diagrams*

$$\begin{tikzcd} \mathrm{D}(H_T) \arrow[r, "\mathrm{Inf}_{H_S}^{H_T}"] \arrow[d, "\mathrm{RInd}_{H_T}^T"'] & \mathrm{D}(H_S) \arrow[d, "\mathrm{RInd}_{H_S}^S"] \\ \mathrm{D}(T) \arrow[r, "\mathrm{Inf}_S^T"'] & \mathrm{D}(S) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{D}(S) \arrow[r, "\mathrm{L}_U"] \arrow[d, "\mathrm{Res}_{H_S}^S"'] & \mathrm{D}(T) \arrow[d, "\mathrm{Res}_{H_T}^T"] \\ \mathrm{D}(H_S) \arrow[r, "\mathrm{L}_U"'] & \mathrm{D}(H_T) \end{tikzcd}$$

commute.

Proof. Note that H_T is closed, since Lemma 3.2.16.(a) implies $U \subseteq H_S$ so that $T \smallsetminus H_T = f(S \smallsetminus H_S)$ is open. Consider the natural transformation

$$\beta\colon \mathrm{Inf}_S^T \mathrm{RInd}_{H_T}^T \Longrightarrow \mathrm{RInd}_{H_S}^S \mathrm{Inf}_{H_S}^{H_T},$$

which arises as the right mate of the natural map $\mathrm{Res}^S_{H_S} \mathrm{Inf}^T_S \xRightarrow{\cong} \mathrm{Inf}^{H_T}_{H_S} \mathrm{Res}^T_{H_T}$ by Proposition 2.1.9. It suffices to show that β is an isomorphism, because then its mate $\mathrm{L}_U \mathrm{Res}^S_{H_S} \Longrightarrow \mathrm{Res}^T_{H_T} \mathrm{L}_U$ is an isomorphism as well by Example 2.1.13, and hence both diagrams commute.

Fix an open profinite subgroup $K_S \subseteq S$ so that $K_T := f(K_S)$ is open in T. Since $\mathrm{Res}^S_{K_S}$ is conservative by Lemma 3.2.2, it suffices to show that $\mathrm{Res}^S_{K_S}(\beta)$ is an isomorphism. We compute the source and target of $\mathrm{Res}^S_{K_S}(\beta)$ using the derived Mackey decomposition, Lemma 3.2.15, and the fact that $\mathrm{Inf}^{K_T}_{K_S}$ preserves small products, Theorem 3.2.3:

$$\begin{aligned} \mathrm{Res}^S_{K_S} \mathrm{Inf}^T_S \mathrm{RInd}^T_{H_T} &\xRightarrow{\cong} \mathrm{Inf}^{K_T}_{K_S} \mathrm{Res}^T_{K_T} \mathrm{RInd}^T_{H_T} \\ &\xRightarrow{\cong} \mathrm{Inf}^{K_T}_{K_S} \prod_{t \in H_T\backslash T/K_T} \mathrm{Ind}^{K_T}_{t^{-1}H_T t \cap K_T} t_*^{-1} \mathrm{Res}^{H_T}_{H_T \cap tK_T t^{-1}} \\ &\xRightarrow{\cong} \prod_{t \in H_T\backslash T/K_T} \mathrm{Inf}^{K_T}_{K_S} \mathrm{Ind}^{K_T}_{t^{-1}H_T t \cap K_T} t_*^{-1} \mathrm{Res}^{H_T}_{H_T \cap tK_T t^{-1}} \end{aligned}$$

and

$$\mathrm{Res}^S_{K_S} \mathrm{RInd}^S_{H_S} \mathrm{Inf}^{H_T}_{H_S} \xRightarrow{\cong} \prod_{s \in H_S\backslash S/K_S} \mathrm{Ind}^{K_S}_{s^{-1}H_S s \cap K_S} s_*^{-1} \mathrm{Res}^{H_S}_{H_S \cap sK_S s^{-1}} \mathrm{Inf}^{H_T}_{H_S}$$

By Lemma 3.2.16 we may identify the double coset spaces $H_S\backslash S/K_S$ and $H_T\backslash T/K_T$.

Let now $s \in S$ and write $\overline{s} := f(s) \in T$. By Lemma 3.2.16.(a), we have $U \cap K_S \subseteq s^{-1}H_S s \cap K_S$, so that $f(s^{-1}H_S s \cap K_S) = \overline{s}^{-1} H_T \overline{s} \cap K_T$. To shorten the notation, we write

$$\Psi_s := \mathrm{Ind}^{K_S}_{s^{-1}H_S s \cap K_S} s_*^{-1} \mathrm{Res}^{H_S}_{H_S \cap sK_S s^{-1}}$$

and

$$\Psi_{\overline{s}} \coloneqq \operatorname{Ind}^{K_T}_{\overline{s}^{-1} H_T \overline{s} \cap K_T} \overline{s}^{-1}_* \operatorname{Res}^{H_T}_{H_T \cap \overline{s} K_T \overline{s}^{-1}}.$$

It follows from Lemma 3.2.17 that the natural map

$$\beta_s \colon \operatorname{Inf}^{K_T}_{K_S} \Psi_{\overline{s}} \overset{\cong}{\Longrightarrow} \Psi_s \operatorname{Inf}^{H_T}_{H_S}$$

is an isomorphism. Let us verify that $\operatorname{Res}^S_{K_S}(\beta)$ corresponds to $\prod_{s \in H_S \backslash S / K_S} \beta_s$ under the above identifications; it then follows that $\operatorname{Res}^S_{K_S}(\beta)$ is an isomorphism, which finishes the proof.

It remains to show that the diagram

$$\begin{tikzcd}
\operatorname{Res}^S_{K_S} \operatorname{Inf}^T_S \operatorname{RInd}^T_{H_T} \arrow[r, Rightarrow, "\cong"] \arrow[dr, Rightarrow, "\operatorname{Res}^S_{K_S}(\beta)"'] & \operatorname{Inf}^{K_T}_{K_S} \operatorname{Res}^T_{K_T} \operatorname{RInd}^T_{H_T} \arrow[r, Rightarrow, "\cong"] \arrow[d, Rightarrow, "\gamma"'] & \prod_{s \in H_S \backslash S / K_S} \operatorname{Inf}^{K_T}_{K_S} \Psi_{\overline{s}} \arrow[d, Rightarrow, "\prod_s \beta_s"] \\
& \operatorname{Res}^S_{K_S} \operatorname{RInd}^S_{H_S} \operatorname{Inf}^{H_T}_{H_S} \arrow[r, Rightarrow, "\cong"] & \prod_{s \in H_S \backslash S / K_S} \Psi_s \operatorname{Inf}^{H_T}_{H_S}
\end{tikzcd}$$

commutes. The left vertical map γ is defined so as to make the triangle commute. The commutativity of the square can be checked componentwise. Therefore, it suffices to show that the diagram

$$\begin{tikzcd}
\operatorname{Inf}^{K_T}_{K_S} \big(\operatorname{Res}^T_{K_T} \operatorname{RInd}^T_{H_T}\big) \arrow[r, Rightarrow, "\operatorname{Inf}^{K_T}_{K_S} \rho_{\overline{s}}"] \arrow[d, Rightarrow, "\gamma"'] & \operatorname{Inf}^{K_T}_{K_S} \Psi_{\overline{s}} \arrow[d, Rightarrow, "\beta_s"] \\
\big(\operatorname{Res}^S_{K_S} \operatorname{RInd}^S_{H_S}\big) \operatorname{Inf}^{H_T}_{H_S} \arrow[r, Rightarrow, "(\rho_s)_{\operatorname{Inf}^{H_T}_{H_S}}"'] & \Psi_s \operatorname{Inf}^{H_T}_{H_S}
\end{tikzcd} \tag{3.2.20}$$

commutes, where $\rho_{\overline{s}}$ and ρ_s are the natural maps coming from the Mackey decomposition. The left mate of ρ_s is the map

$$\lambda_s \colon \Phi_s \coloneqq \operatorname{ind}^{H_S}_{H_S \cap s K_S s^{-1}} s_* \operatorname{Res}^{K_S}_{s^{-1} H_S s \cap K_S} \Longrightarrow \operatorname{Res}^S_{H_S} \operatorname{ind}^S_{K_S}$$

that is the s-th component of the isomorphism in the Mackey decomposition for compact induction. The analogous statement holds for the left mate

$$\lambda_{\overline{s}} \colon \Phi_{\overline{s}} \coloneqq \operatorname{ind}^{H_T}_{H_T \cap \overline{s} K_T \overline{s}^{-1}} \overline{s}_* \operatorname{Res}^{K_T}_{\overline{s}^{-1} H_T \overline{s} \cap K_T} \Longrightarrow \operatorname{Res}^T_{H_T} \operatorname{ind}^T_{K_T}$$

of $\rho_{\overline{s}}$. By Example 2.1.12 the commutativity of (3.2.20) is equivalent to the commutativity of

$$\begin{tikzcd}
\big(\operatorname{Res}^S_{H_S} \operatorname{ind}^S_{K_S}\big) \operatorname{Inf}^{K_T}_{K_S} \arrow[d, Rightarrow, "\alpha"'] & \Phi_s \operatorname{Inf}^{K_T}_{K_S} \arrow[l, Rightarrow, "(\lambda_s)_{\operatorname{Inf}^{K_T}_{K_S}}"'] \arrow[d, Rightarrow, "\alpha_s"] \\
\operatorname{Inf}^{H_T}_{H_S} \big(\operatorname{Res}^T_{H_T} \operatorname{ind}^T_{K_T}\big) & \operatorname{Inf}^{H_T}_{H_S} \Phi_{\overline{s}}, \arrow[l, Rightarrow, "\operatorname{Inf}^{H_T}_{H_S} \lambda_{\overline{s}}"]
\end{tikzcd}$$

where α (resp. α_s) is the left mate of γ (resp. β_s). But this can be checked explicitly on the underived level, because all functors involved are exact. $\square$

Remark 3.2.21. Under the hypotheses of Proposition 3.2.19 it follows that the natural map

$$\mathrm{R}\big(\operatorname{Ind}^S_{H_S} \operatorname{Inf}^{H_T}_{H_S}\big) \overset{\cong}{\Longrightarrow} \operatorname{RInd}^S_{H_S} \operatorname{Inf}^{H_T}_{H_S}$$

is an isomorphism. Thus, if $I \in \operatorname{Rep}_k(H_T)$ is injective, then $\operatorname{Inf}^{H_T}_{H_S} I$ is acyclic for $\operatorname{Ind}^S_{H_S}$.

Remark 3.2.22. One can show the following analog of Proposition 3.2.19: Let $K_S \subseteq S$ be an open subgroup and $K_T \coloneqq f(K_S) \subseteq T$. Assume that f induces a bijection $S/K_S \xrightarrow{\cong} T/K_T$. Then the diagrams

$$\begin{tikzcd} \mathrm{D}(K_T) \arrow[r, "\operatorname{Inf}_{K_S}^{K_T}"] \arrow[d, "\operatorname{ind}_{K_T}^T"'] & \mathrm{D}(K_S) \arrow[d, "\operatorname{ind}_{K_S}^S"] \\ \mathrm{D}(T) \arrow[r, "\operatorname{Inf}_S^T"'] & \mathrm{D}(S) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{D}(S) \arrow[r, "\mathrm{R}\Pi_U"] \arrow[d, "\operatorname{Res}_{K_S}^S"'] & \mathrm{D}(T) \arrow[d, "\operatorname{Res}_{K_T}^T"] \\ \mathrm{D}(K_S) \arrow[r, "\mathrm{R}\Pi_U"'] & \mathrm{D}(K_T) \end{tikzcd}$$

commute. Indeed, the bijection $S/K_S \xrightarrow{\cong} T/K_T$ implies that U is a subgroup of K_S and that the natural map

$$k[S] \otimes_{k[K_S]} \operatorname{Inf}_{K_S}^{K_T}(-) \overset{\cong}{\Longrightarrow} \operatorname{Inf}_S^T \circ (k[T] \otimes_{k[K_T]} -)$$

is an isomorphism of functors $\operatorname{Rep}_k(K_T) \longrightarrow \operatorname{Rep}_k(S)$. As all functors involved are exact, we deduce that the left diagram commutes. The commutativity of the right diagram then follows by passing to the right adjoints, see Example 2.1.13.

Warning 3.2.23. Let $H_S \subseteq S$ be a subgroup and put $H_T := f(H_S)$. Then $H_S\backslash S \xrightarrow[\cong]{f} H_T\backslash T$ does *not* imply (and is not implied by) $S/H_S \xrightarrow[\cong]{f} T/H_T$.

For example, this generally fails for the p-adic monoids considered in §3.4. Let us make this explicit: Consider the p-adic submonoid $S = P^+$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ consisting of upper triangular matrices $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ satisfying $\operatorname{val}_p(a) \geq \operatorname{val}_p(d)$ and $\operatorname{val}_p(b) \geq \operatorname{val}_p(d)$, where val_p denotes the p-adic valuation on $\mathbb{Q}_p$. Let $T = M^+$ be the p-adic submonoid of P^+ consisting of diagonal matrices. The canonical projection $f\colon P^+ \twoheadrightarrow M^+$ (given by forgetting the upper right entry) is surjective with kernel $U = \left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$. Now, $P^+ = UM^+$ and hence $U\backslash P^+ \xrightarrow{\cong} M^+$ via f. On the other hand, for any $m = \operatorname{diag}(a, d)$ in M^+ with $\operatorname{val}_p(a) > \operatorname{val}_p(d)$ we have $mUm^{-1} \subsetneq U$, from which it follows that the map

$$P^+/U = UM^+/U \cong \bigsqcup_{m \in M^+} U/mUm^{-1} \twoheadrightarrow \bigsqcup_{m \in M^+} \{m\} = M^+$$

is (surjective but) not injective.

Corollary 3.2.24. *Assume that U is a group and that f induces a bijection $U\backslash S \xrightarrow{\cong} T$. There is a natural isomorphism*

$$\operatorname{Hom}_{\mathrm{D}(k)}\bigl(\operatorname{Res}_1^T(\mathrm{L}_U X), Y\bigr) \cong \operatorname{Hom}_{\mathrm{D}(U)}\bigl(\operatorname{Res}_U^S X, \operatorname{Inf}_U^1 Y\bigr),$$

for all $X \in \mathrm{D}(S)$ and $Y \in \mathrm{D}(k)$. In particular, whenever $n \in \mathbb{Z}$, $V \in \operatorname{Rep}_k(S)$, and $W \in \operatorname{Vect}_k$, there is a natural isomorphism

$$\operatorname{Hom}_k\bigl(\mathrm{L}_U^{-n}(V), W\bigr) \cong \operatorname{Ext}_U^n(V, W).$$

Proof. Applying Proposition 3.2.19 with $H_S = U$ and $H_T = \{1\}$ and Theorem 3.2.3 yields natural isomorphisms

$$\begin{aligned} \operatorname{Hom}_{\mathrm{D}(k)}\bigl(\operatorname{Res}_1^T(\mathrm{L}_U X), Y\bigr) &\cong \operatorname{Hom}_{\mathrm{D}(k)}\bigl(\mathrm{L}_U(\operatorname{Res}_U^S X), Y\bigr) \\ &\cong \operatorname{Hom}_{\mathrm{D}(U)}\bigl(\operatorname{Res}_U^S X, \operatorname{Inf}_U^1 Y\bigr), \end{aligned}$$

for each $X \in \mathrm{D}(S)$ and $Y \in \mathrm{D}(k)$. The last statement follows from the first by taking $X = V[0]$ and $Y = W[n]$ for $V \in \operatorname{Rep}_k(S)$ and $W \in \operatorname{Vect}_k$. $\square$

Corollary 3.2.25. *Retain the hypotheses of Proposition 3.2.19 and suppose that H_S and H_T are compact, open, and torsion-free. Then $\mathrm{L}_U\colon \mathrm{D}(S) \longrightarrow \mathrm{D}(T)$ preserves small products and, in particular, is also a right adjoint.*

Proof. Note that Lemma 3.2.16 implies that $U \subseteq H_S$ is compact and torsion-free. Let I be a set, and take $X_i \in \mathrm{D}(S)$, for $i \in I$. We show that the natural map

$$\alpha\colon \mathrm{L}_U \prod_{i \in I} X_i \longrightarrow \prod_{i \in I} \mathrm{L}_U(X_i)$$

is an isomorphism in $\mathrm{D}(T)$. By Proposition 3.2.19 there is a natural isomorphism

$$\operatorname{Res}^T_{H_T} \mathrm{L}_U \overset{\cong}{\Longrightarrow} \mathrm{L}_U \operatorname{Res}^S_{H_S}.$$

Now observe that the diagram

$$\begin{tikzcd}
\operatorname{Res}^T_{H_T} \mathrm{L}_U \prod_{i\in I} X_i \arrow[r, "\cong"] \arrow[d, "\operatorname{Res}^T_{H_T}(\alpha)"'] & \mathrm{L}_U \prod_{i\in I} \operatorname{Res}^S_{H_S} X_i \arrow[d, "\alpha'"] \\
\operatorname{Res}^T_{H_T} \prod_{i\in I} \mathrm{L}_U X_i \arrow[r, "\cong"'] & \prod_{i\in I} \mathrm{L}_U \operatorname{Res}^S_{H_S} X_i
\end{tikzcd}$$

is commutative, where α' is the canonical map. Since $\operatorname{Res}^S_{H_S}$ and $\operatorname{Res}^T_{H_T}$ are conservative and preserve small products, by Lemma 3.2.2, it suffices to verify that the natural map

$$\alpha' \colon \mathrm{L}_U \prod_{i\in I} \operatorname{Res}^S_{H_S}(X_i) \longrightarrow \prod_{i\in I} \mathrm{L}_U\big(\operatorname{Res}^S_{H_S} X_i\big)$$

is an isomorphism in $\mathrm{D}(H_T)$. But this follows from the fact that $\mathrm{L}_U \colon \mathrm{D}(H_S) \longrightarrow \mathrm{D}(H_T)$ admits a left adjoint (Corollary 3.1.11), which finishes the proof. $\square$

3.3. **The case of p-adic Lie groups.** We fix a continuous surjection $f \colon P \twoheadrightarrow M$ of p-adic Lie groups with kernel U.

The crucial result we need is the following:

Lemma 3.3.1. *Let G be a locally profinite group and $K \subseteq G$ an open subgroup. The natural map*

$$\operatorname{Res}^G_K\big(\hom_{\mathrm{D}(G)}(X,Y)\big) \xrightarrow{\cong} \hom_{\mathrm{D}(K)}\big(\operatorname{Res}^G_K X, \operatorname{Res}^G_K Y\big) \tag{3.3.2}$$

is an isomorphism, for all $X, Y \in \mathrm{D}(G)$.

Proof. The map (3.3.2) is an isomorphism on cohomology by (2.3.15), hence an isomorphism. $\square$

Proposition 3.3.3. *For all $X, Y \in \mathrm{D}(M)$ the natural map*

$$\alpha_{X,Y} \colon \operatorname{Inf}^M_P \hom_{\mathrm{D}(M)}(X,Y) \xrightarrow{\cong} \hom_{\mathrm{D}(P)}\big(\operatorname{Inf}^M_P X, \operatorname{Inf}^M_P Y\big)$$

is an isomorphism in $\mathrm{D}(P)$.

Proof. The natural transformation $\alpha_{X,-} \colon \operatorname{Inf}^M_P \circ \hom_{\mathrm{D}(M)}(X,-) \Longrightarrow \hom_{D(P)}(\operatorname{Inf}^M_P X, -) \circ \operatorname{Inf}^M_P$ arises as the mate of the canonical isomorphism

$$(\operatorname{Inf}^M_P X \otimes_k -) \circ \operatorname{Inf}^M_P \overset{\cong}{\Longrightarrow} \operatorname{Inf}^M_P \circ (X \otimes_k -),$$

see Proposition 2.1.9.

Let $K_P \subseteq P$ be a compact open subgroup such that $K_M \coloneqq f(K_P)$ is torsion-free. By Lemma 3.2.2 the functor Res^P_K is conservative. Therefore, it suffices to show that $\operatorname{Res}^P_{K_P}(\alpha_{X,Y}) = \alpha_{\operatorname{Res}^M_{K_M} X, \operatorname{Res}^M_{K_M} Y}$ is an isomorphism in $\mathrm{D}(K_P)$.

Therefore, we may assume from the beginning that P is compact and M is torsion-free and compact. Let $\mathscr{T}$ be the full subcategory of $\mathrm{D}(M)$ with objects those $X \in \mathrm{D}(M)$ for which $\alpha_{X,Y}$ is an isomorphism for all $Y \in \mathrm{D}(M)$. It is clear that $\mathscr{T}$ is triangulated. Since Inf^M_P preserves small products, by Theorem 3.2.3, and small coproducts, it easily follows that $\mathscr{T}$ is localizing, *i.e.*, closed under small coproducts. Now, $\hom_{\mathrm{D}(M)}(\mathbf{1}, Y) \cong Y$ and $\hom_{\mathrm{D}(P)}(\operatorname{Inf}^M_P \mathbf{1}, \operatorname{Inf}^M_P Y) \cong \operatorname{Inf}^M_P Y$. Under these identifications we have $\alpha_{\mathbf{1},Y} = \mathrm{id}_{\operatorname{Inf}^M_P Y}$, which shows $\mathbf{1} \in \mathscr{T}$. By Proposition 2.3.8, $\mathrm{D}(M)$ is compactly generated by $\mathbf{1}$. This implies $\mathscr{T} = \mathrm{D}(M)$, which proves the assertion. $\square$

Warning 3.3.4. Despite the result in Proposition 3.3.3, the functor Inf^M_P is in general *not* fully faithful. For example, if P and M are compact and torsion-free, then the counit $\mathrm{L}_U(\mathbf{1}) = \mathrm{L}_U(\operatorname{Inf}^M_P \mathbf{1}) \longrightarrow \mathbf{1}$ is not an isomorphism by Corollary 3.1.7.

Corollary 3.3.5 (Projection formula)**.** *For all $X \in \mathrm{D}(P)$ and $Y \in \mathrm{D}(M)$ the natural map*

$$\mathrm{L}_U\big(\mathrm{Inf}_P^M X \otimes_k Y\big) \xrightarrow{\cong} X \otimes_k \mathrm{L}_U(Y)$$

is an isomorphism in $\mathrm{D}(M)$. In particular, $\mathrm{L}_U \mathrm{Inf}_P^M Y \cong \mathrm{L}_U(\mathbf{1}) \otimes_k Y$ for all $Y \in \mathrm{D}(M)$.

Proof. By Proposition 3.3.3 the natural map

$$\mathrm{Inf}_P^M \circ \mathrm{hom}_{\mathrm{D}(M)}(Y, -) \overset{\cong}{\Longrightarrow} \mathrm{hom}_{\mathrm{D}(P)}(\mathrm{Inf}_P^M Y, -) \circ \mathrm{Inf}_P^M$$

is an isomorphism of functors $\mathrm{D}(M) \longrightarrow \mathrm{D}(P)$. Passing to the left adjoints, see Example 2.1.13, shows that the induced map

$$\mathrm{L}_U \circ (\mathrm{Inf}_P^M Y \otimes_k -) \overset{\cong}{\Longrightarrow} (Y \otimes_k -) \circ \mathrm{L}_U$$

is an isomorphism of functors $\mathrm{D}(P) \longrightarrow \mathrm{D}(M)$. □

Proposition 3.3.6. *Given $X \in \mathrm{D}(P)$ and $Y \in \mathrm{D}(M)$, the natural map*

$$\mathrm{RH}^0\big(U, \mathrm{hom}_{\mathrm{D}(P)}(X, \mathrm{Inf}_P^M Y)\big) \xrightarrow{\cong} \mathrm{hom}_{\mathrm{D}(M)}(\mathrm{L}_U X, Y)$$

is an isomorphism in $\mathrm{D}(M)$.

Proof. The natural map

$$\mathrm{RH}^0(U, -) \circ \mathrm{hom}_{\mathrm{D}(P)}(X, -) \circ \mathrm{Inf}_P^M \Longrightarrow \mathrm{hom}_{\mathrm{D}(M)}\big(\mathrm{L}_U(X), -\big)$$

arises from $- \otimes_k \mathrm{L}_U(X) \overset{\cong}{\Longrightarrow} \mathrm{L}_U \circ (- \otimes_k X) \circ \mathrm{Inf}_P^M$ by passing to the left adjoints, see Example 2.1.13. The latter map is an isomorphism by Corollary 3.3.5, hence so is the first. □

Recall that we denote by $X^\vee = \mathrm{hom}(X, \mathbf{1})$ the dual object, see Notation 2.3.16.

Corollary 3.3.7. *There is a natural isomorphism $\mathrm{RH}^0(U, X^\vee) \cong (\mathrm{L}_U X)^\vee$ for all $X \in \mathrm{D}(P)$.*

Proof. Apply Proposition 3.3.6 with $Y = \mathbf{1}$. □

3.4. **Computing the left adjoint using positive monoids.** We suppose now that $P = U \rtimes M$ is a semidirect product of p-adic Lie groups. Let $f\colon P \twoheadrightarrow M$ be the canonical projection. We fix a compact open torsion-free subgroup K_P of P. As before, we write $K_U = K_P \cap U$ and $K_M = f(K_P)$. We make the following hypothesis:

Hypothesis 3.4.1. *There exists a* strictly positive *element $z \in M$, i.e., an element z in the center of M such that $zK_Uz^{-1} \subseteq K_U$ and*

$$\bigcup_{n>0} z^{-n}K_Uz^n = U.$$

We fix a strictly positive element $z \in M$. Note that the existence of a strictly positive element necessitates that U be the union of its compact open subgroups.

Example 3.4.2. The example one should have in mind is the following: Let $\mathfrak{F}/\mathbb{Q}_p$ be a finite extension and $\mathbf{G}$ a connected reductive group defined over $\mathfrak{F}$. Fix a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ with unipotent radical $\mathbf{U}$. Let $\overline{\mathbf{P}}$ be the opposite parabolic subgroup with respect to $\mathbf{P}$, that is, the unique parabolic subgroup $\overline{\mathbf{P}}$ of $\mathbf{G}$ such that $\mathbf{M} \coloneqq \mathbf{P} \cap \overline{\mathbf{P}}$ is a Levi subgroup for $\mathbf{P}$ and $\overline{\mathbf{P}}$. Denote $\overline{\mathbf{U}}$ the unipotent radical of $\overline{\mathbf{P}}$. Let $K \subseteq \mathbf{G}(\mathfrak{F})$ be a compact open subgroup with an Iwahori decomposition

$$K = (K \cap \mathbf{U}(\mathfrak{F}))(K \cap \mathbf{M}(\mathfrak{F}))(K \cap \overline{\mathbf{U}}(\mathfrak{F})).$$

By [BK98, (6.14)] there exists a strongly positive element for the choice $K_P \coloneqq (K \cap \mathbf{U}(\mathfrak{F}))(K \cap \mathbf{M}(\mathfrak{F}))$. Note that the strongly positive elements in the sense of *op. cit.* are strictly positive in our sense.

The following definition is standard, cf. [BK98, (6.5)], [Vig98, II.4], or [Eme10a, §3.1].

Definition. An element $m \in M$ is called *positive* if $mK_Um^{-1} \subseteq K_U$. We denote M^+ the monoid of positive elements in M. Note that M^+ is a p-adic monoid, since $K_P \cap M \subseteq M^+$.

The p-adic monoid $P^+ := K_U M^+$ satisfies $K_U \backslash P^+ \cong M^+$.

Remark 3.4.3. We allow for U to be compact. In this case, Hypothesis 3.4.1 amounts to saying that $U = K_U$ is torsion-free and $M = M^+$.

Our goal in this section is to give a more precise description of the functor L_U using positive monoids. We first collect some elementary facts.

Lemma 3.4.4. *The following assertions hold:*

(a) *M (resp. P) is generated as a monoid by M^+ and z^{-1} (resp. P^+ and z^{-1}).*

(b) *Let $S = \{z^n\}_{n\geq 0}$ be the multiplicative set generated by z. Then $k[M] = S^{-1}k[M^+]$ is the left Ore localization of $k[M^+]$ at S. In particular, the ring extension $k[M^+] \subseteq k[M]$ is flat.*
Similarly, $k[P] = S^{-1}k[P^+]$ is the left Ore localization of $k[P^+]$ at S, and hence the ring extension $k[P^+] \subseteq k[P]$ is flat.

(c) *The functors $\operatorname{ind}_{M^+}^M \colon \operatorname{Rep}_k(M^+) \longrightarrow \operatorname{Rep}_k(M)$ and $\operatorname{ind}_{P^+}^P \colon \operatorname{Rep}_k(P^+) \longrightarrow \operatorname{Rep}_k(P)$ are exact.*

Proof. (a) Let $m \in M$. Since mK_Um^{-1} is a compact subgroup of $U = \bigcup_{n>0} z^{-n}K_Uz^n$, there exists n such that $mK_Um^{-1} \subseteq z^{-n}K_Uz^n$. Then $z^n m \in M^+$, proving the assertion for M. Similarly, let $u \in U$ and $m \in M$ be arbitrary. Let $n \gg 0$ such that $u \in z^{-n}K_Uz^n$ and $z^n m \in M^+$. Then $z^n um = z^n u z^{-n} \cdot z^n m \in P^+$, proving the assertion for P.

(b) We prove the assertion for $k[P^+]$; the case of $k[M^+]$ is similar but easier. Note that $k[P^+]$ embeds into the bigger ring $k[P]$ in which the elements of S are invertible. Moreover, S satisfies the left Ore condition, meaning that for all $s \in S$ and $r \in k[P^+]$ there exist $s' \in S$ and $r' \in k[P^+]$ such that $s'r = r's$; indeed this is clear for $s' := s$ and $r' := srs^{-1} \in k[P^+]$. Now, [MR01, Lem. 2.1.13.(ii)] shows that S is a left denominator set in $k[P^+]$. Then (a) implies that $k[P]$ is the left Ore localization of $k[P^+]$ with respect to S. The flatness assertion follows from *op. cit.* Proposition 2.1.16.

(c) This follows immediately from (b).

□

Recall that $\operatorname{Res}_{P^+}^P \colon \mathrm{D}(P) \longrightarrow \mathrm{D}(P^+)$ admits a left adjoint $\operatorname{ind}_{P^+}^P$ (by Lemma 3.4.4.(c)) and a right adjoint $\operatorname{RInd}_{P^+}^P$.

Lemma 3.4.5. *The following equivalent assertions hold:*

(i) *The unit $\eta \colon \mathrm{id}_{\mathrm{D}(P)} \xrightarrow{\cong} \operatorname{RInd}_{P^+}^P \operatorname{Res}_{P^+}^P$ is an isomorphism.*

(ii) *$\operatorname{Res}_{P^+}^P \colon \mathrm{D}(P) \longrightarrow \mathrm{D}(P^+)$ is fully faithful.*

(iii) *The counit $\varepsilon \colon \operatorname{ind}_{P^+}^P \operatorname{Res}_{P^+}^P \xrightarrow{\cong} \mathrm{id}_{\mathrm{D}(P)}$ is an isomorphism.*

Proof. The fact that (i), (ii), and (iii) are equivalent follows from Lemma 2.1.5. Since both $\operatorname{ind}_{P^+}^P$ and $\operatorname{Res}_{P^+}^P$ are exact, it suffices for (iii) to show that the counit

$$\varepsilon \colon \operatorname{ind}_{P^+}^P \operatorname{Res}_{P^+}^P \Longrightarrow \mathrm{id}_{\operatorname{Rep}_k(P)}$$

of the underived adjunction is an isomorphism. But this follows from the fact that $\operatorname{ind}_{P^+}^P = k[P] \otimes_{k[P^+]} -$ is a localization functor by Lemma 3.4.4.(b). □

By Theorem 3.2.3 the functor $\operatorname{Inf}_{P^+}^{M^+} \colon \mathrm{D}(M^+) \longrightarrow \mathrm{D}(P^+)$ admits a left adjoint L_{K_U}.

Proposition 3.4.6. *There is a natural isomorphism*

$$\operatorname{ind}_{M^+}^M \mathrm{L}_{K_U} \operatorname{Res}_{P^+}^P \xrightarrow{\cong} \mathrm{L}_U$$

of functors $\mathrm{D}(P) \longrightarrow \mathrm{D}(M)$.

Proof. The functor $\mathrm{ind}_{M^+}^M\colon \mathrm{D}(M^+) \longrightarrow \mathrm{D}(M)$ exists by Lemma 3.4.4.(c) and is left adjoint to $\mathrm{Res}_{M^+}^M$. From Lemma 3.4.5 it follows that the composite

$$\mathrm{Inf}_P^M \underset{\eta}{\overset{\cong}{\Longrightarrow}} \mathrm{RInd}_{P^+}^P \mathrm{Res}_{P^+}^P \mathrm{Inf}_P^M \overset{\cong}{\Longrightarrow} \mathrm{RInd}_{P^+}^P \mathrm{Inf}_{P^+}^{M^+} \mathrm{Res}_{M^+}^M$$

is a natural isomorphism. The assertion follows by passing to the left adjoints, cf. Example 2.1.13. □

Corollary 3.4.7. *The functors* $\mathrm{L}_U^{-n}\colon \mathrm{Rep}_k(P) \longrightarrow \mathrm{Rep}_k(M)$ *vanish for* $n \notin \{0, 1, \ldots, \dim U\}$. *In particular,* L_U *preserves bounded complexes.*

Proof. Let $n \in \mathbb{Z} \smallsetminus \{0, 1, \ldots, \dim U\}$. Since $\dim K_U = \dim U$, Corollary 3.1.8 and Proposition 3.1.10 imply

$$\mathrm{L}_{K_U}^{-n} = \mathrm{H}^{\dim K_U - n}(K_U, -) = 0.$$

Now, Proposition 3.4.6 shows $\mathrm{L}_U^{-n} = \mathrm{ind}_{M^+}^M \mathrm{L}_{K_U}^{-n} \mathrm{Res}_{P^+}^P = 0$. □

Corollary 3.4.8. *The functors* $\mathrm{L}_U\colon \mathrm{D}^{\mathrm{b}}(P) \rightleftarrows \mathrm{D}^{\mathrm{b}}(M) :\mathrm{Inf}_P^M$ *define an adjunction on the derived categories of bounded complexes.*

Proof. This is immediate from Corollary 3.4.7. □

3.4.1. *The M^+-action.* Our next goal is to decribe the M^+-action on the representations

$$\mathrm{L}_{K_U}^n(V)$$

in more detail, for any $n \in \mathbb{Z}$ and $V \in \mathrm{Rep}_k(P^+)$. Recall that $\mathrm{Res}_{K_M}^{M^+} \mathrm{L}_{K_U} \cong \mathrm{L}_{K_U} \mathrm{Res}_{K_P}^{P^+}$, by Proposition 3.2.19.

Lemma 3.4.9. *Let $K_U' \subseteq K_U$ be an open subgroup. The diagram*

$$\begin{tikzcd} \mathrm{D}(K_U') \arrow[r, "\mathrm{L}_{K_U'}"] \arrow[d, "\mathrm{ind}_{K_U'}^{K_U}"'] & \mathrm{D}(k) \\ \mathrm{D}(K_U) \arrow[ur, "\mathrm{L}_{K_U}"'] & \end{tikzcd}$$

is commutative.

Proof. This is a special case of Proposition 3.2.11. □

Definition. Let $K_U' \subseteq K_U$ be an open subgroup and $S \subseteq P$ a closed submonoid with $K_U \subseteq S$. The composite map

$$\mathrm{cores}\colon \mathrm{L}_{K_U'} \mathrm{Res}_{K_U'}^S \overset{\cong}{\Longrightarrow} \mathrm{L}_{K_U} \mathrm{ind}_{K_U'}^{K_U} \mathrm{Res}_{K_U'}^S \xRightarrow{\mathrm{L}_{K_U}\varepsilon} \mathrm{L}_{K_U} \mathrm{Res}_{K_U}^S$$

is called the *corestriction*. (Here, $\varepsilon\colon \mathrm{ind}_{K_U'}^{K_U} \mathrm{Res}_{K_U'}^{K_U} \Longrightarrow \mathrm{id}_{\mathrm{D}(K_U)}$ denotes the counit map.)

Lemma 3.4.10. *The corestriction arises from the natural map* $\iota\colon \mathrm{RInd}_{K_U}^S \mathrm{Inf}_{K_U}^1 \Longrightarrow \mathrm{RInd}_{K_U'}^S \mathrm{Inf}_{K_U'}^1$ *by passing to the left adjoints.*

Proof. From Proposition 2.1.9 it follows that under the natural bijections

$$\begin{array}{ccccc} \mathrm{Nat}\big(\mathrm{ind}_{K_U'}^{K_U} \mathrm{Res}_{K_U'}^{K_U}, \mathrm{id}_{\mathrm{D}(K_U)}\big) & \overset{\cong}{\longleftrightarrow} & \mathrm{Nat}\big(\mathrm{Res}_{K_U'}^{K_U}, \mathrm{Res}_{K_U'}^{K_U}\big) & \overset{\cong}{\longleftrightarrow} & \mathrm{Nat}\big(\mathrm{id}_{\mathrm{D}(K_U)}, \mathrm{Ind}_{K_U'}^{K_U} \mathrm{Res}_{K_U'}^{K_U}\big), \\ \varepsilon & \longleftrightarrow & \mathrm{id}_{\mathrm{Res}_{K_U'}^{K_U}} & \longleftrightarrow & \eta', \end{array}$$

ε corresponds to the unit map $\eta'\colon \mathrm{id}_{\mathrm{D}(K_U)} \Longrightarrow \mathrm{Ind}_{K_U'}^{K_U} \mathrm{Res}_{K_U'}^{K_U}$ of the adjunction $\mathrm{Res}_{K_U'}^{K_U} \dashv \mathrm{Ind}_{K_U'}^{K_U}$. The map

$$\mathrm{RInd}_{K_U}^S \xRightarrow{\mathrm{RInd}_{K_U}^S \eta'} \mathrm{RInd}_{K_U}^S \mathrm{Ind}_{K_U'}^{K_U} \mathrm{Res}_{K_U'}^{K_U} \overset{\cong}{\Longrightarrow} \mathrm{RInd}_{K_U'}^S \mathrm{Res}_{K_U'}^{K_U} \tag{3.4.11}$$

is clearly induced by the "inclusion" of smooth inductions. By construction, we obtain cores from

$$\iota\colon \operatorname{RInd}_{K_U}^{S} \operatorname{Inf}_{K_U}^{1} \xRightarrow{(3.4.11)} \operatorname{RInd}_{K_U'}^{S} \operatorname{Res}_{K_U'}^{K_U} \operatorname{Inf}_{K_U}^{1} \xRightarrow{\cong} \operatorname{RInd}_{K_U'}^{S} \operatorname{Inf}_{K_U'}^{1}$$

by passing to the left adjoints, see Example 2.1.13. This proves the assertion. ☐

Recall Notation 3.2.14 for the definition of $m_*\colon \mathrm{D}(K_U) \longrightarrow \mathrm{D}(mK_Um^{-1})$, for $m \in M$.

Lemma 3.4.12. *The diagram*

$$\begin{tikzcd}
\mathrm{D}(K_U) \arrow[r, "\mathrm{L}_{K_U}"] \arrow[d, "m_*"'] & \mathrm{D}(k) \\
\mathrm{D}(mK_Um^{-1}) \arrow[ur, "\mathrm{L}_{mK_Um^{-1}}"'] &
\end{tikzcd}$$

commutes, for all $m \in M$.

Proof. Let $m \in M$. Since the natural map $m_*^{-1} \operatorname{Inf}_{mK_Um^{-1}}^{1} \xRightarrow{\cong} \operatorname{Inf}_{K_U}^{1}$ is an isomorphism, the claim follows by passing to the left adjoints, see Example 2.1.13. ☐

Given $m \in M^+$, multiplication by m induces two natural transformations

$$\alpha_m\colon m_* \operatorname{Res}_{K_U}^{P^+} \Longrightarrow \operatorname{Res}_{mK_Um^{-1}}^{P^+},$$

and

$$\alpha_m'\colon \operatorname{Res}_1^{M^+} \Longrightarrow \operatorname{Res}_1^{M^+}.$$

The right mates are given, respectively, by

$$\begin{aligned}
\beta_m\colon \operatorname{Ind}_{mK_Um^{-1}}^{P^+} V &\longrightarrow \operatorname{Ind}_{K_U}^{P^+} m_*^{-1} V,\\
\varphi &\longmapsto [g \mapsto \varphi(mg)],
\end{aligned}$$

for $V \in \operatorname{Rep}_k(mK_Um^{-1})$ and

$$\begin{aligned}
\beta_m'\colon \operatorname{Ind}_1^{M^+} W &\longrightarrow \operatorname{Ind}_1^{M^+} W,\\
\varphi &\longmapsto [m^+ \mapsto \varphi(mm^+)],
\end{aligned}$$

for $W \in \operatorname{Vect}_k$. We are now in a position to describe the M^+-action.

Proposition 3.4.13. *Let $m \in M^+$. The diagram*

$$\begin{tikzcd}
\operatorname{Res}_1^{M^+} \mathrm{L}_{K_U} \arrow[r, Rightarrow, "(\alpha_m')_{\mathrm{L}_{K_U}}"] \arrow[d, Rightarrow, "\cong"'] & \operatorname{Res}_1^{M^+} \mathrm{L}_{K_U} \arrow[r, Rightarrow, "\cong"] & \mathrm{L}_{K_U} \operatorname{Res}_{K_U}^{P^+} \\
\mathrm{L}_{K_U} \operatorname{Res}_{K_U}^{P^+} \arrow[r, Rightarrow, "\cong"'] & \mathrm{L}_{mK_Um^{-1}} m_* \operatorname{Res}_{K_U}^{P^+} \arrow[r, Rightarrow, "\mathrm{L}_{mK_Um^{-1}}\alpha_m"'] & \mathrm{L}_{mK_Um^{-1}} \operatorname{Res}_{mK_Um^{-1}}^{P^+} \arrow[u, Rightarrow, "\text{cores}"']
\end{tikzcd} \tag{3.4.14}$$

of functors $\mathrm{D}(P^+) \longrightarrow \mathrm{D}(k)$ is commutative.

Proof. The left vertical and the upper right horizontal maps are isomorphisms by Proposition 3.2.19. The lower left horizontal map is an isomorphism by Lemma 3.4.12. We claim that the diagram (3.4.15)

$$\begin{tikzcd}
\operatorname{Inf}_{P^+}^{M^+} \operatorname{Ind}_1^{M^+} \arrow[dr, phantom, "(*)"] & \operatorname{Inf}_{P^+}^{M^+} \operatorname{Ind}_1^{M^+} \arrow[l, Rightarrow, "\operatorname{Inf}_{P^+}^{M^+} \beta_m'"'] & \operatorname{RInd}_{K_U}^{P^+} \operatorname{Inf}_{K_U}^{1} \arrow[l, Rightarrow, "\cong"'] \arrow[dl, Rightarrow, "(\mathrm{R}\gamma)_{\operatorname{Inf}_{K_U}^1}"] \arrow[d, Rightarrow, "\iota"] \\
\operatorname{RInd}_{K_U}^{P^+} \operatorname{Inf}_{K_U}^{1} \arrow[u, Rightarrow, "\cong"] & \operatorname{RInd}_{K_U}^{P^+} m_*^{-1} \operatorname{Inf}_{mK_Um^{-1}}^{1} \arrow[l, Rightarrow, "\cong"] & \operatorname{RInd}_{mK_Um^{-1}}^{P^+} \operatorname{Inf}_{mK_Um^{-1}}^{1} \arrow[l, Rightarrow, "(\mathrm{R}\beta_m)_{\operatorname{Inf}_{mK_Um^{-1}}^1}"]
\end{tikzcd}$$

of functors $\mathrm{D}(k) \longrightarrow \mathrm{D}(P^+)$ is commutative. Here, γ is defined as follows: Consider the unit

$$\eta' \colon \mathrm{id}_{\mathrm{Rep}_k(K_U)} \Longrightarrow \mathrm{Ind}_{mK_Um^{-1}}^{K_U} \mathrm{Res}_{mK_Um^{-1}}^{K_U} .$$

Then γ is the natural transformation

$$\gamma \coloneqq (\beta_m)_{\mathrm{Res}_{mK_Um^{-1}}^{K_U}} \circ \mathrm{Ind}_{K_U}^{P^+} \eta' \colon \mathrm{Ind}_{K_U}^{P^+} \Longrightarrow \mathrm{Ind}_{K_U}^{P^+} m_*^{-1} \mathrm{Res}_{mK_Um^{-1}}^{K_U}$$

of functors $\mathrm{Rep}_k(K_U) \longrightarrow \mathrm{Rep}_k(P^+)$. It is immediate from the definitions that

$$(\mathrm{R}\beta_m)_{\mathrm{Inf}^1_{mK_Um^{-1}}} \circ \iota = (\mathrm{R}\gamma)_{\mathrm{Inf}^1_{K_U}} ,$$

that is, the lower right triangle in diagram (3.4.15) commutes.

By Remark 3.2.21, we have the natural isomorphism

$$\mathrm{R}\big(\mathrm{Ind}_{K_U}^{P^+} \mathrm{Inf}^1_{K_U}\big) \xRightarrow{\cong} \mathrm{RInd}_{K_U}^{P^+} \mathrm{Inf}^1_{K_U} .$$

It follows that also $\mathrm{R}\big(\mathrm{Ind}_{K_U}^{P^+} m_*^{-1} \mathrm{Inf}^1_{mK_Um^{-1}}\big) \xRightarrow{\cong} \mathrm{RInd}_{K_U}^{P^+} m_*^{-1} \mathrm{Inf}^1_{mK_Um^{-1}}$ is an isomorphism, and we have $(\mathrm{R}\gamma)_{\mathrm{Inf}^1_{K_U}} = \mathrm{R}(\gamma_{\mathrm{Inf}^1_{K_U}})$. An easy computation shows that the diagram

$$\begin{tikzcd}
\mathrm{Inf}_{P^+}^{M^+} \mathrm{Ind}_1^{M^+} & \mathrm{Inf}_{P^+}^{M^+} \mathrm{Ind}_1^{M^+} \arrow[l, Leftarrow, "\mathrm{Inf}_{P^+}^{M^+} \beta'_m"'] & \mathrm{Ind}_{K_U}^{P^+} \mathrm{Inf}^1_{K_U} \arrow[l, Leftarrow, "\cong"'] \arrow[d, Rightarrow, "\gamma_{\mathrm{Inf}^1_{K_U}}"] \\
\mathrm{Ind}_{K_U}^{P^+} \mathrm{Inf}^1_{K_U} \arrow[u, Rightarrow, "\cong"] & & \mathrm{Ind}_{K_U}^{P^+} m_*^{-1} \mathrm{Inf}^1_{mK_Um^{-1}} \arrow[ll, Leftarrow, "\cong"]
\end{tikzcd}$$

of functors $\mathrm{Vect}_k \longrightarrow \mathrm{Rep}_k(P^+)$ commutes. Passing to the derived categories, it follows that the diagram marked $(*)$ is commutative. Therefore, the whole diagram (3.4.15) commutes. By passing to the left adjoints, see Example 2.1.13, we deduce that the diagram in the assertion commutes. ☐

Definition. Given $m \in M^+$, we define

$$\mathrm{conj}_m \colon \mathrm{L}_{K_U} \mathrm{Res}_{K_U}^{P^+} \Longrightarrow \mathrm{L}_{mK_Um^{-1}} \mathrm{Res}_{mK_Um^{-1}}^{P^+}$$

as the composite of the bottom horizontal maps in diagram (3.4.14).

Corollary 3.4.16. *Let $V \in \mathrm{Rep}_k(P^+)$ and $n \in \mathbb{Z}$. The action of $m \in M^+$ on $\mathrm{L}^n_{K_U}(V)$ is given by the composition*

$$\mathrm{L}^n_{K_U}(V) \xrightarrow{\mathrm{conj}_m} \mathrm{L}^n_{mK_Um^{-1}}(V) \xrightarrow{\mathrm{cores}} \mathrm{L}^n_{K_U}(V).$$

3.4.2. *A smooth character.* Write $d \coloneqq \dim K_U$. We use the method in [SS22b, paragraph before Lem. 1.6] to construct a smooth character $\delta_P \colon P \longrightarrow k^\times$. Assume that also K_M is torsion-free so that the right adjoint $F_{K_P}^{K_M} \colon \mathrm{D}(K_M) \longrightarrow \mathrm{D}(K_P)$ of $\mathrm{RH}^0(K_U, -)$ exists (Lemma 3.1.3). We fix a k-linear isomorphism $\mathrm{H}^{-d}(F_{K_P}^{K_M}(\mathbf{1})) \cong k$, which is possible by Proposition 3.1.10. Given any other compact open torsion-free subgroup $K'_P \subseteq P$ such that also K'_M is torsion-free, the natural isomorphisms

$$\mathrm{Res}_{K'_P \cap K_P}^{K'_P} F_{K'_P}^{K'_M}(\mathbf{1}) \xleftarrow{\cong} F_{K'_P \cap K_P}^{K'_M \cap K_M}(\mathbf{1}) \xrightarrow{\cong} \mathrm{Res}_{K'_P \cap K_P}^{K_P} F_{K_P}^{K_M}(\mathbf{1}),$$

see Lemma 3.1.3, allow us to identify $\mathrm{H}^{-d}(F_{K'_P}^{K'_M}(\mathbf{1}))$ with k. We make the following convention:

Notation 3.4.17. Let G be a group. Given a subgroup $H \subseteq G$ and $g \in G$, we write ${}^gH \coloneqq gHg^{-1}$ and $H^g \coloneqq g^{-1}Hg$.

Let now $x \in P$ and write $m \coloneqq f(x) \in M$. The isomorphism $x_* \mathrm{Inf}_{K'_P}^{K'_M} \xRightarrow{\cong} \mathrm{Inf}_{{}^xK'_P}^{{}^mK'_M} m_*$ induces, by passing to the right adjoints twice (cf. Example 2.1.13), a canonical isomorphism

$$x_* F_{K'_P}^{K'_M} \xRightarrow{\cong} F_{{}^xK'_P}^{{}^mK'_M} m_*. \tag{3.4.18}$$

The same method shows the following easy lemma:

Lemma 3.4.19. *Let K'_P, x, and m be as above.*

(a) Let $K''_P \subseteq K'_P$ be an open subgroup. The diagram

$$\begin{tikzcd}
x_* F^{K''_M}_{K''_P} \operatorname{Res}^{K'_M}_{K''_M} \arrow[r, Rightarrow, "\cong"] \arrow[d, Rightarrow, "\cong"'] & F^{{}^m K''_M}_{{}^x K''_P} m_* \operatorname{Res}^{K'_M}_{K''_M} \arrow[d, equal] \\
x_* \operatorname{Res}^{K'_P}_{K''_P} F^{K'_M}_{K'_P} \arrow[d, equal] & F^{{}^m K''_M}_{{}^x K''_P} \operatorname{Res}^{{}^m K'_M}_{{}^m K''_M} m_* \arrow[d, Rightarrow, "\cong"] \\
\operatorname{Res}^{{}^x K'_P}_{{}^x K''_P} x_* F^{K'_M}_{K'_P} \arrow[r, Rightarrow, "\cong"'] & \operatorname{Res}^{{}^x K'_P}_{{}^x K''_P} F^{{}^m K'_M}_{{}^x K'_P} m_*
\end{tikzcd}$$

is commutative. (Here, the upper left and lower right vertical arrows are defined as in Lemma 3.1.3.)

(b) Let $x' \in P$ and put $m' \coloneqq f(x')$. The diagram

$$\begin{tikzcd}
x_* x'_* F^{K'_M}_{K'_P} \arrow[r, Rightarrow, "\cong"] \arrow[d, equal] & x_* F^{{}^{m'} K'_M}_{{}^{x'} K'_P} m'_* \arrow[r, Rightarrow, "\cong"] & F^{{}^{mm'} K'_M}_{{}^{xx'} K'_P} m_* m'_* \arrow[d, equal] \\
(xx')_* F^{K'_M}_{K'_P} \arrow[rr, Rightarrow, "\cong"'] & & F^{{}^{mm'} K'_M}_{{}^{xx'} K'_P} (mm')_*
\end{tikzcd}$$

is commutative.

(c) Suppose $x \in K'_P$. The diagram

$$\begin{tikzcd}
x_* F^{K'_M}_{K'_P} \arrow[rr, Rightarrow, "\cong"] \arrow[dr, Rightarrow, "(\alpha_x)_{F^{K'_M}_{K'_P}}"'] & & F^{K'_M}_{K'_P} m_* \arrow[dl, Rightarrow, "F^{K'_M}_{K'_P} \alpha_m"] \\
& F^{K'_M}_{K'_P} &
\end{tikzcd}$$

is commutative, where $\alpha_x \colon x_* \Longrightarrow \mathrm{id}_{\mathrm{D}(K'_P)}$ and $\alpha_m \colon m_* \Longrightarrow \mathrm{id}_{\mathrm{D}(K'_M)}$ are multiplication with x and m, respectively.

We view $\mathbf{1}$ as the trivial representation of P. The map

$$\begin{aligned}
\operatorname{Res}^{K'_P}_1 F^{K'_M}_{K'_P}(\mathbf{1}) &= \operatorname{Res}^{{}^x K'_P}_1 x_* F^{K'_M}_{K'_P}(\mathbf{1}) \\
&\longrightarrow \operatorname{Res}^{{}^x K'_P}_1 F^{{}^m K'_M}_{{}^x K'_P} m_*(\mathbf{1}) = \operatorname{Res}^{{}^x K'_P}_1 F^{{}^m K'_M}_{{}^x K'_P}(\mathbf{1})
\end{aligned} \tag{3.4.20}$$

induced by (3.4.18) defines a k-linear isomorphism

$$k \cong \mathrm{H}^{-d}\big(F^{K'_M}_{K'_P}(\mathbf{1})\big) \longrightarrow \mathrm{H}^{-d}\big(F^{{}^m K'_M}_{{}^x K'_P}(\mathbf{1})\big) \cong k,$$

given by multiplication with a scalar, say, $\delta_P(x) \in k^\times$. The next lemma is inspired by [SS22b, Lem. 1.6]

Lemma 3.4.21. *The map $\delta_P \colon P \longrightarrow k^\times$ is a smooth character which is independent of K'_P and trivial on every pro-p subgroup of P.*

Proof. Let $x \in P$ be fixed and put $m = f(x)$. We verify that $\delta_P(x)$ is independent of the choice of K'_P. Let K''_P be any other compact open torsion-free subgroup of P such that K''_M is torsion-free.

Without loss of generality we may assume $K''_P \subseteq K'_P$. Consider the diagram

$$\begin{tikzcd}
& \mathrm{H}^{-d}\big(F^{K''_M}_{K''_P}(\mathbf{1})\big) \arrow[r] \arrow[dd, "\cong"] & \mathrm{H}^{-d}\big(F^{{}^m K''_M}_{{}^x K''_P}(\mathbf{1})\big) \arrow[dd, "\cong"] \arrow[dr, equal] & \\
k \arrow[ur, equal] \arrow[dr, equal] & & & k. \\
& \mathrm{H}^{-d}\big(F^{K'_M}_{K'_P}(\mathbf{1})\big) \arrow[r, "\delta_P(x)\cdot"'] & \mathrm{H}^{-d}\big(F^{{}^m K'_M}_{{}^x K'_P}(\mathbf{1})\big) \arrow[ur, equal] &
\end{tikzcd}$$

The left and right triangles commute by the definition of the identifications. The square commutes by Lemma 3.4.19.(a). Therefore, the top map is given by multiplication with $\delta_P(x)$, which proves the independence of K'_P.

We now verify that δ_P is a character. Let $x' \in P$ be another element and put $m' = f(x')$. It then follows from Lemma 3.4.19.(b) and the independence of δ_P from K'_P that the diagram

$$\begin{tikzcd}
\mathrm{H}^{-d}\big(F^{K'_M}_{K'_P}(\mathbf{1})\big) \arrow[r, "\delta_P(x')\cdot"] \arrow[dr, "\delta_P(xx')\cdot"'] & \mathrm{H}^{-d}\big(F^{{}^{m'} K'_M}_{{}^{x'} K'_P}(\mathbf{1})\big) \arrow[d, "\delta_P(x)\cdot"] \\
& \mathrm{H}^{-d}\big(F^{{}^{mm'} K'_M}_{{}^{xx'} K'_P}(\mathbf{1})\big)
\end{tikzcd}$$

is commutative. Therefore, δ_P is a character. If $x \in K'_P$, then the map (3.4.20) is given by the natural action of x on $F^{K'_M}_{K'_P}(\mathbf{1})$ by Lemma 3.4.19.(c). From Proposition 3.1.10 it now follows that δ_P is trivial on K'_P. In particular, δ_P is smooth. Finally, if H is any pro-p subgroup of P, the smoothness of δ_P implies that the image $\delta_P(H)$ is a finite p-subgroup of $k^\times$. But since every finite subgroup of $k^\times$ has order prime to p, we deduce that δ_P is trivial on H. $\square$

Definition. We put $\omega_P \coloneqq \delta_P[\dim K_U] \in \mathrm{D}(P)$. By Lemma 3.4.21 we have $\mathrm{Res}^P_{K_P} \omega_P \cong \omega_{K_P}$. Therefore, the definition $\omega_S \coloneqq \mathrm{Res}^P_S \omega_P$, for any open submonoid $S \subseteq P$, is unambiguous.

Let $V \in \mathrm{Rep}_k(P^+)$ be a smooth P^+-representation. There is an action of M^+ on $\mathrm{H}^0(K_U, V)$ called the *Hecke action*, cf. [Eme10a, 3.1.3. Def.]. Concretely, $m \in M^+$ acts on $\mathrm{H}^0(K_U, V)$ through the composition

$$\begin{array}{ccccc}
\mathrm{H}^0(K_U, V) & \xrightarrow{\mathrm{conj}_m} & \mathrm{H}^0(mK_Um^{-1}, V) & \xrightarrow{\mathrm{cores}} & \mathrm{H}^0(K_U, V) \\
v & \longmapsto & mv & \longmapsto & \sum_{u \in K_U/mK_Um^{-1}} umv.
\end{array}$$

This gives a left exact functor $\mathrm{H}^0(K_U, -)\colon \mathrm{Rep}_k(P^+) \longrightarrow \mathrm{Rep}_k(M^+)$, hence also a functor on the derived categories

$$\mathrm{RH}^0(K_U, -)\colon \mathrm{D}(P^+) \longrightarrow \mathrm{D}(M^+).$$

Lemma 3.4.22. *The diagram*

$$\begin{tikzcd}
\mathrm{D}(P^+) \arrow[r, "\mathrm{RH}^0(K_U,-)"] \arrow[d, "\mathrm{Res}^{P^+}_{K_P}"'] & \mathrm{D}(M^+) \arrow[d, "\mathrm{Res}^{M^+}_{K_M}"] \\
\mathrm{D}(K_P) \arrow[r, "\mathrm{RH}^0(K_U,-)"] \arrow[d, "\mathrm{Res}^{K_P}_{K_U}"'] & \mathrm{D}(K_M) \arrow[d, "\mathrm{Res}^{K_M}_{1}"] \\
\mathrm{D}(K_U) \arrow[r, "\mathrm{RH}^0(K_U,-)"] & \mathrm{D}(k)
\end{tikzcd}$$

is commutative.

Proof. Observe that $\operatorname{Res}^{P^+}_{K_P}$ admits an exact left adjoint, by Lemma 3.4.4.(c), and hence preserves K-injective complexes (Lemma 2.3.3.(a)). Thus, the commutativity of the top square can be verified on the underived level, where it is clear.

Since $\mathrm{H}^0(K_U,-)\colon \operatorname{Rep}_k(K_P) \longrightarrow \operatorname{Rep}_k(K_M)$ and $\mathrm{H}^0(K_U,-)\colon \operatorname{Rep}_k(K_U) \longrightarrow \operatorname{Vect}_k$ have finite cohomological dimension, [Har66, Cor. 5.3 γ] shows that both derived functors can be computed using acyclic resolutions. As $\operatorname{Res}^{K_P}_{K_U}$ preserves $\mathrm{H}^0(K_U,-)$-acyclic objects (see, *e.g.*, [NSW13, (1.3.6) Prop. (ii)]) it follows that also the bottom square commutes. $\square$

Proposition 3.4.23. *For each $n \in \mathbb{Z}$ there exists a natural isomorphism*

$$\mathrm{H}^0 \circ \mathrm{RH}^0(K_U, \omega_P \otimes_k -) \xRightarrow{\cong} \mathrm{H}^0 \circ \mathrm{L}_{K_U}$$

of functors $\mathrm{D}(P^+) \longrightarrow \operatorname{Rep}_k(M^+)$.

Proof. We will abuse notation and write ω_P instead of $\operatorname{Res}^P_H \omega_P$, for each closed subgroup $H \subseteq P$. By Corollary 3.1.8 there exists a natural isomorphism

$$\mathrm{RH}^0(K_U, \omega_P \otimes_k -) \operatorname{Res}^{P^+}_{K_P} \xRightarrow{\cong} \mathrm{L}_{K_U} \operatorname{Res}^{P^+}_{K_P}$$

of functors $\mathrm{D}(P^+) \longrightarrow \mathrm{D}(K_M)$. Applying $\operatorname{Res}^{K_M}_1$ and using Proposition 3.2.19 and Lemma 3.4.22 (twice), we see that it suffices to show that the induced k-linear maps on the 0-th cohomology

$$\mathrm{H}^0 \circ \mathrm{RH}^0(K_U, \omega_P \otimes_k -) \operatorname{Res}^{P^+}_{K_U} \xRightarrow{\cong} \mathrm{H}^0 \circ \mathrm{L}_{K_U} \operatorname{Res}^{P^+}_{K_U} \tag{3.4.24}$$

(which, by construction, are already K_M-equivariant) are in fact M^+-equivariant. Let $m \in M^+$ be arbitrary. Note that the action map

$$\alpha_m \colon m_*\omega_P \longrightarrow \omega_P$$

is given by multiplication with $\delta_P(m)$. The diagram

$$\begin{tikzcd}
m_*\mathrm{RH}^0(K_U, \omega_P \otimes_k -) \operatorname{Res}^{P^+}_{K_U} \arrow[r, Rightarrow, "\cong"] \arrow[d, Rightarrow, "\delta_P(m)\cdot"'] & m_*\mathrm{L}_{K_U} \operatorname{Res}^{P^+}_{K_U} \arrow[dd, Rightarrow, "\cong"] \\
m_*\mathrm{RH}^0(K_U, m_*^{-1}\omega_P \otimes_k -) \operatorname{Res}^{P^+}_{K_U} \arrow[d, Rightarrow, "\cong"'] & \\
\mathrm{RH}^0({}^mK_U, \omega_P \otimes_k -) m_* \operatorname{Res}^{P^+}_{K_U} \arrow[r, Rightarrow, "\cong"] \arrow[d, Rightarrow, "\alpha_m"'] & \mathrm{L}_{{}^mK_U} m_* \operatorname{Res}^{P^+}_{K_U} \arrow[d, Rightarrow, "\alpha_m"] \\
\mathrm{RH}^0({}^mK_U, \omega_P \otimes_k -) \operatorname{Res}^{P^+}_{{}^mK_U} \arrow[r, Rightarrow, "\cong"] \arrow[d, Rightarrow, "\cong"'] & \mathrm{L}_{{}^mK_U} \operatorname{Res}^{P^+}_{{}^mK_U} \arrow[dd, Rightarrow, "\cong"] \\
\mathrm{RH}^0\big(K_U, \operatorname{ind}^{K_U}_{{}^mK_U}(\omega_P \otimes_k -)\big) \operatorname{Res}^{P^+}_{{}^mK_U} \arrow[d, Rightarrow, "\cong"'] & \\
\mathrm{RH}^0(K_U, \omega_P \otimes_k -) \operatorname{ind}^{K_U}_{{}^mK_U} \operatorname{Res}^{P^+}_{{}^mK_U} \arrow[r, Rightarrow, "\cong"] \arrow[d, Rightarrow, "\varepsilon"'] & \mathrm{L}_{K_U} \operatorname{ind}^{K_U}_{{}^mK_U} \operatorname{Res}^{P^+}_{{}^mK_U} \arrow[d, Rightarrow, "\varepsilon"] \\
\mathrm{RH}^0(K_U, \omega_P \otimes_k -) \operatorname{Res}^{P^+}_{K_U} \arrow[r, Rightarrow, "\cong"'] & \mathrm{L}_{K_U} \operatorname{Res}^{P^+}_{K_U}
\end{tikzcd}$$

is commutative. The commutativity of the first diagram follows from Proposition 2.1.19 applied with $c^* = d^* = m_*^{-1}$, $f^* = \operatorname{Inf}^1_{K_U}$, and $g^* = \operatorname{Inf}^1_{{}^mK_U}$. The commutativity of the third diagram follows from Proposition 2.1.19 applied with $c^* = \mathrm{id}_{\mathrm{D}(k)}$, $f^* = \operatorname{Inf}^1_{{}^mK_U}$, $g^* = \operatorname{Inf}^1_{K_U}$, and $d^* = \operatorname{Res}^{K_U}_{{}^mK_U}$.

Since the vertical maps describe the respective M^+-actions, it follows that (3.4.24) is M^+-equivariant. □

Remark 3.4.25. Proposition 3.4.23 implies that δ_P coincides with the (inflation to P of the) character α^u defined in [Eme10b, 3.6].

Combining Propositions 3.4.6 and 3.4.23 yields the following result:

Corollary 3.4.26. *For each $i \in \mathbb{Z}$ there is a natural isomorphism*

$$\operatorname{ind}_{M^+}^{M} \mathrm{H}^{\dim K_U + i}(K_U, \omega_P \otimes -) \operatorname{Res}_{P^+}^{P} \xRightarrow{\cong} \mathrm{L}_U^i$$

of functors $\operatorname{Rep}_k(P) \longrightarrow \operatorname{Rep}_k(M)$.

The above description allows us to explicitly compute a first example:

Corollary 3.4.27. *Assume that the strictly positive element $z \in M$ satisfies $\bigcap_{n \geq 0} z^n K_U z^{-n} = \{1\}$. Let $\chi\colon M \longrightarrow k^\times$ be a smooth character. The counit $\varepsilon_\chi\colon \mathrm{L}_U(\operatorname{Inf}_P^M(\chi)) \xrightarrow{\cong} \chi$ is an isomorphism in $\mathrm{D}(M)$.*

Proof. If U has the discrete topology, the compact subgroup K_U is finite. As z is positive, we must have $z^n K_U z^{-n} = K_U$ for all $n \in \mathbb{Z}$, and hence the hypothesis shows $K_U = \{1\}$. As z is strictly positive (see Hypothesis 3.4.1), it follows that $U = \{1\}$ in which case the assertion is trivial.

If U is not discrete, the hypothesis implies that U contains no compact open subgroup which is normalized by M. In particular, U is not compact. It is clear that $\mathrm{H}^0(\varepsilon_\chi)\colon \mathrm{L}_U^0(\operatorname{Inf}_P^M(\chi)) \xrightarrow{\cong} \chi$ is an isomorphism. It remains to show that $\mathrm{L}_U^i(\operatorname{Inf}_P^M(\chi)) = \{0\}$, for all $i \neq 0$. By Proposition 3.2.12 there is a natural isomorphism $\mathrm{L}_U(\operatorname{Inf}_P^M(\chi)) \xrightarrow{\cong} \chi \otimes \mathrm{L}_U(\mathbf{1})$. We are thus reduced to the case $\chi = \mathbf{1}$. By Propositions 3.4.6 and 3.4.23 there is a natural isomorphism

$$\operatorname{ind}_{M^+}^{M} \mathrm{H}^{d+i}(K_U, \delta_P) \xrightarrow{\cong} \mathrm{L}_U^i(\mathbf{1}),$$

where $d \coloneqq \dim U$. Fix $i \neq 0$. As K_U is a Poincaré group, the k-vector space $\mathrm{H}^{-i}(K_U, \mathbf{1})$ is finite-dimensional, and the cup product $\mathrm{H}^{-i}(K_U, \mathbf{1}) \times \mathrm{H}^{d+i}(K_U, \mathbf{1}) \longrightarrow \mathbf{1}$ is a perfect pairing. Therefore, $\mathrm{H}^{-i}(K_U, \mathbf{1})$ identifies with the k-linear dual of $\mathrm{H}^{d+i}(K_U, \mathbf{1})$, and for each $m \in M^+$ the diagram

$$\begin{tikzcd}
\mathrm{H}^{d+i}(mK_Um^{-1}, \mathbf{1}) \arrow[r, "\cong"] \arrow[d, "\mathrm{cores}"'] & \mathrm{H}^{-i}(mK_Um^{-1}, \mathbf{1})^* \arrow[d, "\mathrm{res}^*"] \\
\mathrm{H}^{d+i}(K_U, \mathbf{1}) \arrow[r, "\cong"'] & \mathrm{H}^{-i}(K_U, \mathbf{1})^*
\end{tikzcd}$$

is commutative. As $\mathrm{H}^{-i}(K_U, \mathbf{1})$ is finite-dimensional and $i \neq 0$, we find (by the hypothesis) an $n \gg 0$ such that for $m \coloneqq z^n$ the right vertical map vanishes. Hence also the left vertical map is zero. As the Hecke action of m on $\mathrm{H}^{d+i}(K_U, \delta_P)$ is given by the composite

$$\mathrm{H}^{d+i}(K_U, \delta_P) \xrightarrow{\delta_P(m)\cdot \mathrm{conj}_m} \mathrm{H}^{d+i}(mK_Um^{-1}, \delta_P) \xrightarrow{\mathrm{cores}} \mathrm{H}^{d+i}(K_U, \delta_P),$$

we deduce that $\operatorname{ind}_{M^+}^{M} \mathrm{H}^{d+i}(K_U, \delta_P) = \{0\}$. Therefore, $\mathrm{L}_U^i(\mathbf{1}) = \{0\}$. □

4. The left adjoint of derived parabolic induction

As in §3 we fix a field k of characteristic $p > 0$.

4.1. **General results.** Let G be a p-adic reductive group, *i.e.*, the group of $\mathfrak{F}$-points of a connected reductive group defined over a finite field extension $\mathfrak{F}/\mathbb{Q}_p$. Let P be a parabolic subgroup of G with Levi quotient M and unipotent radical U.

The functor of *parabolic induction*

$$i_M^G := \mathrm{Ind}_P^G \circ \mathrm{Inf}_P^M \colon \mathrm{Rep}_k(M) \longrightarrow \mathrm{Rep}_k(G)$$

is exact and hence defines a functor $i_M^G\colon \mathrm{D}(M) \longrightarrow \mathrm{D}(G)$ on the derived categories.

Theorem 4.1.1. *The derived parabolic induction i_M^G admits a left adjoint, denoted $\mathrm{L}(U,-)$. In fact, there is a natural isomorphism*

$$\mathrm{RHom}_{\mathrm{Rep}_k(M)}\big(\mathrm{L}(U,X),Y\big) \xrightarrow{\cong} \mathrm{RHom}_{\mathrm{Rep}_k(G)}\big(X, i_M^G(Y)\big) \qquad \text{in } \mathrm{D}(k),$$

for all $X \in \mathrm{D}(G)$ and $Y \in \mathrm{D}(M)$.

Proof. By Lemma 2.1.3 and Theorem 3.2.3 it follows that $\mathrm{L}(U,-) = \mathrm{L}_U \circ \mathrm{Res}_P^G$ is the left adjoint of i_M^G. The last statement is analogous to Corollary 3.2.4 (use that Ind_P^G is exact). □

Fix a compact, open, torsion-free subgroup K of G. For any closed subgroup H of G we write $K_H := K \cap H$. Let M^+ and P^+ be the positive monoids of §3.4 associated with K_P. Then [BK98, (6.14)] shows that Hypothesis 3.4.1 is satisfied.

Proposition 4.1.2. *There is a natural isomorphism*

$$\mathrm{ind}_{M^+}^M \mathrm{L}_{K_U} \mathrm{Res}_{P^+}^G \overset{\cong}{\Longrightarrow} \mathrm{L}(U,-)$$

of functors $\mathrm{D}(G) \longrightarrow \mathrm{D}(M)$.

Proof. Immediate from Proposition 3.4.6. □

Notation 4.1.3. For any $n \in \mathbb{Z}$ we denote by $\mathrm{L}^n(U,-)\colon \mathrm{Rep}_k(G) \longrightarrow \mathrm{Rep}_k(M)$ the n-th cohomology functor of $\mathrm{L}(U,-)$, that is,

$$\mathrm{L}^n(U,V) := \mathrm{H}^n\big(\mathrm{L}(U,V[0])\big), \qquad \text{for } V \in \mathrm{Rep}_k(G).$$

We record the following consequence of Theorem 4.1.1:

Corollary 4.1.4. *Given $V \in \mathrm{Rep}_k(G)$ and $W \in \mathrm{Rep}_k(M)$, there is a convergent first-quadrant spectral sequence*

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathrm{Rep}_k(M)}\big(\mathrm{L}^{-j}(U,V),W\big) \Longrightarrow \mathrm{Ext}^{i+j}_{\mathrm{Rep}_k(G)}\big(V, i_M^G W\big).$$

In particular, there is a five-term exact sequence

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \mathrm{Ext}^1_{\mathrm{Rep}_k(M)}\big(\mathrm{L}^0(U,V),W\big) \arrow[r] & \mathrm{Ext}^1_{\mathrm{Rep}_k(G)}\big(V,i_M^G W\big) \arrow[r] & \mathrm{Hom}_{\mathrm{Rep}_k(M)}\big(\mathrm{L}^{-1}(U,V),W\big) \arrow[dll, "d_2^{0,1}"'] \\
& \mathrm{Ext}^2_{\mathrm{Rep}_k(M)}\big(\mathrm{L}^0(U,V),W\big) \arrow[r] & \mathrm{Ext}^2_{\mathrm{Rep}_k(G)}\big(V,i_M^G W\big). &
\end{tikzcd}$$

Proof. The functor $\mathrm{RHom}_{\mathrm{Rep}_k(M)}(-,W)$ is left t-exact and $\mathrm{L}(U,V)$ is right bounded. Now, Lemma 2.3.20.(b) together with Theorem 4.1.1 yields the spectral sequence. □

Recall the smooth character $\delta_P\colon P \longrightarrow k^\times$ constructed before Lemma 3.4.21.

Proposition 4.1.5. *For each $n \in \mathbb{Z}$ there is a natural isomorphism*

$$\mathrm{ind}_{M^+}^M \mathrm{H}^0 \circ \mathrm{RH}^0\big(K_U, \omega_P \otimes_k \mathrm{Res}_{P^+}^G(-)\big) \overset{\cong}{\Longrightarrow} \mathrm{H}^0 \circ \mathrm{L}(U,-)$$

of functors $\mathrm{D}(G) \longrightarrow \mathrm{Rep}_k(M)$. In particular, $\mathrm{L}^{-n}(U,-)$ vanishes provided $n \notin \{0,1,\dots,\dim K_U\}$.

Proof. Combine the previous proposition with Proposition 3.4.23. □

Corollary 4.1.6. *There are adjunctions*

$$\mathrm{L}(U,-)\colon \mathrm{D}^+(G) \rightleftarrows \mathrm{D}^+(M) : i_M^G,$$
$$\mathrm{L}(U,-)\colon \mathrm{D}^{\mathrm{b}}(G) \rightleftarrows \mathrm{D}^{\mathrm{b}}(M) : i_M^G.$$

Proof. This is immediate from Proposition 4.1.5. □

Proposition 4.1.7. *Let $Q \subseteq P$ be a parabolic subgroup of G with Levi quotient L and unipotent radical U'.*

(a) The diagram

$$\begin{tikzcd} \mathrm{D}(G) \arrow[r, "\mathrm{L}(U,-)"] \arrow[dr, "\mathrm{L}(U',-)"'] & \mathrm{D}(M) \arrow[d, "\mathrm{L}(U'/U,-)"] \\ & \mathrm{D}(L) \end{tikzcd}$$

is commutative.

(b) Given $V \in \mathrm{Rep}_k(G)$, there is a convergent third-quadrant spectral sequence

$$E_2^{i,j} = \mathrm{L}^i\big(U'/U, \mathrm{L}^j(U,V)\big) \Longrightarrow \mathrm{L}^{i+j}(U',V).$$

In particular, there is a five-term exact sequence

$$\begin{tikzcd} \mathrm{L}^{-2}(U',V) \arrow[r] & \mathrm{L}^{-2}\big(U'/U, \mathrm{L}^0(U,V)\big) \arrow[r, "d^{-2,0}"] & \mathrm{L}^0\big(U'/U, \mathrm{L}^{-1}(U,V)\big) \arrow[dll, rounded corners] \\ \mathrm{L}^{-1}(U',V) \arrow[r] & \mathrm{L}^{-1}\big(U'/U, \mathrm{L}^0(U,V)\big) \arrow[r] & 0. \end{tikzcd}$$

Proof. (a) Clearly, there is a natural isomorphism $\mathrm{L}^0(U'/U,-) \circ \mathrm{L}^0(U,-) \xRightarrow{\cong} \mathrm{L}^0(U',-)$ of functors $\mathrm{Rep}_k(G) \longrightarrow \mathrm{Rep}_k(L)$. Passing to the right adjoints, see Example 2.1.13, yields a natural isomorphism $i_L^G \xRightarrow{\cong} i_M^G \circ i_L^M$. But this extends to a natural isomorphism of functors $\mathrm{D}(L) \longrightarrow \mathrm{D}(G)$. Hence, passing to the left adjoints yields a natural isomorphism $\mathrm{L}(U'/U,-) \circ \mathrm{L}(U,-) \xRightarrow{\cong} \mathrm{L}(U',-)$.

(b) The functors $\mathrm{L}(U'/U,-)$ and $\mathrm{L}(U,-)$ are right t-exact (as left adjoints of t-exact functors). Now, Lemma 2.3.20.(a) applied to (a) yields the desired spectral sequence. □

As an immediate consequence of Corollary 3.2.24 we have the following result:

Proposition 4.1.8. *There is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{D}(k)}\big(\mathrm{Res}_1^M \mathrm{L}(U,X), Y\big) \cong \mathrm{Hom}_{\mathrm{D}(U)}\big(\mathrm{Res}_U^G X, \mathrm{Inf}_U^1 Y\big)$$

for all $X \in \mathrm{D}(G)$, $Y \in \mathrm{D}(k)$. In particular, for every $n \in \mathbb{Z}$ there exists a natural k-linear isomorphism

$$\mathrm{Hom}_k\big(\mathrm{L}^{-n}(U,V), W\big) \xrightarrow{\cong} \mathrm{Ext}^n_{\mathrm{Rep}_k(U)}(V, \mathrm{Inf}_U^1 W),$$

for all $V \in \mathrm{Rep}_k(G)$ and $W \in \mathrm{Vect}_k$.

4.2. **Preservation of global admissibility.** Retain the notation of §4.1. Let $\overline{P}$ be the parabolic subgroup of G opposite P with unipotent radical $\overline{U}$ and Levi quotient M. We make the further assumption that the compact, open, torsion-free subgroup $K \subseteq G$ admits an Iwahori decomposition:

$$K = K_{\overline{U}} K_M K_U.$$

Recall from [SS22b, Def. 4.2] the following definition:

Definition. Let H be a p-adic Lie group and $H' \subseteq H$ a compact, open, torsion-free subgroup. A complex $X \in \mathrm{D}(H)$ is called *globally admissible* if for every $i \in \mathbb{Z}$ the k-vector space $\mathrm{H}^i(H', X)$ is finite-dimensional. By [SS22b, Cor. 4.6], this definition is independent of the choice of H'.

The strictly full triangulated subcategory of $\mathrm{D}(H)$ consisting of the globally admissible complexes is denoted $\mathrm{D}(H)^{\mathrm{a}}$.

In this section we prove the following result:

Theorem 4.2.1. *There is an adjunction*

$$\mathrm{L}(U,-)\colon \mathrm{D}(G)^{\mathrm{a}} \rightleftarrows \mathrm{D}(M)^{\mathrm{a}} : i_M^G.$$

We will prove separately that $\mathrm{L}(U,-)$ and i_M^G preserve globally admissible complexes. Before turning to the proofs, let us observe an immediate consequence.

Given a p-adic Lie group H we denote by $\mathrm{D}^+_{\mathrm{adm}}(H)$, resp. $\mathrm{D}^{\mathrm{b}}_{\mathrm{adm}}(H)$, the strictly full subcategory of $\mathrm{D}^+(H)$, resp. $\mathrm{D}^{\mathrm{b}}(H)$, consisting of those complexes X such that $\mathrm{H}^i(X)$ is admissible, for every $i \in \mathbb{Z}$.

Corollary 4.2.2. *There are adjunctions*

$$\mathrm{L}(U,-)\colon \mathrm{D}^+_{\mathrm{adm}}(G) \rightleftarrows \mathrm{D}^+_{\mathrm{adm}}(M) : i_M^G,$$
$$\mathrm{L}(U,-)\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{adm}}(G) \rightleftarrows \mathrm{D}^{\mathrm{b}}_{\mathrm{adm}}(M) : i_M^G.$$

Proof. By [SS22b, Prop. 4.9] we have $\mathrm{D}^+_{\mathrm{adm}}(G) = \mathrm{D}^+(G) \cap \mathrm{D}(G)^{\mathrm{a}}$ and, *a fortiori*, $\mathrm{D}^{\mathrm{b}}_{\mathrm{adm}}(G) = \mathrm{D}^{\mathrm{b}}(G) \cap \mathrm{D}(G)^{\mathrm{a}}$; similarly with G replaced by M. As $\mathrm{L}(U,-)$ and i_M^G preserve bounded complexes, the claim follows from Theorem 4.2.1. $\square$

4.2.1. *Parabolic induction preserves global admissibility.* It is well-known that the parabolic induction functor $i_M^G\colon \mathrm{Rep}_k(M) \longrightarrow \mathrm{Rep}_k(G)$ preserves admissibility, [Vig96, II.2.1]. In order to prove that derived parabolic induction $i_M^G = \mathrm{Ind}_P^G \mathrm{Inf}_P^M$ preserves global admissibility, we show separately that Inf_P^M and Ind_P^G preserve global admissibility.

Lemma 4.2.3. *One has* $\mathrm{Ind}_P^G\big(\mathrm{D}(P)^{\mathrm{a}}\big) \subseteq \mathrm{D}(G)^{\mathrm{a}}$.

Proof. Let $X \in \mathrm{D}(P)^{\mathrm{a}}$ be globally admissible. We need to show that $\mathrm{H}^i(K, \mathrm{Ind}_P^G X)$ is finite-dimensional, for all $i \in \mathbb{Z}$. Note that, for every $g \in G$ the subgroup $P \cap gKg^{-1} \subseteq P$ is compact, open, and torsion-free. Applying the derived Mackey decomposition, Lemma 3.2.15.(a), we deduce

$$\begin{aligned}
\mathrm{RH}^0(K, \mathrm{Ind}_P^G X) &\cong \mathrm{RH}^0\Big(K, \prod_{g \in P\backslash G/K} \mathrm{Ind}^K_{g^{-1}Pg\cap K}\, g_*^{-1} \mathrm{Res}^P_{P\cap gKg^{-1}} X\Big)\\
&\cong \prod_{g \in P\backslash G/K} \mathrm{RH}^0\big(K, \mathrm{Ind}^K_{g^{-1}Pg\cap K}\, g_*^{-1} \mathrm{Res}^P_{P\cap gKg^{-1}} X\big)\\
&\cong \prod_{g \in P\backslash G/K} \mathrm{RH}^0\big(g^{-1}Pg\cap K, g_*^{-1} \mathrm{Res}^P_{P\cap gKg^{-1}} X\big)\\
&\cong \prod_{g \in P\backslash G/K} \mathrm{RH}^0\big(P\cap gKg^{-1}, \mathrm{Res}^P_{P\cap gKg^{-1}} X\big).
\end{aligned}$$

Note that the product is finite, because $P\backslash G$ is compact and K is open. The second isomorphism holds, since $\mathrm{RH}^0(K,-)$ is an additive functor. For the third isomorphism, observe that there is a natural isomorphism $\mathrm{Inf}^1_{g^{-1}Pg\cap K} \xRightarrow{\cong} \mathrm{Res}^K_{g^{-1}Pg\cap K} \mathrm{Inf}^1_K$, so that passing to the right adjoints (see Example 2.1.13) yields an isomorphism $\mathrm{RH}^0(K,-)\,\mathrm{Ind}^K_{g^{-1}Pg\cap K} \xRightarrow{\cong} \mathrm{RH}^0(g^{-1}Pg\cap K,-)$. The last isomorphism is given by conjugation.

As X is globally admissible, the right hand side in the above computation has finite-dimensional cohomology, hence so does the left hand side. Consequently, $\mathrm{Ind}_P^G X$ is globally admissible. $\square$

Before proving that Inf_P^M preserves global admissibility, we recall a well-known general fact.

Lemma 4.2.4. *Let $\mathscr{C}$ be a tensor triangulated category which is compactly generated by the tensor unit $\mathbf{1}$. Let $\mathscr{T}$ be a full triangulated subcategory of $\mathscr{C}$. One has $T \otimes X \in \mathscr{T}$ for every $T \in \mathscr{T}$ and every compact $X \in \mathscr{C}$.*

Proof. Consider the full subcategory $\mathscr{S}$ of $\mathscr{C}$ consisting of those X such that for all $T \in \mathscr{T}$ it holds that $T \otimes X \in \mathscr{T}$. As $\mathscr{T}$ is triangulated, it is clear that $\mathscr{S}$ is triangulated. By [Nee92, Lem. 2.2] the full triangulated subcategory $\langle \mathbf{1} \rangle$ generated by $\mathbf{1}$ contains precisely the compact objects. Since $\mathbf{1} \in \mathscr{S}$, we deduce $\langle \mathbf{1} \rangle \subseteq \mathscr{S}$, that is, $\mathscr{S}$ contains all compact objects. This implies the assertion. $\square$

Lemma 4.2.5. *One has* $\mathrm{Inf}_P^M\big(\mathrm{D}(M)^{\mathrm{a}}\big) \subseteq \mathrm{D}(P)^{\mathrm{a}}$.

Proof. Given $X \in \mathrm{D}(M)$, we observe that $\mathrm{Res}_{K_M}^M(X)$ is globally admissible if and only if X is; similarly with M replaced by P. Therefore, it suffices to prove $\mathrm{Inf}_{K_P}^{K_M}\big(\mathrm{D}(K_M)^{\mathrm{a}}\big) \subseteq \mathrm{D}(K_P)^{\mathrm{a}}$. Let $X \in \mathrm{D}(K_M)^{\mathrm{a}}$ be globally admissible. By Proposition 3.1.6 we know that $\mathrm{RH}^0(K_U, \mathbf{1})$ is compact. Now, Lemma 4.2.4 implies $X \otimes_k \mathrm{RH}^0(K_U, \mathbf{1}) \in \mathrm{D}(K_M)^{\mathrm{a}}$. Applying the projection formula, Lemma 3.1.2, we deduce that

$$\mathrm{RH}^0\big(K_P, \mathrm{Inf}_{K_P}^{K_M} X\big) \cong \mathrm{RH}^0\big(K_M, \mathrm{RH}^0(K_U, \mathrm{Inf}_{K_P}^{K_M} X)\big) \cong \mathrm{RH}^0\big(K_M, X \otimes_k \mathrm{RH}^0(K_U, \mathbf{1})\big)$$

has finite-dimensional cohomology. Therefore, $\mathrm{Inf}_{K_P}^{K_M} X$ is globally admissible. $\square$

Proposition 4.2.6. *Derived parabolic induction restricts to a functor* $i_M^G \colon \mathrm{D}(M)^{\mathrm{a}} \longrightarrow \mathrm{D}(G)^{\mathrm{a}}$.

Proof. Combine Lemmas 4.2.3 and 4.2.5. $\square$

4.2.2. $\mathrm{L}(U,-)$ *preserves global admissibility.* Fix a positive central element $z_0 \in M$ satisfying the conditions in [BK98, (6.14)]. In particular, z_0 is strictly positive and satisfies $z_0 K_{\overline{U}} z_0^{-1} \supsetneq K_{\overline{U}}$. Let C be the central subgroup of M generated by z_0. Then $C^+ \coloneqq M^+ \cap C$ is the submonoid generated by z_0. Note that $K_M C$ is a direct product, since $C \cap K_M = \{1\}$.

Lemma 4.2.7. *The diagram*

$$\begin{tikzcd}
\mathrm{D}(G) \arrow[r, "\mathrm{L}(U,-)"] \arrow[d, "\mathrm{Res}^G_{K_P C^+}"'] & \mathrm{D}(M) \arrow[r, "\mathrm{Res}^M_{K_M C}"] & \mathrm{D}(K_M C) \arrow[d, "\mathrm{L}_{K_M}"] \\
\mathrm{D}(K_P C^+) \arrow[r, "\mathrm{L}_{K_P}"'] & \mathrm{D}(C^+) \arrow[r, "\mathrm{ind}^C_{C^+}"'] & \mathrm{D}(C)
\end{tikzcd}$$

is commutative. In particular, there is a natural k-linear isomorphism

$$\mathrm{H}^i\big(K_M, \mathrm{L}(U,X)\big) \cong \mathrm{ind}_{C^+}^C \mathrm{H}^{i+\dim K_U}(K_P, X)$$

for every $X \in \mathrm{D}(G)$.

Proof. We compute

$$\begin{aligned}
\mathrm{L}_{K_M} \mathrm{Res}^M_{K_M C} \mathrm{L}(U,-) &= \mathrm{L}_{K_M} \mathrm{Res}^M_{K_M C} \mathrm{L}_U \mathrm{Res}^G_P \\
&\cong \mathrm{L}_{K_M} \mathrm{L}_U \mathrm{Res}^P_{UK_M C} \mathrm{Res}^G_P && \text{(by Prop. 3.2.19)} \\
&\cong \mathrm{L}_{K_M} \mathrm{ind}^{K_M C}_{K_M C^+} \mathrm{L}_{K_U} \mathrm{Res}^{UK_M C}_{K_P C^+} \mathrm{Res}^G_{UK_M C} && \text{(by Prop. 3.4.6)} \\
&\cong \mathrm{ind}^C_{C^+} \mathrm{L}_{K_M} \mathrm{L}_{K_U} \mathrm{Res}^G_{K_P C^+} && \text{(by Prop. 3.2.11)} \\
&\cong \mathrm{ind}^C_{C^+} \mathrm{L}_{K_P} \mathrm{Res}^G_{K_P C^+} && \text{(by Prop. 3.2.10)}
\end{aligned}$$

which proves the first assertion. Now, passing to cohomology and applying Proposition 3.4.23, we deduce a C-equivariant natural isomorphism

$$\mathrm{H}^{i+\dim K_M}\big(K_M, \delta_{K_M C} \otimes \mathrm{L}(U,X)\big) \cong \mathrm{ind}^C_{C^+} \mathrm{H}^{i+\dim K_P}\big(K_P, \delta_{UK_M C} \otimes X\big).$$

As $\delta_{K_M C}$ is trivial on K_M, the left hand side becomes $\delta_{K_M C} \otimes \mathrm{H}^{i+\dim K_M}(K_M, \mathrm{L}(U,X))$ by the projection formula, Lemma 3.1.2. As $\delta_{UK_M C}$ is trivial on K_P and defined on C, the right hand side becomes $\delta_{UK_M C} \otimes \operatorname{ind}_{C^+}^{C} \mathrm{H}^{i+\dim K_P}(K_P, X)$. Bearing in mind that $\dim K_U = \dim K_P - \dim K_M$, the last assertion follows. □

We recall the Hecke action of C^+ on $\mathrm{H}^0(K,V)$, for $V \in \operatorname{Rep}_k(G)$: Note first that the inclusion $K_U \subseteq K$ induces, for each $z \in C^+$, a bijection

$$K_U/zK_Uz^{-1} \cong K/K_{\overline{P}}zK_Uz^{-1}, \tag{4.2.8}$$

since $K = K_U K_{\overline{P}}$. (By our choice of z_0, $K_{\overline{P}}zK_Uz^{-1} = K \cap zKz^{-1}$ is an open subgroup of K.) Then the composite

$$\begin{aligned} z \star (-) \colon\ \mathrm{H}^0(K,V) &\xrightarrow{z\cdot} \mathrm{H}^0\big(K_{\overline{P}}zK_Uz^{-1}, V\big) \xrightarrow{\text{cores}} \mathrm{H}^0(K,V), \\ v &\longmapsto zv \longmapsto \textstyle\sum_{u \in K_U/zK_Uz^{-1}} uzv \end{aligned}$$

defines the Hecke action of C^+ on $\mathrm{H}^0(K,V)$ such that the natural inclusion $\mathrm{H}^0(K,V) \subseteq \mathrm{H}^0(K_P,V)$ is C^+-equivariant. We obtain a left exact functor $\mathrm{H}^0(K,-)\colon \operatorname{Rep}_k(G) \longrightarrow \operatorname{Rep}_k(C^+)$ and hence a derived functor

$$\mathrm{RH}^0(K,-)\colon \mathrm{D}(G) \longrightarrow \mathrm{D}(C^+)$$

on the derived categories.

Lemma 4.2.9. *The inclusion $\mathrm{H}^0(K,-) \hookrightarrow \mathrm{H}^0(K_P,-)$ induces a natural isomorphism*

$$\operatorname{ind}_{C^+}^{C} \mathrm{RH}^0(K,-) \xrightarrow{\cong} \operatorname{ind}_{C^+}^{C} \mathrm{RH}^0(K_P,-) \operatorname{Res}^G_{K_P C^+}.$$

Proof. We first show that $\operatorname{Res}^G_{K_P C^+}$ sends injective objects to $\mathrm{H}^0(K_P,-)$-acyclic objects. As K_P is open in $K_P C^+$, it follows that $\operatorname{Res}^{K_P C^+}_{K_P}$ admits an exact left adjoint and hence preserves injective objects (Lemma 2.1.8). As $\mathrm{H}^0(K_P,-)$ has finite cohomological dimension, the functor $\mathrm{RH}^0(K_P,-)$ can be computed using acyclic objects, see [Har66, Cor. 5.3 γ]. Hence, there is an isomorphism

$$\operatorname{Res}^{C^+}_1 \mathrm{RH}^0(K_P,-) \cong \mathrm{RH}^0(K_P,-) \operatorname{Res}^{K_P C^+}_{K_P}.$$

Now, if J in $\operatorname{Rep}_k(G)$ is injective, then so is $\operatorname{Res}^G_{K_P}(J)$ by [Eme10b, Prop. 2.1.11] and therefore,

$$\operatorname{Res}^{C^+}_1 \mathrm{H}^i\big(K_P, \operatorname{Res}^G_{K_P C^+}(J)\big) = \mathrm{H}^i\big(K_P, \operatorname{Res}^G_{K_P}(J)\big) = 0, \qquad \text{for all } i > 0.$$

Hence, $\operatorname{Res}^G_{K_P C^+}(J)$ is $\mathrm{H}^0(K_P,-)$-acyclic.

We are thus reduced to showing that the natural map

$$\operatorname{ind}_{C^+}^{C} \mathrm{H}^0(K,V) \longrightarrow \operatorname{ind}_{C^+}^{C} \mathrm{H}^0(K_P,V) \tag{4.2.10}$$

is bijective, for all $V \in \operatorname{Rep}_k(G)$. As $\operatorname{ind}_{C^+}^{C}$ is exact, the injectivity is clear. The surjectivity follows from the same argument as in [Eme10b, Lem. 3.4.5] which we recall here for the benefit of the reader: Let $v \in \mathrm{H}^0(K_P,V)$ be arbitrary. As V is smooth, there exists $z \in C^+$ such that v is fixed by $z^{-1}K_{\overline{U}}zK_P = z^{-1}Kz \cap K$. Now, we have

$$z \star v = \sum_{u \in K_U/zK_Uz^{-1}} uzv \in \mathrm{H}^0(K,V),$$

because zv is fixed by $K_{\overline{P}}zK_Uz^{-1}$ and u runs through a representing system of $K/K_{\overline{P}}zK_Uz^{-1}$, see (4.2.8). Therefore, $z^{-1} \otimes z \star v$ maps to $1 \otimes v \in \operatorname{ind}_{C^+}^{C} \mathrm{H}^0(K_P,V)$ under (4.2.10). This finishes the proof. □

Lemma 4.2.11. *Let $V \in \operatorname{Rep}_k(C^+)$ be a finite-dimensional representation. The canonical map*

$$\begin{aligned} V &\twoheadrightarrow \operatorname{ind}_{C^+}^{C} V, \\ v &\longmapsto 1 \otimes v \end{aligned}$$

is surjective.

Proof. This is immediate from the proof of [Eme10b, Lem. 3.2.1 (1)]. $\square$

Proposition 4.2.12. *The functor* $\mathrm{L}(U,-)$ *restricts to a functor* $\mathrm{D}(G)^{\mathrm{a}} \longrightarrow \mathrm{D}(M)^{\mathrm{a}}$.

Proof. Let X in $\mathrm{D}(G)^{\mathrm{a}}$ be a globally admissible complex and let $i \in \mathbb{Z}$ be arbitrary. Since $\mathrm{Res}_1^{C^+} \mathrm{RH}^0(K,-) \cong \mathrm{RH}^0(K,-) \mathrm{Res}_K^G$, we know that $\mathrm{H}^i(K,X)$ is a finite-dimensional smooth C^+-representation. Now, the sequence of maps

$$\begin{aligned} \mathrm{H}^i(K,X) &\twoheadrightarrow \mathrm{ind}_{C^+}^{C} \mathrm{H}^i(K,X) && \text{(Lemma 4.2.11)} \\ &\xrightarrow{\cong} \mathrm{ind}_{C^+}^{C} \mathrm{H}^i(K_P,X) && \text{(Lemma 4.2.9)} \\ &\xrightarrow{\cong} \mathrm{H}^{i-\dim K_U}\big(K_M, \mathrm{L}(U,X)\big) && \text{(Lemma 4.2.7)} \end{aligned}$$

shows that $\mathrm{H}^{i-\dim K_U}\big(K_M, \mathrm{L}(U,X)\big)$ is finite-dimensional. Therefore $\mathrm{L}(U,X)$ is globally admissible. $\square$

Finally, Propositions 4.2.6 and 4.2.12 together imply Theorem 4.2.1.

4.3. **On the Satake homomorphism.** The functor $\mathrm{L}(U,-)\colon \mathrm{D}(G) \longrightarrow \mathrm{D}(M)$ can be used to define a derived version of the Satake homomorphism, which we will now describe.

We fix an open compact mod center subgroup $K \subseteq G$ satisfying the Iwasawa decomposition $G = PK$ and $K_P = K_U K_M$. Writing $\mathrm{L}(K_U,-) \coloneqq \mathrm{L}_{K_U} \circ \mathrm{Res}_{K_P}^K \colon \mathrm{D}(K) \longrightarrow \mathrm{D}(K_M)$, there are natural isomorphisms

$$\mathrm{L}(U,-)\,\mathrm{ind}_K^G = \mathrm{L}_U \,\mathrm{Res}_P^G \,\mathrm{ind}_K^G \overset{\cong}{\Longrightarrow} \mathrm{L}_U \,\mathrm{ind}_{K_P}^P \,\mathrm{Res}_{K_P}^K \overset{\cong}{\Longrightarrow} \mathrm{ind}_{K_M}^M \mathrm{L}(K_U,-),$$

where the first isomorphism follows from the Iwasawa decomposition and the last is the one from Proposition 3.2.11.

Definition. Given $X \in \mathrm{D}(K)$, we call the k-algebra homomorphism

$$\mathcal{S}_X\colon \mathrm{End}_{\mathrm{D}(G)}\big(\mathrm{ind}_K^G X\big) \longrightarrow \mathrm{End}_{\mathrm{D}(M)}\big(\mathrm{ind}_{K_M}^M \mathrm{L}(K_U,X)\big)$$

induced by $\mathrm{L}(U,-)$ the *derived Satake homomorphism*.

If $V \in \mathrm{Rep}_k(K)$, composing $\mathcal{S}_V$ with the cohomology functor H^{-n} yields a homomorphism

$$\mathcal{S}_V^n\colon \mathrm{End}_{\mathrm{Rep}_k(G)}\big(\mathrm{ind}_K^G V\big) \longrightarrow \mathrm{End}_{\mathrm{Rep}_k(M)}\big(\mathrm{ind}_{K_M}^M \mathrm{L}^{-n}(K_U,V)\big)$$

of k-algebras, which we call the *n-th Satake homomorphism.* (As usual, we write $\mathrm{L}^{-n}(K_U,V) \coloneqq \mathrm{H}^{-n}(\mathrm{L}(K_U,V[0]))$.) Note that $\mathcal{S}_V^n$ is zero whenever $n \notin \{0,1,\dots,\dim K_U\}$.

Remark 4.3.1. One could also define a variant of the derived Satake homomorphism via

$$\mathrm{RHom}_{\mathrm{Rep}_k(G)}\big(\mathrm{ind}_K^G V, \mathrm{ind}_K^G V\big) \longrightarrow \mathrm{RHom}_{\mathrm{Rep}_k(M)}\big(\mathrm{ind}_{K_M}^M \mathrm{L}(K_U,V), \mathrm{ind}_{K_M}^M \mathrm{L}(K_U,V)\big).$$

It might be interesting to relate the induced map on the cohomology algebras to Ronchetti's derived Satake homomorphism, [Ron19].

The aim of this section is to show that $\mathcal{S}_V^0$ and $\mathcal{S}_V^{\dim K_U}$ are the well-known variants of the Satake homomorphism that were introduced by Herzig, [Her11b], and extensively studied by Herzig and Henniart–Vignéras, [Her11a, HV15, HV12].

Theorem 4.3.2. *Let* $V \in \mathrm{Rep}_k(K)$.

(a) Denote $\eta\colon V \twoheadrightarrow \mathrm{L}^0(K_U,V)$ *the natural projection map. The map* $\mathcal{S}_V^0$ *is explicitly given by*

$$\begin{gathered} \mathrm{End}_{\mathrm{Rep}_k(G)}(\mathrm{ind}_K^G V) \longrightarrow \mathrm{End}_{\mathrm{Rep}_k(M)}\big(\mathrm{ind}_{K_M}^M \mathrm{L}^0(K_U,V)\big), \\ \mathcal{S}_V^0(\Phi)([1,\eta(v)])(m) = \eta\Big(\sum_{u \in K_U\backslash U} \Phi([1,v])(um)\Big) \end{gathered}$$

for all $\Phi \in \mathrm{End}_{\mathrm{Rep}_k(G)}(\mathrm{ind}_K^G V)$, $v \in V$, *and* $m \in M$.

(b) The map $\mathcal{S}_V^{\dim K_U}$ *is explicitly given by*

$$\begin{aligned}\mathrm{End}_{\mathrm{Rep}_k(G)}(\mathrm{ind}_K^G V) &\longrightarrow \mathrm{End}_{\mathrm{Rep}_k(M)}\big(\mathrm{ind}_{K_M}^M \mathrm{H}^0(K_U, V)\big),\\ \mathcal{S}_V^{\dim K_U}(\Phi)([1,v])(m) &= \sum_{u\in U/K_U} \Phi([1,v])(mu)\end{aligned}$$

for all $\Phi \in \mathrm{End}_{\mathrm{Rep}_k(G)}(\mathrm{ind}_K^G V)$*,* $v \in \mathrm{H}^0(K_U, V)$*, and* $m \in M$*.*

Explanation 4.3.3. Since $\delta_P \otimes -$ is an equivalence of categories, there are natural isomorphisms $\mathrm{H}^0(K_U, -) \circ (\delta_P \otimes -) \overset{\cong}{\Longrightarrow} (\delta_P \otimes -) \circ \mathrm{H}^0(K_U, -)$ and $(\delta_P \otimes -) \circ \mathrm{ind}_{K_M}^M \overset{\cong}{\Longrightarrow} \mathrm{ind}_{K_M}^M \circ (\delta_P \otimes -)$. By Proposition 3.4.23 we have a natural isomorphism $\mathrm{L}^{-\dim K_U}(K_U, V) \cong \mathrm{H}^0(K_U, \delta_P \otimes V) \cong \delta_P \otimes \mathrm{H}^0(K_U, V)$. Consequently, the target of $\mathcal{S}_V^{\dim K_U}$ may be identified with

$$\mathrm{End}_{\mathrm{Rep}_k(M)}\big(\mathrm{ind}_{K_M}^M \mathrm{L}^{-\dim K_U}(K_U, V)\big) \cong \mathrm{End}_{\mathrm{Rep}_k(M)}\big(\mathrm{ind}_{K_M}^M \mathrm{H}^0(K_U, V)\big),$$

which justifies the formulation in Theorem 4.3.2.(b).

The proof of Theorem 4.3.2.(b) needs some preparation and is deferred until the end of the section. Statement (a) is well-known. Since I could not find a precise reference to the literature, I will recall the easy proof for the convenience of the reader.

Proof of Theorem 4.3.2.(a). By definition the diagram

$$\begin{tikzcd} \mathrm{ind}_K^G V \arrow[r, two heads, "\eta_U"] \arrow[d, "\Phi"'] & \mathrm{L}^0(U, \mathrm{ind}_K^G V) \arrow[r, "\cong"] \arrow[d, "\mathrm{L}^0(U,\Phi)"] & \mathrm{ind}_{K_M}^M \mathrm{L}^0(K_U, V) \arrow[d, "\mathcal{S}_V^0(\Phi)"] \\ \mathrm{ind}_K^G V \arrow[r, two heads, "\eta_U"'] & \mathrm{L}^0(U, \mathrm{ind}_K^G V) \arrow[r, "\cong"'] & \mathrm{ind}_{K_M}^M \mathrm{L}^0(K_U, V) \end{tikzcd}$$

is commutative, where the horizontal compositions are given explicitly by

$$[g, v] \longmapsto [\mathrm{pr}_M(g), \eta(v)], \qquad \text{for } g \in P \text{ and } v \in V.$$

(Recall that $G = PK$.) Fix a complete representing system $\mathcal{M}$ of $K_M \backslash M$ in M. Then the map

$$\begin{aligned} K_U\backslash U \times \mathcal{M} &\xrightarrow{\cong} K_P\backslash P,\\ (K_U u, m) &\longmapsto K_P um \end{aligned}$$

is bijective. Using $\Phi([1,v]) = \sum_{u\in K_U\backslash U} \sum_{m\in\mathcal{M}} \big[(um)^{-1}, \Phi([1,v])(um)\big]$, we compute

$$\begin{aligned}\mathcal{S}_V^0(\Phi)\big([1,\eta(v)]\big) &= \sum_{u\in K_U\backslash U} \sum_{m\in\mathcal{M}} \big[m^{-1}, \eta\big(\Phi([1,v])(um)\big)\big]\\ &= \sum_{m\in\mathcal{M}} \Big[m^{-1}, \eta\Big(\sum_{u\in K_U\backslash U} \Phi([1,v])(um)\Big)\Big],\end{aligned}$$

which shows $\mathcal{S}_V^0(\Phi)\big([1,\eta(v)]\big)(m) = \eta\big(\sum_{u\in K_U\backslash U} \Phi([1,v])(um)\big)$ for all $m \in M$. □

4.3.1. *Finishing the proof of Theorem 4.3.2.(b).* Fix $W \in \mathrm{Rep}_k(K_P)$ and recall from Proposition 3.4.23 the isomorphism

$$\mathrm{L}_U^{-\dim K_U}(\mathrm{ind}_{K_P}^P W) \cong \mathrm{ind}_{M^+}^M \mathrm{H}^0\big(K_U, \delta_P \otimes \mathrm{ind}_{K_P}^P W\big) \cong \delta_P \otimes \mathrm{ind}_{M^+}^M \mathrm{H}^0(K_U, \mathrm{ind}_{K_P}^P W).$$

As was already remarked, the character δ_P can be neglected for the computation of the Satake homomorphism. The M^+-action on $\mathrm{H}^0(K_U, \mathrm{ind}_{K_P}^P W)$ is given by

$$m \star f := \sum_{u\in K_U/mK_Um^{-1}} umf,$$

for $m \in M^+$ and $f \in \operatorname{ind}_{K_P}^P W$. By the Mackey decomposition, Lemma 3.2.15, we have a natural isomorphism

$$\mathrm{H}^0\big(K_U, \operatorname{ind}_{K_P}^P W\big) \cong \bigoplus_{g \in K_U \backslash P / K_P} \mathrm{H}^0(K_U^g \cap K_P, W), \tag{4.3.4}$$

where $K_U^g \coloneqq g^{-1} K_U g$. We deduce that $\mathrm{H}^0(K_U, \operatorname{ind}_{K_P}^P W)$ is generated by elements of the form

$$[K_U g, w] \coloneqq \sum_{u \in K_U / K_U \cap K_P^{g^{-1}}} [ug, w], \quad \text{for } w \in \mathrm{H}^0(K_U^g \cap K_P, W),$$

where g runs through a representing system for $K_U \backslash P / K_P$. In order to describe the M^+-action on the $[K_U g, w]$ we make the following definition:

Definition. Given $m \in M^+$ and $g \in P$, we define the "projection map"

$$\begin{aligned} \mu_{m,g} \colon \mathrm{H}^0(K_U^g \cap K_P, W) &\longrightarrow \mathrm{H}^0(K_U^{mg} \cap K_P, W), \\ w &\longmapsto \sum_{u \in K_U^{mg} \cap K_P / K_U^g \cap K_P} uw. \end{aligned}$$

As m is positive, we have $K_U^{mg} \cap K_P \supseteq K_U^g \cap K_P$, and hence $\mu_{m,g}$ is well-defined.

Lemma 4.3.5. *Let $g \in P$ and $m, m_1, m_2 \in M^+$.*

- *(a) $\mu_{m,g} = \mathrm{id}_{\mathrm{H}^0(K_U, W)}$ provided $g \in P^+ = K_U M^+$.*
- *(b) $\mu_{m_1,g} = \mu_{m_2,g}$ provided $m_1 g, m_2 g \in P^+$.*
- *(c) $\mu_{1,g} = \mathrm{id}_{\mathrm{H}^0(K_U^g \cap K_P, W)}$.*
- *(d) $\mu_{m_1 m_2, g} = \mu_{m_1, m_2 g} \circ \mu_{m_2, g}$.*
- *(e) $\mu_{m,ugh}(w) = h^{-1}\mu_{m,g}(hw)$ for each $u \in K_U$, $h \in K_P$, $w \in \mathrm{H}^0(K_U^{ugh}, W)$.*

Proof. Note that, if $h \in P^+$, then $K_U^h \cap K_P = K_U$. Now, (a) follows by applying this observation to $h = g$ and $h = mg$, and (b) follows by applying it to $h = m_1 g$ and $h = m_2 g$.

(c) is trivial and (d) is straightforward. For (e) note that $K_U^u = K_U$ and $K_U^{mu} = K_U^m$, because $u, mum^{-1} \in K_U$. □

Notation 4.3.6. Given $g \in P$, we put

$$\mu_g \coloneqq \mu_{m,g} \colon \mathrm{H}^0(K_U^g \cap K_P, W) \longrightarrow \mathrm{H}^0(K_U, W),$$

where $m \in M^+$ is any element satisfying $mg \in P^+$. (Such elements exist by Hypothesis 3.4.1.) By Lemma 4.3.5.(b) the definition of μ_g is independent of the chosen m.

Lemma 4.3.7. *For all $g \in P$, $w \in \mathrm{H}^0(K_U^g \cap K_P, W)$ and $m \in M^+$ we have*

$$m \star [K_U g, w] = [K_U mg, \mu_{m,g}(w)] \qquad \text{in } \mathrm{H}^0(K_U, \operatorname{ind}_{K_P}^P W).$$

Proof. We compute

$$\begin{aligned}
m \star [K_U g, w] &= \sum_{n \in K_U/K_U^{m^{-1}}} \sum_{u \in K_U/K_U \cap K_P^{g^{-1}}} [nmug, w] \\
&= \sum_{n \in K_U^{mg}/K_U^g} \sum_{u \in K_U^g/K_U^g \cap K_P} [mgnu, w] \\
&= \sum_{n \in K_U^{mg}/K_U^g \cap K_P} [mgn, w] \\
&= \sum_{n \in K_U^{mg}/K_U^{mg} \cap K_P} \sum_{u \in K_U^{mg} \cap K_P/K_U^g \cap K_P} [mgnu, w] \\
&= \sum_{n \in K_U/K_U \cap K_P^{(mg)^{-1}}} \Big[nmg, \sum_{u \in K_U^{mg} \cap K_P/K_U^g \cap K_P} uw\Big] \\
&= [K_U mg, \mu_{m,g}(w)].
\end{aligned}$$

□

Proposition 4.3.8. *The natural map*

$$\begin{aligned}
\operatorname{ind}_{M^+}^M \mathrm{H}^0\big(K_U, \operatorname{ind}_{K_P}^P W\big) &\xrightarrow{\cong} \operatorname{ind}_{K_M}^M \mathrm{H}^0(K_U, W), \\
1 \otimes [K_U g, w] &\longmapsto \big[\operatorname{pr}_M(g), \mu_g(w)\big]
\end{aligned}$$

is an M-equivariant isomorphism.

Proof. We first observe that the inclusion $\operatorname{ind}_{K_P}^{P^+} W \subseteq \operatorname{ind}_{K_P}^P W$ induces an isomorphism

$$\operatorname{ind}_{M^+}^M \mathrm{H}^0(K_U, \operatorname{ind}_{K_P}^{P^+} W) \xrightarrow{\cong} \operatorname{ind}_{M^+}^M \mathrm{H}^0(K_U, \operatorname{ind}_{K_P}^P W).$$

Indeed, injectivity is clear, since $\mathrm{H}^0(K_U, -)$ and $\operatorname{ind}_{M^+}^M$ are left exact. The surjectivity follows from the fact that for every $g \in P$ there exists $m \in M^+$ such that $mg \in P^+$. Concretely, the inverse map is given by

$$\begin{aligned}
\operatorname{ind}_{M^+}^M \mathrm{H}^0(K_U, \operatorname{ind}_{K_P}^P W) &\longrightarrow \operatorname{ind}_{M^+}^M \mathrm{H}^0(K_U, \operatorname{ind}_{K_P}^{P^+} W), \\
1 \otimes [K_U g, w] &\longmapsto m^{-1} \otimes [K_U mg, \mu_{m,g}(w)],
\end{aligned} \tag{4.3.9}$$

where $m \in M^+$ is chosen such that $mg \in P^+$. (Note that the same m works if g is replaced by any other element in $K_U g$.)

We now prove that the map

$$\begin{aligned}
\mathrm{H}^0(K_U, \operatorname{ind}_{K_P}^{P^+} W) &\xrightarrow{\cong} \operatorname{ind}_{K_M}^{M^+} H^0(K_U, W), \\
[K_U m, w] &\longmapsto [m, w],
\end{aligned} \tag{4.3.10}$$

where $m \in M^+$, is an M^+-equivariant isomorphism. The M^+-equivariance of the map follows from Lemma 4.3.7 together with $\mu_{m',m} = \mathrm{id}$, for all $m', m \in M^+$ (Lemma 4.3.5.(a)). As for (4.3.4) the Mackey decomposition provides a canonical isomorphism

$$\mathrm{H}^0(K_U, \operatorname{ind}_{K_P}^{P^+} W) \cong \bigoplus_{m \in K_U \backslash P^+/K_P} \mathrm{H}^0(K_U, W).$$

Since the projection map $K_U \backslash P^+/K_P \longrightarrow M^+/K_M$ is bijective, we deduce that (4.3.10) is an isomorphism.

The map in the proposition is now given by (4.3.9) composed with $\operatorname{ind}_{M^+}^M$ (4.3.10) and hence is an isomorphism. □

Lemma 4.3.11. *The following assertions hold:*

(a)
$$K_P g K_U = \bigsqcup_{u \in K_U / K_U \cap g K_U g^{-1}} K_M u g K_U, \qquad \textit{for all } g \in P.$$
(b)
$$K_M \backslash P / K_U \cong K_M \backslash M \times U / K_U.$$

Proof. (a) is easy and (b) is trivial. □

Finally, we finish the proof of Theorem 4.3.2.

Proof of Theorem 4.3.2.(b). Let $V \in \operatorname{Rep}_k(K)$ and $\Phi \in \operatorname{End}_{\operatorname{Rep}_k(G)}(\operatorname{ind}_K^G V)$. Recall that we want to show
$$\mathcal{S}_V^{\dim K_U}(\Phi)([1,v])(m) = \sum_{u \in U/K_U} \Phi([1,v])(mu)$$
for all $v \in \mathrm{H}^0(K_U, V)$ and $m \in M$. Recall also that the Iwasawa decomposition $G = PK$ allows us to identify $\operatorname{ind}_K^G V$ with $\operatorname{ind}_{K_P}^P V$.

Note that $K_P g K_U = \bigsqcup_{u \in K_U \cap K_P^g \backslash K_U} K_P g u$. Therefore, we have
$$\begin{aligned}
\Phi([1,v]) &= \sum_{g \in K_P \backslash P} [g^{-1}, \Phi([1,v])(g)] \\
&= \sum_{g \in K_P \backslash P / K_U} \sum_{u \in K_U \cap K_P^g \backslash K_U} \big[u^{-1} g^{-1}, \Phi([1,v])(gu)\big] \\
&= \sum_{g \in K_P \backslash P / K_U} \sum_{u \in K_U \cap K_P^g \backslash K_U} \big[u^{-1} g^{-1}, \Phi([1,v])(g)\big],
\end{aligned}$$
where the last equality uses $v \in \mathrm{H}^0(K_U, V)$ and the K_U-equivariance of Φ. Viewing $\mathcal{S}_V^{\dim K_U}(\Phi)$ as an endomorphism on $\operatorname{ind}_{M^+}^M \mathrm{H}^0(K_U, \operatorname{ind}_{K_P}^P V)$ we deduce
$$\mathcal{S}_V^{\dim K_U}(\Phi)\big(1 \otimes [K_U, v]\big) = \sum_{g \in K_P \backslash P / K_U} 1 \otimes [K_U g^{-1}, \Phi([1,v])(g)].$$

Now, applying the isomorphism in Proposition 4.3.8, we view $\mathcal{S}_V^{\dim K_U}$ as an endomorphism of $\operatorname{ind}_{K_M}^M \mathrm{H}^0(K_U, V)$ and compute
$$\begin{aligned}
\mathcal{S}_V^{\dim K_U}(\Phi)([1,v]) &= \sum_{g \in K_P \backslash P / K_U} \big[\operatorname{pr}_M(g^{-1}), \mu_{g^{-1}}(\Phi([1,v])(g))\big] \\
&= \sum_{\substack{g \in K_P \backslash P / K_U, \\ u \in K_U / K_U \cap g K_U g^{-1}}} \big[\operatorname{pr}_M(g^{-1}), \Phi([1,v])(ug)\big] && (\text{def. of } \mu_{g^{-1}}) \\
&= \sum_{g \in K_M \backslash P / K_U} \big[\operatorname{pr}_M(g^{-1}), \Phi([1,v])(g)\big] && (\text{Lem. 4.3.11.(a)}) \\
&= \sum_{m \in K_M \backslash M} \Big[m^{-1}, \sum_{u \in U/K_U} \Phi([1,v])(mu)\Big] && (\text{Lem. 4.3.11.(b)}),
\end{aligned}$$
where for the second equality we have used $K_P \cap g K_U g^{-1} = K_U \cap g K_U g^{-1}$ (recall that U is normalized by P). This implies the claim. □

5. The example of $\mathrm{GL}_2(\mathbb{Q}_p)$

In this section we put $G := \mathrm{GL}_2(\mathbb{Q}_p)$ and let B be the Borel subgroup of upper triangular matrices with unipotent radical U and T the Levi subgroup of diagonal matrices. Further, we fix the maximal compact open subgroup $K := \mathrm{GL}_2(\mathbb{Z}_p)$ and denote by Z the center of $\mathrm{GL}_2(\mathbb{Q}_p)$. We fix an algebraically closed coefficient field k of characteristic p.

We will compute $\mathrm{L}^\bullet(U, V)$ for all irreducible smooth representations V of G.

5.1. **The weights of K.** The irreducible smooth representations of K have been classified by Barthel–Livné in [BL94, Prop. 4]. We recall the classification and set up some notation.

Definition. Given $0 \le r \le p-1$ and $0 \le e < p-1$, we consider

$$\mathrm{Sym}^r(k^2) \otimes \det{}^e. \tag{5.1.1}$$

Every irreducible smooth representation of K is isomorphic to some representation of the form (5.1.1). We identify $\mathrm{Sym}^r(k^2)$ with the subvector space of homogeneous polynomials of degree r in $k[x,y]$. Under this identification, K acts through its quotient $\mathrm{GL}_2(\mathbb{F}_p)$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(x,y) \coloneqq f(ax+cy, bx+dy), \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_p).$$

Thus, a k-basis of $\mathrm{Sym}^r(k^2)$ is given by $x^r, x^{r-1}y, \dots, y^r$, and we have

$$\begin{aligned} \mathrm{H}^0\big(K_U, \mathrm{Sym}^r(k^2)\big) &= \langle x^r \rangle_k, \\ \mathrm{L}^0\big(K_U, \mathrm{Sym}^r(k^2)\big) &= \mathrm{Sym}^r(k^2)/\langle x^r, x^{r-1}y, \dots, xy^{r-1}\rangle_k, \end{aligned}$$

where $\langle \dots \rangle_k$ denotes the k-linear span.

5.2. **Irreducible smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$.** We view $\mathrm{Sym}^r(k^2)$ as a smooth ZK-representation by letting p act trivially. Denote by X (resp. Y) the k-linear projection of $\mathrm{Sym}^r(k^2)$ onto $\langle x^r \rangle_k$ (resp. $\langle y \rangle_k$) with respect to the basis $x^r, x^{r-1}y, \dots, y^r$. We define the endomorphism $\Phi \in \mathrm{End}_{\mathrm{Rep}_k(G)}\big(\mathrm{ind}_{ZK}^G \mathrm{Sym}^r(k^2)\big)$ by

$$\Phi([1,v]) \coloneqq \Big[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1}, Y(v)\Big] + \sum_{i=0}^{p-1} \Big[\begin{pmatrix} 1 & i \\ 0 & p \end{pmatrix}^{-1}, X(\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \cdot v)\Big].$$

Then [BL94, Prop. 8] shows that $\mathrm{End}_{\mathrm{Rep}_k(G)}\big(\mathrm{ind}_{ZK}^G \mathrm{Sym}^r(k^2)\big) \cong k[\Phi]$ as k-algebras.

Definition. For $0 \le r \le p-1$ and $\lambda \in k$ we define the smooth G-representation $V(r,\lambda)$ by the short exact sequence

$$0 \longrightarrow \mathrm{ind}_{ZK}^G \mathrm{Sym}^r(k^2) \xrightarrow{\Phi-\lambda} \mathrm{ind}_{ZK}^G \mathrm{Sym}^r(k^2) \longrightarrow V(r,\lambda) \longrightarrow 0.$$

Note: $\Phi - \lambda$ is injective by [BL94, Thm. 19]. Given a smooth character $\chi\colon \mathbb{Q}_p^\times \longrightarrow k^\times$, we put

$$V(r,\lambda,\chi) \coloneqq (\chi \circ \det) \otimes V(r,\lambda).$$

Notation 5.2.1. For each $\lambda \in k$ we consider the characters

$$\begin{aligned} \mu_\lambda\colon \mathbb{Q}_p^\times &\longrightarrow k^\times, & x &\longmapsto \lambda^{\mathrm{val}_p(x)} \quad \text{and} \\ \omega\colon \mathbb{Q}_p^\times \cong p^{\mathbb{Z}} \times \mathbb{Z}_p^\times &\longrightarrow \mathbb{F}_p^\times \subseteq k^\times, & (p^n, x) &\longmapsto x \bmod p. \end{aligned}$$

If $\chi_1, \chi_2\colon \mathbb{Q}_p^\times \longrightarrow k^\times$ are smooth characters, we write

$$\chi_1 \boxtimes \chi_2\colon T \longrightarrow k^\times, \qquad \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \longmapsto \chi_1(a)\chi_2(d).$$

The representations $V(r,\lambda,\chi)$ are now described as follows:

Theorem 5.2.2 ([BL94, Thm. 30], [Bre03, Thm. 1.1])**.** *Let $(r,\lambda) \in \{0,1,\dots,p-1\} \times k$ and let $\chi\colon \mathbb{Q}_p^\times \longrightarrow k^\times$ be a smooth character.*

(a) If $(r,\lambda) \notin \{(0,\pm 1),(p-1,\pm 1)\}$, then $V(r,\lambda,\chi)$ is irreducible.

(b) There are non-split short exact sequences

$$0 \longrightarrow (\chi\mu_{\pm 1} \circ \det) \otimes \mathrm{Sp} \longrightarrow V(0,\pm 1,\chi) \longrightarrow \chi\mu_{\pm 1} \circ \det \longrightarrow 0$$

and

$$0 \longrightarrow \chi\mu_{\pm 1} \circ \det \longrightarrow V(p-1,\pm 1,\chi) \longrightarrow (\chi\mu_{\pm 1} \circ \det) \otimes \mathrm{Sp} \longrightarrow 0,$$

where $\mathrm{Sp} = i_T^G(\mathbf{1}_T)/\mathbf{1}_G$ is the Steinberg representation, and where $\mathbf{1}_T$ and $\mathbf{1}_G$ are the trivial representations of T and G, respectively.

(c) *If* $(r,\lambda)\neq(0,\pm1)$ *and* $\lambda\neq 0$, *there is a* G*-equivariant isomorphism*

$$V(r,\lambda,\chi)\xrightarrow{\cong} i_T^G(\chi\mu_{\lambda^{-1}}\boxtimes\chi\mu_\lambda\omega^r).$$

(d) $V(r,\lambda,\chi)$ *is supersingular if and only if* $\lambda=0$.

By [BL94, Thm. 33], every smooth irreducible representation of G admitting a central character is a quotient of some $V(r,\lambda,\chi)$.

5.3. **Computation of the left adjoint on irreducibles.** Recall the endomorphism Φ on the representation $\operatorname{ind}_{ZK}^G\operatorname{Sym}^r(k^2)$ from the previous section. The character $\delta_B\colon B\longrightarrow k^\times$ is (by Remark 3.4.25) explicitly given by

$$\delta_B=\operatorname{Inf}_B^T\big(\omega\boxtimes\omega^{-1}\big).$$

Lemma 5.3.1. *Let* $0\le r\le p-1$ *and recall the map* $\eta\colon\operatorname{Sym}^r(k^2)\twoheadrightarrow\mathrm{L}^0(K_U,\operatorname{Sym}^r(k^2))\cong\mathbf{1}\boxtimes\omega^r$.

(a) *The endomorphism* $\mathcal{S}^0_{\operatorname{Sym}^r(k^2)}(\Phi)$ *on* $\operatorname{ind}_{ZK_T}^T(\mathbf{1}\boxtimes\omega^r)$ *is given by*

$$\mathcal{S}^0_{\operatorname{Sym}^r(k^2)}(\Phi)\big([1,\eta(y^r)]\big)=\Big[\begin{pmatrix}p&0\\0&1\end{pmatrix}^{-1},\eta(y^r)\Big].$$

(b) *The endomorphism* $\mathcal{S}^1_{\operatorname{Sym}^r(k^2)}(\Phi)$ *on* $\operatorname{ind}_{ZK_T}^T\delta_B\otimes\langle x^r\rangle_k\cong\operatorname{ind}_{ZK_T}^T(\omega^{r+1}\boxtimes\omega^{-1})$ *is given by*

$$\mathcal{S}^1_{\operatorname{Sym}^r(k^2)}(\Phi)\big([1,x^r]\big)=\Big[\begin{pmatrix}1&0\\0&p\end{pmatrix}^{-1},x^r\Big].$$

Proof. By Theorem 4.3.2 we have

$$\begin{aligned}\mathcal{S}^0_{\operatorname{Sym}^r(k^2)}(\Phi)\big([1,\eta(y^r)]\big)&=\sum_{t\in ZK_T\backslash T}\Big[t^{-1},\eta\Big(\sum_{u\in K_U\backslash U}\Phi([1,y^r])(ut)\Big)\Big]\\&=\Big[\begin{pmatrix}p&0\\0&1\end{pmatrix}^{-1},\eta(Y(y^r))\Big]+\Big[\begin{pmatrix}1&0\\0&p\end{pmatrix}^{-1},\eta\Big(\sum_{i=0}^{p-1}X(\begin{pmatrix}1&i\\0&1\end{pmatrix}\cdot y^r)\Big)\Big]\\&=\Big[\begin{pmatrix}p&0\\0&1\end{pmatrix}^{-1},\eta(y^r)\Big],\end{aligned}$$

where we have used that the second summand in the second line is zero: If $r>0$, this follows from $\operatorname{Im}(X)\subseteq\operatorname{Ker}(\eta)$; if $r=0$, we use $\operatorname{char}(k)=p$. This proves (a). For (b) we compute in a similar fashion:

$$\begin{aligned}\mathcal{S}^1_{\operatorname{Sym}^r(k^2)}(\Phi)\big([1,x^r]\big)&=\sum_{t\in ZK_T\backslash T}\Big[t^{-1},\sum_{u\in U/K_U}\Phi([1,x^r])(tu)\Big]\\&=\Big[\begin{pmatrix}p&0\\0&1\end{pmatrix}^{-1},\sum_{i=0}^{p-1}\begin{pmatrix}1&i\\0&1\end{pmatrix}\cdot Y(x^r)\Big]+\Big[\begin{pmatrix}1&0\\0&p\end{pmatrix}^{-1},X(x^r)\Big]\\&=\Big[\begin{pmatrix}1&0\\0&p\end{pmatrix}^{-1},x^r\Big].\end{aligned}$$

□

Proposition 5.3.2. *Let* $(r,\lambda)\in\{0,1,\dots,p-1\}\times k$ *and let* $\chi\colon\mathbb{Q}_p^\times\longrightarrow k^\times$ *be a smooth character. One has*

$$\mathrm{L}^{-i}\big(U,V(r,\lambda,\chi)\big)=\begin{cases}\chi\mu_{\lambda^{-1}}\boxtimes\chi\mu_\lambda\omega^r, & \text{if } i=0 \text{ and } \lambda\neq0,\\ \chi\mu_\lambda\omega^{r+1}\boxtimes\chi\mu_{\lambda^{-1}}\omega^{-1}, & \text{if } i=1 \text{ and } \lambda\neq 0,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. By Proposition 3.2.12 we have $\mathrm{L}\big(U,V(r,\lambda,\chi)\big)\cong(\chi\circ\det)\otimes\mathrm{L}\big(U,V(r,\lambda)\big)$. We may therefore assume from the start that $\chi=\mathbf{1}$ is the trivial character. As $\dim U=1$, we know from Proposition 4.1.5 that $\mathrm{L}^{-i}\big(U,V(r,\lambda)\big)=0$ provided $i\notin\{0,1\}$. Consider the short exact sequence

$$0\longrightarrow\operatorname{ind}_{ZK}^G\operatorname{Sym}^r(k^2)\xrightarrow{\Phi-\lambda}\operatorname{ind}_{ZK}^G\operatorname{Sym}^r(k^2)\longrightarrow V(r,\lambda)\longrightarrow0.\tag{5.3.3}$$

It is immediate from Lemma 5.3.1.(a) that $\mathcal{S}^0_{\mathrm{Sym}^r(k^2)}(\Phi) - \lambda$ is injective. Hence, the long exact sequence associated with $\mathrm{L}(U,-)$ and (5.3.3) splits into the following two short exact sequences:

$$0 \longrightarrow \mathrm{ind}_{ZK_T}^T(\mathbf{1} \boxtimes \omega^r) \xrightarrow{\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} - \lambda} \mathrm{ind}_{ZK_T}^T(\mathbf{1} \boxtimes \omega^r) \longrightarrow \mathrm{L}^0\big(U, V(r,\lambda)\big) \longrightarrow 0,$$

and

$$0 \longrightarrow \mathrm{ind}_{ZK_T}^T(\omega^{r+1} \boxtimes \omega^{-1}) \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)^{-1} - \lambda} \mathrm{ind}_{ZK_T}^T(\omega^{r+1} \boxtimes \omega^{-1}) \longrightarrow \mathrm{L}^{-1}\big(U, V(r,\lambda)\big) \longrightarrow 0.$$

The assertion now follows easily. □

Corollary 5.3.4. *Let $\chi, \chi_1, \chi_2 \colon \mathbb{Q}_p^\times \longrightarrow k^\times$ be smooth characters with $\chi_1 \neq \chi_2$, and let $0 \le r \le p-1$. The T-representations $\mathrm{L}^{-i}(U,V)$, for irreducible smooth G-representations V and $i \in \{-1,0\}$, are as presented in Table 1.*

Type	V	$\mathrm{L}^{-1}(U,V)$	$\mathrm{L}^0(U,V)$
character	$\chi \circ \det$	0	$\chi \boxtimes \chi$
special series	$(\chi \circ \det) \otimes \mathrm{Sp}$	$\chi\omega \boxtimes \chi\omega^{-1}$	0
principal series	$i_T^G(\chi_1 \boxtimes \chi_2)$	$\chi_2\omega \boxtimes \chi_1\omega^{-1}$	$\chi_1 \boxtimes \chi_2$
supersingular	$V(r,0,\chi)$	0	0

TABLE 1. Computing $\mathrm{L}^{-i}(U,V)$ for irreducible V.

Proof. If $V \cong \chi \circ \det$, the assertion follows from Corollary 3.4.27. If V is supersingular, then $\mathrm{L}(U,V) = \{0\}$ by Proposition 5.3.2. Let now $V \cong i_T^G(\chi_1 \boxtimes \chi_2)$ for two (not necessarily distinct) smooth characters $\chi_1, \chi_2 \colon \mathbb{Q}_p^\times \longrightarrow k^\times$. Write $\chi_j = \mu_{\lambda_j}\omega^{r_j}$ for some $\lambda_j \in k^\times$ and $0 \le r_j \le p-1$ and take $\lambda \in k^\times$ such that $\lambda^2 = \lambda_1\lambda_2$. We have $i_T^G(\chi_1 \boxtimes \chi_2) \cong V(r_2 - r_1, \lambda_2\lambda^{-1}, \mu_\lambda\omega^{r_1})$ by Theorem 5.2.2.(c). Applying Proposition 5.3.2 we compute

$$\begin{aligned} \mathrm{L}^0(U,V) &\cong \mu_\lambda\omega^{r_1}\mu_{\lambda\lambda_2^{-1}} \boxtimes \mu_\lambda\omega^{r_1}\mu_{\lambda_2\lambda^{-1}}\omega^{r_2-r_1} \\ &\cong \chi_1 \boxtimes \chi_2 \end{aligned}$$

and

$$\begin{aligned} \mathrm{L}^{-1}(U,V) &\cong \mu_\lambda\omega^{r_1}\mu_{\lambda_2\lambda^{-1}}\omega^{r_2-r_1+1} \boxtimes \mu_\lambda\omega^{r_1}\mu_{\lambda\lambda_2^{-1}}\omega^{-1} \\ &\cong \chi_2\omega \boxtimes \chi_1\omega^{-1}. \end{aligned}$$

This settles the principal series case. Finally, we deduce the special series case as follows: By Proposition 3.2.12 we may assume $\chi \cong \mathbf{1}$. By Theorem 5.2.2.(b) there is a short exact sequence

$$0 \longrightarrow \mathrm{Sp} \longrightarrow V(0,1) \longrightarrow \mathbf{1} \longrightarrow 0.$$

The long exact sequence associated with $\mathrm{L}(U,-)$ yields

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{L}^{-1}(U,\mathrm{Sp}) \arrow[r] & \mathrm{L}^{-1}\big(U,V(0,1)\big) \arrow[r] & \mathrm{L}^{-1}(U,\mathbf{1}) \arrow[dll, rounded corners] \\ & \mathrm{L}^0(U,\mathrm{Sp}) \arrow[r] & \mathrm{L}^0\big(V(0,1)\big) \arrow[r, "\varphi"'] & \mathrm{L}^0(U,\mathbf{1}) \arrow[r] & 0 \end{tikzcd}$$

Since $\mathrm{L}^{-1}(U,\mathbf{1}) = \{0\}$, we deduce $\mathrm{L}^{-1}(U,\mathrm{Sp}) \cong \mathrm{L}^{-1}\big(U,V(0,1)\big) \cong \omega \boxtimes \omega^{-1}$. As $\mathrm{L}^0\big(V(0,1)\big) \cong \mathbf{1} \boxtimes \mathbf{1} \cong \mathrm{L}^0(U,\mathbf{1})$, it follows that φ is an isomorphism, whence $\mathrm{L}^0(U,\mathrm{Sp}) = \{0\}$. □

References

[BDS16] Paul Balmer, Ivo Dell'Ambrogio, and Beren Sanders. Grothendieck–Neeman duality and the Wirthmüller isomorphism. *Compositio Mathematica*, 152(8):1740–1776, May 2016. doi:10.1112/s0010437x16007375.

[Ben91a] David J. Benson. *Representations and Cohomology. Volume I: Basic Representation Theory of Finite Groups and Associative Algebras*, volume 1 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1991. doi:10.1017/CBO9780511623615.

[Ben91b] David J. Benson. *Representations and Cohomology. Volume II: Cohomology of Groups and Modules*, volume 2 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1991. doi:10.1017/CBO9780511623622.

[BH06] Colin J. Bushnell and Guy Henniart. *The Local Langlands Conjecture for GL(2)*. Number 335 in Grundlehren der mathematischen Wissenschaften. Springer-Verlag Berlin Heidelberg, 1 edition, 2006.

[BK98] Colin J. Bushnell and Philip C. Kutzko. Smooth representations of reductive p-adic groups: structure theory via types. *Proceedings of the London Mathematical Society*, 77(03):582–634, 1998. doi:10.1112/S0024611598000574.

[BL94] Laure Barthel and Ron Livné. Irreducible modular representations of GL_2 of a local field. *Duke Mathematical Journal*, 75(2):261–292, 08 1994. doi:10.1215/S0012-7094-94-07508-X.

[Bre03] Christophe Breuil. Sur quelques représentations modulaires et p-adiques de $\mathrm{GL}_2(\mathbb{Q}_p)$: I. *Compositio Mathematica*, 138(2):165–188, 2003. doi:10.1023/A:1026191928449.

[DDMS99] John D. Dixon, Marcus P. F. Du Sautoy, Avinoam Mann, and Daniel Segal. *Analytic pro-p Groups*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2 edition, 1999. doi:10.1017/CBO9780511470882.

[DK21] Nicolas Dupré and Jan Kohlhaase. Model categories and pro-p Iwahori-Hecke modules, 2021. arXiv:2112.03150.

[Eme10a] Matthew Emerton. Ordinary parts of admissible representations of p-adic reductive groups I. Definition and first properties. In *Représentations p-adiques de groupes p-adiques III : méthodes globales et géométriques*, number 331 in Astérisque. Société mathématique de France, 2010. URL: http://www.numdam.org/item/AST_2010__331__355_0/.

[Eme10b] Matthew Emerton. Ordinary parts of admissible representations of p-adic reductive groups II. Derived functors. In *Représentations p-adiques de groupes p-adiques III : méthodes globales et géométriques*, number 331 in Astérisque. Société mathématique de France, 2010. URL: http://www.numdam.org/item/AST_2010__331__403_0/.

[Har66] Robin Hartshorne. *Residues and Duality*, volume 20 of *Lecture Notes in Mathematics*. Springer, 1966.

[Her11a] Florian Herzig. A Satake isomorphism in characteristic p. *Compositio Mathematica*, 147(1):263–283, 2011. doi:10.1112/S0010437X10004951.

[Her11b] Florian Herzig. The classification of irreducible admissible mod p representations of a p-adic GL_n. *Inventiones Mathematicae*, 186(2):373–434, Nov 2011. doi:10.1007/s00222-011-0321-z.

[Her12] Florian Herzig. The mod p Representation Theory of p-Adic Groups. https://www.math.toronto.edu/~herzig/modpreptheory.pdf, 2012. Notes for graduate course at Fields Institute.

[HS71] Peter J. Hilton and Urs Stammbach. *A Course in Homological Algebra*. Graduate Texts in Mathematics 4. Springer New York, 1 edition, 1971.

[HV12] Guy Henniart and Marie-France Vigneras. Comparison of compact induction with parabolic induction. *Pacific J. Math.*, 260(2):457–495, 2012. doi:10.2140/pjm.2012.260.457.

[HV15] Guy Henniart and Marie-France Vignéras. A Satake isomorphism for representations modulo p of reductive groups over local fields. *Journal für die reine und angewandte Mathematik*, 701:33–75, 2015. doi:10.1515/crelle-2013-0021.

[Kra02] Henning Krause. A Brown representability theorem via coherent functors. *Topology*, 41(4):853–861, 2002. doi:10.1016/S0040-9383(01)00010-6.

[KS74] G. M. Kelly and Ross Street. Review of the elements of 2-categories. In Gregory M. Kelly, editor, *Category Seminar*, pages 75–103, Berlin, Heidelberg, 1974. Springer Berlin Heidelberg. doi:10.1007/BFb0063101.

[KS06] Masaki Kashiwara and Pierre Schapira. *Categories and Sheaves*. Grundlehren der Math. 332. Springer-Verlag, Berlin, 2006.

[Laz65] Michel Lazard. Groupes analytiques p-adiques. *Publications Mathématiques de l'IHÉS*, 26:5–219, 1965. URL: http://www.numdam.org/item/PMIHES_1965__26__5_0.

[Lur12] Jacob Lurie. Higher algebra, 2012. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf.

[ML13] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5. Springer Science & Business Media, 2013.

[MR01] John C. McConnell and James Christopher Robson. *Noncommutative noetherian rings*, volume 30. American Mathematical Soc., 2001. doi:10.1090/gsm/030.

[Nee92] Amnon Neeman. The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel. *Annales scientifiques de l'École Normale Supérieure*, Ser. 4, 25(5):547–566, 1992. URL: http://www.numdam.org/articles/10.24033/asens.1659/, doi:10.24033/asens.1659.

[Nee01] Amnon Neeman. *Triangulated categories*. Number 148 in Annals of Mathematics Studies. Princeton University Press, 2001. URL: `https://hopf.math.purdue.edu/Neeman/triangulatedcats.pdf`.

[NSW13] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of number fields*, volume 323. Springer Science & Business Media, 2013. `doi:10.1007/978-3-540-37889-1`.

[Ron19] Niccolò Ronchetti. A Satake homomorphism for the mod p derived Hecke algebra, 2019. `arXiv:1808.06512`.

[Roo06] Jan-Erik Roos. Derived Functors of Inverse Limits Revisited. *Journal of the London Mathematical Society*, 73(1):65–83, 2006. `doi:10.1112/S0024610705022416`.

[Sch15] Peter Schneider. Smooth representations and Hecke modules in characteristic p. *Pacific Journal of Mathematics*, 279, 12 2015. `doi:10.2140/pjm.2015.279.447`.

[Ser65] Jean-Pierre Serre. Sur la dimension cohomologique des groupes profinis. *Topology*, 3(4):413–420, 1965. `doi:10.1016/0040-9383(65)90006-6`.

[Ser03] Christian Serpé. Resolution of unbounded complexes in Grothendieck categories. *Journal of Pure and Applied Algebra*, 177(1):103–112, 2003. `doi:10.1016/S0022-4049(02)00075-0`.

[Ser13] Jean-Pierre Serre. *Galois cohomology*. Springer Science & Business Media, 2013.

[SS22a] Sarah Scherotzke and Peter Schneider. Derived parabolic induction. *Bulletin of the London Mathematical Society*, 54(1):264–274, 2022. `doi:https://doi.org/10.1112/blms.12606`.

[SS22b] Peter Schneider and Claus Sorensen. Duals in natural characteristic, 2022. `arXiv:2202.01800`.

[Vig96] Marie-France Vignéras. Représentations l-modulaires d'un groupe réductif p-adique avec $l \neq p$. *Progress in Mathematics*, 137, Birkhäuser: Boston, 1996.

[Vig98] Marie-France Vignéras. Induced R-representations of p-adic reductive groups. *Selecta Mathematica*, 4(4):549–623, 1998. `doi:10.1007/s000290050040`.

Mathematisches Institut, Westfälische Wilhelms-Universität Münster, Einsteinstrasse 62, D-48149 Münster, Germany

Email address: `cheyer@uni-muenster.de`