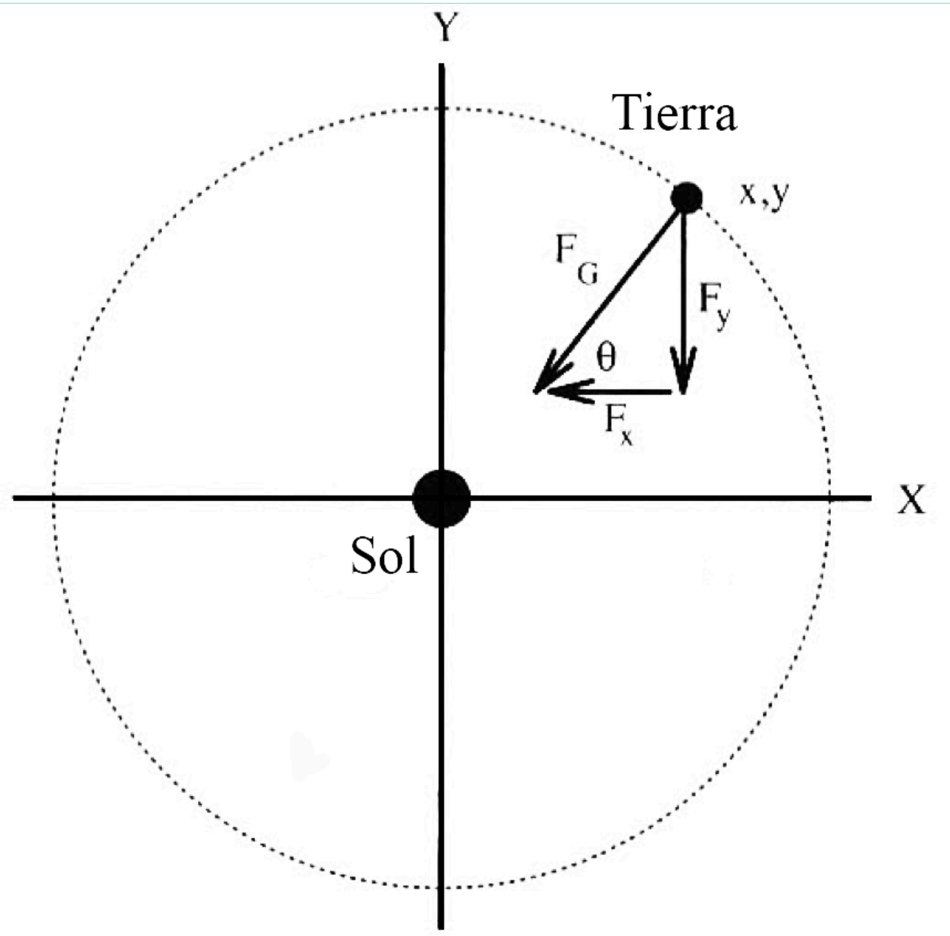
Movimiento Planetario
Puntos de Lagrange

Claudia Zendejas Morales

Física Computacional, 2019-1



Movimiento de un planeta

Ley de Gravitación Universal

$$F_G = \frac{G M_S M_E}{r^2}$$

$$\vec{F} = m\vec{a} = m\frac{d^2\vec{x}}{dt^2}$$

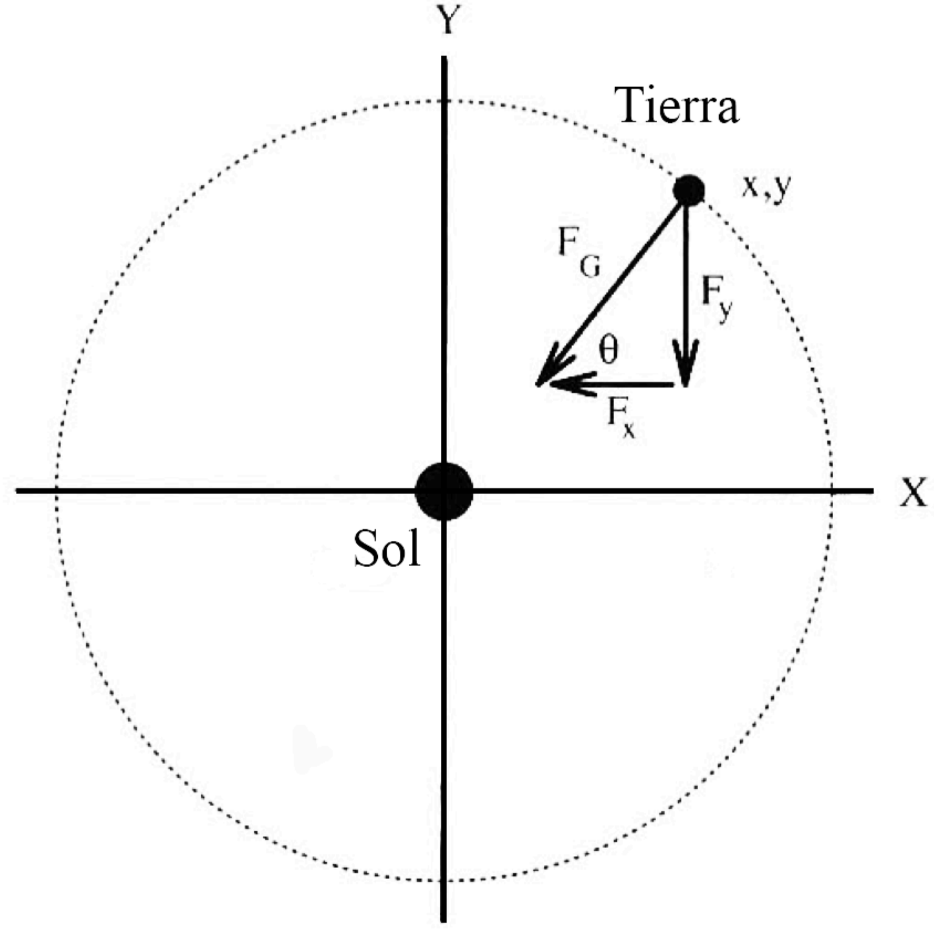
$$F_{G,x} = M_E\frac{d^2x}{dt^2} \qquad F_{G,y} = M_E\frac{d^2y}{dt^2}$$

$$F_{G,x} = -\frac{GM_SM_E}{r^2}\cos\theta = -\frac{GM_SM_Ex}{r^3}$$

$$F_{G,y} = -\frac{GM_SM_E}{r^2}\sin\theta = -\frac{GM_SM_Ey}{r^3}$$

$$\frac{dx}{dt} = v_x \qquad \frac{dy}{dt} = v_y$$

$$\frac{dv_x}{dt} = -\frac{GM_Sx}{r^3} \qquad \frac{dv_y}{dt} = -\frac{GM_Sy}{r^3}$$



Unidades Astronómicas

$$v = 1\,AU/año = 4740\ m/s$$

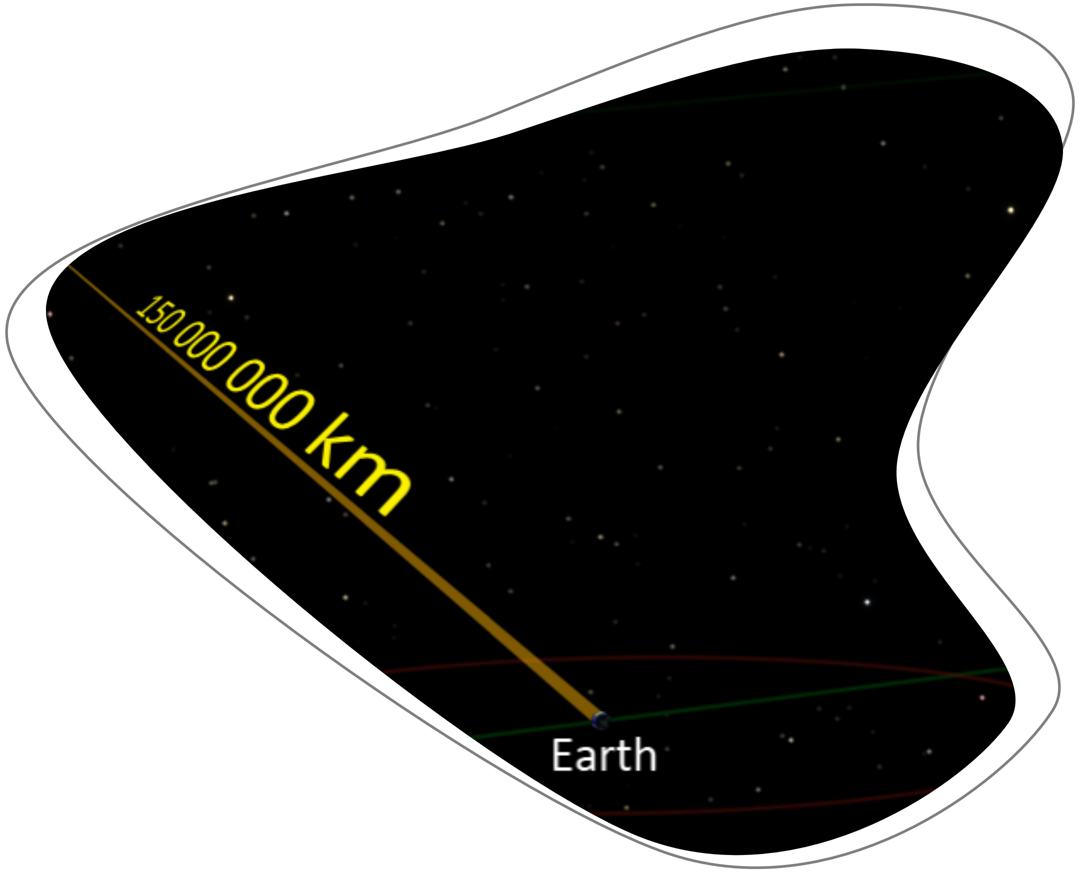
1 AU=149 597 870 700 m

1 año = 365.25 días = 3.156×10[7] s

$$GM_\odot = 1.32814\times10^{20}\,\frac{m^3}{s^2}$$

$$M_\odot = 1.99\times10^{30}\ Kg$$

$$G = 6.67408\times10^{-11}\,\frac{N\cdot m^2}{kg^2}$$



Aproximación a órbita circular:

$$\mu\frac{v^2}{r} = F_G(r) = \frac{GM_SM_E}{r^2}$$

Velocidad tangencial:

$$v = \omega r = \frac{2\pi}{T}r = \frac{2\pi r}{1\ año} = 2\pi\frac{AU}{año} = 29.77\,\frac{km}{s}$$

$$G(M_S + M_E) \simeq GM_S = v^2 r = 4\pi^2\,\frac{AU^3}{año^2}$$

$$M_S \gg M_E$$

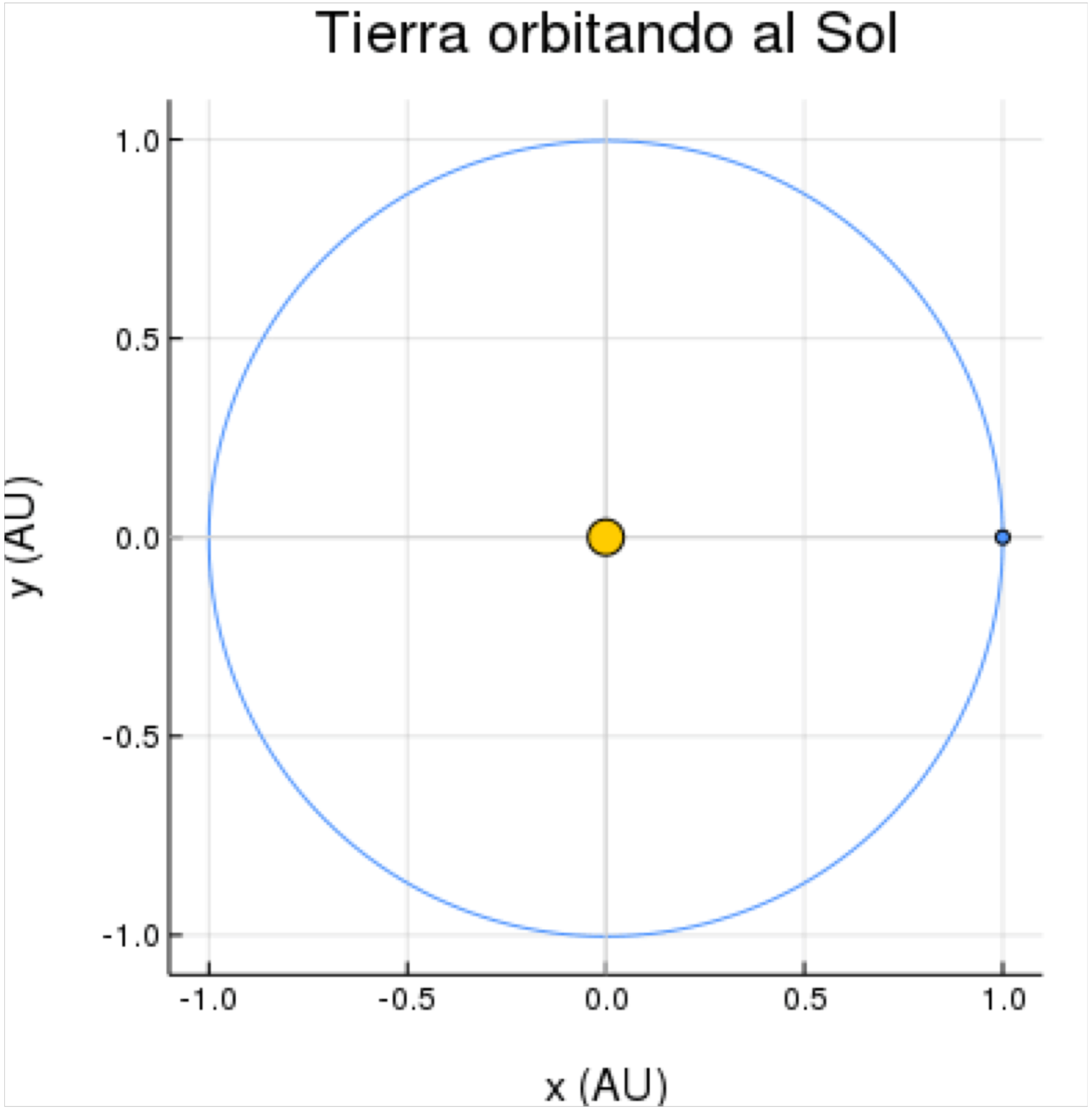
Solución Computacional

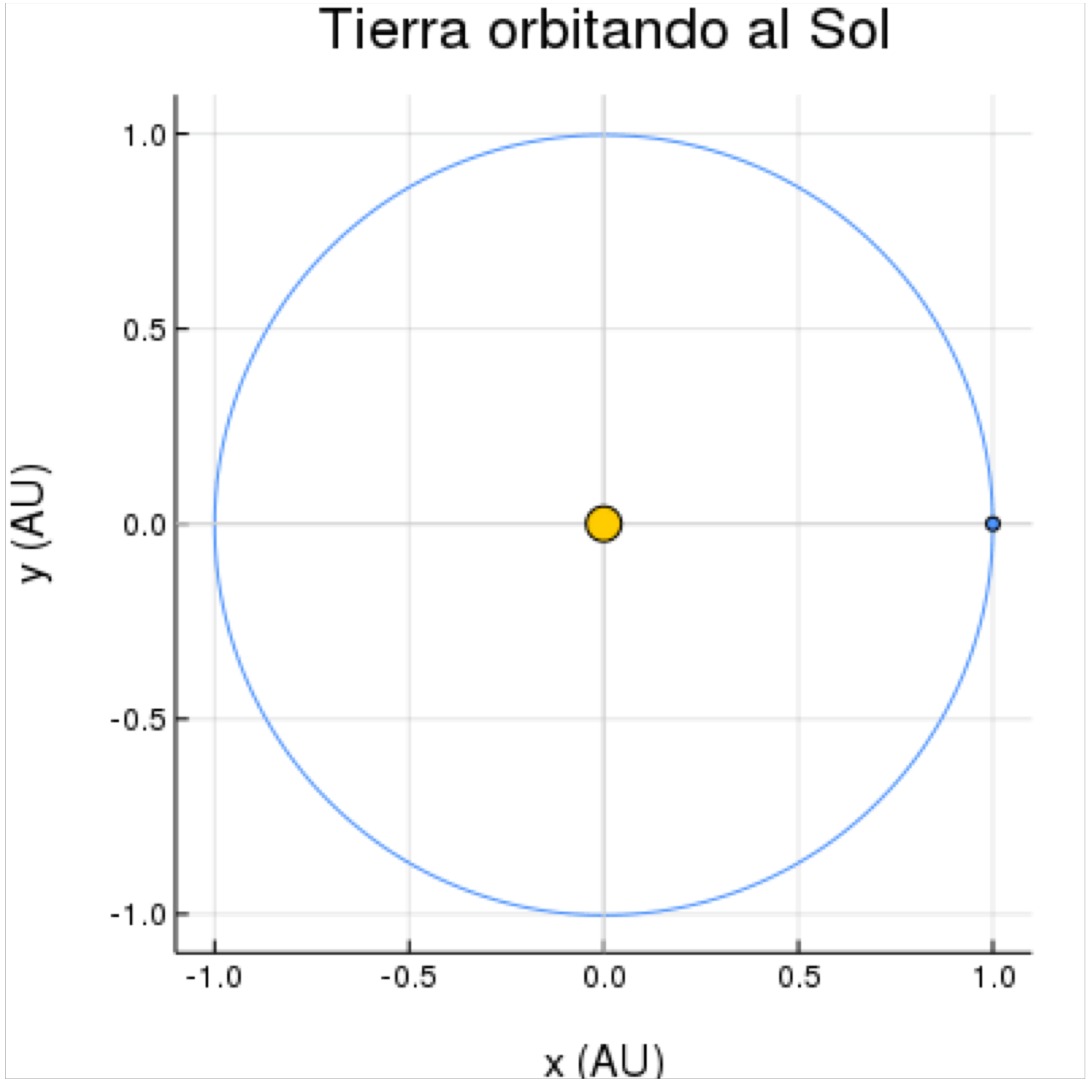
$$\frac{dx}{dt} = v_x$$

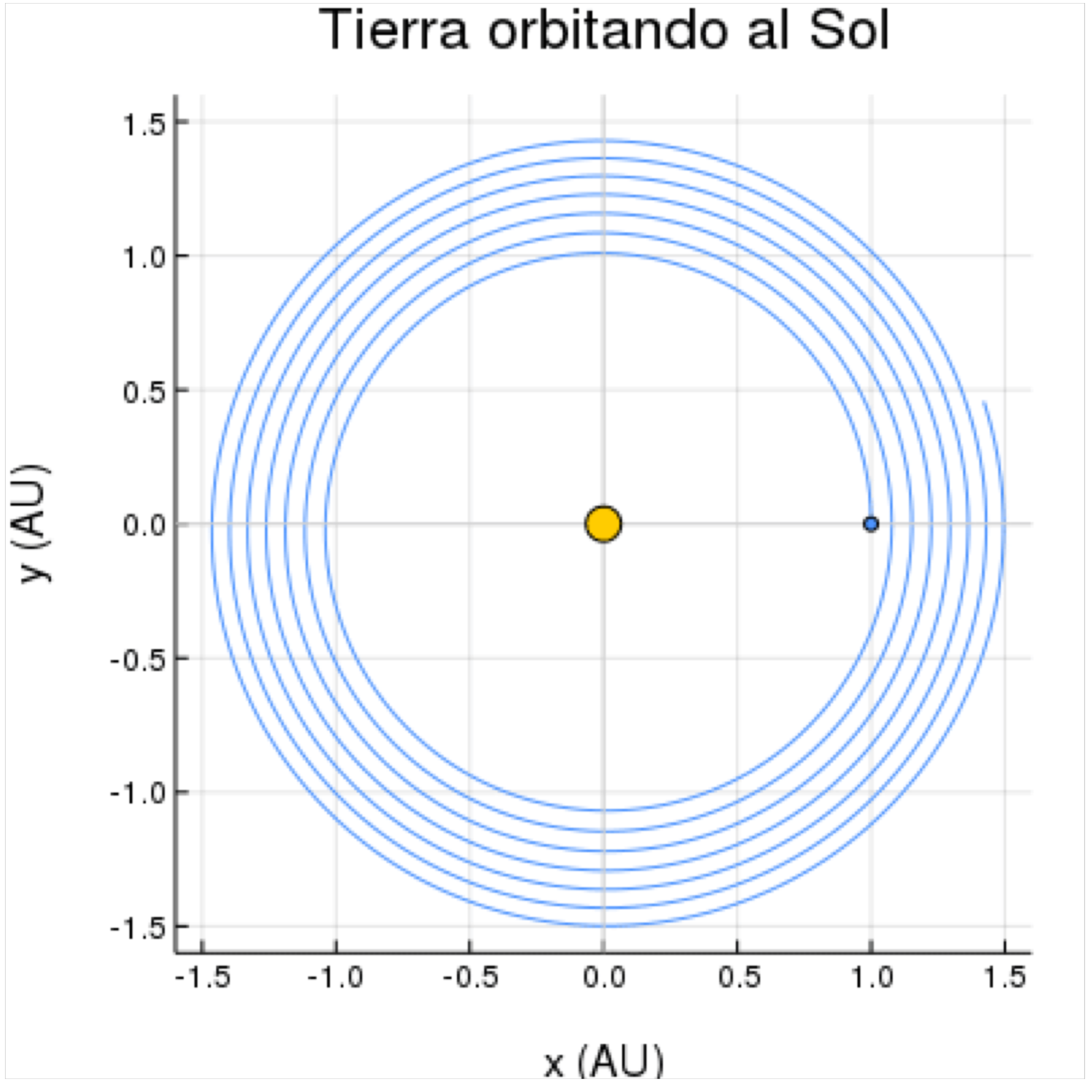
$$\frac{dv_x}{dt} = -\frac{4\pi^2\ x}{r^3}$$

$$\frac{dy}{dt} = v_y$$

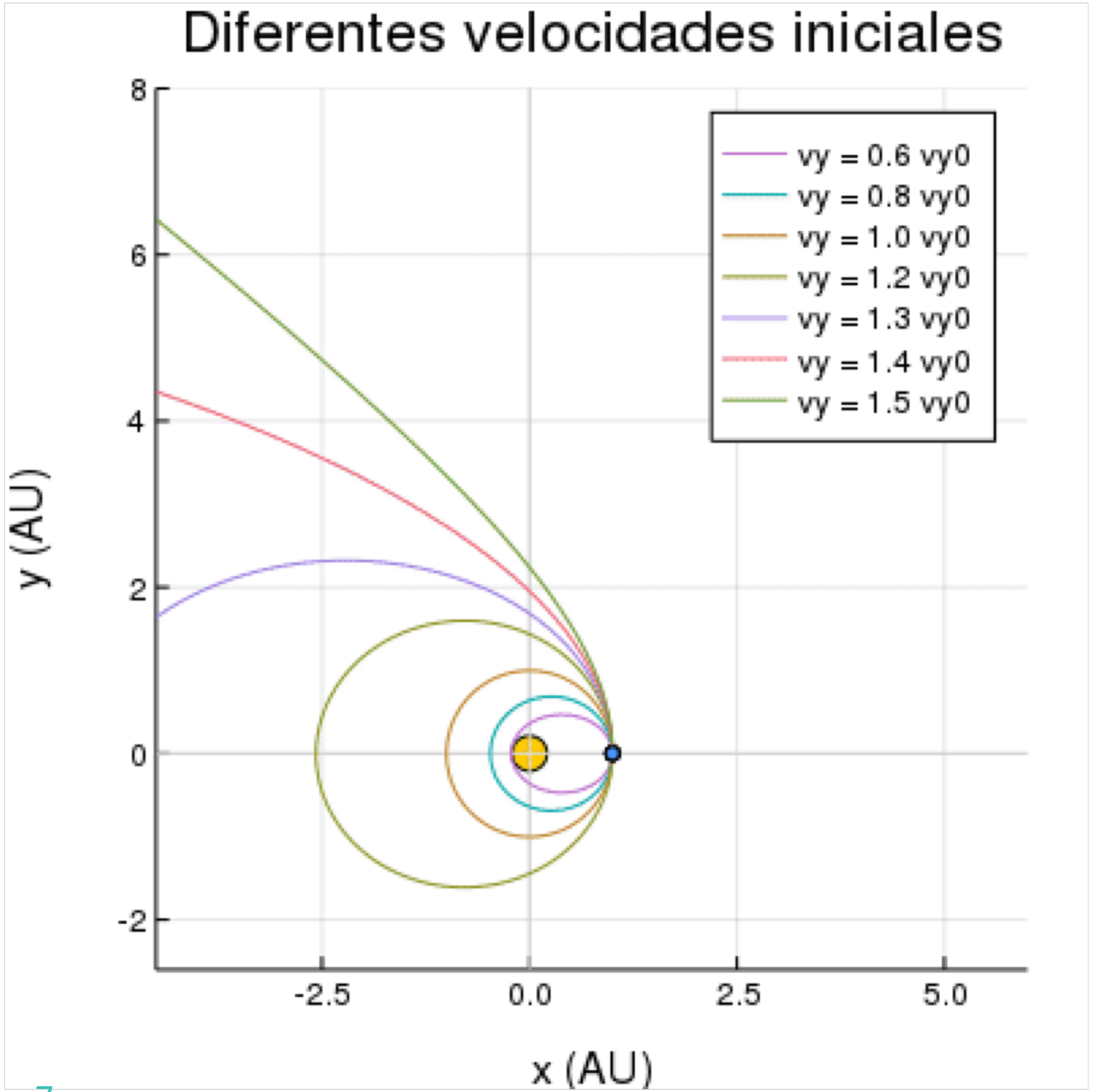
$$\frac{dv_y}{dt} = -\frac{4\pi^2\ y}{r^3}$$

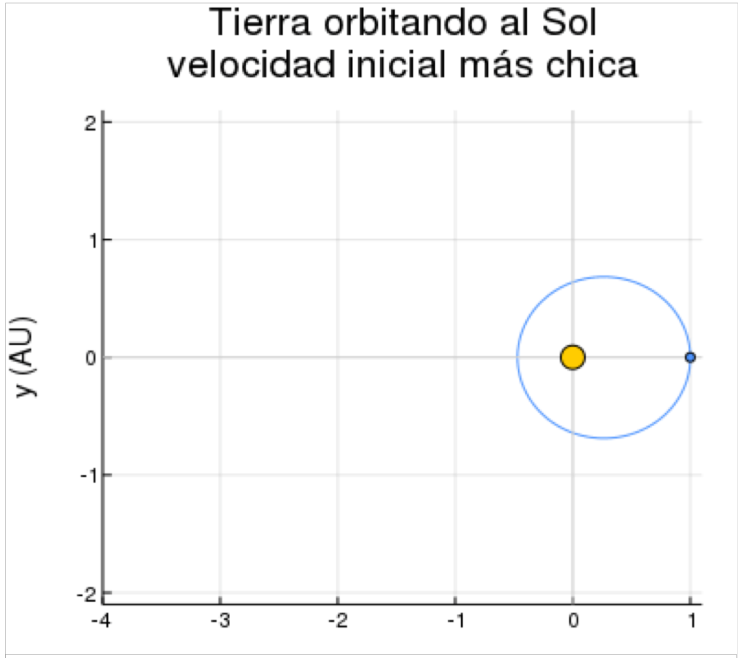


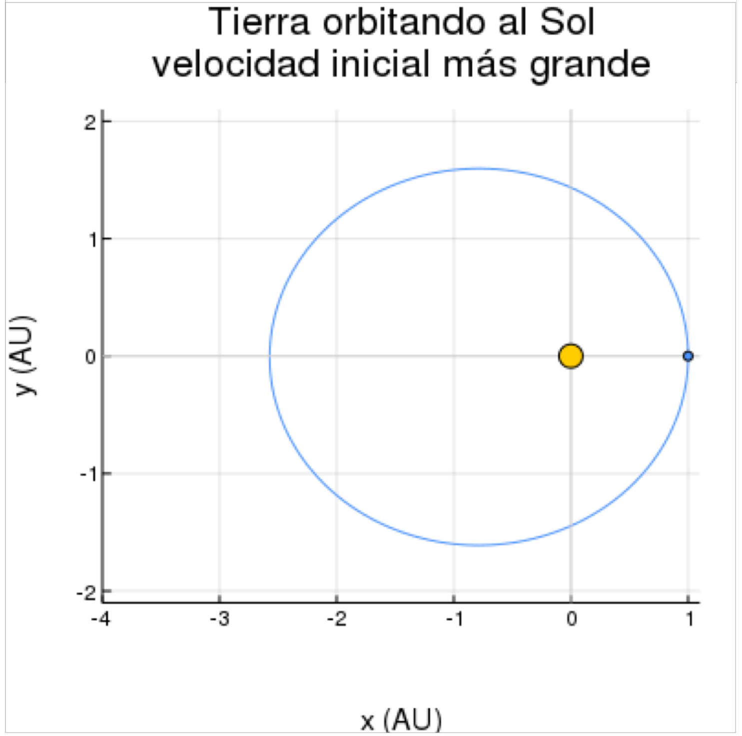


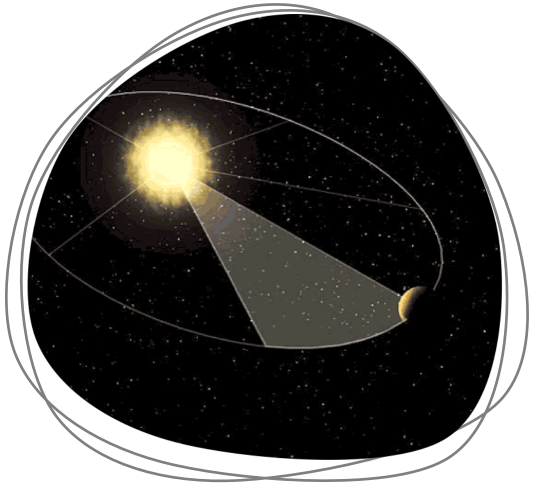


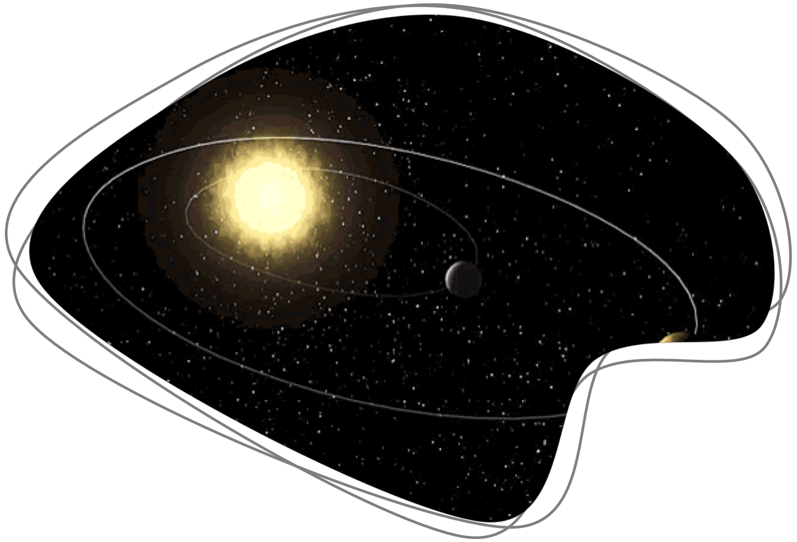
Euler Cromer | Euler

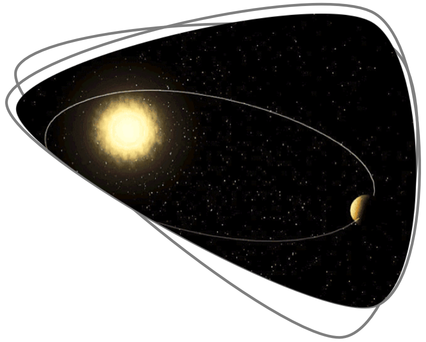

Diferentes velocidades iniciales
vy = 0.6 vy0
vy = 0.8 vy0
vy = 1.0 vy0
vy = 1.2 vy0
vy = 1.3 vy0
vy = 1.4 vy0
vy = 1.5 vy0
y (AU)
x (AU)
8
6
4
2
0
-2
-2.5
0.0
2.5
5.0


Tierra orbitando al Sol
velocidad inicial más chica
y (AU)
2
1
0
-1
-2
-4
-3
-2
-1
0
1


Tierra orbitando al Sol
velocidad inicial más grande
y (AU)
x (AU)
2
1
0
-1
-2
-4
-3
-2
-1
0
1







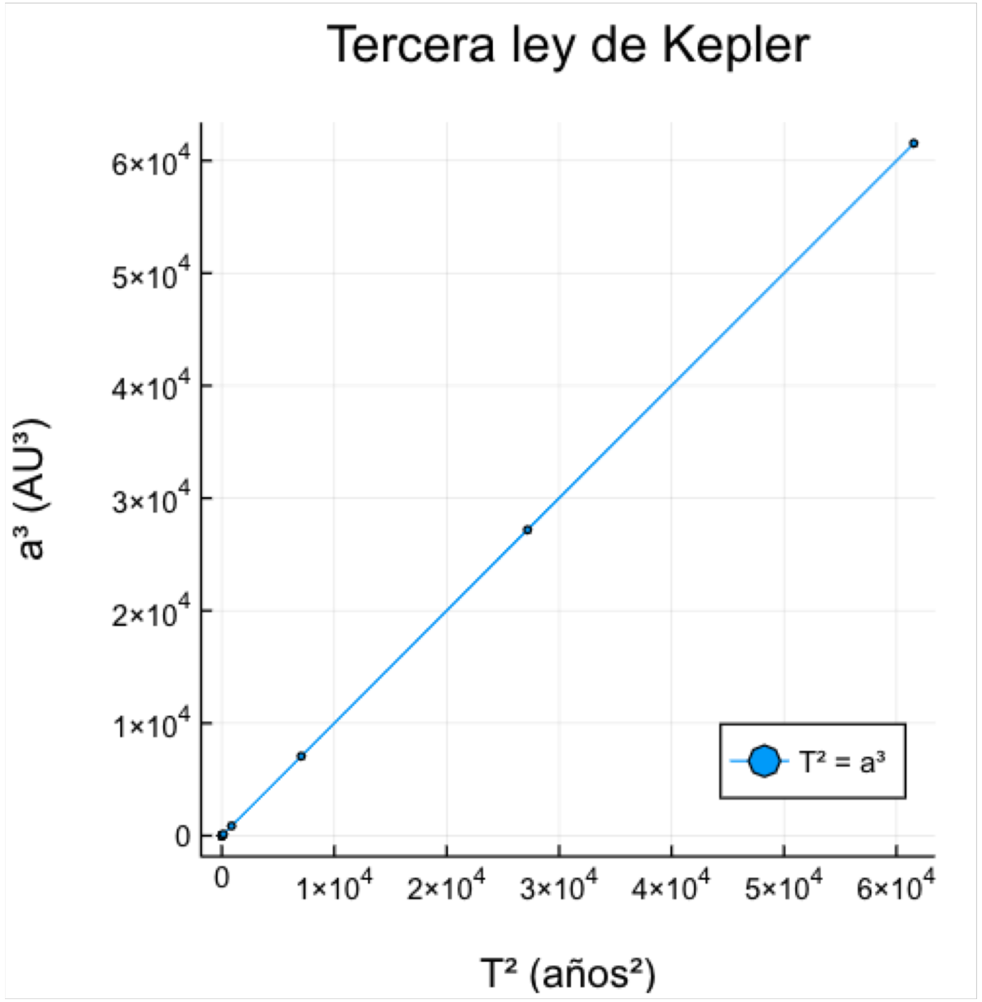
Leyes de Kepler

- Todos los planetas se mueven en órbitas elípticas, con el Sol en un foco
- La línea que une un planeta con el Sol barre áreas iguales en tiempos iguales
- Si T es el periodo y a el semieje mayor de la órbita, entonces T^2/a^3 es constante para todos los planetas

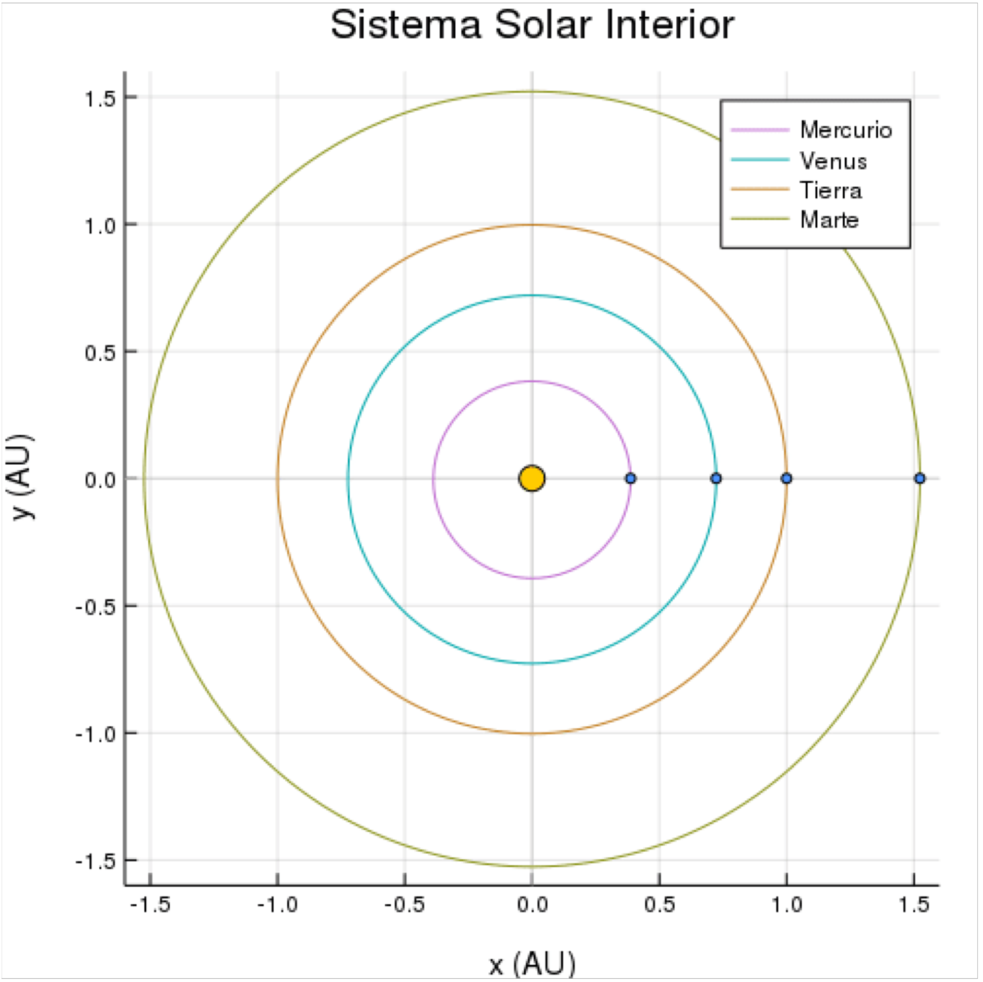
Imágenes: Ref. [5]

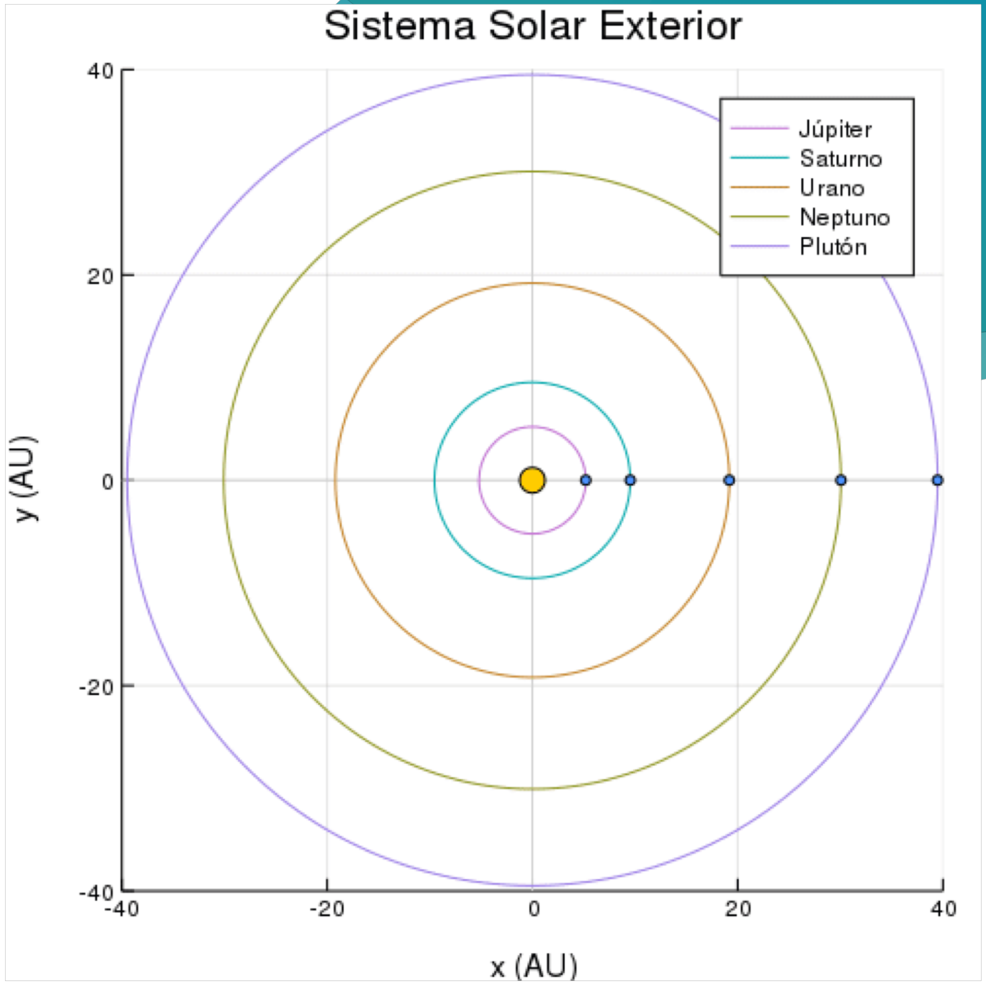
$$\frac{T^2}{a^3} = 1$$

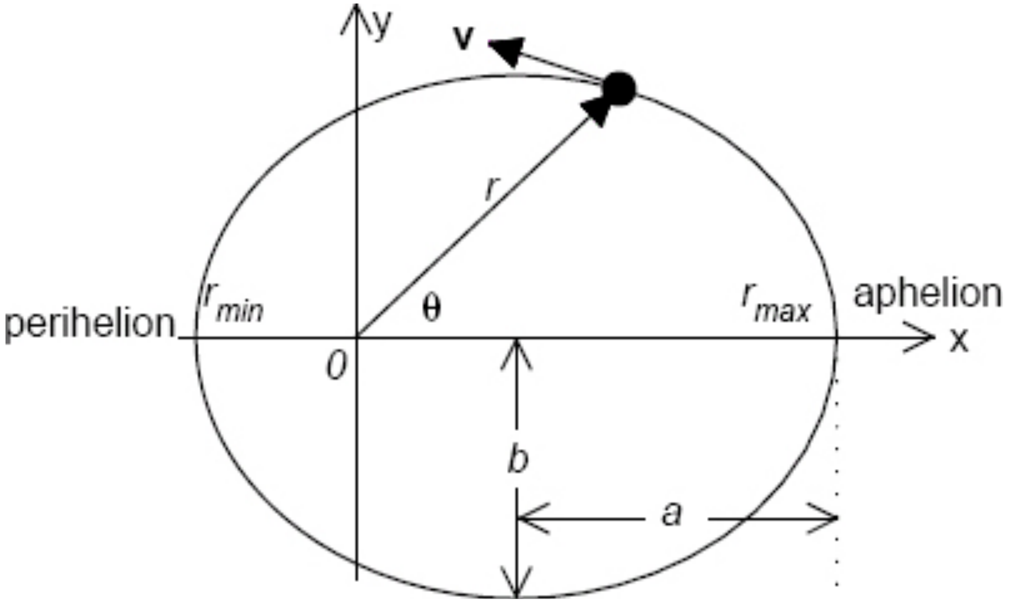
Planeta	radio (AU) [1] [2] [4]	masa ($M_\odot$) [1] [2] [4]	velocidad $v=2\pi/r^{1/2}$
Mercurio	0.3871	1.660×10^{-7}	10.099
Venus	0.7233	2.448×10^{-6}	7.388
Tierra	1.0000	3.004×10^{-6}	6.283
Marte	1.5237	3.227×10^{-7}	5.090
Júpiter	5.2029	9.551×10^{-4}	2.755
Saturno	9.5367	2.859×10^{-4}	2.035
Urano	19.189	4.355×10^{-5}	1.434
Neptuno	30.070	5.178×10^{-5}	1.146
Plutón	39.482	6.582×10^{-9}	0.999



Planeta	Periodo (años)	T^2/a^3	Error (%)
Mercurio	0.2408	0.9996	0.0360
Venus	0.6150	0.9995	0.0472
Tierra	1.0000	1.0000	0.0000
Marte	1.8800	0.9991	0.0881
Júpiter	11.8650	0.9995	0.0463
Saturno	29.4500	0.9999	0.0055
Urano	84.0550	0.9999	0.0069
Neptuno	164.8900	1.0000	0.0027
Plutón	248.0800	1.0000	0.0032


Sistema Solar Interior
Mercurio
Venus
Tierra
Marte
y (AU)
x (AU)
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5


Sistema Solar Exterior
Júpiter
Saturno
Urano
Neptuno
Plutón
y (AU)
x (AU)
40
20
0
-20
-40



Excentricidad

$$r = \frac{a(1 - e^2)}{1 - e\cos\theta}$$

$e>1$	hipérbola (no unido)
$e=1$	parábola (frontera)
$e<1$	elipse (unido, **primera ley de Kepler**)
$e=0$	círculo (unido)

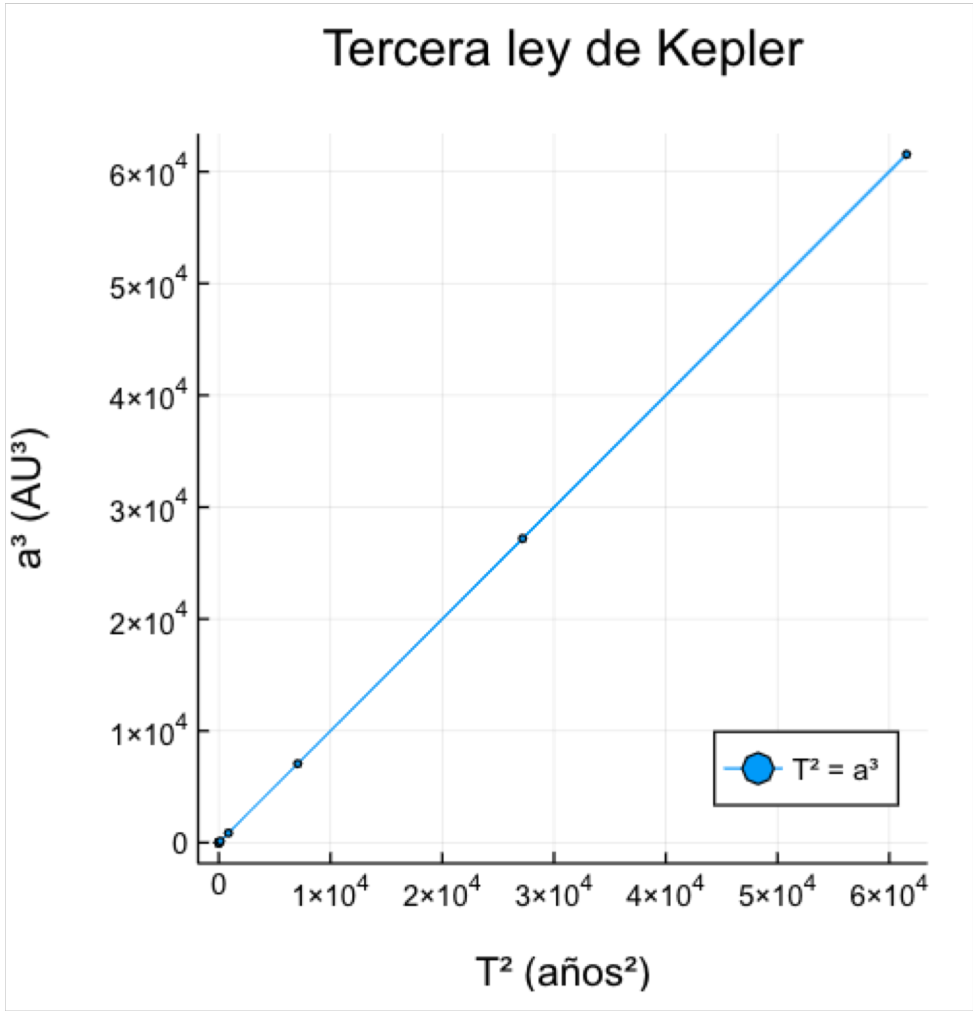
$$r_{max} = a(1+e) \quad (\text{cuando } \theta = 0)$$

$$r_{min} = a(1-e) \quad (\text{cuando } \theta = \pi)$$

$$v_{max} = \sqrt{G\,M_S}\sqrt{\frac{(1+e)}{a(1-e)}\left(1+\frac{M_P}{M_S}\right)} = v_{perihelio}$$

$$v_{min} = \sqrt{G\,M_S}\sqrt{\frac{(1-e)}{a(1+e)}\left(1+\frac{M_P}{M_S}\right)} = v_{afelio}$$

Planeta	e [1] [2]	a (AU) [1] [2]	r_{min} (AU)	r_{max} (AU)	v_{afelio} $(AU/\text{año})$	$v_{perihelio}$ $(AU/\text{año})$
Mercurio	0.2056	0.3871	0.3075	0.4667	8.1976	12.4409
Venus	0.0068	0.7233	0.7184	0.7282	7.3378	7.4383
Tierra	0.0167	1.0000	0.9833	1.0167	6.1791	6.3890
Marte	0.0934	1.5237	1.3814	1.6660	4.6349	5.5900
Júpiter	0.0484	5.2029	4.9511	5.4547	2.6256	2.8927
Saturno	0.0539	9.5367	9.0227	10.0507	1.9280	2.1477
Urano	0.0473	19.189	8.28148	20.0966	1.3681	1.5039
Neptuno	0.0086	30.070	29.8114	30.3286	1.1360	1.1557
Plutón	0.2488	39.482	29.6589	49.3051	0.77550	1.2893



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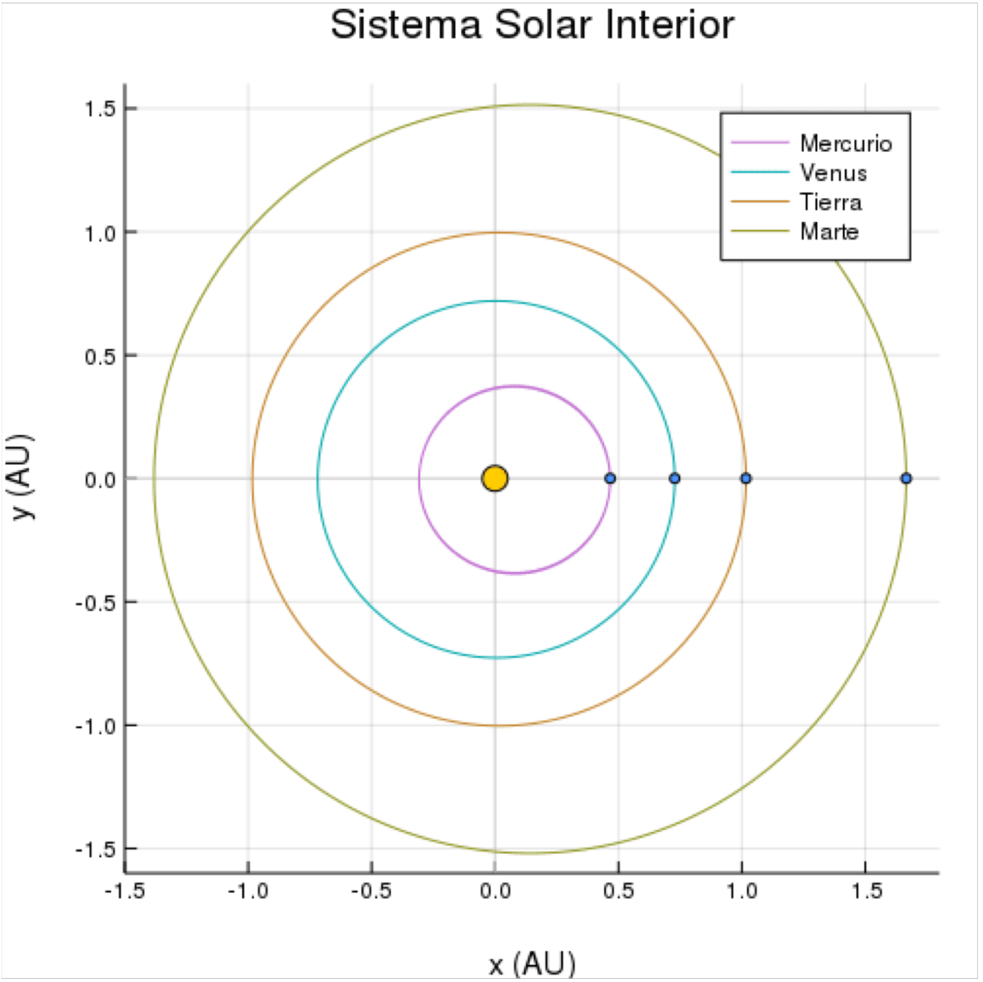
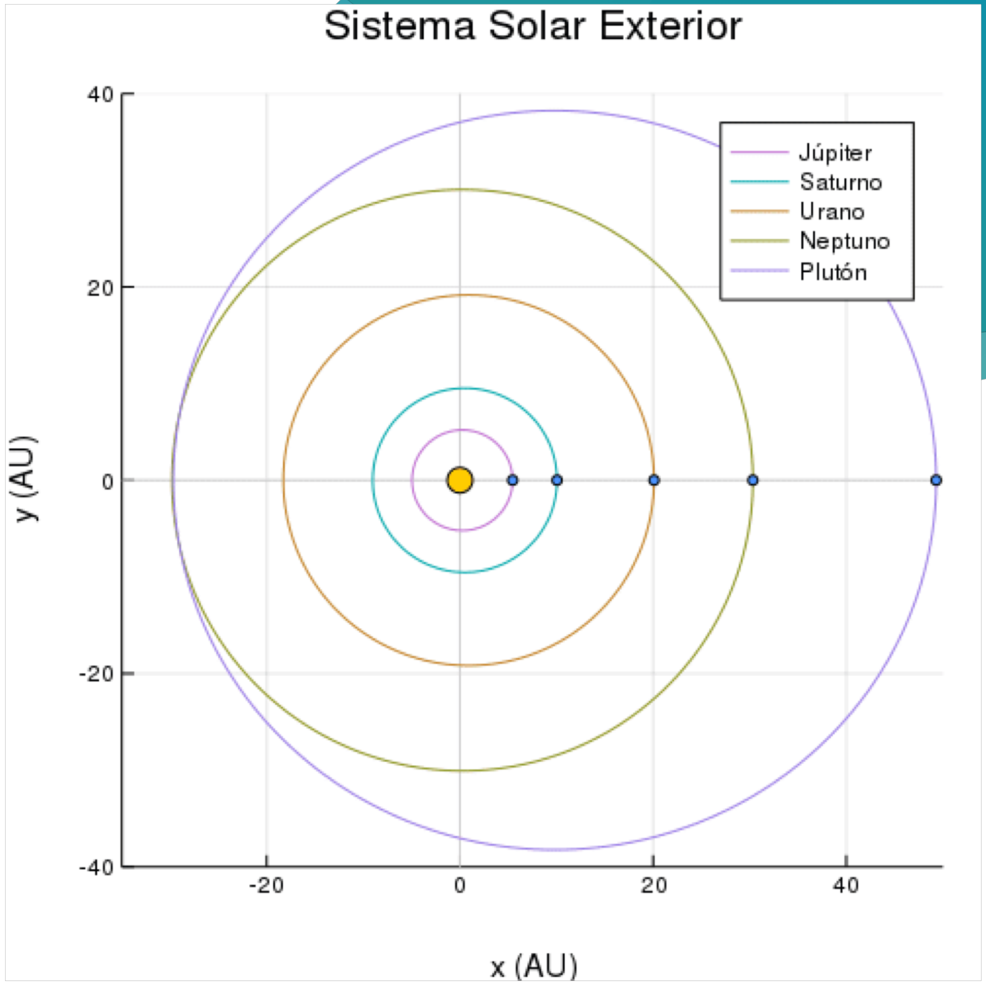
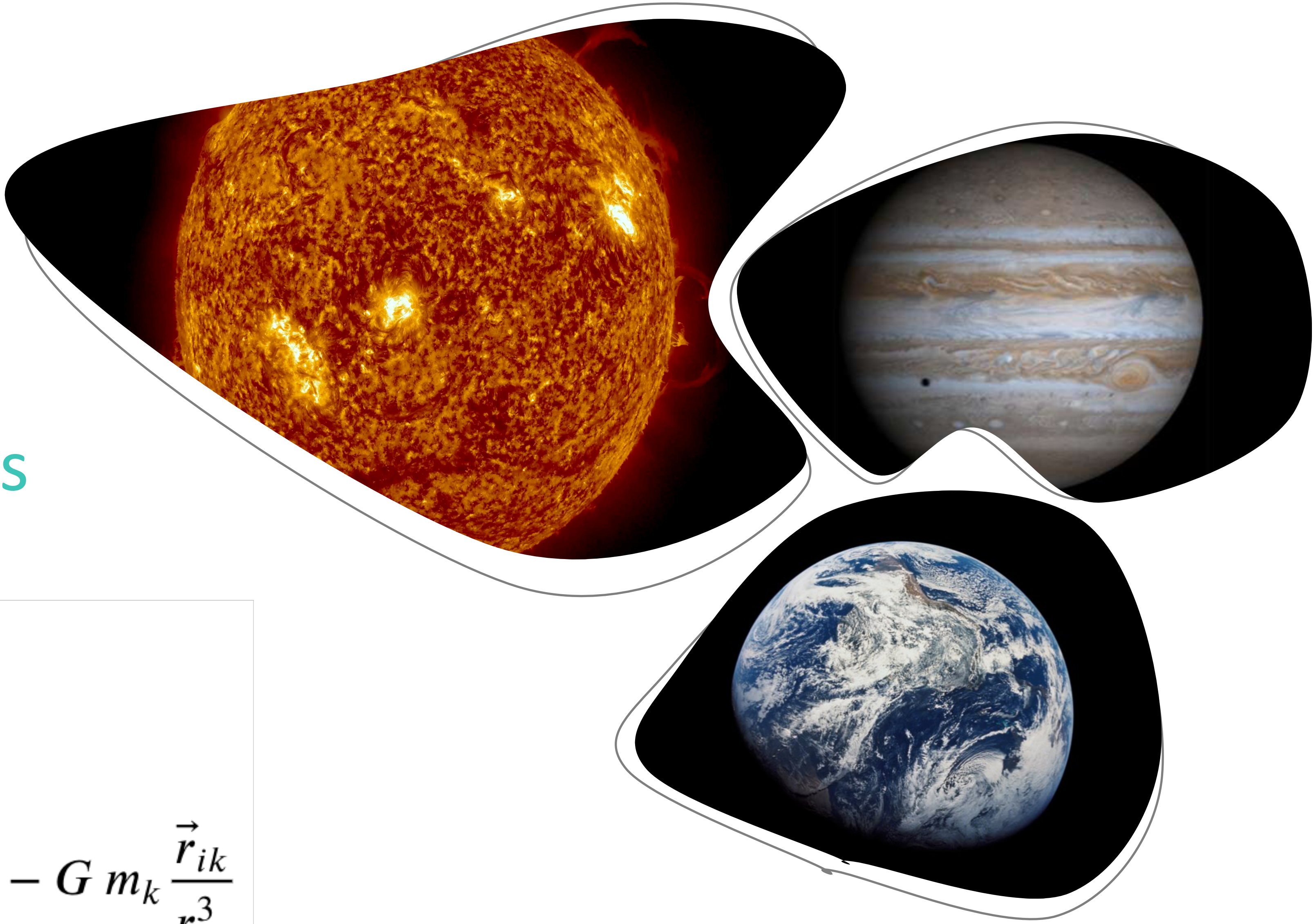
Tierra	1.0000	1.0000	0.0000
Marte	1.8800	0.9991	0.0881
Júpiter	11.8800	1.0021	0.2066
Saturno	29.4600	1.0006	0.0624
Urano	84.0600	1.0001	0.0050
Neptuno	164.9000	1.0001	0.0094
Plutón	248.0800	1.0000	0.0032


Sistema Solar Interior
Mercurio
Venus
Tierra
Marte
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
y (AU)
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
x (AU)


Sistema Solar Exterior
Júpiter
Saturno
Urano
Neptuno
Plutón
40
20
0
-20
-40
y (AU)
-20
0
20
40
x (AU)



Tres cuerpos

$$\frac{d\vec{r}_i}{dt} = \vec{v}_i$$

$$\frac{d\vec{v}_i}{dt} = -G\,m_j\,\frac{\vec{r}_{ij}}{r_{ij}^3} - G\,m_k\,\frac{\vec{r}_{ik}}{r_{ik}^3}$$

Efecto de Júpiter sobre la Tierra

$$\frac{dx_e}{dt} = v_{x,e}$$

$$\frac{dv_{x,e}}{dt} = -\frac{G\,M_S\,x_e}{r^3} - \frac{G\,M_J\,(x_e - x_j)}{r_{EJ}^3}$$

$$\frac{dy_e}{dt} = v_{y,e}$$

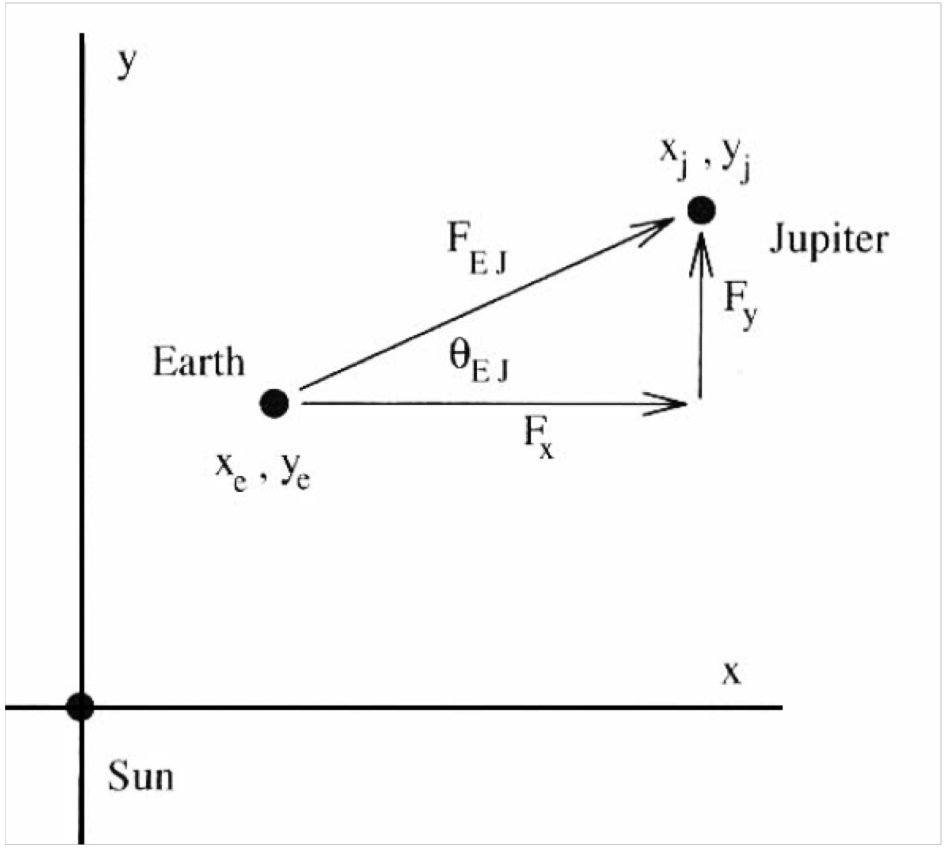
$$\frac{dv_{y,e}}{dt} = -\frac{G\,M_S\,y_e}{r^3} - \frac{G\,M_J\,(y_e - y_j)}{r_{EJ}^3}$$

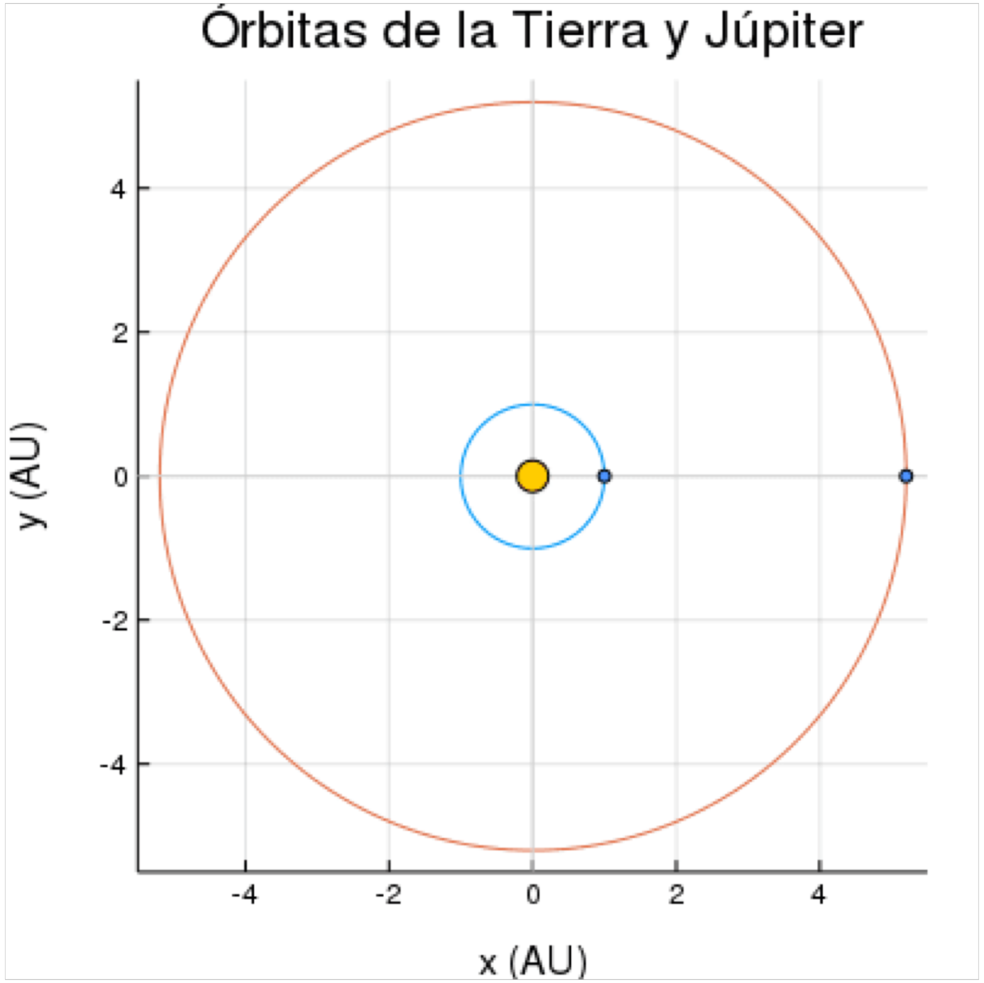
$$\frac{dx_j}{dt} = v_{x,j}$$

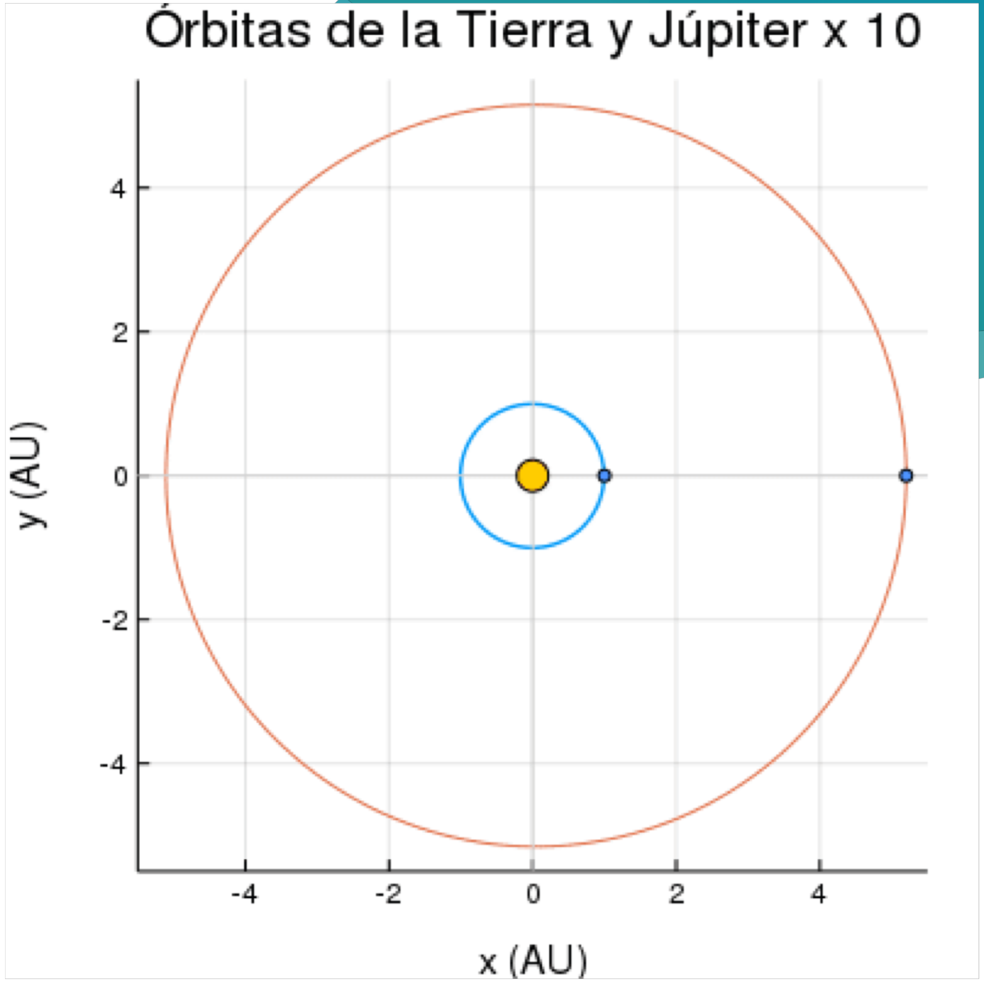
$$\frac{dv_{x,j}}{dt} = -\frac{G\,M_S\,x_j}{r^3} - \frac{G\,M_E\,(x_j - x_e)}{r_{EJ}^3}$$

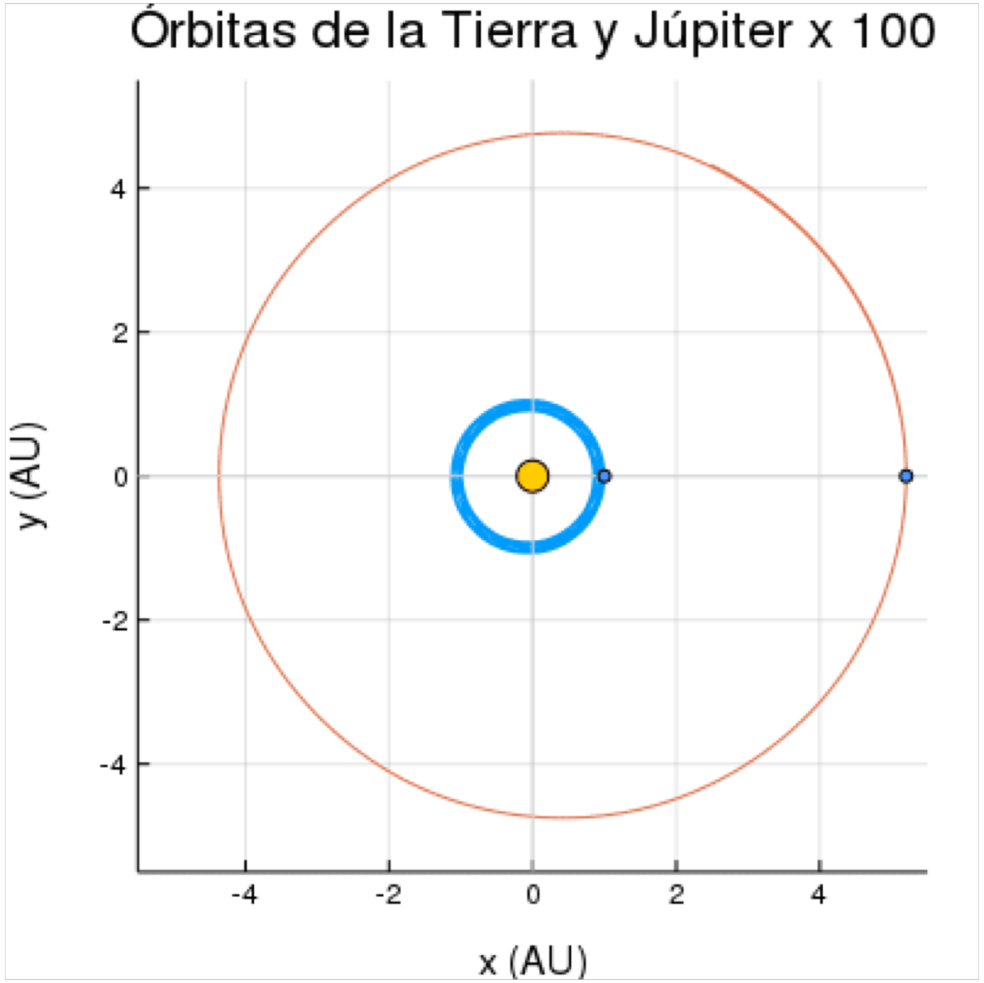
$$\frac{dy_j}{dt} = v_{y,j}$$

$$\frac{dv_{y,j}}{dt} = -\frac{G\,M_S\,y_j}{r^3} - \frac{G\,M_E\,(y_j - y_e)}{r_{EJ}^3}$$

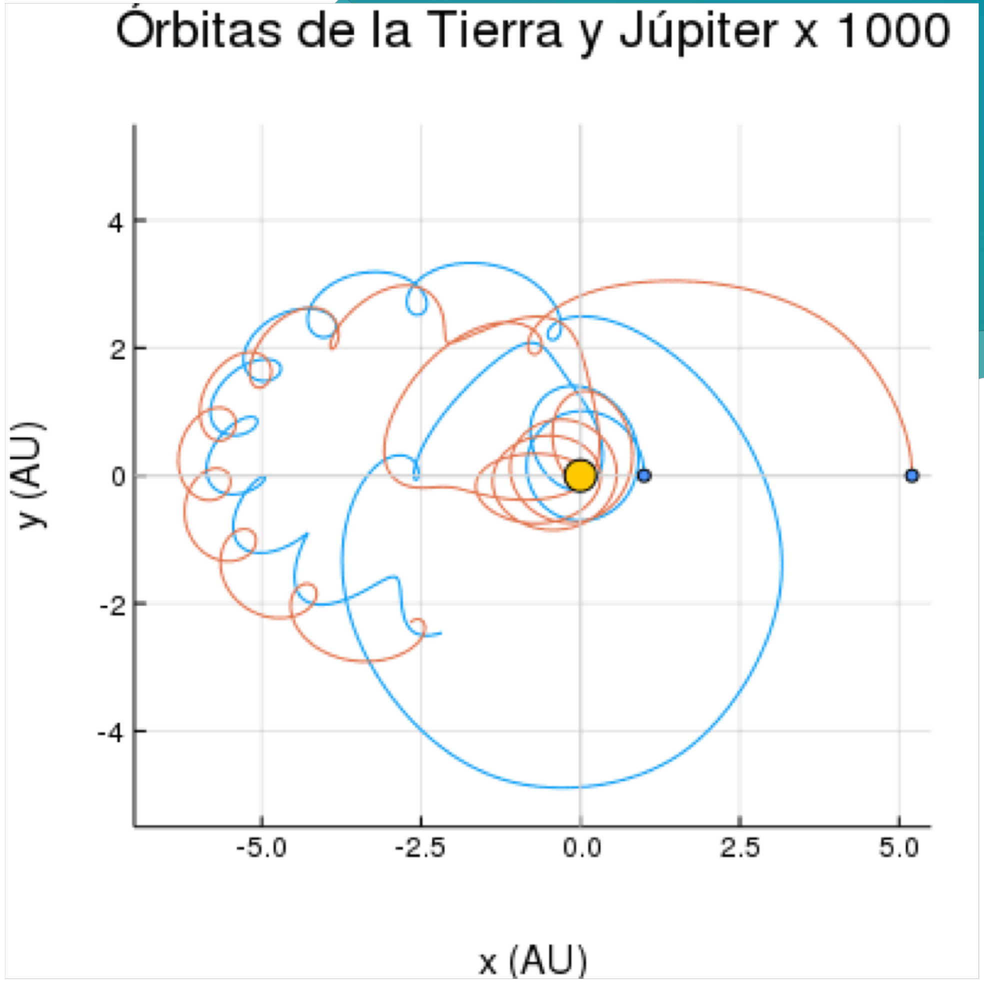



Órbitas de la Tierra y Júpiter
4
2
0
-2
-4
y (AU)
-4
-2
0
2
4
x (AU)

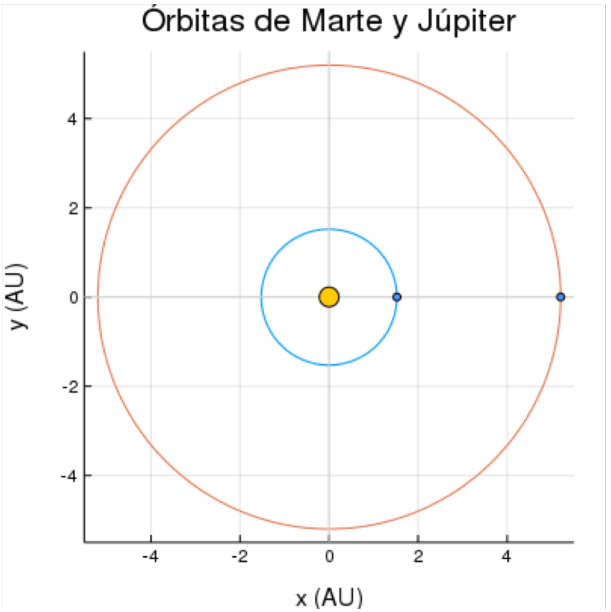

Órbitas de la Tierra y Júpiter x 10
4
2
0
-2
-4
y (AU)
-4
-2
0
2
4
x (AU)

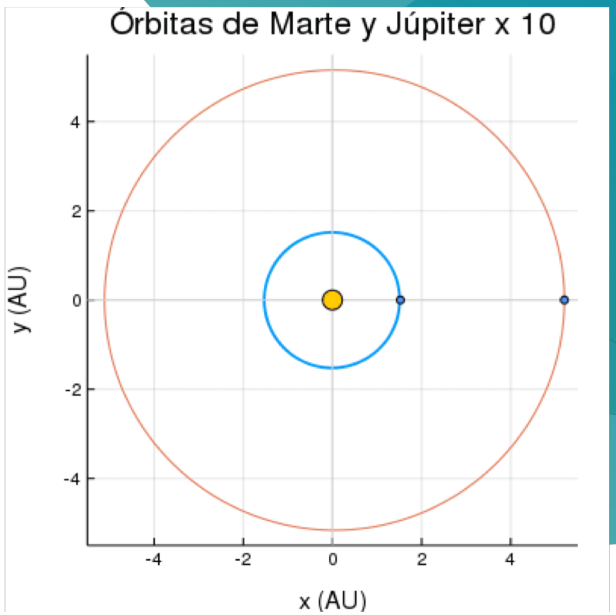


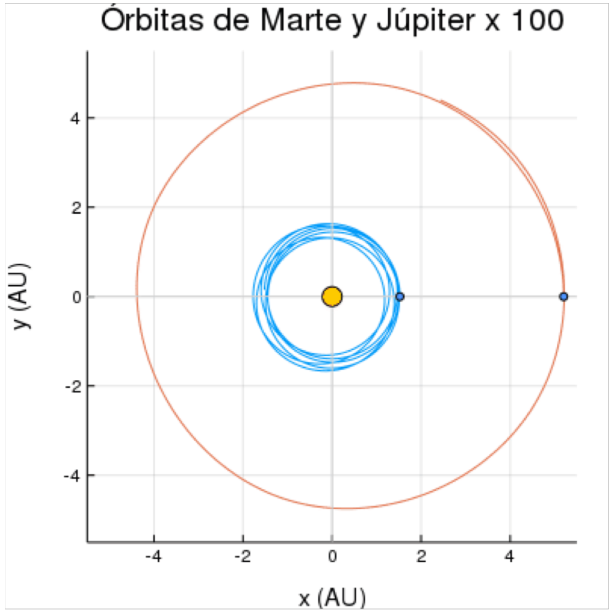
Órbitas de la Tierra y Júpiter x 100
y (AU)
x (AU)
4
2
0
-2
-4
-4
-2
0
2
4

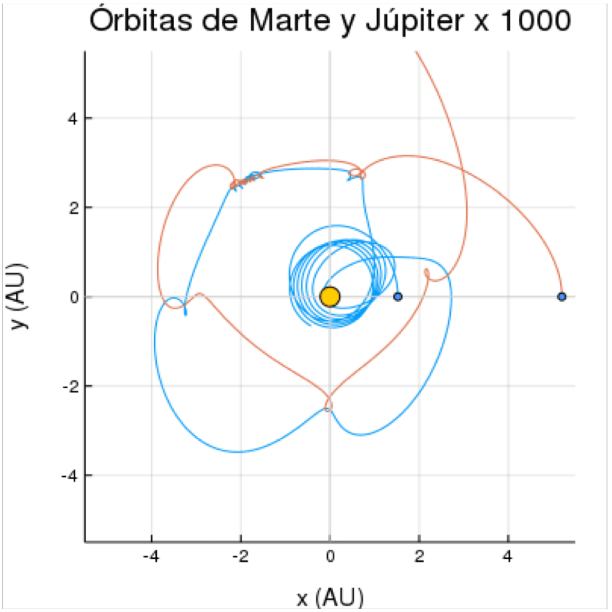


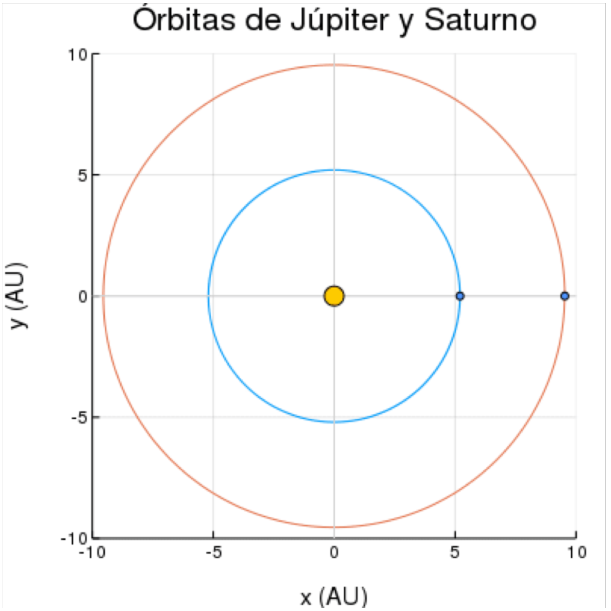
Órbitas de la Tierra y Júpiter x 1000
y (AU)
x (AU)
4
2
0
-2
-4
-5.0
-2.5
0.0
2.5
5.0

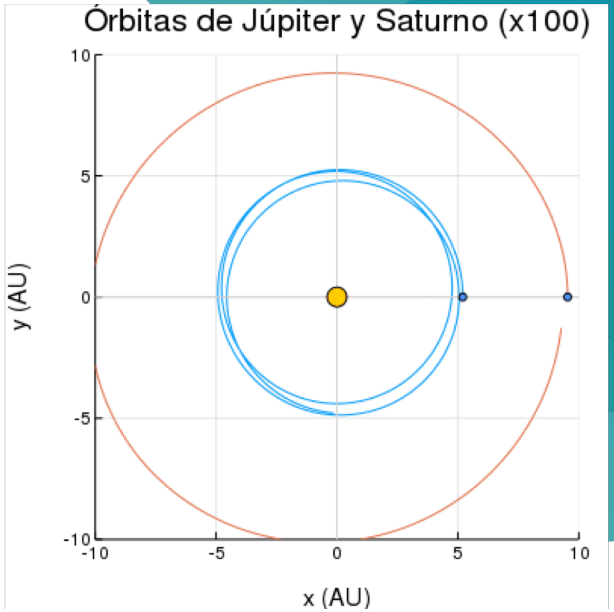

Órbitas de Marte y Júpiter
y (AU)
x (AU)
4
2
0
-2
-4

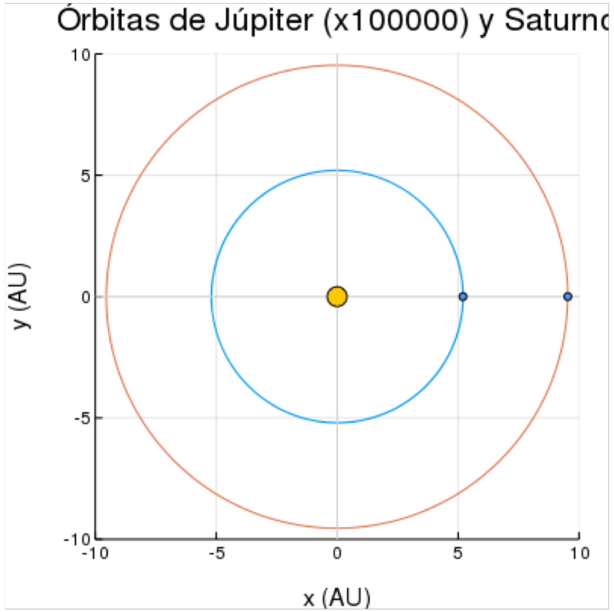

Órbitas de Marte y Júpiter x 10
y (AU)
x (AU)
4
2
0
-2
-4

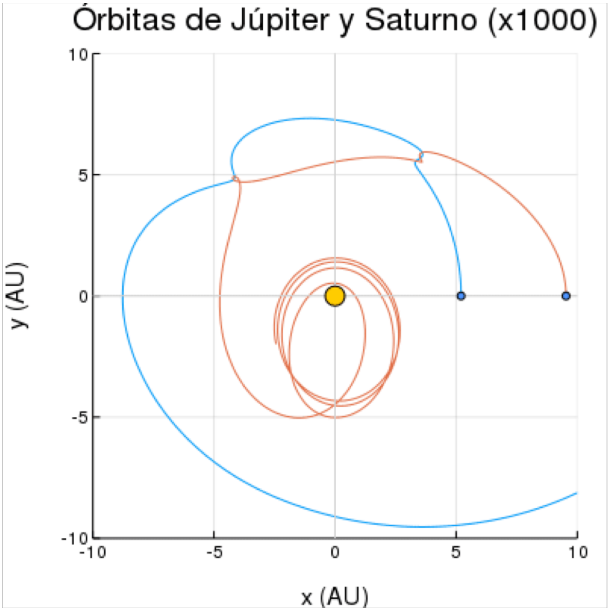

Órbitas de Marte y Júpiter x 100
y (AU)
x (AU)
4
2
0
-2
-4


Órbitas de Marte y Júpiter x 1000
y (AU)
x (AU)
4
2
0
-2
-4

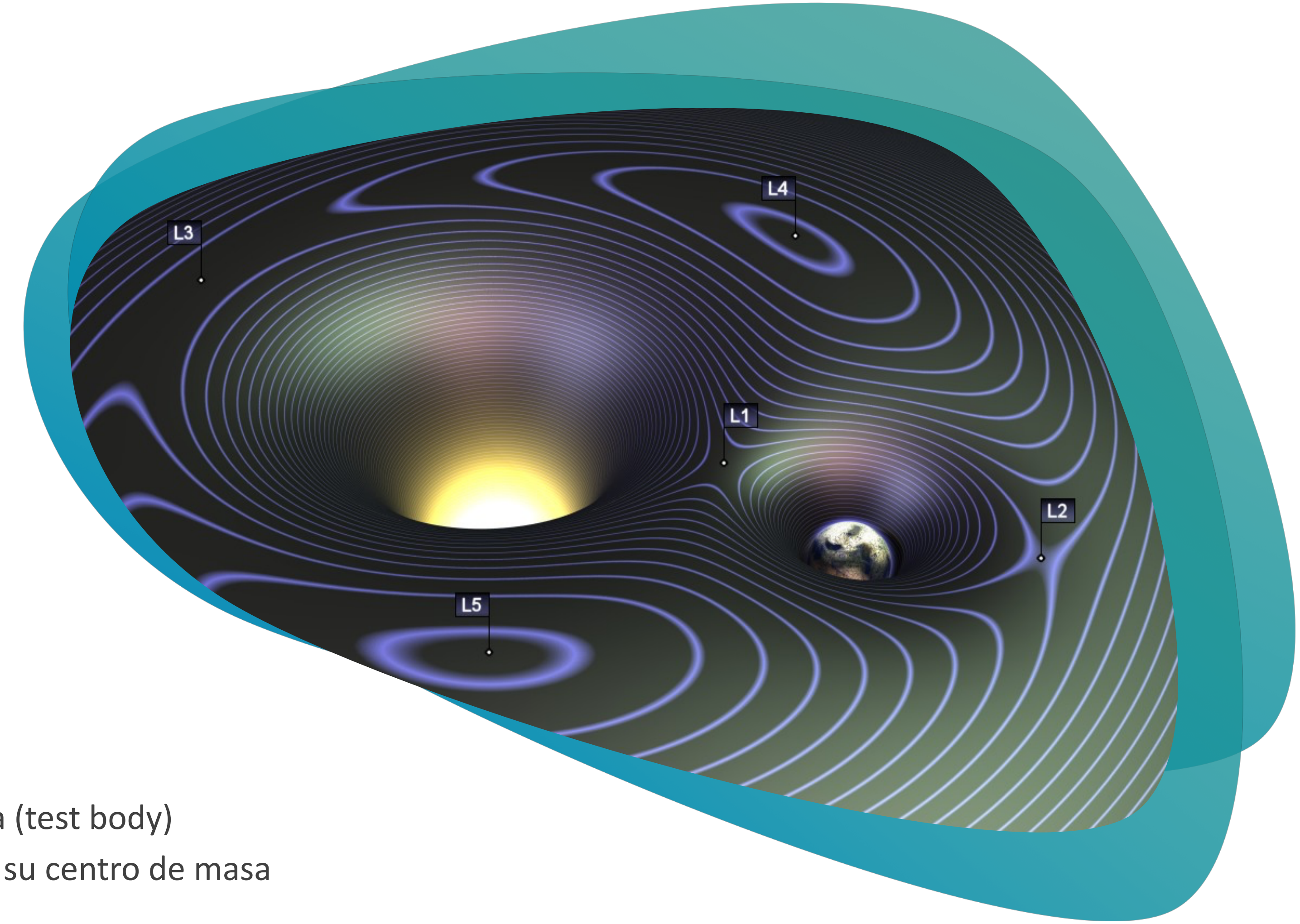

Órbitas de Júpiter y Saturno
x (AU)
y (AU)


Órbitas de Júpiter y Saturno (x100)
x (AU)
y (AU)


Órbitas de Júpiter (x100000) y Saturno
x (AU)
y (AU)


Órbitas de Júpiter y Saturno (x1000)
x (AU)
y (AU)

Problema restringido de tres cuerpos



Caso particular idealizado:

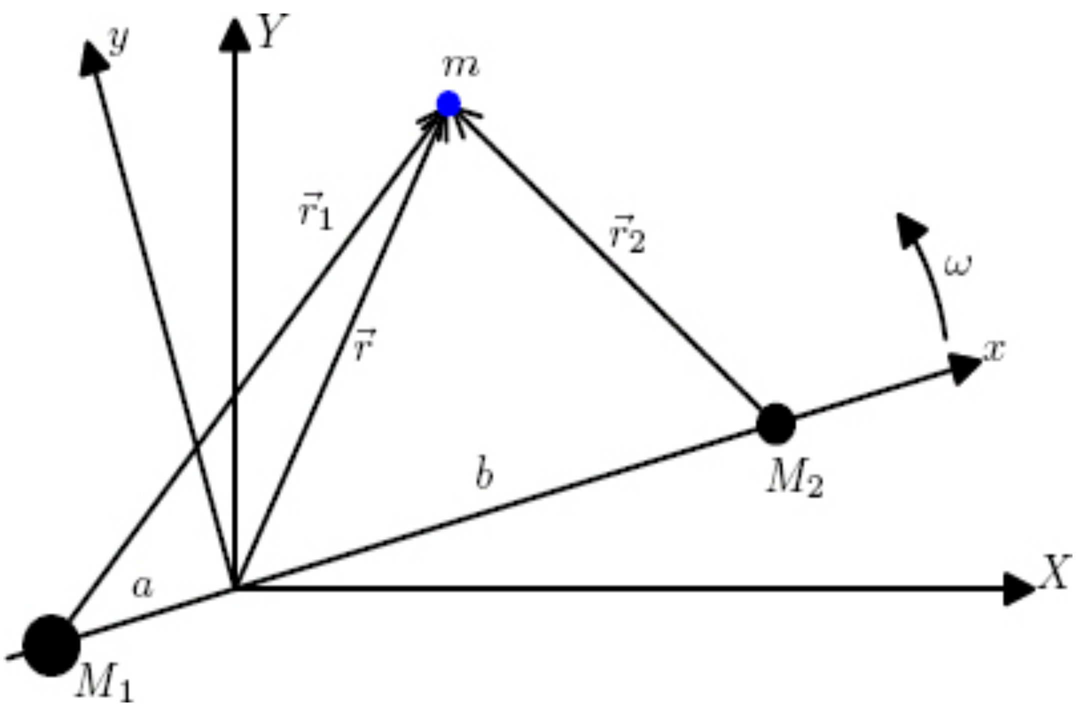
- Dos masas son mucho más grandes que la tercera (test body)
- Las dos masas primarias se mueven alrededor de su centro de masa

$$\vec{F} = -GM_1 m \frac{\vec{r}_1}{r_1^3} - GM_2 m \frac{\vec{r}_2}{r_2^3}$$

$$r_1 = \sqrt{(x+a)^2 + y^2}, \qquad r_2 = \sqrt{(x-b)^2 + y^2}$$

En el sistema de coordendas giratorio de las masas primarias:

$$\vec{a} = -GM_1 \frac{\vec{r}_1}{r_1^3} - GM_2 \frac{\vec{r}_2}{r_2^3} + \omega^2 \vec{r} - 2\vec{\omega} \times \vec{v}$$



El potencial efectivo

$$\vec{a} = -\nabla V - 2\vec{\omega} \times \vec{v}$$

$$V(x,y) = -\frac{GM_1}{r_1} - \frac{GM_2}{r_2} - \frac{1}{2}\omega^2 r^2$$

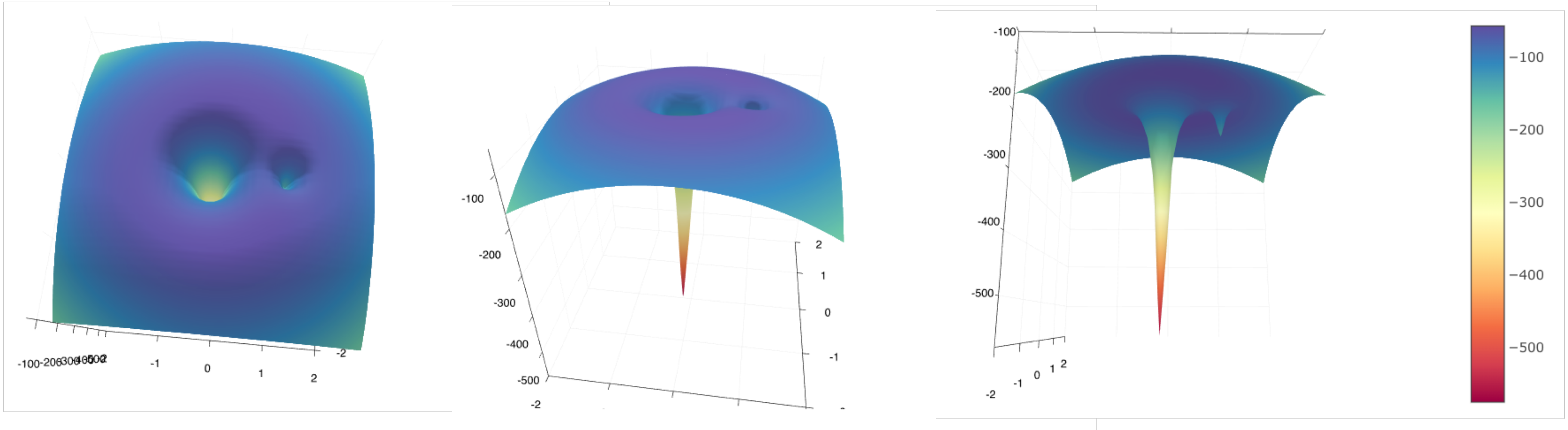
$$= -\frac{GM_1}{\sqrt{(x+a)^2 + y^2}} - \frac{GM_2}{\sqrt{(x-b)^2 + y^2}} - \frac{1}{2}\omega^2 (x^2 + y^2)$$

Unidades para el problema restringido de tres cuerpos (RTB)

Masa	$M = M_1 + M_2$, masa total del sistema
Longitud	$R = a + b$, la separación entre las primarias
Tiempo	T, periodo rotacional de las primarias

$$V(x,y) = -4\pi^2 \left[\frac{1-\alpha}{\sqrt{(x+\alpha)^2 + y^2}} + \frac{\alpha}{\sqrt{(x-1+\alpha)^2 + y^2}} + \frac{x^2+y^2}{2} \right]$$

Parámetro de masa: $\alpha = \dfrac{M_2}{M_1 + M_2}$

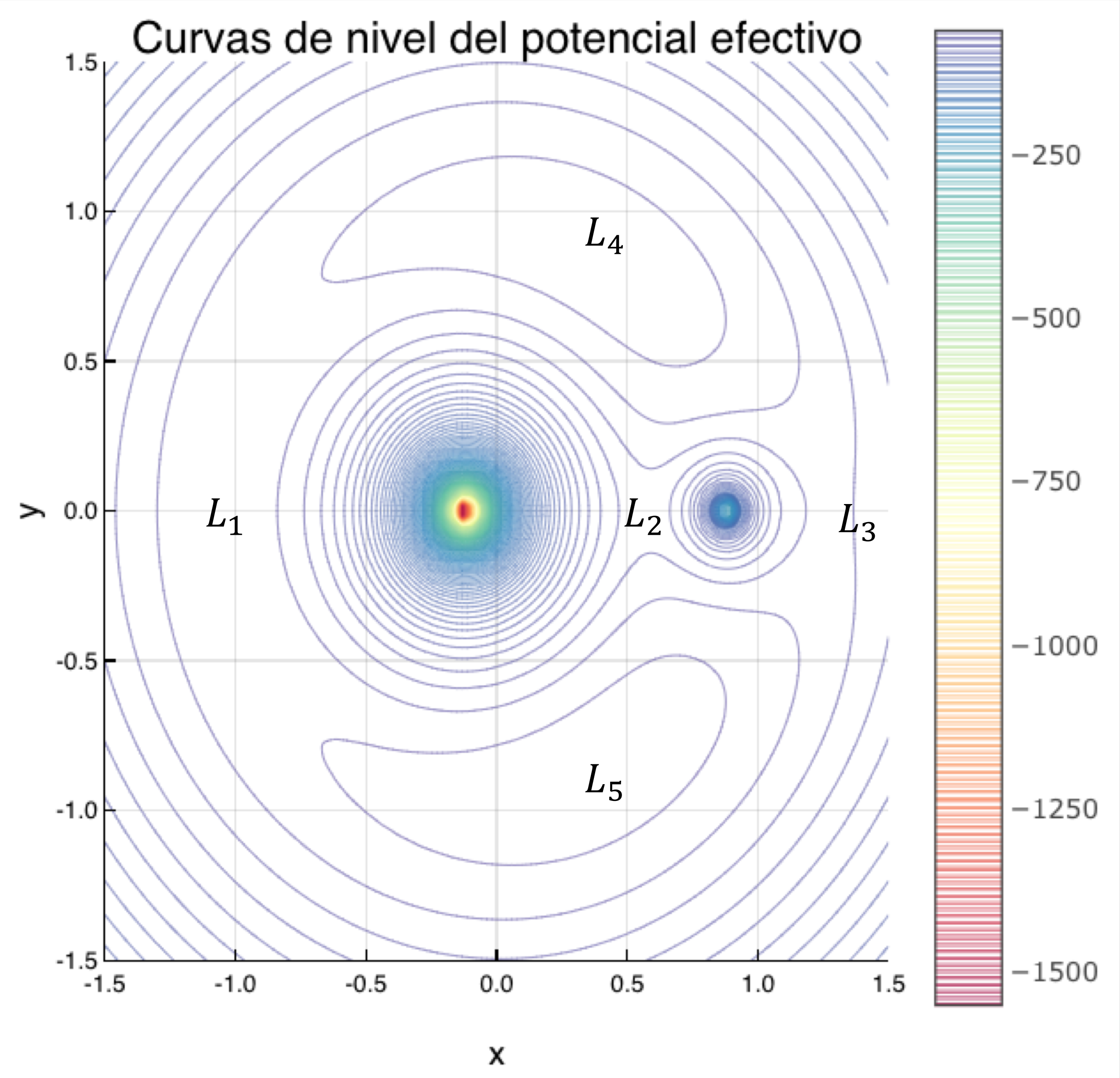


Puntos de Lagrange

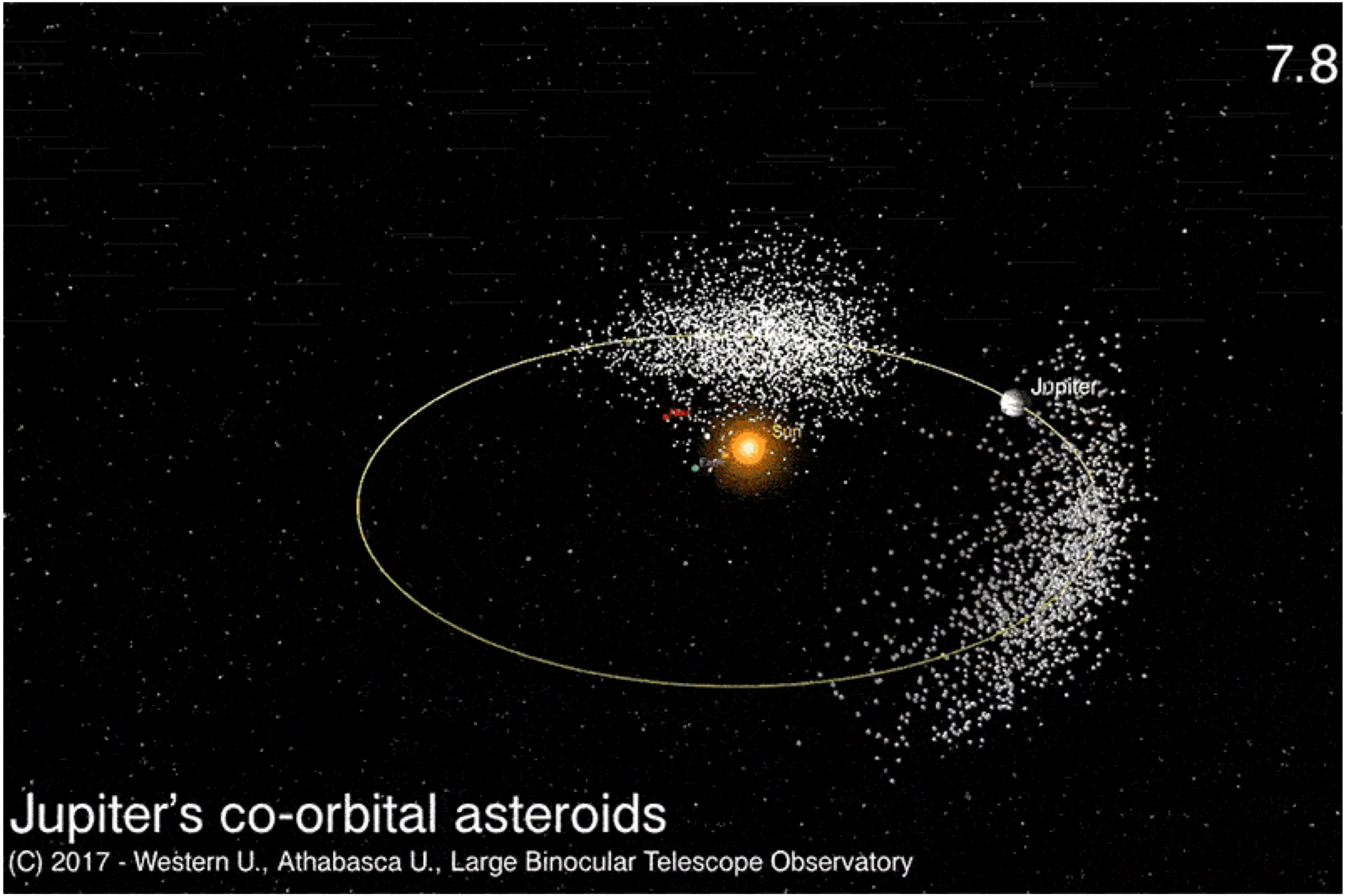
$$L_1 \;:\; \left(-1-\frac{5a}{12},0\right)$$

$$L_2, L_3 \;:\; \left(1\mp\sqrt[3]{\frac{\alpha}{3}},0\right)$$

$$L_4, L_5 \;:\; \left(\frac{1}{2}-\alpha,\pm\frac{\sqrt{3}}{2}\right)$$



$\alpha = 0.121$


7.8
Jupiter
Sun
Jupiter's co-orbital asteroids
(C) 2017 - Western U., Athabasca U., Large Binocular Telescope Observatory

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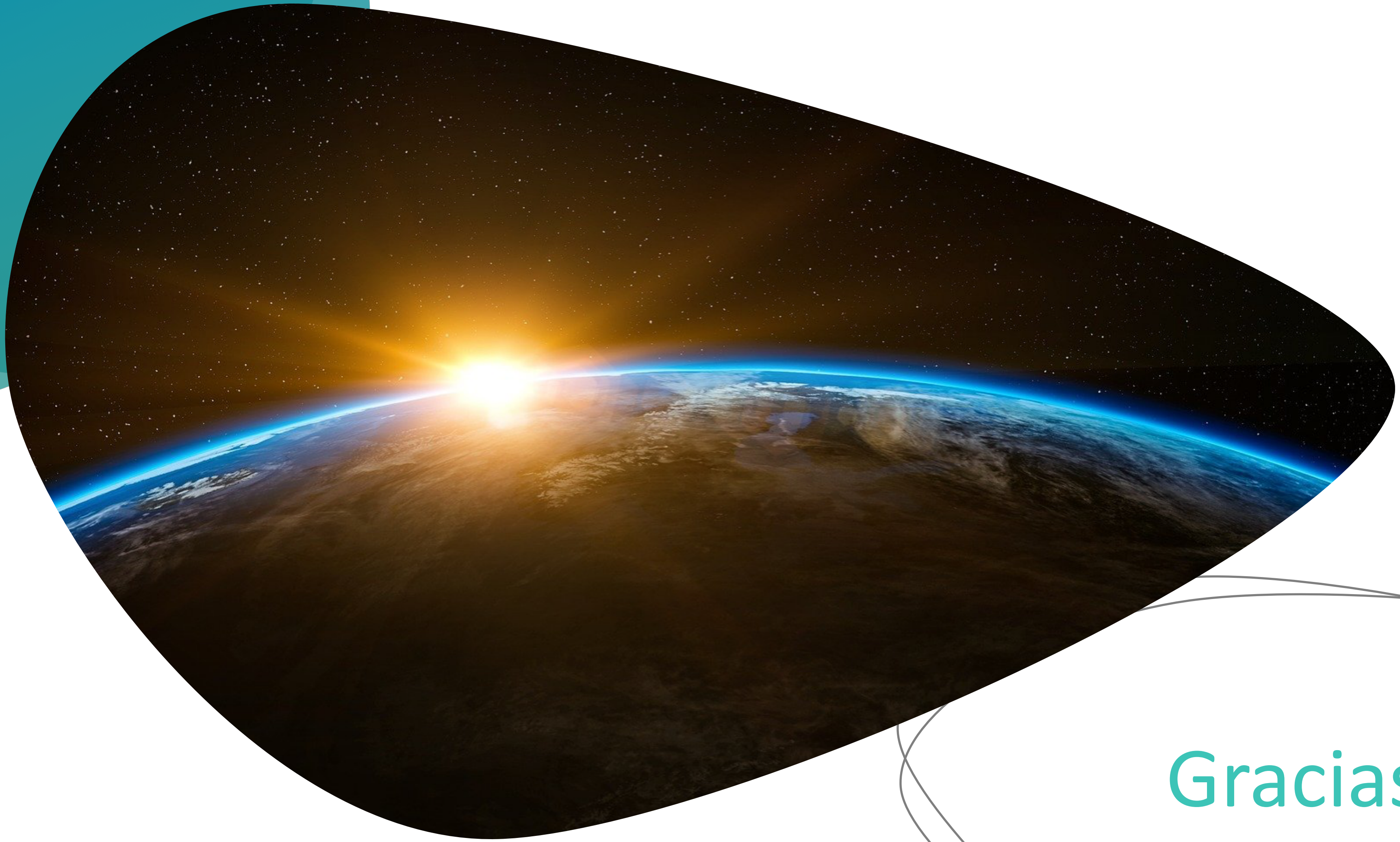

Gracias