Finite Volume Methods for Hyperbolic Problems

Gas Dynamics and Euler Equations

- The Euler equations
- Conservative vs. primitive variables
- Contact discontinuities
- Projecting phase space to p–u plane
- Hugoniot loci and integral curves
- Solving the Riemann problem

Riemann Problems and Jupyter Solutions

Theory and Approximate Solvers for Hyperbolic PDEs

David I. Ketcheson, RJL, and Mauricio del Razo

General information and links to book, Github, Binder, etc.:

bookstore.siam.org/fa16/bonus

View static version of notebooks at:

www.clawpack.org/riemann_book/html/Index.html

In particular see: **Euler.ipynb**

Compressible gas dynamics

Conservation laws:

$$\begin{aligned}\rho_t + (\rho u)_x &= 0\\ (\rho u)_t + (\rho u^2 + p)_x &= 0\end{aligned}$$

Equation of state:

$$p = P(\rho).$$

Same as shallow water if $P(\rho) = \frac{1}{2} g \rho^2$ (with $\rho \equiv h$).

Isothermal: $P(\rho) = a^2 \rho$ (since T proportional to p/ρ).

Isentropic: $P(\rho) = \hat{\kappa} \rho^\gamma$ ($\gamma \approx 1.4$ for air)

Jacobian matrix:

$$f'(q) = \begin{bmatrix} 0 & 1 \\ P'(\rho) - u^2 & 2u \end{bmatrix}, \qquad \lambda = u \pm \sqrt{P'(\rho)}.$$

Gas dynamics variables

$\rho =$ density

$\vec{u} =$ velocity (just u in 1D, $[u, v]$ in 2D, $[u, v, w]$ in 3D)

$h\vec{u} =$ momentum

$p =$ pressure

$e =$ internal energy (vibration, heat) $= \frac{p}{(\gamma-1)\rho}$ for polytropic

$\frac{1}{2}\rho\|\vec{u}\|_2^2 =$ kinetic energy

$E =$ total energy

$c_p, c_v =$ specific heat at constant pressure or volume

$T =$ temperature $= e/c_v = \frac{p}{c_v(\gamma-1)\rho}$ for polytropic

$\gamma = c_p/c_v =$ adiabatic exponent for polytropic, $1 < \gamma \leq 5/3$

$h = e + p/\rho =$ specific enthalpy

$H = \frac{E+p}{\rho} = h + \frac{1}{2}u^2 =$ total specific enthalpy

$s = c_v \log(p/\rho^\gamma) +$ const $=$ specific entropy for polytropic

Equations of state

Polytropic: $E = e + \frac{1}{2}\rho u^2$ and $e = \frac{p}{(\gamma-1)\rho}$, so

$$\begin{aligned} p &= \rho e(\gamma - 1) \\ &= (\gamma - 1)\left(E - \frac{1}{2}\rho u^2\right) \\ &= P(\rho, \rho u, E) \end{aligned}$$

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Isothermal: $T = \frac{p}{c_v(\gamma-1)\rho}$ for polytropic, so

$$p = Tc_v(\gamma - 1)\rho \equiv a^2\rho = P(\rho)$$

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Isentropic: $s = c_v \log(p/\rho^\gamma) + \text{const}$

$$p = \hat{c}\rho^\gamma = P(\rho)$$

Euler equations of gas dynamics

Conservation of mass, momentum, energy: $q_t + f(q)_x = 0$ with

$$q = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \qquad f(q) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{bmatrix}$$

where $E = \rho e + \frac{1}{2}\rho u^2$

Equation of state: $p =$ pressure $= p(\rho, E)$

Ideal gas, polytropic EOS: $p = \rho e(\gamma - 1) = (\gamma - 1)\left(E - \frac{1}{2}\rho u^2\right)$

$\gamma \approx 7/5 = 1.4$ for air, $\quad \gamma = 5/3$ for monatomic gas

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The Jacobian $f'(q)$ has eigenvalues $u - c, \; u, \; u + c$ where

$$c = \sqrt{\left.\frac{dp}{d\rho}\right|_{\text{at constant entropy}}} = \sqrt{\frac{\gamma p}{\rho}} \text{ for polytropic}$$

Euler equations in primitive variables

Can rewrite the conservation laws in quasilinear form:

$$\begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_t + \begin{bmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \gamma p & u \end{bmatrix} \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_x = 0.$$

Eigenvalues and eigenvectors:

$$\begin{aligned} \lambda^1 &= u - c, & \lambda^2 &= u, & \lambda^3 &= u + c, \\ r^1 &= \begin{bmatrix} -\rho/c \\ 1 \\ -\rho c \end{bmatrix}, & r^2 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, & r^3 &= \begin{bmatrix} \rho/c \\ 1 \\ \rho c \end{bmatrix}, \end{aligned}$$

Euler equations in primitive variables

$$
\begin{aligned}
\nabla\lambda^1 &= \begin{bmatrix} -\partial c/\partial\rho \\ 1 \\ -\partial c/\partial p \end{bmatrix} = \begin{bmatrix} c/2\rho \\ 1 \\ -c/2p \end{bmatrix} \quad &\Longrightarrow \quad &\nabla\lambda^1\cdot r^1 = \tfrac{1}{2}(\gamma+1), \\
\nabla\lambda^2 &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad &\Longrightarrow \quad &\nabla\lambda^2\cdot r^2 = 0, \\
\nabla\lambda^3 &= \begin{bmatrix} \partial c/\partial\rho \\ 1 \\ \partial c/\partial p \end{bmatrix} = \begin{bmatrix} -c/2\rho \\ 1 \\ c/2p \end{bmatrix} \quad &\Longrightarrow \quad &\nabla\lambda^3\cdot r^3 = \tfrac{1}{2}(\gamma+1).
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$$\nabla\lambda^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \nabla\lambda^2 \cdot r^2 = 0,$$

$$\nabla\lambda^3 = \begin{bmatrix} \partial c/\partial\rho \\ 1 \\ \partial c/\partial p \end{bmatrix} = \begin{bmatrix} -c/2\rho \\ 1 \\ c/2p \end{bmatrix} \quad \Longrightarrow \quad \nabla\lambda^3 \cdot r^3 = \frac{1}{2}(\gamma+1).$$

1-waves and 3-waves are genuinely nonlinear,
2-waves are linearly degenerate (contact discontinuity).

Contact discontinuities

Consider Riemann problem for conservative variables:

$$q = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \qquad f(q) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{bmatrix}$$

Suppose $p_\ell = p_r$ and $u_\ell = u_r \equiv u$,

Then the Rankine-Hugoniot condition $s\Delta q = \Delta f$ becomes:

$$s \begin{bmatrix} \Delta\rho \\ u\Delta\rho \\ \Delta E \end{bmatrix} = \begin{bmatrix} u\Delta\rho \\ u^2\Delta\rho \\ u\Delta E \end{bmatrix}$$

Satisfied with $s = u$, for any jump in density $\Delta\rho$.

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Satisfied with $s = u$, for any jump in density $\Delta\rho$.

And for any equation of state.

Euler in conservation form

Jacobian:

$$f'(q) = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma-3)u^2 & (3-\gamma)u & (\gamma-1) \\ \frac{1}{2}(\gamma-1)u^3 - uH & H-(\gamma-1)u^2 & \gamma u \end{bmatrix},$$

$$H = \frac{E+p}{\rho} = h + \frac{1}{2}u^2 = \text{ total specific enthalpy}$$

Eigenvalues and eigenvectors:

$$\lambda^1 = u-c, \qquad \lambda^2 = u, \qquad \lambda^3 = u+c,$$

$$r^1 = \begin{bmatrix} 1 \\ u-c \\ H-uc \end{bmatrix}, \quad r^2 = \begin{bmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{bmatrix}, \quad r^3 = \begin{bmatrix} 1 \\ u+c \\ H+uc \end{bmatrix}.$$

Riemann invariants for Euler (polytropic gas)

1-Riemann invariants: $s, \quad u + \dfrac{2}{\gamma - 1}\sqrt{\dfrac{\gamma p}{\rho}},$

2-Riemann invariants: $u, \quad p,$

3-Riemann invariants: $s, \quad u - \dfrac{2}{\gamma - 1}\sqrt{\dfrac{\gamma p}{\rho}}.$

Riemann invariants for Euler (polytropic gas)

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Note: The entropy s is constant through any (smooth) simple 1-wave or 3-wave.

In particular, linear acoustic waves are isentropic.

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In particular, linear acoustic waves are isentropic.

Note: u and p constant across in any simple 2-wave,
and across a contact discontinuity (check R-H condition).

Since $\lambda^2 = u$, this says characteristics are parallel
(the field is linearly degenerate)

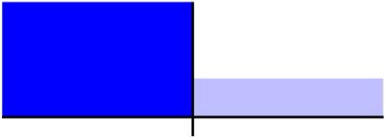
Riemann Problem for Euler equations

Initial data:

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$

Shock tube problem: $u_l = u_r = 0$, jump in ρ and p.

Pressure:



Similar to solution of dam break problem for shallow water equations.

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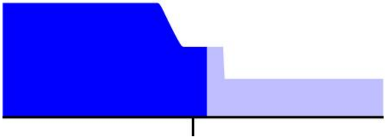

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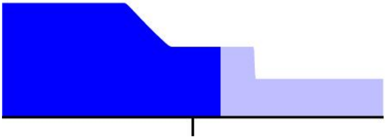

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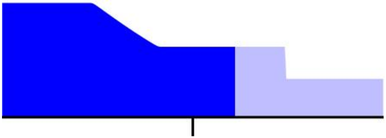

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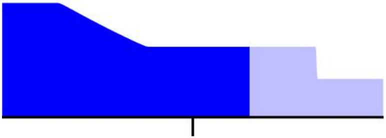

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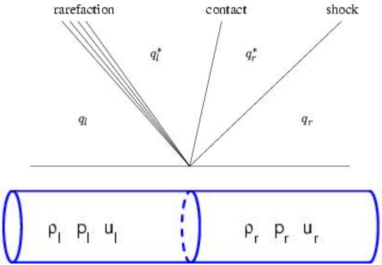
Riemann Problem for gas dynamics

Waves propagating in x–t space:



In primitive variables:

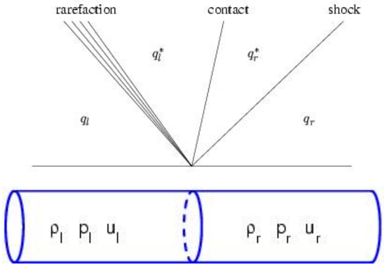
$$q_\ell^* = \left[\begin{array}{c} \rho_l^* \\ p^* \\ u^* \end{array} \right]$$

$$q_r^* = \left[\begin{array}{c} \rho_r^* \\ p^* \\ u^* \end{array} \right]$$

Only ρ jumps across 2-wave

Riemann Problem for gas dynamics

Waves propagating in x–t space:



Similarity solution
(function of x/t alone)

In primitive variables:

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Only ρ jumps across 2-wave

Waves can be approximated by discontinuities:

High-resolution wave-propagation methods

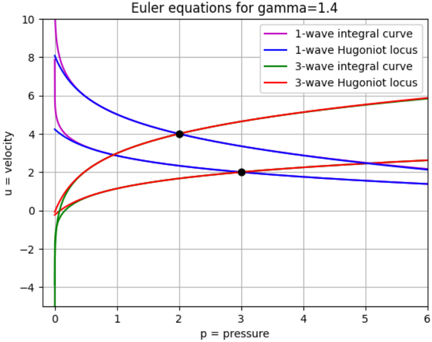
Approximate Riemann solvers

Riemann Problem for gas dynamics

Any jump in ρ is allowed across contact discontinuity

General Riemann solver:

- Project 3D phase space to p–u plane,
  Hugoniot loci and integral curves can be written as
  $u = \phi(p)$, (and $\rho = \rho(p)$)
- Find intersection
  $(p^*,\ u^*)$,
- Compute ρ_ℓ^* and ρ_r^*.



Integral curves for gas dynamics

In 1-wave, we know the Riemann invariants are constant,

$$s = c_v \log(p/\rho^\gamma) \quad \text{and} \quad u + \frac{2}{\gamma - 1} c \quad \text{with } c = \sqrt{\frac{\gamma p}{\rho}}$$

Given values in left state q_ℓ, can then compute integral curve as:

$$u = u_\ell + \left(\frac{2\,c_\ell}{\gamma - 1}\right)\left(1 - (p/p_\ell)^{(\gamma-1)/(2\gamma)}\right) \equiv \phi_\ell(p) \quad \text{for } p \leq p_\ell.$$

Note that ρ does not appear!

Since s is constant, $\rho = (p/p_\ell)^{1/\gamma}\rho_\ell$.

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Since s is constant, $\rho = (p/p_\ell)^{1/\gamma}\rho_\ell$.

Can find similar expression for 3-wave integral curve,

$$u = u_r - \left(\frac{2\,c_r}{\gamma - 1}\right)\left(1 - (p/p_r)^{(\gamma-1)/(2\gamma)}\right) \equiv \phi_r(p) \quad \text{for } p \le p_r.$$

Hugoniot locus for gas dynamics

From Rankine-Hugoniot conditions, can deduce that (1-wave):

$$u = u_\ell + \frac{2\,c_\ell}{\sqrt{2\gamma(\gamma-1)}} \left(\frac{1 - p/p_\ell}{\sqrt{1+\beta p/p_\ell}}\right) \equiv \phi_\ell(p) \quad \text{for } p \geq p_\ell.$$

where $\beta = (\gamma+1)/(\gamma-1)$.

Again note that ρ does not appear!

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For any p on this Hugoniot locus, we also find that:

$$\rho = \left(\frac{1+\beta p/p_\ell}{p/p_\ell+\beta}\right)\rho_\ell.$$

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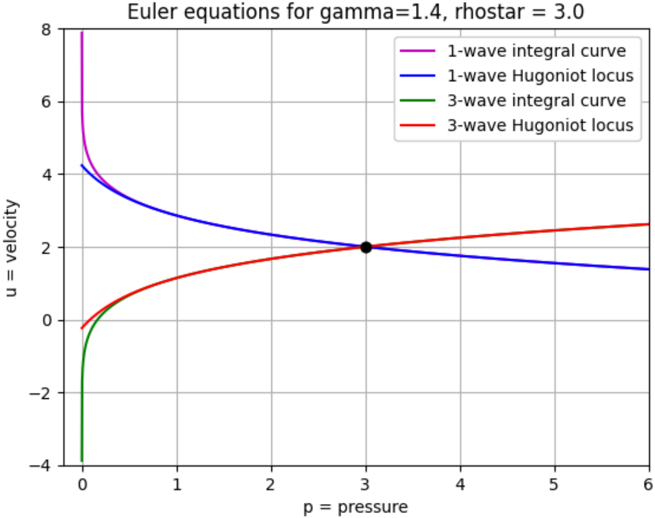
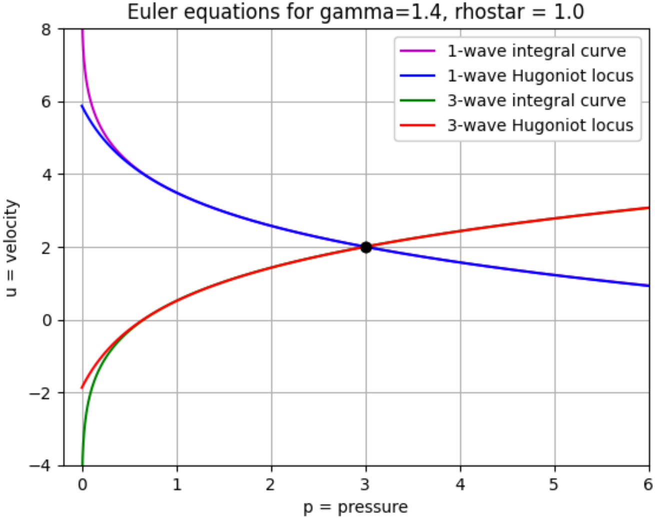
Similar expression for 3-wave, $u = \phi_r(p)$ for $p \geq p_r$.

Euler equations phase plane



Note these are curves in (p, u, ρ) space projected to plane.

Euler equations phase plane

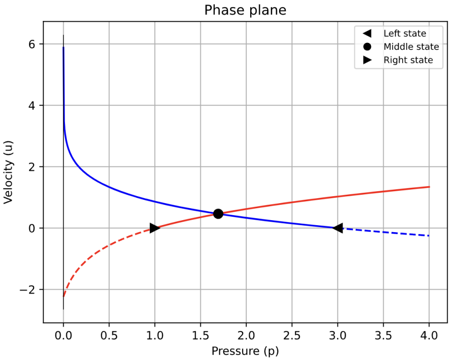


Note these are curves in (p, u, ρ) space projected to plane.

Solving the Euler Riemann problem

```
In [71]: left_state  = State(Density = 3., Velocity = 0., Pressure = 3.)
         right_state = State(Density = 1.,Velocity = 0.,Pressure = 1.)

         euler.phase_plane_plot(left_state, right_state)
         grid(True)
```



blue = integral curve, red = Hugoniot locus, dashed = nonphysical

Solving the Euler Riemann problem

```
In [71]: left_state  = State(Density = 3., Velocity = 0., Pressure = 3.)
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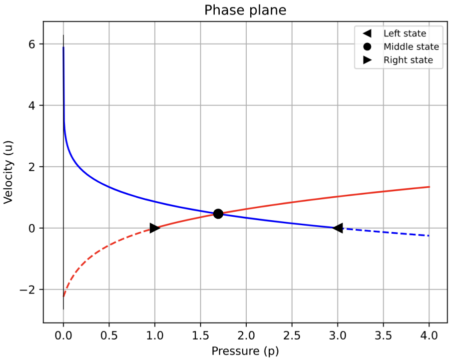


Solve $\phi_l(p) - \phi_r(p) = 0$ for p_m

$u_m = \phi_l(p_m) = \phi_r(p_m)$

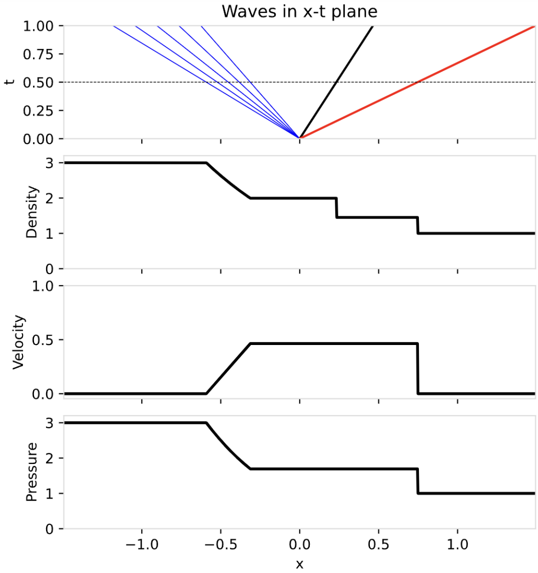
$\rho_{m\ell} = \rho(p_m)$ across 1-wave

$\rho_{mr} = \rho(p_m)$ across 2-wave

Red curve is displaced from blue in ρ direction (into page).

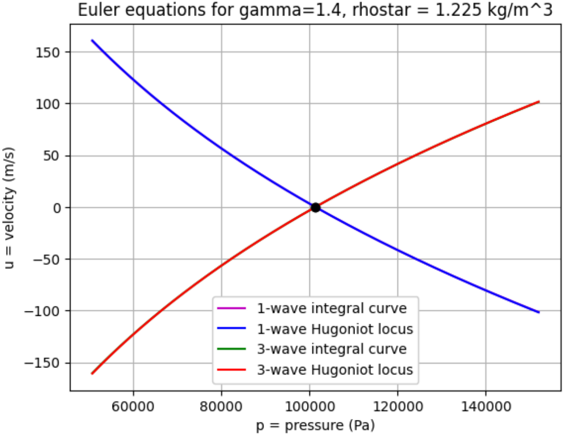
blue = integral curve, red = Hugoniot locus, dashed = nonphysical

Solving the Euler Riemann problem



Euler equations at atmospheric conditions

With parameters for air at $T^* = 20°$ C, Density $\rho^* = 1.225$ kg/m^3.
Pressure $p^* = 101,325$ Pa $= 1$ atm, Speed of sound: $c^* = 340.3$ m/s



from ≈ 0.5 atm to 2 atm

Euler equations at atmospheric conditions

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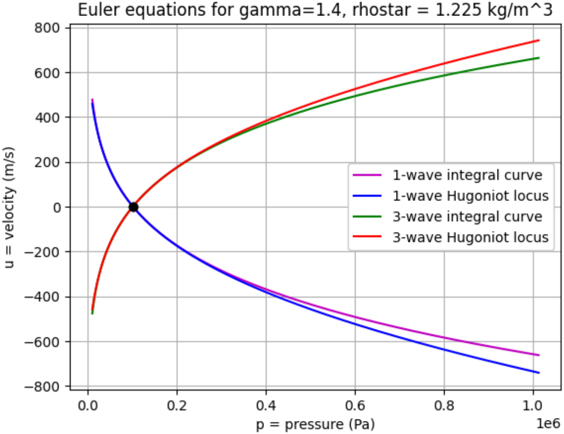
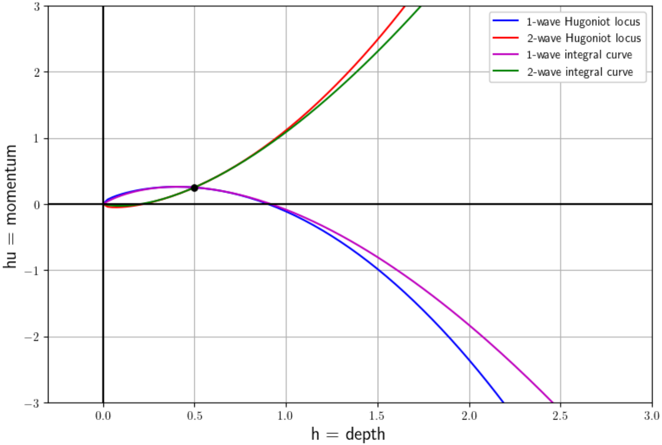


from ≈ 0.1 atm to 10 atm

Shallow water equations phase plane

In the h–hu phase plane (the conserved quantities):



Shallow water equations phase plane

Replot in the h–u phase plane (primitive variables):