Finite Volume Methods for Hyperbolic Problems

Accuracy, Consistency, Stability, CFL Condition

- Order of accuracy, local and global error
- Consistent numerical flux functions
- Stability
- CFL Condition

For more details see e.g. Chapter 10 of
Finite Difference Methods for ODEs and PDEs

Finite differences vs. finite volumes

Finite difference Methods

- Pointwise values $Q_i^n \approx q(x_i, t_n)$
- Approximate derivatives by finite differences
- Assumes smoothness

Finite volume Methods

- Approximate cell averages: $Q_i^n \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t_n)\, dx$
- Integral form of conservation law,

$$\frac{\partial}{\partial t} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)\, dx = f(q(x_{i-1/2}, t)) - f(q(x_{i+1/2}, t))$$

leads to conservation law $q_t + f_x = 0$ but also directly to numerical method.

Order of Accuracy — upwind method

Upwind method for advection $q_t + uq_x = 0$ with $u > 0$:

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n)$$

Written in form that mimics PDE:

$$\left(\frac{Q_i^{n+1} - Q_i^n}{\Delta t}\right) + u\left(\frac{Q_i^n - Q_{i-1}^n}{\Delta x}\right) = 0$$

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Local truncation error:

Insert true solution $u(x,t)$ into difference equation

$$\tau(x,t) = \left(\frac{q(x_i, t_{n+1}) - q(x_i, t_n)}{\Delta t}\right) + u\left(\frac{q(x_i, t_n) - q(x_{i-1}, t_n)}{\Delta x}\right)$$

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Assume smoothness and expand in Taylor series:

$$q(x_i, t_{n+1}) = q(x_i, t_n) + \Delta t q_t(x_i, t_n) + \frac{1}{2}\Delta t^2 q_{tt}(x_i, t_n) + \cdots$$

$$q(x_{i-1}, t_n) = q(x_i, t_n) - \Delta x q_x(x_i, t_n) + \frac{1}{2}\Delta x^2 q_{xx}(x_i, t_n) + \cdots$$

Order of Accuracy — upwind method

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gives (with everything evaluated at (x_i, t_n)):

$$\begin{aligned}
\tau(x_i,t_n) &= \left(\frac{\Delta t q_t + \frac{1}{2}\Delta t^2 q_{tt} + \cdots}{\Delta t}\right) + u\left(\frac{\Delta x q_x + \frac{1}{2}\Delta x^2 q_{xx} + \cdots}{\Delta x}\right) \\
&= (q_t + uq_x) + \frac{1}{2}(\Delta t q_{tt} - u\Delta x q_{xx}) + \mathcal{O}(\Delta x^2, \Delta t^2)
\end{aligned}$$

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Since q is the exact solution, $q_t + uq_x = 0$ and $q_{tt} = u^2 q_{xx}$, so

$$\tau(x_i,t_n) = \frac{1}{2}\Delta x\left(\frac{u\Delta t}{\Delta x} - 1\right)uq_{xx} + \mathcal{O}(\Delta x^2)$$

Order of Accuracy — upwind method

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Assuming $\Delta t/\Delta x$ is constant as we refine the grid.

The method is said to be first order accurate.

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Can show that if the method is also stable as $\Delta x \to 0$ then the global error will also be first order for smooth enough solutions.

$$E(x,t) \equiv Q(x,t) - q(x,t) = \mathcal{O}(\Delta x)$$

where we fix (x,t) and let $Q(x,t)$ denote the numerical approximation at this point as the grid is refined.

Order of Accuracy — upwind method

Global error: $E(x,t) \equiv Q(x,t) - q(x,t)$

Discontinuous solutions?

If $q(x,t)$ has a discontinuity then we cannot expect convergence pointwise or in the max-norm

$$\|E(\cdot,t)\|_\infty = \max_{a\le x\le b} |E(x,t)|.$$

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Best we can hope for is convergence in some norm like

$$\|E(\cdot,t)\|_1 = \int_a^b |E(x,t)|\,dx \quad \approx \Delta x \sum_i |Q_i^n - q(x_i,t_n)|.$$

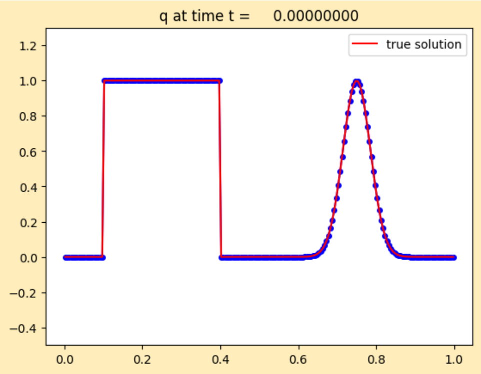
For upwind on discontinuous data, we expect

$$\|E(\cdot,t)\|_1 = \mathcal{O}(\Delta x^{1/2}).$$

Advection tests

$q_t + q_x = 0$ with periodic BCs
Solution at $t = 1$ should agree with initial data.

Initial data with 200 cells:



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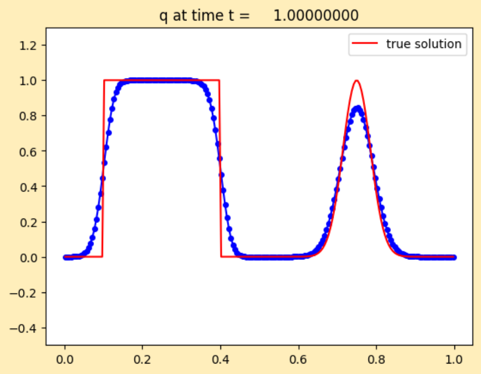
Upwind solution with 200 cells:



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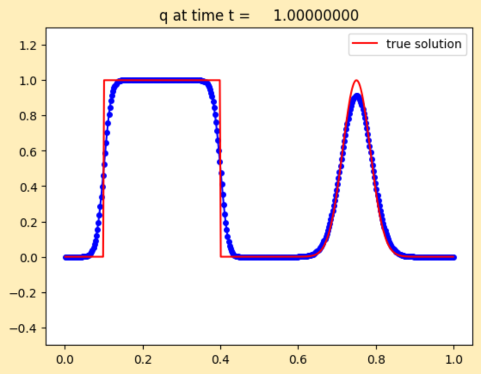
Upwind solution with 400 cells:



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Consistency

A method is consistent if $\tau \to 0$ as $\Delta t,\ \Delta x \to 0$.

The one-step error is $\Delta t\, \tau$:

$$\Delta t\, \tau = q(x, t+\Delta t) - \left(q(x,t) - \frac{u \Delta t}{\Delta x}(q(x,t) - q(x-\Delta x, t)) \right).$$

An error of this magnitude is made in each of $T/\Delta t$ time steps.

This suggests $E \approx (T/\Delta t)(\Delta t\, \tau) = T\tau$:

$\tau = O(\Delta x^p + \Delta t^p) \implies$ global error is $O(\Delta x^p + \Delta t^p)$

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This is valid provided the method is stable!

Consistency $+$ stability $=$ convergence

Consistency for conservation law

For $q_t + f(q)_x = 0$, consider a method in conservation form,

$$Q_i^{n+1} = Q_i^n + \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n).$$

The method is consistent with the PDE if

$$F_{i-1/2} = \mathcal{F}(Q_{i-1}, Q_i) \quad \text{with } \mathcal{F}(\bar{q}, \bar{q}) = f(\bar{q})$$

and the numerical flux function is Lipschitz continuous,

$$|\mathcal{F}(q_\ell, q_r) - f(\bar{q})| \leq C \max(|q_\ell - \bar{q}|,\ |q_r - \bar{q}|).$$

for all q_ℓ, q_r in a neighborhood of $\bar{q}$.

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Example: $\mathcal{F}(q_\ell, q_r) = uq_\ell$ for upwind, with $C = u$.

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Consistent if $\mathcal{F}(\bar{q}, \bar{q}) = f(\bar{q})$ and Lipschitz continuous.

Upwind for $u > 0$: $f(q) = uq, \;\; \mathcal{F}(q_\ell, q_r) = uq_\ell$, with $C = u$.

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For nonlinear problems, C can depend on $\bar{q}$, e.g.
Burgers': $f(q) = \frac{1}{2}q^2, \;\; \mathcal{F}(q_\ell, q_r) = \frac{1}{2}q_\ell^2$, can take $C = \bar{q} + \epsilon$.

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Godunov's method (upwind) for $q_t + Aq_x = 0$:

$$\mathcal{F}(q_\ell, q_r) = A^+ q_\ell + A^- q_r \implies \mathcal{F}(\bar{q},\bar{q}) = A^+\bar{q} + A^-\bar{q} = A\bar{q} = f(\bar{q})$$

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Centered flux: $\mathcal{F}(q_\ell, q_r) = \frac{1}{2}A(q_\ell + q_r)$
Centered flux for $q_t + f(q)_x = 0$: $\mathcal{F}(q_\ell, q_r) = \frac{1}{2}(f(q_\ell) + f(q_r))$

Consistent provided $f(q)$ is Lipschitz, but unstable!

Fundamental Theorem

$$\boxed{\text{Consistency} + \text{Stability} = \text{Convergence}}$$

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ODE: zero-stability, stability on $q'(t) = 0$ is enough.
Dahlquist Theorem.

Linear PDE: Lax-Richtmyer stability
Uniform power boundedness of a family of matrices
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Scalar conservation law: total variation stability, entropy stability

Systems of conservation laws: few convergence proofs

Stability of the upwind method

Upwind method for advection $q_t + uq_x = 0$ with $u > 0$:

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The quantity

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is called the Courant number or the CFL number after Courant, Friedrichs, and Lewy (1928 paper on existence and uniqueness of PDE solutions).

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Can prove that the upwind method is stable provided

$$0 \leq \frac{u\Delta t}{\Delta x} \leq 1.$$

Then the method converges in the 1-norm as $\Delta x \to 0$.

The CFL Condition (Courant-Friedrichs-Lewy)

Domain of dependence: The solution $q(X,T)$ depends on the data $q(x,0)$ over some set of x values, $x \in \mathcal{D}(X,T)$.

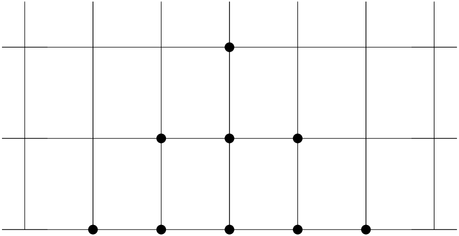
Advection: $q(X,T) = q(X - uT, 0)$ and so $\mathcal{D}(X,T) = \{X - uT\}$.

The CFL Condition: A numerical method can be convergent only if its numerical domain of dependence contains the true domain of dependence of the PDE, at least in the limit as Δt and Δx go to zero.

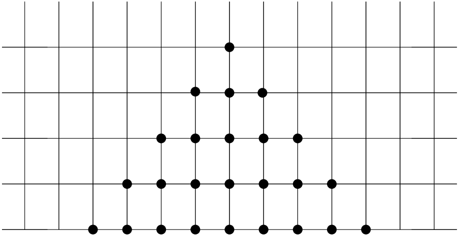
Note: Necessary but not sufficient for stability!

Numerical domain of dependence

With a 3-point explicit method:



On a finer grid with $\Delta t/\Delta x$ fixed:



The CFL Condition

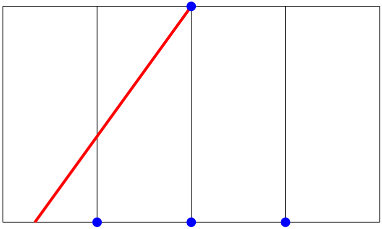
For the method to be stable, the numerical domain of dependence must include the true domain of dependence.

For advection, the solution is constant along characteristics,

$$q(x,t) = q(x - ut, 0)$$

For a 3-point method, CFL condition requires $\left|\frac{u\Delta t}{\Delta x}\right| \leq 1.$

If this is violated:
True solution is determined by data at a point $x - ut$ that is ignored by the numerical method, even as the grid is refined.



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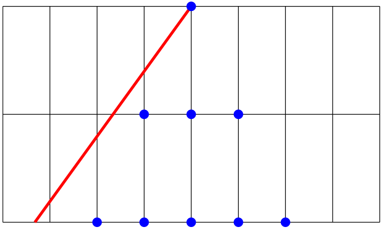

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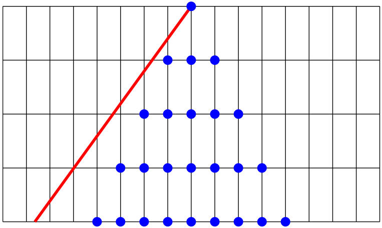

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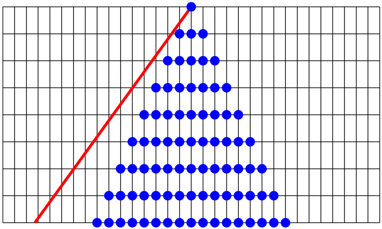
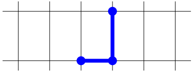
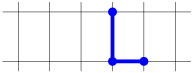


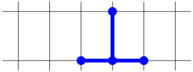
Stencil CFL Condition



$$0 \le \frac{u\Delta t}{\Delta x} \le 1$$



$$-1 \le \frac{u\Delta t}{\Delta x} \le 0$$



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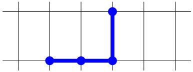
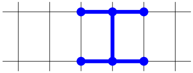


$$0 \le \frac{u\Delta t}{\Delta x} \le 2$$



$$-\infty < \frac{u\Delta t}{\Delta x} < \infty$$

Parabolic equations

Examples: Heat equation $q_t = \beta q_{xx}$,
Advection-diffusion equation $q_t + uq_x = \beta q_{xx}$,
Fluid dynamics with viscosity

Domain of dependence for any point (x, t) with $t > 0$ is:
Entire axis $-\infty < x < \infty$ for Cauchy problem,
All initial and boundary data up to time t for IBVP.

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CFL condition requires either:
Implicit method, or
Explicit method with $\Delta t/\Delta x \to 0$ as grid is refined,
e.g. $\Delta t = (\Delta x)^2$.

Linear hyperbolic systems

Linear system of m equations: $q(x,t) \in \mathbb{R}^m$ for each (x,t) and

$$q_t + Aq_x = 0, \qquad -\infty < x, \infty, \;\; t \ge 0.$$

A is $m \times m$ with eigenvalues λ^p and eigenvectors r^p,
for $p = 1,\ 2,\ ,\ldots,\ m$:

$$Ar^p = \lambda^p r^p.$$

Combining these for $p = 1,\ 2,\ ,\ldots,\ m$ gives

$$AR = R\Lambda$$

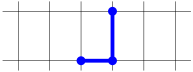
where

$$R = [r^1 \;\; r^2 \;\; \cdots \;\; r^m], \qquad \Lambda = \text{diag}(\lambda^1,\ \lambda^2,\ \ldots,\ \lambda^m).$$

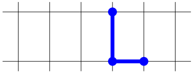
The system is hyperbolic if the eigenvalues are real and
R is invertible. Then A can be diagonalized:

$$R^{-1}AR = \Lambda$$

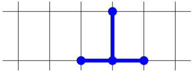
Stencil CFL Condition



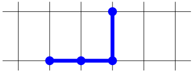
$$0 \le \frac{\lambda_p \Delta t}{\Delta x} \le 1, \quad \forall p$$



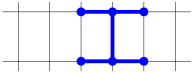
$$-1 \le \frac{\lambda_p \Delta t}{\Delta x} \le 0, \quad \forall p$$



$$-1 \le \frac{\lambda_p \Delta t}{\Delta x} \le 1, \quad \forall p$$



$$0 \le \frac{\lambda_p \Delta t}{\Delta x} \le 2, \quad \forall p$$



$$-\infty < \frac{\lambda_p \Delta t}{\Delta x} < \infty, \quad \forall p$$