Finite Volume Methods for Hyperbolic Problems

TVD Methods and Limiters

- Slope limiters vs. flux limiters
- Total variation for scalar problems
- Proving TVD in flux-limiter form
- Design of TVD limiters
- Sweby Region

High-Resolution methods

- Methods that give good accuracy for smooth solutions
  Clawpack methods: at best second-order accuracy
- Do not have oscillations around discontinuities
  Not only ugly but can lead to nonlinear instabilities

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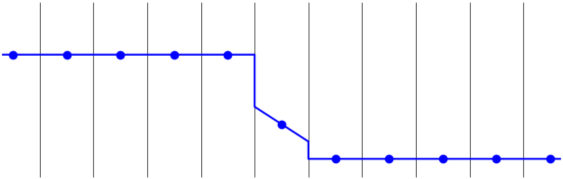
- Godunov-type methods — based on Riemann solvers
  Wave-propagation algorithms with "limiters"

Limiters can eliminate oscillations

Step function data with minmod slope:



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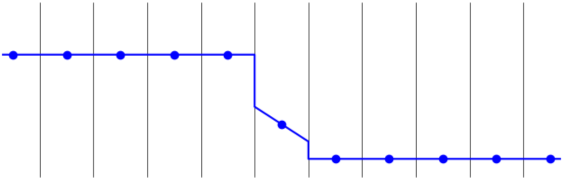
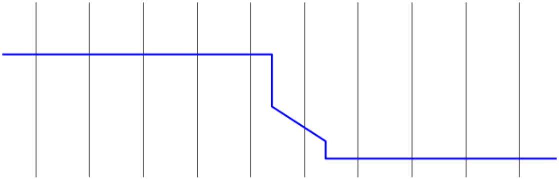


Evolving solution and averaging maintains monotonicity:



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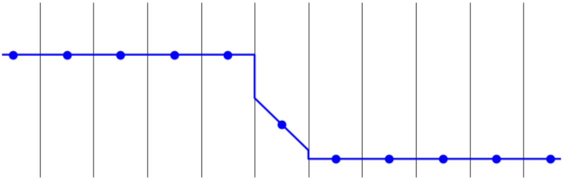

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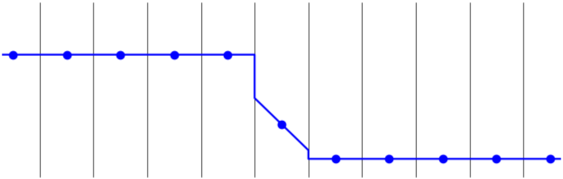

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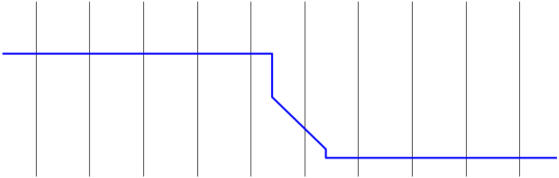

Monotonized centered (MC) limiter

Using the centered slope $(Q^n_{i+1} - Q^n_{i-1})/(2\Delta x)$ gives second-order accuracy (Fromm's method) but not monotonicity.

Limit this slope based on twice the one-sided slopes.

$$\sigma^n_i = \text{minmod}\left(\left(\frac{Q^n_{i+1} - Q^n_{i-1}}{2\Delta x}\right),\ 2\left(\frac{Q^n_i - Q^n_{i-1}}{\Delta x}\right),\ 2\left(\frac{Q^n_{i+1} - Q^n_i}{\Delta x}\right)\right).$$

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Rationale:

- Where solution is smooth, centered slope is smaller and chosen, hence maintains accuracy.
- Near jumps in solution, don't expect second-order but want to resolve discontinuities as sharply as possible.

TVD REA Algorithm

1. Reconstruct a piecewise linear function $\tilde{q}^n(x, t_n)$ defined for all x, from the cell averages Q_i^n.

$$\tilde{q}^n(x, t_n) = Q_i^n + \sigma_i^n(x - x_i) \quad \text{for all } x \in \mathcal{C}_i$$

   with the property that $TV(\tilde{q}^n) \le TV(Q^n)$.

2. Evolve the hyperbolic equation exactly (or approximately) with this initial data to obtain $\tilde{q}^n(x, t_{n+1})$ a time k later.

3. Average this function over each grid cell to obtain new cell averages

$$Q_i^{n+1} = \frac{1}{\Delta x} \int_{\mathcal{C}_i} \tilde{q}^n(x, t_{n+1})\, dx.$$

Note: Steps 2 and 3 are always TVD.

MC slopes are not always a TVD reconstruction

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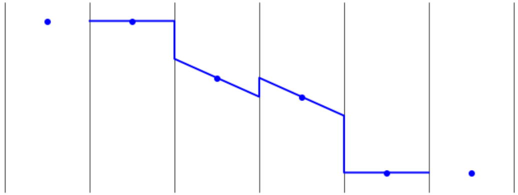

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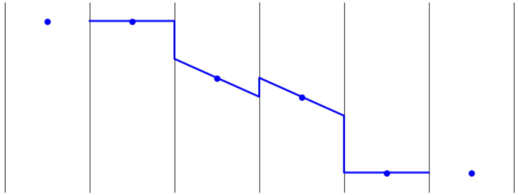
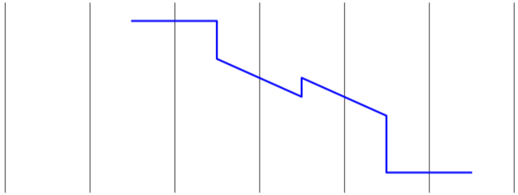


But evolving and averaging still maintains monotonicity (TVD):



Slope limiters and flux limiters

Slope limiter formulation for advection:

$$\tilde{q}^n(x, t_n) = Q_i^n + \sigma_i^n(x - x_i) \qquad \text{for } x_{i-1/2} \le x < x_{i+1/2}.$$

Applying REA algorithm gives (for $u > 0$):

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) - \frac{1}{2}\frac{u\Delta t}{\Delta x}\left(\Delta x - u\Delta t\right)\left(\sigma_i^n - \sigma_{i-1}^n\right)$$

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Flux limiter formulation:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

with flux

$$F_{i-1/2}^n = uQ_{i-1}^n + \frac{1}{2}u(\Delta x - u\Delta t)\sigma_{i-1}^n$$

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$$F_{i-1/2}^n = uQ_{i-1}^n + \frac{1}{2}u(\Delta x - u\Delta t)\sigma_{i-1}^n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} u\tilde{q}(x_{i-1/2}, t)\,dt.$$

Lax-Wendroff and flux limiters

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Flux limiters and wave limiters

Flux limiter method: Replace $\Delta Q^n_{i-1/2}$ by limited version $\delta^n_{i-1/2}$

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For systems of equations:

- Solve Riemann problem to decompose $\Delta Q^n_{i-1/2}$ into waves

$$\Delta Q_{i-1/2} = \sum_p \mathcal{W}^p_{i-1/2} = \sum_p \alpha^p_{i-1/2} r^p$$

- Use wave propagation form of Godunov (first-order) update
- Apply limiters to waves to get $\widetilde{\mathcal{W}}^p_{i-1/2} = \tilde{\alpha}^p_{i-1/2} r^p$
- Use limited waves in “second-order” corrections

Flux limiters for scalar problem

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Limiter based on the ratio

$$\theta^n_{i-1/2} = \frac{Q_I - Q_{I-1}}{Q_i - Q_{i-1}}$$

where I denotes the cell in the upwind direction:

$$I = \begin{cases} i-1 & \text{if } u > 0 \\ i+1 & \text{if } u < 0. \end{cases}$$

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Note that:

- $\theta \approx 1 + \mathcal{O}(\Delta x)$ where the solution is smooth,
- $\theta < 0$ if slopes have different sign.

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Desirable properties:

- $\phi(\theta) = 0$ for $\theta \leq 0$ (zero slope at extrema)
- $\phi(1) = 1$ so nearly using Lax-Wendroff where smooth

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Note that:

- $\phi(\theta) \equiv 0$ for all $\theta \implies$ upwind method
- $\phi(\theta) \equiv 1$ for all $\theta \implies$ Lax-Wendroff

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- $\phi(\theta) \equiv 1$ for all $\theta \implies$ Lax-Wendroff
- $\phi(\theta) = \theta \implies$ Beam-Warming: $\delta^n_{i-1/2} = Q_I - Q_{I-1}$

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- $\phi(\theta) =$ minmod$(1, \theta) \implies$ Minmod method

TVD flux limiter methods

For $q_t + uq_x = 0$ with $u > 0$ and $\nu = u\Delta t/\Delta x$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

$$\begin{aligned} F_{i-1/2}^n &= uQ_{i-1}^n + \frac{1}{2}u(1 - u\Delta t/\Delta x)\delta_{i-1/2}^n \\ &= uQ_{i-1}^n + \frac{1}{2}u(1-\nu)[\phi(\theta_{i-1/2})(Q_i - Q_{i-1})] \end{aligned}$$

Can be written as:

$$Q_i^{n+1} = Q_i^n - \left[\nu + \frac{1}{2}\nu(1-\nu)\left(\frac{\phi(\theta_{i+1/2})}{\theta_{i+1/2}} - \phi(\theta_{i-1/2})\right)\right](Q_i^n - Q_{i-1}^n)$$

since $Q_{i+1} - Q_i = (1/\theta_{i+1/2})(Q_i - Q_{i-1})$.

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Use this part of Theorem 6.1 (Harten):

The method

$$Q_i^{n+1} = Q_i^n - C_{i-1}^n(Q_i^n - Q_{i-1}^n)$$

is TVD provided $0 \le C_i^n \le 1$ for all i, regardless of how these coefficients depend on $Q^n,\ \Delta x,\ \Delta t$.

TVD flux limiter methods

$$Q_i^{n+1} = Q_i - C_{i-1}(Q_i - Q_{i-1}), \qquad TV(Q) = \sum |Q_{i+1} - Q_i|$$

Proof that method is TVD provided $0 \le C_i \le 1$ for all i:

$$\begin{aligned} Q_{i+1}^{n+1} - Q_i^{n+1} &= (Q_{i+1} - Q_i) - C_i(Q_{i+1} - Q_i) + C_{i-1}(Q_i - Q_{i-1}) \\ &= (1 - C_i)(Q_{i+1} - Q_i) + C_{i-1}(Q_{i+1} - Q_i) \end{aligned}$$

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$$|Q_{i+1}^{n+1} - Q_i^{n+1}| \le (1 - C_i)|Q_{i+1} - Q_i| + C_{i-1}|Q_i - Q_{i-1}|$$

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$$|Q_{i+1}^{n+1} - Q_i^{n+1}| \le (1 - C_i)|Q_{i+1} - Q_i| + C_{i-1}|Q_i - Q_{i-1}|$$

$$\sum |Q_{i+1}^{n+1} - Q_i^{n+1}| \le \sum (1 - C_i)|Q_{i+1} - Q_i| + \sum C_{i-1}|Q_i - Q_{i-1}|$$

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$$\begin{aligned} |Q_{i+1}^{n+1} - Q_i^{n+1}| &\le (1 - C_i)|Q_{i+1} - Q_i| + C_{i-1}|Q_i - Q_{i-1}| \\ \sum |Q_{i+1}^{n+1} - Q_i^{n+1}| &\le \sum (1 - C_i)|Q_{i+1} - Q_i| + \sum C_{i-1}|Q_i - Q_{i-1}| \\ &\le \sum (1 - C_i)|Q_{i+1} - Q_i| + \sum C_i|Q_{i+1} - Q_i| \end{aligned}$$

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$$Q_i^{n+1} = Q_i - C_{i-1}(Q_i - Q_{i-1}), \qquad TV(Q) = \sum |Q_{i+1} - Q_i|$$

Proof that method is TVD provided $0 \leq C_i \leq 1$ for all i:

$$\begin{aligned} Q_{i+1}^{n+1} - Q_i^{n+1} &= (Q_{i+1} - Q_i) - C_i(Q_{i+1} - Q_i) + C_{i-1}(Q_i - Q_{i-1}) \\ &= (1 - C_i)(Q_{i+1} - Q_i) + C_{i-1}(Q_{i+1} - Q_i) \end{aligned}$$

$$\begin{aligned} |Q_{i+1}^{n+1} - Q_i^{n+1}| &\leq (1 - C_i)|Q_{i+1} - Q_i| + C_{i-1}|Q_i - Q_{i-1}| \\ \sum |Q_{i+1}^{n+1} - Q_i^{n+1}| &\leq \sum (1 - C_i)|Q_{i+1} - Q_i| + \sum C_{i-1}|Q_i - Q_{i-1}| \\ &\leq \sum (1 - C_i)|Q_{i+1} - Q_i| + \sum C_i|Q_{i+1} - Q_i| \\ &\leq \sum (1 - C_i + C_i)|Q_{i+1} - Q_i| = TV(Q^n) \end{aligned}$$

TVD flux limiter methods

The method

$$Q_i^{n+1} = Q_i^n - \left[\nu + \frac{1}{2}\nu(1-\nu)\left(\frac{\phi(\theta_{i+1/2})}{\theta_{i+1/2}} - \phi(\theta_{i-1/2})\right)\right](Q_i^n - Q_{i-1}^n)$$

is TVD provided

$$0 \le \left[\nu + \frac{1}{2}\nu(1-\nu)\left(\frac{\phi(\theta_1)}{\theta_1} - \phi(\theta_2)\right)\right] \le 1$$

for all values of θ_1 and θ_2 (provided $0 \le \nu \le 1$).

TVD flux limiter methods

The method

$$Q_i^{n+1} = Q_i^n - \left[\nu + \frac{1}{2}\nu(1-\nu)\left(\frac{\phi(\theta_{i+1/2})}{\theta_{i+1/2}} - \phi(\theta_{i-1/2})\right)\right](Q_i^n - Q_{i-1}^n)$$

is TVD provided

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for all values of θ_1 and θ_2 (provided $0 \leq \nu \leq 1$).

This is true if

$$-2 \leq \left(\frac{\phi(\theta_1)}{\theta_1} - \phi(\theta_2)\right) \leq 2$$

for all values of θ_1 and θ_2

TVD flux limiter methods

So the method

$$Q_i^{n+1} = Q_i^n - \left[\nu + \frac{1}{2}\nu(1-\nu)\left(\frac{\phi(\theta_{i+1/2})}{\theta_{i+1/2}} - \phi(\theta_{i-1/2})\right)\right](Q_i^n - Q_{i-1}^n)$$

is TVD provided

$$-2 \leq \left(\frac{\phi(\theta_1)}{\theta_1} - \phi(\theta_2)\right) \leq 2$$

for all values of θ_1 and θ_2.

Satisfied provided $\phi(\theta)$ satisfies:

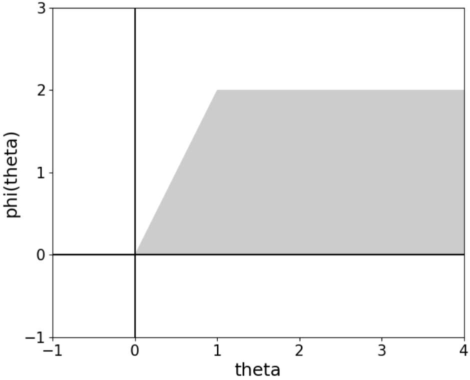
$$0 \leq \frac{\phi(\theta)}{\theta} \leq 2, \qquad 0 \leq \phi(\theta) \leq 2,$$

or

$$0 \leq \phi(\theta) \leq \text{minmod}(2, 2\theta).$$

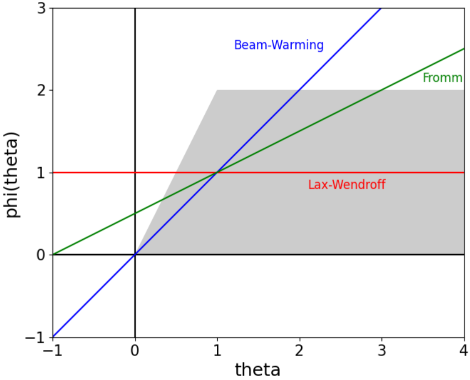
Sweby diagram

If we plot $\phi(\theta)$, the curve must lie in the shaded region:



Sweby diagram

If we plot $\phi(\theta)$, the curve must lie in the shaded region:



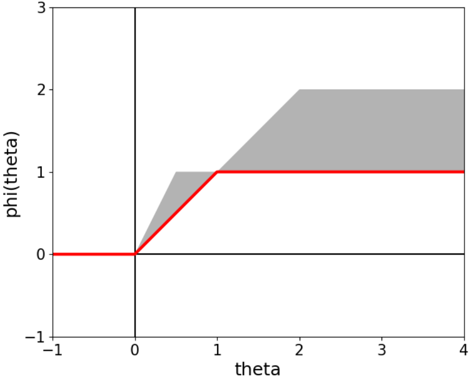
Standard second order methods go outside this region.

Recall we want $\phi(1) = 1$ for good accuracy of smooth solutions.

Sweby diagram

Sweby's investigation suggested best methods lie between Lax-Wendroff and Beam-Warming (and inside the TVD region).

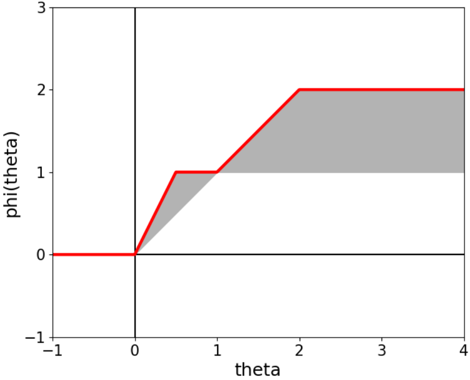
Sweby region:



$\phi(\theta) = \text{minmod}(1, \theta)$ follows the lower limit of this region.

Superbee method

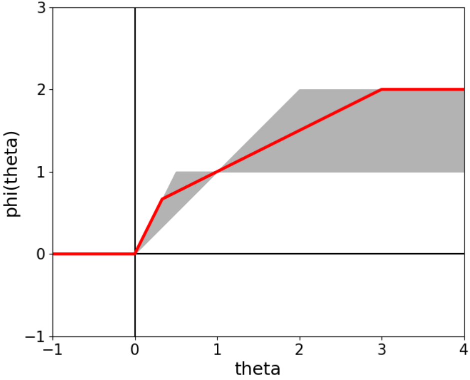
The superbee limiter follows the upper limit:



$$\phi(\theta) = \max(0,\ \min(1, 2\theta),\ \min(2, \theta))$$

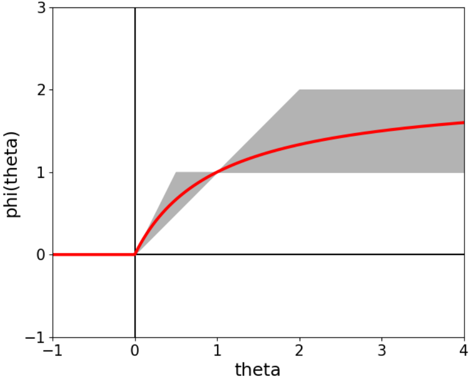
MC method

The Monotonized Centered (MC) limiter follows Fromm's method near $\theta = 1$, and is smooth at $\theta = 1$:



$$\phi(\theta) = \max(0,\ \min((1+\theta)/2,\ 2,\ 2\theta))$$

The van Leer limiter is a smoother version of MC



$\phi(\theta) = \frac{\theta+|\theta|}{1+|\theta|}$.

Some popular limiters

Linear methods:

$$\begin{aligned}
\text{upwind}: &\quad \phi(\theta) = 0 \\
\text{Lax-Wendroff}: &\quad \phi(\theta) = 1 \\
\text{Beam-Warming}: &\quad \phi(\theta) = \theta \\
\text{Fromm}: &\quad \phi(\theta) = \frac{1}{2}(1+\theta)
\end{aligned}$$

High-resolution limiters:

$$\begin{aligned}
\text{minmod}: &\quad \phi(\theta) = \text{minmod}(1,\theta) \\
\text{superbee}: &\quad \phi(\theta) = \max(0,\ \min(1,2\theta),\ \min(2,\theta)) \\
\text{MC}: &\quad \phi(\theta) = \max(0,\ \min((1+\theta)/2,\ 2,\ 2\theta)) \\
\text{van Leer}: &\quad \phi(\theta) = \frac{\theta+|\theta|}{1+|\theta|}
\end{aligned}$$