Finite Volume Methods for Hyperbolic Problems

Derivation of Conservation Laws

- Integral form in one space dimension
- Advection
- Compressible gas – mass and momentum
- Source terms
- Diffusion

First order hyperbolic PDE in 1 space dimension

Linear: $q_t + Aq_x = 0, \qquad q(x,t) \in \mathbb{R}^m,\ A \in \mathbb{R}^{m\times m}$

Conservation law: $q_t + f(q)_x = 0, \qquad f : \mathbb{R}^m \to \mathbb{R}^m$ (flux)

Quasilinear form: $q_t + f'(q)q_x = 0$

Hyperbolic if A or $f'(q)$ is diagonalizable with real eigenvalues.

Models wave motion or advective transport.

Eigenvalues are wave speeds.

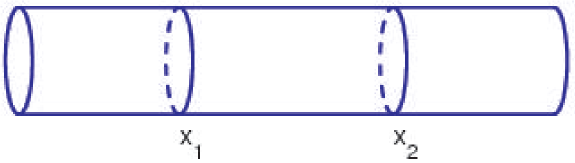
Note: Second order wave equation $p_{tt} = c^2 p_{xx}$ can be written as a first-order system (acoustics).

Derivation of Conservation Laws

$q(x,t) =$ density function for some conserved quantity, so

$$\int_{x_1}^{x_2} q(x,t)\,dx = \text{total mass in interval}$$

changes only because of fluxes at left or right of interval.



Derivation of Conservation Laws

$q(x,t) =$ density function for some conserved quantity.

Integral form:

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = F_1(t) - F_2(t)$$

where

$$F_j = f(q(x_j,t)), \qquad f(q) = \text{flux function.}$$

Derivation of Conservation Laws

If q is smooth enough, we can rewrite

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = f(q(x_1,t)) - f(q(x_2,t))$$

as

$$\int_{x_1}^{x_2} q_t\,dx = -\int_{x_1}^{x_2} f(q)_x\,dx$$

or

$$\int_{x_1}^{x_2} (q_t + f(q)_x)\,dx = 0$$

True for all $x_1,\ x_2 \implies$ differential form:

$$q_t + f(q)_x = 0.$$

Advective flux

If $\rho(x,t)$ is the density (mass per unit length),

$$\int_{x_1}^{x_2} \rho(x,t)\,dx = \text{total mass in } [x_1, x_2]$$

and $u(x,t)$ is the velocity, then the advective flux is

$$\rho(x,t)u(x,t)$$

Units: mass/length $\times$ length/time $=$ mass/time.

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Continuity equation (conservation of mass):

$$\rho_t + (\rho u)_x = 0$$

Advection equation

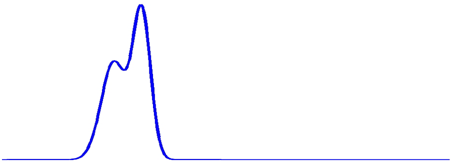
Flow in a pipe at constant velocity

$u =$ constant flow velocity

$q(x,t) =$ tracer concentration, $\quad f(q) = uq$

$\implies \quad q_t + uq_x = 0, \quad$ with initial condition $q(x,0) = \mathring{q}(x)$.

True solution: $q(x,t) = q(x-ut,0) = \mathring{q}(x-ut)$



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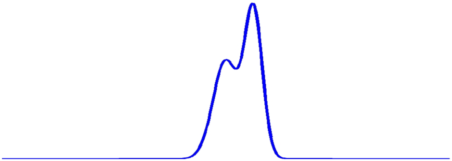

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Compressible gas dynamics

In one space dimension (e.g. in a pipe).
$\rho(x,t) =$ density, $\quad u(x,t) =$ velocity,
$p(x,t) =$ pressure, $\quad \rho(x,t)u(x,t) =$ momentum.

Conservation of:

mass:	ρ	flux:	ρu
momentum:	ρu	flux:	$(\rho u)u + p$
(energy)			

Conservation laws:

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

Equation of state:

$$p = P(\rho).$$

(Later: p may also depend on internal energy / temperature)

Compressible gas dynamics

Conservation laws:

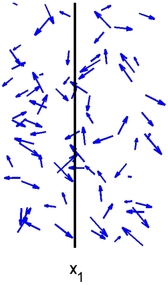
$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

Momentum flux:
$\rho u^2 = (\rho u)u =$ advective flux

p term in flux?

- $-p_x =$ force in Newton's second law,
- as momentum flux: microscopic motion of gas molecules.

Momentum flux arising from pressure



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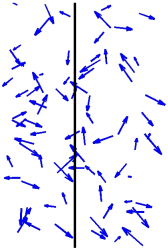

Note that:

- molecules with positive x-velocity crossing x_1 to right increase the momentum in $[x_1, x_2]$
- molecules with negative x-velocity crossing x_1 to left also increase the momentum in $[x_1, x_2]$

Hence momentum flux increases with pressure $p(x_1, t)$ even if macroscopic (average) velocity is zero.

Source terms (balance laws)

$$q_t + f(q)_x = \psi(q)$$

Results from integral form

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} q(x,t)\,dx = f(q(x_1,t)) - f(q(x_2,t)) + \int_{x_1}^{x_2} \psi(q(x,t))\,dx$$

Examples:

- Reacting flow, e.g. combustion,
- External forces such as gravity
- Viscosity, drag
- Radiative heat transfer
- Geometric source terms (e.g., quasi-1d problems)
- Bottom topography in shallow water

Source term example: advection with decay

$q(x,t) =$ mass / unit length
First suppose no advection,
but at each point, exponential decay occurs:

$$q(x,t)_t = -\lambda q(x,t) \equiv \psi(q(x,t)).$$

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With advection:

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With advection:

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = uq(x_1,t) - uq(x_2,t) + \int_{x_1}^{x_2} \psi(q(x,t))\,dx.$$

$$\int_{x_1}^{x_2} q_t + (uq)_x - \psi(q)\,dx = 0 \quad \text{holds for all } \ x_1,\ x_2$$

Diffusive flux

$q(x,t) =$ concentration
$\beta =$ diffusion coefficient ($\beta > 0$)

diffusive flux $= -\beta q_x(x,t)$

$q_t + f_x = 0 \implies$ diffusion equation:

$$q_t = (\beta q_x)_x = \beta q_{xx} \text{ (if } \beta = \text{const).}$$

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Heat equation: Same form, where

$q(x,t) =$ density of thermal energy $= \kappa T(x,t)$,
$T(x,t) =$ temperature, $\kappa =$ heat capacity,
flux $= -\beta T(x,t) = -(\beta/\kappa) q(x,t) \implies$

$$q_t(x,t) = (\beta/\kappa) q_{xx}(x,t).$$

Advection-diffusion

$q(x,t) =$ concentration that advects with velocity u
and diffuses with coefficient β:

$$\text{flux} = uq - \beta q_x.$$

Advection-diffusion equation:

$$q_t + uq_x = \beta q_{xx}.$$

If $\beta > 0$ then this is a parabolic equation.

Advection dominated if u/β (the Péclet number) is large.

Fluid dynamics: “parabolic terms” arise from

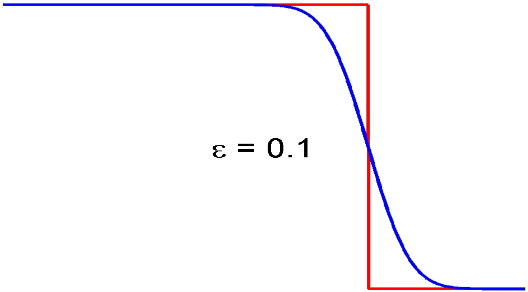
- thermal diffusion and
- diffusion of momentum, where the diffusion parameter is the viscosity.

Discontinuous solutions

Vanishing Viscosity solution: The Riemann solution $q(x,t)$ is the limit as $\epsilon \to 0$ of the solution $q^\epsilon(x,t)$ of the parabolic advection-diffusion equation

$$q_t + uq_x = \epsilon q_{xx}.$$

For any $\epsilon > 0$ this has a classical smooth solution:



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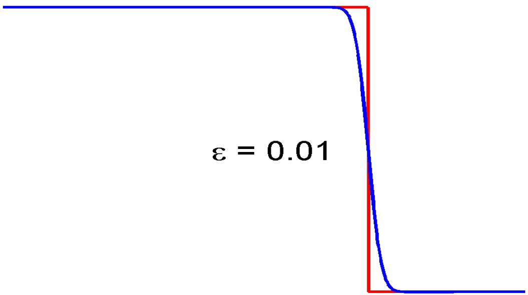

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