Finite Volume Methods for Hyperbolic Problems

Nonlinear Systems

Shock Waves and the Hugoniot Locus

- Shallow water equations
- Rankine-Hugoniot condition
- Hugoniot locus in phase space
- All-shock Riemann solutions

Riemann Problems and Jupyter Solutions

Theory and Approximate Solvers for Hyperbolic PDEs

David I. Ketcheson, RJL, and Mauricio del Razo

General information and links to book, Github, Binder, etc.:

bookstore.siam.org/fa16/bonus

View static version of notebooks at:

www.clawpack.org/riemann_book/html/Index.html

Shallow water equations

$h(x,t) =$ depth
$u(x,t) =$ velocity (depth averaged, varies only with x)

Conservation of mass and momentum hu gives system of two equations.

mass flux $= hu$,
momentum flux $= (hu)u + p$ where $p =$ hydrostatic pressure

$$h_t + (hu)_x = 0$$
$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$

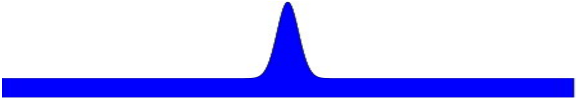
Jacobian matrix:

$$f'(q) = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix}, \qquad \lambda = u \pm \sqrt{gh}.$$

Shock formation

For nonlinear problems wave speed generally depends on q.

Waves can steepen up and form shocks
$\Longrightarrow$ even smooth data can lead to discontinuous solutions.



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Note:

- System of two equations gives rise to 2 waves.
- Each wave behaves like solution of nonlinear scalar equation.

Not quite... no linear superposition. Nonlinear interaction!

The Riemann problem

Dam break problem for shallow water equations

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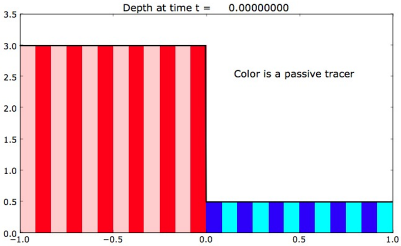
$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$



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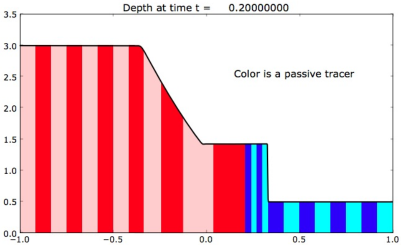

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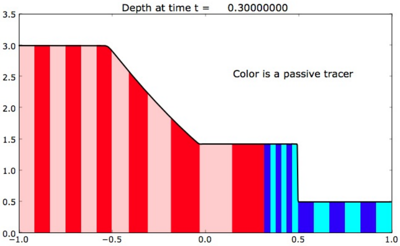

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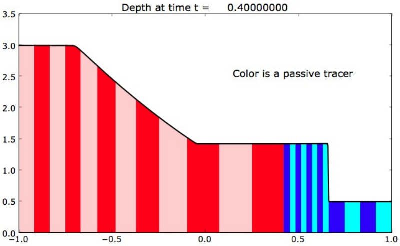

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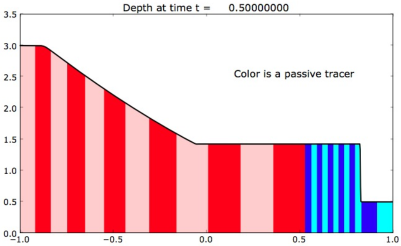
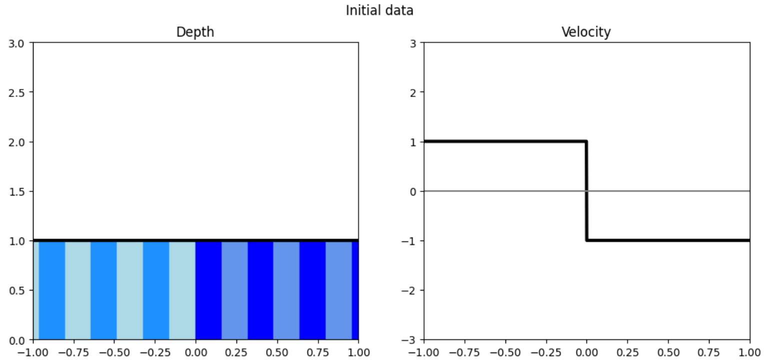


Two-shock Riemann solution

With $h_\ell = h_r$ and $u_\ell = -u_r > 0$



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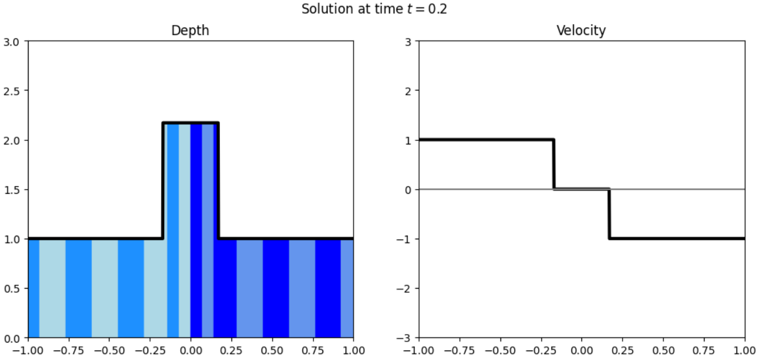

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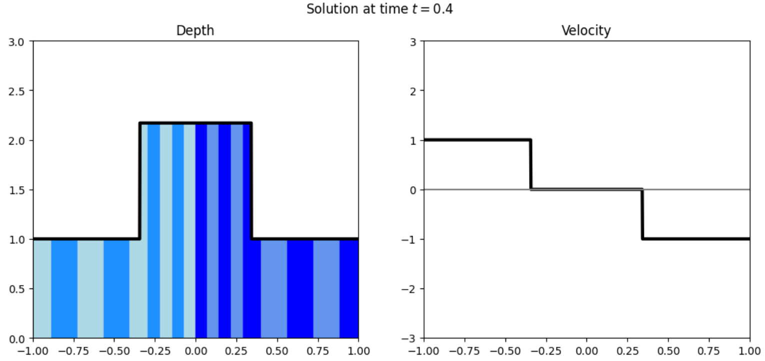

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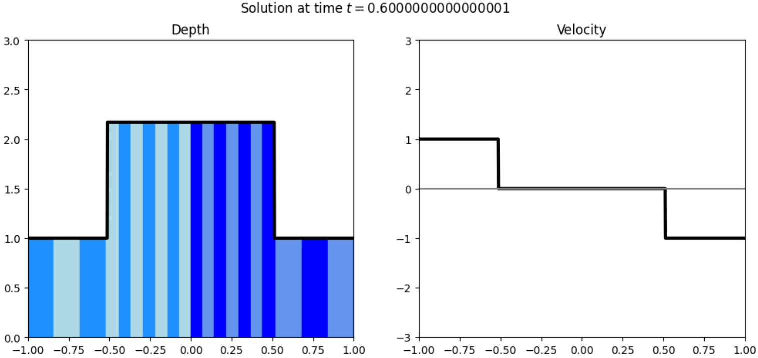
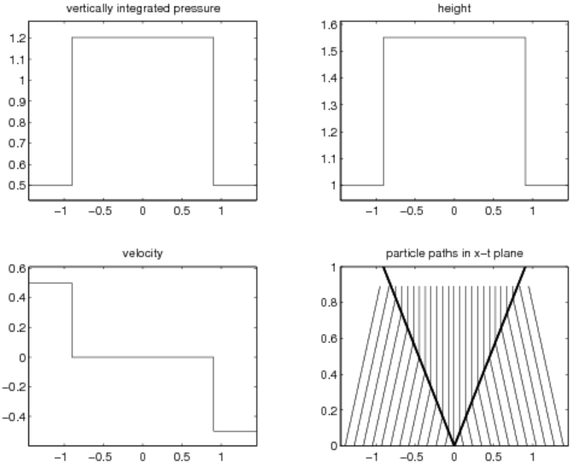


Two-shock Riemann solution for shallow water

Initially $h_l = h_r = 1, \quad u_l = -u_r = 0.5 > 0$

Solution at later time:



Characteristics for scalar nonlinear problem

Scalar hyperbolic equation in quasi-linear form: $q_t + f'(q)q_x = 0$.

Characteristic curve in x–t plane: $X(t)$ satisfying

$$X'(t) = f'(q(X(t), t)).$$

Along this curve,

$$\frac{d}{dt}\, q(X(t), t) = X'(t)q_x + q_t = 0$$

So for a scalar equation,

$q(x, t)$ is constant along characteristic curves.

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Advection: Characteristics satisfy $X'(t) = u$, so $X(t) = x_0 + ut$ are parallel straight lines.

Nonlinear: Characteristcs are straight since $f'(q(X(t), t))$ is constant, but not parallel. Crossing $\Longrightarrow$ shock formation.

Characteristics for nonlinear systems

Hyperbolic system in quasi-linear form: $q_t + f'(q)q_x = 0$.

Eigenvalues of Jacobian: $\lambda^p(q)$ with $f'(q)r^p(q) = \lambda^p(q)r^p(q)$.

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\end{aligned}$$

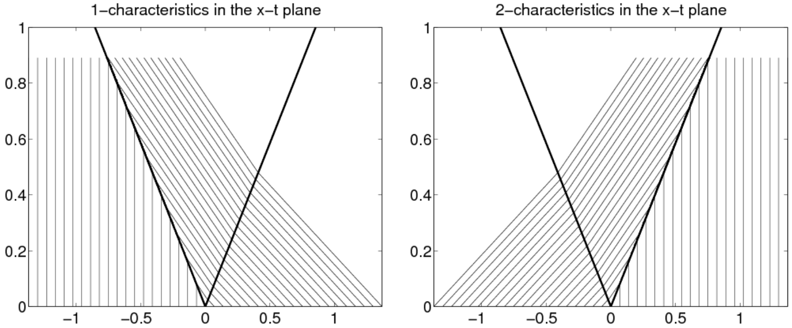
This $= 0$ if we choose $X'(t) = \lambda^p(q(X(t),t))$.

So in the simple wave case, $q(X(t),t)$ is constant along each ray with $X'(t) = \lambda^p(q(X(t),t))$ (as long as these don't cross).

Two-shock Riemann solution for shallow water

Characteristic curves $X'(t) = u(X(t),t) \pm \sqrt{gh(X(t),t)}$

Slope of characteristic is constant in regions where q is constant. (Shown for $g = 1$ so $\sqrt{gh} = 1$ everywhere initially.)



Note that 1-characteristics impinge on 1-shock,
2-characteristics impinge on 2-shock.

An isolated shock

If an isolated shock with left and right states q_l and q_r is propagating at speed s

then the Rankine-Hugoniot condition must be satisfied:

$$f(q_r) - f(q_l) = s(q_r - q_l)$$

For a system $q \in \mathbb{R}^m$ this can only hold for certain pairs q_l, q_r:

For a linear system, $f(q_r) - f(q_l) = Aq_r - Aq_l = A(q_r - q_l)$.
So $q_r - q_l$ must be an eigenvector of $f'(q) = A$.

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$A \in \mathbb{R}^{m \times m} \implies$ there will be m rays through q_l in state space in the eigen-directions, and q_r must lie on one of these.

For a nonlinear system, there will be m curves through q_l called the Hugoniot loci.

Hugoniot loci for shallow water

$$q = \begin{bmatrix} h \\ hu \end{bmatrix}, \qquad f(q) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}.$$

Fix $q_* = (h_*, u_*)$.

What states q can be connected to q_* by an isolated shock?

The Rankine-Hugoniot condition $s(q - q_*) = f(q) - f(q_*)$ gives:

$$\begin{aligned} s(h_* - h) &= h_*u_* - hu, \\ s(h_*u_* - hu) &= h_*u_*^2 - hu^2 + \frac{1}{2}g(h_*^2 - h^2). \end{aligned}$$

Two equations with 3 unknowns (h, u, s), so we expect
1-parameter families of solutions.

Hugoniot loci for shallow water

Rankine-Hugoniot conditions:

$$s(h_* - h) = h_* u_* - hu,$$
$$s(h_* u_* - hu) = h_* u_*^2 - hu^2 + \frac{1}{2} g(h_*^2 - h^2).$$

For any $h > 0$ we can solve for

$$u(h) = u_* \pm \sqrt{\frac{g}{2}\left(\frac{h_*}{h} - \frac{h}{h_*}\right)(h_* - h)}$$
$$s(h) = (h_* u_* - hu)/(h_* - h).$$

This gives 2 curves in h–hu space (one for $+$, one for $-$).

Hugoniot loci for shallow water

For any $h > 0$ we have a possible shock state. Set

$$h = h_* + \alpha,$$

so that $h = h_*$ at $\alpha = 0$, to obtain

$$hu = h_*u_* + \alpha \left[u_* \pm \sqrt{gh_* + \frac{1}{2} g\alpha(3 + \alpha/h_*)} \right].$$

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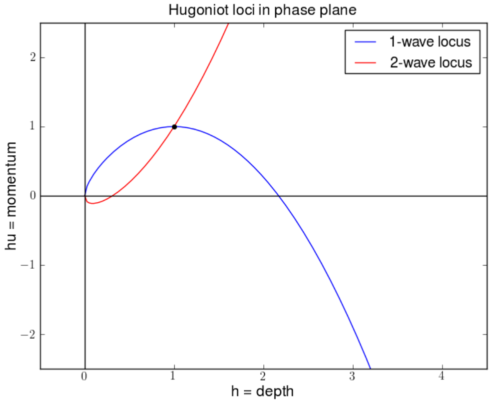
Hence we have

$$\begin{bmatrix} h \\ hu \end{bmatrix} = \begin{bmatrix} h_* \\ h_* u_* \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ u_* \pm \sqrt{g h_* + \mathcal{O}(\alpha)} \end{bmatrix} \qquad \text{as } \alpha \to 0.$$

Close to q_* the curves are tangent to eigenvectors of $f'(q_*)$
Expected since $f(q) - f(q_*) \approx f'(q_*)(q - q_*)$.

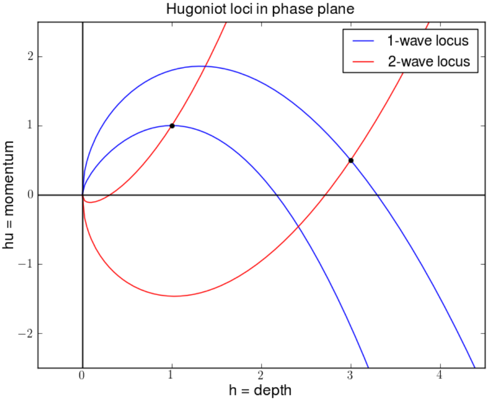
Hugoniot loci for one particular q_*

States that can be connected to q_* by a "shock"



Note: Might not satisfy entropy condition.

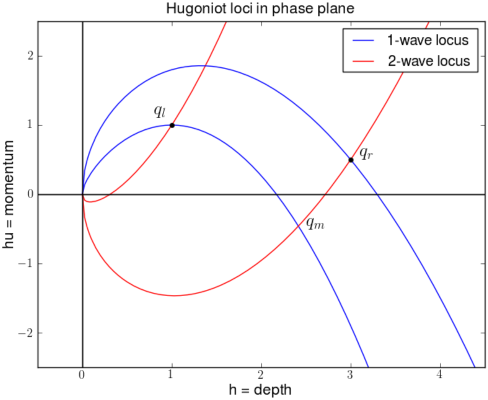
Hugoniot loci for two different states



“All-shock” Riemann solution:

From q_l along 1-wave locus to q_m,
From q_r along 2-wave locus to q_m,

All-shock Riemann solution



From q_l along 1-wave locus to q_m,
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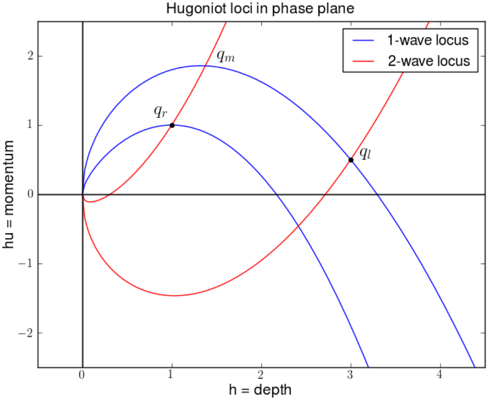


From q_l along 1-wave locus to q_m,
From q_r along 2-wave locus to q_m,

2-shock Riemann solution for shallow water

Given arbitrary states q_l and q_r, we can solve the Riemann problem with two shocks.

Choose q_m so that q_m is on the 1-Hugoniot locus of q_l and also q_m is on the 2-Hugoniot locus of q_r.

This requires

$$u_m = u_r + (h_m - h_r)\sqrt{\frac{g}{2}\left(\frac{1}{h_m} + \frac{1}{h_r}\right)}$$

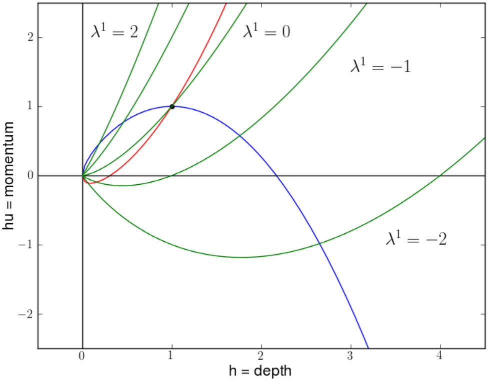
and

$$u_m = u_l - (h_m - h_l)\sqrt{\frac{g}{2}\left(\frac{1}{h_m} + \frac{1}{h_l}\right)}.$$

Equate and solve single nonlinear equation for h_m.

Hugoniot loci for one particular q_*

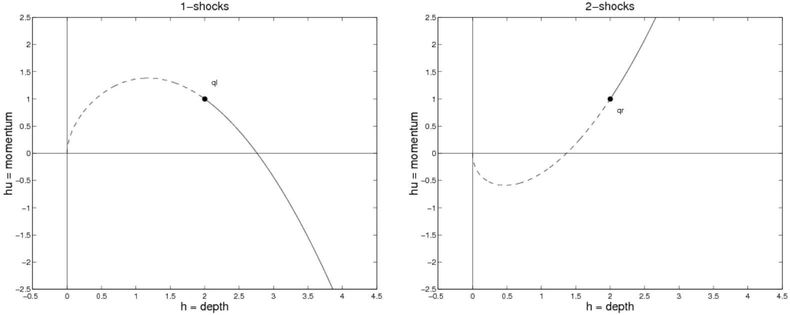
Green curves are contours of $\lambda^1 = u - \sqrt{gh}$



Note: Increases in one direction only along blue curve.

Hugoniot locus for shallow water

States that can be connected to the given state by a 1-wave or 2-wave satisfying the R-H conditions:

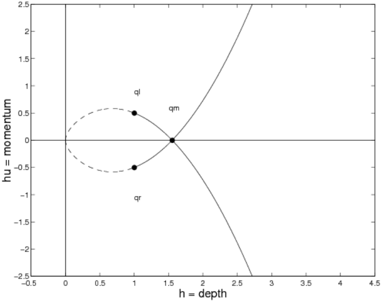


Solid portion: states that can be connected by shock satisfying entropy condition.

Dashed portion: states that can be connected with R-H condition satisfied but not the physically correct solution.

2-shock Riemann solution for shallow water

Colliding with $u_l = -u_r > 0$:



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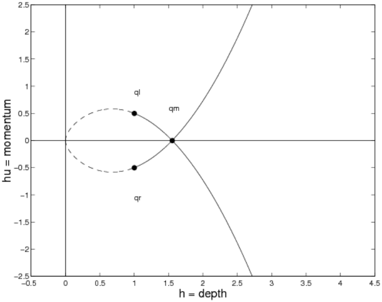

Entropy condition: Characteristics should impinge on shock:

λ^1 should decrease going from q_l to q_m,

λ^2 should increase going from q_r to q_m,

This is satisfied along solid portions of Hugoniot loci above, not satisfied on dashed portions (entropy-violating shocks).

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Characteristic curves $X'(t) = u(X(t),t) \pm \sqrt{gh(X(t),t)}$

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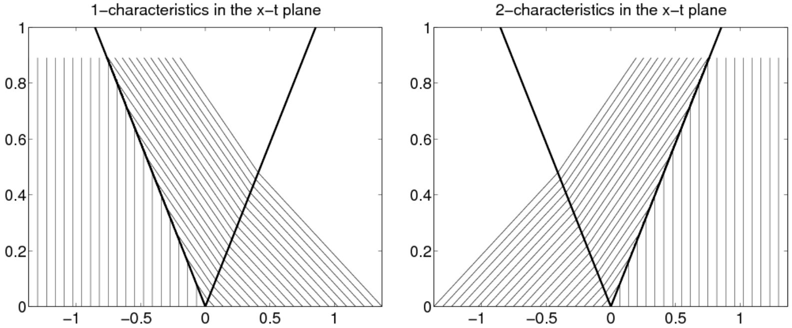

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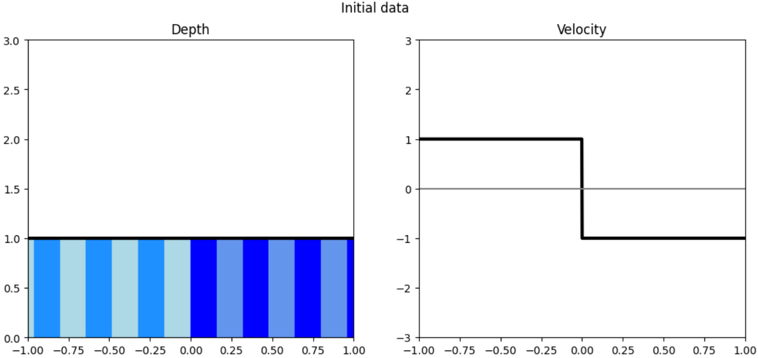

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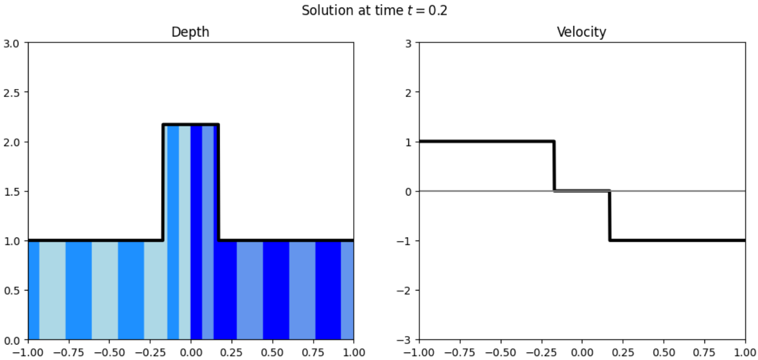

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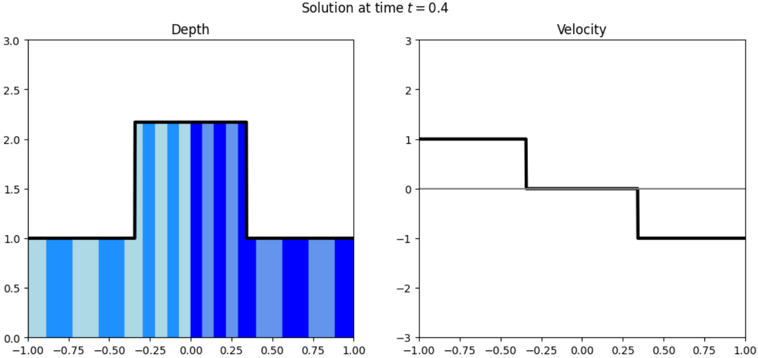

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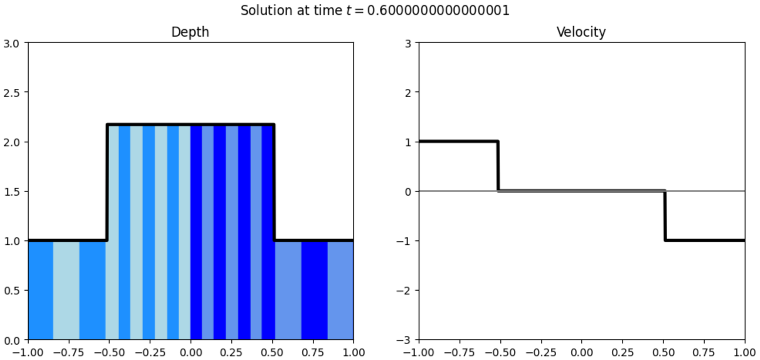
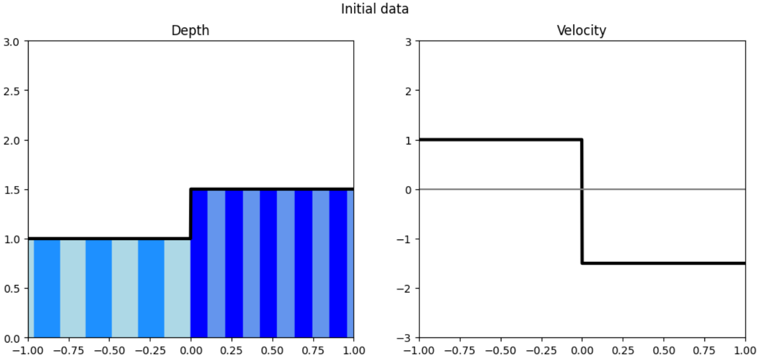


Two-shock Riemann solution

With non-equal states, but $u_\ell > 0$ and $u_r < 0$:



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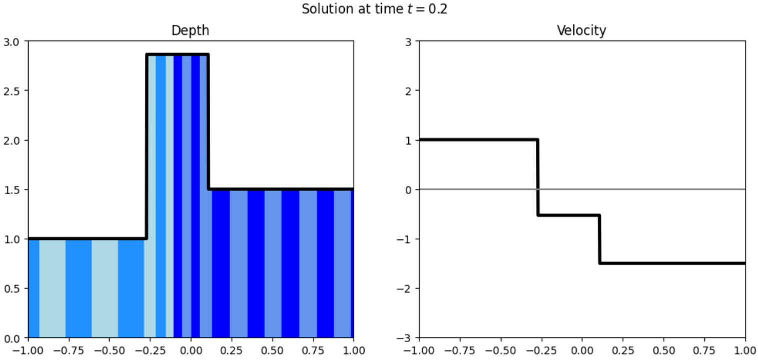

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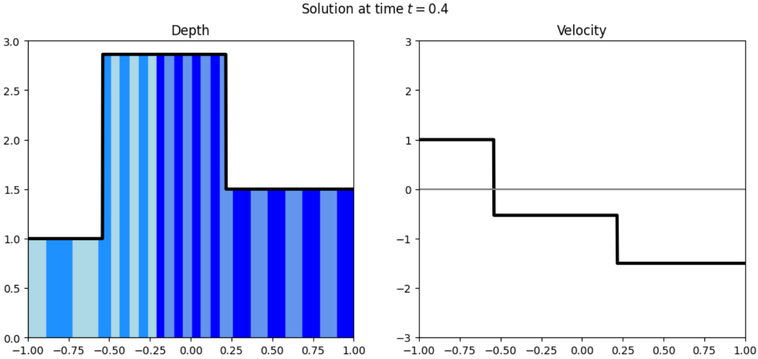

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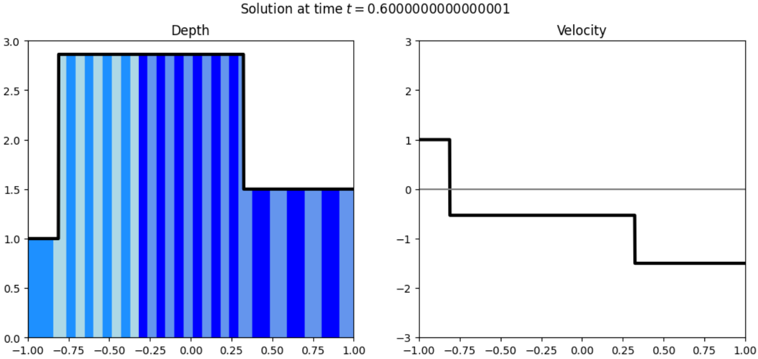
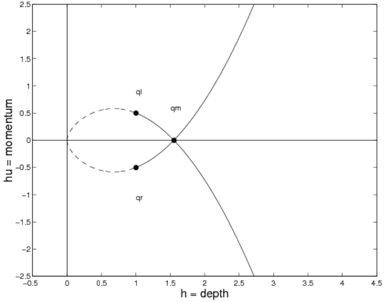
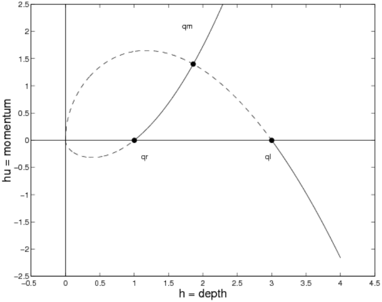


2-shock Riemann solution for shallow water

Colliding with $u_l = -u_r > 0$:



Dam break:



2-shock Riemann solution for shallow water

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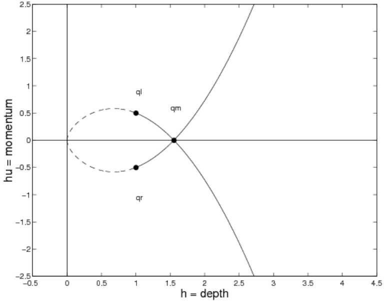
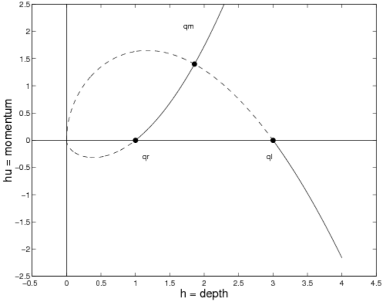


Dam break:



Entropy condition: Characteristics should impinge on shock:

λ^1 should decrease going from q_l to q_m,

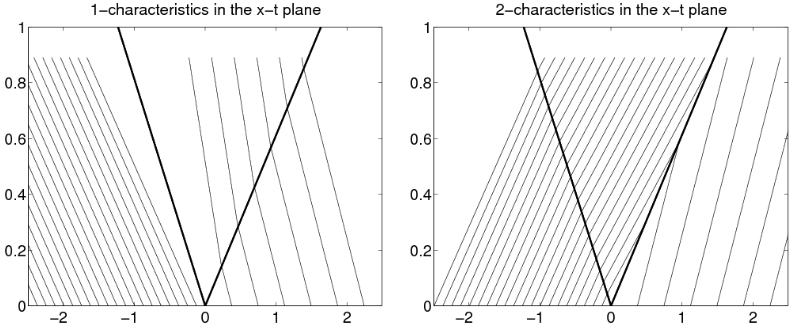
λ^2 should increase going from q_r to q_m,

This is satisfied along solid portions of Hugoniot loci above, not satisfied on dashed portions (entropy-violating shocks).

Entropy-violatiing Riemann solution for dam break

Characteristic curves $X'(t) = u(X(t), t) \pm \sqrt{gh(X(t), t)}$

Slope of characteristic is constant in regions where q is constant.

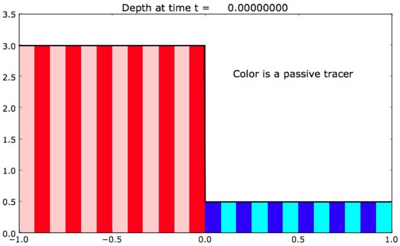


Note that 1-characteristics do not impinge on 1-shock, 2-characteristics impinge on 2-shock.

The Riemann problem

Dam break problem for shallow water equations

$$h_t + (hu)_x = 0$$
$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$



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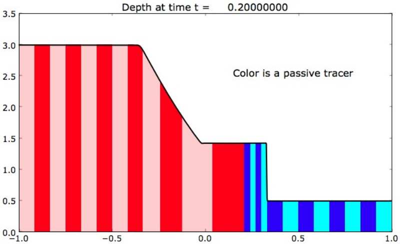

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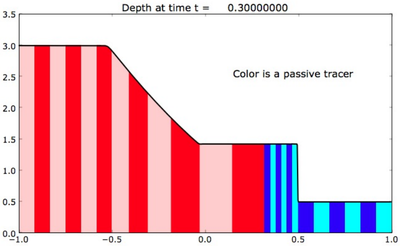

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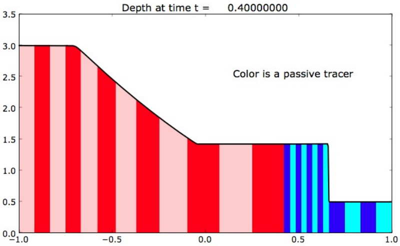

The Riemann problem

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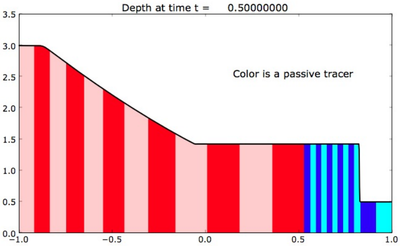

Riemann Problems and Jupyter Solutions

Theory and Approximate Solvers for Hyperbolic PDEs

David I. Ketcheson, RJL, and Mauricio del Razo

General information and links to book, Github, Binder, etc.:

bookstore.siam.org/fa16/bonus

View static version of notebooks at:

www.clawpack.org/riemann_book/html/Index.html