Finite Volume Methods for Hyperbolic Problems

Linearization of Nonlinear Systems

- General form, Jacobian matrix
- Scalar Burgers' equation
- Compressible gas dynamics
- Linear acoustics equations

Linearization

General nonlinear conservation law: $q_t + f(q)_x = 0$

Suppose $q(x,t) = q_0 + \tilde{q}(x,t)$ where $\|\tilde{q}(x,t)\| = \epsilon$ is small.

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Then

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Linearization: $\tilde{q}_t + A\tilde{q}_x = 0$ where $A = f'(q_0) =$ Jacobian matrix

Scalar: Advection equation

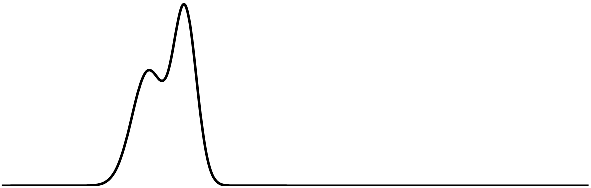
Nonlinear Burgers' equation

Conservation form: $u_t + \left(\frac{1}{2}u^2\right)_x = 0, \qquad f(u) = \frac{1}{2}u^2$.

Quasi-linear form: $\qquad u_t + uu_x = 0.$

This looks like an advection equation with u advected with speed u.

True solution: u is constant along characteristic with speed u until the wave "breaks".



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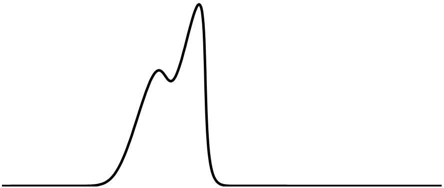

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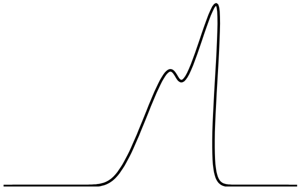

After breaking, the weak solution contains a shock wave.

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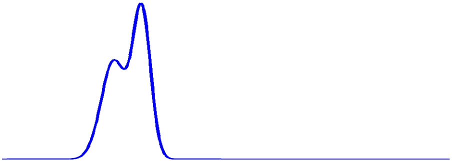
Linearization about u_0:

$$f(u) = \frac{1}{2}u^2 \implies f'(u_0) = u_0$$

So if $u(x,0) = u_0 + \tilde{u}(x,0)$ with $\|\tilde{u}\|$ small, then $\tilde{u}(x,t)$ approximately satisfies advection equation

$$\tilde{u}_t + u_0 u_x = 0.$$



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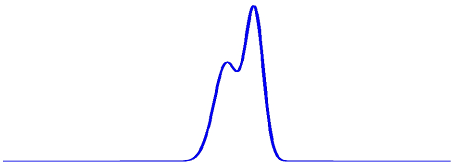

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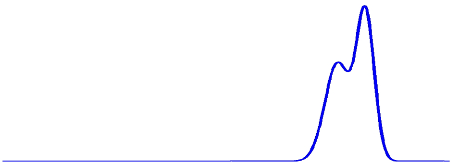

Compressible gas dynamics (simple case)

In one space dimension (e.g. in a pipe).
$\rho(x,t) =$ density, $\quad u(x,t) =$ velocity,
$p(x,t) =$ pressure, $\quad \rho(x,t)u(x,t) =$ momentum.

Conservation of:

mass:	ρ	flux:	ρu
momentum:	ρu	flux:	$(\rho u)u + p$

Conservation laws:

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

Equation of state:

$$p = P(\rho).$$

Linearization of gas dynamics

Suppose $\rho(x,t) \approx \rho_0$ and $u(x,t) \approx u_0$.

Model small perturbations to this steady state (sound waves).

$$\left[\begin{array}{c} \rho(x,t) \\ (\rho u)(x,t) \end{array}\right] = \left[\begin{array}{c} \rho_0 \\ \rho_0 u_0 \end{array}\right] + \left[\begin{array}{c} \widetilde{\rho}(x,t) \\ (\widetilde{\rho u})(x,t) \end{array}\right]$$

or $q(x,t) = q_0 + \tilde{q}(x,t)$ where $\|\tilde{q}(x,t)\| = \epsilon$ is small.

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Then nonlinear equation $q_t + f(q)_x = 0$ leads to

$$\begin{aligned} \tilde{q}_t &= q_t \\ &= -f(q)_x \\ &= -f'(q)q_x \\ &= -f'(q_0 + \tilde{q})\tilde{q}_x \\ &= -f'(q_0)\tilde{q}_x + \mathcal{O}(\epsilon^2). \end{aligned}$$

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Linearization: $\tilde{q}_t + A\tilde{q}_x = 0$ where $A = f'(q_0)$.

Linearization of gas dynamics

$$\begin{aligned} \rho_t + (\rho u)_x &= 0 \\ (\rho u)_t + (\rho u^2 + P(\rho))_x &= 0 \end{aligned}$$

so

$$q = \left[\begin{array}{c} \rho \\ \rho u \end{array}\right] = \left[\begin{array}{c} q^1 \\ q^2 \end{array}\right],$$

$$f(q) = \left[\begin{array}{c} \rho u \\ \rho u^2 + P(\rho) \end{array}\right] = \left[\begin{array}{c} f^1(q) \\ f^2(q) \end{array}\right] = \left[\begin{array}{c} q^2 \\ (q^2)^2/q^1 + P(q^1) \end{array}\right].$$

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Jacobian:

$$f'(q) = \begin{bmatrix} \partial f^1/\partial q^1 & \partial f^1/\partial q^2 \\ \partial f^2/\partial q^1 & \partial f^2/\partial q^2 \end{bmatrix}.$$

$$f'(q_0) = \begin{bmatrix} 0 & 1 \\ -u_0^2 + P'(\rho_0) & 2u_0 \end{bmatrix}.$$

Linearization of gas dynamics

Linearization: $\tilde{q}_t + A\tilde{q}_x = 0$ where $A = f'(q_0)$.

$$A = f'(q_0) = \left[\begin{array}{cc} 0 & 1 \\ -u_0^2 + P'(\rho_0) & 2u_0 \end{array}\right].$$

This can be written out as (2.47):

$$\begin{aligned} \tilde{\rho}_t + (\widetilde{\rho u})_x &= 0 \\ (\widetilde{\rho u})_t + (-u_0^2 + P'(\rho_0))\tilde{\rho}_x + 2u_0(\widetilde{\rho u})_x &= 0. \end{aligned}$$

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Rewrite in terms of p and u perturbations (Exer. 2.1):

$$\begin{aligned} \tilde{p}_t + u_0\tilde{p}_x + K_0\tilde{u}_x &= 0, \\ \rho_0\tilde{u}_t + \tilde{p}_x + \rho_0 u_0\tilde{u}_x &= 0, \end{aligned}$$

where $K_0 = \rho_0 P'(\rho_0)$ is the bulk modulus.

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gives the system $q_t + Aq_x = 0$ (Drop tildes)

$$q(x,t) = \begin{bmatrix} p(x,t) \\ u(x,t) \end{bmatrix}, \qquad A = \begin{bmatrix} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{bmatrix}$$

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Eigenvalues: $\lambda = u_0 \pm c_0$

where $c_0 = \sqrt{K_0/\rho_0} = \sqrt{P'(\rho_0)}$ is the linearized sound speed.

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Usually $u_0 = 0$ for linear acoustics. Then $\lambda^1 = -c_0, \;\; \lambda^2 = +c_0$.

Hyperbolicity

A system of m conservation laws $q_t + f(q)_x = 0$ is called hyperbolic at some point $\bar{q}$ is state space if

> The $m \times m$ Jacobian matrix $f'(\bar{q})$ is diagonalizable with real eigenvalues $\lambda^1(q),\ \ldots,\ \lambda^m(q)$.

Then small disturbances about the steady state $q = \bar{q}$ satisfy a linear hyperbolic system and propagate as waves.

- Shallow water equations are hyperbolic for $h > 0$.
- Nonlinear elasticity hyperbolic if $\sigma'(\epsilon) > 0$.
- Gas dynamics hyperbolic if $P'(\rho) > 0$.

Quasi-linear form: $\quad q_t + f'(q)q_x = 0$
Usually want to use conservation form!