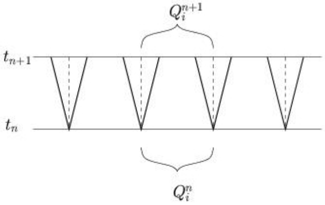
Finite Volume Methods for Scalar Conservation Laws

- Godunov's method
- Fluxes, cell averages, and wave propagation form
- Transonic rarefactions waves
- Approximate Riemann solver with entropy fix

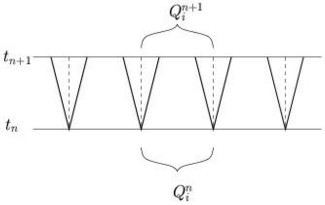
Godunov's Method for $q_t + f(q)_x = 0$



1. Solve Riemann problems at all interfaces, yielding waves $\mathcal{W}^p_{i-1/2}$ and speeds $s^p_{i-1/2}$, for $p = 1,\ 2,\ \ldots,\ m$.

Riemann problem: Original equation with piecewise constant data.

Godunov's Method for $q_t + f(q)_x = 0$



Then either:

1. Compute new cell averages by integrating over cell at t_{n+1},

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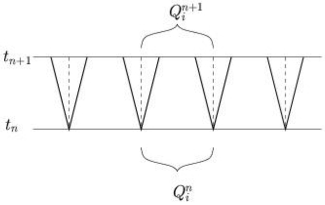
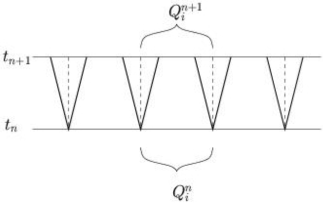


Then either:

1. Compute new cell averages by integrating over cell at t_{n+1},
2. Compute fluxes at interfaces and flux-difference:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[F_{i+1/2}^n - F_{i-1/2}^n]$$

Godunov's Method for $q_t + f(q)_x = 0$



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2. Compute fluxes at interfaces and flux-difference:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[F_{i+1/2}^n - F_{i-1/2}^n]$$

3. Update cell averages by contributions from all waves entering cell:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}]$$

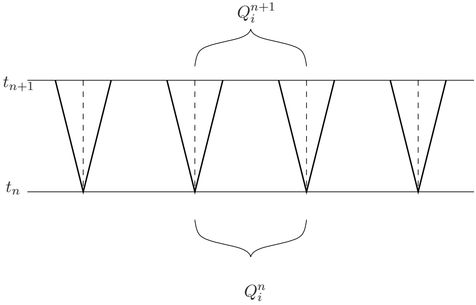
where $\mathcal{A}^\pm \Delta Q_{i-1/2} = \sum_{i=1}^{m} (s_{i-1/2}^p)^\pm \mathcal{W}_{i-1/2}^p$.

Godunov's method with flux differencing

Q_i^n defines a piecewise constant function

$$\tilde{q}^n(x, t_n) = Q_i^n \text{ for } x_{i-1/2} < x < x_{i+1/2}$$

Discontinuities at cell interfaces $\implies$ Riemann problems.



$$\tilde{q}^n(x_{i-1/2}, t) \equiv q^{\Downarrow}(Q_{i-1}, Q_i) \quad \text{for } t > t_n.$$

$$F_{i-1/2}^n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q^{\Downarrow}(Q_{i-1}^n, Q_i^n))\,dt = f(q^{\Downarrow}(Q_{i-1}^n, Q_i^n)).$$

Riemann problem for scalar nonlinear problem

$q_t + f(q)_x = 0$ with data

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x \geq 0 \end{cases}$$

Similarity solution: $q(x,t) = \tilde{q}(x/t)$ so $q(0,t) =$ constant.

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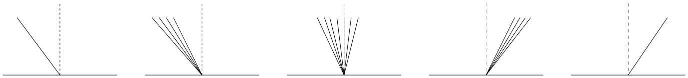
For convex flux (e.g. Burgers' or traffic flow with quadratic flux), the Riemann solution consists of:

- Shock wave if $f'(q_l) > f'(q_r)$,
- Rarefaction wave if $f'(q_l) < f'(q_r)$.

Riemann problem for scalar convex flux

$q_t + f(q)_x = 0$ with $f''(q)$ of one sign, so $f'(q)$ is monotone.

6 possible cases:



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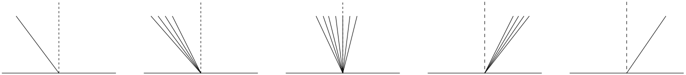

Case 3: Transonic rarefaction $\quad f'(q_l) < 0 < f'(q_r)$

Convex $\implies$ there is at most one point q_s where $f'(q_s) = 0$.
q_s is called the sonic point or stagnation point.

Terminology from gas dynamics: wave speeds $u \pm c$
$\implies$ sonic points where $|u| = c$, supersonic if $|u| > c$.

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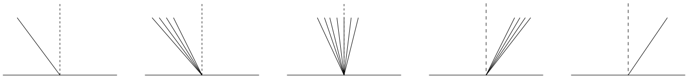

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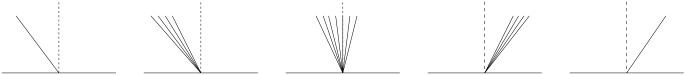
Case 6: Shock moving at speed 0. Then $f(q_l) = f(q_r)$

Godunov flux for scalar problem



The Godunov flux function for the case $f''(q) > 0$ is

$$F^n_{i-1/2} = \begin{cases} f(Q_i) & \text{if } s \leq 0 \text{ and } Q_i < q_s \\ f(Q_{i-1}) & \text{if } s \geq 0 \text{ and } Q_{i-1} > q_s \\ f(q_s) & \text{if } Q_{i-1} \leq q_s \leq Q_i. \end{cases}$$

where $s = \frac{f(Q_i)-f(Q_{i-1})}{Q_i - Q_{i-1}}$ is the Rankine-Hugoniot shock speed.

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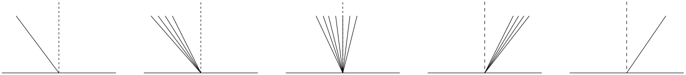

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A more general formula: (for any continuous $f(q)$)

$$F^n_{i-1/2} = \begin{cases} \displaystyle\min_{Q_{i-1} \leq q \leq Q_i} f(q) & \text{if } Q_{i-1} \leq Q_i \\ \displaystyle\max_{Q_i \leq q \leq Q_{i-1}} f(q) & \text{if } Q_i \leq Q_{i-1}, \end{cases}$$

Upwind wave-propagation algorithm

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right].$$

Fluctuations:

$$\mathcal{A}^-\Delta Q_{i-1/2} = \sum_{p=1}^{m}(\lambda^p)^-\mathcal{W}^p_{i-1/2} \quad = A^-\left(Q_i - Q_{i-1}\right),$$

$$\mathcal{A}^+\Delta Q_{i-1/2} = \sum_{p=1}^{m}(\lambda^p)^+\mathcal{W}^p_{i-1/2} \quad = A^+\left(Q_i - Q_{i-1}\right),$$

For a linear system, $s^p = \lambda^p$ and waves $\mathcal{W}^p$ are eigenvectors.

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For scalar advection $m = 1$, only one wave.

$\mathcal{W}_{i-1/2} = \Delta Q_{i-1/2} = Q_i - Q_{i-1}$ and $s = u$,

$$\mathcal{A}^-\Delta Q_{i-1/2} = u^-\mathcal{W}_{i-1/2},$$
$$\mathcal{A}^+\Delta Q_{i-1/2} = u^+\mathcal{W}_{i-1/2}.$$

Godunov for scalar nonlinear in terms of fluctuations

Flux-differencing formula:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+1/2} - F_{i-1/2}\right].$$

Can be rewritten in terms of fluctuations as

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If we define

$$\mathcal{A}^-\Delta Q_{i-1/2} = F_{i-1/2} - f(Q_{i-1}) \quad \text{left-going fluctuation}$$
$$\mathcal{A}^+\Delta Q_{i-1/2} = f(Q_i) - F_{i-1/2} \quad \text{right-going fluctuation}$$

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Agrees with previous definition for linear systems.

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$\mathcal{A}^-\Delta Q_{i-1/2} = F_{i-1/2} - f(Q_{i-1})$ left-going fluctuation

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For high-resolution method, we also need to define a wave $\mathcal{W}$ and speed s,

$$\mathcal{W}_{i-1/2} = Q_i - Q_{i-1},$$

$$s_{i-1/2} = \begin{cases} (f(Q_i) - f(Q_{i-1}))/(Q_i - Q_{i-1}) & \text{if } Q_{i-1} \neq Q_i \\ f'(Q_i) & \text{if } Q_{i-1} = Q_i. \end{cases}$$

Approximate Riemann solver

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For scalar nonlinear: Use same formulas with

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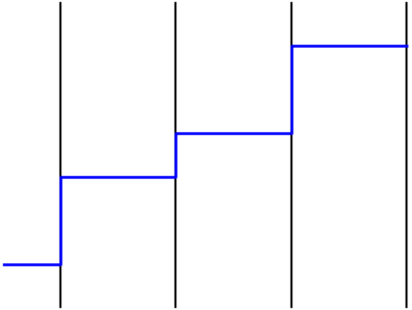
Replacing rarefaction with shock: also exact (after averaging), except in case of transonic rarefaction.

Rarefaction waves in wave propagation method

Initial data giving rarefaction waves (Burgers' equation):



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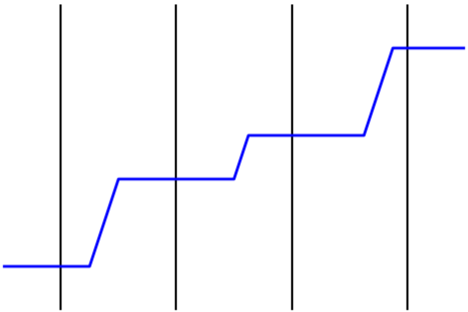
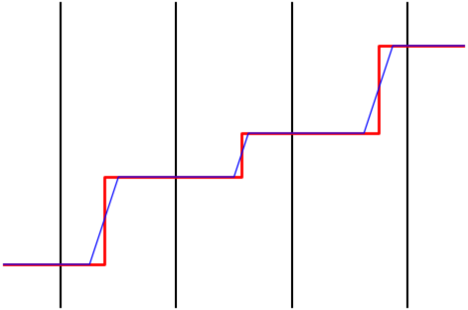


Rarefaction waves in wave propagation method

Initial data giving rarefaction waves (Burgers' equation):

Approximating rarefaction with shock gives same cell average.



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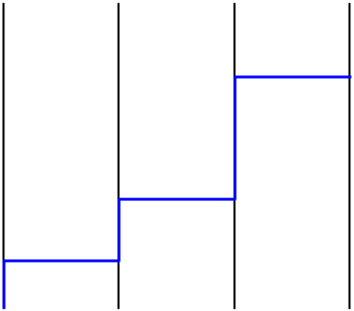

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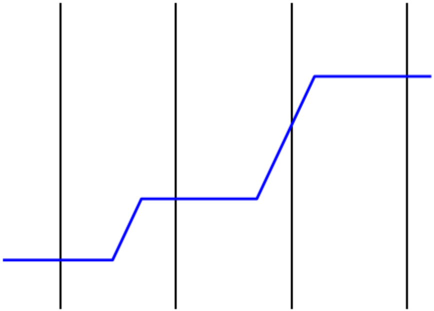
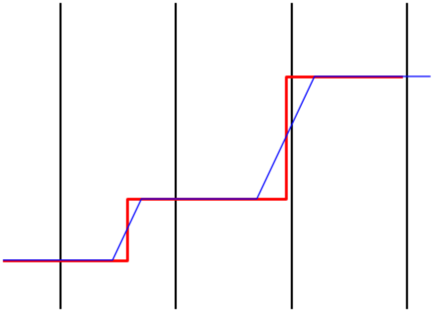


Rarefaction waves in wave propagation method

Initial data with a transonic rarefaction (Burgers' equation):

Approximating rarefaction with shock gives poor approximation!



Entropy fix

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right].$$

Define wave $\mathcal{W}$ and speed s using Rankine-Hugoniot:
(both for $\mathcal{A}^+\Delta Q_{i-1/2}$, $\mathcal{A}^-\Delta Q_{i+1/2}$ and for corrections)

$$\mathcal{W}_{i-1/2} = Q_i - Q_{i-1},$$
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Fix for transonic rarefaction: But if $f'(Q_{i-1}) < 0 < f'(Q_i)$, use:

$$\mathcal{A}^-\Delta Q_{i-1/2} = f(q_s) - f(Q_{i-1}) \qquad \text{left-going fluctuation}$$

$$\mathcal{A}^+\Delta Q_{i-1/2} = f(Q_i) - f(q_s) \qquad \text{right-going fluctuation}$$

Wave limiters for scalar nonlinear

For $q_t + f(q)_x = 0$, just one wave: $\mathcal{W}_{i-1/2} = Q_i^n - Q_{i-1}^n$.

Godunov:

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"Lax-Wendroff":

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right] - \frac{\Delta t}{\Delta x}(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2})$$

$$\tilde{F}_{i-1/2} = \frac{1}{2}\left(1 - \left|\frac{s_{i-1/2}\Delta t}{\Delta x}\right|\right)|s_{i-1/2}|\mathcal{W}_{i-1/2}$$

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High-resolution method:

$$\tilde{F}_{i-1/2} = \frac{1}{2}\left(1 - \left|\frac{s_{i-1/2}\Delta t}{\Delta x}\right|\right)|s_{i-1/2}|\widetilde{\mathcal{W}}_{i-1/2}$$

$\widetilde{\mathcal{W}}_{i-1/2} = \phi(\theta)\,\mathcal{W}_{i-1/2}$, where $\theta_{i-1/2} = \mathcal{W}_{I-1/2}/\mathcal{W}_{i-1/2}$.

Entropy-violating numerical solutions

Riemann problem for Burgers' equation with $q_l = -1$ and $q_r = 2$: