Finite Volume Methods for Hyperbolic Problems

Nonlinear Scalar Conservation Laws
Rarefaction Waves

- Form of centered rarefaction wave
- Non-uniqueness of weak solutions
- Entropy conditions

Weak solutions to $q_t + f(q)_x = 0$

$q(x,t)$ is a weak solution if it satisfies the integral form of the conservation law over all rectangles in space-time,

$$\int_{x_1}^{x_2} q(x,t_2)\,dx - \int_{x_1}^{x_2} q(x,t_1)\,dx$$
$$= \int_{t_1}^{t_2} f(q(x_1,t))\,dt - \int_{t_1}^{t_2} f(q(x_2,t))\,dt$$

Obtained by integrating

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = f(q(x_1,t)) - f(q(x_2,t))$$

from t_n to t_{n+1}.

Rankine-Hugoniot jump condition

$$s(q_r - q_\ell) = f(q_r) - f(q_\ell).$$

This must hold for any discontinuity propagating with speed s, even for systems of conservation laws.

For scalar problem, any jump allowed with speed:

$$s = \frac{f(q_r) - f(q_l)}{q_r - q_l}.$$

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For systems, $q_r - q_l$ and $f(q_r) - f(q_l)$ are vectors, s scalar,

R-H condition: $f(q_r) - f(q_l)$ must be scalar multiple of $q_r - q_l$.

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For linear system, $f(q) = Aq$, this says

$$s(q_r - q_l) = A(q_r - q_l),$$

Jump must be an eigenvector, speed s the eigenvalue.

Weak solutions to $q_t + f(q)_x = 0$

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Other admissibility conditions needed to pick out the physically correct weak solution, e.g.

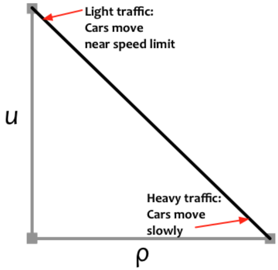
- Vanishing viscosity limit,
- “Entropy conditions”

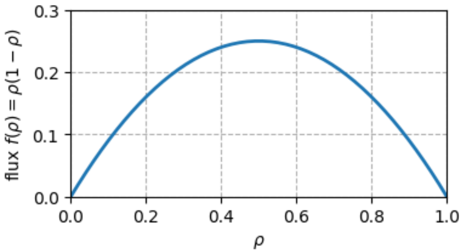
Traffic flow — LWR model

First models due to Lighthill, Whitham, Richards in 1950's

Density of cars (per carlength): $q(x,t), \qquad 0 \le q \le 1.$

Desired driving speed: $U(q) = u_{\max}(1-q), \quad 0 \le U(q) \le u_{\max}.$





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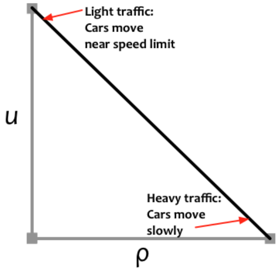

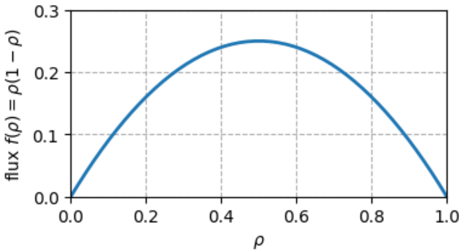

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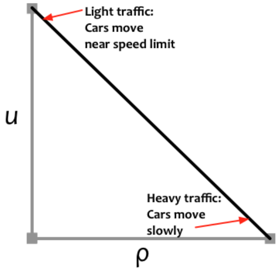
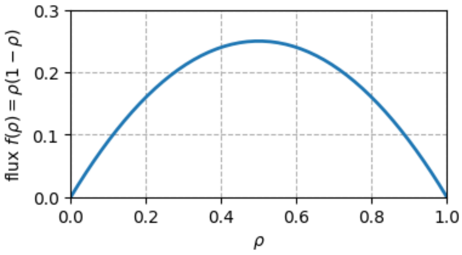
Characteristic speed: $f'(q) = u_{\max}(1-2q), \quad -u_{\max} \le f'(q) \le u_{\max}$





Jupyter Notebook for Traffic Flow

Chapter on Traffic Flow in the book

Riemann Problems and Jupyter Solutions

View static version of notebook at:

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Notebook on nonconvex scalar problems also may be useful:

www.clawpack.org/riemann_book/html/Nonconvex_scalar.html

Convex flux functions

The scalar conservation law $q_t + f(q)_x = 0$ has a convex flux if $f''(q)$ has the same sign for all q:

$$f''(q) > 0 \;\; \forall q \quad \text{or} \quad f''(q) < 0 \;\; \forall q.$$

This means that the characteristic speed $f'(q)$ is either strictly increasing or strictly decreasing as q increases.

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Generalization of convexity for systems:
Each characteristic field must be genuinely nonlinear.

Riemann problem for traffic flow

Initial data of the form

$$q(x,0) = \begin{cases} q_\ell & \text{if } x < 0 \\ q_r & \text{if } x \geq 0 \end{cases}$$

$$U(q) = u_{\max}(1-q), \quad f(q) = qU(q), \quad 0 \leq q \leq 1$$

Case 1: $q_\ell < q_r$, so $U(q_\ell) > U(q_r)$, $f'(q_\ell) > f'(q_r)$.

Fast moving cars approaching traffic jam
Expect shock wave.

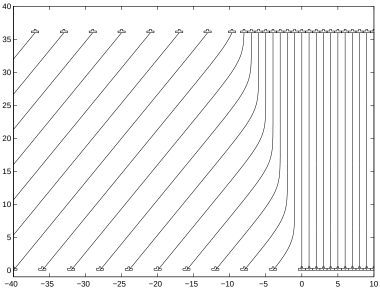
Case 2: $q_\ell > q_r$, so $U(q_\ell) < U(q_r)$, $f'(q_\ell) < f'(q_r)$.

Slow moving cars can accelerate
Expect rarefaction wave.

Cars approaching red light $(q_\ell < 1, \;\; q_r = 1)$

Shock speed:

$$s = \frac{f(q_r) - f(q_\ell)}{q_r - q_\ell} = \frac{-2u_{\max} q_\ell}{1 - q_\ell} < 0 \quad \text{(for this data, could be } > 0\text{)}$$



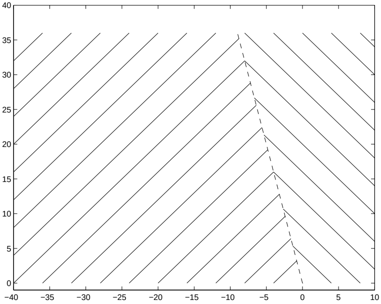
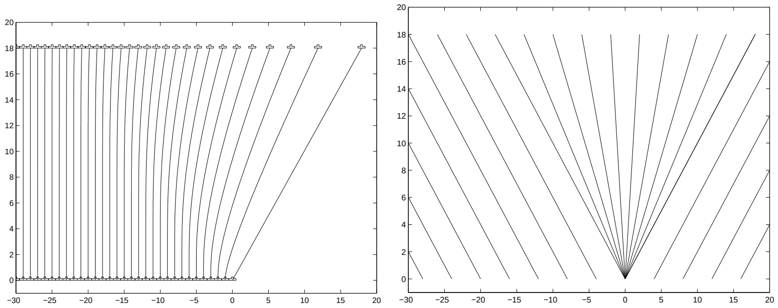


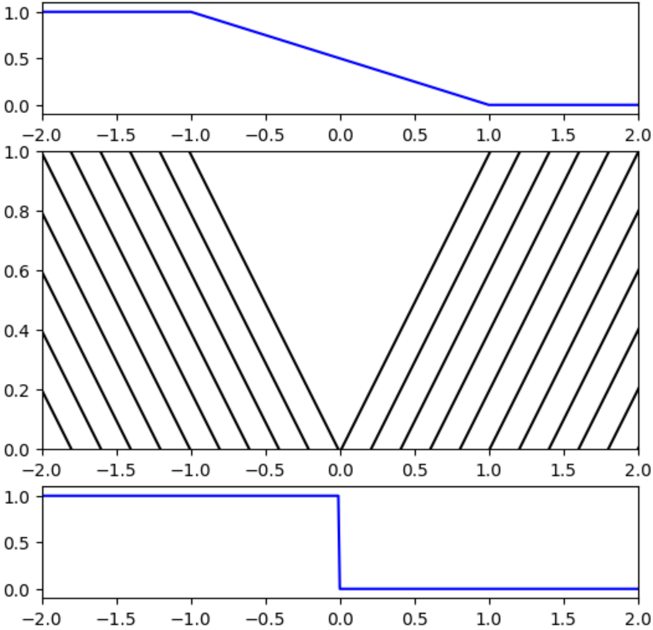
Figure 11.3 — Rarefaction wave

Cars accelerating at green light $(q_\ell = 1, \;\; q_r = 0)$

Characteristic speed $f'(q) = u_{\max}(1 - 2q)$

varies from $f'(q_\ell) = -u_{\max}$ to $f'(q_r) = u_{\max}$.

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Centered rarefaction waves

Similarity solution with piecewise constant initial data:

$$q(x,t) = \begin{cases} q_\ell & \text{if } x/t \le f'(q_\ell) \\ \tilde{q}(x/t) & \text{if } f'(q_\ell) \le x/t \le f'(q_r) \\ q_r & \text{if } x/t \ge f'(q_r), \end{cases}$$

solves the Riemann problem for convex f, provided

$$f'(q_\ell) < f'(q_r),$$

so that characteristics spread out as time advances.

Rarefaction waves

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Determining $\tilde{q}(\xi)$:

$$\begin{aligned} q(x,t) = \tilde{q}(x/t) \implies q_t(x,t) &= -(x/t^2)\tilde{q}'(x/t), \\ q_x(x,t) &= (1/t)\tilde{q}'(x/t). \end{aligned}$$

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Quasilinear form: $q_t(x,t) + f'(q(x,t))q_x(x,t) = 0$ leads to

$$-(x/t^2)\tilde{q}'(x/t) + f'(\tilde{q}(x/t))(1/t)\tilde{q}'(x/t) = 0$$

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Cancel $(1/t)\tilde{q}'(x/t)$ to get:

$$-(x/t) + f'(\tilde{q}(x/t)) = 0 \quad \text{or} \quad f'(\tilde{q}(\xi)) = \xi.$$

Centered rarefaction for traffic flow

Take $u_{\max} = 1$.

$$f(q) = q(1-q) \implies f'(q) = (1-2q).$$

Solving $f'(\tilde{q}(\xi)) = \xi$ gives

$$(1 - 2\tilde{q}(\xi)) = \xi \implies \tilde{q}(\xi) = \frac{1}{2}(1-\xi)$$

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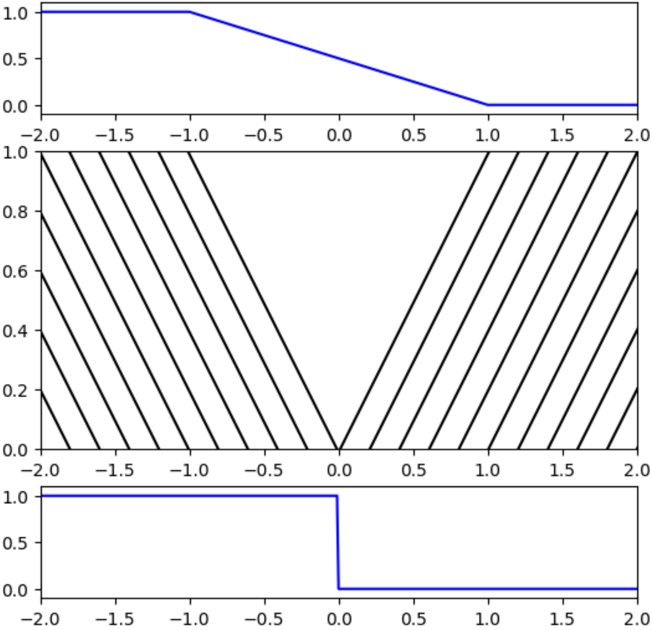
Solution is linear in x at each t.

(Since $f(q)$ was quadratic, not true in general.)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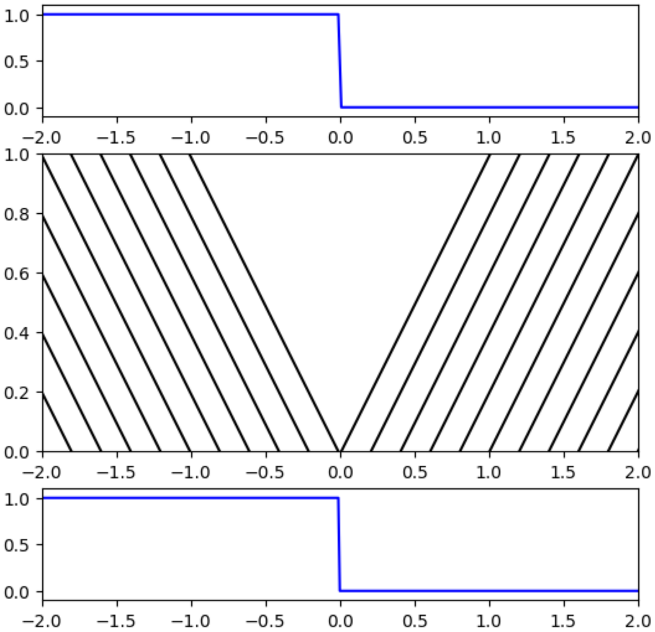
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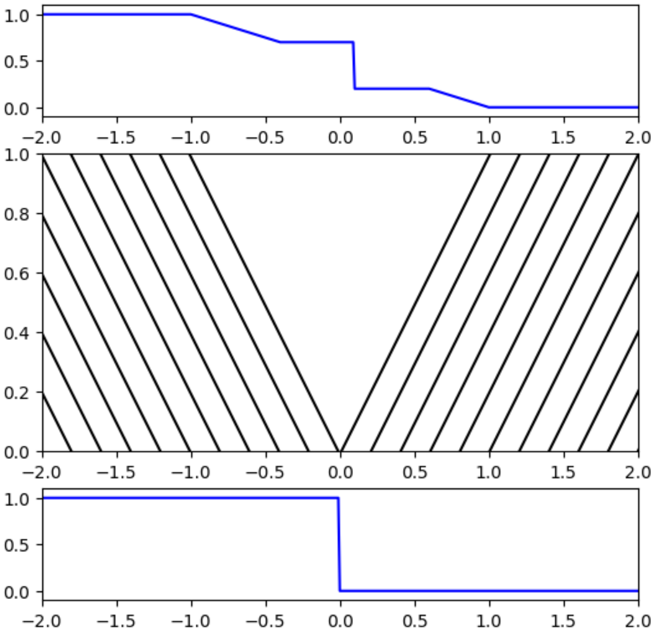
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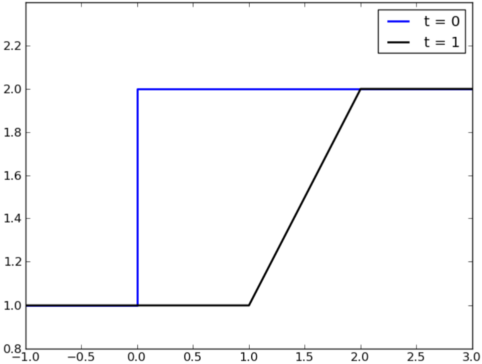
Weak solutions to Burgers' equation

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0, \qquad u_\ell = 1, \;\; u_r = 2$$

Characteristic speed: u Rankine-Hugoniot speed: $\frac{1}{2}(u_\ell + u_r)$.

"Physically correct" rarefaction wave solution:



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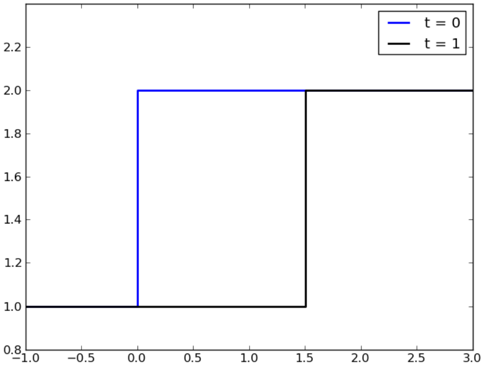
Entropy violating weak solution:



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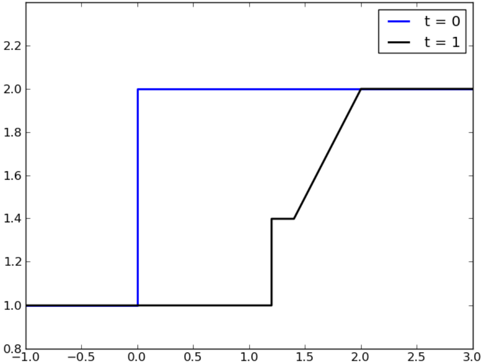
Another Entropy violating weak solution:



Vanishing viscosity solution

We want $q(x,t)$ to be the limit as $\epsilon \to 0$ of solution to

$$q_t + f(q)_x = \epsilon q_{xx}.$$

This selects a unique weak solution:

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Note: This means characteristics must approach shock from both sides as t advances, not move away from shock!