Finite Volume Methods for Hyperbolic Problems

Derivation of Conservation Laws

- Integral form in one space dimension
- Advection
- Compressible gas – mass and momentum
- Source terms
- Diffusion

First order hyperbolic PDE in 1 space dimension

Linear: $q_t + Aq_x = 0, \qquad q(x,t) \in \mathbb{R}^m,\ A \in \mathbb{R}^{m\times m}$

Conservation law: $q_t + f(q)_x = 0, \qquad f : \mathbb{R}^m \to \mathbb{R}^m$ (flux)

Quasilinear form: $q_t + f'(q)q_x = 0$

Hyperbolic if A or $f'(q)$ is diagonalizable with real eigenvalues.

Models wave motion or advective transport.

Eigenvalues are wave speeds.

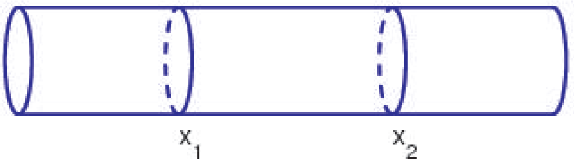
Note: Second order wave equation $p_{tt} = c^2 p_{xx}$ can be written as a first-order system (acoustics).

Derivation of Conservation Laws

$q(x,t) =$ density function for some conserved quantity, so

$$\int_{x_1}^{x_2} q(x,t)\,dx = \text{total mass in interval}$$

changes only because of fluxes at left or right of interval.



Derivation of Conservation Laws

$q(x,t) =$ density function for some conserved quantity.

Integral form:

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = F_1(t) - F_2(t)$$

where

$$F_j = f(q(x_j,t)), \qquad f(q) = \text{flux function.}$$

Derivation of Conservation Laws

If q is smooth enough, we can rewrite

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = f(q(x_1,t)) - f(q(x_2,t))$$

as

$$\int_{x_1}^{x_2} q_t\,dx = -\int_{x_1}^{x_2} f(q)_x\,dx$$

or

$$\int_{x_1}^{x_2} (q_t + f(q)_x)\,dx = 0$$

True for all $x_1,\ x_2 \implies$ differential form:

$$q_t + f(q)_x = 0.$$

Advective flux

If $\rho(x,t)$ is the density (mass per unit length),

$$\int_{x_1}^{x_2} \rho(x,t)\,dx = \text{total mass in } [x_1, x_2]$$

and $u(x,t)$ is the velocity, then the advective flux is

$$\rho(x,t)u(x,t)$$

Units: mass/length $\times$ length/time $=$ mass/time.

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Units: mass/length $\times$ length/time $=$ mass/time.

Continuity equation (conservation of mass):

$$\rho_t + (\rho u)_x = 0$$

Advection equation

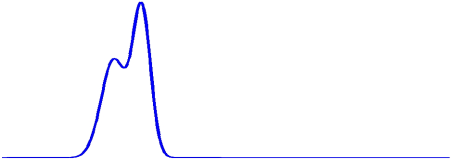
Flow in a pipe at constant velocity

$u =$ constant flow velocity

$q(x,t) =$ tracer concentration, $\quad f(q) = uq$

$\implies \quad q_t + uq_x = 0, \quad$ with initial condition $q(x,0) = \overset{\circ}{q}(x)$.

True solution: $q(x,t) = q(x-ut,0) = \overset{\circ}{q}(x-ut)$



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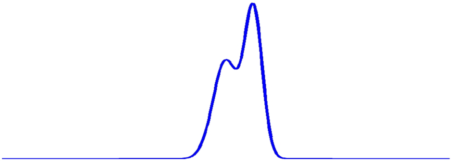

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Compressible gas dynamics

In one space dimension (e.g. in a pipe).
$\rho(x,t) =$ density, $u(x,t) =$ velocity,
$p(x,t) =$ pressure, $\rho(x,t)u(x,t) =$ momentum.

Conservation of:

mass:	ρ	flux:	ρu
momentum:	ρu	flux:	$(\rho u)u + p$
(energy)			

Conservation laws:

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

Equation of state:

$$p = P(\rho).$$

(Later: p may also depend on internal energy / temperature)

Compressible gas dynamics

Conservation laws:

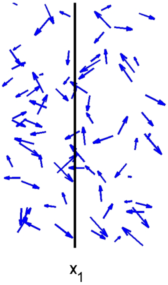
$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

Momentum flux:
$\rho u^2 = (\rho u)u =$ advective flux

p term in flux?

- $-p_x =$ force in Newton's second law,
- as momentum flux: microscopic motion of gas molecules.

Momentum flux arising from pressure



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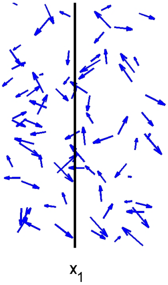

Note that:

- molecules with positive x-velocity crossing x_1 to right increase the momentum in $[x_1, x_2]$
- molecules with negative x-velocity crossing x_1 to left also increase the momentum in $[x_1, x_2]$

Hence momentum flux increases with pressure $p(x_1, t)$ even if macroscopic (average) velocity is zero.

Source terms (balance laws)

$$q_t + f(q)_x = \psi(q)$$

Results from integral form

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} q(x,t)\,dx = f(q(x_1,t)) - f(q(x_2,t)) + \int_{x_1}^{x_2} \psi(q(x,t))\,dx$$

Examples:

- Reacting flow, e.g. combustion,
- External forces such as gravity
- Viscosity, drag
- Radiative heat transfer
- Geometric source terms (e.g., quasi-1d problems)
- Bottom topography in shallow water

Source term example: advection with decay

$q(x,t) =$ mass / unit length
First suppose no advection,
but at each point, exponential decay occurs:

$$q(x,t)_t = -\lambda q(x,t) \equiv \psi(q(x,t)).$$

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With advection:

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = uq(x_1,t) - uq(x_2,t) + \int_{x_1}^{x_2} \psi(q(x,t))\,dx.$$

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With advection:

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = uq(x_1,t) - uq(x_2,t) + \int_{x_1}^{x_2} \psi(q(x,t))\,dx.$$

$$\int_{x_1}^{x_2} q_t + (uq)_x - \psi(q)\,dx = 0 \quad \text{holds for all } \ x_1,\ x_2$$

Diffusive flux

$q(x,t) =$ concentration
$\beta =$ diffusion coefficient ($\beta > 0$)

diffusive flux $= -\beta q_x(x,t)$

$q_t + f_x = 0 \implies$ diffusion equation:

$$q_t = (\beta q_x)_x = \beta q_{xx} \text{ (if } \beta = \text{const).}$$

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Heat equation: Same form, where

$q(x,t) =$ density of thermal energy $= \kappa T(x,t)$,
$T(x,t) =$ temperature, $\kappa =$ heat capacity,
flux $= -\beta T(x,t) = -(\beta/\kappa)q(x,t) \implies$

$$q_t(x,t) = (\beta/\kappa)q_{xx}(x,t).$$

Advection-diffusion

$q(x,t) =$ concentration that advects with velocity u
and diffuses with coefficient β:

$$\text{flux} = uq - \beta q_x.$$

Advection-diffusion equation:

$$q_t + uq_x = \beta q_{xx}.$$

If $\beta > 0$ then this is a parabolic equation.

Advection dominated if u/β (the Péclet number) is large.

Fluid dynamics: "parabolic terms" arise from

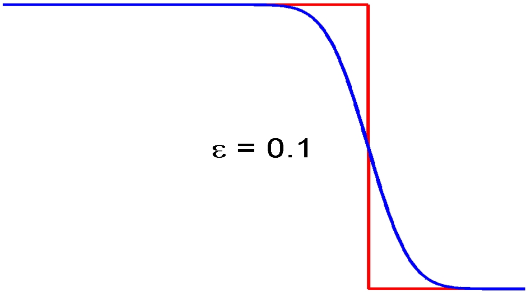
- thermal diffusion and
- diffusion of momentum, where the diffusion parameter is the viscosity.

Discontinuous solutions

Vanishing Viscosity solution: The Riemann solution $q(x,t)$ is the limit as $\epsilon \to 0$ of the solution $q^\epsilon(x,t)$ of the parabolic advection-diffusion equation

$$q_t + uq_x = \epsilon q_{xx}.$$

For any $\epsilon > 0$ this has a classical smooth solution:



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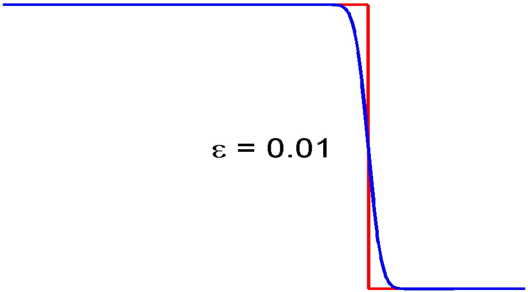

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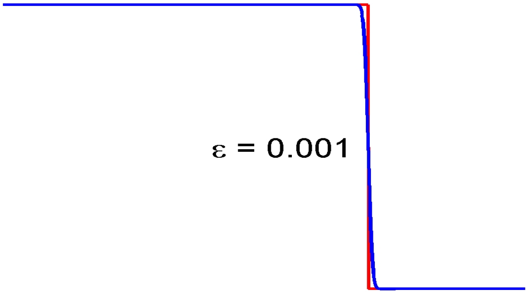

Variable Coefficient Advection

- Quasi-1D pipe
- Units in one space dimension
- Conservative form: $q_t + (u(x)q)_x = 0$
- Advective form: $q_t + u(x)q_x = 0$ (color equation)

Variable-coefficient advection

Incompressible flow in 1D pipe with constant cross section
$\Longrightarrow u \equiv$constant in space.

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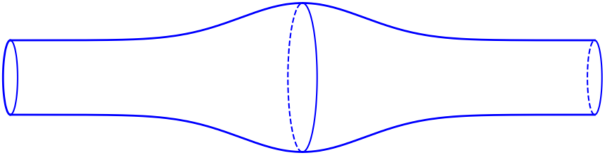
$\implies u \equiv$constant in space.

If cross-sectional area $\kappa(x)$ varies, then so does $u(x)$.



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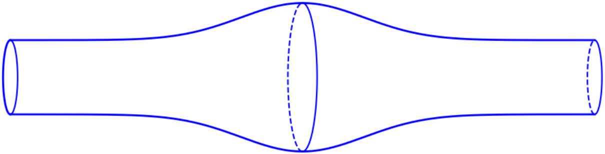

Incompressible $\implies$ flux of fluid must be constant, so

$$\kappa(x)u(x) \equiv U = \text{constant.}$$

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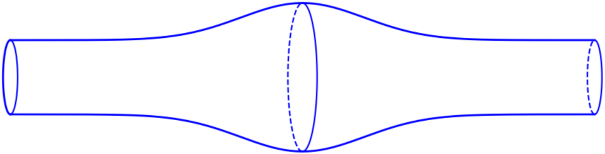

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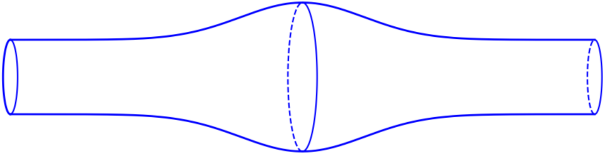
PDE for concentration of a passive tracer advected with flow?

Variable-coefficient advection



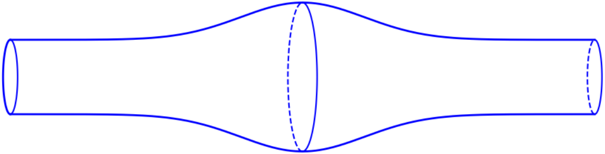
Incompressible $\implies$ flux of fluid must be constant, so

$$\kappa(x)u(x) \equiv U \implies u(x) = U/\kappa(x).$$

Concentration of passive tracer: $q(x,t)$
If units of q are mass / unit length, then q is conserved quantity with flux uq, and we obtain the conservation law

$$q_t(x,t) + (u(x)q(x,t))_x = 0.$$

Variable-coefficient advection



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$$q_t(x,t) + (u(x)q(x,t))_x = 0.$$

However, if q is in units of mass / unit volume, then:

$$q_t(x,t) + u(x)q_x(x,t) = 0. \qquad \text{(color equation)}$$

Variable-coefficient advection

Derivation of color equation:

Incompressible $\implies$ flux of fluid must be constant, so

$$\kappa(x)u(x) \equiv U \implies u(x) = U/\kappa(x).$$

If $q(x,t)$ in units of mass/volume, the mass/length is $\kappa(x)q(x,t)$. This is now the conserved quantity.

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Conservation law is:

$$(\kappa(x)q(x,t))_t + (Uq(x,t))_x = 0,$$

$$\kappa(x)q_t(x,t) + Uq_x(x,t)) = 0,$$

$$q_t(x,t) + u(x)q_x(x,t)) = 0.$$

Variable-coefficient advection

Color equation:

$$q_t(x,t) + u(x)q_x(x,t) = 0.$$

Can be rewriten as a balance law
(conservation law plus source term):

$$q_t(x,t) + (u(x)q(x,t))_x = u'(x)q(x,t)$$

Will revisit different forms when studying numerical methods.

Finite Volume Methods for Hyperbolic Problems

Linearization of Nonlinear Systems

- General form, Jacobian matrix
- Scalar Burgers' equation
- Compressible gas dynamics
- Linear acoustics equations

Linearization

General nonlinear conservation law: $q_t + f(q)_x = 0$

Suppose $q(x,t) = q_0 + \tilde{q}(x,t)$ where $\|\tilde{q}(x,t)\| = \epsilon$ is small.

Linearization

General nonlinear conservation law: $q_t + f(q)_x = 0$

Suppose $q(x,t) = q_0 + \tilde{q}(x,t)$ where $\|\tilde{q}(x,t)\| = \epsilon$ is small.

Then

$$\begin{aligned}
\tilde{q}_t &= q_t \\
&= -f(q)_x \\
&= -f'(q)q_x \\
&= -f'(q_0 + \tilde{q})\tilde{q}_x \\
&= -f'(q_0)\tilde{q}_x + \mathcal{O}(\epsilon^2).
\end{aligned}$$

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Linearization: $\tilde{q}_t + A\tilde{q}_x = 0$ where $A = f'(q_0) =$ Jacobian matrix

Scalar: Advection equation

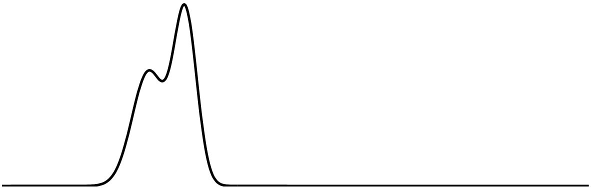
Nonlinear Burgers' equation

Conservation form: $u_t + \left(\frac{1}{2}u^2\right)_x = 0, \qquad f(u) = \frac{1}{2}u^2$.

Quasi-linear form: $\qquad u_t + uu_x = 0.$

This looks like an advection equation with u advected with speed u.

True solution: u is constant along characteristic with speed u until the wave "breaks".



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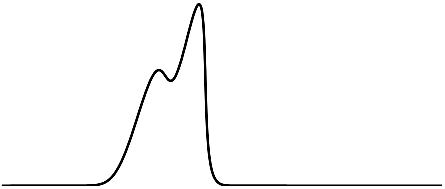

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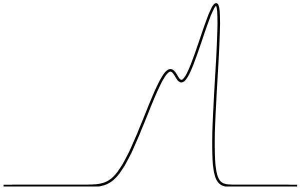

After breaking, the weak solution contains a shock wave.

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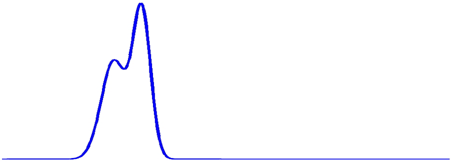
Linearization about u_0:

$$f(u) = \frac{1}{2}u^2 \implies f'(u_0) = u_0$$

So if $u(x,0) = u_0 + \tilde{u}(x,0)$ with $\|\tilde{u}\|$ small, then $\tilde{u}(x,t)$ approximately satisfies advection equation

$$\tilde{u}_t + u_0 u_x = 0.$$



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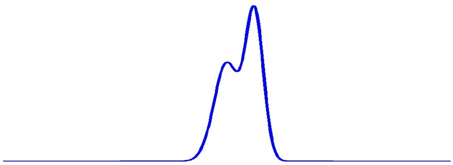

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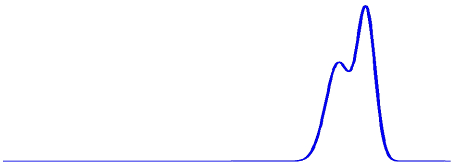

Compressible gas dynamics (simple case)

In one space dimension (e.g. in a pipe).
$\rho(x,t) =$ density, $\quad u(x,t) =$ velocity,
$p(x,t) =$ pressure, $\quad \rho(x,t)u(x,t) =$ momentum.

Conservation of:

mass:	ρ	flux:	ρu
momentum:	ρu	flux:	$(\rho u)u + p$

Conservation laws:

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

Equation of state:

$$p = P(\rho).$$

Linearization of gas dynamics

Suppose $\rho(x,t) \approx \rho_0$ and $u(x,t) \approx u_0$.

Model small perturbations to this steady state (sound waves).

$$\begin{bmatrix} \rho(x,t) \\ (\rho u)(x,t) \end{bmatrix} = \begin{bmatrix} \rho_0 \\ \rho_0 u_0 \end{bmatrix} + \begin{bmatrix} \widetilde{\rho}(x,t) \\ (\widetilde{\rho u})(x,t) \end{bmatrix}$$

or $q(x,t) = q_0 + \tilde{q}(x,t)$ where $\|\tilde{q}(x,t)\| = \epsilon$ is small.

Linearization of gas dynamics

Suppose $\rho(x,t) \approx \rho_0$ and $u(x,t) \approx u_0$.

Model small perturbations to this steady state (sound waves).

$$\left[\begin{array}{c} \rho(x,t) \\ (\rho u)(x,t) \end{array}\right] = \left[\begin{array}{c} \rho_0 \\ \rho_0 u_0 \end{array}\right] + \left[\begin{array}{c} \widetilde{\rho}(x,t) \\ (\widetilde{\rho u})(x,t) \end{array}\right]$$

or $q(x,t) = q_0 + \tilde{q}(x,t)$ where $\|\tilde{q}(x,t)\| = \epsilon$ is small.

Then nonlinear equation $q_t + f(q)_x = 0$ leads to

$$\begin{aligned} \tilde{q}_t &= q_t \\ &= -f(q)_x \\ &= -f'(q)q_x \\ &= -f'(q_0+\tilde{q})\tilde{q}_x \\ &= -f'(q_0)\tilde{q}_x + \mathcal{O}(\epsilon^2). \end{aligned}$$

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Linearization: $\tilde{q}_t + A\tilde{q}_x = 0$ where $A = f'(q_0)$.

Linearization of gas dynamics

$$\begin{aligned}\rho_t + (\rho u)_x &= 0\\ (\rho u)_t + (\rho u^2 + P(\rho))_x &= 0\end{aligned}$$

so

$$q = \begin{bmatrix} \rho \\ \rho u \end{bmatrix} = \begin{bmatrix} q^1 \\ q^2 \end{bmatrix},$$

$$f(q) = \begin{bmatrix} \rho u \\ \rho u^2 + P(\rho) \end{bmatrix} = \begin{bmatrix} f^1(q) \\ f^2(q) \end{bmatrix} = \begin{bmatrix} q^2 \\ (q^2)^2/q^1 + P(q^1) \end{bmatrix}.$$

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Jacobian:

$$f'(q) = \begin{bmatrix} \partial f^1/\partial q^1 & \partial f^1/\partial q^2 \\ \partial f^2/\partial q^1 & \partial f^2/\partial q^2 \end{bmatrix}.$$

$$f'(q_0) = \begin{bmatrix} 0 & 1 \\ -u_0^2 + P'(\rho_0) & 2u_0 \end{bmatrix}.$$

Linearization of gas dynamics

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$$A = f'(q_0) = \begin{bmatrix} 0 & 1 \\ -u_0^2 + P'(\rho_0) & 2u_0 \end{bmatrix}.$$

This can be written out as (2.47):

$$\begin{aligned} \tilde{\rho}_t + (\widetilde{\rho u})_x &= 0 \\ (\widetilde{\rho u})_t + (-u_0^2 + P'(\rho_0))\tilde{\rho}_x + 2u_0(\widetilde{\rho u})_x &= 0. \end{aligned}$$

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Rewrite in terms of p and u perturbations (Exer. 2.1):

$$\begin{aligned} \tilde{p}_t + u_0\tilde{p}_x + K_0\tilde{u}_x &= 0, \\ \rho_0\tilde{u}_t + \tilde{p}_x + \rho_0 u_0\tilde{u}_x &= 0, \end{aligned}$$

where $K_0 = \rho_0 P'(\rho_0)$ is the bulk modulus.

Linearization of gas dynamics

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gives the system $q_t + Aq_x = 0$ (Drop tildes)

$$q(x,t) = \begin{bmatrix} p(x,t) \\ u(x,t) \end{bmatrix}, \qquad A = \begin{bmatrix} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{bmatrix}$$

Linearization of gas dynamics

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Eigenvalues: $\lambda = u_0 \pm c_0$

where $c_0 = \sqrt{K_0/\rho_0} = \sqrt{P'(\rho_0)}$ is the linearized sound speed.

Linearization of gas dynamics

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where $c_0 = \sqrt{K_0/\rho_0} = \sqrt{P'(\rho_0)}$ is the linearized sound speed.

Usually $u_0 = 0$ for linear acoustics. Then $\lambda^1 = -c_0, \;\; \lambda^2 = +c_0$.

Hyperbolicity

A system of m conservation laws $q_t + f(q)_x = 0$ is called hyperbolic at some point $\bar{q}$ is state space if

> The $m \times m$ Jacobian matrix $f'(\bar{q})$ is diagonalizable with real eigenvalues $\lambda^1(q),\ \ldots,\ \lambda^m(q)$.

Then small disturbances about the steady state $q = \bar{q}$ satisfy a linear hyperbolic system and propagate as waves.

- Shallow water equations are hyperbolic for $h > 0$.
- Nonlinear elasticity hyperbolic if $\sigma'(\epsilon) > 0$.
- Gas dynamics hyperbolic if $P'(\rho) > 0$.

Quasi-linear form: $\quad q_t + f'(q)q_x = 0$
Usually want to use conservation form!

Finite Volume Methods for Hyperbolic Problems

Linear Hyperbolic Systems

- General form, coefficient matrix, hyperbolicity
- Scalar advection equation
- Linear acoustics equations
- Eigen decomposition
- Characteristics and general solution
- Boundary conditions

Linear hyperbolic systems

Linear system of m equations: $q(x,t) \in \mathbb{R}^m$ for each (x,t) and

$$q_t + Aq_x = 0, \qquad -\infty < x, \infty, \;\; t \geq 0.$$

A is $m \times m$ matrix (constant for now, independent of x,t)

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$\exists$ nonsingular $R:\;\; R^{-1}AR = \Lambda$ diagonal with $\lambda^p \geq 0$.

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$\exists$ nonsingular R : $R^{-1}AR = \Lambda$ diagonal with $\lambda^p \geq 0$.

Eigenvalues are wave speeds.

Eigenvectors used to split arbibrary data into waves.
So matrix of eigenvectors must be nonsingular.

Advection equation as a linear system

$$q_t + uq_x = 0$$

with u a constant (real) velocity. $(1 \times 1$ diagonalizable, $\lambda^1 = u)$

Initial condition:

$$q(x,0) = \overset{\circ}{q}(x), \quad -\infty < x < \infty.$$

The solution to this Cauchy problem is:

$$q(x,t) = \overset{\circ}{q}(x - ut)$$

It is constant along each characteristic curve

$$X(t) = x_0 + ut$$

Characteristics for advection

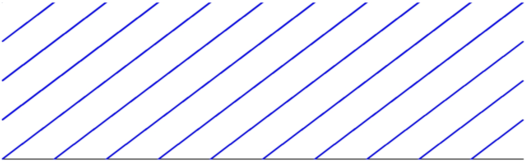
$q(x,t) = \overset{\circ}{q}(x-ut) \implies q$ is constant along lines

$$X(t) = x_0 + ut, \quad t \geq 0.$$

Can also see that q is constant along $X(t)$ from:

$$\begin{aligned}\frac{d}{dt}q(X(t),t) &= q_x(X(t),t)X'(t) + q_t(X(t),t)\\ &= q_x(X(t),t)u + q_t(X(t),t)\\ &= 0.\end{aligned}$$

In x–t plane:



Diagonalization of linear system

Consider constant coefficient linear system $q_t + Aq_x = 0$.

Suppose hyperbolic:

- Real eigenvalues $\lambda^1 \leq \lambda^2 \leq \cdots \leq \lambda^m$,
- Linearly independent eigenvectors $r^1,\ r^2,\ \ldots,\ r^m$.

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Let $R = [r^1|r^2|\cdots|r^m] \quad m \times m$ matrix of eigenvectors.

Then $Ar^p = \lambda^p r^p$ means that $AR = R\Lambda$ where

$$\Lambda = \begin{bmatrix} \lambda^1 & & & \\ & \lambda^2 & & \\ & & \ddots & \\ & & & \lambda^m \end{bmatrix} \equiv \text{diag}(\lambda^1, \lambda^2, \ldots, \lambda^m).$$

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$AR = R\Lambda \implies A = R\Lambda R^{-1}$ and $R^{-1}AR = \Lambda$.
Similarity transformation with R diagonalizes A.

Diagonalization of linear system

Consider constant coefficient linear system $q_t + Aq_x = 0$.

Multiply system by R^{-1}:

$$R^{-1}q_t(x,t) + R^{-1}Aq_x(x,t) = 0.$$

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Introduce $RR^{-1} = I$:

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Use $R^{-1}AR = \Lambda$ and define $w(x,t) = R^{-1}q(x,t)$:

$$w_t(x,t) + \Lambda w_x(x,t) = 0.$$

Since R is constant!

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$$w_t(x,t) + \Lambda w_x(x,t) = 0.$$ Since R is constant!

This decouples to m independent scalar advection equations:

$$w^p_t(x,t) + \lambda^p w^p_x(x,t) = 0. \qquad p = 1,\ 2,\ \ldots,\ m.$$

Solution to Cauchy problem

Suppose $q(x,0) = \overset{\circ}{q}(x)$ for $-\infty < x < \infty$.

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The solution to the decoupled equation $w_t^p + \lambda^p w_x^p = 0$ is

$$w^p(x,t) = w^p(x - \lambda^p t, 0) = \overset{\circ}{w}{}^p(x - \lambda^p t).$$

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We can rewrite this as

$$q(x,t) = \sum_{p=1}^{m} w^p(x,t)\, r^p = \sum_{p=1}^{m} \overset{\circ}{w}{}^p(x - \lambda^p t)\, r^p$$

Linear acoustics

Example: Linear acoustics in a 1d gas tube

$$q = \begin{bmatrix} p \\ u \end{bmatrix} \qquad \begin{array}{l} p(x,t) = \text{pressure perturbation} \\ u(x,t) = \text{velocity} \end{array}$$

Equations:

$$\begin{aligned} p_t + K_0 u_x &= 0 \qquad \text{Change in pressure due to compression} \\ \rho_0 u_t + p_x &= 0 \qquad \text{Newton's second law, } F = ma \end{aligned}$$

where $K_0 =$ bulk modulus, and $\rho_0 =$ unperturbed density.

Hyperbolic system:

$$\begin{bmatrix} p \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & K_0 \\ 1/\rho_0 & 0 \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_x = 0.$$

Eigenvectors for acoustics

$$A = \begin{bmatrix} 0 & K_0 \\ 1/\rho_0 & 0 \end{bmatrix}$$

Eigenvectors:

$$r^1 = \begin{bmatrix} -\rho_0 c_0 \\ 1 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} \rho_0 c_0 \\ 1 \end{bmatrix}.$$

Check that $Ar^p = \lambda^p r^p$, where

$$\lambda^1 = -c_0, \qquad \lambda^2 = +c_0.$$

with $c_0 = \sqrt{K_0/\rho_0} \implies K_0 = \rho_0 c_0^2$.

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Let $Z_0 = \rho_0 c_0 = \sqrt{K_0 \rho_0} =$ impedance.

Physical meaning of eigenvectors

Eigenvectors for acoustics:

$$r^1 = \begin{bmatrix} -\rho_0 c_0 \\ 1 \end{bmatrix} = \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} \rho_0 c_0 \\ 1 \end{bmatrix} = \begin{bmatrix} Z_0 \\ 1 \end{bmatrix}.$$

Consider a pure 1-wave (simple wave), at speed $\lambda^1 = -c_0$,
If $\overset{\circ}{q}(x) = \bar{q} + \overset{\circ}{w}^1(x) r^1$ then

$$q(x,t) = \bar{q} + \overset{\circ}{w}^1(x - \lambda^1 t) r^1$$

Variation of q, as measured by q_x or $\Delta q = q(x + \Delta x) - q(x)$ is proportional to eigenvector r^1, e.g.

$$q_x(x,t) = \overset{\circ}{w}^1_x(x - \lambda^1 t) r^1$$

Physical meaning of eigenvectors

Eigenvectors for acoustics:

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In a simple 1-wave (propagating at speed $\lambda^1 = -c_0$),

$$\begin{bmatrix} p_x \\ u_x \end{bmatrix} = \beta(x) \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix}$$

The pressure variation is $-Z_0$ times the velocity variation.

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Similarly, in a simple 2-wave ($\lambda^2 = c_0$),

$$\begin{bmatrix} p_x \\ u_x \end{bmatrix} = \beta(x) \begin{bmatrix} Z_0 \\ 1 \end{bmatrix}$$

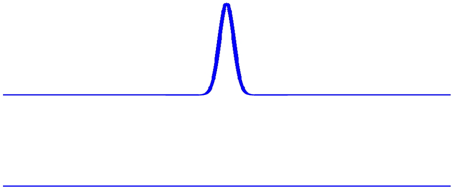
The pressure variation is Z_0 times the velocity variation.

Acoustic waves

$$q(x,0) = \begin{bmatrix} p(x) \\ 0 \end{bmatrix} = -\frac{p(x)}{2Z}\begin{bmatrix} -Z \\ 1 \end{bmatrix} + \frac{p(x)}{2Z}\begin{bmatrix} Z \\ 1 \end{bmatrix}$$

$$= w^1(x,0)r^1 + w^2(x,0)r^2$$

$$= \begin{bmatrix} p(x)/2 \\ -p(x)/(2Z) \end{bmatrix} + \begin{bmatrix} p(x)/2 \\ p(x)/(2Z) \end{bmatrix}.$$



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\begin{aligned}
q(x,0) = \begin{bmatrix} p(x) \\ 0 \end{bmatrix} \quad &= \quad -\frac{p(x)}{2Z}\begin{bmatrix} -Z \\ 1 \end{bmatrix} \quad + \quad \frac{p(x)}{2Z}\begin{bmatrix} Z \\ 1 \end{bmatrix} \\
&= \quad w^1(x,0)r^1 \quad + \quad w^2(x,0)r^2 \\
&= \begin{bmatrix} p(x)/2 \\ -p(x)/(2Z) \end{bmatrix} + \begin{bmatrix} p(x)/2 \\ p(x)/(2Z) \end{bmatrix}.
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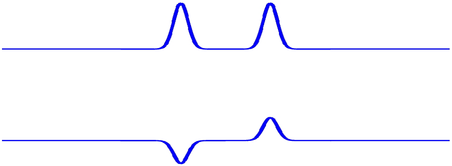

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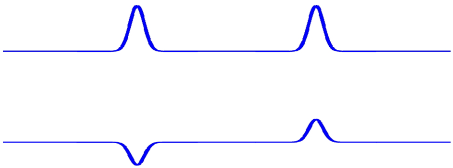

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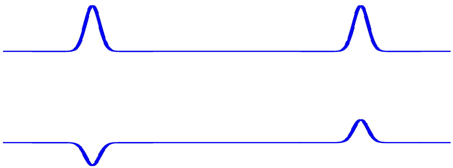

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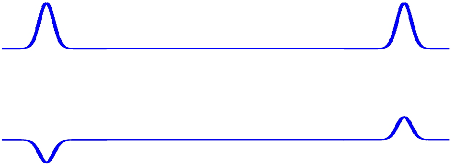

Solution by tracing back on characteristics

The general solution for acoustics:

$$\begin{aligned} q(x,t) &= w^1(x-\lambda^1 t, 0)r^1 + w^2(x-\lambda^2 t, 0)r^2 \\ &= w^1(x+c_0 t, 0)r^1 + w^2(x-c_0 t, 0)r^2 \end{aligned}$$

Recall that $w(x,0) = R^{-1}q(x,0)$, i.e.

$$w^1(x,0) = \ell^1 q(x,0), \qquad w^2(x,0) = \ell^2 q(x,0)$$

where ℓ^1 and ℓ^2 are rows of R^{-1}.

$$R^{-1} = \left[\begin{array}{c} \ell^1 \\ \ell^2 \end{array} \right]$$

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The general solution for acoustics:

$$\begin{aligned} q(x,t) &= w^1(x-\lambda^1 t, 0)r^1 + w^2(x-\lambda^2 t, 0)r^2 \\ &= w^1(x+c_0 t, 0)r^1 + w^2(x-c_0 t, 0)r^2 \end{aligned}$$

Recall that $w(x,0) = R^{-1}q(x,0)$, i.e.

$$w^1(x,0) = \ell^1 q(x,0), \qquad w^2(x,0) = \ell^2 q(x,0)$$

where ℓ^1 and ℓ^2 are rows of R^{-1}.

$$R^{-1} = \left[\begin{array}{c} \ell^1 \\ \ell^2 \end{array} \right]$$

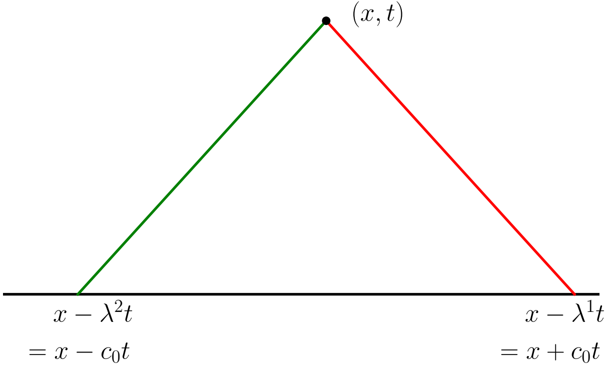
Note: ℓ^1 and ℓ^2 are left-eigenvectors of A:

$$\ell^p A = \lambda^p \ell^p \qquad \text{since} \;\; R^{-1}A = \Lambda R^{-1}.$$

Solution by tracing back on characteristics

The general solution for acoustics:

$$\begin{aligned} q(x,t) &= w^1(x-\lambda^1 t,0)r^1 + w^2(x-\lambda^2 t,0)r^2 \\ &= w^1(x+c_0 t,0)r^1 + w^2(x-c_0 t,0)r^2 \end{aligned}$$



Solution by tracing back on characteristics

The general solution for acoustics:

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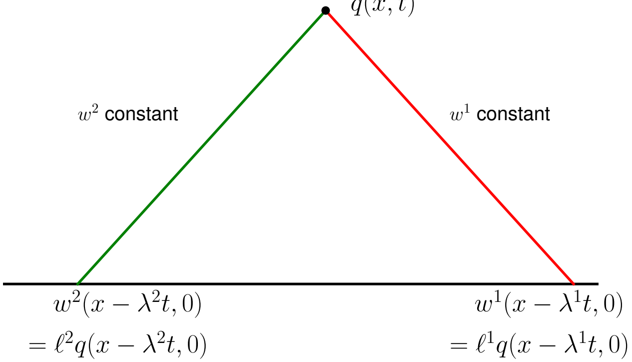

Linear acoustics

Example: Linear acoustics in a 1d gas tube,
linearized about $p = p_0,\ u = u_0$

$$q = \begin{bmatrix} p \\ u \end{bmatrix} \qquad \begin{array}{l} p(x,t) = \text{pressure perturbation} \\ u(x,t) = \text{velocity perturbation} \end{array}$$

Equations include advective transport at speed u_0:

$$\begin{aligned} p_t + u_0 p_x + K_0 u_x &= 0 \qquad \text{Change in pressure due to compressio} \\ \rho_0 u_t + p_x + u_0 u_x &= 0 \qquad \text{Newton's second law, } F = ma \end{aligned}$$

where K_0 = bulk modulus, and ρ_0 = unperturbed density.

Hyperbolic system:

$$\begin{bmatrix} p \\ u \end{bmatrix}_t + \begin{bmatrix} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_x = 0.$$

Eigenvectors for acoustics

$$A = \begin{bmatrix} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{bmatrix}$$

Eigenvectors:

$$r^1 = \begin{bmatrix} -\rho_0 c_0 \\ 1 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} \rho_0 c_0 \\ 1 \end{bmatrix}.$$

Check that $Ar^p = \lambda^p r^p$, where

$$\lambda^1 = u_0 - c_0, \qquad \lambda^2 = u_0 + c_0.$$

with $c_0 = \sqrt{K_0/\rho_0} \implies K_0 = \rho_0 c_0^2$.

Eigenvectors for acoustics

$$A = \begin{bmatrix} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{bmatrix} \qquad = u_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & K_0 \\ 1/\rho_0 & 0 \end{bmatrix}$$

Eigenvectors:

$$r^1 = \begin{bmatrix} -\rho_0 c_0 \\ 1 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} \rho_0 c_0 \\ 1 \end{bmatrix}.$$

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Note: Eigenvectors are independent of u_0.

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Note: Eigenvectors are independent of u_0.

Let $Z_0 = \rho_0 c_0 = \sqrt{K_0 \rho_0} =$ impedance.

Initial–boundary value problem (IBVP) for advection

Advection equation on finite 1D domain:

$$q_t + uq_x = 0 \qquad a < x < b, \;\; t \geq 0,$$

with initial data

$$q(x, 0) = \eta(x) \qquad a < x < b.$$

and boundary data at the inflow boundary:

If $u > 0$, need data at $x = a$:

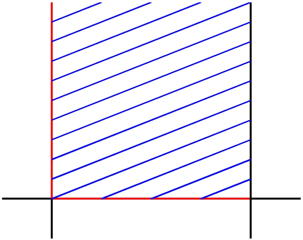
$$q(a, t) = g(t), \qquad t \geq 0,$$

If $u < 0$, need data at $x = b$:

$$q(b, t) = g(t), \qquad t \geq 0,$$

Characteristics for IBVP

In x–t plane for the case $u > 0$:



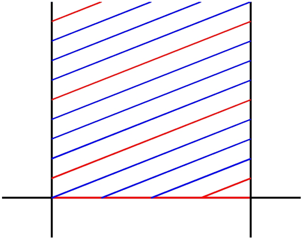
Solution:

$$q(x,t) = \begin{cases} \eta(x - ut) & \text{if } a \le x - ut \le b, \\ g((x-a)/u) & \text{otherwise} \end{cases}.$$

Periodic boundary conditions

$q(a,t) = q(b,t), \qquad t \geq 0.$

In x–t plane for the case $u > 0$:



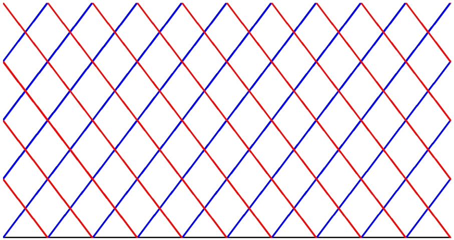
Solution:

$$q(x,t) = \eta(X_0(x,t)),$$

where $X_0(x,t) = a + \text{mod}(x - ut - a,\ b - a)$.

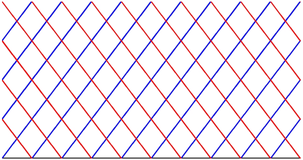
Linear acoustics — characteristics

$$
\begin{aligned}
q(x,t) &= w^1(x+ct,0)r^1 + w^2(x-ct,0)r^2 \\
&= \frac{-\overset{\circ}{p}(x+ct)}{2Z_0}\begin{bmatrix} -Z_0 \\ 1 \end{bmatrix} + \frac{\overset{\circ}{p}(x-ct)}{2Z_0}\begin{bmatrix} Z_0 \\ 1 \end{bmatrix}.
\end{aligned}
$$



For IBVP on $a < x < b$, must specify one incoming boundary condition at each side: $w^2(a,t)$ and $w^1(b,t)$

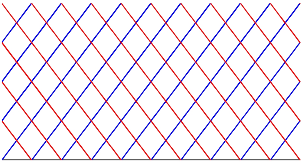
Acoustics boundary conditions



Outflow (non-reflecting, absorbing) boundary conditions:

$$w^2(a, t) = 0, \qquad w^1(b, t) = 0.$$

Acoustics boundary conditions



Outflow (non-reflecting, absorbing) boundary conditions:

$$w^2(a,t) = 0, \qquad w^1(b,t) = 0.$$

Periodic boundary conditions:

$$w^2(a,t) = w^2(b,t), \qquad w^1(b,t) = w^1(a,t),$$

or simply

$$q(a,t) = q(b,t).$$

Acoustics boundary conditions

Solid wall (reflecting) boundary conditions:

$$u(a,t) = 0, \qquad u(b,t) = 0.$$

which can be written in terms of characteristic variables as:

$$w^2(a,t) = -w^1(a,t), \qquad w^1(b,t) = -w^2(a,t)$$

since $u = w^1 + w^2$.

$$q(a,t) = w^1(a,t)\begin{bmatrix} -Z_0 \\ 1 \end{bmatrix} + w^2(a,t)\begin{bmatrix} Z_0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} p(a,t) \\ u(a,t) \end{bmatrix} = \begin{bmatrix} (-w^1(a,t) + w^2(a,t))Z_0 \\ w^1(a,t) + w^2(a,t) \end{bmatrix} = \begin{bmatrix} -2w^1(a,t)Z_0 \\ 0 \end{bmatrix}.$$

Figure 3.1

Figure 3.1 illustrates the acoustics solution with $u(x, 0) \equiv 0$.

An animation can be found in the **Clawpack Gallery**

Gallery of fvmbook applications $\longrightarrow$ Chapter 3

$\longrightarrow$ **animation of Pressure and Velocity**

Shows solution computed numerically on a fine grid, with:

- Solid wall boundary condition at the left,
- Outflow boundary condition at the right.

Finite Volume Methods for Hyperbolic Problems

Linear Systems – Riemann Problems

- Riemann problems
- Riemann problem for advection
- Riemann problem for acoustics
- Phase plane

The Riemann problem

The Riemann problem consists of the hyperbolic equation under study together with initial data of the form

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x \geq 0 \end{cases}$$

Piecewise constant with a single jump discontinuity from q_l to q_r.

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Piecewise constant with a single jump discontinuity from q_l to q_r.

The Riemann problem is fundamental to understanding

- The mathematical theory of hyperbolic problems,
- Godunov-type finite volume methods

Why? Even for nonlinear systems of conservation laws, the Riemann problem can often be solved for general q_l and q_r, and consists of a set of waves propagating at constant speeds.

The Riemann problem for advection

The Riemann problem for the advection equation $q_t + uq_x = 0$ with

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x \geq 0 \end{cases}$$

has solution

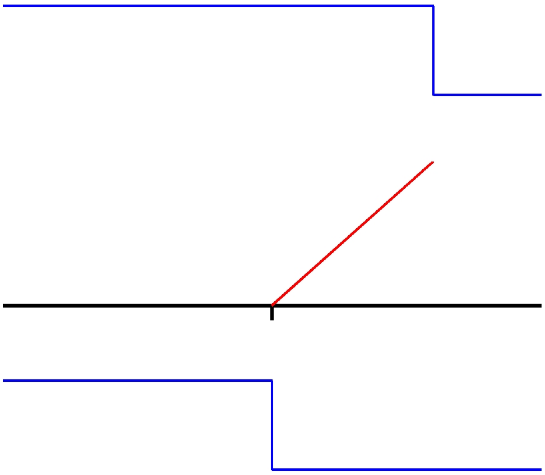
$$q(x,t) = q(x-ut,0) = \begin{cases} q_l & \text{if } x < ut \\ q_r & \text{if } x \geq ut \end{cases}$$

consisting of a single wave of strength $\mathcal{W}^1 = q_r - q_l$ propagating with speed $s^1 = u$.

Riemann solution for advection

$q(x, T)$

x–t plane



$q(x, 0)$

Discontinuous solutions

Note: The Riemann solution is not a classical solution of the PDE $q_t + uq_x = 0$, since q_t and q_x blow up at the discontinuity.

Integral form:

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = uq(x_1,t) - uq(x_2,t)$$

Integrate in time from t_1 to t_2 to obtain

$$\begin{aligned}\int_{x_1}^{x_2} q(x,t_2)\,dx - \int_{x_1}^{x_2} q(x,t_1)\,dx \\ = \int_{t_1}^{t_2} uq(x_1,t)\,dt - \int_{t_1}^{t_2} uq(x_2,t)\,dt.\end{aligned}$$

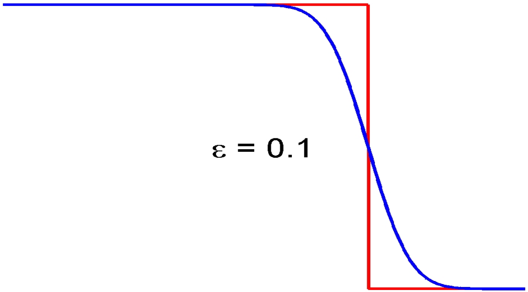
The Riemann solution satisfies the given initial conditions and this integral form for all $x_2 > x_1$ and $t_2 > t_1 \geq 0$.

Discontinuous solutions

Vanishing Viscosity solution: The Riemann solution $q(x,t)$ is the limit as $\epsilon \to 0$ of the solution $q^\epsilon(x,t)$ of the parabolic advection-diffusion equation

$$q_t + uq_x = \epsilon q_{xx}.$$

For any $\epsilon > 0$ this has a classical smooth solution:



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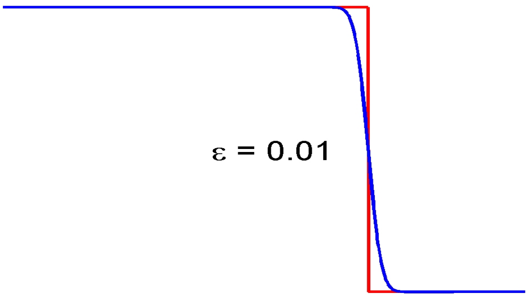

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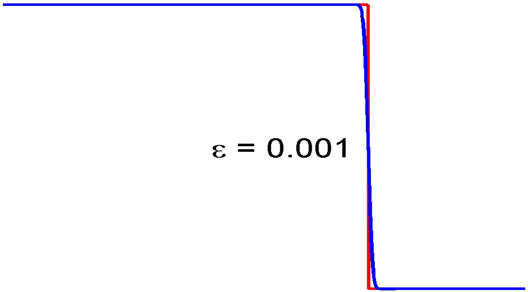

Riemann Problems and Jupyter Solutions

Theory and Approximate Solvers for Hyperbolic PDEs

David I. Ketcheson, RJL, and Mauricio del Razo

General information and links to book, Github, Binder, etc.:
bookstore.siam.org/fa16/bonus

View static version of notebooks at:
www.clawpack.org/riemann_book/html/Index.html

Eigenvectors for acoustics

$$A = \begin{bmatrix} 0 & K_0 \\ 1/\rho_0 & 0 \end{bmatrix}$$

Eigenvectors:

$$r^1 = \begin{bmatrix} -\rho_0 c_0 \\ 1 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} \rho_0 c_0 \\ 1 \end{bmatrix}.$$

Check that $Ar^p = \lambda^p r^p$, where

$$\lambda^1 = -c_0, \qquad \lambda^2 = +c_0.$$

with $c_0 = \sqrt{K_0/\rho_0} \implies K_0 = \rho_0 c_0^2$.

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Let $Z_0 = \rho_0 c_0 = \sqrt{K_0 \rho_0} =$ impedance.

Physical meaning of eigenvectors

Eigenvectors for acoustics:

$$r^1 = \begin{bmatrix} -\rho_0 c_0 \\ 1 \end{bmatrix} = \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} \rho_0 c_0 \\ 1 \end{bmatrix} = \begin{bmatrix} Z_0 \\ 1 \end{bmatrix}.$$

Consider a pure 1-wave (simple wave), at speed $\lambda^1 = -c_0$,
If $\overset{\circ}{q}(x) = \bar{q} + \overset{\circ}{w}^1(x) r^1$ then

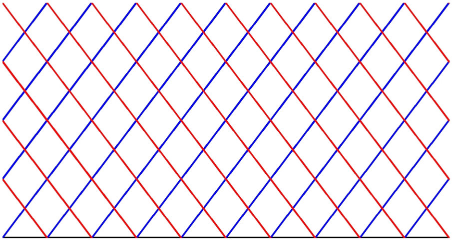
$$q(x,t) = \bar{q} + \overset{\circ}{w}^1(x - \lambda^1 t) r^1$$

Variation of q, as measured by q_x or $\Delta q = q(x + \Delta x) - q(x)$ is proportional to eigenvector r^1, e.g.

$$q_x(x,t) = \overset{\circ}{w}^1_x(x - \lambda^1 t) r^1$$

Linear acoustics — characteristics

$$q(x,t) = w^1(x+ct,0)r^1 + w^2(x-ct,0)r^2$$
$$= \frac{-\overset{\circ}{p}(x+ct)}{2Z_0}\begin{bmatrix} -Z_0 \\ 1 \end{bmatrix} + \frac{\overset{\circ}{p}(x-ct)}{2Z_0}\begin{bmatrix} Z_0 \\ 1 \end{bmatrix}.$$



For IBVP on $a < x < b$, must specify one incoming boundary condition at each side: $w^2(a,t)$ and $w^1(b,t)$

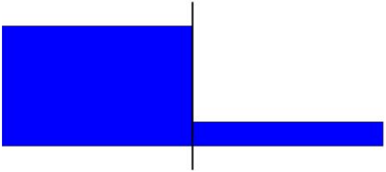
Riemann Problem

Special initial data:

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$

Example: Acoustics with bursting diaphram ($u_l = u_r = 0$)

Pressure:



Acoustic waves propagate with speeds $\pm c$.

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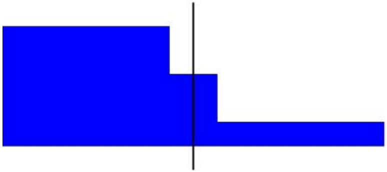

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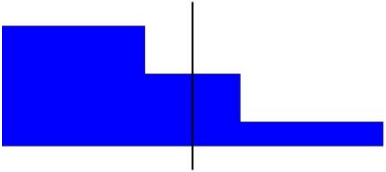

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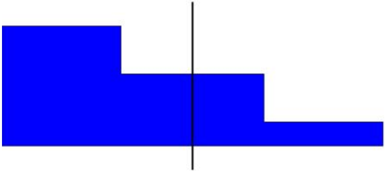
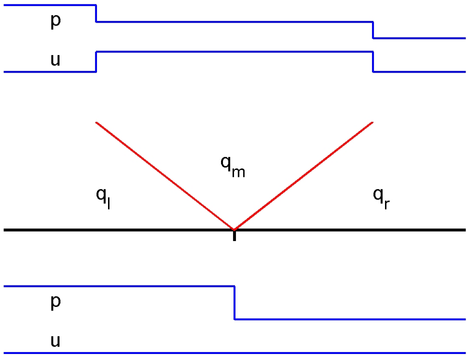


Acoustic waves propagate with speeds $\pm c$.

Riemann Problem for acoustics

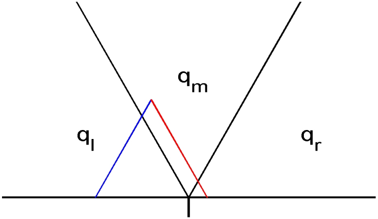
Waves propagating in x–t space:



Left-going wave $\mathcal{W}^1 = q_m - q_l$ and
right-going wave $\mathcal{W}^2 = q_r - q_m$ are eigenvectors of A.

Riemann Problem for acoustics

In x–t plane:



$$q(x,t) = w^1(x+ct,0)r^1 + w^2(x-ct,0)r^2$$

Decompose q_l and q_r into eigenvectors:

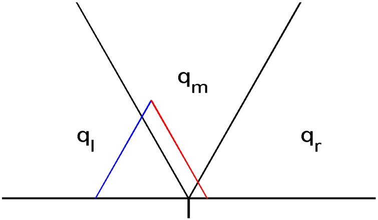
$$q_l = w_l^1 r^1 + w_l^2 r^2$$
$$q_r = w_r^1 r^1 + w_r^2 r^2$$

Then

$$q_m = w_r^1 r^1 + w_l^2 r^2$$

Riemann Problem for acoustics

In x–t plane:



Decompose $q_r - q_l$ into eigenvectors: Solve $R\alpha = \Delta q$

$$\begin{aligned} q_r - q_l &= (w_r^1 - w_r^1)r^1 + (w_r^2 - w_r^2)r^2 \\ &= \alpha^1 r^1 + \alpha^2 r^2 \qquad = \mathcal{W}^1 + \mathcal{W}^2. \end{aligned}$$

Then

$$\begin{aligned} q_m &= w_r^1 r^1 + w_l^2 r^2 \\ &= q_l + \alpha^1 r^1 \qquad = q_r - \alpha^2 r^2. \end{aligned}$$

Riemann solution for acoustics

$$r^1 = \begin{bmatrix} -\rho c \\ 1 \end{bmatrix} = \begin{bmatrix} -Z \\ 1 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} \rho c \\ 1 \end{bmatrix} = \begin{bmatrix} Z \\ 1 \end{bmatrix}.$$

Solving $R\alpha = \Delta q$ gives:

$$\alpha^1 = \frac{-\Delta p + Z\Delta u}{2Z}, \qquad \alpha^2 = \frac{\Delta p + Z\Delta u}{2Z},$$

so

$$q_m = q_l + \alpha^1 r^1 = \frac{1}{2}\begin{bmatrix} (p_l + p_r) - Z(u_r - u_l) \\ (u_l + u_r) - (p_r - p_l)/Z \end{bmatrix}.$$

Riemann solution for acoustics

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so

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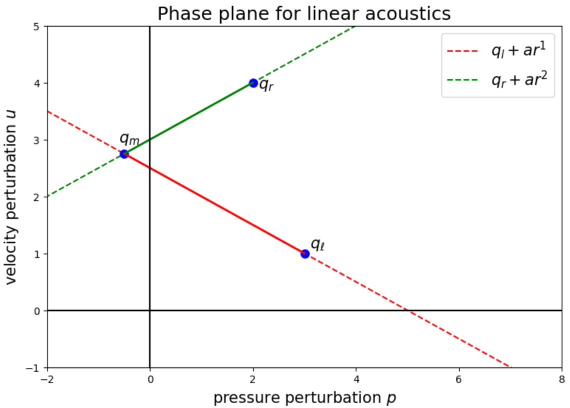
Ex: shock tube with $u_l = u_r = 0$:

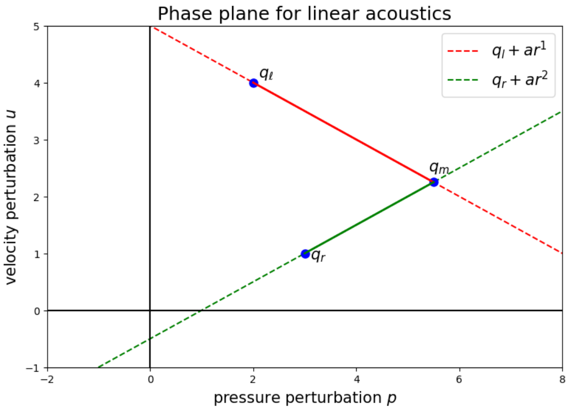
$$q_m = q_l + \alpha^1 r^1 = \frac{1}{2} \begin{bmatrix} (p_l + p_r) \\ -(p_r - p_l)/Z \end{bmatrix}.$$

Phase plane solution to Riemann problem



q_ℓ and q_m are connected by a multiple of r^1
q_m and q_r are connected by a multiple of r^2

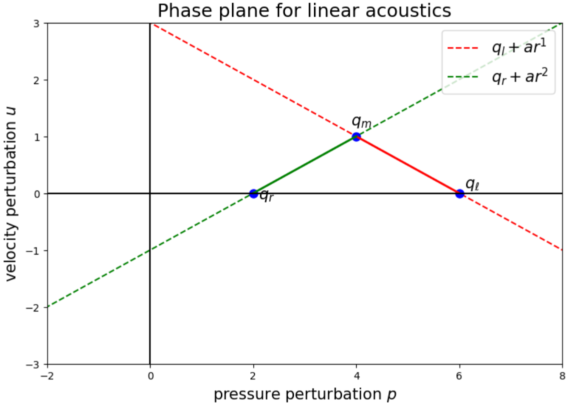
Phase plane solution to Riemann problem



q_ℓ and q_m are connected by a multiple of r^1
q_m and q_r are connected by a multiple of r^2

Note that swapping q_ℓ and q_r changes the solution!

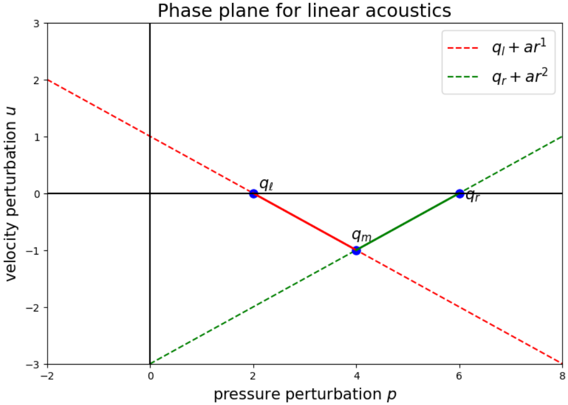
Phase plane solution to Riemann problem



"Shock tube" solution with $u_\ell = u_r = 0$.

q_ℓ and q_m are connected by a multiple of r^1
q_m and q_r are connected by a multiple of r^2

Phase plane solution to Riemann problem



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Riemann Problems and Jupyter Solutions

Theory and Approximate Solvers for Hyperbolic PDEs

David I. Ketcheson, RJL, and Mauricio del Razo

General information and links to book, Github, Binder, etc.:

bookstore.siam.org/fa16/bonus

View static version of notebooks at:

www.clawpack.org/riemann_book/html/Index.html

Riemann solution for a linear system

Linear hyperbolic system: $q_t + Aq_x = 0$ with $A = R\Lambda R^{-1}$.
General Riemann problem data $q_l,\ q_r \in \mathbb{R}^m$.

Decompose jump in q into eigenvectors:

$$q_r - q_l = \sum_{p=1}^{m} \alpha^p r^p$$

Note: the vector α of eigen-coefficients is

$$\alpha = R^{-1}(q_r - q_l) = R^{-1}q_r - R^{-1}q_l = w_r - w_l.$$

Riemann solution consists of m waves $\mathcal{W}^p \in \mathbb{R}^m$:

$$\mathcal{W}^p = \alpha^p r^p, \qquad \text{propagating with speed } \ s^p = \lambda^p.$$

Phase space

For a system of m equations, phase space is m-dimensional.

Solving the Riemann problem finds a path from q_ℓ to q_r that generally has m segments, each in the direction of an eigenvector (for a linear system; curves more generally).

If $\lambda^1 \leq \lambda^2 \leq \cdots \leq \lambda^m$, then first segment from q_ℓ to $q_\ell + \alpha^1 r^1$, next segment goes to $q_\ell + \alpha^1 r^1 + \alpha^2 r^2$, etc.

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Unique such path provided eigenvectors are linearly independent. $q_\ell + \alpha^1 r^1 + \alpha^2 r^2 + \cdots + \alpha^m r^m = q_r$.

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Visualization is most useful when $m = 2$ (phase plane).

But sometimes illuminating to project phase space onto a two-dimensional plane.

Finite Volume Methods for Hyperbolic Problems

Linear Systems – Nonhyperbolic Cases

- Acoustics equations if $K_0 < 0$ (eigenvalues complex)
- Acoustics equations if $K_0 = 0$ (not diagonalizable)
- Coupled advection equations

Linear acoustics

Example: Linear acoustics in a 1d gas tube

$$q = \begin{bmatrix} p \\ u \end{bmatrix} \qquad \begin{array}{l} p(x,t) = \text{pressure perturbation} \\ u(x,t) = \text{velocity} \end{array}$$

Equations:

$$\begin{aligned} p_t + K_0 u_x &= 0 \qquad && \text{Change in pressure due to compression} \\ \rho_0 u_t + p_x &= 0 && \text{Newton's second law, } F = ma \end{aligned}$$

where $K_0 =$ bulk modulus, and $\rho_0 =$ unperturbed density.

Hyperbolic system:

$$\begin{bmatrix} p \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & K_0 \\ 1/\rho_0 & 0 \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_x = 0.$$

Acoustics equations when hyperbolicity fails

Eigenvalues are $\pm\sqrt{K_0/\rho}$ (wave speeds),
real and distinct provided $K_0 > 0$ and $\rho_0 > 0$.

Now suppose $K_0 < 0$. Then eigenvalues pure imaginary.

Recall $K_0 = \rho_0 P'(\rho_0)$ from linearization.

Physically we expect pressure to increase as density increases.

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Physically we expect pressure to increase as density increases.

Otherwise, mass flowing in leads to decreased pressure and hence greater mass flow, with mass growing exponentially without bound.

Second-order PDE form of acoustics

$$\begin{aligned} p_t + K_0 u_x = 0 \quad &\Longrightarrow \; p_{tt} = -K_0 u_{xt} \\ u_t + (1/\rho_0) u_x = 0 \quad &\Longrightarrow \; u_{tx} = -(1/\rho_0) p_{xx} \end{aligned}$$

Combining gives

$$p_{tt} = c_0^2 p_{xx}$$

with $c_0^2 = K_0/\rho_0$. This is the wave equation provided $c_0^2 > 0$.

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has positive coefficients and is an elliptic equation.

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has positive coefficients and is an elliptic equation.

To solve for $x_1 \le x \le x_2$ and $t_0 \le t \le T$, the elliptic equation requires BCs on all four sides, including at $t = T$.

The initial-boundary value problem is ill-posed.

Acoustics equations when hyperbolicity fails

Eigenvalues are $\pm\sqrt{K_0/\rho_0}$ (wave speeds).

Now suppose $K_0 = 0$. Then eigenvalues are $\lambda^1 = \lambda^2 = 0$.
Wave speeds are 0, not necessarily a problem.

But the matrix is a Jordan block, not diagonalizable:

$$A = \begin{bmatrix} 0 & 0 \\ 1/\rho_0 & 0 \end{bmatrix}.$$

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$p(x,t) = \overset{\circ}{p}(x)$ for all time
u_t can grow arbitrarily quickly depending on $\overset{\circ}{p}_x$. Ill-posed.

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In particular, Riemann problem can have infinite p_x at origin.

Acoustics equations in limit $K_0 = K \to 0$

$$A = \begin{bmatrix} 0 & K \\ 1/\rho & 0 \end{bmatrix}, \quad \text{Eigenvalues: } \lambda = \pm\sqrt{K/\rho} \to 0.$$

Impedance $Z = \sqrt{K\rho} \to 0$.

$$q_m = q_l + \alpha^1 r^1 = \frac{1}{2} \begin{bmatrix} (p_l + p_r) - Z(u_r - u_l) \\ (u_l + u_r) - (p_r - p_l)/Z \end{bmatrix}.$$

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So if $p_r \neq p_l$, then $u_m \to \infty$ as $K \to 0$

Another non-diagonalizable example (Sec. 16.3.1)

$$
\begin{aligned}
q_t^1 + uq_x^1 + \beta q_x^2 &= 0, \\
q_t^2 \qquad\qquad + vq_x^2 &= 0,
\end{aligned}
$$

has

$$A = \begin{bmatrix} u & \beta \\ 0 & v \end{bmatrix}.$$

Eigenvalues and eigenvectors (if $v \leq u$ and $\beta \neq 0$):

$$\lambda^1 = v, \qquad \lambda^2 = u,$$

$$r^1 = \begin{bmatrix} \beta \\ v - u \end{bmatrix}, \qquad r^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

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r^1 = \begin{bmatrix} \beta \\ v - u \end{bmatrix}, \qquad r^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
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As $u \to v$ the eigenvector r^1 becomes colinear with r^2 and the eigenvector matrix R becomes singular (unless $\beta = 0$).

Introduction to Finite Volume Methods

- Comparsion to finite differences
- Conservation form, importance for shocks
- Godunov's method, wave propagation view
- Upwind for advection
- REA Algorithm
- Godunov applied to acoustics

Finite difference method

Based on point-wise approximations:

$$Q_i^n \approx q(x_i, t_n), \quad \text{with} \;\; x_i = i\Delta x, \;\; t_n = n\Delta t.$$

Approximate derivatives by finite differences.

Ex: Upwind method for advection equation if $u > 0$:

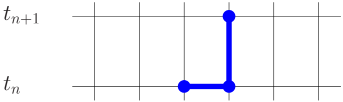
$$q_t + uq_x = 0$$

$$\frac{Q_i^{n+1} - Q_i^n}{\Delta t} + u\left(\frac{Q_i^n - Q_{i-1}^n}{\Delta x}\right) = 0$$

or

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} u(Q_i^n - Q_{i-1}^n).$$

Stencil:



Finite differences vs. finite volumes

Finite difference Methods

- Pointwise values $Q_i^n \approx q(x_i, t_n)$
- Approximate derivatives by finite differences
- Assumes smoothness

Finite volume Methods

- Approximate cell averages: $Q_i^n \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t_n)\, dx$
- Integral form of conservation law,

$$\frac{\partial}{\partial t} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)\, dx \quad = \quad f(q(x_{i-1/2}, t)) - f(q(x_{i+1/2}, t))$$

leads to conservation law $q_t + f_x = 0$ but also directly to numerical method.

Finite volume method

$Q_i^n \approx \frac{1}{h}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t_n)\,dx$

Integral form: $$\frac{\partial}{\partial t}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)\,dx = f(q(x_{i-1/2},t)) - f(q(x_{i+1/2},t))$$

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Integrate from t_n to t_{n+1} $\implies$

$$\int q(x,t_{n+1})\,dx = \int q(x,t_n)\,dx + \int_{t_n}^{t_{n+1}} f(q(x_{i-1/2},t)) - f(q(x_{i+1/2},t))\,dt$$

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Integrate from t_n to t_{n+1} $\Longrightarrow$

$$\int q(x,t_{n+1})\,dx = \int q(x,t_n)\,dx + \int_{t_n}^{t_{n+1}} f(q(x_{i-1/2},t)) - f(q(x_{i+1/2},t))\,dt$$

$$\begin{aligned}\frac{1}{\Delta x}\int q(x,t_{n+1})\,dx = \frac{1}{\Delta x}\int q(x,t_n)\,dx \\ - \frac{\Delta t}{\Delta x}\left(\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q(x_{i+1/2},t)) - f(q(x_{i-1/2},t))\,dt\right)\end{aligned}$$

Numerical method: $$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

Numerical flux: $F_{i-1/2}^n \approx \dfrac{1}{\Delta t}\displaystyle\int_{t_n}^{t_{n+1}} f(q(x_{i-1/2},t))\,dt.$

Upwind for advection as a finite volume method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

$$F_{i-1/2} \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} uq(x_{i-1/2}, t)\, dt.$$

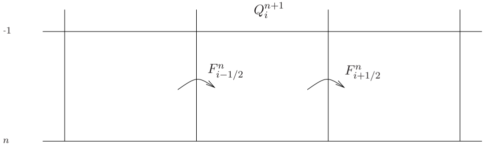
For $u > 0$:

$$F_{i-1/2}^n = uQ_{i-1}^n, \qquad F_{i+1/2}^n = uQ_i^n$$

so

$$\begin{aligned} Q_i^{n+1} &= Q_i^n + \frac{\Delta t(uQ_{i-1}^n - uQ_i^n)}{\Delta x} \\ &= Q_i^n - \frac{\Delta t u}{\Delta x}(Q_i^n - Q_{i-1}^n) \end{aligned}$$

Stencil:
(x-t plane)



Upwind method for advection

Flux: $f(q) = uq$

Numerical flux: $F^n_{i-1/2} \approx \dfrac{1}{\Delta t} \displaystyle\int_{t_n}^{t_{n+1}} f(q(x_{i-1/2}, t))\, dt$.

If $q(x, t_n)$ is piecewise constant in each cell, then

$$F^n_{i-1/2} = \begin{cases} uQ^n_{i-1} & \text{if } u > 0, \\ uQ^n_i & \text{if } u < 0. \end{cases}$$

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This gives the upwind method:

$$Q^{n+1}_i = Q^n_i - \frac{u\Delta t}{\Delta x}(Q^n_i - Q^n_{i-1}) \qquad \text{if } u > 0$$

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Conservation form

The method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

is in conservation form.

The total mass is conserved up to fluxes at the boundaries:

$$\Delta x \sum_i Q_i^{n+1} = \Delta x \sum_i Q_i^n - \Delta t(F_{+\infty} - F_{-\infty}).$$

Note: an isolated shock must travel at the right speed!

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} q(x,t)\,dx = F(x_1) - F(x_2).$$

Nonlinear scalar conservation laws

Burgers' equation: $u_t + \left(\frac{1}{2}u^2\right)_x = 0$.

Quasilinear form: $u_t + uu_x = 0$.

These are equivalent for smooth solutions, not for shocks!

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Upwind methods for $u > 0$:

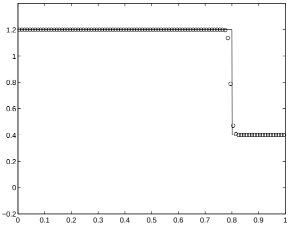
Conservative: $U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}((U_i^n)^2 - (U_{i-1}^n)^2)\right)$

Quasilinear: $U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}U_i^n(U_i^n - U_{i-1}^n)$.

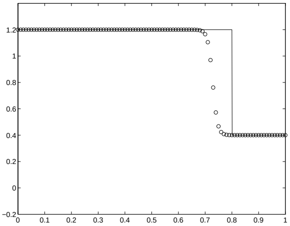
Ok for smooth solutions, not for shocks!

Importance of conservation form

Solution to Burgers' equation using conservative upwind:



Solution to Burgers' equation using quasilinear upwind:



Weak solutions depend on the conservation law

The conservation laws

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0$$

and

$$(u^2)_t + \left(\frac{2}{3}u^3\right)_x = 0 \qquad \text{i.e.} \qquad q = u^2,\ f(q) = \frac{2}{3}q^{3/2}$$

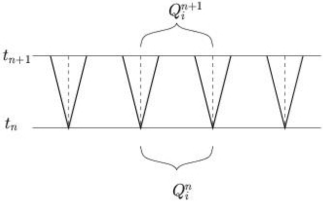
both have the same quasilinear form

$$u_t + uu_x = 0$$

but have different weak solutions,

different shock speeds!

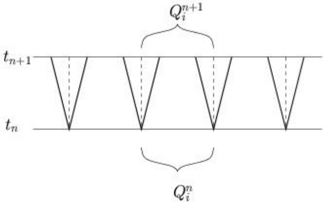
Godunov's Method for $q_t + f(q)_x = 0$



1. Solve Riemann problems at all interfaces, yielding waves $\mathcal{W}^p_{i-1/2}$ and speeds $s^p_{i-1/2}$, for $p = 1,\ 2,\ \ldots,\ m$.

Riemann problem: Original equation with piecewise constant data.

Godunov's Method for $q_t + f(q)_x = 0$



Then either:

1. Compute new cell averages by integrating over cell at t_{n+1},

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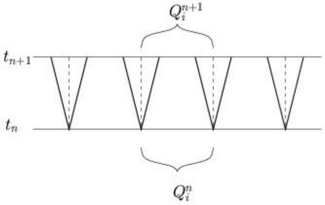

Then either:

1. Compute new cell averages by integrating over cell at t_{n+1},
2. Compute fluxes at interfaces and flux-difference:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[F_{i+1/2}^n - F_{i-1/2}^n]$$

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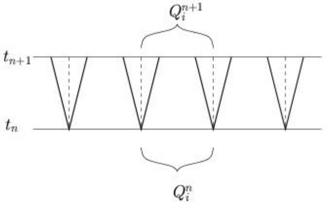

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3. Update cell averages by contributions from all waves entering cell:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}]$$

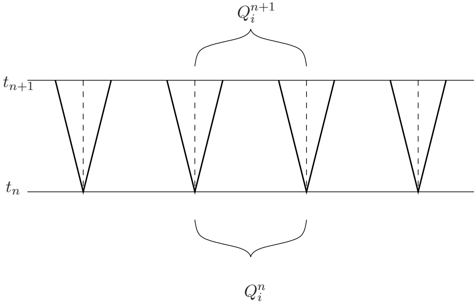
where $\mathcal{A}^\pm \Delta Q_{i-1/2} = \sum_{i=1}^{m} (s_{i-1/2}^p)^\pm \mathcal{W}_{i-1/2}^p$.

Godunov's method with flux differencing

Q_i^n defines a piecewise constant function

$$\tilde{q}^n(x, t_n) = Q_i^n \text{ for } x_{i-1/2} < x < x_{i+1/2}$$

Discontinuities at cell interfaces $\implies$ Riemann problems.



$\tilde{q}^n(x_{i-1/2}, t) \equiv q^{\Downarrow}(Q_{i-1}, Q_i)$ for $t > t_n$.

$$F_{i-1/2}^n = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(q^{\Downarrow}(Q_{i-1}^n, Q_i^n))\, dt = f(q^{\Downarrow}(Q_{i-1}^n, Q_i^n)).$$

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$$Q^{n+1}_i = Q^n_i - \frac{u\Delta t}{\Delta x}(Q^n_{i+1} - Q^n_i) \qquad \text{if } u < 0$$

First-order REA Algorithm

1. Reconstruct a piecewise constant function $\tilde{q}^n(x, t_n)$ defined for all x, from the cell averages Q_i^n.

$$\tilde{q}^n(x, t_n) = Q_i^n \quad \text{for all } x \in \mathcal{C}_i.$$

2. Evolve the hyperbolic equation exactly (or approximately) with this initial data to obtain $\tilde{q}^n(x, t_{n+1})$ a time Δt later.

3. Average this function over each grid cell to obtain new cell averages

$$Q_i^{n+1} = \frac{1}{\Delta x} \int_{\mathcal{C}_i} \tilde{q}^n(x, t_{n+1})\, dx.$$

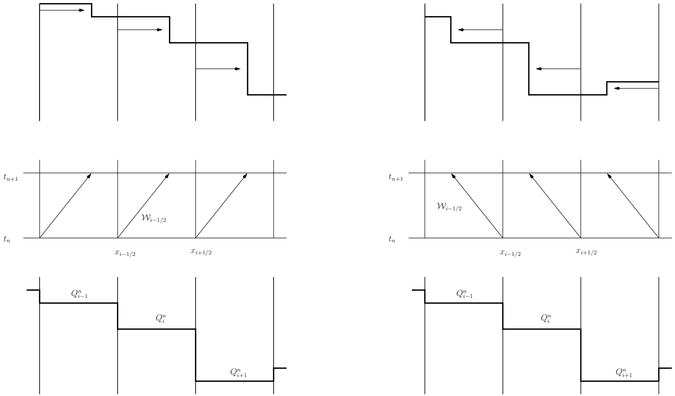
Godunov's method for advection

Q_i^n defines a piecewise constant function

$$\tilde{q}^n(x, t_n) = Q_i^n \text{ for } x_{i-1/2} < x < x_{i+1/2}$$

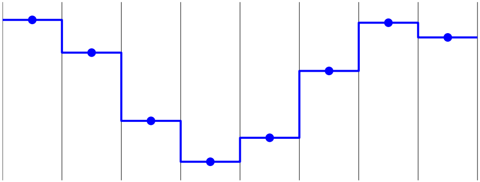
Discontinuities at cell interfaces $\implies$ Riemann problems.

$u > 0$ $\qquad\qquad$ $u < 0$

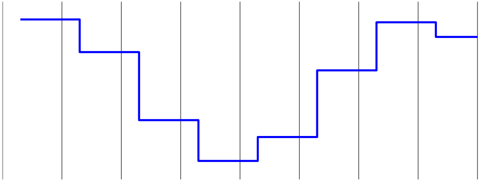


First-order REA Algorithm

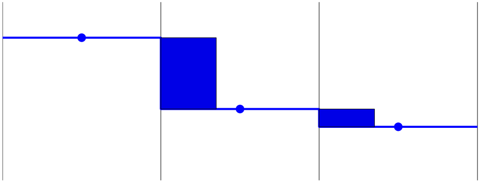
Cell averages and piecewise constant reconstruction:



After evolution:



Cell update



The cell average is modified by

$$\frac{u\Delta t \cdot (Q_{i-1}^n - Q_i^n)}{\Delta x}$$

So we obtain the upwind method

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n).$$

Wave propagation form of cell update

The cell average is modified by

$$\frac{u\Delta t \cdot (Q_{i-1}^n - Q_i^n)}{\Delta x} = -\frac{\Delta t}{\Delta x} s\mathcal{W}_{i-1/2}$$

where $\mathcal{W}_{i-1/2} = (Q_i^n - Q_{i-1}^n)$ is the wave strength and $s = u$ is the wave speed.

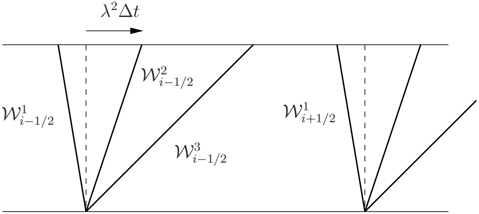
The general upwind method for $u < 0$ or $u > 0$:

$$\begin{aligned} Q_i^{n+1} &= Q_i^n - \frac{\Delta t}{\Delta x}\left[u^+(Q_i^n - Q_{i-1}^n) + u^-(Q_{i+1}^n - Q_i^n)\right] \\ &= \frac{\Delta t}{\Delta x}\left[s^+\mathcal{W}_{i-1/2} + s^-\mathcal{W}_{i-1/2}\right] \end{aligned}$$

where $u^+ = \max(u, 0), \;\; u^- = \min(u, 0)$.
This is the wave propagation form of upwind.

Wave-propagation viewpoint

For linear system $q_t + Aq_x = 0$, the Riemann solution consists of waves $\mathcal{W}^p$ propagating at constant speed λ^p.



$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha^p_{i-1/2} r^p \equiv \sum_{p=1}^{m} \mathcal{W}^p_{i-1/2}.$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\lambda^2 \mathcal{W}^2_{i-1/2} + \lambda^3 \mathcal{W}^3_{i-1/2} + \lambda^1 \mathcal{W}^1_{i+1/2}\right].$$

Godunov (upwind) for a linear system

$q_t + Aq_x = 0$ where $A = R\Lambda R^{-1}$. Define the matrices

$$\Lambda^+ = \begin{bmatrix} (\lambda^1)^+ & & & \\ & (\lambda^2)^+ & & \\ & & \ddots & \\ & & & (\lambda^m)^+ \end{bmatrix}, \qquad \Lambda^- = \begin{bmatrix} (\lambda^1)^- & & & \\ & (\lambda^2)^- & & \\ & & \ddots & \\ & & & (\lambda^m)^- \end{bmatrix}.$$

and

$$A^+ = R\Lambda^+ R^{-1}, \qquad \text{and} \qquad A^- = R\Lambda^- R^{-1}.$$

Note:

$$A^+ + A^- = R(\Lambda^+ + \Lambda^-)R^{-1} = R\Lambda R^{-1} = A.$$

Then Godunov's method becomes

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[A^+(Q_i - Q_{i-1}) + A^-(Q_{i+1} - Q_i)\right].$$

Matrix splitting for upwind method

For $q_t + Aq_x = 0$, the upwind method (Godunov) is:

$$
\begin{aligned}
Q_i^{n+1} &= Q_i^n + \frac{\Delta t}{\Delta x}\left[\sum_{p=1}^{m} (\lambda^p)^+ \alpha_{i-1/2}^p r^p + \sum_{p=1}^{m} (\lambda^p)^- \alpha_{i+1/2}^p r^p\right] \\
&= Q_i^n + \frac{\Delta t}{\Delta x}\left[A^+ \Delta Q_{i-1/2} + A^- \Delta Q_{i+1/2}\right] \\
&= Q_i^n + \frac{\Delta t}{\Delta x}\left[A^+ (Q_i^n - Q_{i-1}^n) + A^- (Q_{i+1}^n - Q_i^n)\right]
\end{aligned}
$$

Matrix splitting for upwind method

For $q_t + Aq_x = 0$, the upwind method (Godunov) is:

$$Q_i^{n+1} = Q_i^n + \frac{\Delta t}{\Delta x}\left[\sum_{p=1}^{m}(\lambda^p)^+\alpha_{i-1/2}^p r^p + \sum_{p=1}^{m}(\lambda^p)^-\alpha_{i+1/2}^p r^p\right]$$

$$= Q_i^n + \frac{\Delta t}{\Delta x}\left[A^+\Delta Q_{i-1/2} + A^-\Delta Q_{i+1/2}\right]$$

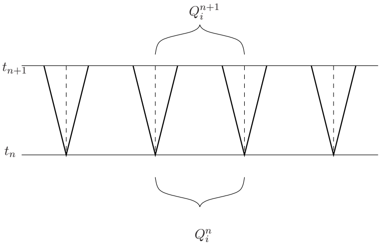
$$= Q_i^n + \frac{\Delta t}{\Delta x}\left[A^+(Q_i^n - Q_{i-1}^n) + A^-(Q_{i+1}^n - Q_i^n)\right]$$

Natural generalization of upwind to a system.

If all eigenvalues are positive, then $A^+ = A$ and $A^- = 0$,

If all eigenvalues are negative, then $A^+ = 0$ and $A^- = A$.

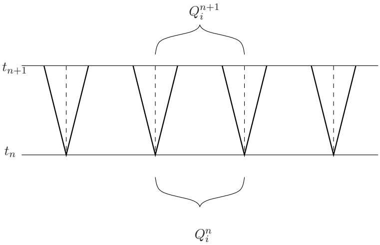
Godunov (upwind) on acoustics



Data at time t_n : $\tilde{q}^n(x, t_n) = Q_i^n$ for $x_{i-1/2} < x < x_{i+1/2}$
Solving Riemann problems for small Δt gives solution:

$$\tilde{q}^n(x, t_{n+1}) = \begin{cases} Q^*_{i-1/2} & \text{if } x_{i-1/2} - c\Delta t < x < x_{i-1/2} + c\Delta t, \\ Q_i^n & \text{if } x_{i-1/2} + c\Delta t < x < x_{i+1/2} - c\Delta t, \\ Q^*_{i+1/2} & \text{if } x_{i+1/2} - c\Delta t < x < x_{i+1/2} + c\Delta t, \end{cases}$$

Godunov (upwind) on acoustics



Data at time t_n : $\tilde{q}^n(x, t_n) = Q_i^n$ for $x_{i-1/2} < x < x_{i+1/2}$
Solving Riemann problems for small Δt gives solution:

$$\tilde{q}^n(x, t_{n+1}) = \begin{cases} Q^*_{i-1/2} & \text{if } x_{i-1/2} - c\Delta t < x < x_{i-1/2} + c\Delta t, \\ Q_i^n & \text{if } x_{i-1/2} + c\Delta t < x < x_{i+1/2} - c\Delta t, \\ Q^*_{i+1/2} & \text{if } x_{i+1/2} - c\Delta t < x < x_{i+1/2} + c\Delta t, \end{cases}$$

So computing cell average gives:

$$Q_i^{n+1} = \frac{1}{\Delta x}\left[c\Delta t Q^*_{i-1/2} + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t Q^*_{i+1/2}\right].$$

Godunov (upwind) on acoustics

$$Q_i^{n+1} = \frac{1}{\Delta x}\left[c\Delta t Q^*_{i-1/2} + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t Q^*_{i+1/2}\right].$$

Solve Riemann problems:

$$Q_i^n - Q_{i-1}^n = \Delta Q_{i-1/2} = \mathcal{W}^1_{i-1/2} + \mathcal{W}^2_{i-1/2} = \alpha^1_{i-1/2} r^1 + \alpha^2_{i-1/2} r^2,$$
$$Q_{i+1}^n - Q_i^n = \Delta Q_{i+1/2} = \mathcal{W}^1_{i+1/2} + \mathcal{W}^2_{i+1/2} = \alpha^1_{i+1/2} r^1 + \alpha^2_{i+1/2} r^2,$$

Godunov (upwind) on acoustics

$$Q_i^{n+1} = \frac{1}{\Delta x}\left[c\Delta t Q^*_{i-1/2} + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t Q^*_{i+1/2}\right].$$

Solve Riemann problems:

$$Q_i^n - Q_{i-1}^n = \Delta Q_{i-1/2} = \mathcal{W}^1_{i-1/2} + \mathcal{W}^2_{i-1/2} = \alpha^1_{i-1/2}r^1 + \alpha^2_{i-1/2}r^2,$$
$$Q_{i+1}^n - Q_i^n = \Delta Q_{i+1/2} = \mathcal{W}^1_{i+1/2} + \mathcal{W}^2_{i+1/2} = \alpha^1_{i+1/2}r^1 + \alpha^2_{i+1/2}r^2,$$

The intermediate states are:

$$Q^*_{i-1/2} = Q_i^n - \mathcal{W}^2_{i-1/2}, \qquad Q^*_{i+1/2} = Q_i^n + \mathcal{W}^1_{i+1/2},$$

Godunov (upwind) on acoustics

$$Q_i^{n+1} = \frac{1}{\Delta x}\left[c\Delta t Q^*_{i-1/2} + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t Q^*_{i+1/2}\right].$$

Solve Riemann problems:

$$Q_i^n - Q_{i-1}^n = \Delta Q_{i-1/2} = \mathcal{W}^1_{i-1/2} + \mathcal{W}^2_{i-1/2} = \alpha^1_{i-1/2} r^1 + \alpha^2_{i-1/2} r^2,$$

$$Q_{i+1}^n - Q_i^n = \Delta Q_{i+1/2} = \mathcal{W}^1_{i+1/2} + \mathcal{W}^2_{i+1/2} = \alpha^1_{i+1/2} r^1 + \alpha^2_{i+1/2} r^2,$$

The intermediate states are:

$$Q^*_{i-1/2} = Q_i^n - \mathcal{W}^2_{i-1/2}, \qquad Q^*_{i+1/2} = Q_i^n + \mathcal{W}^1_{i+1/2},$$

So,

$$\begin{aligned} Q_i^{n+1} &= \frac{1}{\Delta x}\left[c\Delta t(Q_i^n - \mathcal{W}^2_{i-1/2}) + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t(Q_i^n + \mathcal{W}^1_{i+1/2})\right] \\ &= Q_i^n - \frac{c\Delta t}{\Delta x}\mathcal{W}^2_{i-1/2} + \frac{c\Delta t}{\Delta x}\mathcal{W}^1_{i+1/2}. \end{aligned}$$

Godunov (upwind) on acoustics

$$
\begin{aligned}
Q_i^{n+1} &= \frac{1}{\Delta x}\left[c\Delta t Q^*_{i-1/2} + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t Q^*_{i+1/2}\right] \\
&= \frac{1}{\Delta x}\left[c\Delta t(Q_i^n - \mathcal{W}^2_{i-1/2}) + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t(Q_i^n + \mathcal{W}^1_{i+1/2})\right] \\
&= Q_i^n - \frac{c\Delta t}{\Delta x}\mathcal{W}^2_{i-1/2} + \frac{c\Delta t}{\Delta x}\mathcal{W}^1_{i+1/2} \\
&= Q_i^n - \frac{\Delta t}{\Delta x}(c\mathcal{W}^2_{i-1/2} + (-c)\mathcal{W}^1_{i+1/2}).
\end{aligned}
$$

Godunov (upwind) on acoustics

$$
\begin{aligned}
Q_i^{n+1} &= \frac{1}{\Delta x}\left[c\Delta t Q^*_{i-1/2} + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t Q^*_{i+1/2}\right] \\
&= \frac{1}{\Delta x}\left[c\Delta t(Q_i^n - \mathcal{W}^2_{i-1/2}) + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t(Q_i^n + \mathcal{W}^1_{i+1/2})\right] \\
&= Q_i^n - \frac{c\Delta t}{\Delta x}\mathcal{W}^2_{i-1/2} + \frac{c\Delta t}{\Delta x}\mathcal{W}^1_{i+1/2} \\
&= Q_i^n - \frac{\Delta t}{\Delta x}(c\mathcal{W}^2_{i-1/2} + (-c)\mathcal{W}^1_{i+1/2}).
\end{aligned}
$$

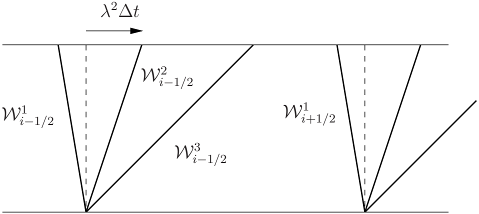
General form for linear system with m equations:

$$
\begin{aligned}
Q_i^{n+1} &= Q_i^n - \frac{\Delta t}{\Delta x}\left[\sum_{p:\lambda^p>0}\lambda^p\mathcal{W}^p_{i-1/2} + \sum_{p:\lambda^p<0}\lambda^p\mathcal{W}^p_{i+1/2}\right] \\
&= Q_i^n - \frac{\Delta t}{\Delta x}\left[\sum_{m=1}^{p}(\lambda^p)^+\mathcal{W}^p_{i-1/2} + \sum_{m=1}^{p}(\lambda^p)^-\mathcal{W}^p_{i+1/2}\right]
\end{aligned}
$$

Wave-propagation viewpoint

For linear system $q_t + Aq_x = 0$, the Riemann solution consists of

waves $\mathcal{W}^p$ propagating at constant speed λ^p.



$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha^p_{i-1/2} r^p \equiv \sum_{p=1}^{m} \mathcal{W}^p_{i-1/2}.$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\lambda^2 \mathcal{W}^2_{i-1/2} + \lambda^3 \mathcal{W}^3_{i-1/2} + \lambda^1 \mathcal{W}^1_{i+1/2}\right].$$

Accuracy, Consistency, Stability, CFL Condition

- Order of accuracy, local and global error
- Consistent numerical flux functions
- Stability
- CFL Condition

For more details see e.g. Chapter 10 of
Finite Difference Methods for ODEs and PDEs

Finite differences vs. finite volumes

Finite difference Methods

- Pointwise values $Q_i^n \approx q(x_i, t_n)$
- Approximate derivatives by finite differences
- Assumes smoothness

Finite volume Methods

- Approximate cell averages: $Q_i^n \approx \dfrac{1}{\Delta x} \displaystyle\int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t_n)\, dx$
- Integral form of conservation law,

$$\frac{\partial}{\partial t} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)\, dx \quad = \quad f(q(x_{i-1/2},t)) - f(q(x_{i+1/2},t))$$

leads to conservation law $q_t + f_x = 0$ but also directly to numerical method.

Order of Accuracy — upwind method

Upwind method for advection $q_t + uq_x = 0$ with $u > 0$:

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n)$$

Written in form that mimics PDE:

$$\left(\frac{Q_i^{n+1} - Q_i^n}{\Delta t}\right) + u\left(\frac{Q_i^n - Q_{i-1}^n}{\Delta x}\right) = 0$$

Order of Accuracy — upwind method

Upwind method for advection $q_t + uq_x = 0$ with $u > 0$:

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$$\left(\frac{Q_i^{n+1} - Q_i^n}{\Delta t}\right) + u\left(\frac{Q_i^n - Q_{i-1}^n}{\Delta x}\right) = 0$$

Local truncation error:

Insert true solution $u(x,t)$ into difference equation

$$\tau(x,t) = \left(\frac{q(x_i, t_{n+1}) - q(x_i, t_n)}{\Delta t}\right) + u\left(\frac{q(x_i, t_n) - q(x_{i-1}, t_n)}{\Delta x}\right)$$

Order of Accuracy — upwind method

Upwind method for advection $q_t + uq_x = 0$ with $u > 0$:

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n)$$

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Local truncation error:

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$$\tau(x,t) = \left(\frac{q(x_i,t_{n+1}) - q(x_i,t_n)}{\Delta t}\right) + u\left(\frac{q(x_i,t_n) - q(x_{i-1},t_n)}{\Delta x}\right)$$

Assume smoothness and expand in Taylor series:

$$q(x_i,t_{n+1}) = q(x_i,t_n) + \Delta t q_t(x_i,t_n) + \frac{1}{2}\Delta t^2 q_{tt}(x_i,t_n) + \cdots$$

$$q(x_{i-1},t_n) = q(x_i,t_n) - \Delta x q_x(x_i,t_n) + \frac{1}{2}\Delta x^2 q_{xx}(x_i,t_n) + \cdots$$

Order of Accuracy — upwind method

Insert Taylor series into

$$\tau(x,t) = \left(\frac{q(x_i,t_{n+1}) - q(x_i,t_n)}{\Delta t}\right) + u\left(\frac{q(x_i,t_n) - q(x_{i-1},t_n)}{\Delta x}\right)$$

gives (with everything evaluated at (x_i,t_n)):

$$\begin{aligned}\tau(x_i,t_n) &= \left(\frac{\Delta t q_t + \frac{1}{2}\Delta t^2 q_{tt} + \cdots}{\Delta t}\right) + u\left(\frac{\Delta x q_x + \frac{1}{2}\Delta x^2 q_{xx} + \cdots}{\Delta x}\right) \\ &= (q_t + uq_x) + \frac{1}{2}(\Delta t q_{tt} - u\Delta x q_{xx}) + \mathcal{O}(\Delta x^2, \Delta t^2)\end{aligned}$$

Order of Accuracy — upwind method

Insert Taylor series into

$$\tau(x,t) = \left(\frac{q(x_i,t_{n+1}) - q(x_i,t_n)}{\Delta t}\right) + u\left(\frac{q(x_i,t_n) - q(x_{i-1},t_n)}{\Delta x}\right)$$

gives (with everything evaluated at (x_i, t_n)):

$$\begin{aligned}\tau(x_i,t_n) &= \left(\frac{\Delta t q_t + \frac{1}{2}\Delta t^2 q_{tt} + \cdots}{\Delta t}\right) + u\left(\frac{\Delta x q_x + \frac{1}{2}\Delta x^2 q_{xx} + \cdots}{\Delta x}\right)\\ &= (q_t + uq_x) + \frac{1}{2}(\Delta t q_{tt} - u\Delta x q_{xx}) + \mathcal{O}(\Delta x^2, \Delta t^2)\end{aligned}$$

Since q is the exact solution, $q_t + uq_x = 0$ and $q_{tt} = u^2 q_{xx}$, so

$$\tau(x_i,t_n) = \frac{1}{2}\Delta x\left(\frac{u\Delta t}{\Delta x} - 1\right)uq_{xx} + \mathcal{O}(\Delta x^2)$$

Order of Accuracy — upwind method

Local truncation error:

$$\tau(x_i, t_n) = \frac{1}{2}\Delta x \left(\frac{u\Delta t}{\Delta x} - 1 \right) u q_{xx} + \mathcal{O}(\Delta x^2)$$

Assuming $\Delta t / \Delta x$ is constant as we refine the grid.

The method is said to be first order accurate.

Order of Accuracy — upwind method

Local truncation error:

$$\tau(x_i, t_n) = \frac{1}{2}\Delta x \left(\frac{u\Delta t}{\Delta x} - 1 \right) u q_{xx} + \mathcal{O}(\Delta x^2)$$

Assuming $\Delta t/\Delta x$ is constant as we refine the grid.

The method is said to be first order accurate.

Can show that if the method is also stable as $\Delta x \to 0$ then the global error will also be first order for smooth enough solutions.

$$E(x,t) \equiv Q(x,t) - q(x,t) = \mathcal{O}(\Delta x)$$

where we fix (x,t) and let $Q(x,t)$ denote the numerical approximation at this point as the grid is refined.

Order of Accuracy — upwind method

Global error: $E(x,t) \equiv Q(x,t) - q(x,t)$

Discontinuous solutions?

If $q(x,t)$ has a discontinuity then we cannot expect convergence pointwise or in the max-norm

$$\|E(\cdot,t)\|_\infty = \max_{a \le x \le b} |E(x,t)|.$$

The numerical method is almost always smeared out.

Order of Accuracy — upwind method

Global error: $E(x,t) \equiv Q(x,t) - q(x,t)$

Discontinuous solutions?

If $q(x,t)$ has a discontinuity then we cannot expect convergence pointwise or in the max-norm

$$\|E(\cdot,t)\|_\infty = \max_{a\le x\le b} |E(x,t)|.$$

The numerical method is almost always smeared out.

Best we can hope for is convergence in some norm like

$$\|E(\cdot,t)\|_1 = \int_a^b |E(x,t)|\,dx \quad \approx \Delta x \sum_i |Q_i^n - q(x_i,t_n)|.$$

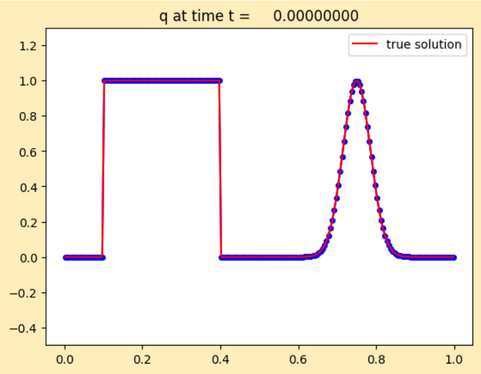
For upwind on discontinuous data, we expect

$$\|E(\cdot,t)\|_1 = \mathcal{O}(\Delta x^{1/2}).$$

Advection tests

$q_t + q_x = 0$ with periodic BCs
Solution at $t = 1$ should agree with initial data.

Initial data with 200 cells:



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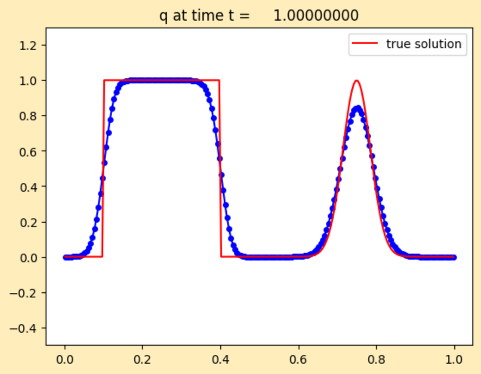
Advection tests

$q_t + q_x = 0$ with periodic BCs
Solution at $t = 1$ should agree with initial data.

Upwind solution with 200 cells:



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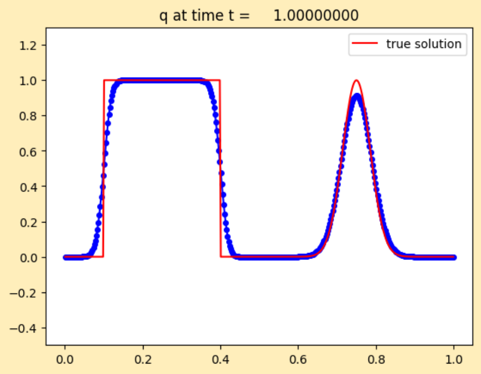
Advection tests

$q_t + q_x = 0$ with periodic BCs
Solution at $t = 1$ should agree with initial data.

Upwind solution with 400 cells:



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Consistency

A method is consistent if $\tau \to 0$ as $\Delta t,\ \Delta x \to 0$.

The one-step error is $\Delta t\,\tau$:

$$\Delta t\,\tau = q(x, t + \Delta t) - \left(q(x,t) - \frac{u\Delta t}{\Delta x}(q(x,t) - q(x - \Delta x, t)) \right).$$

An error of this magnitude is made in each of $T/\Delta t$ time steps.

This suggests $E \approx (T/\Delta t)(\Delta t\,\tau) = T\tau$:

$\tau = O(\Delta x^p + \Delta t^p) \implies$ global error is $O(\Delta x^p + \Delta t^p)$

The method is pth order accurate

Consistency

A method is consistent if $\tau \to 0$ as $\Delta t,\ \Delta x \to 0$.

The one-step error is $\Delta t\, \tau$:

$$\Delta t\, \tau = q(x, t + \Delta t) - \left(q(x,t) - \frac{u \Delta t}{\Delta x} (q(x,t) - q(x - \Delta x, t)) \right).$$

An error of this magnitude is made in each of $T/\Delta t$ time steps.

This suggests $E \approx (T/\Delta t)(\Delta t\, \tau) = T\tau$:

$\tau = O(\Delta x^p + \Delta t^p) \implies$ global error is $O(\Delta x^p + \Delta t^p)$

The method is pth order accurate

This is valid provided the method is stable!

Consistency $+$ stability $=$ convergence

Consistency for conservation law

For $q_t + f(q)_x = 0$, consider a method in conservation form,

$$Q_i^{n+1} = Q_i^n + \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n).$$

The method is consistent with the PDE if

$$F_{i-1/2} = \mathcal{F}(Q_{i-1}, Q_i) \quad \text{with } \mathcal{F}(\bar{q}, \bar{q}) = f(\bar{q})$$

and the numerical flux function is Lipschitz continuous,

$$|\mathcal{F}(q_\ell, q_r) - f(\bar{q})| \le C \max(|q_\ell - \bar{q}|,\ |q_r - \bar{q}|).$$

for all q_ℓ, q_r in a neighborhood of $\bar{q}$.

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Example: $\mathcal{F}(q_\ell, q_r) = u q_\ell$ for upwind, with $C = u$.

Consistency for conservation law

$$Q_i^{n+1} = Q_i^n + \frac{\Delta t}{\Delta x}(\mathcal{F}(Q_i^n, Q_{i+1}^n) - \mathcal{F}(Q_{i-1}^n, Q_i^n))$$

Consistent if $\mathcal{F}(\bar{q}, \bar{q}) = f(\bar{q})$ and Lipschitz continuous.

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Burgers': $f(q) = \frac{1}{2}q^2, \;\; \mathcal{F}(q_\ell, q_r) = \frac{1}{2}q_\ell^2$, can take $C = \bar{q} + \epsilon$.

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Godunov's method (upwind) for $q_t + Aq_x = 0$:

$$\mathcal{F}(q_\ell, q_r) = A^+ q_\ell + A^- q_r \implies \mathcal{F}(\bar{q}, \bar{q}) = A^+ \bar{q} + A^- \bar{q} = A\bar{q} = f(\bar{q})$$

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Centered flux: $\mathcal{F}(q_\ell, q_r) = \frac{1}{2}A(q_\ell + q_r)$
Centered flux for $q_t + f(q)_x = 0$: $\mathcal{F}(q_\ell, q_r) = \frac{1}{2}(f(q_\ell) + f(q_r))$

Consistent provided $f(q)$ is Lipschitz, but unstable!

Fundamental Theorem

$$\boxed{\text{Consistency} + \text{Stability} = \text{Convergence}}$$

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ODE: zero-stability, stability on $q'(t) = 0$ is enough.
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Uniform power boundedness of a family of matrices
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Scalar conservation law: total variation stability, entropy stability

Systems of conservation laws: few convergence proofs

Stability of the upwind method

Upwind method for advection $q_t + uq_x = 0$ with $u > 0$:

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n)$$

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$$\frac{u\Delta t}{\Delta x}$$

is called the Courant number or the CFL number after Courant, Friedrichs, and Lewy (1928 paper on existence and uniqueness of PDE solutions).

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Can prove that the upwind method is stable provided

$$0 \le \frac{u\Delta t}{\Delta x} \le 1.$$

Then the method converges in the 1-norm as $\Delta x \to 0$.

The CFL Condition (Courant-Friedrichs-Lewy)

Domain of dependence: The solution $q(X,T)$ depends on the data $q(x,0)$ over some set of x values, $x \in \mathcal{D}(X,T)$.

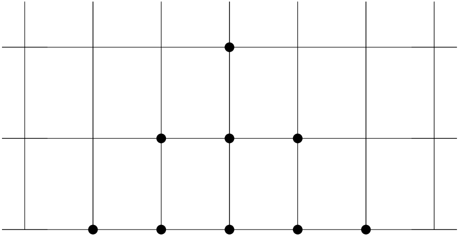
Advection: $q(X,T) = q(X-uT,0)$ and so $\mathcal{D}(X,T) = \{X-uT\}$.

The CFL Condition: A numerical method can be convergent only if its numerical domain of dependence contains the true domain of dependence of the PDE, at least in the limit as Δt and Δx go to zero.

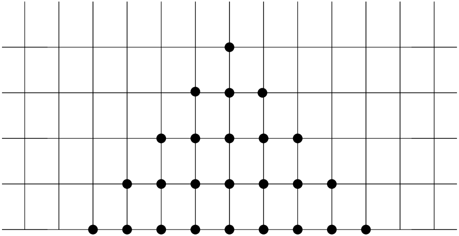
Note: Necessary but not sufficient for stability!

Numerical domain of dependence

With a 3-point explicit method:



On a finer grid with $\Delta t/\Delta x$ fixed:



The CFL Condition

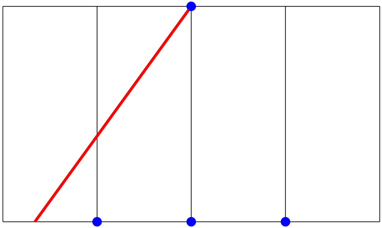
For the method to be stable, the numerical domain of dependence must include the true domain of dependence.

For advection, the solution is constant along characteristics,

$$q(x,t) = q(x - ut, 0)$$

For a 3-point method, CFL condition requires $\left|\frac{u\Delta t}{\Delta x}\right| \le 1.$

If this is violated:
True solution is determined by data at a point $x - ut$ that is ignored by the numerical method, even as the grid is refined.



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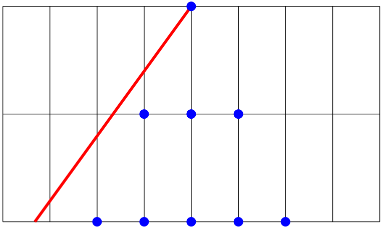

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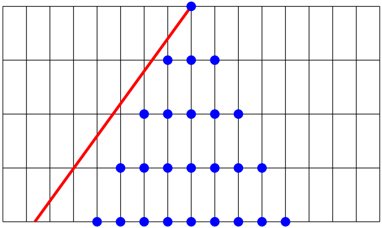

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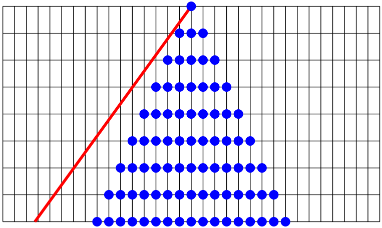
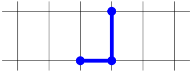
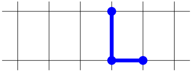
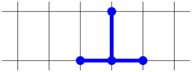




$$0 \le \frac{u\Delta t}{\Delta x} \le 1$$



$$-1 \le \frac{u\Delta t}{\Delta x} \le 0$$



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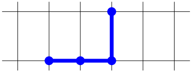
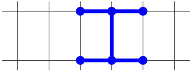


$$0 \le \frac{u\Delta t}{\Delta x} \le 2$$



$$-\infty < \frac{u\Delta t}{\Delta x} < \infty$$

Parabolic equations

Examples: Heat equation $q_t = \beta q_{xx}$,
Advection-diffusion equation $q_t + uq_x = \beta q_{xx}$,
Fluid dynamics with viscosity

Domain of dependence for any point (x, t) with $t > 0$ is:
Entire axis $-\infty < x < \infty$ for Cauchy problem,
All initial and boundary data up to time t for IBVP.

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Solution to Cauchy problem is $q(x, t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-x^2/4t\right)$.

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CFL condition requires either:
Implicit method, or
Explicit method with $\Delta t/\Delta x \to 0$ as grid is refined,
e.g. $\Delta t = (\Delta x)^2$.

Linear hyperbolic systems

Linear system of m equations: $q(x,t) \in \mathbb{R}^m$ for each (x,t) and

$$q_t + Aq_x = 0, \qquad -\infty < x, \infty, \;\; t \geq 0.$$

A is $m \times m$ with eigenvalues λ^p and eigenvectors r^p,
for $p = 1,\ 2,\ ,\ldots,\ m$:

$$Ar^p = \lambda^p r^p.$$

Combining these for $p = 1,\ 2,\ ,\ldots,\ m$ gives

$$AR = R\Lambda$$

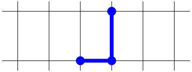
where

$$R = [r^1 \;\; r^2 \;\; \cdots \;\; r^m], \qquad \Lambda = \text{diag}(\lambda^1,\ \lambda^2,\ \ldots,\ \lambda^m).$$

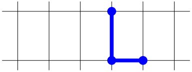
The system is hyperbolic if the eigenvalues are real and
R is invertible. Then A can be diagonalized:

$$R^{-1}AR = \Lambda$$

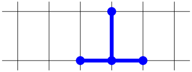
Stencil CFL Condition



$$0 \leq \frac{\lambda_p \Delta t}{\Delta x} \leq 1, \quad \forall p$$



$$-1 \leq \frac{\lambda_p \Delta t}{\Delta x} \leq 0, \quad \forall p$$



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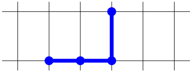
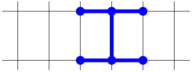


$$0 \leq \frac{\lambda_p \Delta t}{\Delta x} \leq 2, \quad \forall p$$



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Dissipation, Dispersion, Modified Equations

- Upwind, Lax-Friedrichs
- Lax-Wendroff and Beam-Warming
- Numerical dissipation and dispersion
- Modified equations

Symmetric methods

Centered in space, forward in time:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x} A(Q_{i+1}^n - Q_{i-1}^n)$$

Flux differencing with $\mathcal{F}(Q_{i-1}, Q_i) = \frac{1}{2}(AQ_{i-1} + AQ_i)$ for $f(q) = Aq$.

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Lax-Friedrichs:

$$Q_i^{n+1} = \frac{1}{2}(Q_{i-1}^n + Q_{i+1}^n) - \frac{\Delta t}{2\Delta x} A(Q_{i+1}^n - Q_{i-1}^n)$$

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Numerical dissipation

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This can be rewritten as

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x} A(Q_{i+1}^n - Q_{i-1}^n) + \frac{1}{2}(Q_{i-1}^n - 2Q_i^n + Q_{i+1}^n)$$

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The unstable method with the addition of artificial viscosity,

Approximates $q_t + Aq_x = \epsilon q_{xx}$ (modified equation)

with $\epsilon = \frac{\Delta x^2}{2\Delta t} = \mathcal{O}(\Delta x)$ if $\Delta t/\Delta x$ is fixed as $\Delta x \to 0$.

Modified Equations

The upwind method

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gives a first-order accurate approximation to $q_t + uq_x = 0$.

But it gives a second-order approximation to

$$q_t + uq_x = \frac{u\Delta x}{2}\left(1 - \frac{u\Delta t}{\Delta x}\right) q_{xx}.$$

This is an advection-diffusion equation.

Indicates that the numerical solution will diffuse.

Note: coefficient of diffusive term is $O(\Delta x)$.

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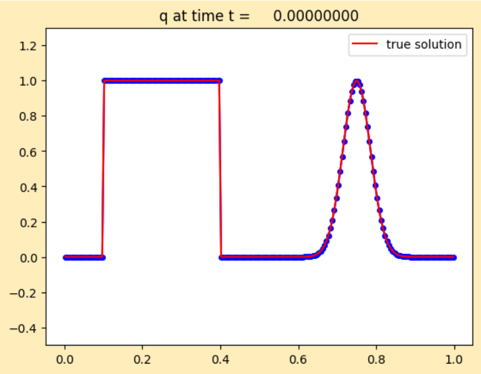
Note: No diffusion if $\frac{u\Delta t}{\Delta x} = 1 \quad (Q_i^{n+1} = Q_{i-1}^n$ exactly).

Advection tests

$q_t + q_x = 0$ with periodic BCs
Solution at $t = 1$ should agree with initial data.

Initial data with 200 cells:



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Advection tests

$q_t + q_x = 0$ with periodic BCs
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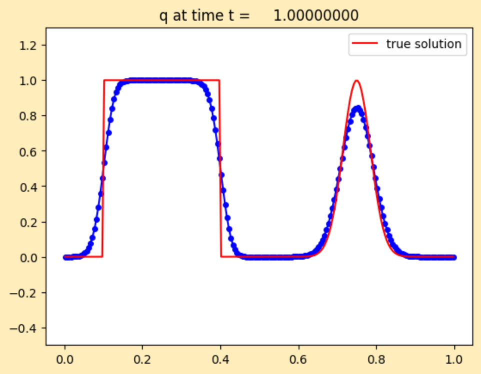
Upwind solution with 200 cells:



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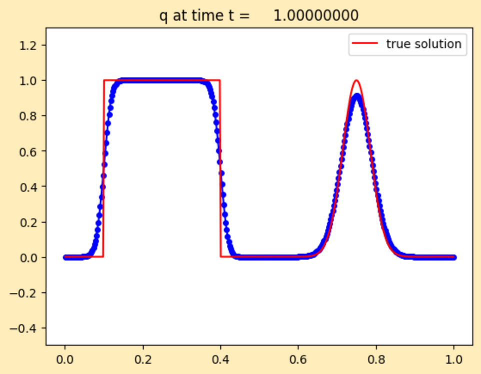
Upwind solution with 400 cells:



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Lax-Wendroff

Second-order accuracy?

Taylor series:

$$q(x, t + \Delta t) = q(x, t) + \Delta t q_t(x, t) + \frac{1}{2}\Delta t^2 q_{tt}(x, t) + \cdots$$

From $q_t = -Aq_x$ we find $q_{tt} = A^2 q_{xx}$.

$$q(x, t + \Delta t) = q(x, t) - \Delta t A q_x(x, t) + \frac{1}{2}\Delta t^2 A^2 q_{xx}(x, t) + \cdots$$

Replace q_x and q_{xx} by centered differences:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x} A(Q_{i+1}^n - Q_{i-1}^n) + \frac{1}{2}\frac{\Delta t^2}{\Delta x^2} A^2(Q_{i-1}^n - 2Q_i^n + Q_{i+1}^n)$$

Modified Equation for Lax-Wendroff

The Lax-Wendroff method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x} A(Q_{i+1}^n - Q_{i-1}^n) + \frac{1}{2}\frac{\Delta t^2}{\Delta x^2} A^2 (Q_{i-1}^n - 2Q_i^n + Q_{i+1}^n)$$

gives a second-order accurate approximation to $q_t + uq_x = 0$.

But it gives a third-order approximation to

$$q_t + uq_x = -\frac{u\Delta x^2}{6}\left(1 - \left(\frac{u\Delta t}{\Delta x}\right)^2\right) q_{xxx}.$$

This has a dispersive term with $O(\Delta x^2)$ coefficient.

Indicates that the numerical solution will become oscillatory.

Dispersion relation

Consider a single Fourier mode:

$$q(x,0) = e^{i\xi x} \implies q(x,t) = e^{i(\xi x - \omega t)}$$

Determine $\omega(\xi)$ based on the PDE (dispersion relation)

$q_t = -i\omega q, \quad q_x = i\xi q,$

$q_t + uq_x = 0 \implies \omega(\xi) = u\xi, \qquad q(x,t) = e^{i\xi(x-ut)}$
(translates at speed u for all ξ)

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$$q_t + uq_x = \epsilon q_{xx} \implies \qquad q(x,t) = e^{-\epsilon\xi^2 t} e^{i\xi(x-ut)}$$

(decays)

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$$q_t + uq_x = \epsilon q_{xx} \implies \qquad q(x,t) = e^{-\epsilon\xi^2 t} e^{i\xi(x-ut)}$$

(decays)

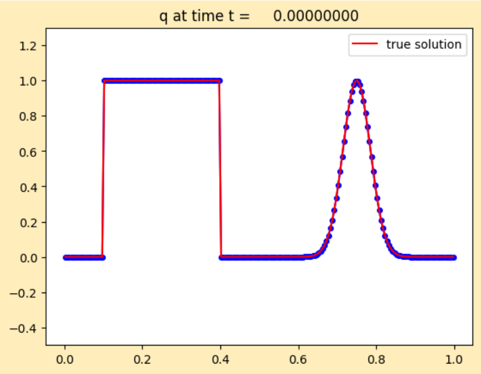
$$q_t + uq_x = \beta q_{xxx} \implies \quad q(x,t) = e^{i\xi(x-(u+\beta\xi^2)t)}$$

(translates at speed $u + \beta\xi^2$ that depends on wave number!)

Advection tests

$q_t + q_x = 0$ with periodic BCs
Solution at $t = 1$ should agree with initial data.

Initial data with 200 cells:



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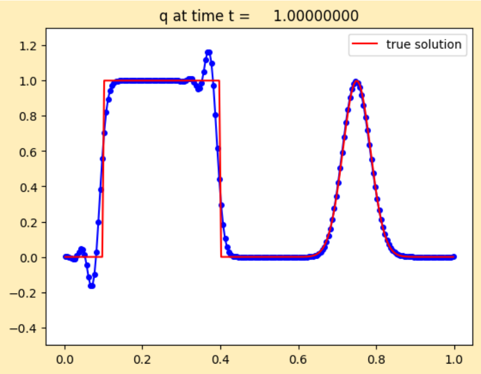
Lax-Wendroff solution with 200 cells:



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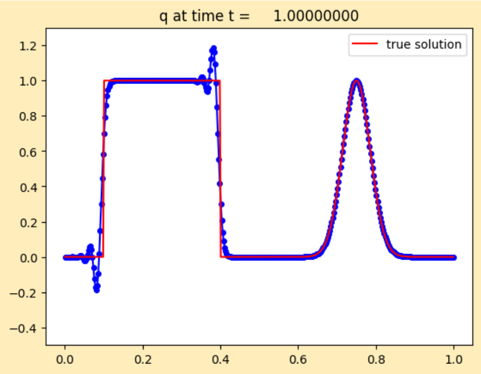

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Beam-Warming method

Taylor series for second order accuracy:

$$q(x, t + \Delta t) = q(x, t) - \Delta t A q_x(x, t) + \frac{1}{2}\Delta t^2 A^2 q_{xx}(x, t) + \cdots$$

Replace q_x and q_{xx} by one-sided differences:

$$\begin{aligned} Q_i^{n+1} = Q_i^n &- \frac{\Delta t}{2\Delta x} A(3Q_i^n - 4Q_{i-1}^n + Q_{i-2}^n) \\ &+ \frac{1}{2}\frac{\Delta t^2}{\Delta x^2} A^2(Q_i^n - 2Q_{i-1}^n + Q_{i-2}^n) \end{aligned}$$

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CFL condition: $0 \leq \frac{\lambda^p \Delta t}{\Delta x} \leq 2$ for all eigenvalues.

This is also the stability limit (von Neumann analysis).

Advection tests

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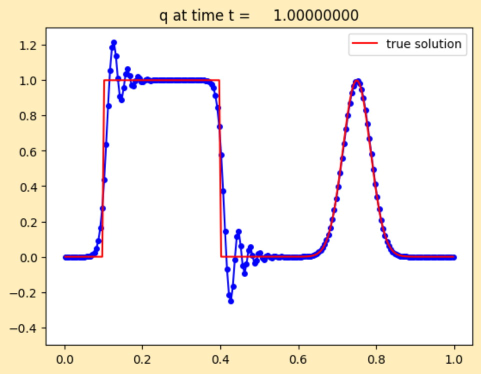

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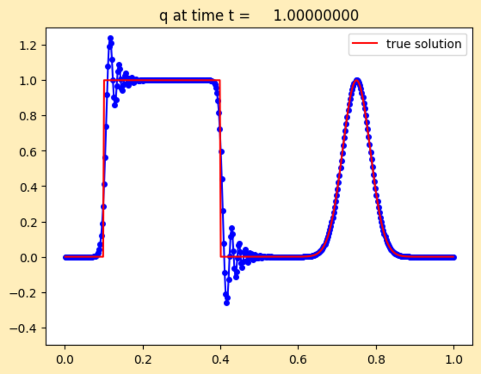

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Finite Volume Methods for Hyperbolic Problems

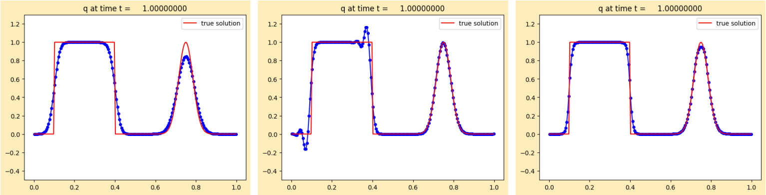
High-Resolution TVD Methods

- Godunov: wave-propagation and REA algorithms
- Extension of REA to piecewise linear
- Relation to Lax-Wendroff, Beam-Warming
- Limiters and minmod
- Monotonicity and Total Variation

Advection tests with periodic BCs

Compare Upwind, Lax-Wendroff, minmod...

With 200 cells:



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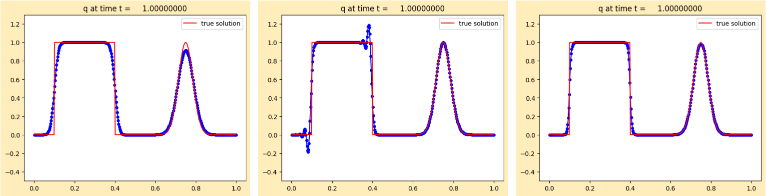

High-Resolution methods

- Methods that give good accuracy for smooth solutions
  Clawpack methods: at best second-order accuracy
- Do not have oscillations around discontinuities
  Not only ugly but can lead to nonlinear instabilities

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  "Shock capturing" methods for nonlinear problems

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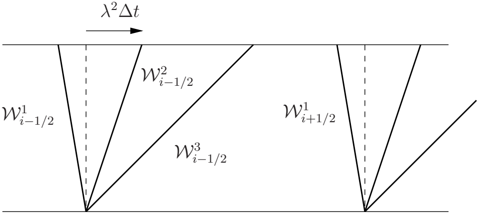
- Godunov-type methods — based on Riemann solvers
  Wave-propagation algorithms with "limiters"

Wave-propagation viewpoint

For linear system $q_t + Aq_x = 0$, the Riemann solution consists of

waves $\mathcal{W}^p$ propagating at constant speed λ^p.



$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha^p_{i-1/2} r^p \equiv \sum_{p=1}^{m} \mathcal{W}^p_{i-1/2}.$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\lambda^2 \mathcal{W}^2_{i-1/2} + \lambda^3 \mathcal{W}^3_{i-1/2} + \lambda^1 \mathcal{W}^1_{i+1/2}\right].$$

First-order REA Algorithm

1. Reconstruct a piecewise constant function $\tilde{q}^n(x, t_n)$ defined for all x, from the cell averages Q_i^n.

$$\tilde{q}^n(x, t_n) = Q_i^n \quad \text{for all } x \in \mathcal{C}_i.$$

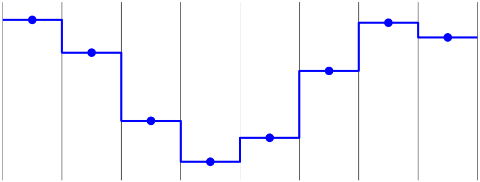
2. Evolve the hyperbolic equation exactly (or approximately) with this initial data to obtain $\tilde{q}^n(x, t_{n+1})$ a time Δt later.

3. Average this function over each grid cell to obtain new cell averages

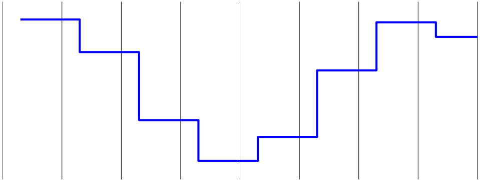
$$Q_i^{n+1} = \frac{1}{\Delta x} \int_{\mathcal{C}_i} \tilde{q}^n(x, t_{n+1})\, dx.$$

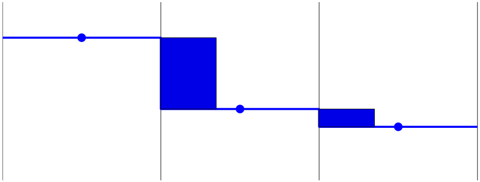
First-order REA Algorithm

Cell averages and piecewise constant reconstruction:



After evolution:





The cell average is modified by

$$\frac{u\Delta t \cdot (Q_{i-1}^n - Q_i^n)}{\Delta x}$$

So we obtain the upwind method

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n).$$

Second-order REA Algorithm

1. Reconstruct a piecewise linear function $\tilde{q}^n(x, t_n)$ defined for all x, from the cell averages Q_i^n.

$$\tilde{q}^n(x, t_n) = Q_i^n + \sigma_i^n(x - x_i) \quad \text{for all } x \in \mathcal{C}_i.$$

2. Evolve the hyperbolic equation exactly (or approximately) with this initial data to obtain $\tilde{q}^n(x, t_{n+1})$ a time Δt later.

3. Average this function over each grid cell to obtain new cell averages

$$Q_i^{n+1} = \frac{1}{\Delta x} \int_{\mathcal{C}_i} \tilde{q}^n(x, t_{n+1})\, dx.$$

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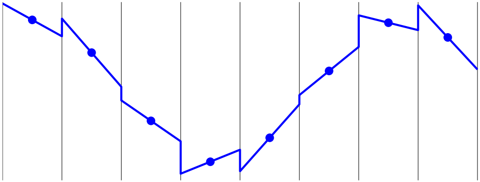
Note: Conservative for any choice of slopes σ_i^n.

Second-order REA Algorithm

Cell averages and piecewise linear reconstruction:



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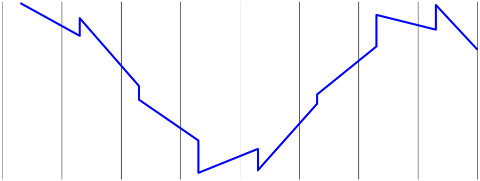

Choice of slopes

$$\tilde{Q}^n(x, t_n) = Q_i^n + \sigma_i^n(x - x_i) \qquad \text{for } x_{i-1/2} \le x < x_{i+1/2}.$$

Applying REA algorithm gives:

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) - \frac{1}{2}\frac{u\Delta t}{\Delta x}\left(\Delta x - u\Delta t\right)\left(\sigma_i^n - \sigma_{i-1}^n\right)$$

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Choice of slopes:

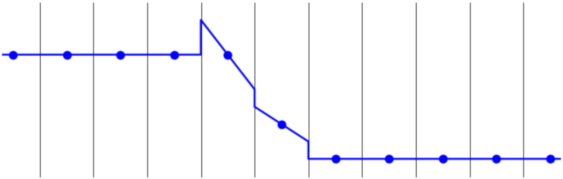
Centered slope: $\sigma_i^n = \dfrac{Q_{i+1}^n - Q_{i-1}^n}{2\Delta x}$ (Fromm)

Upwind slope: $\sigma_i^n = \dfrac{Q_i^n - Q_{i-1}^n}{\Delta x}$ (Beam-Warming)

Downwind slope: $\sigma_i^n = \dfrac{Q_{i+1}^n - Q_i^n}{\Delta x}$ (Lax-Wendroff)

Slopes can create oscillations

Step function data with Lax-Wendroff slope:



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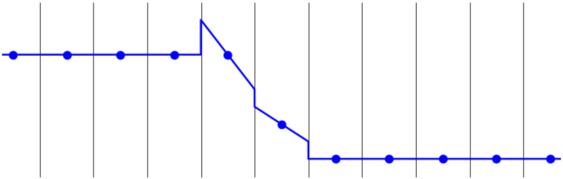
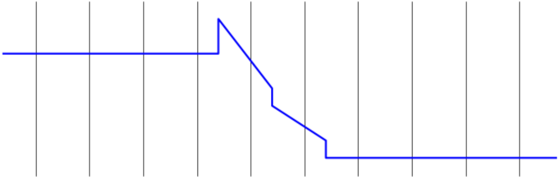
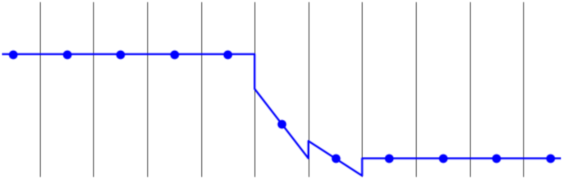


Evolving solution and averaging can result in overshoot:



Slopes can create oscillations

Step function data with Beam-Warming slope:



Slopes can create oscillations

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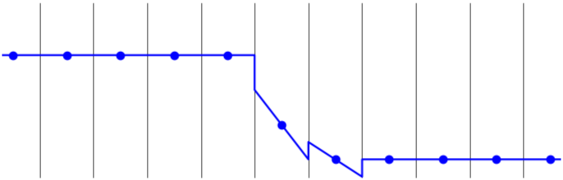

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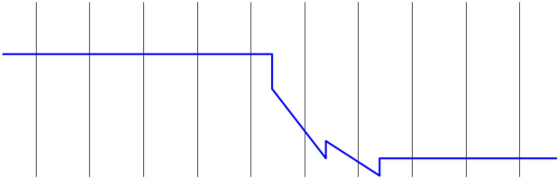


High-resolution methods

Want to use slope where solution is smooth for “second-order” accuracy.

Where solution is not smooth, adding slope corrections gives oscillations.

Limit the slope based on the behavior of the solution, e.g.,

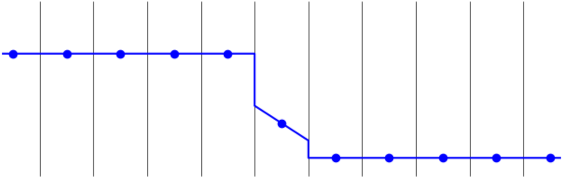
$$\sigma_i^n = \text{minmod}\left(\left(\frac{Q_i^n - Q_{i-1}^n}{\Delta x}\right), \left(\frac{Q_{i+1}^n - Q_i^n}{\Delta x}\right)\right)$$

where

$$\text{minmod}(a,b) = \begin{cases} a & \text{if } |a| < |b| \text{ and } ab > 0 \\ b & \text{if } |b| < |a| \text{ and } ab > 0 \\ 0 & \text{if } ab \le 0. \end{cases}$$

Limiters can eliminate oscillations

Step function data with minmod slope:



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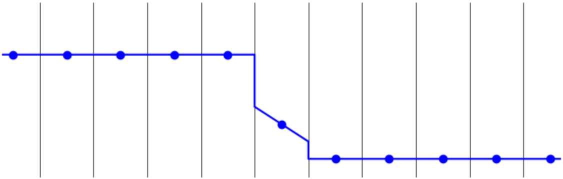
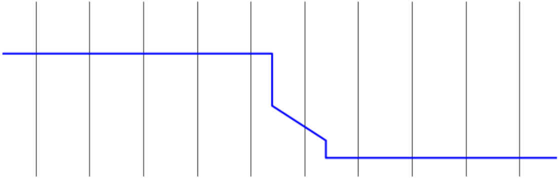
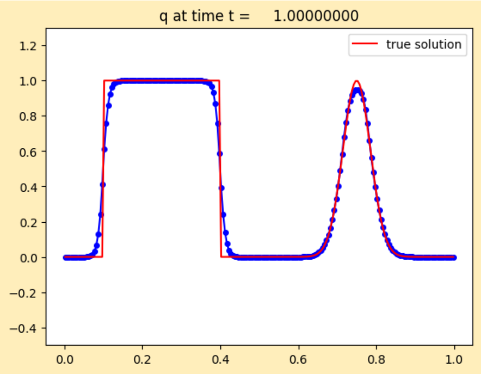


Evolving solution and averaging maintains monotonicity:



$q_t + q_x = 0$ with periodic BCs
Solution at $t = 1$ should agree with initial data.

Minmod solution with 200 cells:



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Advection tests

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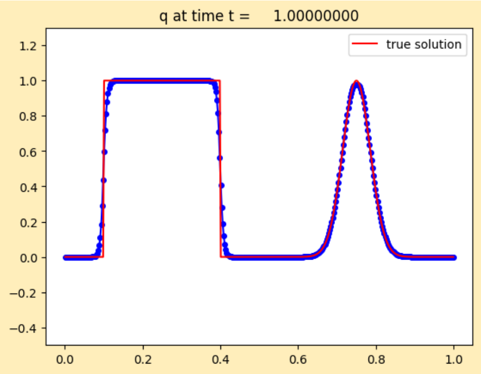

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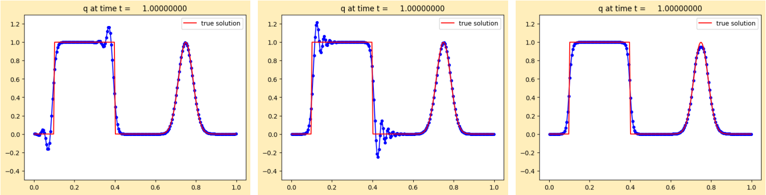
Advection tests with periodic BCs

Compare Lax-Wendroff, Beam-Warming, minmod...

With 200 cells:



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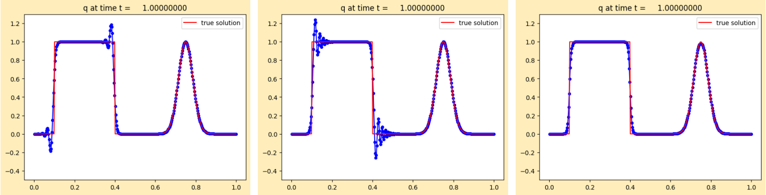
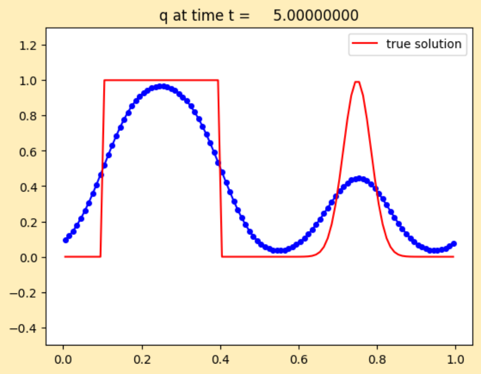


Advection tests

$q_t + q_x = 0$ with periodic BCs
Solution at $t = 1, 2, 3, 4, 5, \ldots$ should agree with initial data.

Upwind solution with 100 cells at $t = 5$:



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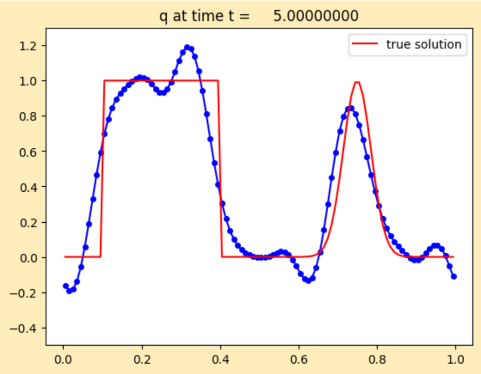
Lax-Wendroff solution with 100 cells at $t = 5$:



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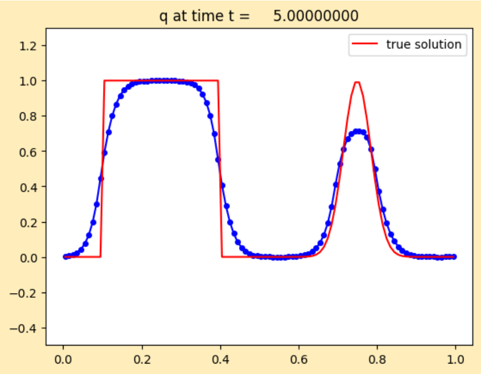
Minmod limiter solution with 100 cells at $t = 5$:



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Advection tests

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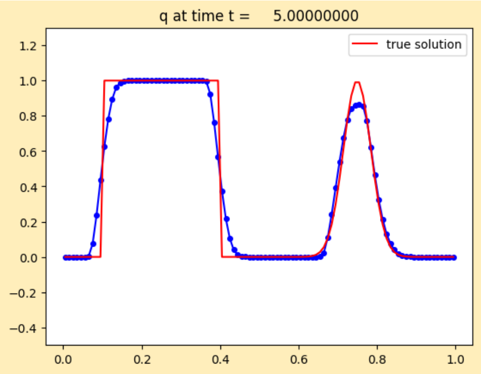
Monotonized Central limiter solution with 100 cells at $t = 5$:



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Advection tests

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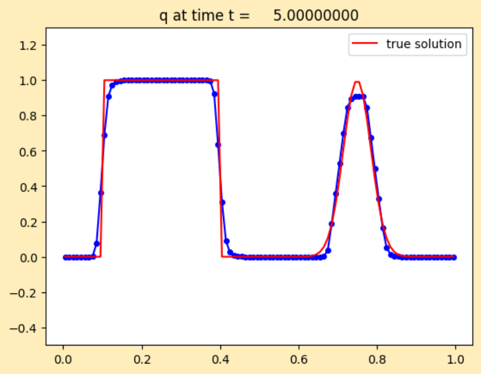
Solution at $t = 1, 2, 3, 4, 5, \ldots$ should agree with initial data.

Superbee limiter solution with 100 cells at $t = 5$:



$CLAW/apps/fvmbook/chap6/compareadv

Monotonicity Preserving methods

A scalar method is said to be monotonicity preserving if:

Given any data Q_i^n that satisfies

$$Q_{i-1}^n \geq Q_i^n \quad \text{for all } i.$$

Taking one time step preserves this property:

$$Q_{i-1}^{n+1} \geq Q_i^{n+1} \quad \text{for all } i.$$

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And similarly if $\geq$ replaced by $\leq$.

In particular:

An isolated discontinuity propagates without any oscillations.

TVD Methods

Total variation:

$$TV(Q) = \sum_i |Q_i - Q_{i-1}|$$

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A method is Total Variation Diminishing (TVD) if

$$TV(Q^{n+1}) \leq TV(Q^n).$$

Gives a form of stability useful for proving convergence, also for nonlinear scalar conservation laws.

TVD implies monotonicity preserving

Any TVD method for a scalar PDE is monotonicity preserving.

Prove the contrapositive:

Suppose

$$Q_{i-1}^n \geq Q_i^n \quad \text{for all } i$$

but after one step we do **not** have $Q_{i-1}^{n+1} \geq Q_i^{n+1}$ for all i.

Then the total variation of the solution must have increased.

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Since TV is a global property, how do we derive methods that we can prove are TVD for any data?

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Since TV is a global property, how do we derive methods that we can prove are TVD for any data?

Use these facts (for scalar conservation law):

- Exact solution is TVD
- If we average $q(x,t)$ over grid cells to compute Q_i, then $TV(Q_i) \le TV(q(\cdot,t))$.

$$TV(Q) = \sum_i |Q_i - Q_{i-1}|, \qquad TV(q) = \int |q_x(x)|\,dx$$

TVD REA Algorithm

1. Reconstruct a piecewise linear function $\tilde{q}^n(x, t_n)$ defined for all x, from the cell averages Q_i^n.

$$\tilde{q}^n(x, t_n) = Q_i^n + \sigma_i^n (x - x_i) \quad \text{for all } x \in \mathcal{C}_i$$

   with the property that $TV(\tilde{q}^n) \leq TV(Q^n)$.

2. Evolve the hyperbolic equation exactly (or approximately) with this initial data to obtain $\tilde{q}^n(x, t_{n+1})$ a time k later.

3. Average this function over each grid cell to obtain new cell averages

$$Q_i^{n+1} = \frac{1}{\Delta x} \int_{\mathcal{C}_i} \tilde{q}^n(x, t_{n+1})\, dx.$$

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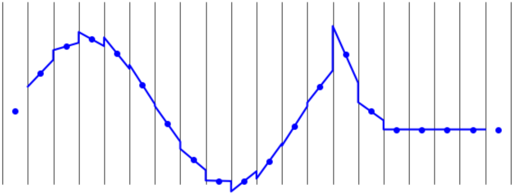
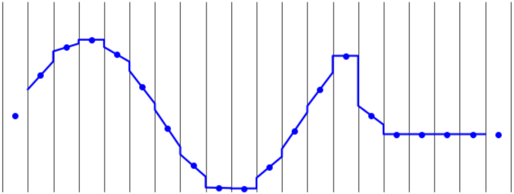
So $\quad TV(Q^{n+1}) \le TV(\tilde{q}^n(\cdot, t_{n+1})) \le TV(\tilde{q}^n(\cdot, t_n)) \le TV(Q^n)$

Reconstruction step

Lax-Wendroff slopes do not give TVD reconstruction:



Minmod slopes do give TVD reconstruction:



Finite Volume Methods for Hyperbolic Problems

TVD Methods and Limiters

- Slope limiters vs. flux limiters
- Total variation for scalar problems
- Proving TVD in flux-limiter form
- Design of TVD limiters
- Sweby Region

High-Resolution methods

- Methods that give good accuracy for smooth solutions
  Clawpack methods: at best second-order accuracy
- Do not have oscillations around discontinuities
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  "Shock capturing" methods for nonlinear problems
- Easy to combine with adaptive mesh refinement (AMR)
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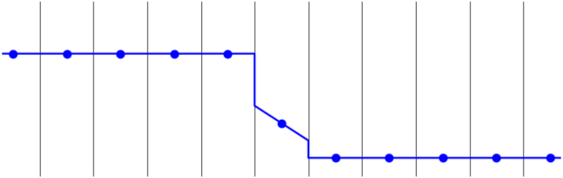
- Godunov-type methods — based on Riemann solvers
  Wave-propagation algorithms with "limiters"

Limiters can eliminate oscillations

Step function data with minmod slope:



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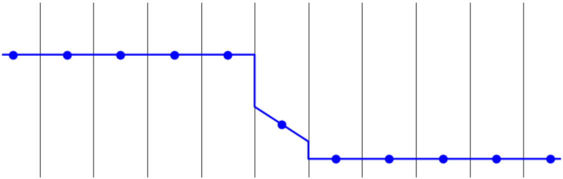
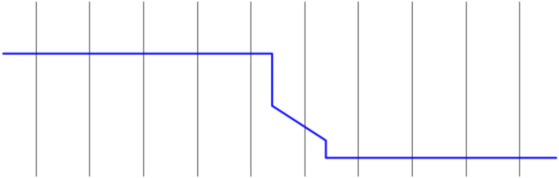


Evolving solution and averaging maintains monotonicity:



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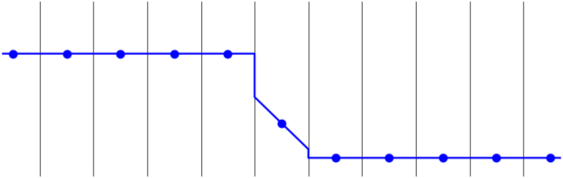

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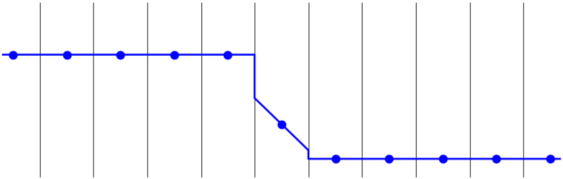

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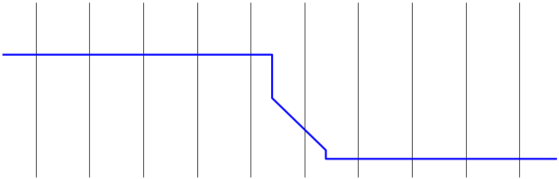

Monotonized centered (MC) limiter

Using the centered slope $(Q^n_{i+1} - Q^n_{i-1})/(2\Delta x)$ gives second-order accuracy (Fromm's method) but not monotonicity.

Limit this slope based on twice the one-sided slopes.

$$\sigma^n_i = \text{minmod}\left(\left(\frac{Q^n_{i+1} - Q^n_{i-1}}{2\Delta x}\right),\ 2\left(\frac{Q^n_i - Q^n_{i-1}}{\Delta x}\right),\ 2\left(\frac{Q^n_{i+1} - Q^n_i}{\Delta x}\right)\right).$$

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Using the centered slope $(Q_{i+1}^n - Q_{i-1}^n)/(2\Delta x)$ gives second-order accuracy (Fromm's method) but not monotonicity.

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Rationale:

- Where solution is smooth, centered slope is smaller and chosen, hence maintains accuracy.
- Near jumps in solution, don't expect second-order but want to resolve discontinuities as sharply as possible.

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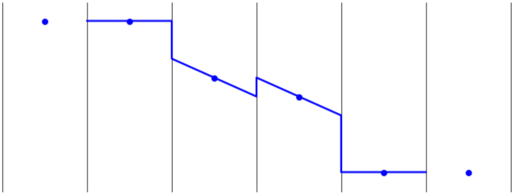

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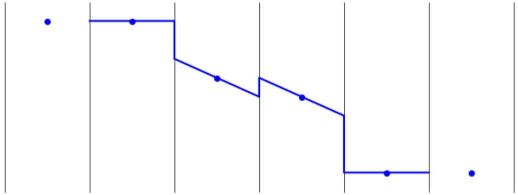
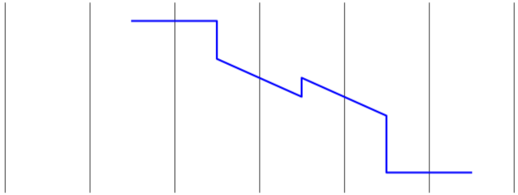


But evolving and averaging still maintains monotonicity (TVD):



Slope limiters and flux limiters

Slope limiter formulation for advection:

$$\tilde{q}^n(x, t_n) = Q_i^n + \sigma_i^n(x - x_i) \qquad \text{for } x_{i-1/2} \le x < x_{i+1/2}.$$

Applying REA algorithm gives (for $u > 0$):

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) - \frac{1}{2}\frac{u\Delta t}{\Delta x}\left(\Delta x - u\Delta t\right)\left(\sigma_i^n - \sigma_{i-1}^n\right)$$

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Lax-Wendroff and flux limiters

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Flux limiters and wave limiters

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$$F^n_{i-1/2} = u^+ Q^n_{i-1} + u^- Q^n_i + \frac{1}{2}|u|(1 - |u|\Delta t/\Delta x)\delta^n_{i-1/2}$$

For systems of equations:

- Solve Riemann problem to decompose $\Delta Q^n_{i-1/2}$ into waves

$$\Delta Q_{i-1/2} = \sum_p \mathcal{W}^p_{i-1/2} = \sum_p \alpha^p_{i-1/2} r^p$$

- Use wave propagation form of Godunov (first-order) update
- Apply limiters to waves to get $\widetilde{\mathcal{W}}^p_{i-1/2} = \tilde{\alpha}^p_{i-1/2} r^p$
- Use limited waves in “second-order” corrections

Flux limiters for scalar problem

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Limiter based on the ratio

$$\theta^n_{i-1/2} = \frac{Q_I - Q_{I-1}}{Q_i - Q_{i-1}}$$

where I denotes the cell in the upwind direction:

$$I = \begin{cases} i-1 & \text{if } u > 0 \\ i+1 & \text{if } u < 0. \end{cases}$$

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Note that:

- $\theta \approx 1 + \mathcal{O}(\Delta x)$ where the solution is smooth,
- $\theta < 0$ if slopes have different sign.

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Desirable properties:

- $\phi(\theta) = 0$ for $\theta \leq 0$ (zero slope at extrema)
- $\phi(1) = 1$ so nearly using Lax-Wendroff where smooth

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Note that:

- $\phi(\theta) \equiv 0$ for all $\theta \implies$ upwind method
- $\phi(\theta) \equiv 1$ for all $\theta \implies$ Lax-Wendroff

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Note that:

- $\phi(\theta) \equiv 0$ for all $\theta \implies$ upwind method
- $\phi(\theta) \equiv 1$ for all $\theta \implies$ Lax-Wendroff
- $\phi(\theta) = \theta \implies$ Beam-Warming: $\delta_{i-1/2}^n = Q_I - Q_{I-1}$

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Flux limiter method:

$$F^n_{i-1/2} = u^+ Q^n_{i-1} + u^- Q^n_i + \frac{1}{2}|u|(1 - |u|\Delta t/\Delta x)\delta^n_{i-1/2}$$

$$\delta^n_{i-1/2} = \phi(\theta^n_{i-1/2})\Delta Q^n_{i-1/2} \qquad \text{where} \quad \theta^n_{i-1/2} = \frac{Q_I - Q_{I-1}}{Q_i - Q_{i-1}}$$

Note that:

- $\phi(\theta) \equiv 0$ for all $\theta \implies$ upwind method
- $\phi(\theta) \equiv 1$ for all $\theta \implies$ Lax-Wendroff
- $\phi(\theta) = \theta \implies$ Beam-Warming: $\delta^n_{i-1/2} = Q_I - Q_{I-1}$
- $\phi(\theta) = \frac{1}{2}(1+\theta) \implies$ Fromm's method
- $\phi(\theta) = \text{minmod}(1, \theta) \implies$ Minmod method

TVD flux limiter methods

For $q_t + uq_x = 0$ with $u > 0$ and $\nu = u\Delta t/\Delta x$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

$$\begin{aligned} F_{i-1/2}^n &= uQ_{i-1}^n + \frac{1}{2}u(1 - u\Delta t/\Delta x)\delta_{i-1/2}^n \\ &= uQ_{i-1}^n + \frac{1}{2}u(1-\nu)[\phi(\theta_{i-1/2})(Q_i - Q_{i-1})] \end{aligned}$$

Can be written as:

$$Q_i^{n+1} = Q_i^n - \left[\nu + \frac{1}{2}\nu(1-\nu)\left(\frac{\phi(\theta_{i+1/2})}{\theta_{i+1/2}} - \phi(\theta_{i-1/2})\right)\right](Q_i^n - Q_{i-1}^n)$$

since $Q_{i+1} - Q_i = (1/\theta_{i+1/2})(Q_i - Q_{i-1})$.

TVD flux limiter methods

$$Q_i^{n+1} = Q_i^n - \left[\nu + \frac{1}{2}\nu(1-\nu)\left(\frac{\phi(\theta_{i+1/2})}{\theta_{i+1/2}} - \phi(\theta_{i-1/2})\right)\right](Q_i^n - Q_{i-1}^n)$$

Use this part of Theorem 6.1 (Harten):

The method

$$Q_i^{n+1} = Q_i^n - C_{i-1}^n(Q_i^n - Q_{i-1}^n)$$

is TVD provided $0 \le C_i^n \le 1$ for all i, regardless of how these coefficients depend on $Q^n,\ \Delta x,\ \Delta t$.

TVD flux limiter methods

$$Q_i^{n+1} = Q_i - C_{i-1}(Q_i - Q_{i-1}), \qquad TV(Q) = \sum |Q_{i+1} - Q_i|$$

Proof that method is TVD provided $0 \le C_i \le 1$ for all i:

$$\begin{aligned} Q_{i+1}^{n+1} - Q_i^{n+1} &= (Q_{i+1} - Q_i) - C_i(Q_{i+1} - Q_i) + C_{i-1}(Q_i - Q_{i-1}) \\ &= (1 - C_i)(Q_{i+1} - Q_i) + C_{i-1}(Q_{i+1} - Q_i) \end{aligned}$$

TVD flux limiter methods

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$$|Q_{i+1}^{n+1} - Q_i^{n+1}| \le (1 - C_i)|Q_{i+1} - Q_i| + C_{i-1}|Q_i - Q_{i-1}|$$

TVD flux limiter methods

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$$|Q_{i+1}^{n+1} - Q_i^{n+1}| \le (1 - C_i)|Q_{i+1} - Q_i| + C_{i-1}|Q_i - Q_{i-1}|$$

$$\sum |Q_{i+1}^{n+1} - Q_i^{n+1}| \le \sum (1 - C_i)|Q_{i+1} - Q_i| + \sum C_{i-1}|Q_i - Q_{i-1}|$$

TVD flux limiter methods

$$Q_i^{n+1} = Q_i - C_{i-1}(Q_i - Q_{i-1}), \qquad TV(Q) = \sum |Q_{i+1} - Q_i|$$

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$$\begin{aligned} |Q_{i+1}^{n+1} - Q_i^{n+1}| &\le (1 - C_i)|Q_{i+1} - Q_i| + C_{i-1}|Q_i - Q_{i-1}| \\ \sum |Q_{i+1}^{n+1} - Q_i^{n+1}| &\le \sum (1 - C_i)|Q_{i+1} - Q_i| + \sum C_{i-1}|Q_i - Q_{i-1}| \\ &\le \sum (1 - C_i)|Q_{i+1} - Q_i| + \sum C_i|Q_{i+1} - Q_i| \end{aligned}$$

TVD flux limiter methods

$$Q_i^{n+1} = Q_i - C_{i-1}(Q_i - Q_{i-1}), \qquad TV(Q) = \sum |Q_{i+1} - Q_i|$$

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$$\begin{aligned} |Q_{i+1}^{n+1} - Q_i^{n+1}| &\le (1 - C_i)|Q_{i+1} - Q_i| + C_{i-1}|Q_i - Q_{i-1}| \\ \sum |Q_{i+1}^{n+1} - Q_i^{n+1}| &\le \sum(1 - C_i)|Q_{i+1} - Q_i| + \sum C_{i-1}|Q_i - Q_{i-1}| \\ &\le \sum(1 - C_i)|Q_{i+1} - Q_i| + \sum C_i|Q_{i+1} - Q_i| \\ &\le \sum(1 - C_i + C_i)|Q_{i+1} - Q_i| = TV(Q^n) \end{aligned}$$

TVD flux limiter methods

The method

$$Q_i^{n+1} = Q_i^n - \left[\nu + \frac{1}{2}\nu(1-\nu)\left(\frac{\phi(\theta_{i+1/2})}{\theta_{i+1/2}} - \phi(\theta_{i-1/2})\right)\right](Q_i^n - Q_{i-1}^n)$$

is TVD provided

$$0 \le \left[\nu + \frac{1}{2}\nu(1-\nu)\left(\frac{\phi(\theta_1)}{\theta_1} - \phi(\theta_2)\right)\right] \le 1$$

for all values of θ_1 and θ_2 (provided $0 \le \nu \le 1$).

TVD flux limiter methods

The method

$$Q_i^{n+1} = Q_i^n - \left[\nu + \frac{1}{2}\nu(1-\nu)\left(\frac{\phi(\theta_{i+1/2})}{\theta_{i+1/2}} - \phi(\theta_{i-1/2})\right)\right](Q_i^n - Q_{i-1}^n)$$

is TVD provided

$$0 \le \left[\nu + \frac{1}{2}\nu(1-\nu)\left(\frac{\phi(\theta_1)}{\theta_1} - \phi(\theta_2)\right)\right] \le 1$$

for all values of θ_1 and θ_2 (provided $0 \le \nu \le 1$).

This is true if

$$-2 \le \left(\frac{\phi(\theta_1)}{\theta_1} - \phi(\theta_2)\right) \le 2$$

for all values of θ_1 and θ_2

So the method

$$Q_i^{n+1} = Q_i^n - \left[\nu + \frac{1}{2}\nu(1-\nu)\left(\frac{\phi(\theta_{i+1/2})}{\theta_{i+1/2}} - \phi(\theta_{i-1/2})\right)\right](Q_i^n - Q_{i-1}^n)$$

is TVD provided

$$-2 \le \left(\frac{\phi(\theta_1)}{\theta_1} - \phi(\theta_2)\right) \le 2$$

for all values of θ_1 and θ_2.

Satisfied provided $\phi(\theta)$ satisfies:

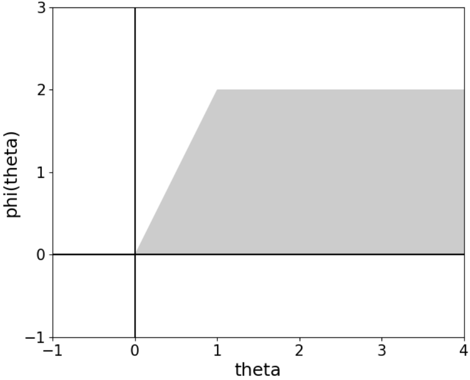
$$0 \le \frac{\phi(\theta)}{\theta} \le 2, \qquad 0 \le \phi(\theta) \le 2,$$

or

$$0 \le \phi(\theta) \le \text{minmod}(2, 2\theta).$$

Sweby diagram

If we plot $\phi(\theta)$, the curve must lie in the shaded region:



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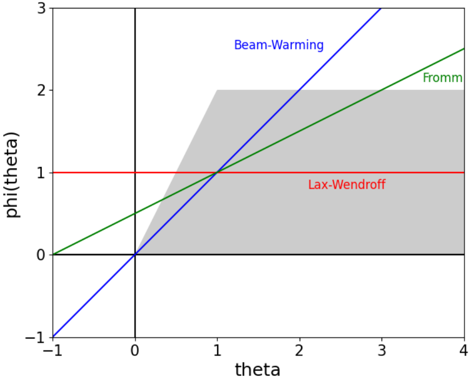


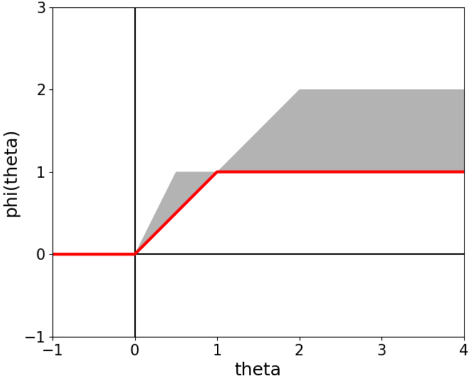
Standard second order methods go outside this region.

Recall we want $\phi(1) = 1$ for good accuracy of smooth solutions.

Sweby diagram

Sweby's investigation suggested best methods lie between Lax-Wendroff and Beam-Warming (and inside the TVD region).

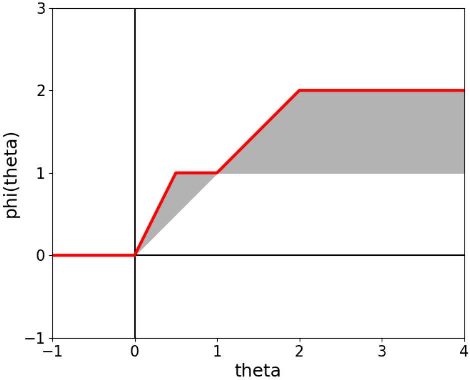
Sweby region:



$\phi(\theta) = \text{minmod}(1, \theta)$ follows the lower limit of this region.

Superbee method

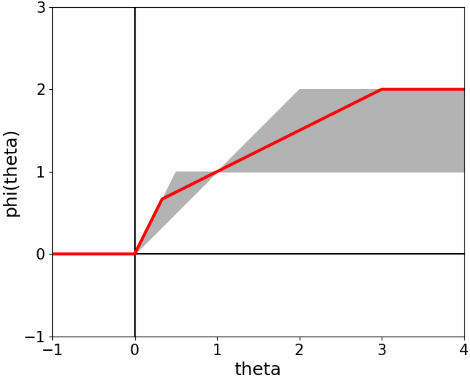
The superbee limiter follows the upper limit:



$$\phi(\theta) = \max(0,\ \min(1, 2\theta),\ \min(2, \theta))$$

MC method

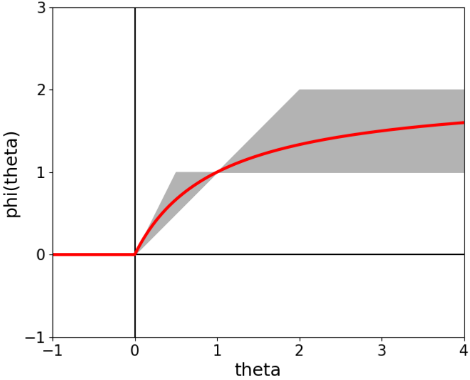
The Monotonized Centered (MC) limiter follows Fromm's method near $\theta = 1$, and is smooth at $\theta = 1$:



$$\phi(\theta) = \max(0,\ \min((1+\theta)/2,\ 2,\ 2\theta))$$

van Leer method

The van Leer limiter is a smoother version of MC



$\phi(\theta) = \frac{\theta+|\theta|}{1+|\theta|}$.

Some popular limiters

Linear methods:

$$
\begin{aligned}
\text{upwind}: &\quad \phi(\theta) = 0 \\
\text{Lax-Wendroff}: &\quad \phi(\theta) = 1 \\
\text{Beam-Warming}: &\quad \phi(\theta) = \theta \\
\text{Fromm}: &\quad \phi(\theta) = \frac{1}{2}(1+\theta)
\end{aligned}
$$

High-resolution limiters:

$$
\begin{aligned}
\text{minmod}: &\quad \phi(\theta) = \text{minmod}(1,\theta) \\
\text{superbee}: &\quad \phi(\theta) = \max(0,\ \min(1,2\theta),\ \min(2,\theta)) \\
\text{MC}: &\quad \phi(\theta) = \max(0,\ \min((1+\theta)/2,\ 2,\ 2\theta)) \\
\text{van Leer}: &\quad \phi(\theta) = \frac{\theta+|\theta|}{1+|\theta|}
\end{aligned}
$$

Finite Volume Methods for Hyperbolic Problems

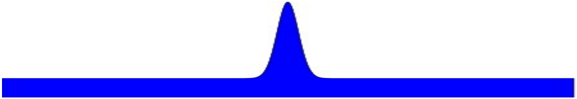
Nonlinear Scalar PDEs – Traffic flow

- Traffic flow — car following models
- Traffic flow — conservation law
- Shock formation
- Rankine-Hugoniot jump conditions
- Riemann problems

Shock formation

For nonlinear problems wave speed generally depends on q.

Waves can steepen up and form shocks
$\Longrightarrow$ even smooth data can lead to discontinuous solutions.



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Note:

- System of two equations gives rise to 2 waves.
- Each wave behaves like solution of nonlinear scalar equation.

Not quite... no linear superposition. Nonlinear interaction!

Shocks in traffic flow

Car following model

$X_j(t) =$ location of jth car at time t on one-lane road.

$$\frac{dX_j(t)}{dt} = V_j(t).$$

Velocity $V_j(t)$ of jth car varies with j and t.

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Simple model: Driver adjusts speed (instantly) depending on distance to car ahead.

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$$\frac{dX_j(t)}{dt} = V_j(t).$$

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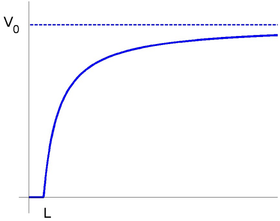
Simple model: Driver adjusts speed (instantly) depending on distance to car ahead.

$$V_j(t) = v\big(X_{j+1}(t) - X_j(t)\big)$$

for some function $v(s)$ that defines speed as a function of separation s.

Simulations: `http://www.traffic-simulation.de/`
Select ring road and watch for shock to develop.

Function $v(s)$ (speed as function of separation)



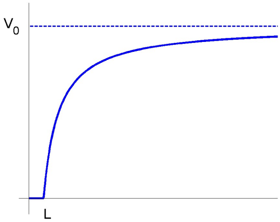
$$v(s) = \begin{cases} u_{\max}\left(1 - \frac{L}{s}\right) & \text{if } s \geq L, \\ 0 & \text{if } s \leq L. \end{cases}$$

where:

$L =$ car length

$u_{\max} =$ maximum velocity

Function $v(s)$ (speed as function of separation)



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where:

$L =$ car length

$u_{\max} =$ maximum velocity

Local density: $0 < L/s \leq 1$ $\;(s = L \implies$ bumper-to-bumper)

Continuum model

Switch to density function:

Let $q(x,t) =$ density of cars, normalized so:

Units for x: carlengths, so $x = 10$ is 10 carlengths from $x = 0$.

Units for q: cars per carlength, so $0 \leq q \leq 1$.

Total number of cars in interval $x_1 \leq x \leq x_2$ at time t is

$$\int_{x_1}^{x_2} q(x,t)\,dx$$

Flux function for traffic

$q(x,t) =$ density, $u(x,t) =$ velocity $= U(q(x,t))$.

flux: $f(q) = uq$ Conservation law: $q_t + f(q)_x = 0$.

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Constant velocity $u_{\max}$ independent of density:

$$f(q) = u_{\max} q \implies q_t + u_{\max} q_x = 0 \quad \text{(advection)}$$

Flux function for traffic

$q(x,t) =$ density, $u(x,t) =$ velocity $= U(q(x,t))$.

flux: $f(q) = uq$ Conservation law: $q_t + f(q)_x = 0$.

Constant velocity $u_{\max}$ independent of density:

$$f(q) = u_{\max} q \implies q_t + u_{\max} q_x = 0 \quad \text{(advection)}$$

Velocity varying with density:

$$V(s) = u_{\max}(1 - L/s) \implies U(q) = u_{\max}(1-q),$$

$$f(q) = u_{\max} q(1-q) \qquad \text{(quadratic nonlinearity)}$$

Characteristics for a scalar problem

$q_t + f(q)_x = 0 \implies q_t + f'(q)q_x = 0$ (if solution is smooth).

Characteristic curves satisfy $X'(t) = f'(q(X(t),t)), \;\; X(0) = x_0$.

How does solution vary along this curve?

$$\begin{aligned}\frac{d}{dt}q(X(t),t) &= q_x(X(t),t)X'(t) + q_t(X(t),t) \\ &= q_x(X(t),t)f'(q(X(t),t)) + q_t(X(t),t) \\ &= 0\end{aligned}$$

Characteristics for a scalar problem

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&= q_x(X(t),t)f'(q(X(t),t)) + q_t(X(t),t) \\
&= 0
\end{aligned}
$$

So solution is constant on characteristic
as long as solution stays smooth.

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So solution is constant on characteristic
as long as solution stays smooth.

$q(X(t),t) =$ constant $\implies X'(t)$ is constant on characteristic,
so characteristics are straight lines!

Nonlinear Burgers' equation

Conservation form: $u_t + \left(\frac{1}{2}u^2\right)_x = 0, \qquad f(u) = \frac{1}{2}u^2.$

Quasi-linear form: $u_t + uu_x = 0.$

Nonlinear Burgers' equation

Conservation form: $u_t + \left(\frac{1}{2}u^2\right)_x = 0, \qquad f(u) = \frac{1}{2}u^2$.

Quasi-linear form: $\qquad u_t + uu_x = 0.$

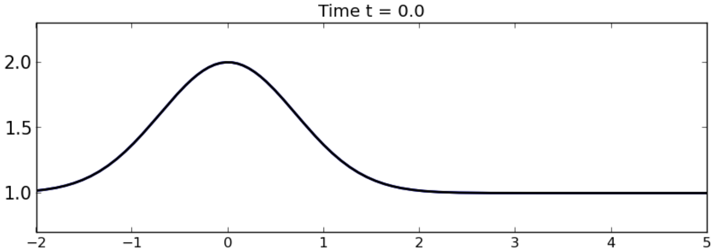
Like an advection equation with u advected with speed u.

True solution: u is constant along characteristic with speed $f'(u) = u$ until the wave "breaks" (shock forms).

Burgers' equation

Quasi-linear form: $u_t + uu_x = 0$

The solution is constant on characteristics so each value advects at constant speed equal to the value...



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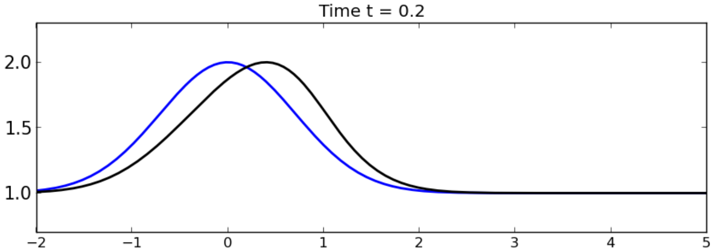

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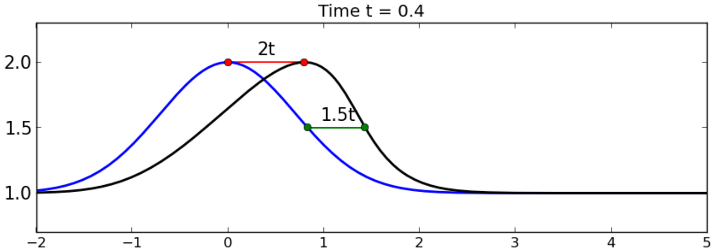

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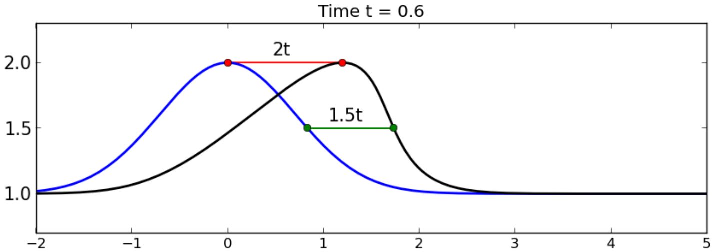

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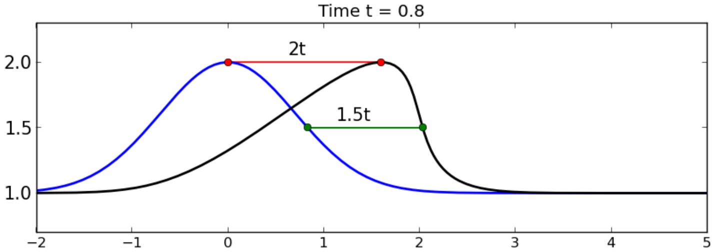

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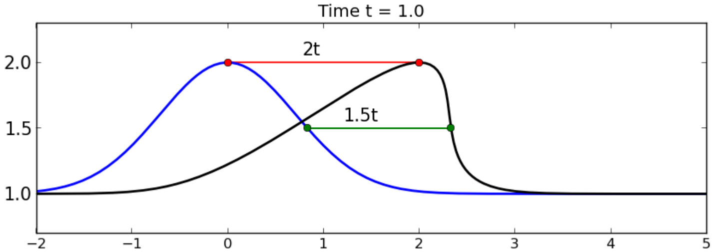

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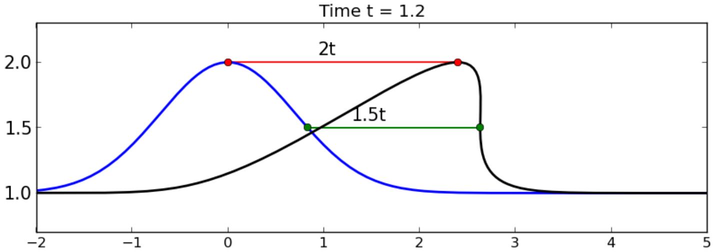

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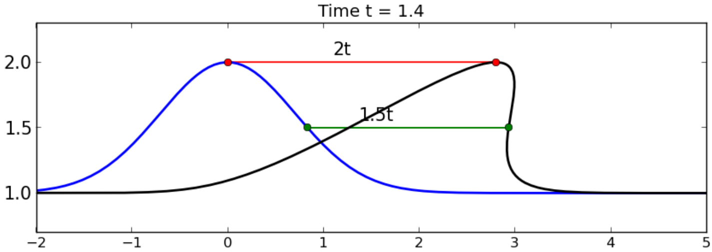

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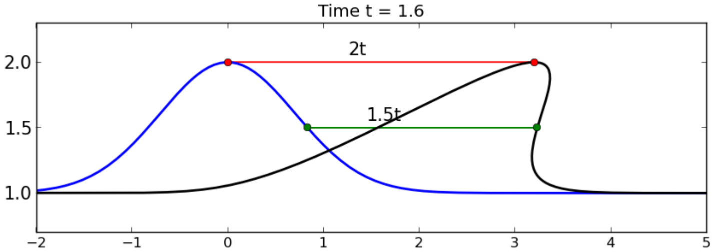

Burgers' equation

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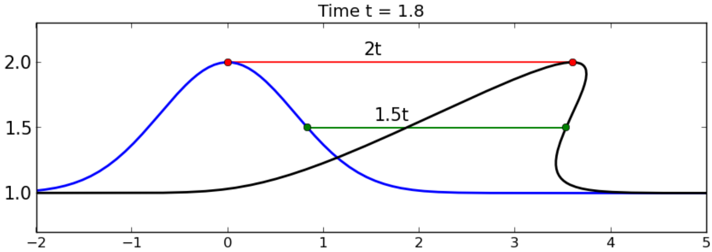

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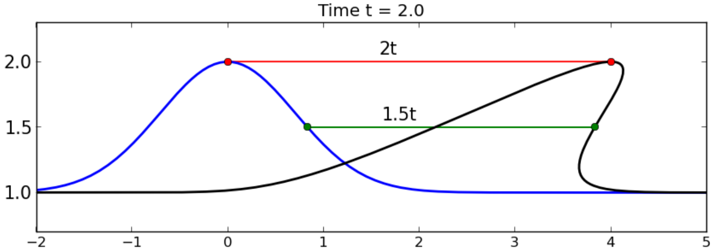


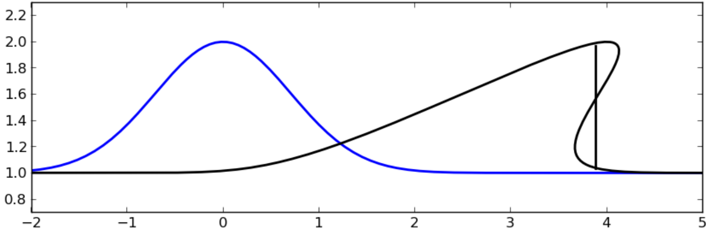
Burgers' equation

Triple valued solution is not physically possible for density.

Equal-area rule:

The area "under" the curve is conserved with time,

We must insert a shock so the two areas cut off are equal.



Vanishing Viscosity solution

Viscous Burgers' equation: $u_t + \left(\frac{1}{2}u^2\right)_x = \epsilon u_{xx}.$

This parabolic equation has a smooth C^∞ solution for all $t > 0$ for any initial data.

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Why try to solve hyperbolic equation?

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Limiting solution as $\epsilon \to 0$ gives the shock-wave solution.

Why try to solve hyperbolic equation?

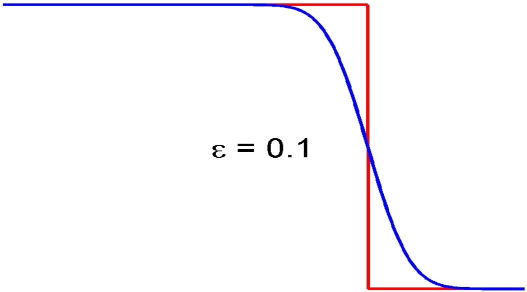
- Solving parabolic equation requires implicit method,
- Often correct value of physical "viscosity" is very small, shock profile that cannot be resolved on the desired grid
  $\Longrightarrow$ smoothness of exact solution doesn't help!

Discontinuous solutions

Vanishing Viscosity solution: The Riemann solution $q(x,t)$ is the limit as $\epsilon \to 0$ of the solution $q^\epsilon(x,t)$ of the parabolic advection-diffusion equation

$$q_t + uq_x = \epsilon q_{xx}.$$

For any $\epsilon > 0$ this has a classical smooth solution:



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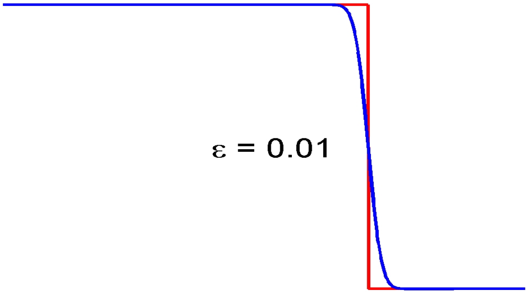

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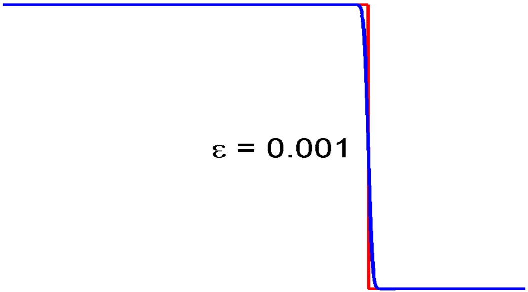

Weak solutions to $q_t + f(q)_x = 0$

$q(x,t)$ is a weak solution if it satisfies the integral form of the conservation law over all rectangles in space-time,

$$\int_{x_1}^{x_2} q(x,t_2)\,dx - \int_{x_1}^{x_2} q(x,t_1)\,dx$$
$$= \int_{t_1}^{t_2} f(q(x_1,t))\,dt - \int_{t_1}^{t_2} f(q(x_2,t))\,dt$$

Obtained by integrating

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = f(q(x_1,t)) - f(q(x_2,t))$$

from t_n to t_{n+1}.

Weak solutions to $q_t + f(q)_x = 0$

Alternatively, multiply PDE by smooth test function $\phi(x,t)$, with compact support ($\phi(x,t) \equiv 0$ for $|x|$ and t sufficiently large), and then integrate over rectangle,

$$\int_0^\infty \int_{-\infty}^\infty (q_t + f(q)_x)\phi(x,t)\,dx\,dt$$

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$$\int_0^\infty \int_{-\infty}^\infty (q_t + f(q)_x)\phi(x,t)\,dx\,dt$$

Then we can integrate by parts to get

$$\int_0^\infty \int_{-\infty}^\infty (q\phi_t + f(q)\phi_x)\,dx\,dt = -\int_{-\infty}^\infty q(x,0)\phi(x,0)\,dx.$$

$q(x,t)$ is a weak solution if this holds for all such ϕ.

Weak solutions to $q_t + f(q)_x = 0$

A function $q(x,t)$ that is piecewise smooth with jump discontinuities is a weak solution only if:

- The PDE is satisfied where q is smooth,
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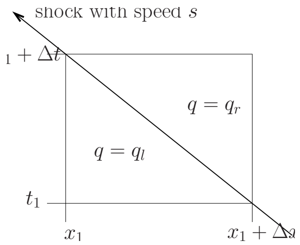
Note: The weak solution may not be unique!

Shock speed with states q_l and q_r at instant t_1



Then

$$\int_{x_1}^{x_1+\Delta x} q(x, t_1 + \Delta t)\,dx - \int_{x_1}^{x_1+\Delta x} q(x, t_1)\,dx$$
$$= \int_{t_1}^{t_1+\Delta t} f(q(x_1, t))\,dt - \int_{t_1}^{t_1+\Delta t} f(q(x_1 + \Delta x, t))\,dt.$$

Since q is essentially constant along each edge, this becomes

$$\Delta x\, q_\ell - \Delta x\, q_r = \Delta t f(q_\ell) - \Delta t f(q_r) + \mathcal{O}(\Delta t^2),$$

Taking the limit as $\Delta t \to 0$ gives

$$s(q_r - q_\ell) = f(q_r) - f(q_\ell).$$

Rankine-Hugoniot jump condition

$$s(q_r - q_\ell) = f(q_r) - f(q_\ell).$$

This must hold for any discontinuity propagating with speed s, even for systems of conservation laws.

For scalar problem, any jump allowed with speed:

$$s = \frac{f(q_r) - f(q_l)}{q_r - q_l}.$$

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For linear system, $f(q) = Aq$, this says

$$s(q_r - q_l) = A(q_r - q_l),$$

Jump must be an eigenvector, speed s the eigenvalue.

Figure 11.1 — Shock formation in traffic

Discrete cars: Continuum model: $f'(q) = u_{\max}(1-2q)$

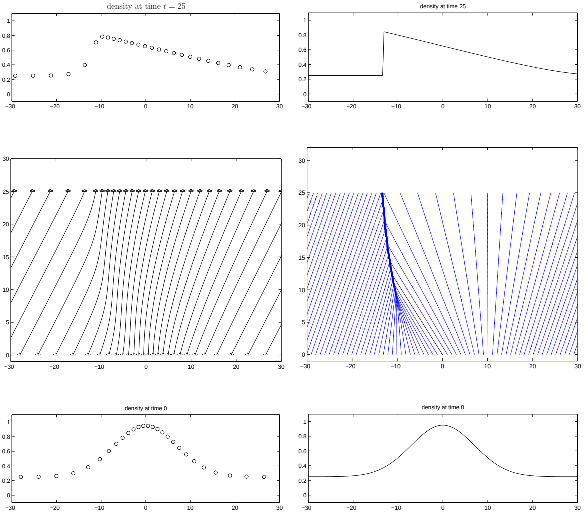


Figure 11.1 — Shock formation

(a) particle paths (car trajectories) $u(x,t) = u_{\max}(1 - q(x,t))$

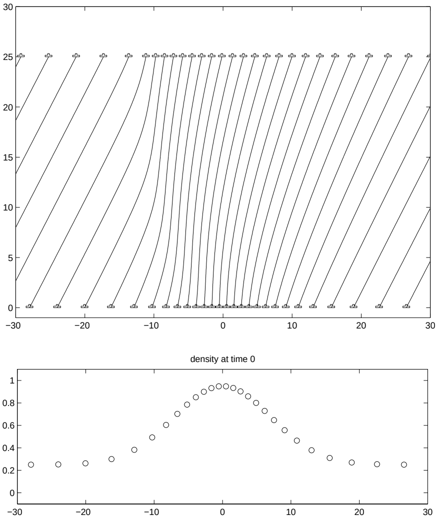
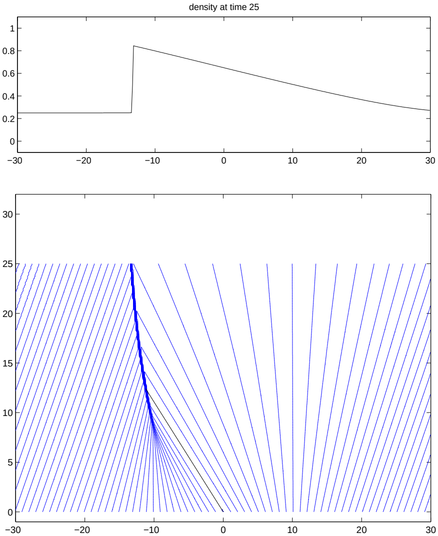


Figure 11.1 — Shock formation

(b) characteristics: $f'(q) = u_{\text{max}}(1 - 2q)$



Riemann problem for traffic flow

Initial data of the form

$$q(x,0) = \begin{cases} q_\ell & \text{if } x < 0 \\ q_r & \text{if } x \geq 0 \end{cases}$$

$$U(q) = u_{\max}(1-q), \quad f(q) = qU(q), \quad 0 \leq q \leq 1$$

Case 1: $q_\ell < q_r$, so $U(q_\ell) > U(q_r)$.

Fast moving cars approaching traffic jam
Expect shock wave.

Case 2: $q_\ell > q_r$, so $U(q_\ell) < U(q_r)$.

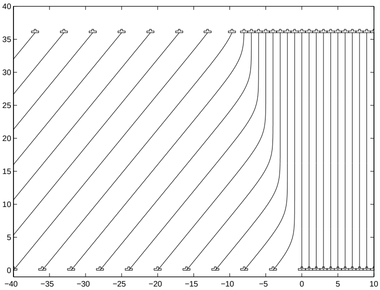
Slow moving cars can accelerate
Expect rarefaction wave.

Figure 11.2 — Traffic jam shock wave

Cars approaching red light $(q_\ell < 1, \;\; q_r = 1)$

Shock speed:

$$s = \frac{f(q_r) - f(q_\ell)}{q_r - q_\ell} = \frac{-2u_{\max} q_\ell}{1 - q_\ell} < 0 \quad \text{(for this data, could be } > 0\text{)}$$



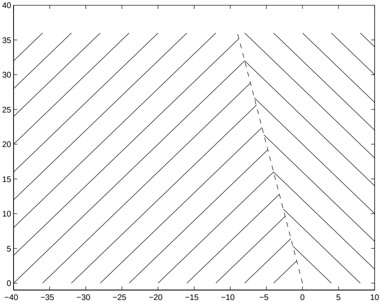
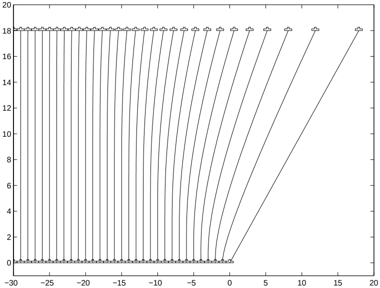


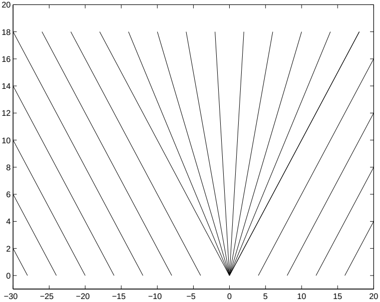
Figure 11.3 — Rarefaction wave

Cars accelerating at green light $(q_\ell = 1, \;\; q_r = 0)$

Characteristic speed $f'(q) = u_{\max}(1 - 2q)$

varies from $f'(q_\ell) = -u_{\max}$ to $f'(q_r) = u_{\max}$.





Finite Volume Methods for Hyperbolic Problems

Nonlinear Scalar Conservation Laws
Rarefaction Waves

- Form of centered rarefaction wave
- Non-uniqueness of weak solutions
- Entropy conditions

Weak solutions to $q_t + f(q)_x = 0$

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Note: The weak solution may not be unique!

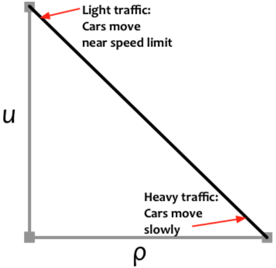
Other admissibility conditions needed to pick out the physically correct weak solution, e.g.

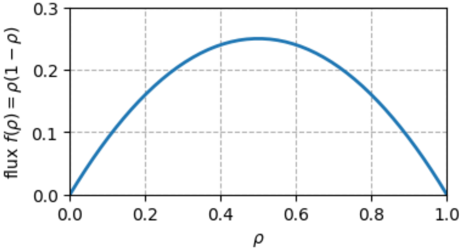
- Vanishing viscosity limit,
- “Entropy conditions”

First models due to Lighthill, Whitham, Richards in 1950's

Density of cars (per carlength): $q(x,t), \qquad 0 \leq q \leq 1.$

Desired driving speed: $U(q) = u_{\max}(1-q), \quad 0 \leq U(q) \leq u_{\max}.$





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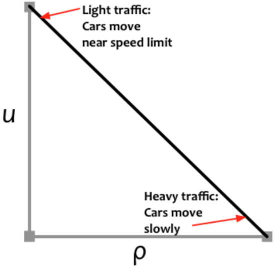
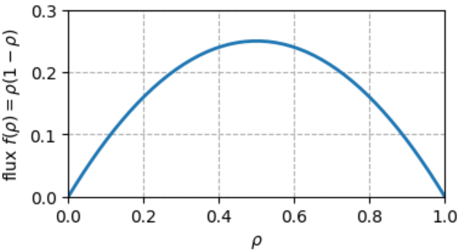
Flux: $f(q) = qU(q) = u_{\max}q(1-q), \quad 0 \le f(q) \le \frac{1}{4}u_{\max}$





Traffic flow — LWR model

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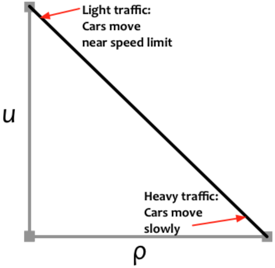
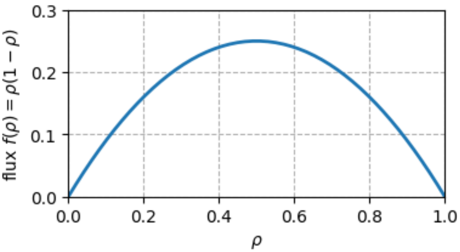
Characteristic speed: $f'(q) = u_{\max}(1-2q), \quad -u_{\max} \le f'(q) \le u_{\max}$





Jupyter Notebook for Traffic Flow

Chapter on Traffic Flow in the book

Riemann Problems and Jupyter Solutions

View static version of notebook at:

www.clawpack.org/riemann_book/html/Traffic_flow.html

Jupyter Notebook for Traffic Flow

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Riemann Problems and Jupyter Solutions

View static version of notebook at:

www.clawpack.org/riemann_book/html/Traffic_flow.html

Notebook on nonconvex scalar problems also may be useful:

www.clawpack.org/riemann_book/html/Nonconvex_scalar.html

Convex flux functions

The scalar conservation law $q_t + f(q)_x = 0$ has a convex flux if $f''(q)$ has the same sign for all q:

$$f''(q) > 0 \ \ \forall q \quad \text{or} \quad f''(q) < 0 \ \ \forall q.$$

This means that the characteristic speed $f'(q)$ is either strictly increasing or strictly decreasing as q increases.

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Consequence: The Riemann problem solution consists of either a single shock wave or a rarefaction wave.

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Nonconvex flux: The Riemann solution can consist of multiple shocks with rarefaction waves in between.

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Nonconvex flux: The Riemann solution can consist of multiple shocks with rarefaction waves in between.

Generalization of convexity for systems:
Each characteristic field must be genuinely nonlinear.

Riemann problem for traffic flow

Initial data of the form

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Fast moving cars approaching traffic jam
Expect shock wave.

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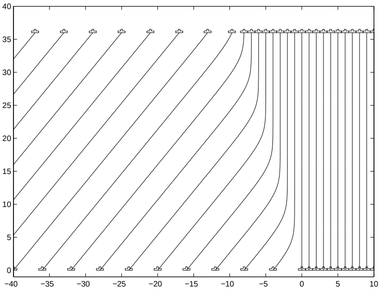

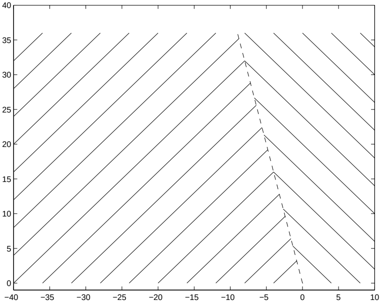

Figure 11.3 — Rarefaction wave

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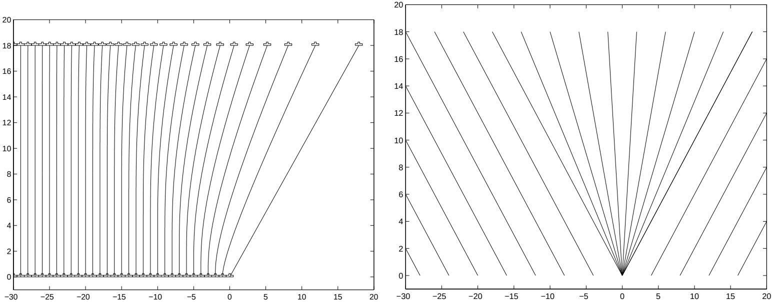
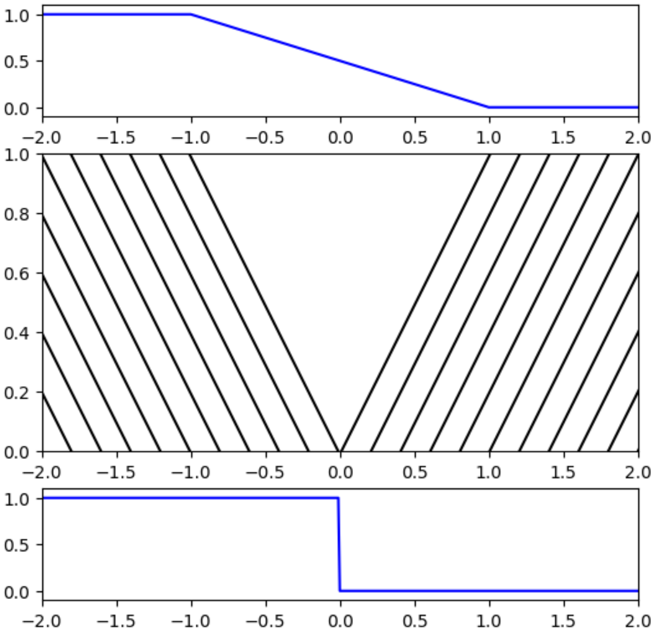
varies from $f'(q_\ell) = -u_{\max}$ to $f'(q_r) = u_{\max}$.




1.0
0.5
0.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
0.8
0.6
0.4
0.2

Centered rarefaction waves

Similarity solution with piecewise constant initial data:

$$q(x,t) = \begin{cases} q_\ell & \text{if } x/t \le f'(q_\ell) \\ \tilde{q}(x/t) & \text{if } f'(q_\ell) \le x/t \le f'(q_r) \\ q_r & \text{if } x/t \ge f'(q_r), \end{cases}$$

solves the Riemann problem for convex f, provided

$$f'(q_\ell) < f'(q_r),$$

so that characteristics spread out as time advances.

Rarefaction waves

$$q(x,t) = \begin{cases} q_\ell & \text{if } x/t \le f'(q_\ell) \\ \tilde{q}(x/t) & \text{if } f'(q_\ell) \le x/t \le f'(q_r) \\ q_r & \text{if } x/t \ge f'(q_r), \end{cases}$$

Determining $\tilde{q}(\xi)$:

$$\begin{aligned} q(x,t) = \tilde{q}(x/t) \implies q_t(x,t) &= -(x/t^2)\tilde{q}'(x/t), \\ q_x(x,t) &= (1/t)\tilde{q}'(x/t). \end{aligned}$$

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Cancel $(1/t)\tilde{q}'(x/t)$ to get:

$$-(x/t) + f'(\tilde{q}(x/t)) = 0 \quad \text{or} \quad f'(\tilde{q}(\xi)) = \xi.$$

Centered rarefaction for traffic flow

Take $u_{\max} = 1$.

$$f(q) = q(1-q) \implies f'(q) = (1-2q).$$

Solving $f'(\tilde{q}(\xi)) = \xi$ gives

$$(1-2\tilde{q}(\xi)) = \xi \implies \tilde{q}(\xi) = \frac{1}{2}(1-\xi)$$

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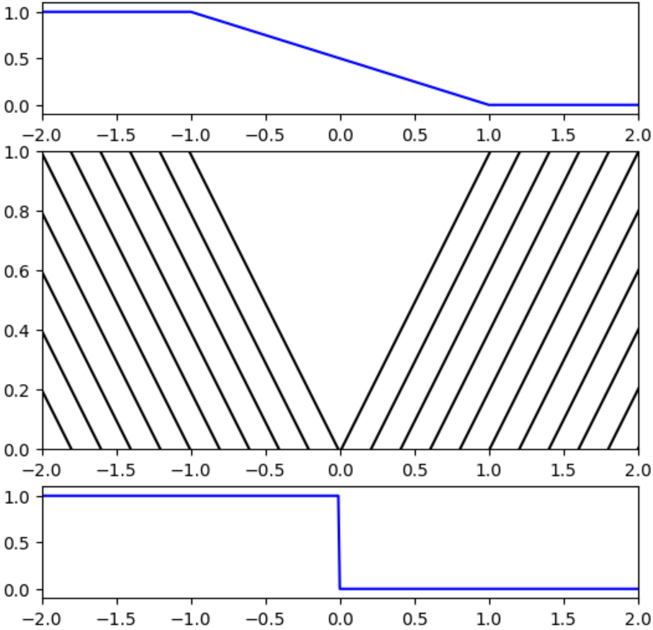
Solution is linear in x at each t.

(Since $f(q)$ was quadratic, not true in general.)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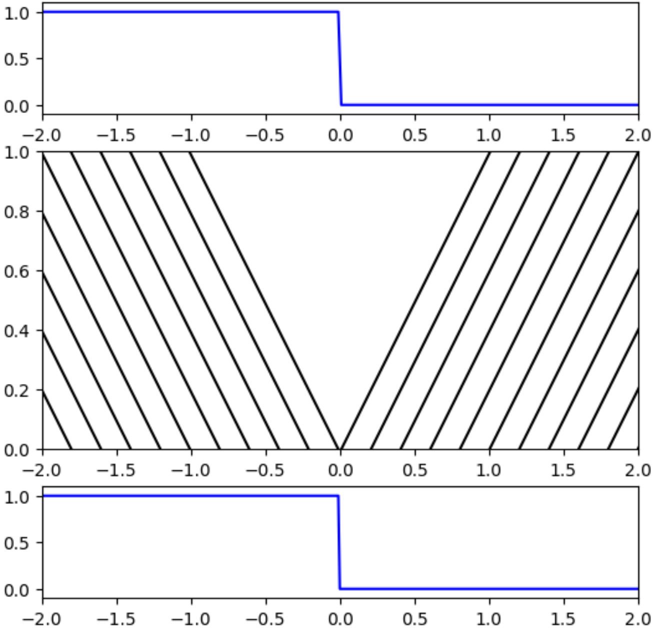
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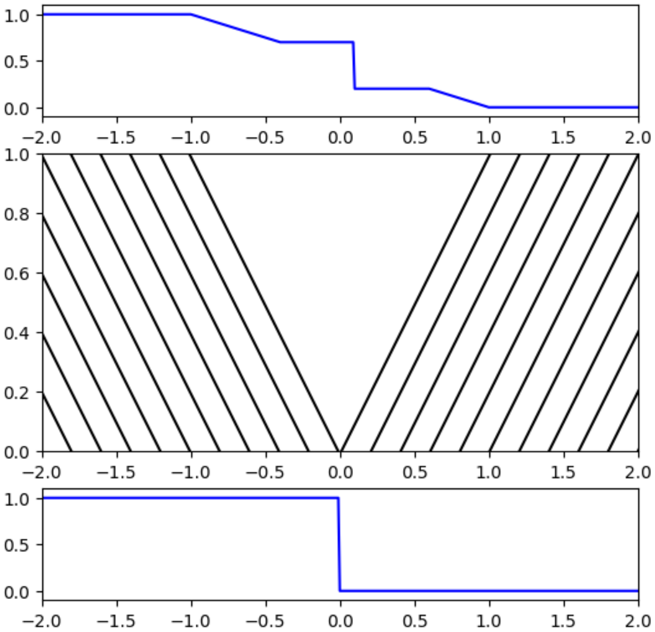
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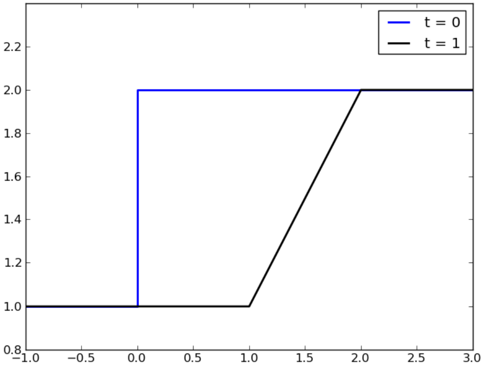
Weak solutions to Burgers' equation

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0, \qquad u_\ell = 1, \;\; u_r = 2$$

Characteristic speed: u Rankine-Hugoniot speed: $\frac{1}{2}(u_\ell + u_r)$.

"Physically correct" rarefaction wave solution:



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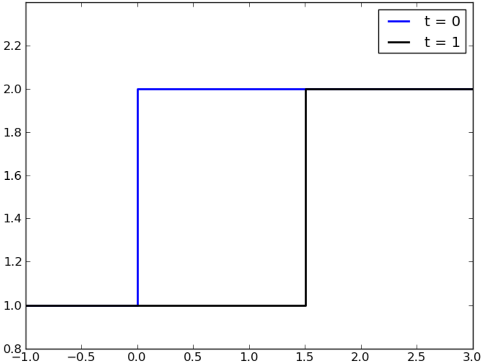
Entropy violating weak solution:



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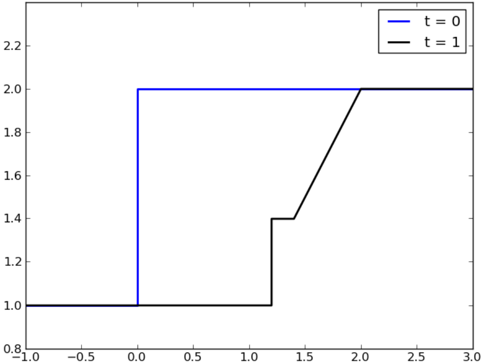
Another Entropy violating weak solution:



Vanishing viscosity solution

We want $q(x,t)$ to be the limit as $\epsilon \to 0$ of solution to

$$q_t + f(q)_x = \epsilon q_{xx}.$$

This selects a unique weak solution:

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Lax Entropy Condition:

A discontinuity propagating with speed s in the solution of a convex scalar conservation law is admissible only if $f'(q_\ell) > s > f'(q_r)$, where $s = (f(q_r) - f(q_\ell))/(q_r - q_\ell)$.

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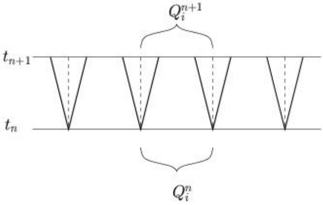
Note: This means characteristics must approach shock from both sides as t advances, not move away from shock!

Finite Volume Methods for Scalar Conservation Laws

- Godunov's method
- Fluxes, cell averages, and wave propagation form
- Transonic rarefactions waves
- Approximate Riemann solver with entropy fix

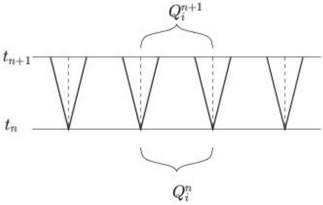
Godunov's Method for $q_t + f(q)_x = 0$



1. Solve Riemann problems at all interfaces, yielding waves $\mathcal{W}^p_{i-1/2}$ and speeds $s^p_{i-1/2}$, for $p = 1,\ 2,\ \ldots,\ m$.

Riemann problem: Original equation with piecewise constant data.

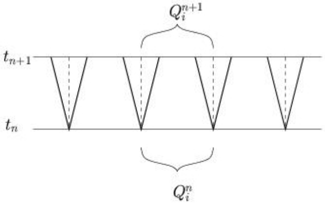
Godunov's Method for $q_t + f(q)_x = 0$



Then either:

1. Compute new cell averages by integrating over cell at t_{n+1},

Godunov's Method for $q_t + f(q)_x = 0$

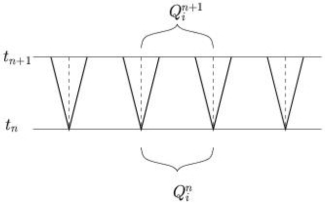


Then either:

1. Compute new cell averages by integrating over cell at t_{n+1},
2. Compute fluxes at interfaces and flux-difference:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[F_{i+1/2}^n - F_{i-1/2}^n]$$

Godunov's Method for $q_t + f(q)_x = 0$



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$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[F_{i+1/2}^n - F_{i-1/2}^n]$$

3. Update cell averages by contributions from all waves entering cell:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}]$$

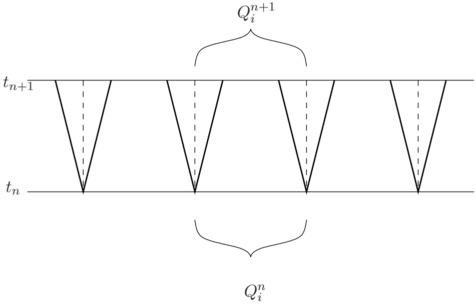
where $\mathcal{A}^\pm \Delta Q_{i-1/2} = \sum_{i=1}^{m} (s_{i-1/2}^p)^\pm \mathcal{W}_{i-1/2}^p$.

Godunov's method with flux differencing

Q_i^n defines a piecewise constant function

$$\tilde{q}^n(x, t_n) = Q_i^n \text{ for } x_{i-1/2} < x < x_{i+1/2}$$

Discontinuities at cell interfaces $\implies$ Riemann problems.



$$\tilde{q}^n(x_{i-1/2}, t) \equiv q^{\psi}(Q_{i-1}, Q_i) \quad \text{for } t > t_n.$$

$$F_{i-1/2}^n = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(q^{\psi}(Q_{i-1}^n, Q_i^n))\, dt = f(q^{\psi}(Q_{i-1}^n, Q_i^n)).$$

Riemann problem for scalar nonlinear problem

$q_t + f(q)_x = 0$ with data

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x \geq 0 \end{cases}$$

Similarity solution: $q(x,t) = \tilde{q}(x/t)$ so $q(0,t) =$ constant.

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Similarity solution: $q(x,t) = \tilde{q}(x/t)$ so $q(0,t) =$ constant.

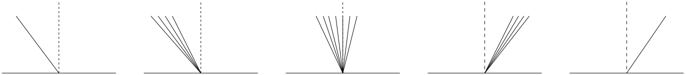
For convex flux (e.g. Burgers' or traffic flow with quadratic flux), the Riemann solution consists of:

- Shock wave if $f'(q_l) > f'(q_r)$,
- Rarefaction wave if $f'(q_l) < f'(q_r)$.

Riemann problem for scalar convex flux

$q_t + f(q)_x = 0$ with $f''(q)$ of one sign, so $f'(q)$ is monotone.

6 possible cases:



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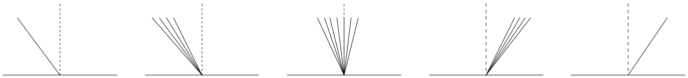

Case 3: Transonic rarefaction $\quad f'(q_l) < 0 < f'(q_r)$

Convex $\implies$ there is at most one point q_s where $f'(q_s) = 0$.
q_s is called the sonic point or stagnation point.

Terminology from gas dynamics: wave speeds $u \pm c$
$\implies$ sonic points where $|u| = c$, supersonic if $|u| > c$.

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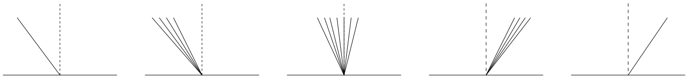

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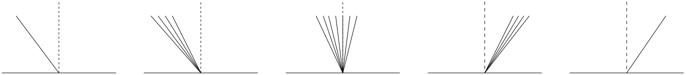
Case 6: Shock moving at speed 0. Then $f(q_l) = f(q_r)$

Godunov flux for scalar problem



The Godunov flux function for the case $f''(q) > 0$ is

$$F^n_{i-1/2} = \begin{cases} f(Q_i) & \text{if } s \le 0 \text{ and } Q_i < q_s \\ f(Q_{i-1}) & \text{if } s \ge 0 \text{ and } Q_{i-1} > q_s \\ f(q_s) & \text{if } Q_{i-1} \le q_s \le Q_i. \end{cases}$$

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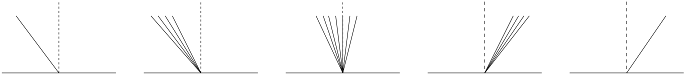

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where $s = \frac{f(Q_i)-f(Q_{i-1})}{Q_i - Q_{i-1}}$ is the Rankine-Hugoniot shock speed.

A more general formula: (for any continuous $f(q)$)

$$F^n_{i-1/2} = \begin{cases} \displaystyle\min_{Q_{i-1} \le q \le Q_i} f(q) & \text{if } Q_{i-1} \le Q_i \\ \displaystyle\max_{Q_i \le q \le Q_{i-1}} f(q) & \text{if } Q_i \le Q_{i-1}, \end{cases}$$

Upwind wave-propagation algorithm

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right].$$

Fluctuations:

$$\mathcal{A}^-\Delta Q_{i-1/2} = \sum_{p=1}^{m}(\lambda^p)^-\mathcal{W}^p_{i-1/2} \quad = A^-\,(Q_i - Q_{i-1}),$$

$$\mathcal{A}^+\Delta Q_{i-1/2} = \sum_{p=1}^{m}(\lambda^p)^+\mathcal{W}^p_{i-1/2} \quad = A^+\,(Q_i - Q_{i-1}),$$

For a linear system, $s^p = \lambda^p$ and waves $\mathcal{W}^p$ are eigenvectors.

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For scalar advection $m = 1$, only one wave.
$\mathcal{W}_{i-1/2} = \Delta Q_{i-1/2} = Q_i - Q_{i-1}$ and $s = u$,

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Godunov for scalar nonlinear in terms of fluctuations

Flux-differencing formula:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+1/2} - F_{i-1/2}\right].$$

Can be rewritten in terms of fluctuations as

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If we define

$$\mathcal{A}^-\Delta Q_{i-1/2} = F_{i-1/2} - f(Q_{i-1}) \quad \text{left-going fluctuation}$$

$$\mathcal{A}^+\Delta Q_{i-1/2} = f(Q_i) - F_{i-1/2} \quad \text{right-going fluctuation}$$

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Agrees with previous definition for linear systems.

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$\mathcal{A}^-\Delta Q_{i-1/2} = F_{i-1/2} - f(Q_{i-1})$ left-going fluctuation

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For high-resolution method, we also need to define a wave $\mathcal{W}$ and speed s,

$$\mathcal{W}_{i-1/2} = Q_i - Q_{i-1},$$

$$s_{i-1/2} = \begin{cases} (f(Q_i) - f(Q_{i-1}))/(Q_i - Q_{i-1}) & \text{if } Q_{i-1} \neq Q_i \\ f'(Q_i) & \text{if } Q_{i-1} = Q_i. \end{cases}$$

Approximate Riemann solver

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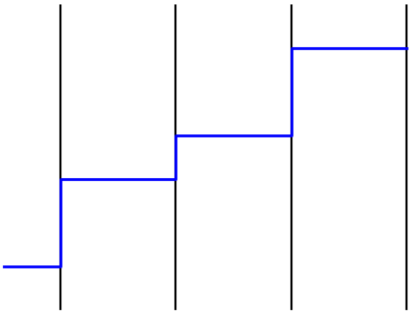
This is exact solution for shock.

Replacing rarefaction with shock: also exact (after averaging), except in case of transonic rarefaction.

Rarefaction waves in wave propagation method

Initial data giving rarefaction waves (Burgers' equation):



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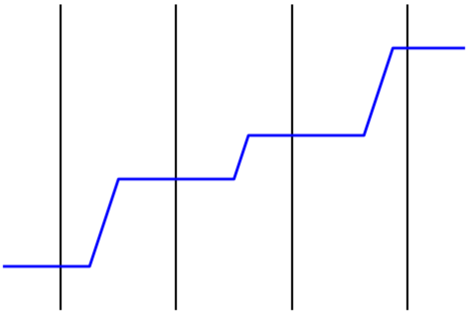
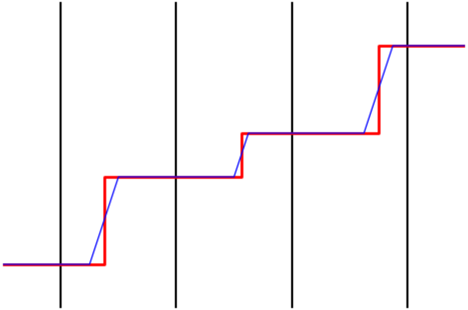


Rarefaction waves in wave propagation method

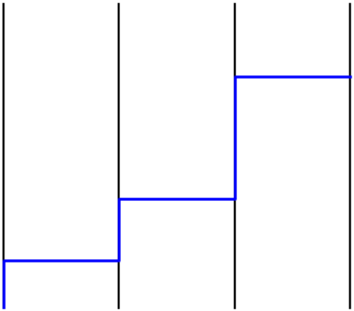
Initial data giving rarefaction waves (Burgers' equation):

Approximating rarefaction with shock gives same cell average.



Rarefaction waves in wave propagation method

Initial data with a transonic rarefaction (Burgers' equation):



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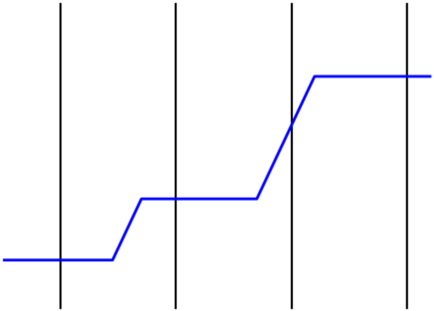
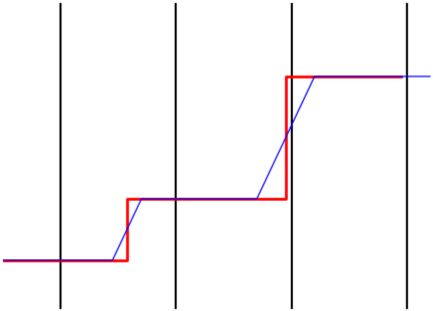


Rarefaction waves in wave propagation method

Initial data with a transonic rarefaction (Burgers' equation):

Approximating rarefaction with shock gives poor approximation!



Entropy fix

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right].$$

Define wave $\mathcal{W}$ and speed s using Rankine-Hugoniot:
(both for $\mathcal{A}^+\Delta Q_{i-1/2}$, $\mathcal{A}^-\Delta Q_{i+1/2}$ and for corrections)

$$\mathcal{W}_{i-1/2} = Q_i - Q_{i-1},$$

$$s_{i-1/2} = \begin{cases} (f(Q_i) - f(Q_{i-1}))/(Q_i - Q_{i-1}) & \text{if } Q_{i-1} \neq Q_i \\ f'(Q_i) & \text{if } Q_{i-1} = Q_i. \end{cases}$$

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Fix for transonic rarefaction: But if $f'(Q_{i-1}) < 0 < f'(Q_i)$, use:

$$\mathcal{A}^-\Delta Q_{i-1/2} = f(q_s) - f(Q_{i-1}) \quad \text{left-going fluctuation}$$
$$\mathcal{A}^+\Delta Q_{i-1/2} = f(Q_i) - f(q_s) \quad \text{right-going fluctuation}$$

Wave limiters for scalar nonlinear

For $q_t + f(q)_x = 0$, just one wave: $\mathcal{W}_{i-1/2} = Q_i^n - Q_{i-1}^n$.

Godunov:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right].$$

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"Lax-Wendroff":

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right] - \frac{\Delta t}{\Delta x}(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2})$$

$$\tilde{F}_{i-1/2} = \frac{1}{2}\left(1 - \left|\frac{s_{i-1/2}\Delta t}{\Delta x}\right|\right)|s_{i-1/2}|\mathcal{W}_{i-1/2}$$

Wave limiters for scalar nonlinear

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$$\tilde{F}_{i-1/2} = \frac{1}{2}\left(1 - \left|\frac{s_{i-1/2}\Delta t}{\Delta x}\right|\right)|s_{i-1/2}|\mathcal{W}_{i-1/2}$$

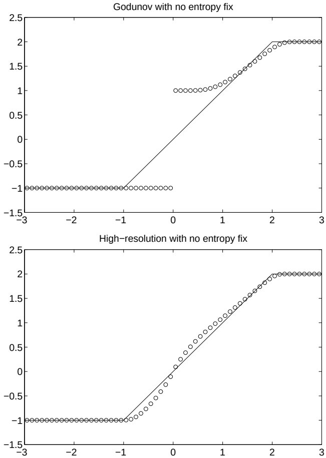
High-resolution method:

$$\tilde{F}_{i-1/2} = \frac{1}{2}\left(1 - \left|\frac{s_{i-1/2}\Delta t}{\Delta x}\right|\right)|s_{i-1/2}|\widetilde{\mathcal{W}}_{i-1/2}$$

$\widetilde{\mathcal{W}}_{i-1/2} = \phi(\theta)\,\mathcal{W}_{i-1/2}$, where $\theta_{i-1/2} = \mathcal{W}_{I-1/2}/\mathcal{W}_{i-1/2}$.

Entropy-violating numerical solutions

Riemann problem for Burgers' equation with $q_l = -1$ and $q_r = 2$:

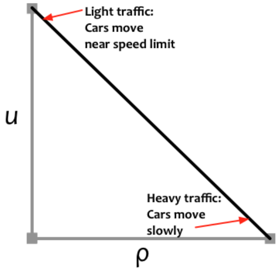


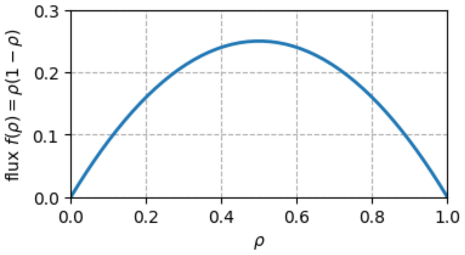
Traffic flow — LWR model

First models due to Lighthill, Whitham, Richards in 1950's

Density of cars (per carlength): $q(x, t), \qquad 0 \leq q \leq 1.$

Desired driving speed: $U(q) = u_{\max}(1 - q), \quad 0 \leq U(q) \leq u_{\max}.$





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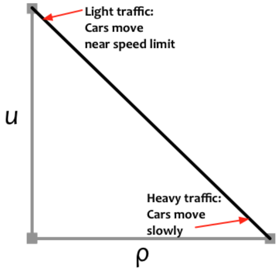
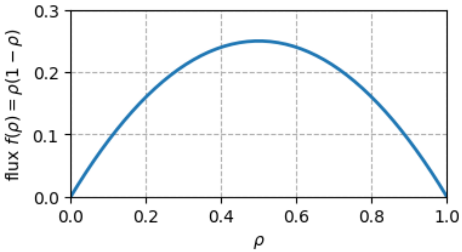
Flux: $f(q) = qU(q) = u_{\max}q(1-q), \quad 0 \le f(q) \le \frac{1}{4}u_{\max}$





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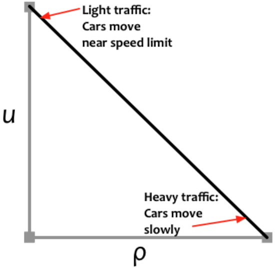
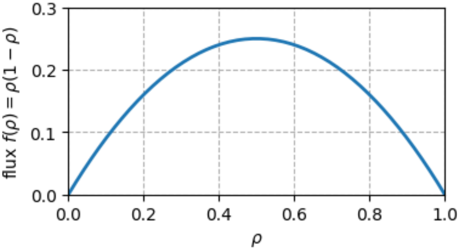
Characteristic speed: $f'(q) = u_{\max}(1-2q), \quad -u_{\max} \le f'(q) \le u_{\max}$





Jupyter Notebook for Traffic Flow

Chapter on Traffic Flow in the book

Riemann Problems and Jupyter Solutions

View static version of notebook at:

www.clawpack.org/riemann_book/html/Traffic_flow.html

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Notebook on nonconvex scalar problems also may be useful:

www.clawpack.org/riemann_book/html/Nonconvex_scalar.html

Convex flux functions

The scalar conservation law $q_t + f(q)_x = 0$ has a convex flux if $f''(q)$ has the same sign for all q:

$$f''(q) > 0 \;\; \forall q \quad \text{or} \quad f''(q) < 0 \;\; \forall q.$$

This means that the characteristic speed $f'(q)$ is either strictly increasing or strictly decreasing as q increases.

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Consequence: The Riemann problem solution consists of either a single shock wave or a rarefaction wave.

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Nonconvex flux: The Riemann solution can consist of multiple shocks with rarefaction waves in between.

Generalization of convexity for systems:
Each characteristic field must be genuinely nonlinear.

Riemann problem for traffic flow

Initial data of the form

$$q(x,0) = \begin{cases} q_\ell & \text{if } x < 0 \\ q_r & \text{if } x \geq 0 \end{cases}$$

$$U(q) = u_{\max}(1-q), \quad f(q) = qU(q), \quad 0 \leq q \leq 1$$

Case 1: $q_\ell < q_r$, so $U(q_\ell) > U(q_r)$, $f'(q_\ell) > f'(q_r)$.

Fast moving cars approaching traffic jam
Expect shock wave.

Case 2: $q_\ell > q_r$, so $U(q_\ell) < U(q_r)$, $f'(q_\ell) < f'(q_r)$.

Slow moving cars can accelerate
Expect rarefaction wave.

Figure 11.2 — Traffic jam shock wave

Cars approaching red light $(q_\ell < 1, \quad q_r = 1)$

Shock speed:

$$s = \frac{f(q_r) - f(q_\ell)}{q_r - q_\ell} = \frac{-2u_{\max} q_\ell}{1 - q_\ell} < 0 \quad \text{(for this data, could be } > 0\text{)}$$

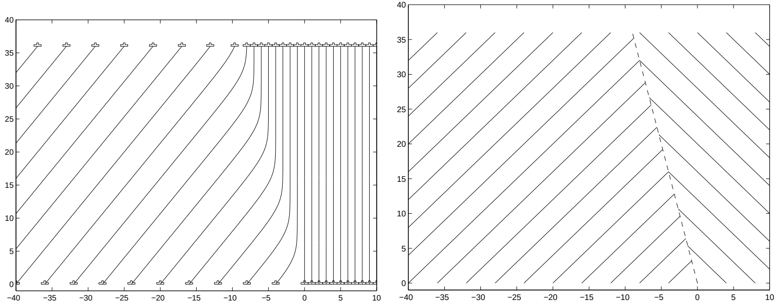
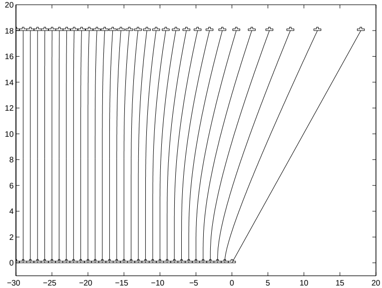


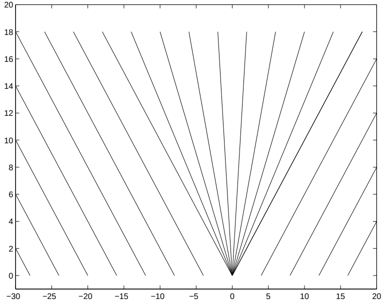
Figure 11.3 — Rarefaction wave

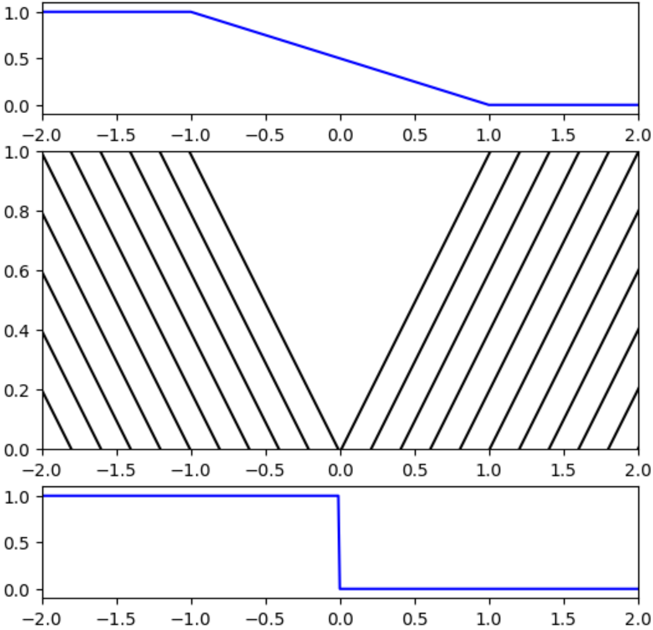
Cars accelerating at green light $(q_\ell = 1, \;\; q_r = 0)$

Characteristic speed $f'(q) = u_{\max}(1 - 2q)$

varies from $f'(q_\ell) = -u_{\max}$ to $f'(q_r) = u_{\max}$.



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Centered rarefaction waves

Similarity solution with piecewise constant initial data:

$$q(x,t) = \begin{cases} q_\ell & \text{if } x/t \le f'(q_\ell) \\ \tilde{q}(x/t) & \text{if } f'(q_\ell) \le x/t \le f'(q_r) \\ q_r & \text{if } x/t \ge f'(q_r), \end{cases}$$

solves the Riemann problem for convex f, provided

$$f'(q_\ell) < f'(q_r),$$

so that characteristics spread out as time advances.

Rarefaction waves

$$q(x,t) = \begin{cases} q_\ell & \text{if } x/t \le f'(q_\ell) \\ \tilde{q}(x/t) & \text{if } f'(q_\ell) \le x/t \le f'(q_r) \\ q_r & \text{if } x/t \ge f'(q_r), \end{cases}$$

Determining $\tilde{q}(\xi)$:

$$\begin{aligned} q(x,t) = \tilde{q}(x/t) \implies q_t(x,t) &= -(x/t^2)\tilde{q}'(x/t), \\ q_x(x,t) &= (1/t)\tilde{q}'(x/t). \end{aligned}$$

Rarefaction waves

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Quasilinear form: $q_t(x,t) + f'(q(x,t))q_x(x,t) = 0$ leads to

$$-(x/t^2)\tilde{q}'(x/t) + f'(\tilde{q}(x/t))(1/t)\tilde{q}'(x/t) = 0$$

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Cancel $(1/t)\tilde{q}'(x/t)$ to get:

$$-(x/t) + f'(\tilde{q}(x/t)) = 0 \quad \text{or} \quad f'(\tilde{q}(\xi)) = \xi.$$

Centered rarefaction for traffic flow

Take $u_{\max} = 1$.

$$f(q) = q(1-q) \implies f'(q) = (1-2q).$$

Solving $f'(\tilde{q}(\xi)) = \xi$ gives

$$(1 - 2\tilde{q}(\xi)) = \xi \implies \tilde{q}(\xi) = \frac{1}{2}(1-\xi)$$

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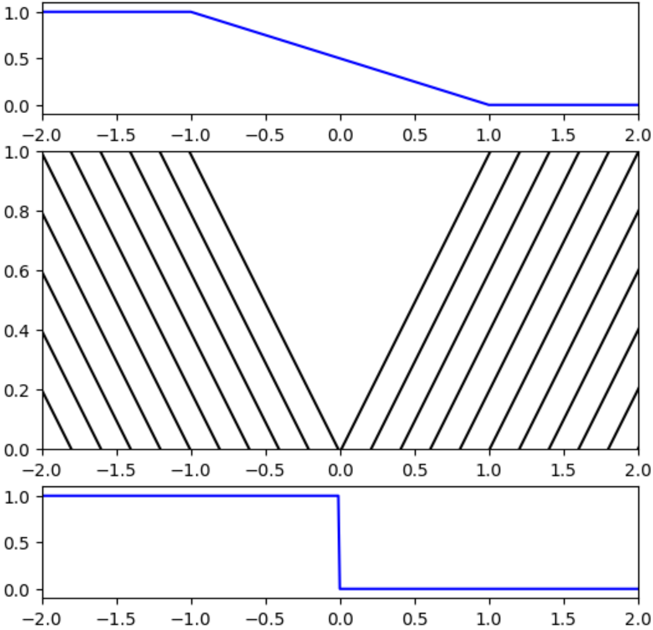
$$q(x,t) = \begin{cases} q_\ell & \text{if } x/t \leq f'(q_\ell) \\ \frac{1}{2}(1 - x/t) & \text{if } f'(q_\ell) \leq x/t \leq f'(q_r) \\ q_r & \text{if } x/t \geq f'(q_r), \end{cases}$$

Solution is linear in x at each t.

(Since $f(q)$ was quadratic, not true in general.)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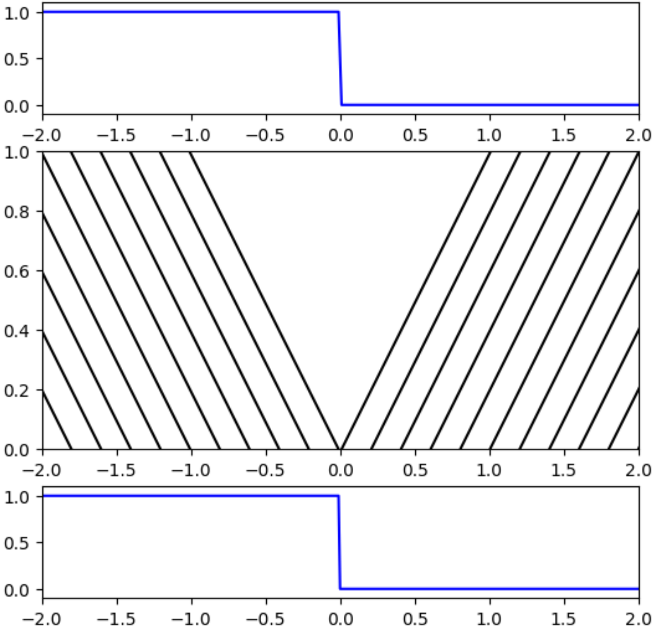
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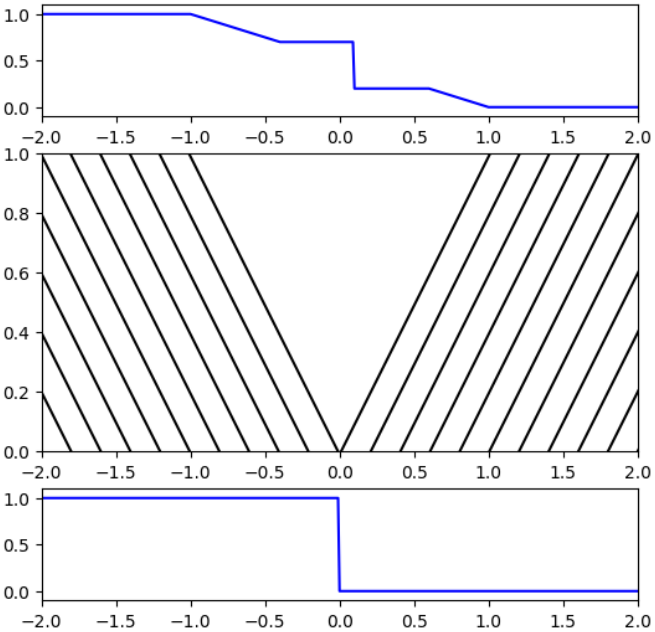
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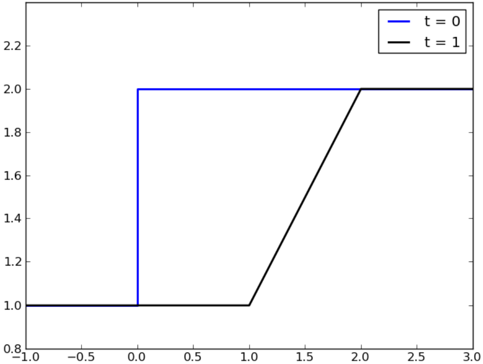
Weak solutions to Burgers' equation

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0, \qquad u_\ell = 1, \;\; u_r = 2$$

Characteristic speed: u Rankine-Hugoniot speed: $\frac{1}{2}(u_\ell + u_r)$.

"Physically correct" rarefaction wave solution:



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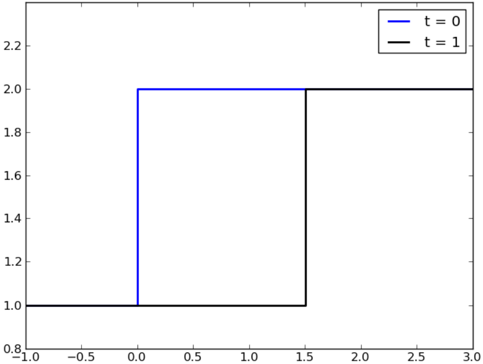
Entropy violating weak solution:



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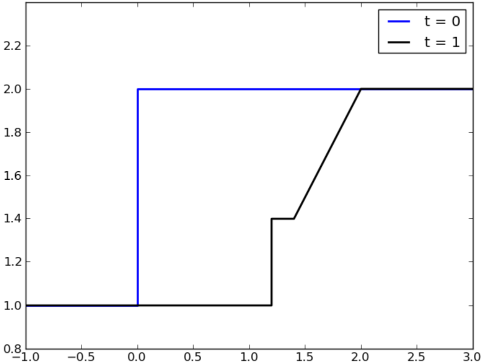
Another Entropy violating weak solution:



Vanishing viscosity solution

We want $q(x,t)$ to be the limit as $\epsilon \to 0$ of solution to

$$q_t + f(q)_x = \epsilon q_{xx}.$$

This selects a unique weak solution:

- Shock if $f'(q_l) > f'(q_r)$,
- Rarefaction if $f'(q_l) < f'(q_r)$.

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Lax Entropy Condition:

A discontinuity propagating with speed s in the solution of a convex scalar conservation law is admissible only if $f'(q_\ell) > s > f'(q_r)$, where $s = (f(q_r) - f(q_\ell))/(q_r - q_\ell)$.

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Note: This means characteristics must approach shock from both sides as t advances, not move away from shock!

Admissible Solutions and Entropy Functions

- Weak solutions and conservation form
- Admissibility / entropy conditions
- Entropy functions
- Weak form of entropy condition
- Relation to vanishing viscosity solution

Weak solutions to $q_t + f(q)_x = 0$

$q(x,t)$ is a weak solution if it satisfies the integral form of the conservation law over all rectangles in space-time,

$$\int_{x_1}^{x_2} q(x,t_2)\,dx - \int_{x_1}^{x_2} q(x,t_1)\,dx$$
$$= \int_{t_1}^{t_2} f(q(x_1,t))\,dt - \int_{t_1}^{t_2} f(q(x_2,t))\,dt$$

Obtained by integrating

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = f(q(x_1,t)) - f(q(x_2,t))$$

from t_n to t_{n+1}.

Weak solutions to $q_t + f(q)_x = 0$

Alternatively, multiply PDE by smooth test function $\phi(x,t)$, with compact support ($\phi(x,t) \equiv 0$ for $|x|$ and t sufficiently large), and then integrate over rectangle,

$$\int_0^\infty \int_{-\infty}^\infty \big(q_t + f(q)_x\big)\phi(x,t)\,dx\,dt$$

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$$\int_0^\infty \int_{-\infty}^\infty (q_t + f(q)_x)\phi(x,t)\,dx\,dt$$

Then we can integrate by parts to get

$$\int_0^\infty \int_{-\infty}^\infty (q\phi_t + f(q)\phi_x)\,dx\,dt = -\int_{-\infty}^\infty q(x,0)\phi(x,0)\,dx.$$

$q(x,t)$ is a weak solution if this holds for all such ϕ.

Weak solutions to $q_t + f(q)_x = 0$

A function $q(x,t)$ that is piecewise smooth with jump discontinuities is a weak solution only if:

- The PDE is satisfied where q is smooth,
- The jump discontinuities all satisfy the Rankine-Hugoniot conditions.

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Note: The weak solution may not be unique!

Other admissibility conditions needed to pick out the physically correct weak solution, e.g.

- Vanishing viscosity limit,
- “Entropy conditions”

Conservation form

The method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

is in conservation form.

The total mass is conserved up to fluxes at the boundaries:

$$\Delta x \sum_i Q_i^{n+1} = \Delta x \sum_i Q_i^n - \Delta t(F_{+\infty} - F_{-\infty}).$$

Note: an isolated shock must travel at the right speed!

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} q(x,t)\,dx = F(x_1) - F(x_2).$$

Nonlinear scalar conservation laws

Burgers' equation: $u_t + \left(\frac{1}{2}u^2\right)_x = 0$.

Quasilinear form: $u_t + uu_x = 0$.

These are equivalent for smooth solutions, not for shocks!

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These are equivalent for smooth solutions, not for shocks!

Upwind methods for $u > 0$:

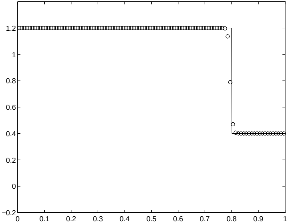
Conservative: $U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}((U_i^n)^2 - (U_{i-1}^n)^2)\right)$

Quasilinear: $U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}U_i^n(U_i^n - U_{i-1}^n)$.

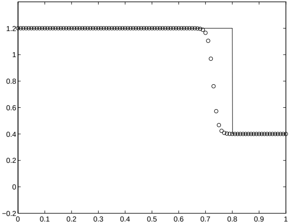
Ok for smooth solutions, not for shocks!

Importance of conservation form

Solution to Burgers' equation using conservative upwind:



Solution to Burgers' equation using quasilinear upwind:



Weak solutions depend on the conservation law

The conservation laws

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0$$

and

$$\left(u^2\right)_t + \left(\frac{2}{3}u^3\right)_x = 0$$

both have the same quasilinear form

$$u_t + uu_x = 0$$

but have different weak solutions,

different shock speeds!

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$$u_t + \left(\frac{1}{2}u^2\right)_x = 0 \quad \Longrightarrow \quad s = \frac{1}{2}\frac{u_r^2 - u_\ell^2}{u_r - u_l} = \frac{1}{2}(u_\ell + u_r).$$

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Speeds are different in general $\Longrightarrow$ different weak solutions.

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Suppose the method is conservative and consistent with $q_t + f(q)_x = 0$,

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and Lipschitz continuity of $\mathcal{F}$.

If a sequence of discrete approximations converge to a function $q(x, t)$ as the grid is refined, then this function is a weak solution of the conservation law.

Note:

Does not guarantee a sequence converges (need stability).

Two sequences might converge to different weak solutions.

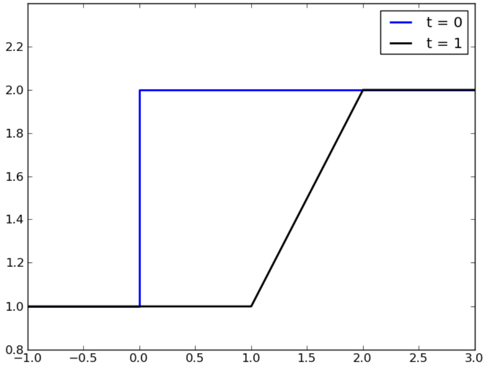
Also need to satisfy an entropy condition.

Weak solutions to Burgers' equation

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0, \qquad u_\ell = 1, \;\; u_r = 2$$

Characteristic speed: u Rankine-Hugoniot speed: $\frac{1}{2}(u_\ell + u_r)$.

"Physically correct" rarefaction wave solution:



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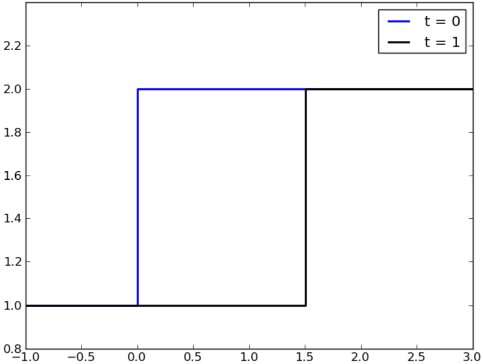
Entropy violating weak solution:



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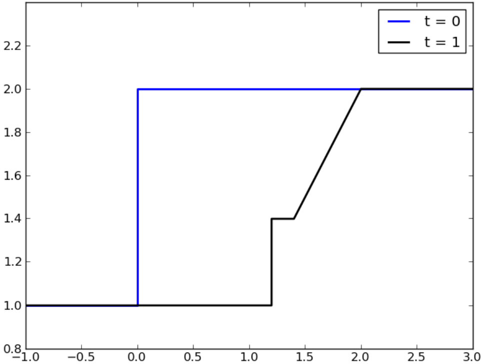
Another Entropy violating weak solution:



Vanishing viscosity solution

We want $q(x,t)$ to be the limit as $\epsilon \to 0$ of solution to

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This selects a unique weak solution:

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Entropy (admissibility) conditions

We generally require additional conditions on a weak solution to a conservation law, to pick out the unique solution that is physically relevant.

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Entropy functions: Function of q that "behaves like" physical entropy for the conservation law being studied.

NOTE: Mathematical entropy functions generally chosen to
decrease for admissible solutions,
increase for entropy-violating solutions.

Entropy functions for convex scalar problems

Entropy function: $\eta : \mathbb{R} \to \mathbb{R}$ Entropy flux: $\psi : \mathbb{R} \to \mathbb{R}$

chosen so that $\eta(q)$ is convex ($\eta''(q) > 0$) (not < 0) and:

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- Entropy decreases across an admissible shock wave.

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Weak form:

$$\begin{aligned}\int_{x_1}^{x_2} \eta(q(x,t_2))\,dx \;\; \leq \;\; & \int_{x_1}^{x_2} \eta(q(x,t_1))\,dx \\ & + \int_{t_1}^{t_2} \psi(q(x_1,t))\,dt - \int_{t_1}^{t_2} \psi(q(x_2,t))\,dt\end{aligned}$$

with equality where solution is smooth.

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with equality where solution is smooth. $\eta(q)_t + \psi(q)_t \le 0$

Entropy functions

How to find η and ψ satisfying this?

$$\eta(q)_t + \psi(q)_x = 0$$

For smooth solutions gives

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Example: Burgers' equation, $f'(u) = u$ and take $\eta(u) = u^2$.

Then $\psi'(u) = 2u^2 \implies$ Entropy function: $\psi(u) = \frac{2}{3}u^3$.

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$$u_t + \left(\frac{1}{2}u^2\right)_x = 0 \quad \Longrightarrow \quad s = \frac{1}{2}\frac{u_r^2 - u_\ell^2}{u_r - u_l} = \frac{1}{2}(u_\ell + u_r).$$

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$$\left(u^2\right)_t + \left(\frac{2}{3}u^3\right)_x = 0 \quad \Longrightarrow \quad s^* = \frac{2}{3}\frac{u_r^3 - u_\ell^3}{u_r^2 - u_\ell^2}.$$

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If $u_\ell > u_r$ (correct shock) then $\frac{\partial}{\partial t}\int u^2\,dx < \psi(u_r^2) - \psi(u_\ell^2)$

If $u_\ell < u_r$ (entropy-violating) then $\frac{\partial}{\partial t}\int u^2\,dx > \psi(u_r^2) - \psi(u_\ell^2)$

Entropy condition for LWR traffic equations

$f(q) = q(1-q)$. Note that $q_t + (1-2q)q_x = 0$ where smooth

Again take entropy function $\eta(q) = q^2$ (we need $\eta''(q) > 0$)
Determine entropy flux by solving

$$\psi'(q) = \eta'(q)f'(q) = 2q(1-2q) \quad \Longrightarrow \quad \psi(q) = q^2 - \frac{4}{3}q^3.$$

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which $= 0$ from the original equation.
Why ≤ 0 for correct shock? Consider vanishing viscosity...

Entropy functions

$$\int_{x_1}^{x_2} \eta(q(x,t_2))\,dx \le \int_{x_1}^{x_2} \eta(q(x,t_1))\,dx + \int_{t_1}^{t_2} \psi(q(x_1,t))\,dt - \int_{t_1}^{t_2} \psi(q(x_2,t))\,dt$$

comes from considering the vanishing viscosity solution:

$$q_t^\epsilon + f(q^\epsilon)_x = \epsilon q_{xx}^\epsilon \qquad \text{with } \epsilon > 0$$

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Multiply by $\eta'(q^\epsilon)$ to obtain:

$$\eta(q^\epsilon)_t + \psi(q^\epsilon)_x = \epsilon \eta'(q^\epsilon) q_{xx}^\epsilon.$$

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Multiply by $\eta'(q^\epsilon)$ to obtain:

$$\eta(q^\epsilon)_t + \psi(q^\epsilon)_x = \epsilon \eta'(q^\epsilon) q_{xx}^\epsilon.$$

Manipulate further to get

$$\eta(q^\epsilon)_t + \psi(q^\epsilon)_x = \epsilon \big(\eta'(q^\epsilon) q_x^\epsilon\big)_x - \epsilon \eta''(q^\epsilon)\,(q_x^\epsilon)^2.$$

Entropy functions

Smooth solution to viscous equation satisfies

$$\eta(q^\epsilon)_t + \psi(q^\epsilon)_x = \epsilon\big(\eta'(q^\epsilon) q_x^\epsilon\big)_x - \epsilon\eta''(q^\epsilon)\,(q_x^\epsilon)^2.$$

Integrating over rectangle $[x_1, x_2] \times [t_1, t_2]$ gives

$$\begin{aligned}
\int_{x_1}^{x_2} \eta(q^\epsilon(x,t_2))\,dx &= \int_{x_1}^{x_2} \eta(q^\epsilon(x,t_1))\,dx \\
&\quad - \left(\int_{t_1}^{t_2} \psi(q^\epsilon(x_2,t))\,dt - \int_{t_1}^{t_2} \psi(q^\epsilon(x_1,t))\,dt\right) \\
&\quad + \epsilon \int_{t_1}^{t_2} \left[\eta'(q^\epsilon(x_2,t))\, q_x^\epsilon(x_2,t) - \eta'(q^\epsilon(x_1,t))\, q_x^\epsilon(x_1,t)\right] dt \\
&\quad - \epsilon \int_{t_1}^{t_2} \int_{x_1}^{x_2} \eta''(q^\epsilon)\,(q_x^\epsilon)^2\,dx\,dt.
\end{aligned}$$

Let $\epsilon \to 0$ to get result:

Term on third line goes to 0,

Term of fourth line is always ≤ 0.

Entropy functions

Weak form of entropy condition:

$$\int_0^\infty \int_{-\infty}^\infty \left[\phi_t \eta(q) + \phi_x \psi(q)\right] dx\, dt + \int_{-\infty}^\infty \phi(x,0)\eta(q(x,0))\, dx \geq 0$$

for all $\phi \in C_0^1(\mathbb{R} \times \mathbb{R})$ with $\phi(x,t) \geq 0$ for all $x,\ t$.

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Informally we may write

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and Lipschitz continuity of $\mathcal{F}$.

If a sequence of discrete approximations converge to a function $q(x, t)$ as the grid is refined, then this function is a weak solution of the conservation law.

Note:

Does not guarantee a sequence converges (need stability).

Can also use FV version of entropy condition in weak form to show that limit must be correct weak solution.

And entropy stability can also be used to prove convergence.

Convergence to Weak Solutions and Nonlinear Stability

- Lax-Wendroff Theorem
- Entropy consistent finite volume methods
- Nonlinear stability
- Total Variation stability

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$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

is in conservation form.

The total mass is conserved up to fluxes at the boundaries:

$$\Delta x \sum_i Q_i^{n+1} = \Delta x \sum_i Q_i^n - \Delta t(F_{+\infty} - F_{-\infty}).$$

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$$\int_0^\infty \int_{-\infty}^\infty (q_t + f(q)_x)\phi(x,t)\,dx\,dt$$

Then we can integrate by parts to get

$$\int_0^\infty \int_{-\infty}^\infty (q\phi_t + f(q)\phi_x)\,dx\,dt = -\int_{-\infty}^\infty q(x,0)\phi(x,0)\,dx.$$

$q(x,t)$ is a weak solution if this holds for all such ϕ.

Lax-Wendroff Theorem

Suppose the method is conservative and consistent with $q_t + f(q)_x = 0$,

$$F_{i-1/2} = \mathcal{F}(Q_{i-1}, Q_i) \quad \text{with } \mathcal{F}(\bar{q}, \bar{q}) = f(\bar{q})$$

and Lipschitz continuity of $\mathcal{F}$.

If a sequence of discrete approximations converge to a function $q(x,t)$ as the grid is refined, then this function is a weak solution of the conservation law.

Note:

Does not guarantee a sequence converges (need stability).

Two sequences might converge to different weak solutions.

Also need to satisfy an entropy condition.

Sketch of proof of Lax-Wendroff Theorem

Conservative numerical method:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

Multiply by Φ_i^n: (cell-averaged version of test function $\phi(x,t)$)

$$\Phi_i^n Q_i^{n+1} = \Phi_i^n Q_i^n - \frac{\Delta t}{\Delta x}\Phi_i^n(F_{i+1/2}^n - F_{i-1/2}^n).$$

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This is true for all values of i and n on each grid.
Now sum over all i and $n \geq 0$ to obtain

$$\sum_{n=0}^{\infty} \sum_{i=-\infty}^{\infty} \Phi_i^n(Q_i^{n+1} - Q_i^n) = -\frac{\Delta t}{\Delta x}\sum_{n=0}^{\infty} \sum_{i=-\infty}^{\infty} \Phi_i^n(F_{i+1/2}^n - F_{i-1/2}^n).$$

Use summation by parts to transfer differences to Φ terms.

Summation by parts

Integration by parts:

$$\int_a^b u(x)v'(x)\,dx = u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)\,dx.$$

Consider sum:

$$\sum_{i=1}^{N} u_i(v_i - v_{i-1})$$

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&\qquad\qquad + u_{N-1}(v_{N-1} - v_{N-2}) + u_N(v_N - v_{N-1}) \\
&\qquad = -u_1 v_0 - (u_2 - u_1)v_1 - (u_3 - u_2)v_2 + \cdots \\
&\qquad\qquad - (u_N - u_{N-1})v_{N-1} + u_N v_N
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&= u_{N-1}v_{N-1} - u_1 v_0 - \sum_{i=1}^{N-1} (u_{i+1} - u_i)v_i
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$$\Delta x \Delta t \left[\sum_{n=1}^{\infty} \sum_{i=-\infty}^{\infty} \left(\frac{\Phi_i^n - \Phi_i^{n-1}}{\Delta t} \right) Q_i^n \right.$$
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Can show that any limiting function

$$Q_i^n \to q(X,T) \quad \text{almost everywhere, as } \Delta x, \Delta t \to 0$$

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Must use $F_{i-1/2}^n \to f(Q_i^n)$ almost everywhere, using consistency of numerical flux $F_{i-1/2} = \mathcal{F}(Q_{i-1}, Q_i)$.

Analog of Lax-Wendroff proof for entropy

Suppose the numerical flux function $\mathcal{F}(Q_{i-1}, Q_i)$ leads to a numerical entropy flux $\Psi(Q_{i-1}, Q_i)$

such that the following discrete entropy inequality holds:

$$\eta(Q_i^{n+1}) \le \eta(Q_i^n) - \frac{\Delta t}{\Delta x}\left[\Psi_{i+1/2}^n - \Psi_{i-1/2}^n\right].$$

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$\Longrightarrow$ If numerical approximations converge to some function, then the limiting function satisfies the entropy condition.

Entropy consistency of Godunov's method

For Godunov's method, $F(Q_{i-1}, Q_i) = f(Q^{\Downarrow}_{i-1/2})$

where $Q^{\Downarrow}_{i-1/2}$ is the constant value along $x_{i-1/2}$ in the Riemann solution.

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If we use exact solution satisfying the entropy condition, then

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \eta\big(\tilde{q}^n(x,t_{n+1})\big)\,dx \;\;\leq\;\; \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \eta\big(\tilde{q}^n(x,t_n)\big)\,dx$$
$$+\,\frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} \psi(\tilde{q}^n(x_{i-1/2},t)\,dt - \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} \psi(\tilde{q}^n(x_{i+1/2},t)\,dt$$

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Follows from Jensen's inequality for convex functions:

If $\eta''(q) \ge 0$ then The value of $\eta(q(x))$ evaluated at the average value of $q(x)$ is less than or equal to the average value of $\eta(q(x))$, i.e.,

$$\eta\left(\int q(x)\,dx\right) \le \int \eta(q(x))\,dx.$$

Convergence and stability

Let q^n be cell averages of exact solution at time t_n

$$Q^n = q^n + E^n.$$

We apply the numerical method to obtain Q^{n+1}:

$$Q^{n+1} = \mathcal{N}(Q^n) = \mathcal{N}(q^n + E^n)$$

and the global error is now

$$\begin{aligned}
E^{n+1} &= Q^{n+1} - q^{n+1} \\
&= \mathcal{N}(q^n + E^n) - q^{n+1} \\
&= \mathcal{N}(q^n + E^n) - \mathcal{N}(q^n) + \mathcal{N}(q^n) - q^{n+1} \\
&= [\mathcal{N}(q^n + E^n) - \mathcal{N}(q^n)] + \Delta t\, \tau^n.
\end{aligned}$$

where τ^n is the local trucation error introduced in this step.

Convergence and stability

$$E^{n+1} = [\mathcal{N}(q^n + E^n) - \mathcal{N}(q^n)] + \Delta t\, \tau^n.$$

so

$$\|E^{n+1}\| \le \|\mathcal{N}(q^n + E^n) - \mathcal{N}(q^n)\| + \Delta t\, \|\tau^n\|$$

If

$$\|\mathcal{N}(q^n + E^n) - \mathcal{N}(q^n)\| \le \|E^n\|$$

then

$$\begin{aligned} \|E^N\| &\le \|E^0\| + \Delta t \sum_{n=1}^{N-1} \|\tau\| \\ &\le (\|E^0\| + T\|\tau\|) \qquad \text{(for } N\Delta t = T\text{).} \end{aligned}$$

Nonlinear stability

Would like to show

$$\|\mathcal{N}(q^n + E^n) - \mathcal{N}(q^n)\| \leq \|E^n\|$$

If method is linear, $\mathcal{N}(q^n + E^n) = \mathcal{N}(q^n) + \mathcal{N}(E^n)$, then enough to show:

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But in nonlinear case we need contractivity,

$$\|\mathcal{N}(P) - \mathcal{N}(Q)\| \leq \|P - Q\|.$$

Nonlinear stability

Entropy stability $\eta(\mathcal{N}(Q)) \le \eta(Q)$ analogous to

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Kružkov's Theorem (1970): Entropy stability for $\eta(q) = |q - k|$,

$$(|q-k|)_t + ((f(q) - f(k))\text{sgn}(q-k))_x \le 0$$

for all constants k implies

$$\|q(\cdot,t) - w(\cdot,t)\|_1 \le \|q(\cdot,0) - w(\cdot,0)\|_1$$

for all $t \ge 0$. (1-norm contractivity)

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Numerical methods with this property are at best first order.

TV Stability

A numerical method is Total Variation Bounded (TVB) if

$$TV(Q^n) \leq R \quad \text{for all } n \text{ with } n\Delta t \leq T$$

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Function space BV: A set of functions such as

$$\{v \in L_1 : \ TV(v) \leq R \text{ and } \mathrm{Supp}(v) \subset [-M, M]\}$$

is a compact set, so any sequence of functions has a convergent subsequence.

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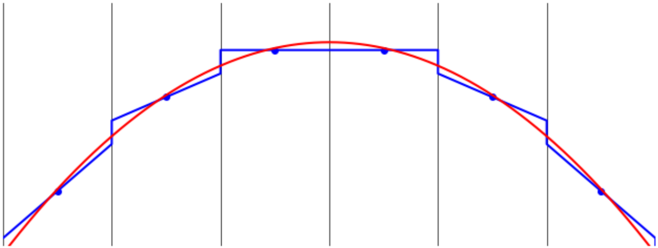
But then Lax-Wendroff Theorem $\implies$ q is a weak solution.
Contradiction.

Accuracy at local extrema

TVD methods must clip local extrema:



Clipping by Δx^2 each time step can lead to $\mathcal{O}(\Delta x)$ global errors

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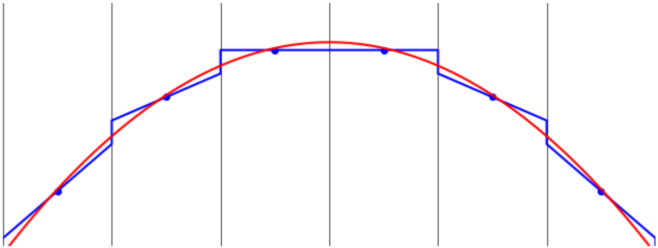

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TVB methods: Only require $TV(Q^{n+1}) \leq (1 + \Delta t)TV(Q^n)$.

Essentially nonoscillatory (ENO) methods

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Order s: $\|E^N\| = C(\Delta x)^s +$ higher order terms.

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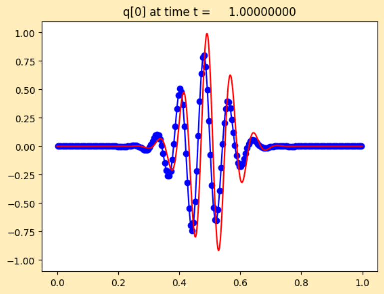
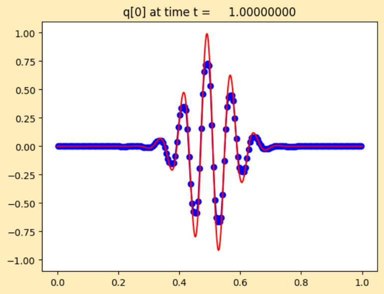
Comparison of Lax-Wendroff and a high-resolution method on linear advection equation with smooth wave packet data.





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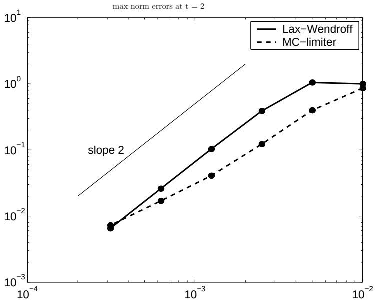
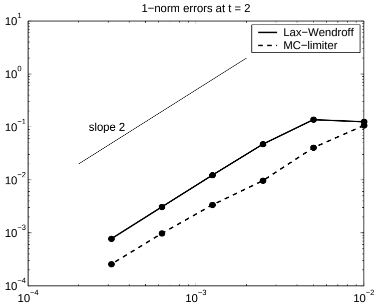
The high-resolution method is not formally second-order accurate, but is more accurate on realistic grids.

Crossover in the max-norm is at 2800 grid points.





Finite Volume Methods for Hyperbolic Problems

Nonlinear Systems

Shock Waves and the Hugoniot Locus

- Shallow water equations
- Rankine-Hugoniot condition
- Hugoniot locus in phase space
- All-shock Riemann solutions

Riemann Problems and Jupyter Solutions

Theory and Approximate Solvers for Hyperbolic PDEs

David I. Ketcheson, RJL, and Mauricio del Razo

General information and links to book, Github, Binder, etc.:

bookstore.siam.org/fa16/bonus

View static version of notebooks at:

www.clawpack.org/riemann_book/html/Index.html

Shallow water equations

$h(x, t) =$ depth
$u(x, t) =$ velocity (depth averaged, varies only with x)

Conservation of mass and momentum hu gives system of two equations.

mass flux $= hu$,
momentum flux $= (hu)u + p$ where $p =$ hydrostatic pressure

$$h_t + (hu)_x = 0$$
$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$

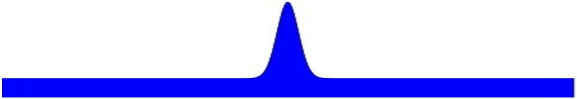
Jacobian matrix:

$$f'(q) = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix}, \qquad \lambda = u \pm \sqrt{gh}.$$

Shock formation

For nonlinear problems wave speed generally depends on q.

Waves can steepen up and form shocks
$\Longrightarrow$ even smooth data can lead to discontinuous solutions.



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Note:

- System of two equations gives rise to 2 waves.
- Each wave behaves like solution of nonlinear scalar equation.

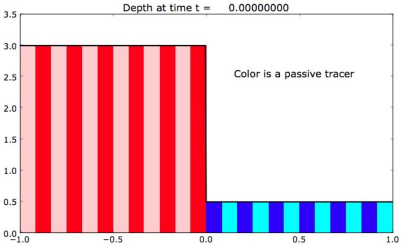
Not quite... no linear superposition. Nonlinear interaction!

The Riemann problem

Dam break problem for shallow water equations

$$h_t + (hu)_x = 0$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$



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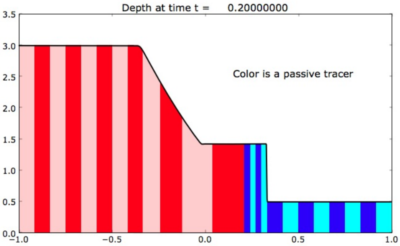

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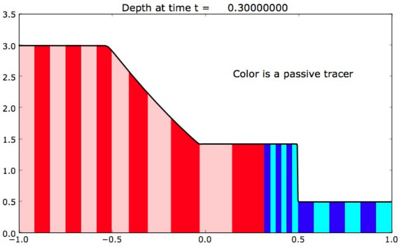

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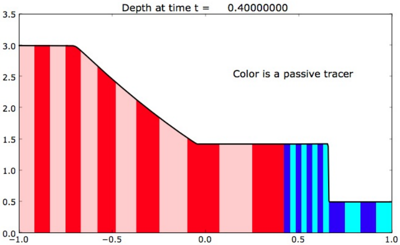

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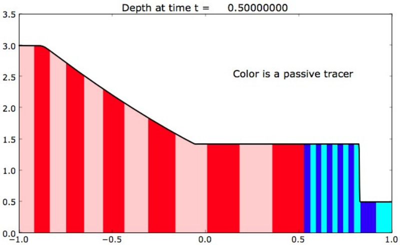
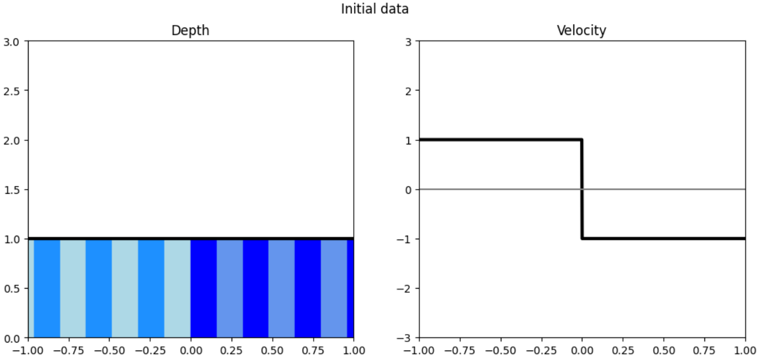


Two-shock Riemann solution

With $h_\ell = h_r$ and $u_\ell = -u_r > 0$



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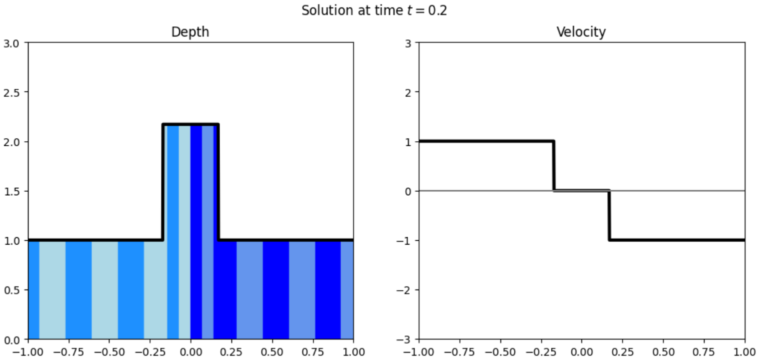

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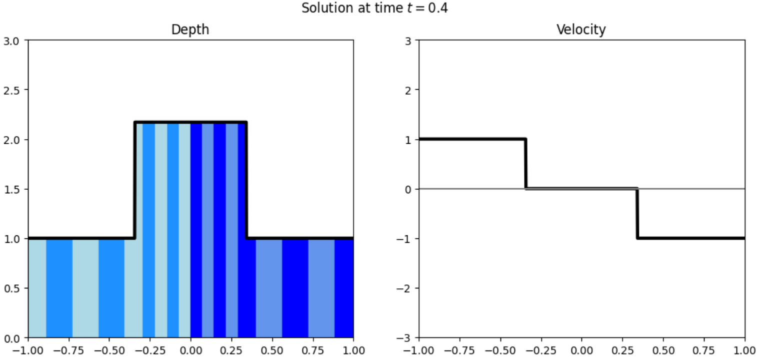

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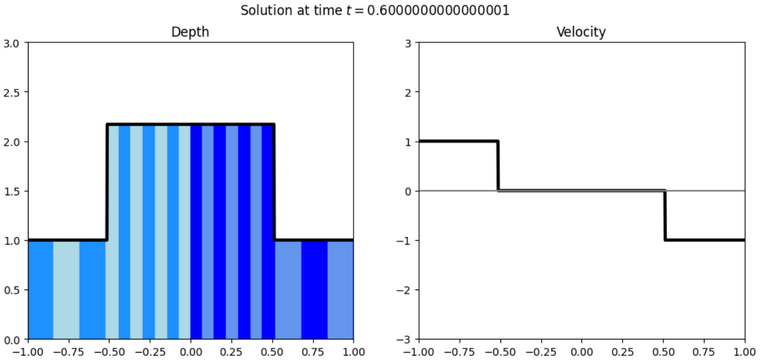
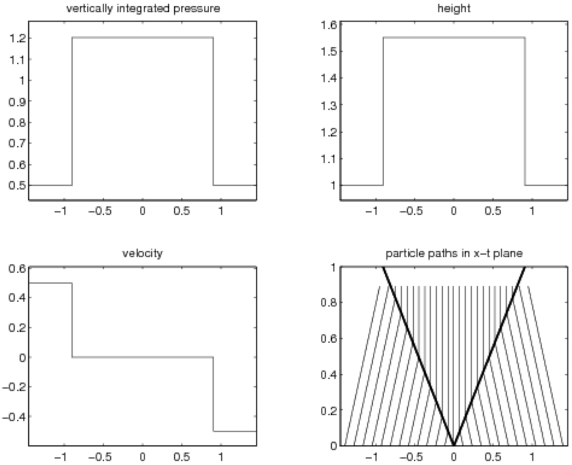


Two-shock Riemann solution for shallow water

Initially $h_l = h_r = 1, \quad u_l = -u_r = 0.5 > 0$

Solution at later time:



Characteristics for scalar nonlinear problem

Scalar hyperbolic equation in quasi-linear form: $q_t + f'(q)q_x = 0$.

Characteristic curve in x–t plane: $X(t)$ satisfying

$$X'(t) = f'(q(X(t), t)).$$

Along this curve,

$$\frac{d}{dt}\, q(X(t), t) = X'(t)q_x + q_t = 0$$

So for a scalar equation,

$q(x, t)$ is constant along characteristic curves.

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Advection: Characteristics satisfy $X'(t) = u$, so $X(t) = x_0 + ut$ are parallel straight lines.

Nonlinear: Characteristcs are straight since $f'(q(X(t), t))$ is constant, but not parallel. Crossing $\Longrightarrow$ shock formation.

Characteristics for nonlinear systems

Hyperbolic system in quasi-linear form: $q_t + f'(q)q_x = 0$.

Eigenvalues of Jacobian: $\lambda^p(q)$ with $f'(q)r^p(q) = \lambda^p(q)r^p(q)$.

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Simple wave in pth family: Suppose we choose $q(x, 0)$
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Let $X(t)$ be a smooth curve and compute

$$\begin{aligned}\frac{d}{dt}q(X(t),t) &= X'(t)q_x(X(t),t) + q_t(X(t),t) \\ &= X'(t)\,q_x(X(t),t) - f'(q(X(t),t))\,q_x(X(t),t) \\ &= w^p(x)\,X'(t)\,r^p(q(X(t),t)) \\ &\qquad - w^p(x)\,f'(q(X(t),t))\,r^p(q(X(t),t)\end{aligned}$$

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\end{aligned}$$

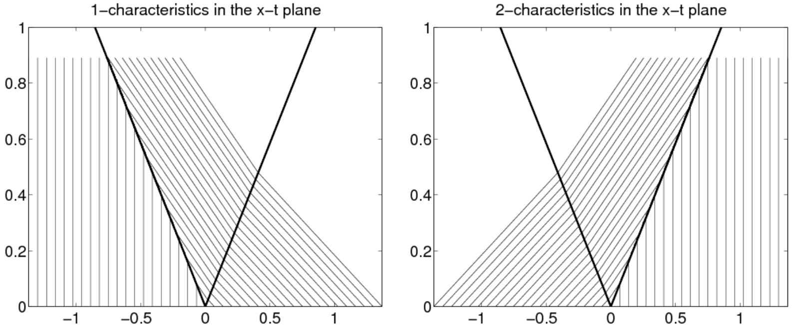
This $= 0$ if we choose $X'(t) = \lambda^p(q(X(t),t))$.

So in the simple wave case, $q(X(t),t)$ is constant along each ray with $X'(t) = \lambda^p(q(X(t),t))$ (as long as these don't cross).

Two-shock Riemann solution for shallow water

Characteristic curves $X'(t) = u(X(t), t) \pm \sqrt{gh(X(t), t)}$

Slope of characteristic is constant in regions where q is constant. (Shown for $g = 1$ so $\sqrt{gh} = 1$ everywhere initially.)



Note that 1-characteristics impinge on 1-shock, 2-characteristics impinge on 2-shock.

An isolated shock

If an isolated shock with left and right states q_l and q_r is propagating at speed s

then the Rankine-Hugoniot condition must be satisfied:

$$f(q_r) - f(q_l) = s(q_r - q_l)$$

For a system $q \in \mathbb{R}^m$ this can only hold for certain pairs q_l, q_r:

For a linear system, $f(q_r) - f(q_l) = Aq_r - Aq_l = A(q_r - q_l)$.
So $q_r - q_l$ must be an eigenvector of $f'(q) = A$.

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$A \in \mathbb{R}^{m \times m} \implies$ there will be m rays through q_l in state space in the eigen-directions, and q_r must lie on one of these.

For a nonlinear system, there will be m curves through q_l called the Hugoniot loci.

Hugoniot loci for shallow water

$$q = \begin{bmatrix} h \\ hu \end{bmatrix}, \qquad f(q) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}.$$

Fix $q_* = (h_*, u_*)$.

What states q can be connected to q_* by an isolated shock?

The Rankine-Hugoniot condition $s(q - q_*) = f(q) - f(q_*)$ gives:

$$\begin{aligned} s(h_* - h) &= h_* u_* - hu, \\ s(h_* u_* - hu) &= h_* u_*^2 - hu^2 + \frac{1}{2}g(h_*^2 - h^2). \end{aligned}$$

Two equations with 3 unknowns (h, u, s), so we expect 1-parameter families of solutions.

Hugoniot loci for shallow water

Rankine-Hugoniot conditions:

$$s(h_* - h) = h_* u_* - hu,$$
$$s(h_* u_* - hu) = h_* u_*^2 - hu^2 + \frac{1}{2} g(h_*^2 - h^2).$$

For any $h > 0$ we can solve for

$$u(h) = u_* \pm \sqrt{\frac{g}{2}\left(\frac{h_*}{h} - \frac{h}{h_*}\right)(h_* - h)}$$

$$s(h) = (h_* u_* - hu)/(h_* - h).$$

This gives 2 curves in h–hu space (one for $+$, one for $-$).

Hugoniot loci for shallow water

For any $h > 0$ we have a possible shock state. Set

$$h = h_* + \alpha,$$

so that $h = h_*$ at $\alpha = 0$, to obtain

$$hu = h_* u_* + \alpha \left[u_* \pm \sqrt{g h_* + \frac{1}{2} g \alpha (3 + \alpha / h_*)} \right].$$

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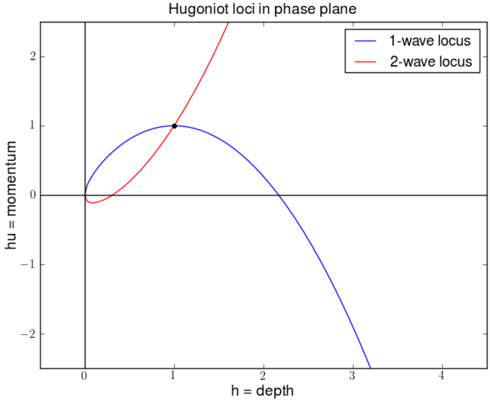
Hence we have

$$\begin{bmatrix} h \\ hu \end{bmatrix} = \begin{bmatrix} h_* \\ h_* u_* \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ u_* \pm \sqrt{g h_* + \mathcal{O}(\alpha)} \end{bmatrix} \qquad \text{as } \alpha \to 0.$$

Close to q_* the curves are tangent to eigenvectors of $f'(q_*)$
Expected since $f(q) - f(q_*) \approx f'(q_*)(q - q_*)$.

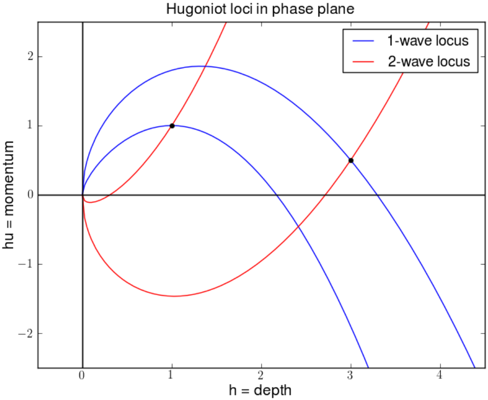
Hugoniot loci for one particular q_*

States that can be connected to q_* by a "shock"



Note: Might not satisfy entropy condition.

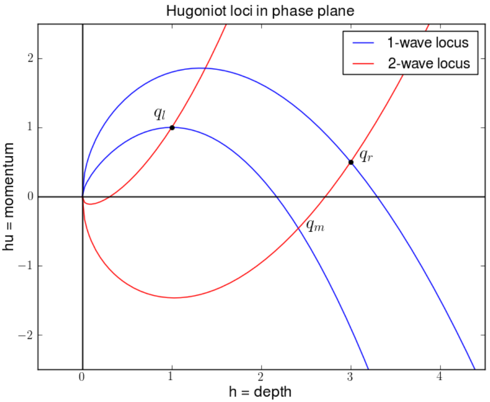
Hugoniot loci for two different states



“All-shock” Riemann solution:

From q_l along 1-wave locus to q_m,
From q_r along 2-wave locus to q_m,

All-shock Riemann solution



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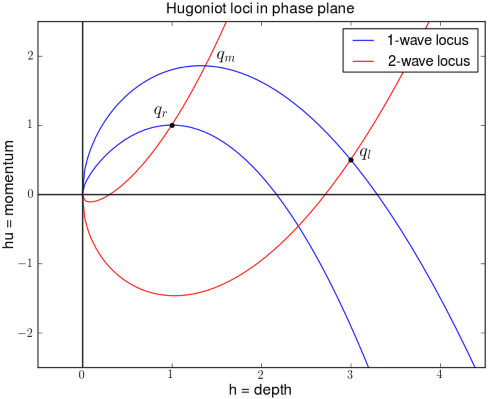


From q_l along 1-wave locus to q_m,
From q_r along 2-wave locus to q_m,

2-shock Riemann solution for shallow water

Given arbitrary states q_l and q_r, we can solve the Riemann problem with two shocks.

Choose q_m so that q_m is on the 1-Hugoniot locus of q_l and also q_m is on the 2-Hugoniot locus of q_r.

This requires

$$u_m = u_r + (h_m - h_r)\sqrt{\frac{g}{2}\left(\frac{1}{h_m} + \frac{1}{h_r}\right)}$$

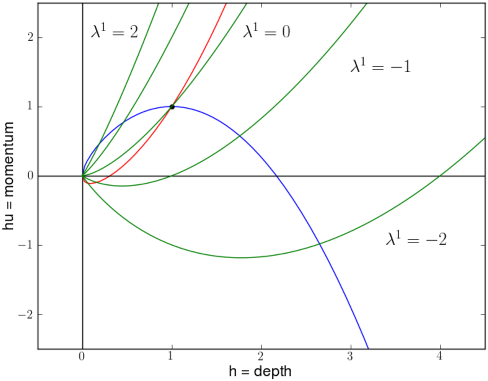
and

$$u_m = u_l - (h_m - h_l)\sqrt{\frac{g}{2}\left(\frac{1}{h_m} + \frac{1}{h_l}\right)}.$$

Equate and solve single nonlinear equation for h_m.

Hugoniot loci for one particular q_*

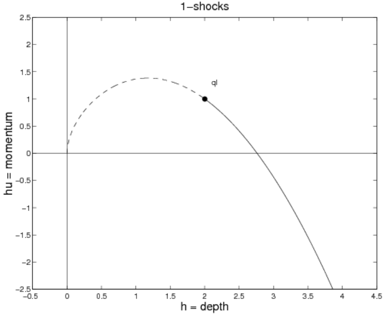
Green curves are contours of $\lambda^1 = u - \sqrt{gh}$

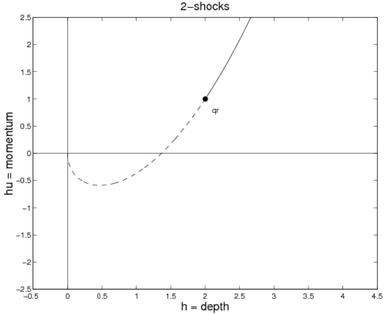


Note: Increases in one direction only along blue curve.

Hugoniot locus for shallow water

States that can be connected to the given state by a 1-wave or 2-wave satisfying the R-H conditions:



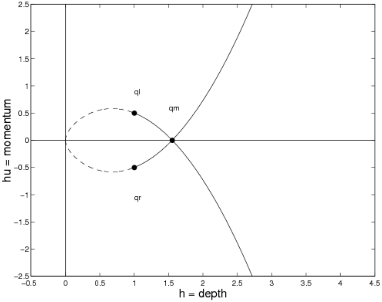


Solid portion: states that can be connected by shock satisfying entropy condition.

Dashed portion: states that can be connected with R-H condition satisfied but not the physically correct solution.

2-shock Riemann solution for shallow water

Colliding with $u_l = -u_r > 0$:



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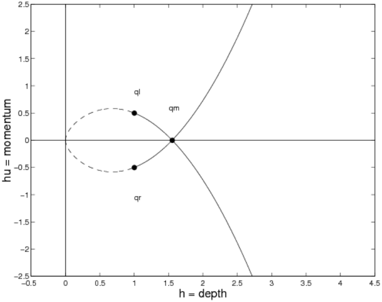


Entropy condition: Characteristics should impinge on shock:

λ^1 should decrease going from q_l to q_m,

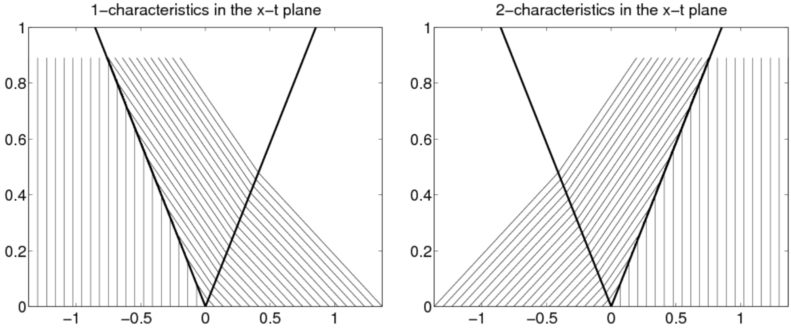
λ^2 should increase going from q_r to q_m,

This is satisfied along solid portions of Hugoniot loci above, not satisfied on dashed portions (entropy-violating shocks).

Two-shock Riemann solution for shallow water

Characteristic curves $X'(t) = u(X(t), t) \pm \sqrt{gh(X(t), t)}$

Slope of characteristic is constant in regions where q is constant. (Shown for $g = 1$ so $\sqrt{gh} = 1$ everywhere initially.)



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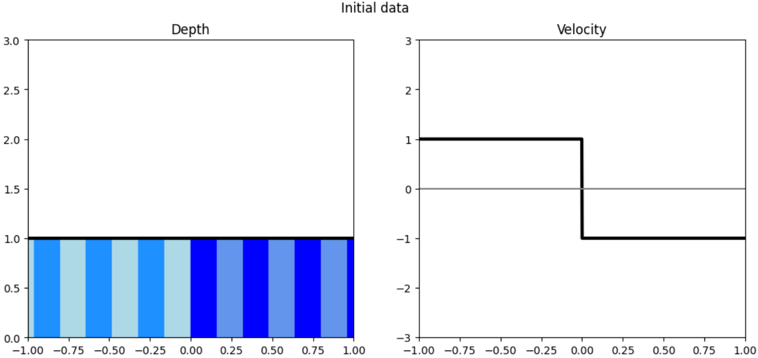

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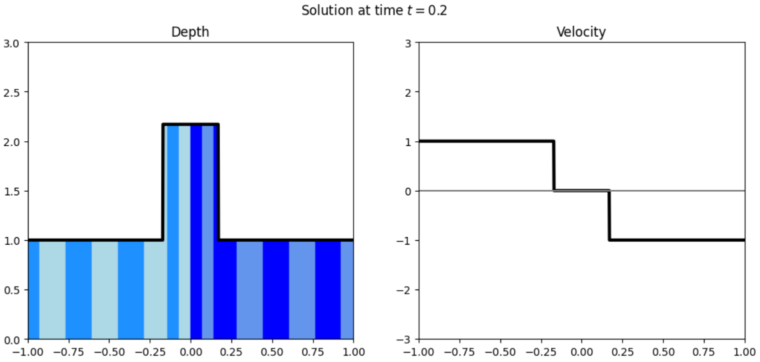

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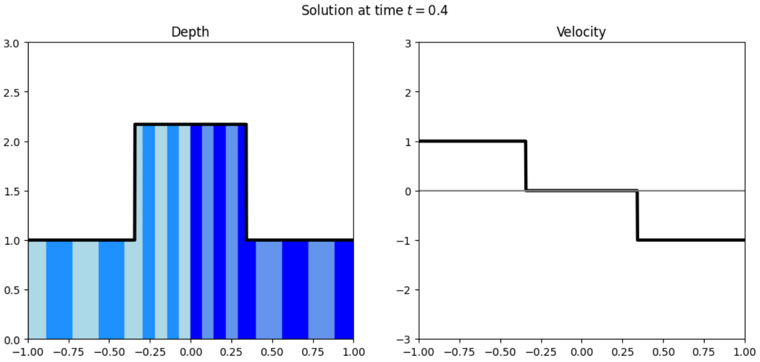

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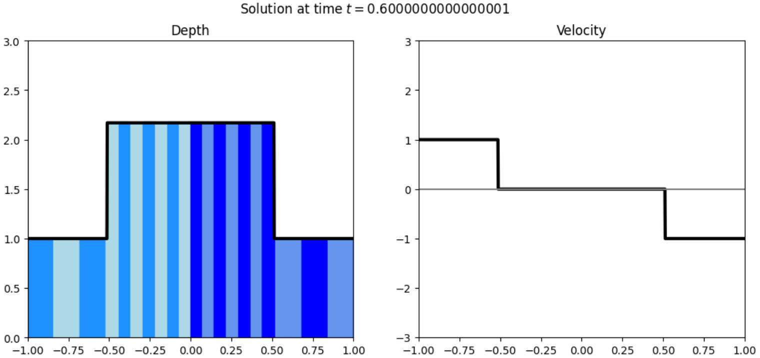
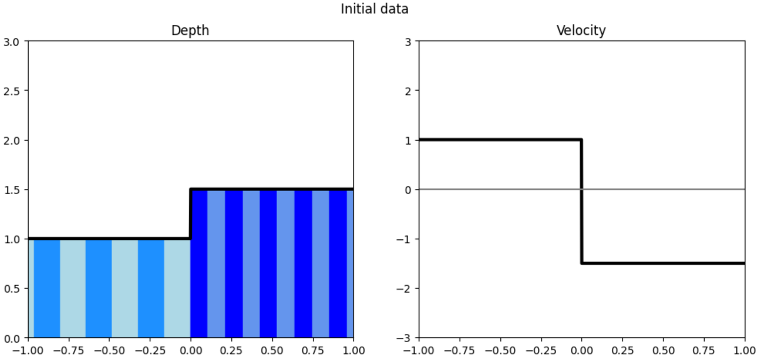


Two-shock Riemann solution

With non-equal states, but $u_\ell > 0$ and $u_r < 0$:



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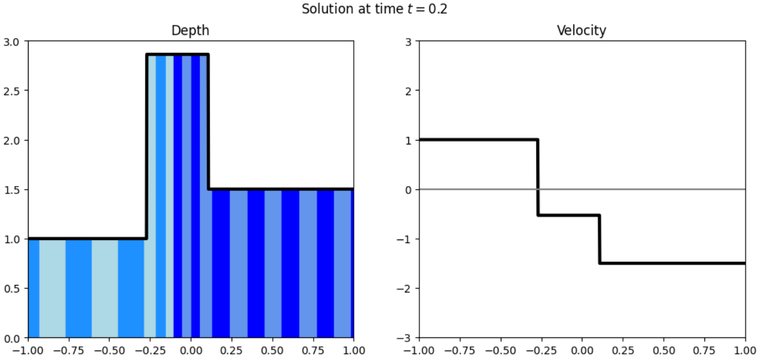

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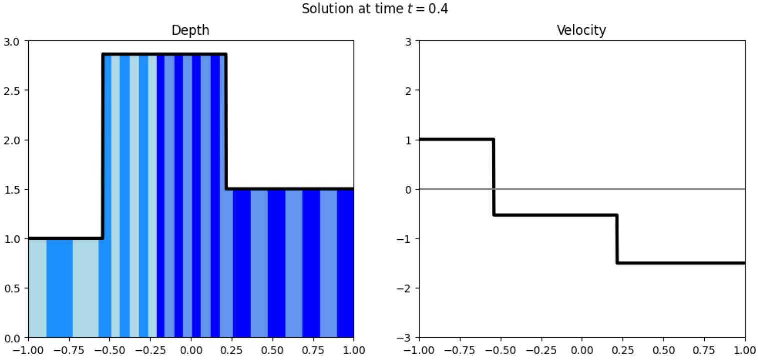

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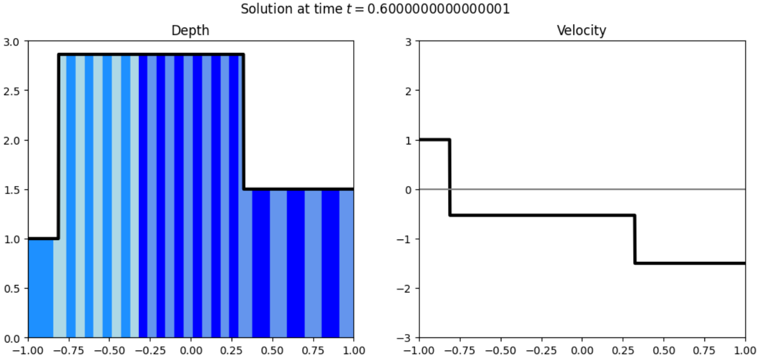
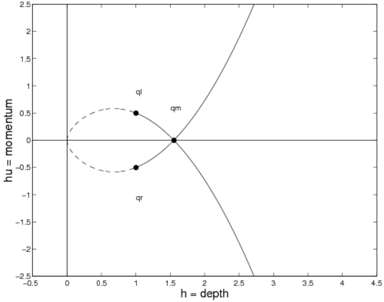
With non-equal states, but $u_\ell > 0$ and $u_r < 0$:

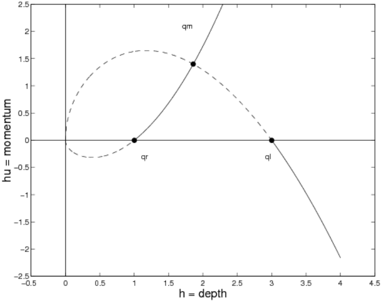


2-shock Riemann solution for shallow water

Colliding with $u_l = -u_r > 0$:



Dam break:



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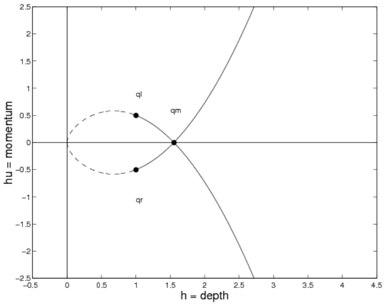

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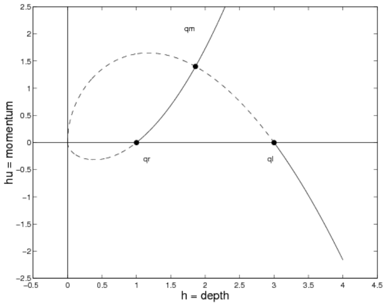


Entropy condition: Characteristics should impinge on shock:

λ^1 should decrease going from q_l to q_m,

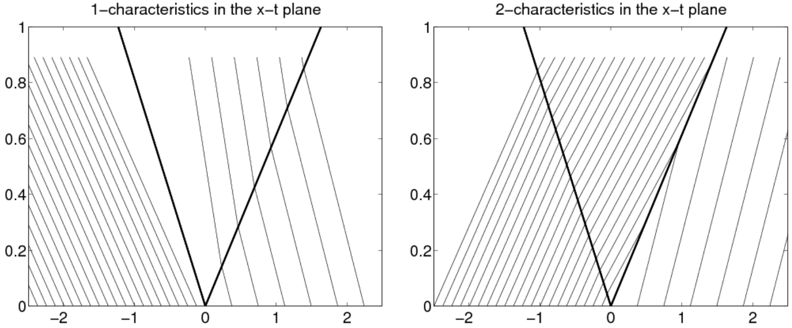
λ^2 should increase going from q_r to q_m,

This is satisfied along solid portions of Hugoniot loci above, not satisfied on dashed portions (entropy-violating shocks).

Entropy-violatiing Riemann solution for dam break

Characteristic curves $X'(t) = u(X(t), t) \pm \sqrt{gh(X(t), t)}$

Slope of characteristic is constant in regions where q is constant.

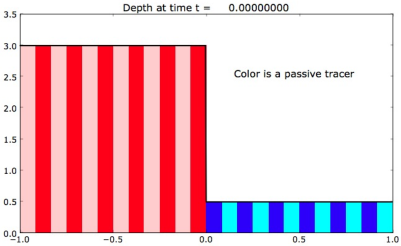


Note that 1-characteristics do not impinge on 1-shock, 2-characteristics impinge on 2-shock.

The Riemann problem

Dam break problem for shallow water equations

$$h_t + (hu)_x = 0$$
$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$



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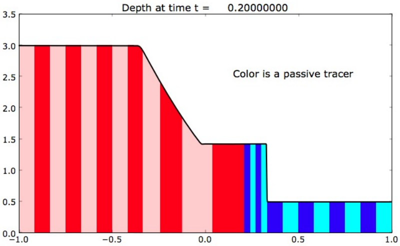

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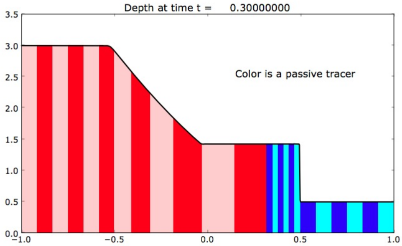

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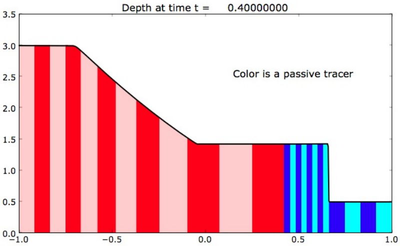

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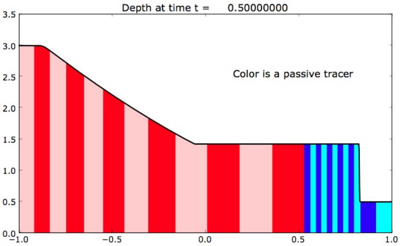


Riemann Problems and Jupyter Solutions

Theory and Approximate Solvers for Hyperbolic PDEs

David I. Ketcheson, RJL, and Mauricio del Razo

General information and links to book, Github, Binder, etc.:

bookstore.siam.org/fa16/bonus

View static version of notebooks at:

www.clawpack.org/riemann_book/html/Index.html

Nonlinear Systems

Rarefaction Waves and Integral Curves

- Integral curves
- Genuine nonlinearity and rarefaction waves
- General Riemann solution for shallow water
- Riemann invariants
- Linear degeneracy and contact discontinuities

Shallow water equations

$$h_t + (hu)_x = 0 \implies h_t + \mu_x = 0$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0 \implies \mu_t + \phi(h,\mu)_x = 0$$

where $\mu = hu$ and $\phi = hu^2 + \frac{1}{2}gh^2 = \mu^2/h + \frac{1}{2}gh^2$.

Jacobian matrix:

$$f'(q) = \begin{bmatrix} \partial\mu/\partial h & \partial\mu/\partial\mu \\ \partial\phi/\partial h & \partial\phi/\partial\mu \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix},$$

Eigenvalues:

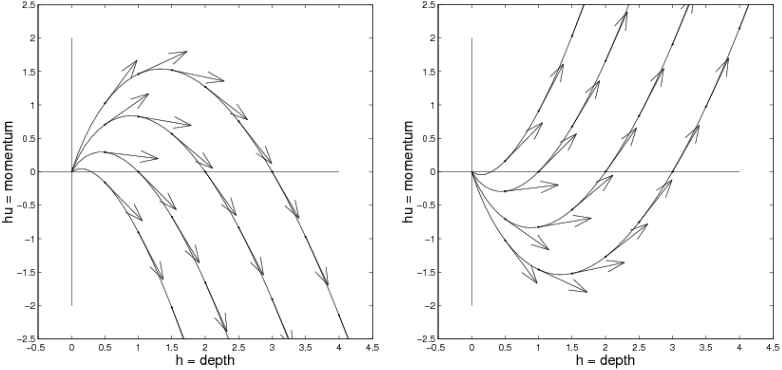
$$\lambda^1 = u - \sqrt{gh}, \qquad \lambda^2 = u + \sqrt{gh}.$$

Eigenvectors:

$$r^1 = \begin{bmatrix} 1 \\ u - \sqrt{gh} \end{bmatrix}, \qquad r^2 = \begin{bmatrix} 1 \\ u + \sqrt{gh} \end{bmatrix}.$$

Integral curves of r^p

Curves in phase plane that are tangent to $r^p(q)$ at each q.



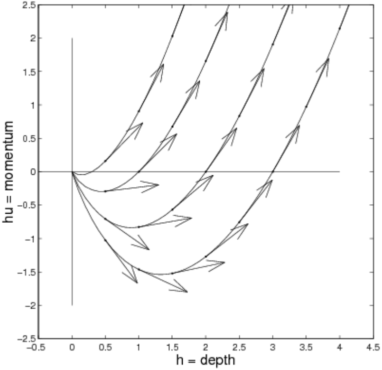
$\tilde{q}(\xi)$: curve through phase space parameterized by $\xi \in \mathbb{R}$.

Satisfying $\tilde{q}'(\xi) = \alpha(\xi) r^p(\tilde{q}(\xi))$ for some scalar $\alpha(\xi)$.

Simple waves

In a simple wave, the values $q(x, t)$ always lie along a single integral curve in some particular pth family.

As initial data, can choose arbitrary smooth $h(x, 0)$,
but then $u(x, 0)$ is determined.



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Integral curve parameterized by $\tilde{q}(\xi)$.

So $q(x,t) = \tilde{q}(\xi(x,t))$ for some $\xi(x,t)$.

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So $q(x,t) = \tilde{q}(\xi(x,t))$ for some $\xi(x,t)$.
Not any $\xi(x,t)$ works. When is the PDE satisfied?

Assuming smooth, require $q_t + f'(q)q_x = 0$:

$$q_t(x,t) = \tilde{q}'(\xi(x,t))\xi_t(x,t)$$
$$q_x(x,t) = \tilde{q}'(\xi(x,t))\xi_x(x,t)$$

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$$\begin{aligned} f'(q(x,t))q_x(x,t) &= f'(q(\xi(x,t))\,\tilde{q}'(\xi(x,t))\xi_x(x,t) \\ &= \lambda^p(q(\xi(x,t))\,\tilde{q}'(\xi(x,t))\xi_x(x,t) \end{aligned}$$

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So $q_t + f'(q)q_x = 0 \implies$

$$[\xi_t(x,t) + \lambda^p(\tilde{q}(\xi(x,t)))\,\xi_x(x,t)]\,\tilde{q}'(\xi(x,t)) = 0.$$

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This is a scalar equation and $\tilde{q}(\xi(x,t))$ is constant along characteristic curves $X'(t) = \lambda^p(\tilde{q}(\xi(x,t)))$ as long as the solution stays smooth.

Converging characteristics $\implies$ shock formation.

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Converging characteristics $\implies$ shock formation.

Once a shock forms, no longer a simple wave in general (waves in other families can be generated).

Centered rarefaction waves

Similarity solution with piecewise constant initial data:

$$q(x,t) = \begin{cases} q_\ell & \text{if } x/t \le \lambda^p(q_\ell) \\ \tilde{q}(x/t) & \text{if } \lambda^p(q_\ell) \le x/t \le \lambda^p(q_r) \\ q_r & \text{if } x/t \ge \lambda^p(q_r), \end{cases}$$

where q_ℓ and q_r are on same integral curve and $\lambda^p(q_\ell) < \lambda^p(q_r)$.

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Special case of simple wave with $\xi(x,t) = x/t$.
Then $\xi_t(x,t) + \lambda^p(\tilde{q}(\xi(x,t)))\, \xi_x(x,t) = 0$ becomes

$$-\frac{x}{t^2} + \lambda^p(\tilde{q}(x/t))\frac{1}{t} = 0 \qquad \Longrightarrow \; \lambda^p(\tilde{q}(x/t)) = x/t$$

Centered rarefaction waves

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So we need to solve $\lambda^p(\tilde{q}(\xi)) = \xi$ for $\tilde{q}(\xi)$.

Generalizes the equation $f'(\tilde{q}(\xi)) = \xi$ for scalar PDE.

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Required so that characteristics spread out as time advances.

Centered rarefaction waves

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where q_ℓ and q_r are on same integral curve and $\lambda^p(q_\ell) < \lambda^p(q_r)$.

Required so that characteristics spread out as time advances.

Also want $\lambda^p(q)$ monotonically increasing from q_ℓ to q_r.

Genuine nonlinearity: generalization of convexity for scalar flux.

Genuine nonlinearity

For scalar problem $q_t + f(q)_x = 0$, want $f''(q) \neq 0 \;\; \forall q$ of interest.

This implies that $f'(q)$ is monotonically increasing or decreasing between q_l and q_r.

Shock if decreasing,   Rarefaction if increasing.

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For system we want $\lambda^p(q)$ to be monotonically varying along integral curve of $r^p(q)$.

If so then this field is genuinely nonlinear.

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For system we want $\lambda^p(q)$ to be monotonically varying along integral curve of $r^p(q)$.

If so then this field is genuinely nonlinear.

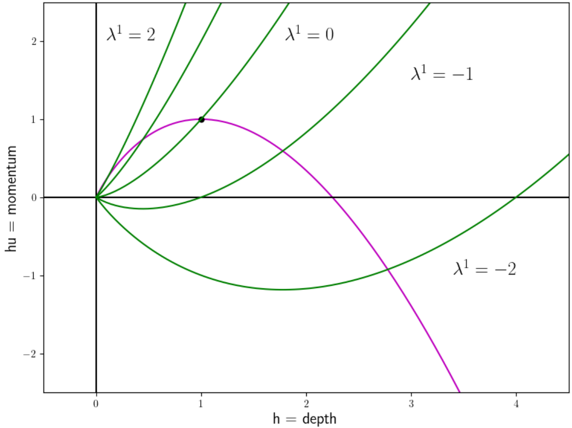
This requires: $\nabla\lambda^p(q) \cdot r^p(q) \neq 0$ for all q in region of interest.

since

$$\frac{d}{d\xi}\lambda^p(\tilde{q}(\xi)) = \nabla\lambda^p(\tilde{q}(\xi)) \cdot \tilde{q}'(\xi).$$

Integral curve for one particular q_*

Green curves are contours of $\lambda^1 = u - \sqrt{gh}$



Note: Increases monotonically in one direction along integral curve.

Shallow water equations

$$h_t + (hu)_x = 0 \implies h_t + \mu_x = 0$$

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$$r^1 = \begin{bmatrix} 1 \\ u - \sqrt{gh} \end{bmatrix}, \qquad r^2 = \begin{bmatrix} 1 \\ u + \sqrt{gh} \end{bmatrix}.$$

Genuine nonlinearity of shallow water equations

1-waves: Requires $\nabla\lambda^1(q)\cdot r^1(q) \neq 0$.

$$\lambda^1 = u - \sqrt{gh} = q^2/q^1 - \sqrt{gq^1},$$

$$\nabla\lambda^1 = \begin{bmatrix} -q^2/(q^1)^2 - \frac{1}{2}\sqrt{g/q^1} \\ 1/q^1 \end{bmatrix} = \begin{bmatrix} -u/h - \frac{1}{2}\sqrt{g/h} \\ 1/h \end{bmatrix}$$

$$r^1 = \begin{bmatrix} 1 \\ q^2/q^1 - \sqrt{gq^1} \end{bmatrix} = \begin{bmatrix} 1 \\ u - \sqrt{gh} \end{bmatrix}$$

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$$r^1 = \begin{bmatrix} 1 \\ q^2/q^1 - \sqrt{gq^1} \end{bmatrix} = \begin{bmatrix} 1 \\ u - \sqrt{gh} \end{bmatrix}$$

and hence

$$\nabla\lambda^1\cdot r^1 = -\frac{3}{2}\sqrt{g/q^1} = -\frac{3}{2}\sqrt{g/h}$$
$$< 0 \quad \text{for all} \quad h > 0.$$

1-waves: integral curves of r^1

$\tilde{q}(\xi)$: curve through phase space parameterized by $\xi \in \mathbb{R}$.

Satisfies $\tilde{q}'(\xi) = \alpha(\xi) r^1(\tilde{q}(\xi))$ for some scalar $\alpha(\xi)$.

Choose $\alpha(\xi) \equiv 1$ and obtain

$$\begin{bmatrix} (\tilde{q}^1)' \\ (\tilde{q}^2)' \end{bmatrix} = \tilde{q}'(\xi) = r^1(\tilde{q}(\xi)) = \begin{bmatrix} 1 \\ \tilde{q}^2/\tilde{q}^1 - \sqrt{g\tilde{q}^1} \end{bmatrix}$$

This is a system of 2 ODEs

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First equation: $\tilde{q}^1(\xi) = \xi \implies \xi = h$.

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This is a system of 2 ODEs

First equation: $\tilde{q}^1(\xi) = \xi \implies \xi = h$.
Second equation $\implies (\tilde{q}^2)' = \tilde{q}^2(\xi)/\xi - \sqrt{g\xi}$.

Require $\tilde{q}^2(h_*) = h_* u_* \implies$

$$\tilde{q}^2(\xi) = \xi u_* + 2\xi \left(\sqrt{gh_*} - \sqrt{g\xi}\right).$$

1-wave integral curves of r^p

So

$$
\begin{aligned}
\tilde{q}^1(\xi) &= \xi, \\
\tilde{q}^2(\xi) &= \xi u_* + 2\xi \left(\sqrt{gh_*} - \sqrt{g\xi}\right).
\end{aligned}
$$

and hence integral curve through $(h_*,\ h_* u_*)$ satisfies

$$
hu = hu_* + 2h\left(\sqrt{gh_*} - \sqrt{gh}\right) \quad \text{for } 0 < h < \infty.
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1-wave integral curves of r^p

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$$\tilde{q}^1(\xi) = \xi,$$
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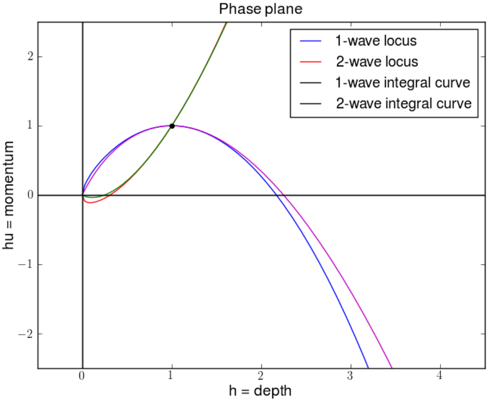
and hence integral curve through $(h_*,\ h_* u_*)$ satisfies

$$hu = hu_* + 2h \left(\sqrt{gh_*} - \sqrt{gh}\right) \quad \text{for } 0 < h < \infty.$$

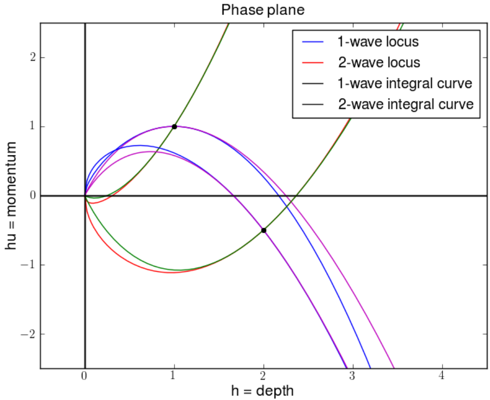
Similarly, 2-wave integral curve through $(h_*,\ h_* u_*)$ satisfies

$$hu = hu_* - 2h \left(\sqrt{gh_*} - \sqrt{gh}\right).$$

Integral curves of r^p versus Hugoniot loci



Solving the shallow water Riemann problem



Solution to Riemann problem depends on which state is q_l, q_r.

Also need to choose correct curve from each state.

The Riemann problem

Dam break problem for shallow water equations

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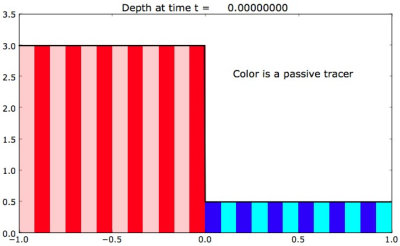

The Riemann problem

Dam break problem for shallow water equations

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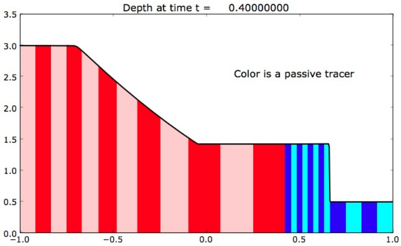

Solving the dam break Riemann problem

$h_\ell > h_r$ and $u_\ell = u_r = 0 \implies$ 1-rarefaction and 2-shock

So the intermediate state q_m lies on:
1-wave integral curve through q_ℓ, and on
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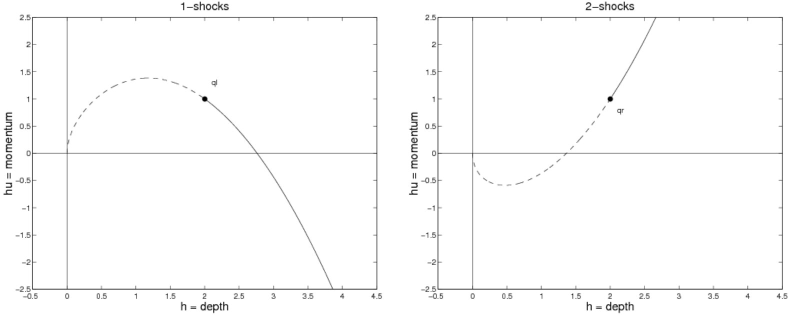
Equate to obtain a single nonlinear equation for h_m:

$$u_l + 2\left(\sqrt{gh_l} - \sqrt{gh_m}\right) = u_r + (h_m - h_r)\sqrt{\frac{g}{2}\left(\frac{1}{h_m} + \frac{1}{h_r}\right)}$$

Hugoniot locus for shallow water

States that can be connected to the given state by a 1-wave or 2-wave satisfying the R-H conditions:



Solid portion: states that can be connected by shock satisfying entropy condition.

Dashed portion: states that can be connected with R-H condition satisfied but not the physically correct solution.

Solving the general Riemann problem

For general data $q_\ell,\ q_r$, the shallow water Riemann solution could have a shock or rarefaction in each family.

Use the fact that across a shock we always expect deeper water "behind" the shock to define 1-wave curve through q_ℓ:

$$\phi_\ell(h) = \begin{cases} u_\ell + 2\left(\sqrt{gh_\ell} - \sqrt{gh}\right) & \text{if } h < h_\ell \\ u_\ell - (h - h_\ell)\sqrt{\frac{g}{2}\left(\frac{1}{h} + \frac{1}{h_\ell}\right)} & \text{if } h \ge h_\ell \end{cases}$$

and 2-wave curve through q_r:

$$\phi_r(h) = \begin{cases} u_r - 2\left(\sqrt{gh_r} - \sqrt{gh}\right) & \text{if } h < h_r \\ u_r + (h - h_r)\sqrt{\frac{g}{2}\left(\frac{1}{h} + \frac{1}{h_r}\right)} & \text{if } h \ge h_r \end{cases}$$

Then determine h_m by using a numerical root finder on

$$\phi(h) = \phi_\ell(h) - \phi_r(h).$$

Riemann invariants

Along a 1-wave integral curve,

$$u = u_* + 2\left(\sqrt{gh_*} - \sqrt{gh}\right)$$

and hence

$$u + 2\sqrt{gh} = u_* + 2\sqrt{gh_*}.$$

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and hence

$$u + 2\sqrt{gh} = u_* + 2\sqrt{gh_*}.$$

So at every point on the integral curve through $(h_*, h_* u_*)$

$$w^1(q) = u + 2\sqrt{gh}$$

has the constant value $w^1(q) \equiv w^1(q_*) = u_* + 2\sqrt{gh_*}$.

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The function $w^1(q)$ is a 1-Riemann invariant for this system.

Riemann invariants

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The integral curves are contour lines of $w^1(q)$.

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2-Riemann invariants:

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Linearly degenerate fields

Scalar advection: $q_t + uq_x = 0$ with $u =$ constant.

Characteristics $X(t) = x_0 + ut$ are parallel.

Discontinuity propagates along a characteristic curve.

Characteristics on either side are parallel so not a shock!

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For the flux $f(q) = uq$, we have $f'(q) = u \; \forall q$ and $f''(q) \equiv 0$.

For a system the analogous property arises if

$$\nabla \lambda^p(q) \cdot r^p(q) \equiv 0$$

holds for all q, in which case

$$\frac{d}{d\xi} \lambda^p(\tilde{q}(\xi)) = \nabla \lambda^p(\tilde{q}(\xi)) \cdot \tilde{q}'(\xi) \equiv 0.$$

So λ^p is constant along each integral curve.

Then pth field is said to be linearly degenerate.

The Riemann problem

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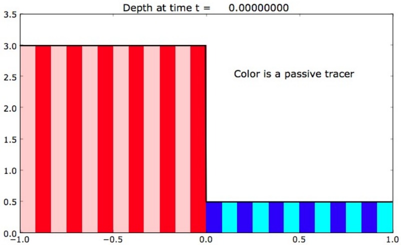

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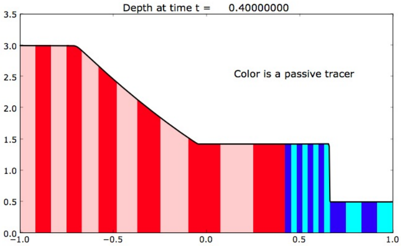

Shallow water with passive tracer

Let $\phi(x,t)$ be tracer concentration and add equation

$$\phi_t + u\phi_x = 0 \implies (h\phi)_t + (uh\phi)_x = 0 \quad \text{(since } h_t + (hu)_x = 0\text{)}.$$

Gives:

$$q = \begin{bmatrix} h \\ hu \\ h\phi \end{bmatrix} = \begin{bmatrix} q^1 \\ q^2 \\ q^3 \end{bmatrix}, \quad f(q) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ uh\phi \end{bmatrix} = \begin{bmatrix} q^2 \\ (q^2)/q^1 + \frac{1}{2}g(q^1)^2 \\ q^2q^3/q^1 \end{bmatrix}.$$

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Jacobian:

$$f'(q) = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + gh & 2u & 0 \\ -u\phi & \phi & u \end{bmatrix}.$$

Shallow water with passive tracer

$$f'(q) = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + gh & 2u & 0 \\ -u\phi & \phi & u \end{bmatrix}.$$

$$\lambda^1 = u - \sqrt{gh}, \qquad \lambda^2 = u, \qquad \lambda^3 = u + \sqrt{gh},$$

$$r^1 = \begin{bmatrix} 1 \\ u - \sqrt{gh} \\ \phi \end{bmatrix}, \quad r^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad r^3 = \begin{bmatrix} 1 \\ u + \sqrt{gh} \\ \phi \end{bmatrix}.$$

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$$\lambda^2 = u = (hu)/h \implies \nabla\lambda^2 = \begin{bmatrix} -u/h \\ 1/h \\ 0 \end{bmatrix} \implies \lambda^2 \cdot r^2 \equiv 0.$$

So 2nd field is linearly degenerate.

(Fields 1 and 3 are genuinely nonlinear.)

Finite Volume Methods for Hyperbolic Problems

Gas Dynamics and Euler Equations

- The Euler equations
- Conservative vs. primitive variables
- Contact discontinuities
- Projecting phase space to p–u plane
- Hugoniot loci and integral curves
- Solving the Riemann problem

Riemann Problems and Jupyter Solutions

Theory and Approximate Solvers for Hyperbolic PDEs

David I. Ketcheson, RJL, and Mauricio del Razo

General information and links to book, Github, Binder, etc.:

bookstore.siam.org/fa16/bonus

View static version of notebooks at:

www.clawpack.org/riemann_book/html/Index.html

In particular see: **Euler.ipynb**

Compressible gas dynamics

Conservation laws:

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

Equation of state:

$$p = P(\rho).$$

Same as shallow water if $P(\rho) = \frac{1}{2} g \rho^2$ (with $\rho \equiv h$).

Isothermal: $P(\rho) = a^2 \rho$ (since T proportional to p/ρ).

Isentropic: $P(\rho) = \hat{\kappa} \rho^\gamma$ ($\gamma \approx 1.4$ for air)

Jacobian matrix:

$$f'(q) = \begin{bmatrix} 0 & 1 \\ P'(\rho) - u^2 & 2u \end{bmatrix}, \qquad \lambda = u \pm \sqrt{P'(\rho)}.$$

Gas dynamics variables

$\rho =$ density

$\vec{u} =$ velocity (just u in 1D, $[u, v]$ in 2D, $[u, v, w]$ in 3D)

$h\vec{u} =$ momentum

$p =$ pressure

$e =$ internal energy (vibration, heat) $= \frac{p}{(\gamma-1)\rho}$ for polytropic

$\frac{1}{2}\rho\|\vec{u}\|_2^2 =$ kinetic energy

$E =$ total energy

$c_p, c_v =$ specific heat at constant pressure or volume

$T =$ temperature $= e/c_v = \frac{p}{c_v(\gamma-1)\rho}$ for polytropic

$\gamma = c_p/c_v =$ adiabatic exponent for polytropic, $1 < \gamma \leq 5/3$

$h = e + p/\rho =$ specific enthalpy

$H = \frac{E+p}{\rho} = h + \frac{1}{2}u^2 =$ total specific enthalpy

$s = c_v \log(p/\rho^\gamma) +$ const $=$ specific entropy for polytropic

Equations of state

Polytropic: $E = e + \frac{1}{2}\rho u^2$ and $e = \frac{p}{(\gamma-1)\rho}$, so

$$\begin{aligned} p &= \rho e(\gamma - 1) \\ &= (\gamma - 1)\left(E - \frac{1}{2}\rho u^2\right) \\ &= P(\rho, \rho u, E) \end{aligned}$$

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Isothermal: $T = \frac{p}{c_v(\gamma-1)\rho}$ for polytropic, so

$$p = Tc_v(\gamma - 1)\rho \equiv a^2\rho = P(\rho)$$

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$$p = \hat{c}\rho^\gamma = P(\rho)$$

Euler equations of gas dynamics

Conservation of mass, momentum, energy: $q_t + f(q)_x = 0$ with

$$q = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \qquad f(q) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{bmatrix}$$

where $E = \rho e + \frac{1}{2}\rho u^2$

Equation of state: $p =$ pressure $= p(\rho, E)$

Ideal gas, polytropic EOS: $p = \rho e(\gamma - 1) = (\gamma - 1)\left(E - \frac{1}{2}\rho u^2\right)$

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The Jacobian $f'(q)$ has eigenvalues $u - c, \;\; u, \;\; u + c$ where

$$c = \sqrt{\left.\frac{dp}{d\rho}\right|_{\text{at constant entropy}}} = \sqrt{\frac{\gamma p}{\rho}} \text{ for polytropic}$$

Euler equations in primitive variables

Can rewrite the conservation laws in quasilinear form:

$$\begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_t + \begin{bmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \gamma p & u \end{bmatrix} \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_x = 0.$$

Eigenvalues and eigenvectors:

$$\begin{aligned} \lambda^1 &= u - c, & \lambda^2 &= u, & \lambda^3 &= u + c, \\ r^1 &= \begin{bmatrix} -\rho/c \\ 1 \\ -\rho c \end{bmatrix}, & r^2 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, & r^3 &= \begin{bmatrix} \rho/c \\ 1 \\ \rho c \end{bmatrix}, \end{aligned}$$

Euler equations in primitive variables

$$\nabla\lambda^1 = \begin{bmatrix} -\partial c/\partial\rho \\ 1 \\ -\partial c/\partial p \end{bmatrix} = \begin{bmatrix} c/2\rho \\ 1 \\ -c/2p \end{bmatrix} \implies \nabla\lambda^1\cdot r^1 = \frac{1}{2}(\gamma+1),$$

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1-waves and 3-waves are genuinely nonlinear,
2-waves are linearly degenerate (contact discontinuity).

Contact discontinuities

Consider Riemann problem for conservative variables:

$$q = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \qquad f(q) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{bmatrix}$$

Suppose $p_\ell = p_r$ and $u_\ell = u_r \equiv u$,

Then the Rankine-Hugoniot condition $s\Delta q = \Delta f$ becomes:

$$s \begin{bmatrix} \Delta\rho \\ u\Delta\rho \\ \Delta E \end{bmatrix} = \begin{bmatrix} u\Delta\rho \\ u^2\Delta\rho \\ u\Delta E \end{bmatrix}$$

Satisfied with $s = u$, for any jump in density $\Delta\rho$.

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Satisfied with $s = u$, for any jump in density $\Delta\rho$.

And for any equation of state.

Euler in conservation form

Jacobian:

$$f'(q) = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma-3)u^2 & (3-\gamma)u & (\gamma-1) \\ \frac{1}{2}(\gamma-1)u^3 - uH & H-(\gamma-1)u^2 & \gamma u \end{bmatrix},$$

$$H = \frac{E+p}{\rho} = h + \frac{1}{2}u^2 = \text{ total specific enthalpy}$$

Eigenvalues and eigenvectors:

$$\lambda^1 = u - c, \qquad \lambda^2 = u, \qquad \lambda^3 = u + c,$$

$$r^1 = \begin{bmatrix} 1 \\ u-c \\ H-uc \end{bmatrix}, \quad r^2 = \begin{bmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{bmatrix}, \quad r^3 = \begin{bmatrix} 1 \\ u+c \\ H+uc \end{bmatrix}.$$

Riemann invariants for Euler (polytropic gas)

$$\text{1-Riemann invariants:} \quad s, \quad u + \frac{2}{\gamma - 1}\sqrt{\frac{\gamma p}{\rho}},$$

$$\text{2-Riemann invariants:} \quad u, \quad p,$$

$$\text{3-Riemann invariants:} \quad s, \quad u - \frac{2}{\gamma - 1}\sqrt{\frac{\gamma p}{\rho}}.$$

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In particular, linear acoustic waves are isentropic.

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In particular, linear acoustic waves are isentropic.

Note: u and p constant across in any simple 2-wave,
and across a contact discontinuity (check R-H condition).

Since $\lambda^2 = u$, this says characteristics are parallel
(the field is linearly degenerate)

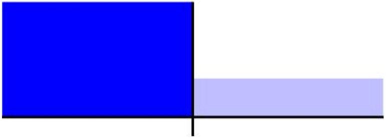
Riemann Problem for Euler equations

Initial data:

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$

Shock tube problem: $u_l = u_r = 0$, jump in ρ and p.

Pressure:



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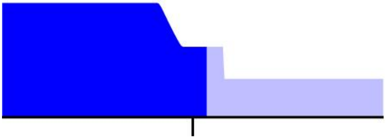

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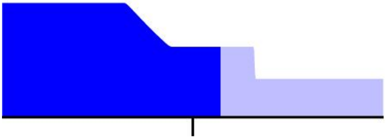

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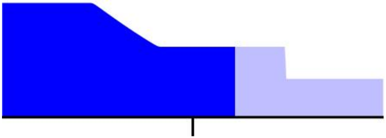

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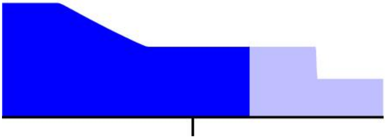

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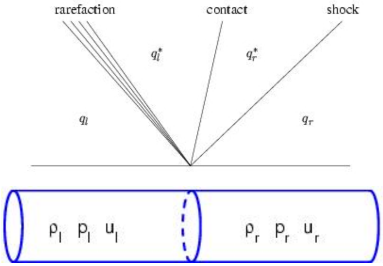
Riemann Problem for gas dynamics

Waves propagating in x–t space:



In primitive variables:

$$q_\ell^* = \begin{bmatrix} \rho_l^* \\ p^* \\ u^* \end{bmatrix}$$

$$q_r^* = \begin{bmatrix} \rho_r^* \\ p^* \\ u^* \end{bmatrix}$$

Only ρ jumps across 2-wave

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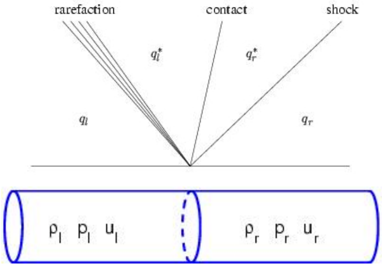

Similarity solution
(function of x/t alone)

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Waves can be approximated by discontinuties:

High-resolution wave-propagation methods

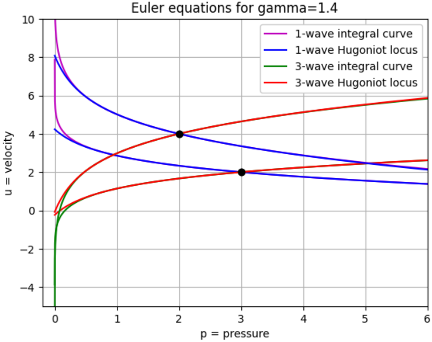
Approximate Riemann solvers

Riemann Problem for gas dynamics

Any jump in ρ is allowed across contact discontinuity

General Riemann solver:

- Project 3D phase space to p–u plane,
  Hugoniot loci and integral curves can be written as
  $u = \phi(p)$, (and $\rho = \rho(p)$)
- Find intersection
  $(p^*,\ u^*)$,
- Compute ρ_ℓ^* and ρ_r^*.



Integral curves for gas dynamics

In 1-wave, we know the Riemann invariants are constant,

$$s = c_v \log(p/\rho^\gamma) \quad \text{and} \quad u + \frac{2}{\gamma - 1} c \quad \text{with } c = \sqrt{\frac{\gamma p}{\rho}}$$

Given values in left state q_ℓ, can then compute integral curve as:

$$u = u_\ell + \left(\frac{2\,c_\ell}{\gamma - 1}\right)\left(1 - (p/p_\ell)^{(\gamma-1)/(2\gamma)}\right) \equiv \phi_\ell(p) \quad \text{for } p \le p_\ell.$$

Note that ρ does not appear!

Since s is constant, $\rho = (p/p_\ell)^{1/\gamma}\rho_\ell$.

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Can find similar expression for 3-wave integral curve,

$$u = u_r - \left(\frac{2\,c_r}{\gamma - 1}\right) \left(1 - (p/p_r)^{(\gamma-1)/(2\gamma)}\right) \equiv \phi_r(p) \quad \text{for } p \le p_r.$$

Hugoniot locus for gas dynamics

From Rankine-Hugoniot conditions, can deduce that (1-wave):

$$u = u_\ell + \frac{2\,c_\ell}{\sqrt{2\gamma(\gamma-1)}} \left(\frac{1 - p/p_\ell}{\sqrt{1+\beta p/p_\ell}}\right) \equiv \phi_\ell(p) \quad \text{for } p \geq p_\ell.$$

where $\beta = (\gamma+1)/(\gamma-1)$.

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For any p on this Hugoniot locus, we also find that:

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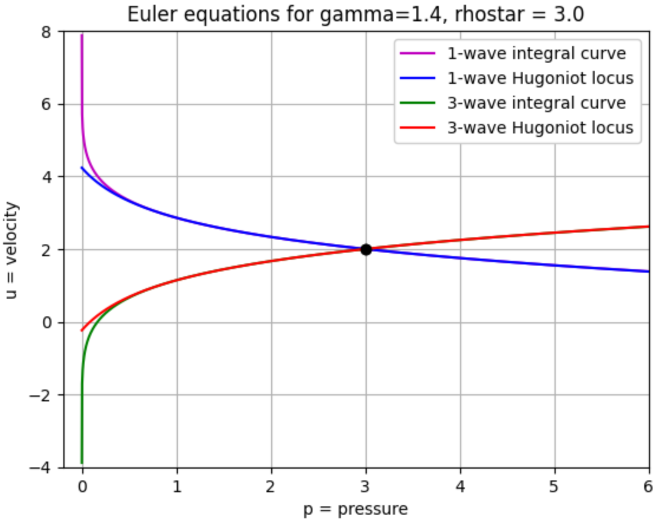
Similar expression for 3-wave, $u = \phi_r(p)$ for $p \geq p_r$.

Euler equations phase plane



Note these are curves in (p, u, ρ) space projected to plane.

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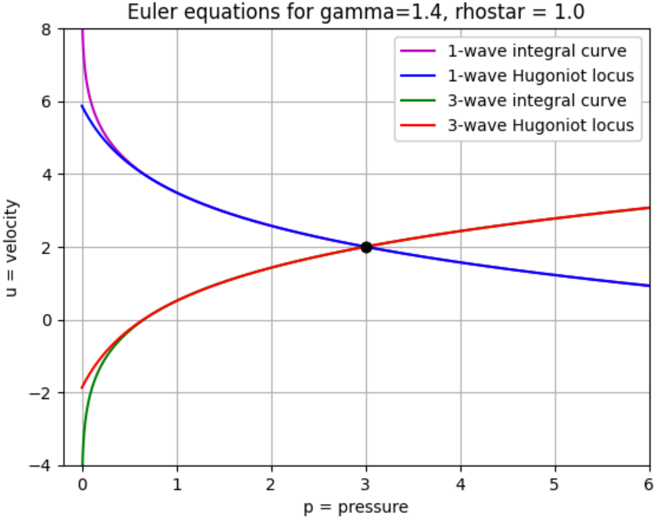

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Solving the Euler Riemann problem

```
In [71]: left_state  = State(Density = 3., Velocity = 0., Pressure = 3.)
         right_state = State(Density = 1.,Velocity = 0.,Pressure = 1.)

         euler.phase_plane_plot(left_state, right_state)
         grid(True)
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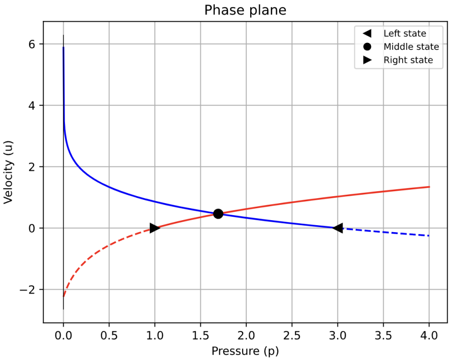

blue = integral curve, red = Hugoniot locus, dashed = nonphysical

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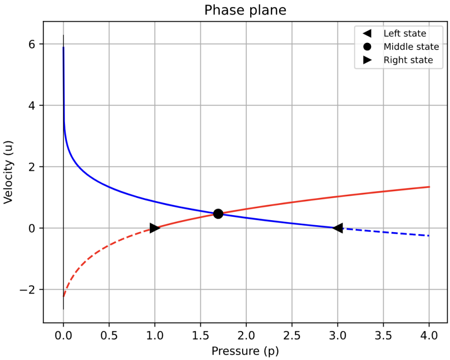


Solve $\phi_l(p) - \phi_r(p) = 0$ for p_m

$u_m = \phi_l(p_m) = \phi_r(p_m)$

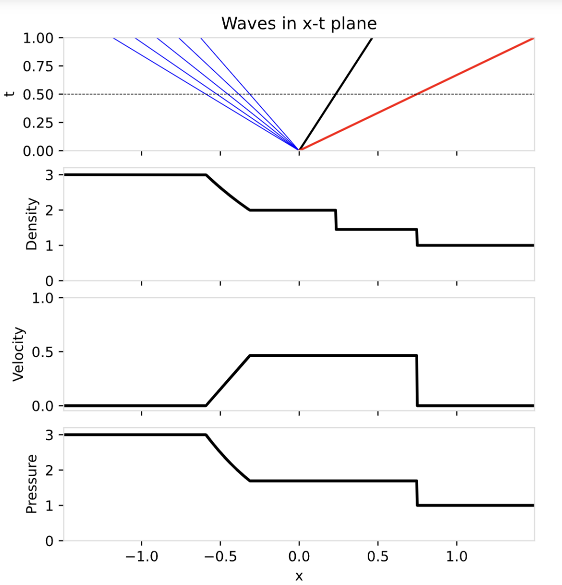
$\rho_{m\ell} = \rho(p_m)$ across 1-wave

$\rho_{mr} = \rho(p_m)$ across 2-wave

Red curve is displaced from blue in ρ direction (into page).

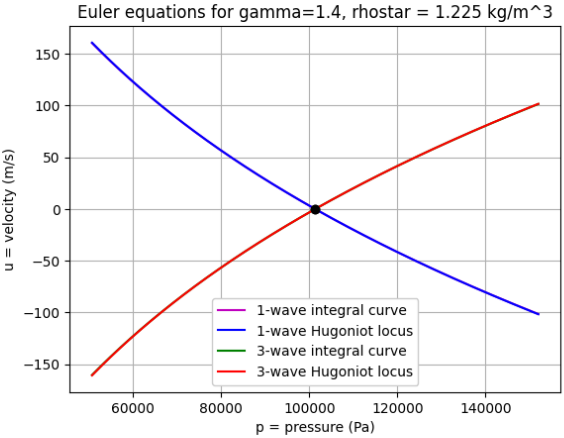
blue = integral curve, red = Hugoniot locus, dashed = nonphysical

Solving the Euler Riemann problem



Euler equations at atmospheric conditions

With parameters for air at $T^* = 20^\circ$ C, Density $\rho^* = 1.225$ kg/m^3.
Pressure $p^* = 101,325$ Pa $= 1$ atm, Speed of sound: $c^* = 340.3$ m/s



from ≈ 0.5 atm to 2 atm

Euler equations at atmospheric conditions

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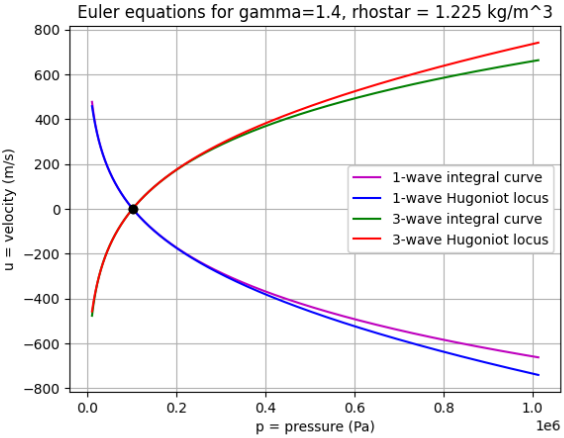
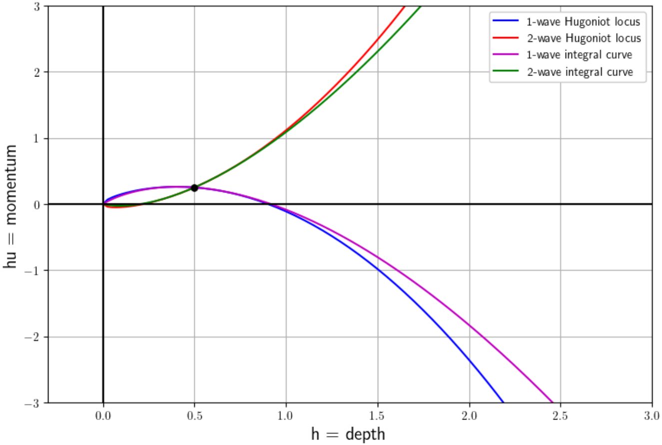


from ≈ 0.1 atm to 10 atm

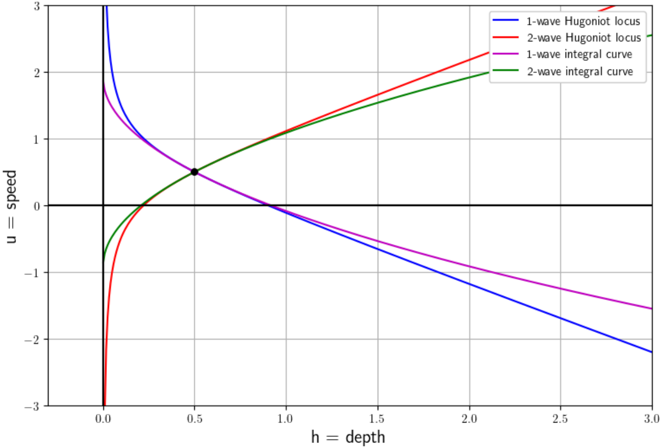
Shallow water equations phase plane

In the h–hu phase plane (the conserved quantities):



Shallow water equations phase plane

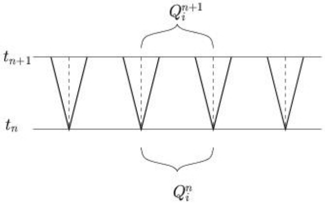
Replot in the h–u phase plane (primitive variables):



Finite Volume Methods for Nonlinear Systems

- Wave propagation method for systems
- High-resolution methods using wave limiters
- Example for shallow water equations

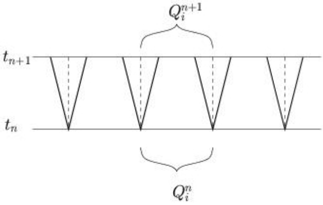
Godunov's Method for $q_t + f(q)_x = 0$



1. Solve Riemann problems at all interfaces, yielding waves $\mathcal{W}^p_{i-1/2}$ and speeds $s^p_{i-1/2}$, for $p = 1,\ 2,\ \ldots,\ m$.

Riemann problem: Original equation with piecewise constant data.

Godunov's Method for $q_t + f(q)_x = 0$



Then either:

1. Compute new cell averages by integrating over cell at t_{n+1},

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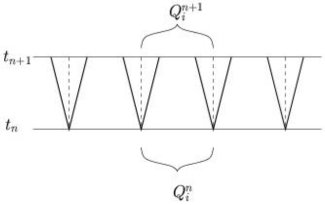

Then either:

1. Compute new cell averages by integrating over cell at t_{n+1},
2. Compute fluxes at interfaces and flux-difference:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[F_{i+1/2}^n - F_{i-1/2}^n]$$

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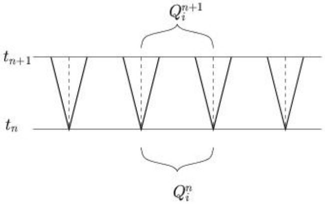

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3. Update cell averages by contributions from all waves entering cell:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}]$$

where $\mathcal{A}^\pm \Delta Q_{i-1/2} = \sum_{i=1}^{m} (s_{i-1/2}^p)^\pm \mathcal{W}_{i-1/2}^p$.

Approximate Riemann solver

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right].$$

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Replacing rarefaction with shock: also exact (after averaging), except in case of transonic rarefaction.

Wave limiters for scalar nonlinear

For $q_t + f(q)_x = 0$, just one wave: $\mathcal{W}_{i-1/2} = Q_i^n - Q_{i-1}^n$.

Godunov:

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"Lax-Wendroff":

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right] - \frac{\Delta t}{\Delta x}(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2})$$

$$\tilde{F}_{i-1/2} = \frac{1}{2}\left(1 - \left|\frac{s_{i-1/2}\Delta t}{\Delta x}\right|\right)|s_{i-1/2}|\mathcal{W}_{i-1/2}$$

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High-resolution method:

$$\tilde{F}_{i-1/2} = \frac{1}{2}\left(1 - \left|\frac{s_{i-1/2}\Delta t}{\Delta x}\right|\right)|s_{i-1/2}|\widetilde{\mathcal{W}}_{i-1/2}$$

$\widetilde{\mathcal{W}}_{i-1/2} = \phi(\theta)\,\mathcal{W}_{i-1/2}$, where $\theta_{i-1/2} = \mathcal{W}_{I-1/2}/\mathcal{W}_{i-1/2}$.

Extension to constant coefficient linear systems

Approach 1: Diagonalize the system to

$$q_t + Aq_x \implies w_t + \Lambda w_x = 0, \quad q = Rw$$

$W^n = R^{-1}Q^n$, Apply scalar algorithm, Set $Q^{n+1} = RW^{n+1}$.

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Solve the linear Riemann problem to decompose $Q_i^n - Q_{i-1}^n$ into waves $\mathcal{W}_{i-1/2}^p = \alpha_{i-1/2}^p r^p$.

Apply a wave limiter to each wave (comparing scalars $\alpha_{i-1/2}^p$).

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Note: Limiters are applied to waves or characteristic components, not to original variables.

Wave-propagation form of high-resolution method

$$
\begin{aligned}
Q_i^{n+1} \;&=\; Q_i^n - \frac{\Delta t}{\Delta x}\left[\sum_{p=1}^{m}(s_{i-1/2}^p)^+\mathcal{W}_{i-1/2}^p + \sum_{p=1}^{m}(s_{i+1/2}^p)^-\mathcal{W}_{i+1/2}^p\right] \\
&\qquad - \frac{\Delta t}{\Delta x}(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2})
\end{aligned}
$$

Correction flux:

$$
\tilde{F}_{i-1/2} = \frac{1}{2}\sum_{p=1}^{M_w}|s_{i-1/2}^p|\left(1 - \frac{\Delta t}{\Delta x}|s_{i-1/2}^p|\right)\widetilde{\mathcal{W}}_{i-1/2}^p
$$

where $\widetilde{\mathcal{W}}_{i-1/2}^p$ is a limited version of $\mathcal{W}_{i-1/2}^p$ to avoid oscillations.

(Unlimited $\widetilde{\mathcal{W}}^p = \mathcal{W}^p \implies$ Lax-Wendroff for a linear system.)

Approximate Riemann Solvers

Some approaches to approximating Riemann solution
by a set of jump discontinuities:

All-shock Riemann solution: Ignore integral curves and use only Hugoniot loci to construct weak solution.

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Local linearization: Replace $q_t + f(q)_x = 0$ by

$$q_t + \hat{A} q_x = 0, \qquad \text{where } \hat{A} = \hat{A}(q_l, q_r) \approx f'(q_{ave}).$$

Eigenvectors give waves. Roe solver $\Longrightarrow$ conservative

Wave limiters for linear system

$Q_i - Q_{i-1}$ is split into waves $\mathcal{W}^p_{i-1/2} = \alpha^p_{i-1/2} r^p \in \mathbb{R}^m$.

For constant coefficient linear system: r^p is constant vector,
Only the scalar α^p varies.

Replace by $\widetilde{\mathcal{W}}^p_{i-1/2} = \Phi(\theta^p_{i-1/2})\mathcal{W}^p_{i-1/2}$ where

$$\theta^p_{i-1/2} = \frac{\alpha^p_{I-1/2}}{\alpha^p_{i-1/2}}$$

where

$$I = \begin{cases} i-1 & \text{if } s^p_{i-1/2} > 0 \\ i+1 & \text{if } s^p_{i-1/2} < 0. \end{cases}$$

In the scalar case this reduces to

$$\theta^1_{i-1/2} = \frac{\mathcal{W}^1_{I-1/2}}{\mathcal{W}^1_{i-1/2}} = \frac{Q_I - Q_{I-1}}{Q_i - Q_{i-1}}$$

Wave limiters for system

$Q_i - Q_{i-1}$ is split into waves $\mathcal{W}^p_{i-1/2} \in \mathbb{R}^m$ with speeds $s^p_{i-1/2}$.
Upwind cell in family p:

$$I = \begin{cases} i-1 & \text{if } s^p_{i-1/2} > 0 \\ i+1 & \text{if } s^p_{i-1/2} < 0. \end{cases}$$

To compare $\mathcal{W}^p_{i-1/2}$ to $\mathcal{W}^p_{I-1/2}$ we want to reduce to a scalar
$\theta^p_{i-1/2} \approx 1$ where the solution is smooth,
negative near extreme points of this wave component.

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Use projection of $\mathcal{W}^p_{I-1/2}$ onto $\mathcal{W}^p_{i-1/2}$:

$$\left(\frac{\mathcal{W}^p_{i-1/2} \cdot \mathcal{W}^p_{I-1/2}}{\mathcal{W}^p_{i-1/2} \cdot \mathcal{W}^p_{i-1/2}}\right) \mathcal{W}^p_{i-1/2} \quad \text{compared to} \quad \mathcal{W}^p_{i-1/2}$$

Wave limiters for system

$Q_i - Q_{i-1}$ is split into waves $\mathcal{W}^p_{i-1/2} \in \mathbb{R}^m$ with speeds $s^p_{i-1/2}$.
Upwind cell in family p:

$$I = \begin{cases} i-1 & \text{if } s^p_{i-1/2} > 0 \\ i+1 & \text{if } s^p_{i-1/2} < 0. \end{cases}$$

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Ratio of coefficients: $\theta^p_{i-1/2} = \dfrac{\mathcal{W}^p_{i-1/2} \cdot \mathcal{W}^p_{I-1/2}}{\mathcal{W}^p_{i-1/2} \cdot \mathcal{W}^p_{i-1/2}}$

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Replace $\mathcal{W}^p_{i-1/2}$ by $\widetilde{\mathcal{W}}^p_{i-1/2} = \phi(\theta^p_{i-1/2})\mathcal{W}^p_{i-1/2}$. ($\phi(\theta)$ = limiter)

Wave limiters for system with eigendecomposition

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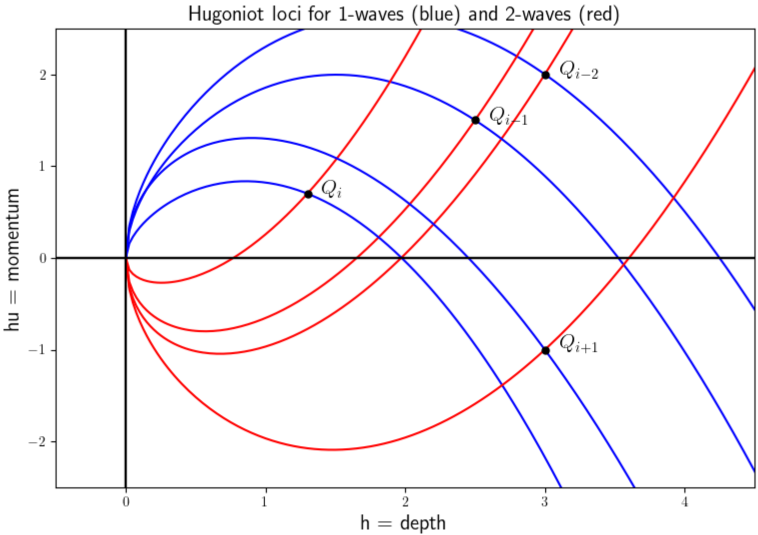
Scalar case: this reduces to

$$\theta^1_{i-1/2} = \frac{\mathcal{W}^1_{I-1/2}}{\mathcal{W}^1_{i-1/2}} = \frac{Q_I - Q_{I-1}}{Q_i - Q_{i-1}}$$

Limiters – shallow water equation



Note that speeds are $s = \Delta(hu)/\Delta(h) =$ slope between states.

Wave propagation methods

- Solving Riemann problem gives waves $\mathcal{W}^p_{i-1/2}$,

$$Q_i - Q_{i-1} = \sum_p \mathcal{W}^p_{i-1/2}$$

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- These waves update neighboring cell averages depending on sign of s^p (Godunov's method) via fluctuations.

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Finite Volume Methods for Hyperbolic Problems

Approximate Riemann Solvers

- HLL method
- Linearized Jacobian approach
- Roe solvers
- Shallow water example
- HLLE method and positivity
- Harten-Hyman entropy fix

Riemann Problems and Jupyter Solutions

Theory and Approximate Solvers for Hyperbolic PDEs

David I. Ketcheson, RJL, and Mauricio del Razo

General information and links to book, Github, Binder, etc.:

bookstore.siam.org/fa16/bonus

View static version of notebooks at:

www.clawpack.org/riemann_book/html/Index.html

In particular see: **Shallow_water_approximate.ipynb**

Approximate Riemann Solvers

For flux-differencing methods: Compute approximation to flux at interface between cells.

Obtain high resolution via higher-order time stepping with flux limiter.

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May require entropy fix if a wave should be transonic rarefaction.

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$$q_t + \hat{A} q_x = 0, \qquad \text{where } \hat{A} = \hat{A}(q_l, q_r) \approx f'(q_{ave}).$$

Eigenvectors give waves. Roe solver $\Longrightarrow$ conservative

HLL Solver

Harten – Lax – van Leer (1983): Given $Q_\ell, Q_r \in \mathbb{R}^m$ for $m \geq 2$,
Use only 2 waves with a single interediate state Q^*.

$s^1 \approx$ minimum characteristic speed
$s^2 \approx$ maximum characteristic speed

$$\mathcal{W}^1 = Q^* - Q_\ell, \qquad \mathcal{W}^2 = Q_r - Q^*$$

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$$s^1\mathcal{W}^1 + s^2\mathcal{W}^2 = f(Q_r) - f(Q_\ell)$$

$$\implies Q^* = \frac{f(Q_r) - f(Q_\ell) - s^2 Q_r + s^1 Q_\ell}{s^1 - s^2}.$$

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Choice of speeds:

- Max and min of expected speeds over entire problem,
- Max and min of eigenvalues of $f'(Q_\ell)$ and $f'(Q_r)$.

HLL Solver for Shallow Water Equations

$$
\begin{aligned}
h_t + (hu)_x &= 0 \\
(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x &= 0
\end{aligned}
$$

Choose e.g.

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\begin{aligned}
s^1 &= u_\ell - \sqrt{gh_\ell}, \\
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Then

$$
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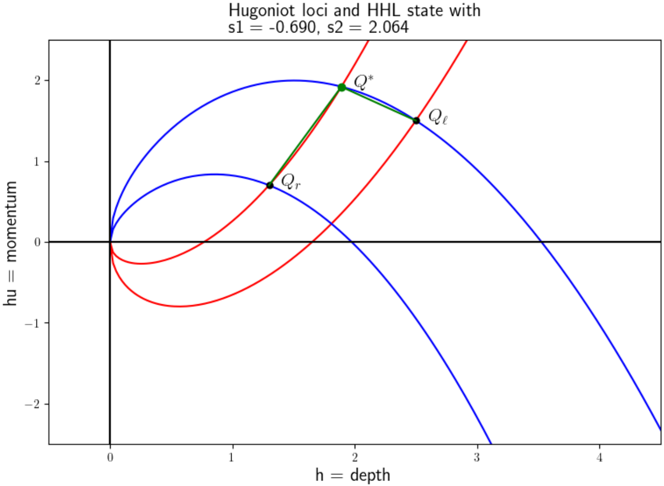
Q^* &= \frac{f(Q_r) - f(Q_\ell) - s^2 Q_r + s^1 Q_\ell}{s^1 - s^2} \\
&= \frac{1}{s^1 - s^2} \begin{bmatrix} h_r u_r - h_\ell u_\ell - s^2 h_r + s^1 h_\ell \\ \left(h_r u_r^2 + \frac{1}{2}gh_r^2\right) - \left(h_\ell u_\ell^2 + \frac{1}{2}gh_\ell^2\right) - s^2 h_r u_r + s^1 h_\ell u_\ell \end{bmatrix}
\end{aligned}
$$

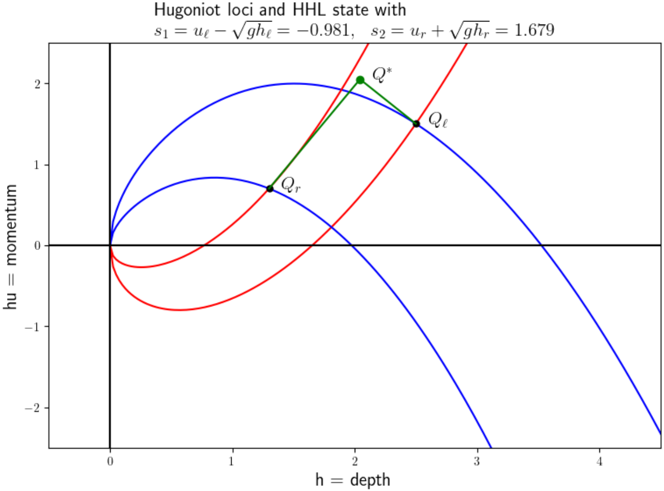
HLL solver for shallow water

If we use the shock speeds from the exact two-shock solution, looks perfect:



HLL solver for shallow water

Using $s_1 = \lambda^1(q_\ell)$ and $s_2 = \lambda^2(q_r)$:



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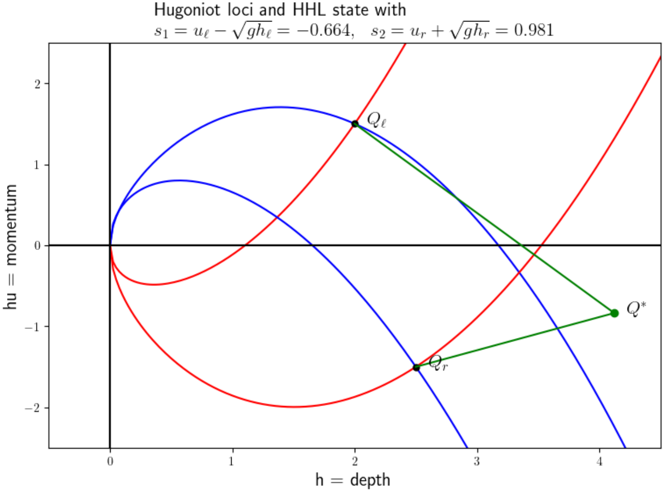
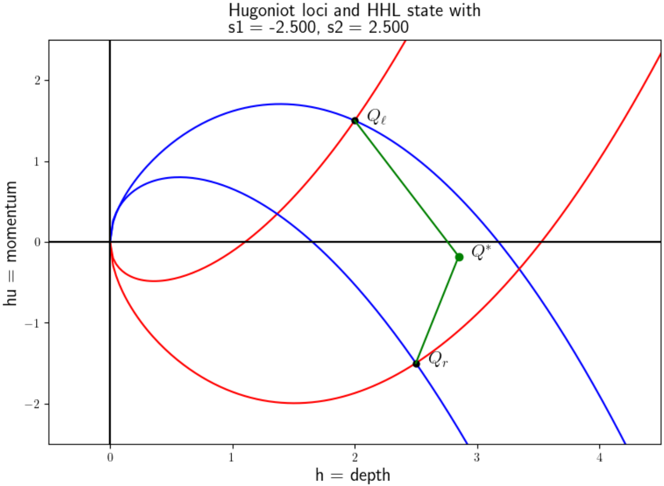


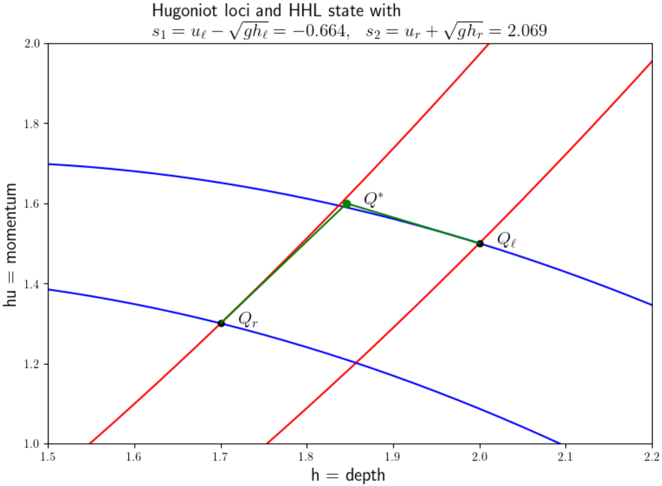
HLL solver for shallow water

Using different choice of $s_1,\ s_2$:



HLL solver for shallow water

If $\Delta Q = Q_r - Q_\ell$ is small, then eigenvalues nearly constant.
For smooth flow, HLL is very accurate (for $m = 2$).



Approximate Riemann Solvers — Local Linearization

Approximate true Riemann solution by set of waves consisting of finite jumps propagating at constant speeds.

Local linearization:

Replace $q_t + f(q)_x = 0$ by

$$q_t + \hat{A} q_x = 0,$$

where $\hat{A} = \hat{A}(q_l, q_r) \approx f'(q_{ave})$.

Then decompose

$$q_r - q_l = \alpha^1 \hat{r}^1 + \cdots \alpha^m \hat{r}^m$$

to obtain waves $\mathcal{W}^p = \alpha^p \hat{r}^p$ with speeds $s^p = \hat{\lambda}^p$.

Approximate Riemann Solvers

How to use?

One approach: determine $Q^* =$ state along $x/t = 0$,

$$Q^* = Q_{i-1} + \sum_{p:s^p<0} \mathcal{W}^p, \quad F_{i-1/2} = f(Q^*),$$

$$\mathcal{A}^-\Delta Q = F_{i-1/2} - f(Q_{i-1}), \quad \mathcal{A}^+\Delta Q = f(Q_i) - F_{i-1/2}.$$

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Or, sometimes can use:

$$\mathcal{A}^-\Delta Q = \sum_{p:s^p<0} s^p\mathcal{W}^p, \qquad \mathcal{A}^+\Delta Q = \sum_{p:s^p>0} s^p\mathcal{W}^p.$$

Conservative only if $\mathcal{A}^-\Delta Q + \mathcal{A}^+\Delta Q = f(Q_i) - f(Q_{i-1})$.

This holds for Roe solver.

Roe Solver

Given q_ℓ, q_r, solve $q_t + \hat{A} q_x = 0$ where $\hat{A}$ chosen to satisfy

$$\hat{A}(q_r - q_\ell) = f(q_r) - f(q_\ell).$$

Then:

- Good approximation for weak waves (smooth flow)

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- Single shock captured exactly:

  $f(q_r) - f(q_\ell) = s(q_r - q_\ell) \implies q_r - q_\ell$ is an eigenvector of $\hat{A}$

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$$\mathcal{A}^- \Delta Q_{i-1/2} = \sum (s^p_{i-1/2})^- \mathcal{W}^p_{i-1/2},$$
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$$\begin{aligned} \mathcal{A}^- \Delta Q_{i-1/2} + \mathcal{A}^+ \Delta Q_{i-1/2} &= \sum s^p_{i-1/2} \mathcal{W}^p_{i-1/2} = \hat{A} \sum \mathcal{W}^p_{i-1/2} \\ &= \hat{A}(q_r - q_\ell) = f(q_r) - f(q_\ell). \end{aligned}$$

Shallow water equations

$h(x,t) =$ depth
$u(x,t) =$ velocity (depth averaged, varies only with x)

Conservation of mass and momentum hu gives system of two equations.

mass flux $= hu$,
momentum flux $= (hu)u + p$ where $p =$ hydrostatic pressure

$$h_t + (hu)_x = 0$$
$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$

Jacobian matrix:

$$f'(q) = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix}, \qquad \lambda = u \pm \sqrt{gh}.$$

Roe solver for Shallow Water

Given $h_\ell,\ u_\ell,\ h_r,\ u_r$, define

$$\bar{h} = \frac{h_\ell + h_r}{2}, \quad \hat{u} = \frac{\sqrt{h_\ell}u_\ell + \sqrt{h_r}u_r}{\sqrt{h_\ell} + \sqrt{hr}}$$

Then

$$\hat{A} = \text{Jacobian matrix evaluated at this average state}$$

satisfies

$$\hat{A}(q_r - q_\ell) = f(q_r) - f(q_\ell).$$

- Roe condition is satisfied,
- Isolated shock modeled well,
- Wave propagation algorithm is conservative,
- High resolution methods obtained using corrections with limited waves.

Roe solver for Shallow Water

Given $h_l,\ u_l,\ h_r,\ u_r$, define

$$\bar{h} = \frac{h_l + h_r}{2}, \quad \hat{u} = \frac{\sqrt{h_l}u_l + \sqrt{h_r}u_r}{\sqrt{h_l} + \sqrt{hr}}$$

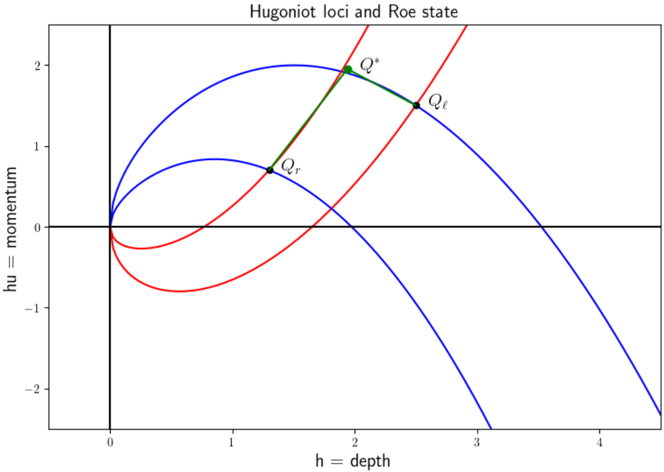
Eigenvalues of $\hat{A} = f'(\hat{q})$ are:

$$\hat{\lambda}^1 = \hat{u} - \hat{c}, \quad \hat{\lambda}^2 = \hat{u} + \hat{c}, \quad \hat{c} = \sqrt{g\bar{h}}.$$

Eigenvectors:

$$\hat{r}^1 = \begin{bmatrix} 1 \\ \hat{u} - \hat{c} \end{bmatrix}, \qquad \hat{r}^2 = \begin{bmatrix} 1 \\ \hat{u} + \hat{c} \end{bmatrix}.$$

Roe solver for shallow water



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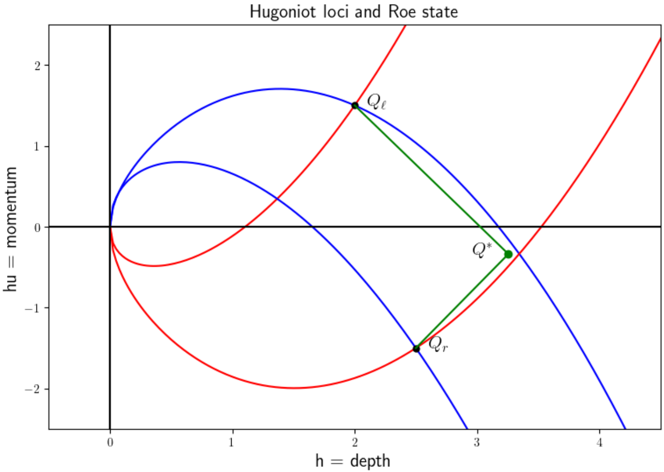
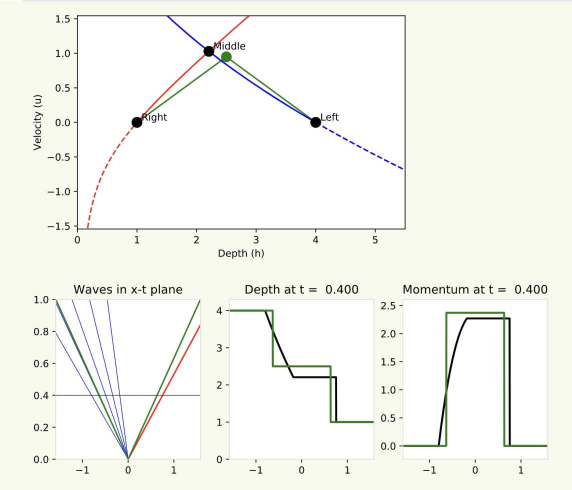


Dam break problem with Roe solver

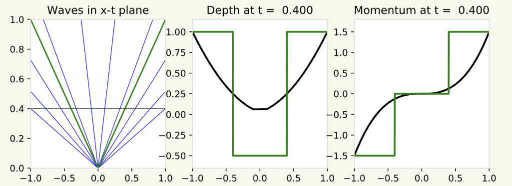


Note that rarefaction replaced by jump.

Example from **RpJs**

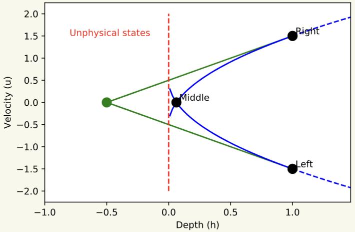
Widgets can be used to experiment.

Nonphyiscal solution with Roe solver




```
In [12]: sw.phase_plane_plot(q_l,q_r,g=1.,y_axis='u',
                             approx_states=states, hmin = -1)
         plt.plot([0,0], [-2,2], 'r--')
         plt.text(-0.8,1.5,'Unphysical states',color='red')

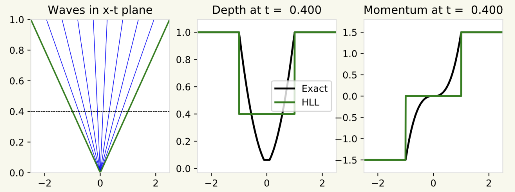
Out[12]: Text(-0.8,1.5,'Unphysical states')
```



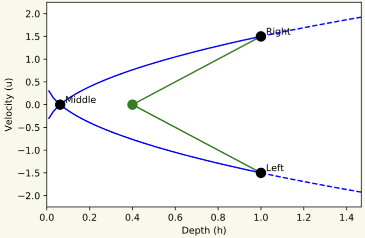
For data that gives near dry state in Q_m, Roe solver may give negative depth.

Example from **RpJs**

HLLE preserves positivity




```
In [25]: sw.phase_plane_plot(q_l,q_r,g=1.,y_axis='u',
                            approx_states=states['hll'])
```



For data that gives near dry state in Q_m, Roe solver may give negative depth.

Choosing s_1, s_2 as characteristic speeds in HLL does much better in this case.

Example from **RpJs**

HLLE Solver

Einfeldt: Choice of speeds for gas dynamics (or shallow water) that guarantees positivity.

Based on characteristic speeds and Roe averages:

$$s^1_{i-1/2} = \min_p(\min(\lambda^p_{i-1}, \hat{\lambda}^p_{i-1/2})),$$
$$s^2_{i-1/2} = \max_p(\max(\lambda^p_{i}, \hat{\lambda}^p_{i-1/2})).$$

where

λ^p_i is the pth eigenvalue of the Jacobian $f'(Q_i)$,

$\hat{\lambda}^p_{i-1/2}$ is the pth eigenvalue using Roe average $f'(\hat{Q}_{i-1/2})$

HLLE Solver

Einfeldt: Choice of speeds for gas dynamics (or shallow water) that guarantees positivity.

Can also show that:

- If Riemann solution consists of single shock,
  then Roe speed is used $\Longrightarrow$ exact solution in this case.

HLLE Solver

Einfeldt: Choice of speeds for gas dynamics (or shallow water) that guarantees positivity.

Can also show that:

- If Riemann solution consists of single shock,
  then Roe speed is used $\Longrightarrow$ exact solution in this case.
- No entropy fix needed.
  (More diffusive than Roe solver.)

Harten-Hyman entropy fix

For any wave splitting $Q_i - Q_{i-1} = \sum \mathcal{W}^p$, with speeds $\hat{\lambda}^p$.

Define

$$q_\ell^k = Q_{i-1} + \sum_{p=1}^{k-1} \mathcal{W}^p, \qquad q_r^k = q_\ell^k + \mathcal{W}^k$$

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If $\lambda_\ell^k \equiv \lambda^k(q_\ell^k) < 0 < \lambda^k(q_r^k) \equiv \lambda_r^k$ then replace $\mathcal{W}^k$ by

$$\begin{aligned} \mathcal{W}_\ell^k &= \beta \mathcal{W}^k, & \text{speed} = \lambda_\ell^k < 0, \\ \mathcal{W}_r^k &= (1-\beta)\mathcal{W}^k, & \text{speed} = \lambda_r^k > 0. \end{aligned}$$

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Conservation requires:

$$\lambda_\ell^k \mathcal{W}_\ell^k + \lambda_r^k \mathcal{W}_r^k = \hat\lambda^k \mathcal{W}^k, \qquad \Longrightarrow \beta = \frac{\lambda_r^k - \hat\lambda^k}{\lambda_r^k - \lambda_\ell^k}$$

Harten-Hyman entropy fix

In wave propagation algorithm, leave $\mathcal{W}^k$ alone for high-resolution correction terms (with limiters).

Similar to entopy fix for scalar problem:

Only need to modify the fluctuations in the "Godunov update"

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Similar to entopy fix for scalar problem:

Only need to modify the fluctuations in the "Godunov update"

$$\mathcal{A}^-\Delta Q = \sum_{p=1}^{m} (\lambda^p)^- \mathcal{W}^p, \qquad \mathcal{A}^+\Delta Q = \sum_{p=1}^{m} (\lambda^p)^+ \mathcal{W}^p,$$

Usually $\quad (\lambda^p)^- = \min(\lambda^p, 0), \qquad (\lambda^p)^+ = \max(\lambda^p, 0)$.

Modify for field k:

$$(\lambda^k)^- = \beta\lambda_\ell^k < 0, \qquad (\lambda^k)^+ = (1-\beta)\lambda_r^k > 0,$$

so that

$$(\lambda^k)^-\mathcal{W}^k = \lambda_\ell^k \beta \mathcal{W}^k \qquad (\lambda^k)^+\mathcal{W}^k = \lambda_r^k (1-\beta)\mathcal{W}^k$$

Finite Volume Methods for Hyperbolic Problems

Multidimensional Hyperbolic Problems

- Derivation of conservation law
- Hyperbolicity
- Advection
- Gas dynamics and acoustics
- Shear waves

Derivation of conservation law

$$\frac{d}{dt} \iint_\Omega q(x,y,t)\,dx\,dy = \text{net flux across } \partial\Omega.$$

Net flux is determined by integrating the flux of q normal to $\partial\Omega$ around this boundary.

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$f(q) =$ flux of q in the x-direction,
$g(q) =$ flux of q in the y-direction,
(both per unit length in orthog direction, per unit time),

$$\vec{f}(q) = (f(q),\, g(q))$$

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$$\vec{f}(q) = (f(q),\ g(q))$$

$\vec{n}(s) = (n^x(s), n^y(s))$ outward-pointing unit normal $(x(s), y(s))$.

Flux at $(x(s), y(s))$ in the direction $\vec{n}(s)$:

$$\vec{n}(s)\cdot\vec{f}(q(x(s),y(s))) = f(q)n^x(s) + g(q)n^y(s),$$

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If q is smooth: divergence theorem $\implies$

$$\frac{d}{dt}\iint_\Omega q(x,y,t)\,dx\,dy = -\iint_\Omega \vec{\nabla}\cdot\vec{f}(q)\,dx\,dy,$$

where the divergence of $\vec{f}$ is

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True for any $\Omega \Longrightarrow \quad q_t + \vec{\nabla}\cdot\vec{f}(q) = 0.$ (PDE form)

First order hyperbolic PDE in 2 space dimensions

General conservation law: $q_t + f(q)_x + g(q)_y = 0$

Quasi-linear form: $q_t + f'(q)q_x + g'(q)q_y = 0$

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where $q \in \mathbb{R}^m,\ f(q) = Aq,\ g(q) = Bq$ and $A, B \in \mathbb{R}^{m\times m}$.

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Then plane wave propagating in any direction satisfies 1D hyperbolic equation.

Plane wave solutions

Suppose

$$\begin{aligned} q(x, y, t) &= \breve{q}(x \cos\theta + y \sin\theta, t) \\ &= \breve{q}(\xi, t). \end{aligned}$$

Plane wave solutions

Suppose

$$q(x,y,t) = \breve{q}(x\cos\theta + y\sin\theta, t)$$
$$= \breve{q}(\xi, t).$$

Then:

$$q_x(x,y,t) = \cos\theta\, \breve{q}_\xi(\xi,t)$$
$$q_y(x,y,t) = \sin\theta\, \breve{q}_\xi(\xi,t)$$

so

$$q_t + Aq_x + Bq_y = \breve{q}_t + (A\cos\theta + B\sin\theta)\breve{q}_\xi$$

and the 2d problem reduces to the 1d hyperbolic equation

$$\breve{q}_t(\xi,t) + (A\cos\theta + B\sin\theta)\breve{q}_\xi(\xi,t) = 0.$$

Advection in 2 dimensions

Constant coefficient: $q_t + uq_x + vq_y = 0$

In this case solution for arbitrary initial data is easy:

$$q(x,y,t) = q(x-ut,\ y-vt,\ 0).$$

Data simply shifts at constant velocity (u,v) in x-y plane.

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Variable coefficient:

Conservation form: $q_t + (u(x, y, t)q)_x + (v(x, y, t)q)_y = 0$

Advective form (color eqn): $q_t + u(x, y, t)q_x + v(x, y, t)q_y = 0$

Advection in 2 dimensions

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Equivalent only if flow is divergence-free (incompressible):

$$\nabla \cdot \vec{u} = u_x(x, y, t) + v_y(x, y, t) = 0 \qquad \forall t \geq 0.$$

Gas dynamics in 2D

$\rho(x, y, t) =$ mass density
$\rho(x, y, t)u(x, y, t) = x$-momentum density
$\rho(x, y, t)v(x, y, t) = y$-momentum density

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If pressure $= P(\rho)$, e.g. isothermal or isentropic:

$$\begin{aligned}
\rho_t + (\rho u)_x + (\rho v)_y &= 0 \\
(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y &= 0 \\
(\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y &= 0
\end{aligned}$$

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For any θ, the matrix $f'(q)\cos\theta + g'(q)\sin\theta$ has eigenvalues

$$\breve{u} - c,\ \breve{u},\ \breve{u} + c$$

where $c = \sqrt{P'(\rho)}$ and $\breve{u} = u\cos\theta + v\sin\theta$.

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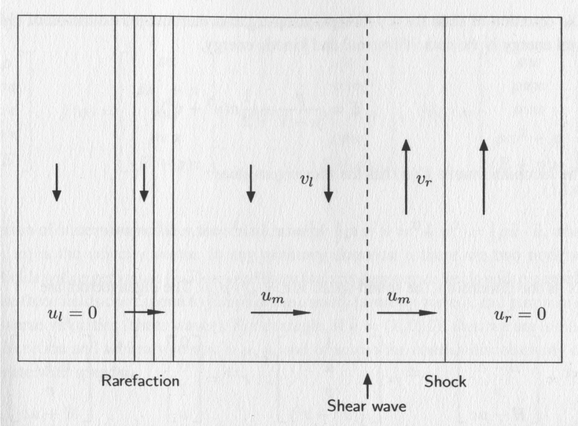
Full Euler equations: 1 more equation for Energy

For any θ, the matrix $f'(q)\cos\theta + g'(q)\sin\theta$ has eigenvalues

$\breve{u} - c,\ \breve{u},\ \breve{u} + c$ Euler: another wave with $\lambda = \breve{u}$

where $c = \sqrt{P'(\rho)}$ and $\breve{u} = u\cos\theta + v\sin\theta$.

Solution of plane wave Riemann problem in 2D



Jump in v from v_ℓ to v_r propagates with the contact discontinuity

Acoustics in 2 dimensions

Linearize about $u = 0,\ v = 0$ and $p =$ perturbation in pressure:

$$\begin{aligned} p_t + K_0(u_x + v_y) &= 0 \\ \rho_0 u_t + p_x &= 0 \\ \rho_0 v_t + p_y &= 0 \end{aligned}$$

Note: pressure responds to compression or expansion and so p_t is proportional to divergence of velocity.

Second and third equations are $F = ma$.

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Second and third equations are $F = ma$.

Gives hyperbolic system $q_t + Aq_x + Bq_y = 0$ with

$$q = \begin{bmatrix} p \\ u \\ v \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & K_0 & 0 \\ 1/\rho_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & K_0 \\ 0 & 0 & 0 \\ 1/\rho_0 & 0 & 0 \end{bmatrix}.$$

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Plane waves:

$$A\cos\theta + B\sin\theta = \begin{bmatrix} 0 & K_0\cos\theta & K_0\sin\theta \\ \cos\theta/\rho_0 & 0 & 0 \\ \sin\theta/\rho_0 & 0 & 0 \end{bmatrix}.$$

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Eigenvalues: $\lambda^1 = -c_0, \;\; \lambda^2 = 0, \;\; \lambda^3 = +c_0$

where $c_0 = \sqrt{K_0/\rho_0}$ is independent of angle θ.

Isotropic: sound propagates at same speed in any direction.

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Isotropic: sound propagates at same speed in any direction.

Note: Zero wave speed for “shear wave” with variation only in velocity in direction $(-\sin\theta, \; \cos\theta)$.

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In this case, decouples into scalar advection equation for each component of w:

$$w_t^p + \lambda^p w_x^p + \mu^p w_y^p = 0 \implies w^p(x, y, t) = w^p(x - \lambda^p t,\ y - \mu^p t,\ 0).$$

Note: In this case information propagates only in a finite number of directions $(\lambda^p,\ \mu^p)$ for $p = 1,\ \ldots,\ m$.

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This is not true for most coupled systems, e.g. acoustics.

Acoustics in 2 dimensions

$$\begin{aligned} p_t + K_0(u_x + v_y) &= 0 \\ \rho_0 u_t + p_x &= 0 \\ \rho_0 v_t + p_y &= 0 \end{aligned}$$

$$A = \begin{bmatrix} 0 & K_0 & 0 \\ 1/\rho_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad R^x = \begin{bmatrix} -Z_0 & 0 & Z_0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Solving $q_t + Aq_x = 0$ gives pressure waves in (p, u).

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$$B = \begin{bmatrix} 0 & 0 & K_0 \\ 0 & 0 & 0 \\ 1/\rho_0 & 0 & 0 \end{bmatrix} \qquad R^y = \begin{bmatrix} -Z_0 & 0 & Z_0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Solving $q_t + Bq_y = 0$ gives pressure waves in (p, v).

Finite Volume Methods for Hyperbolic Problems

Fractional Step Methods

- Dimensional splitting (Chapter 19)
- Fractional steps for source terms (Chapter 17)
- Godunov and Strang splitting
- Cross-derivatives in 2D hyperbolic problems
- Upwind splitting of ABq_{yx} and BAq_{xy}

Fractional steps for source terms

Conservation law with source term (balance law):

$$q_t(x,t) + f(q(x,t))_x = \psi(q(x,t))$$

ψ could depend on (x,t) explicitly too.

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Fractional step (time splitting) method:

To advance full solution by Δt, alternate between:

- $q_t(x,t) + f(q(x,t))_x = 0$ with high-resolution method,
- $q_t(x,t) = \psi(q(x,t)$, an ODE in each grid cell

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Fractional step (time splitting) method:

To advance full solution by Δt, alternate between:

- $q_t(x,t) + f(q(x,t))_x = 0$ with high-resolution method,
- $q_t(x,t) = \psi(q(x,t)$, an ODE in each grid cell

Source term in Clawpack: Provide `src1.f90` in 1d or `src2.f90` in 2d that advances Q in each cell by time Δt.

Set `clawdata.src_split = 1` (or `= 2` for Strang splitting)

Dimensional Splitting

Hyperbolic system in 2d: $q_t + Aq_x + Bq_y = 0$

Use Cartesian grid and alternate between:

x-sweeps : $q_t + Aq_x = 0$

y-sweeps : $q_t + Bq_y = 0.$

Use one-dimensional high-resolution methods for each.

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Use one-dimensional high-resolution methods for each.

- Easy to extend good 1D methods to 2D or 3D.
- Often very effective and efficient.
- May suffer from lack of isotropy.
- May be hard to use with AMR, complex geometry.

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Use one-dimensional high-resolution methods for each.

- Easy to extend good 1D methods to 2D or 3D.
- Often very effective and efficient.
- May suffer from lack of isotropy.
- May be hard to use with AMR, complex geometry.

Alternative: Unsplit methods.

Fractional step method for a linear PDE

$q_t = (\mathcal{A} + \mathcal{B})q$ dimensional splitting: $\mathcal{A} = A\partial_x, \;\; \mathcal{B} = B\partial_y.$

Then

$$q_{tt} = (\mathcal{A} + \mathcal{B})q_t = (\mathcal{A} + \mathcal{B})^2 q,$$

and so

$$\begin{aligned} q(x, \Delta t) &= q(x,0) + \Delta t(\mathcal{A} + \mathcal{B})q(x,0) + \frac{1}{2}\Delta t^2(\mathcal{A} + \mathcal{B})^2 q(x,0) + \cdots \\ &= \left(I + \Delta t(\mathcal{A} + \mathcal{B}) + \frac{1}{2}\Delta t^2(\mathcal{A} + \mathcal{B})^2 + \cdots\right) q(x,0) \end{aligned}$$

Solution operator: $q(x, \Delta t) = e^{\Delta t(\mathcal{A}+\mathcal{B})} q(x,0).$

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Solution operator: $q(x, \Delta t) = e^{\Delta t(\mathcal{A}+\mathcal{B})} q(x,0)$.

With the fractional step method, we instead compute

$$q^*(x, \Delta t) = e^{\Delta t \mathcal{A}} q(x,0),$$

and then

$$q^{**}(x, \Delta t) = e^{\Delta t \mathcal{B}} q^*(x, \Delta t) \;\; = e^{\Delta t \mathcal{B}} e^{\Delta t \mathcal{A}} q(x,0).$$

Splitting error

$$q(x,\Delta t) - q^{**}(x,\Delta t) = \left(e^{\Delta t(\mathcal{A}+\mathcal{B})} - e^{\Delta t\mathcal{B}}e^{\Delta t\mathcal{A}}\right) q(x,0)$$

Combining 2 steps gives:

$$\begin{aligned} q^{**}(x,\Delta t) &= \left(I + \Delta t\mathcal{B} + \frac{1}{2}\Delta t^2\mathcal{B}^2 + \cdots\right)\left(I + \Delta t\mathcal{A} + \frac{1}{2}\Delta t^2\mathcal{A}^2 + \cdots\right) q(x,0) \\ &= \left(I + \Delta t(\mathcal{A}+\mathcal{B}) + \frac{1}{2}\Delta t^2(\mathcal{A}^2 + 2\mathcal{B}\mathcal{A} + \mathcal{B}^2) + \cdots\right) q(x,0). \end{aligned}$$

In true solution operator,

$$\begin{aligned} (\mathcal{A}+\mathcal{B})^2 &= (\mathcal{A}+\mathcal{B})(\mathcal{A}+\mathcal{B}) \\ &= \mathcal{A}^2 + \mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A} + \mathcal{B}^2. \end{aligned}$$

Splitting error

$$\begin{aligned} q(x,\Delta t) - q^{**}(x,\Delta t) &= \left(e^{\Delta t(\mathcal{A}+\mathcal{B})} - e^{\Delta t\mathcal{B}}e^{\Delta t\mathcal{A}}\right) q(x,0) \\ &= \frac{1}{2}\Delta t^2(\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A})q(x,0) + O(\Delta t^3). \end{aligned}$$

There is a splitting error unless the two operators commute.

Splitting error

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No splitting error for constant coefficient advection:

$$\mathcal{A} = u\partial_x, \;\; \mathcal{B} = v\partial_y \quad \mathcal{A}\mathcal{B}q = \mathcal{B}\mathcal{A}q = uvq_{xy}$$

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$$

There is a splitting error if u, v are varying:

$$
\begin{aligned}
\mathcal{A}\mathcal{B}q &= u(x,y)\partial_x\left(v(x,y)\partial_y q\right) = uvq_{xy} + uv_xq_y, \\
\mathcal{B}\mathcal{A}q &= v(x,y)\partial_y\left(u(x,y)\partial_x q\right) = vuq_{xy} + vu_yq_x.
\end{aligned}
$$

Splitting error

$$
\begin{aligned}
q(x,\Delta t) - q^{**}(x,\Delta t) &= \left(e^{\Delta t(\mathcal{A}+\mathcal{B})} - e^{\Delta t\mathcal{B}} e^{\Delta t\mathcal{A}}\right) q(x,0) \\
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\mathcal{B}\mathcal{A}q &= v(x,y)\partial_y \left(u(x,y)\partial_x q\right) = vuq_{xy} + vu_y q_x.
\end{aligned}
$$

There is a splitting error for acoustics since $ABq_{xy} \neq BAq_{xy}$.

Commuting operators

Note that if A and B are simultaneously diagonalizable,

$$A = R\Lambda R^{-1}, \qquad B = RMR^{-1},$$

then

$$AB = R\Lambda MR^{-1} = RM\Lambda R^{-1} = BA$$

Commuting operators

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and conversely, if two diagonalizable matrices commute, then they are simultaneously diagonalizable.

So matrices arising from isotropic PDEs do not commute.

Splitting error

$$q(x, \Delta t) - q^{**}(x, \Delta t) = \left(e^{\Delta t(\mathcal{A}+\mathcal{B})} - e^{\Delta t \mathcal{B}} e^{\Delta t \mathcal{A}}\right) q(x, 0)$$

Combining 2 steps gives:

$$\begin{aligned} q^{**}(x, \Delta t) &= \left(I + \Delta t \mathcal{B} + \frac{1}{2}\Delta t^2 \mathcal{B}^2 + \cdots\right)\left(I + \Delta t \mathcal{A} + \frac{1}{2}\Delta t^2 \mathcal{A}^2 + \cdots\right) q(x, 0) \\ &= \left(I + \Delta t(\mathcal{A} + \mathcal{B}) + \frac{1}{2}\Delta t^2(\mathcal{A}^2 + 2\mathcal{B}\mathcal{A} + \mathcal{B}^2) + \cdots\right) q(x, 0). \end{aligned}$$

In true solution operator,

$$\begin{aligned} (\mathcal{A} + \mathcal{B})^2 &= (\mathcal{A} + \mathcal{B})(\mathcal{A} + \mathcal{B}) \\ &= \mathcal{A}^2 + \mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A} + \mathcal{B}^2. \end{aligned}$$

Strang splitting

Advance the PDE by time step Δt by...

- Time step $\Delta t/2$ on A-problem,
- Time step Δt on B-problem,
- Time step $\Delta t/2$ on A-problem.

Formally second order if each solution method is.

$$\left(e^{\Delta t(\mathcal{A}+\mathcal{B})} - e^{\frac{1}{2}\Delta t\mathcal{A}}e^{\Delta t\mathcal{B}}e^{\frac{1}{2}\Delta t\mathcal{A}}\right)q(x,0) = O(\Delta t^3).$$

Strang splitting

Advance the PDE by time step Δt by...

- Time step $\Delta t/2$ on A-problem,
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In practice often little difference from "first order Godunov splitting" since after N steps,

$$q^N = e^{\frac{1}{2}\Delta t\mathcal{A}}e^{\Delta t\mathcal{B}}e^{\frac{1}{2}\Delta t\mathcal{A}}\;\; e^{\frac{1}{2}\Delta t\mathcal{A}}e^{\Delta t\mathcal{B}}e^{\frac{1}{2}\Delta t\mathcal{A}}\;\; e^{\frac{1}{2}\Delta t\mathcal{A}}e^{\Delta t\mathcal{B}}e^{\frac{1}{2}\Delta t\mathcal{A}}\;\cdots$$
$$e^{\frac{1}{2}\Delta t\mathcal{A}}e^{\Delta t\mathcal{B}}e^{\frac{1}{2}\Delta t\mathcal{A}}\;\; e^{\frac{1}{2}\Delta t\mathcal{A}}e^{\Delta t\mathcal{B}}e^{\frac{1}{2}\Delta t\mathcal{A}}\; q^0$$

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$$e^{\frac{1}{2}\Delta t\mathcal{A}} e^{\Delta t\mathcal{B}} e^{\frac{1}{2}\Delta t\mathcal{A}} \; e^{\frac{1}{2}\Delta t\mathcal{A}} e^{\Delta t\mathcal{B}} e^{\frac{1}{2}\Delta t\mathcal{A}} \; q^0$$

$$= e^{\frac{1}{2}\Delta t\mathcal{A}} \left(e^{\Delta t\mathcal{B}} e^{\Delta t\mathcal{A}}\right)^N e^{\frac{1}{2}\Delta t\mathcal{A}} \; q^0$$

Example of splitting error for source term

Advection-reaction equation: $q_t + uq_x = -\beta(x)q$

Then

$$\frac{d}{dt}q(X(t),t) = -\beta(X(t))\,q(X(t),t) \qquad \text{(exponential decay)}$$

along characteristic $X(t) = x_0 + ut$.

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along characteristic $X(t) = x_0 + ut$.

Splitting: Take $\mathcal{A} = -u\partial_x$ and $\mathcal{B} = -\beta(x)$.

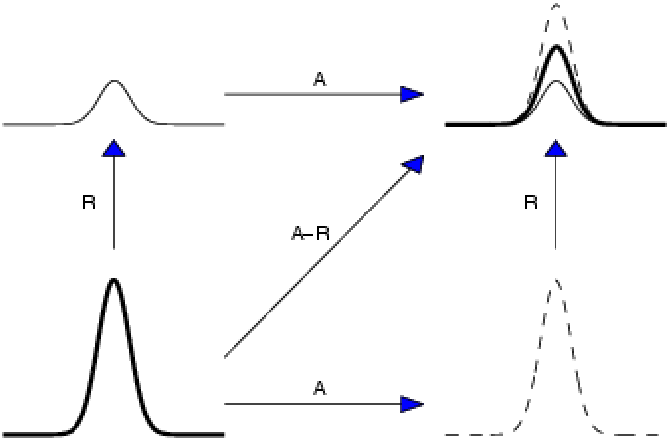
Then:

$$\begin{aligned} \mathcal{AB}q &= u\partial_x(\beta(x)q) = u\beta(x)q_x + u\beta'(x)q \\ \mathcal{BA}q &= \beta(x)uq_x \end{aligned}$$

Splitting error unless $\beta(x) =$ constant

Splitting error in advection-reaction (decay)

$q_t + uq_x = -\beta(x)q$ with $\beta(x)$ decreasing as x increases



β large $\quad\quad$ β smaller

Taylor series in 2d for dimensional splitting

Consider $q_t + Aq_x + Bq_y = 0$.

$$q_{tt} = -Aq_{tx} - Bq_{ty} \quad = \; A^2 q_{xx} + ABq_{yx} + BAq_{xy} + B^2 q_{yy}$$

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$$\begin{aligned} q(x, y, t+\Delta t) &= q + \Delta t\, q_t + \frac{1}{2}\Delta t^2 q_{tt} + \cdots \\ &= q - \Delta t(Aq_x + Bq_y) \\ &\quad + \frac{1}{2}\Delta t^2 \left[A^2 q_{xx} + ABq_{yx} + BAq_{xy} + B^2 q_{yy}\right] + \cdots \end{aligned}$$

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&\quad + \frac{1}{2}\Delta t^2 \left[A^2 q_{xx} + ABq_{yx} + BAq_{xy} + B^2 q_{yy}\right] + \cdots \\
&= q - \Delta t\, Aq_x + \frac{1}{2}\Delta t^2 A^2 q_{xx} \\
&\quad - \Delta t\, Bq_y + \frac{1}{2}\Delta t^2 B^2 q_{yy} \\
&\quad + \frac{1}{2}\Delta t^2 [ABq_{yx} + BAq_{xy}] \quad \text{cross derivatives}
\end{aligned}$$

Dimensional splitting of upwind on $q_t + Aq_x + Bq_y = 0$

$$Q_{ij}^* = Q_{ij}^n - \frac{\Delta t}{\Delta x}[B^+(Q_{ij}^n - Q_{i,j-1}^n) + B^-(Q_{i,j+1}^n - Q_{ij}^n)]$$

$$Q_{ij}^{n+1} = Q_{ij}^* - \frac{\Delta t}{\Delta x}[A^+(Q_{ij}^* - Q_{i-1,j}^*) + A^-(Q_{i+1,j}^* - Q_{ij}^*)$$

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$$Q^{n+1}_{ij} = Q^*_{ij} - \frac{\Delta t}{\Delta x}[A^+(Q^*_{ij} - Q^*_{i-1,j}) + A^-(Q^*_{i+1,j} - Q^*_{ij})$$

Consider one term, e.g. the one in blue above

$$\begin{aligned}\frac{\Delta t}{\Delta x}A^+(Q^*_{ij} - Q^*_{i-1,j}) = \frac{\Delta t}{\Delta x}A^+ &\left[Q^n_{ij} - \frac{\Delta t}{\Delta x}\left(B^+(Q^n_{ij} - Q^n_{i,j-1}) + B^-(Q^n_{i,j+1} - Q^n_{ij})\right)\right] \\ - A^+ &\left[Q^n_{i-1,j} - \frac{\Delta t}{\Delta x}\left(B^+(Q^n_{i-1,j} - Q^n_{i-1,j-1}) + B^-(Q^n_{i-1,j+1} - Q^n_{i-1,j})\right)\right]\end{aligned}$$

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Includes, e.g.:

$$(\tfrac{\Delta t}{\Delta x})^2 A^+B^-(Q^n_{i,j+1} - Q^n_{ij} - Q^n_{i-1,j+1} + Q^n_{i-1,j}) \approx \tfrac{\Delta t^2 \Delta y}{\Delta x \Delta y} A^+B^- q_{xy}(x_i, y_j)$$

Upwind splitting of matrix product

In 1D, the upwind method is

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[A^+(Q_i^n - Q_{i-1}^n) + A^-(Q_{i+1}^n - Q_i^n)]$$

where

$$A = R\Lambda R^{-1} = R\Lambda^+ R^{-1} + R\Lambda^- R^{-1} = A^+ \; + \; A^-$$

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In 2D the unsplit generalization uses

$$\begin{aligned}AB &= (A^+ + A^-)(B^+ + B^-) = A^+B^+ \ + \ A^+B^- \ + \ A^-B^+ \ + \ B^-A^-,\\ BA &= (B^+ + A^-)(B^+ + A^-) = B^+A^+ \ + \ B^+A^- \ + \ B^-A^+ \ + \ B^-A^-.\end{aligned}$$

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$$BA = (B^+ + A^-)(B^+ + A^-) = B^+A^+ \ + \ B^+A^- \ + \ B^-A^+ \ + \ B^-A^-.$$

Scalar advection: only one term is nonzero in each product,
e.g. $u > 0, \ v < 0 \implies uv = vu = u^+v^-$

Multidimensional Finite Volume Methods

- Integral form on a rectangular grid cell
- Flux differencing form
- Scalar advection: donor cell upwind
- Corner transport upwind and transverse waves
- Wave propagation algorithms for systems
- Transverse Riemann solver

Derivation of conservation law

$$\frac{d}{dt}\iint_{\Omega} q(x,y,t)\,dx\,dy = -\int_{\partial\Omega} \vec{n}\cdot\vec{f}(q)\,ds.$$

where $\vec{f}(q) = (f(q),\ g(q))$, fluxes in x- and y-directions.

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If Ω is a rectangular grid cell $[x_{i-1/2},\ x_{i+1/2}] \times [y_{j-1/2},\ y_{j+1/2}]$

Then flux in normal directioni $\vec{n}$ is

$$\vec{n}\cdot\vec{f}(q) = \begin{cases} \mp f(q) & \text{at } x_{i\pm 1/2}, \\ \mp g(q) & \text{at } y_{j\pm 1/2}. \end{cases}$$

2D finite volume method for $q_t + f(q)_x + g(q)_y = 0$

Evolution of total mass due to fluxes through cell edges:

$$
\begin{aligned}
\frac{d}{dt}\iint_{\mathcal{C}_{ij}} q(x,y,t)\,dx\,dy = &\int_{y_{j-1/2}}^{y_{j+1/2}} f(q(x_{i+1/2},y,t)\,dy \\
&- \int_{y_{j-1/2}}^{y_{j+1/2}} f(q(x_{i-1/2},y,t)\,dy \\
&+ \int_{x_{i-1/2}}^{x_{i+1/2}} g(q(x,y_{j+1/2},t)\,dx \\
&- \int_{x_{i-1/2}}^{x_{i+1/2}} g(q(x,y_{j-1/2},t)\,dx.
\end{aligned}
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Suggests:

$$\begin{aligned}\frac{\Delta x \Delta y Q_{ij}^{n+1} - \Delta x \Delta y Q_{ij}^{n}}{\Delta t} = &-\Delta y[F_{i+1/2,j}^{n} - F_{i-1/2,j}^{n}] \\ &- \Delta x[G_{i,j+1/2}^{n} - G_{i,j-1/2}^{n}],\end{aligned}$$

2D finite volume method for $q_t + f(q)_x + g(q)_y = 0$

$$\Delta x \Delta y Q_{ij}^{n+1} = \Delta x \Delta y Q_{ij}^{n} - \Delta t \Delta y [F_{i+1/2,j}^{n} - F_{i-1/2,j}^{n}] - \Delta t \Delta x [G_{i,j+1/2}^{n} - G_{i,j-1/2}^{n}],$$

Where we define numerical fluxes:

$$F_{i-1/2,j}^{n} \approx \frac{1}{\Delta t \Delta y} \int_{t_n}^{t_{n+1}} \int_{y_{j-1/2}}^{y_{j+1/2}} f(q(x_{i-1/2}, y, t))\, dy\, dt,$$

$$G_{i,j-1/2}^{n} \approx \frac{1}{\Delta t \Delta x} \int_{t_n}^{t_{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} g(q(x, y_{j-1/2}, t))\, dx\, dt.$$

2D finite volume method for $q_t + f(q)_x + g(q)_y = 0$

$$\Delta x \Delta y Q_{ij}^{n+1} = \Delta x \Delta y Q_{ij}^n - \Delta t \Delta y [F_{i+1/2,j}^n - F_{i-1/2,j}^n] - \Delta t \Delta x [G_{i,j+1/2}^n - G_{i,j-1/2}^n],$$

Where we define numerical fluxes:

$$F_{i-1/2,j}^n \approx \frac{1}{\Delta t \Delta y} \int_{t_n}^{t_{n+1}} \int_{y_{j-1/2}}^{y_{j+1/2}} f(q(x_{i-1/2}, y, t))\, dy\, dt,$$

$$G_{i,j-1/2}^n \approx \frac{1}{\Delta t \Delta x} \int_{t_n}^{t_{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} g(q(x, y_{j-1/2}, t))\, dx\, dt.$$

Rewrite by dividing by $\Delta x \Delta y \implies$ FV method in conservation form:

$$Q_{ij}^{n+1} = Q_{ij}^n - \frac{\Delta t}{\Delta x}[F_{i+1/2,j}^n - F_{i-1/2,j}^n] - \frac{\Delta t}{\Delta y}[G_{i,j+1/2}^n - G_{i,j-1/2}^n].$$

Dimensional splitting vs. unsplit FV method

Hyperbolic system in 2d: $q_t + f(q)_x + g(q)_y = 0$

Split method:

$$Q^*_{ij} = Q^n_{ij} - \frac{\Delta t}{\Delta x}[F^n_{i+1/2,j} - F^n_{i-1/2,j}]$$
$$Q^{n+1}_{ij} = Q^*_{ij} - \frac{\Delta t}{\Delta y}[G^*_{i,j+1/2} - G^*_{i,j-1/2}].$$

Dimensional splitting vs. unsplit FV method

Hyperbolic system in 2d: $q_t + f(q)_x + g(q)_y = 0$

Split method:

$$
\begin{aligned}
Q_{ij}^* &= Q_{ij}^n - \frac{\Delta t}{\Delta x}[F_{i+1/2,j}^n - F_{i-1/2,j}^n] \\
Q_{ij}^{n+1} &= Q_{ij}^* - \frac{\Delta t}{\Delta y}[G_{i,j+1/2}^* - G_{i,j-1/2}^*].
\end{aligned}
$$

Unsplit method:

$$
\begin{aligned}
Q_{ij}^{n+1} = Q_{ij}^n &- \frac{\Delta t}{\Delta x}[F_{i+1/2,j}^n - F_{i-1/2,j}^n] \\
&- \frac{\Delta t}{\Delta y}[G_{i,j+1/2}^n - G_{i,j-1/2}^n].
\end{aligned}
$$

2D finite volume method for $q_t + f(q)_x + g(q)_y = 0$

$$\begin{aligned} Q_{ij}^{n+1} = Q_{ij}^n &- \frac{\Delta t}{\Delta x}[F_{i+1/2,j}^n - F_{i-1/2,j}^n] \\ &- \frac{\Delta t}{\Delta y}[G_{i,j+1/2}^n - G_{i,j-1/2}^n]. \end{aligned}$$

Fluctuation form:

$$\begin{aligned} Q_{ij}^{n+1} = Q_{ij} &- \frac{\Delta t}{\Delta x}(\mathcal{A}^+\Delta Q_{i-1/2,j} + \mathcal{A}^-\Delta Q_{i+1/2,j}) \\ &- \frac{\Delta t}{\Delta y}(\mathcal{B}^+\Delta Q_{i,j-1/2} + \mathcal{B}^-\Delta Q_{i,j+1/2}) \\ &- \frac{\Delta t}{\Delta x}(\tilde{F}_{i+1/2,j} - \tilde{F}_{i-1/2,j}) - \frac{\Delta t}{\Delta y}(\tilde{G}_{i,j+1/2} - \tilde{G}_{i,j-1/2}). \end{aligned}$$

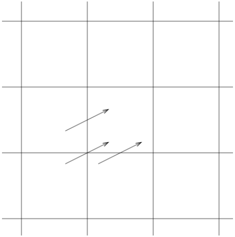
The $\tilde{F}$ and $\tilde{G}$ are correction fluxes to go beyond Godunov's upwind method.

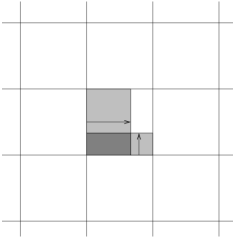
Incorporate approximations to second derivative terms in each direction (q_{xx} and q_{yy}) and mixed term q_{xy}.

Advection: Donor Cell Upwind

With no correction fluxes, Godunov's method for advection is Donor Cell Upwind:

$$Q_{ij}^{n+1} = Q_{ij} - \frac{\Delta t}{\Delta x}[u^+(Q_{ij} - Q_{i-1,j}) + u^-(Q_{i+1,j} - Q_{ij})] \\ - \frac{\Delta t}{\Delta y}[v^+(Q_{ij} - Q_{i,j-1}) + v^-(Q_{i,j+1} - Q_{ij})].$$



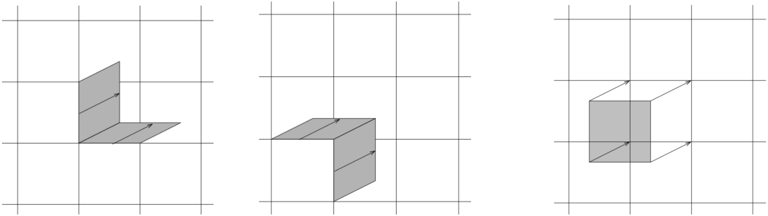


Stable only if $\left|\frac{u\Delta t}{\Delta x}\right| + \left|\frac{v\Delta t}{\Delta y}\right| \le 1.$

Advection: Corner Transport Upwind (CTU)

Correction fluxes can be added to advect waves correctly.

Corner Transport Upwind:



Stable for $\max\left(\left|\frac{u\Delta t}{\Delta x}\right|, \left|\frac{v\Delta t}{\Delta y}\right|\right) \le 1.$

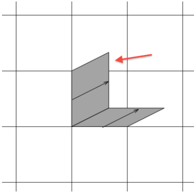
Advection: Corner Transport Upwind (CTU)

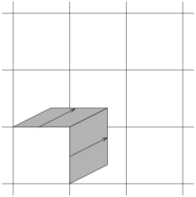
Need to transport triangular region from cell (i, j) to $(i, j + 1)$:

$$\text{Area} = \frac{1}{2}(u\Delta t)(v\Delta t) \implies \left(\frac{\frac{1}{2}uv(\Delta t)^2}{\Delta x \Delta y}\right)(Q_{ij} - Q_{i-1,j}).$$

Accomplished by correction flux:

$$\tilde{G}_{i,j+1/2} = -\frac{1}{2}\frac{\Delta t}{\Delta x}\,uv(Q_{ij} - Q_{i-1,j})$$





$\frac{\Delta t}{\Delta y}(\tilde{G}_{i,j+1/2} - \tilde{G}_{i,j-1/2})$ gives approximation to $\frac{1}{2}\Delta t^2 uvq_{xy}$.

$\frac{\Delta t}{\Delta x}(\tilde{F}_{i+1/2,j} - \tilde{F}_{i-1/2,j})$ gives similar approximation.

Upwind splitting of matrix product

In 1D, the upwind method is

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[A^+(Q_i^n - Q_{i-1}^n) + A^-(Q_{i+1}^n - Q_i^n)]$$

where

$$A = R\Lambda R^{-1} = R\Lambda^+ R^{-1} + R\Lambda^- R^{-1} = A^+ \ + \ A^-$$

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In 2D the unsplit generalization uses

$$\begin{aligned} AB &= (A^+ + A^-)(B^+ + B^-) = A^+B^+ \ + \ A^+B^- \ + \ A^-B^+ \ + \ B^-A^-, \\ BA &= (B^+ + A^-)(B^+ + A^-) = B^+A^+ \ + \ B^+A^- \ + \ B^-A^+ \ + \ B^-A^-. \end{aligned}$$

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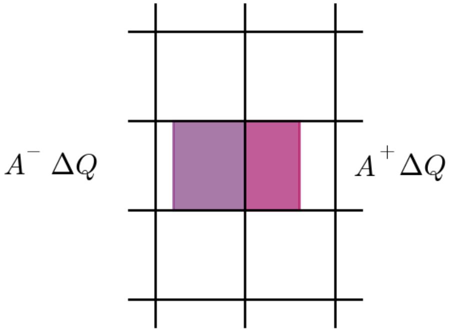
Scalar advection: only one term is nonzero in each product,
e.g. $u > 0, \ v < 0 \implies uv = vu = u^+v^-$

Wave propagation algorithm for $q_t + Aq_x + Bq_y = 0$

Decompose $A = A^+ + A^-$ and $B = B^+ + B^-$.

For $\Delta Q = Q_{ij} - Q_{i-1,j}$:



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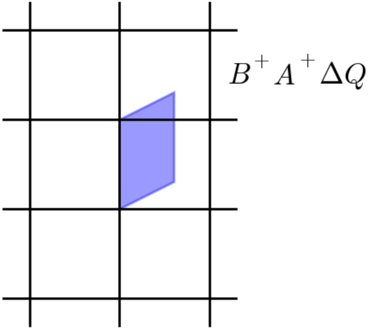

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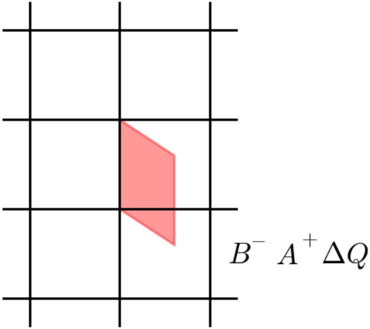

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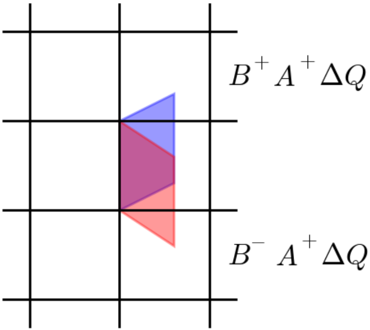

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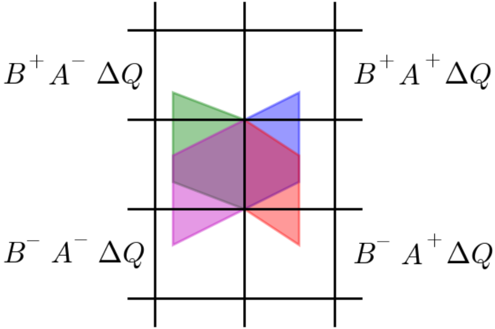
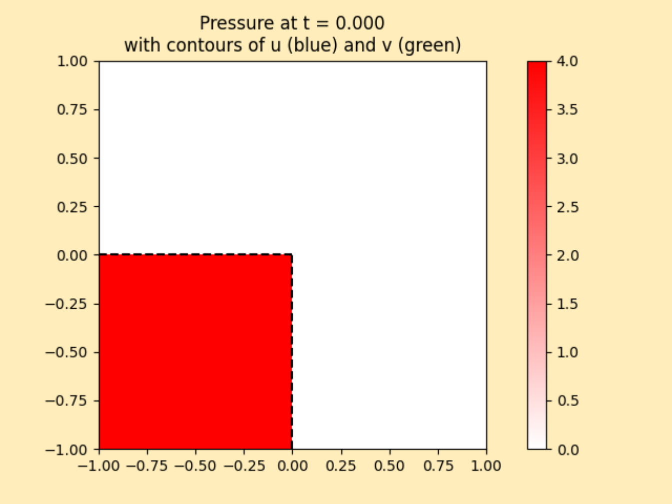


Acoustics near a cell corner

Solve 2D acoustics with $\rho = K = c = Z = 1$

$u = v \equiv 0$ and $p = 4$ in lower left quadrant, $p = 0$ elsewhere



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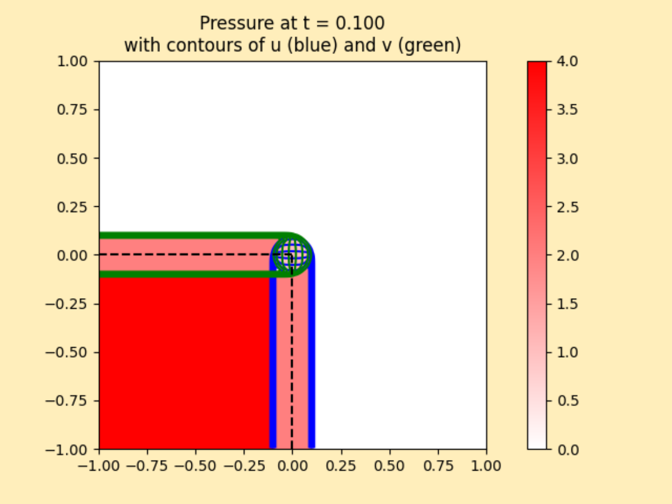

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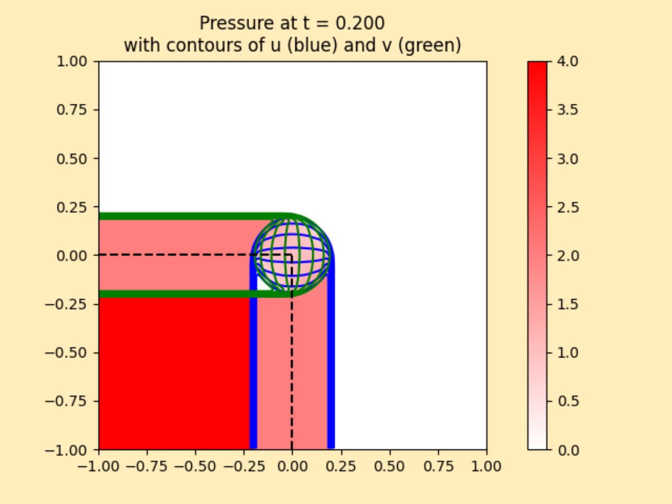

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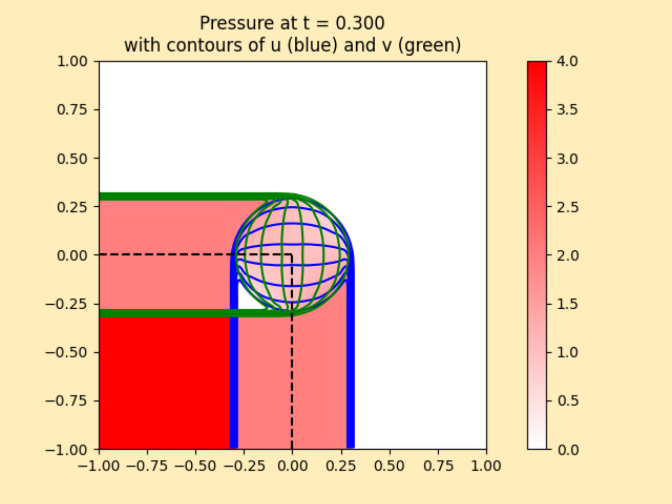

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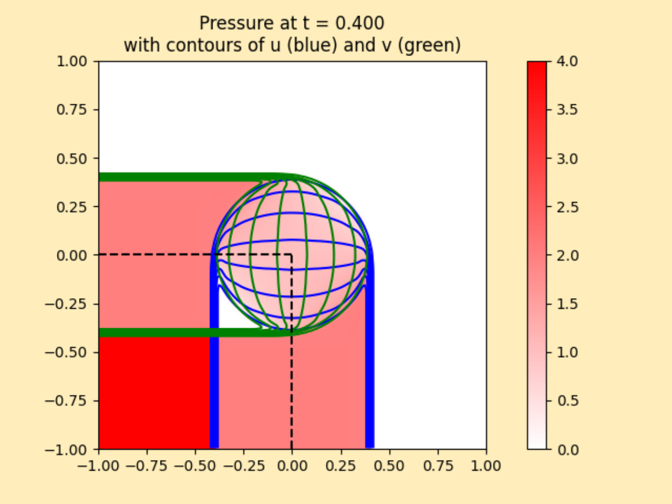

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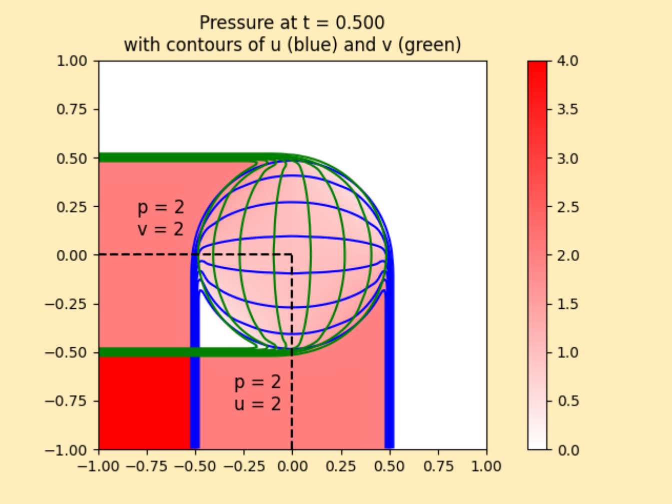
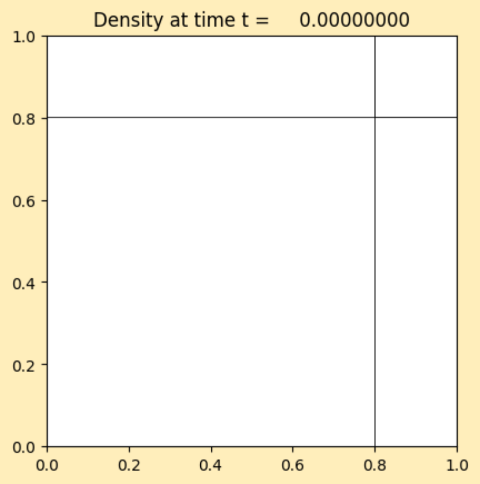


2D Riemann problem for Euler

Values in 4 quadrants chosen to give single shock between each



Clawpack Gallery: euler_2d_quadrants

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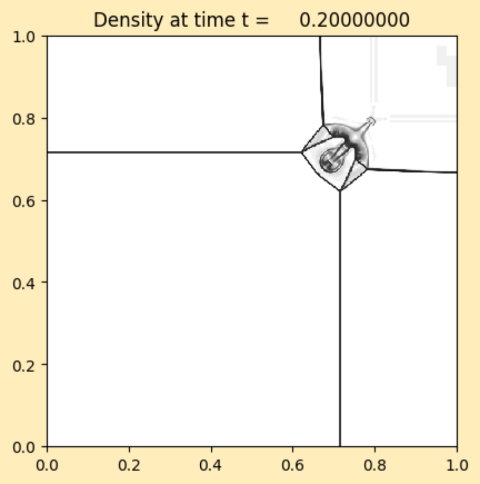

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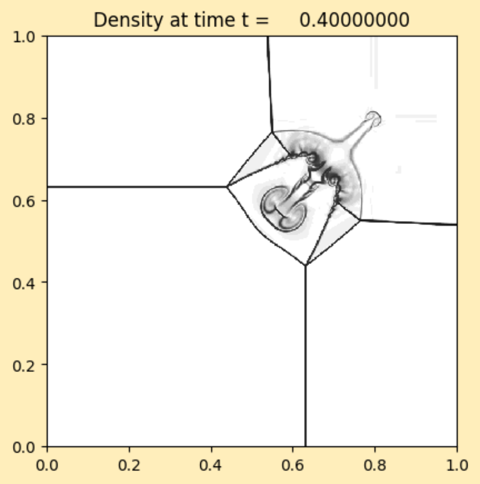

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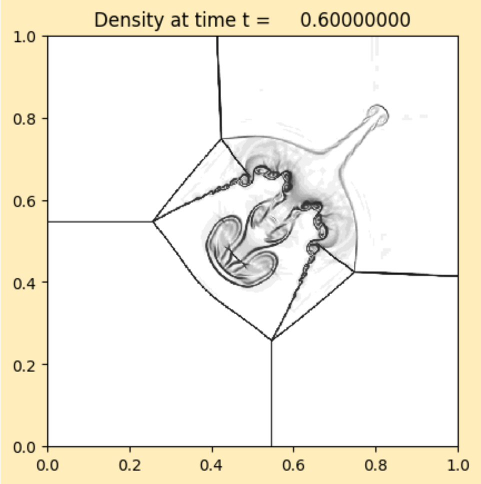

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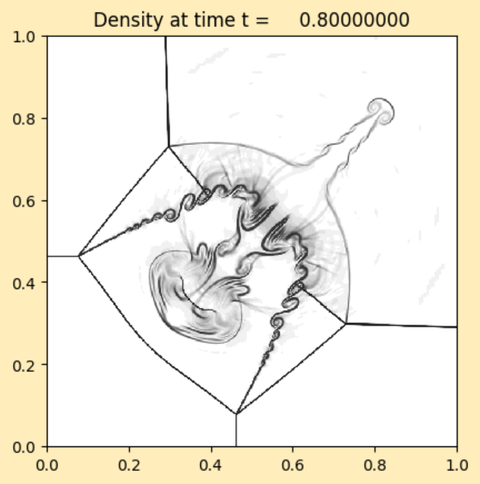

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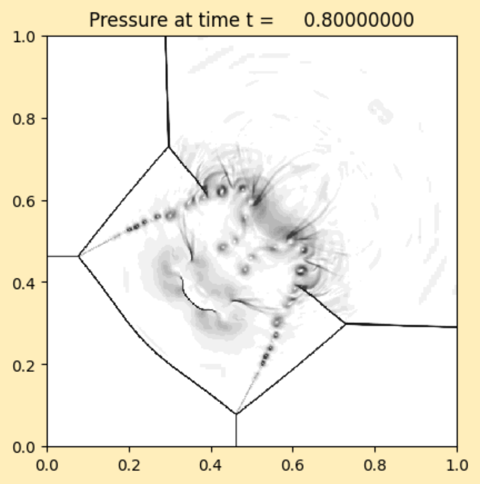

Clawpack Gallery: euler_2d_quadrants

Wave propagation algorithms in 2D

Clawpack requires:

Normal Riemann solver `rpn2.f`
Solves 1d Riemann problem $q_t + Aq_x = 0$
Decomposes $\Delta Q = Q_{ij} - Q_{i-1,j}$ into $\mathcal{A}^+\Delta Q$ and $\mathcal{A}^-\Delta Q$.
For $q_t + Aq_x + Bq_y = 0$, split using eigenvalues, vectors:

$$A = R\Lambda R^{-1} \implies A^- = R\Lambda^- R^{-1}, A^+ = R\Lambda^+ R^{-1}$$

Input parameter `ixy` determines if it's in x or y direction.
In latter case splitting is done using B instead of A.
This is all that's required for dimensional splitting.

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Input parameter `ixy` determines if it's in x or y direction.
In latter case splitting is done using B instead of A.
This is all that's required for dimensional splitting.

Transverse Riemann solver `rpt2.f`
Decomposes $\mathcal{A}^+\Delta Q$ into $\mathcal{B}^-\mathcal{A}^+\Delta Q$ and $\mathcal{B}^+\mathcal{A}^+\Delta Q$ by splitting this vector into eigenvectors of B.

(Or splits vector into eigenvectors of A if `ixy=2`.)

Transverse Riemann solver in Clawpack

`rpt2` takes vector `asdq` and returns `bmasdq` and `bpasdq` where

`asdq` $= \mathcal{A}^*\Delta Q$ represents either
$\mathcal{A}^-\Delta Q$ if `imp = 1`, or
$\mathcal{A}^+\Delta Q$ if `imp = 2`.

Returns $\mathcal{B}^-\mathcal{A}^*\Delta Q$ and $\mathcal{B}^+\mathcal{A}^*\Delta Q$.

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Note: there is also a parameter `ixy`:

`ixy = 1` means normal solve was in x-direction,

`ixy = 2` means normal solve was in y-direction,
In this case `asdq` represents $\mathcal{B}^-\Delta Q$ or $\mathcal{B}^+\Delta Q$ and the routine must return $\mathcal{A}^-\mathcal{B}^*\Delta Q$ and $\mathcal{A}^+\mathcal{B}^*\Delta Q$.

Gas dynamics in 2D

$\rho(x, y, t) =$ mass density
$\rho(x, y, t)u(x, y, t) = x$-momentum density
$\rho(x, y, t)v(x, y, t) = y$-momentum density

If pressure $= P(\rho)$, e.g. isothermal or isentropic:

$$\begin{aligned}
\rho_t + (\rho u)_x + (\rho v)_y &= 0 \\
(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y &= 0 \\
(\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y &= 0
\end{aligned}$$

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1D equation in x: $q_t + f(q)_x = 0$ is:

$$\begin{aligned} \rho_t + (\rho u)_x &= 0 \\ (\rho u)_t + (\rho u^2 + p)_x &= 0 \\ (\rho v)_t + (\rho uv)_x &= 0 \implies v_t + uv_x = 0 \end{aligned}$$

These are just 1D equations for $(\rho,\ \rho u)$
along with an advected quantity v

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These are just 1D equations for $(\rho,\ \rho v)$
along with an advected quantity u

Gas dynamics in 2D – transverse solver

If Roe solver is used for normal Riemann problems:

Eigenvectors of $\hat{A} \approx f'(q)$ are used for splitting in x,

$$\hat{\rho} = \frac{1}{2}(\rho_{i-1,j} + \rho_{i,j}), \qquad \hat{u} = \frac{\sqrt{\rho_{i-1,j}}u_{i-1,j} + \sqrt{\rho_{i,j}}u_{i,j}}{\sqrt{\rho_{i-1,j}} + \sqrt{\rho_{i,j}}}$$

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Use the same Roe averages for this interface to also define $\hat{B} \approx g'(q)$ near this interface.

Split $\mathcal{A}^*\Delta Q$ into eigenvectors of $\hat{B}$ to define $\mathcal{B}^-\mathcal{A}^*\Delta Q$ and $\mathcal{B}^+\mathcal{A}^*\Delta Q$

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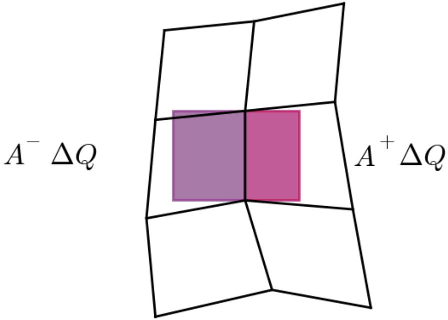
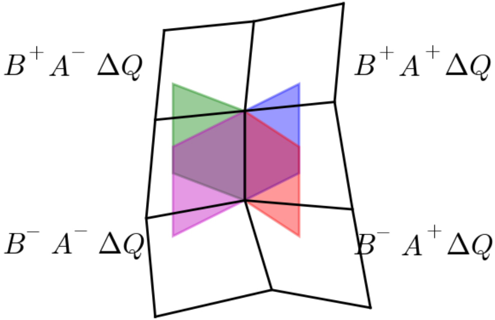
Many normal and transverse solvers available in **$CLAW/riemann/src**

Wave propagation algorithm on a quadrilateral grid



Example: **$CLAW/amrclaw/examples/advection_2d_annulus**

Wave propagation algorithm on a quadrilateral grid



Example: **$CLAW/amrclaw/examples/advection_2d_annulus**

Acoustics in Heterogeneous Media

- One space dimension
- Reflection and transmission at interfaces
- Non-conservative form, Riemann problems
- Two space dimensions
- Transverse Riemann solver
- Some examples

One-dimensional Elasticity

Compressional waves similar to acoustic waves in gas.

Notation:

$$\begin{aligned} X(x,t) &= \text{ location of particle indexed by } x \text{ in the} \\ &\quad\ \text{reference (undeformed) configuration} \\ X(x,0) &= x \ \text{ if initially undeformed} \\ \epsilon(x,t) &= X_x(x,t) - 1 = \text{ strain} \\ u(x,t) &= \text{ velocity of particle indexed by } x \\ \sigma(\epsilon) &= \text{ stress–strain relation} \\ \rho &= \text{ density} \end{aligned}$$

Linear elasticity

Hyperbolic conservation law:

$$\begin{array}{ll} \epsilon_t - u_x = 0 & \text{since } \epsilon_t = X_{xt} = X_{tx} = u_x \\ \rho u_t - \sigma_x = 0 & \text{conservation of momentum, } F = ma \end{array}$$

Linear stress-strain relation (Hooke's law):

$$\sigma(\epsilon) = K\epsilon$$

where K is the bulk modulus of compressibility.

Then

$$\begin{array}{l} \sigma_t - K u_x = 0 \\ u_t - (1/\rho)\sigma_x = 0 \end{array} \qquad A = \begin{bmatrix} 0 & -K \\ -1/\rho & 0 \end{bmatrix}$$

Eigenvalues: $\lambda = \pm\sqrt{K/\rho}$ as in acoustics.

(Equivalent to acoustics with $\sigma = -p$)

Elasticity in heterogeneous material

Suppose $\rho(x),\ \sigma(\epsilon, x)$ vary with x

Conservative form:

$$\begin{aligned} &\epsilon_t - u_x = 0 \\ &(\rho(x)u)_t - \sigma(\epsilon, x)_x = 0 \end{aligned}$$

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Linear stress-strain relation (Hooke's law):

$$\sigma(\epsilon, x) = K(x)\epsilon$$

Non-conservative variable-coefficient linear system:

$$\begin{aligned} &\sigma_t - K(x)u_x = 0 \\ &u_t - (1/\rho(x))\sigma_x = 0 \end{aligned} \qquad A = \begin{bmatrix} 0 & -K(x) \\ -1/\rho(x) & 0 \end{bmatrix}$$

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Variable coefficient acoustics: $p = -\sigma$

Wave propagation in heterogeneous medium

Multiply system

$$q_t + A(x)q_x = 0$$

by $R^{-1}(x)$ on left to obtain

$$R^{-1}(x)q_t + R^{-1}(x)A(x)R(x)\,R^{-1}(x)q_x = 0$$

or

$$(R^{-1}(x)q)_t + \Lambda(x)\ \left[(R^{-1}(x)q)_x - R_x^{-1}(x)q\right] = 0$$

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$$(R^{-1}(x)q)_t + \Lambda(x)\,\left[(R^{-1}(x)q)_x - R_x^{-1}(x)q\right] = 0$$

Let $w(x,t) = R^{-1}(x)q(x,t)$ (characteristic variable).

There is a coupling term on the right: Note typo in (9.51)

$$w_t + \Lambda(x)\,w_x = \Lambda(x)R_x^{-1}(x)R(x)w$$

If the eigenvectors vary with x (i.e. if $R_x \neq 0$)
then waves in other families are generated (e.g. reflections)

Wave propagation in heterogeneous medium

Linear system $q_t + A(x)q_x = 0$. For acoustics:

$$A = \begin{bmatrix} 0 & K(x) \\ 1/\rho(x) & 0 \end{bmatrix} \qquad q = \begin{bmatrix} p \\ u \end{bmatrix}.$$

eigenvalues: $\lambda^1 = -c(x), \qquad \lambda^2 = +c(x),$

where $c(x) = \sqrt{K(x)/\rho(x)} =$ local speed of sound.

eigenvectors: $r^1(x) = \begin{bmatrix} -Z(x) \\ 1 \end{bmatrix}, \qquad r^2(x) = \begin{bmatrix} Z(x) \\ 1 \end{bmatrix}$

where $Z(x) = \rho c = \sqrt{\rho K} =$ impedance.

Transmission and reflection coefficients

Consider an interface between two materials with constant properties in each.

$$\rho_\ell,\ K_\ell \implies c_\ell = \sqrt{\rho_\ell/K_\ell},\ Z_\ell = \sqrt{\rho_\ell K_\ell}$$

$$\rho_r\ K_r \implies c_r = \sqrt{\rho_r/K_r},\ Z_r = \sqrt{\rho_r K_r}$$

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If impedance $Z_\ell = Z_r$ then $r_\ell^p = r_r^p$ and waves are transmitted through interface with no generation of other waves

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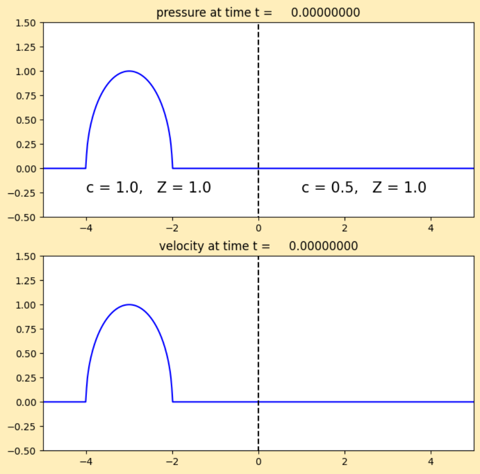
More generally, wave is partly transmitted and partly reflected,

$$C_T = \frac{2Z_r}{Z_\ell + Z_r}, \qquad C_R = \frac{Z_r - Z_\ell}{Z_\ell + Z_r}.$$

Right-going simple wave with $Z_\ell = Z_r$



Note p and u are not conserved, but they are always continuous.

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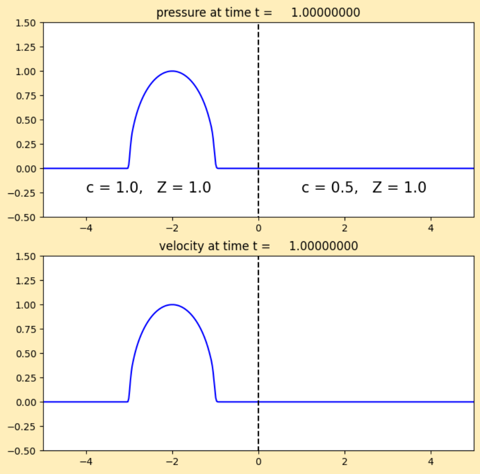

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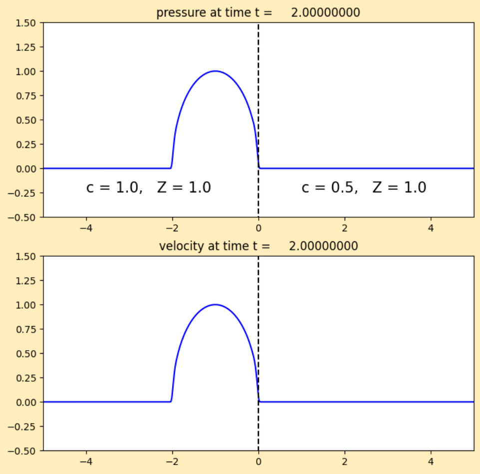

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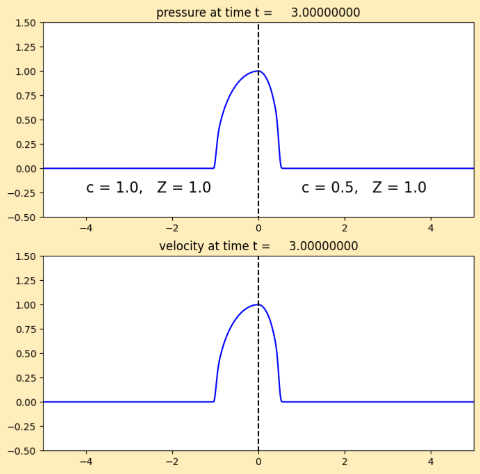

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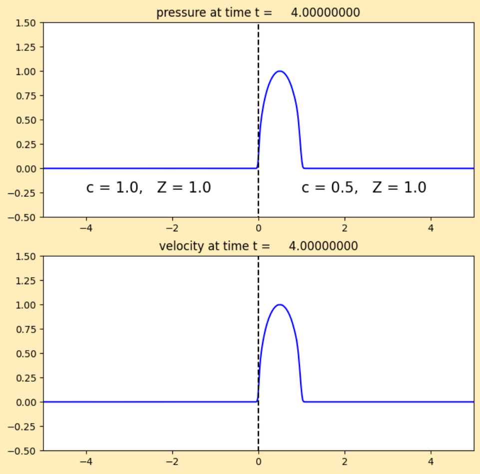

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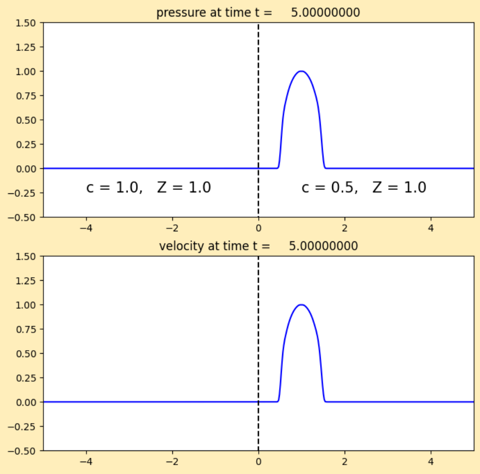

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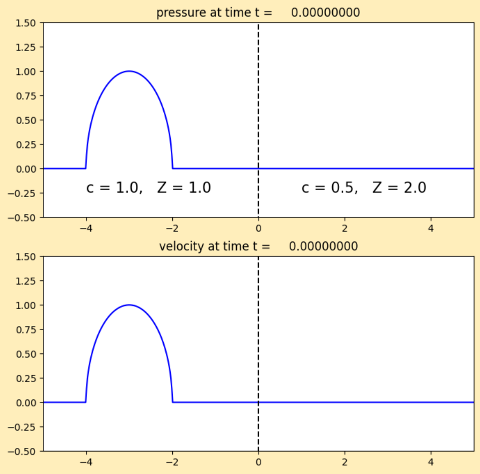
Transmitted/reflected wave with $Z_\ell \neq Z_r$



$$C_T = \frac{2Z_r}{Z_\ell + Z_r} = \frac{4}{3}$$

$$C_R = \frac{Z_r - Z_\ell}{Z_\ell + Z_r} = \frac{1}{3}$$

Note that p and u remain continuous at the interface.

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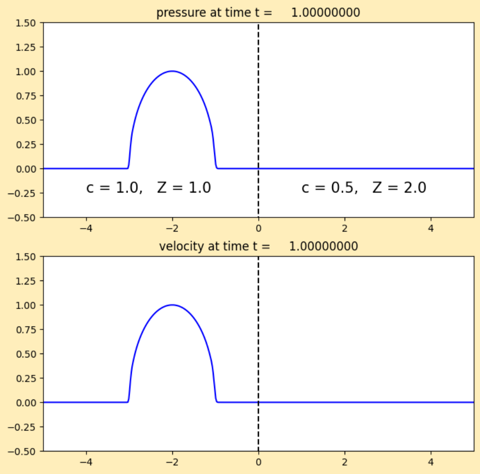

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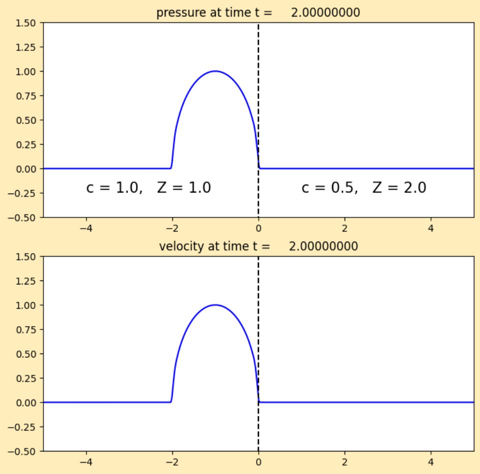

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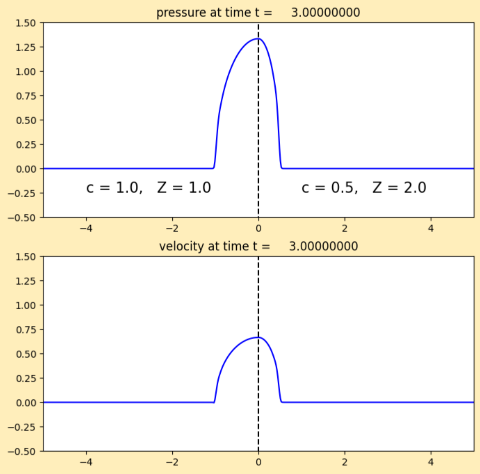

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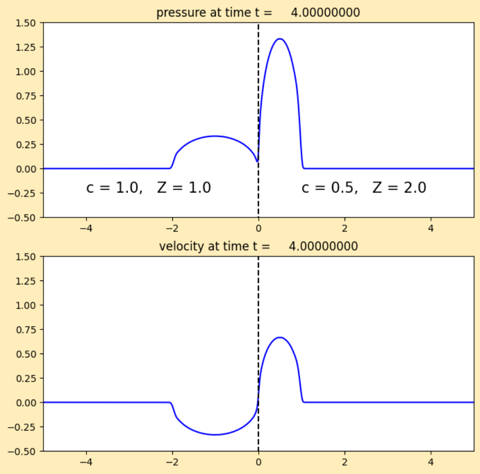

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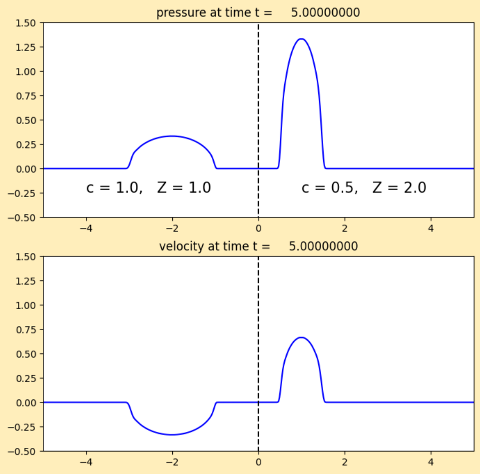

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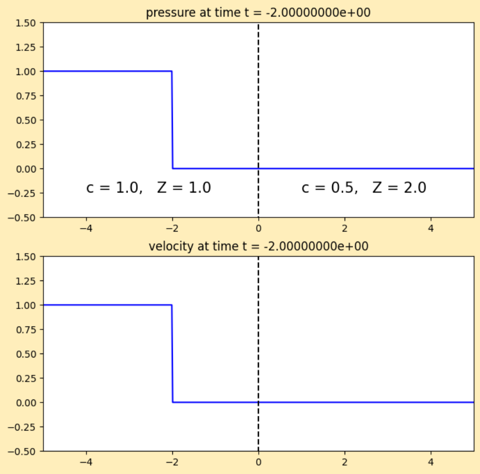

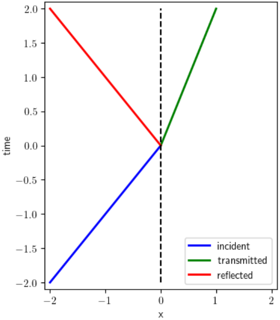

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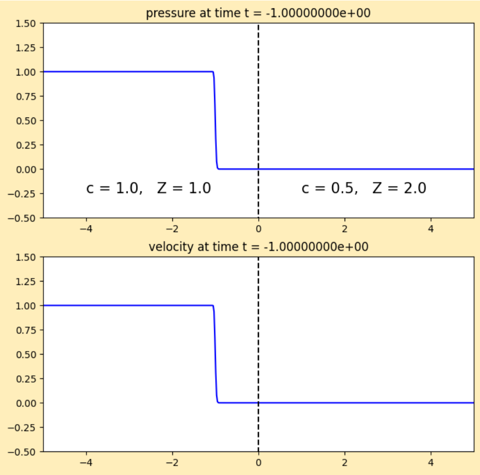

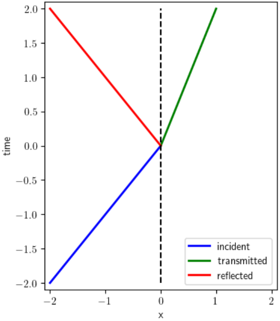

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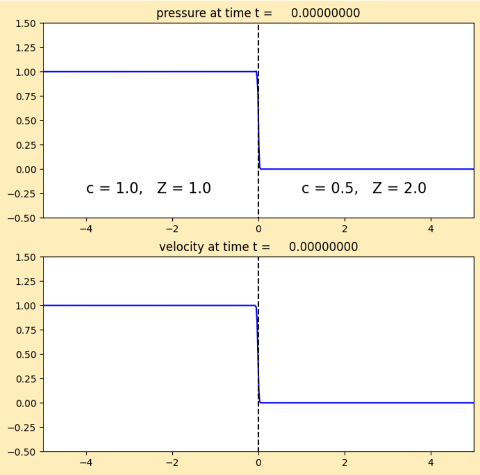

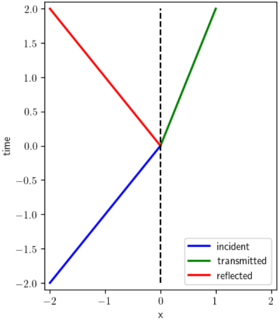

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Looks like Riemann problem data at $t = 0$

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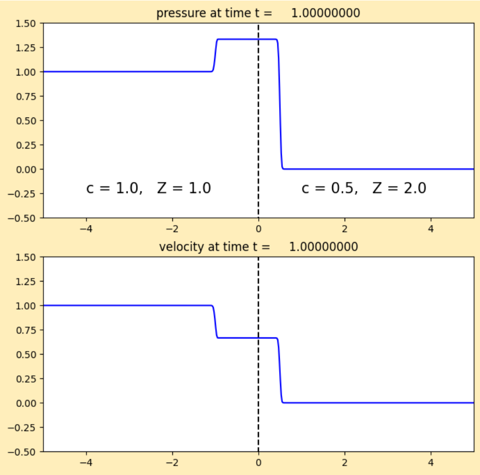

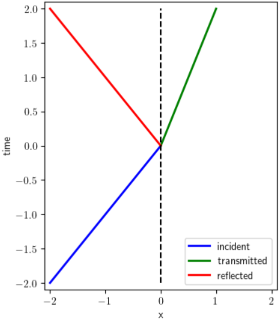

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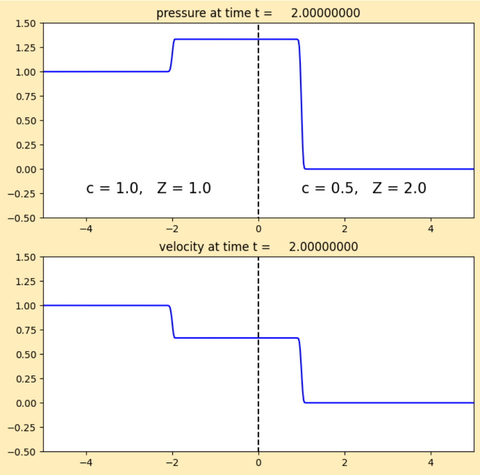

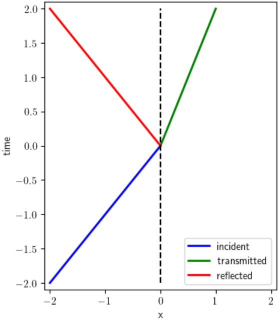

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Riemann problem for heterogeneous medium

Jump discontinuity in $q(x, 0)$ and in $K(x)$ and $\rho(x)$.

Decompose jump in q as linear combination of eigenvectors:

- left-going waves: eigenvectors for material on left,
- right-going waves: eigenvectors for material on right.

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$$R(x) = \begin{bmatrix} -Z(x) & Z(x) \\ 1 & 1 \end{bmatrix}, \qquad R^{-1}(x) = \frac{1}{2Z(x)} \begin{bmatrix} -1 & Z(x) \\ 1 & Z(x) \end{bmatrix}.$$

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Riemann solution: decompose

$$q_r - q_l = \alpha^1 \begin{bmatrix} -Z_l \\ 1 \end{bmatrix} + \alpha^2 \begin{bmatrix} Z_r \\ 1 \end{bmatrix} = \mathcal{W}^1 + \mathcal{W}^2$$

The waves propagate with speeds $s^1 = -c_l$ and $s^2 = c_r$.

Wave propagation in heterogeneous medium

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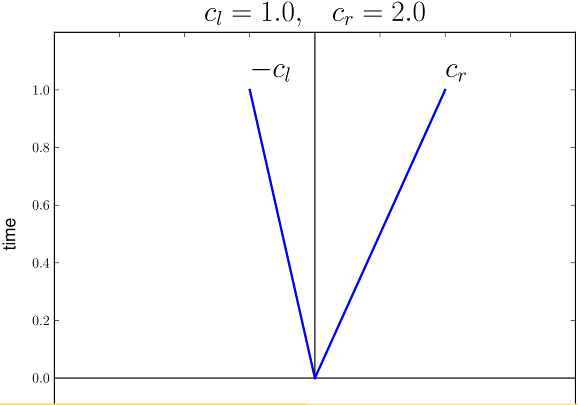

$$q_r - q_\ell = \alpha^1 \begin{bmatrix} -Z_\ell \\ 1 \end{bmatrix} + \alpha^2 \begin{bmatrix} Z_r \\ 1 \end{bmatrix}.$$

gives the linear system

$$R_{\ell r}\, \alpha = q_r - q_\ell,$$

where

$$R_{\ell r} = \begin{bmatrix} -Z_\ell & Z_r \\ 1 & 1 \end{bmatrix} \qquad \Longrightarrow \; R_{\ell r}^{-1} = \frac{1}{Z_\ell + Z_r} \begin{bmatrix} -1 & Z_r \\ 1 & Z_\ell \end{bmatrix}$$

So

$$\begin{bmatrix} \alpha^1 \\ \alpha^2 \end{bmatrix} = \frac{1}{Z_\ell + Z_r} \begin{bmatrix} -1 & Z_r \\ 1 & Z_\ell \end{bmatrix} \begin{bmatrix} p_r - p_\ell \\ u_r - u_\ell \end{bmatrix}.$$

2-wave hitting interface as a Riemann problem

Incident wave:

$$q_r - q_\ell = \beta r_\ell^2 = \beta \begin{bmatrix} Z_\ell \\ 1 \end{bmatrix},$$

then Riemann solution gives

$$\begin{aligned} \alpha &= R_{lr}^{-1}(q_r - q_\ell) \\ &= \frac{\beta}{Z_\ell + Z_r} \begin{bmatrix} -1 & Z_r \\ 1 & Z_\ell \end{bmatrix} \begin{bmatrix} Z_\ell \\ 1 \end{bmatrix} \\ &= \frac{\beta}{Z_\ell + Z_r} \begin{bmatrix} Z_r - Z_\ell \\ 2Z_\ell \end{bmatrix}. \end{aligned}$$

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$$\implies \alpha^1 = \left(\frac{Z_r - Z_\ell}{Z_\ell + Z_r}\right)\beta \quad \text{and} \quad \alpha^2 = \left(\frac{2Z_\ell}{Z_\ell + Z_r}\right)\beta$$

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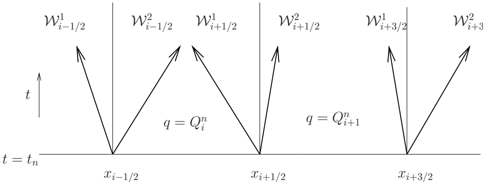
$$\Longrightarrow \alpha^1 = \left(\frac{Z_r - Z_\ell}{Z_\ell + Z_r}\right)\beta \quad \text{and} \quad \alpha^2 = \left(\frac{2Z_\ell}{Z_\ell + Z_r}\right)\beta$$

Pressure jump in reflected wave: $c_R \beta Z_\ell$

Pressure jump in transmitted wave: $c_T \beta Z_\ell$

Godunov's method — variable coefficient acoustics



$$
\begin{aligned}
Q_i - Q_{i-1} &= \begin{bmatrix} p_i - p_{i-1} \\ u_i - u_{i-1} \end{bmatrix} \\
&= \alpha^1_{i-1/2} \begin{bmatrix} -\rho_{i-1}c_{i-1} \\ 1 \end{bmatrix} + \alpha^2_{i-1/2} \begin{bmatrix} \rho_i c_i \\ 1 \end{bmatrix} \\
&= \alpha^1_{i-1/2} r^1_{i-1} + \alpha^2_{i-1/2} r^2_i \\
&= \mathcal{W}^1_{i-1/2} + \mathcal{W}^2_{i-1/2}
\end{aligned}
$$

2D Acoustics in Heterogeneous Media

$$q_t + A(x,y)q_x + B(x,y)q_y = 0,$$

$$q = \begin{bmatrix} p \\ u \\ v \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & K(x,y) & 0 \\ 1/\rho(x,y) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & K(x,y) \\ 0 & 0 & 0 \\ 1/\rho(x,y) & 0 & 0 \end{bmatrix}.$$

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Riemann problem in x:

$$\mathcal{W}^1 = \alpha^1 \begin{bmatrix} -Z_{i-1,j} \\ 1 \\ 0 \end{bmatrix}, \qquad \mathcal{W}^2 = \alpha^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathcal{W}^3 = \alpha^3 \begin{bmatrix} Z_{ij} \\ 1 \\ 0 \end{bmatrix},$$

$$\begin{aligned} \alpha^1 &= (-\Delta Q^1 + Z_{ij}\Delta Q^2)/(Z_{i-1,j} + Z_{ij}), \\ \alpha^2 &= \Delta Q^3, \\ \alpha^3 &= (\Delta Q^1 + Z_{i-1,j}\Delta Q^2)/(Z_{i-1,j} + Z_{ij}). \end{aligned}$$

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$$\begin{aligned} \alpha^1 &= (-\Delta Q^1 + Z_{ij}\Delta Q^2)/(Z_{i-1,j} + Z_{ij}), \\ \alpha^2 &= \Delta Q^3, \\ \alpha^3 &= (\Delta Q^1 + Z_{i-1,j}\Delta Q^2)/(Z_{i-1,j} + Z_{ij}). \end{aligned}$$

Wave speeds: $\quad s^1 = -c_{i-1,j}, \quad s^2 = 0, \quad s^3 = c_{ij}$

Only need to propagate and apply limiters to $\mathcal{W}^1,\ \mathcal{W}^3$.

Wave propagation algorithms in 2D

Clawpack requires:

Normal Riemann solver `rpn2.f`
Solves 1d Riemann problem $q_t + Aq_x = 0$
Decomposes $\Delta Q = Q_{ij} - Q_{i-1,j}$ into $\mathcal{A}^+\Delta Q$ and $\mathcal{A}^-\Delta Q$.
For $q_t + Aq_x + Bq_y = 0$, split using eigenvalues, vectors:

$$A = R\Lambda R^{-1} \implies A^- = R\Lambda^- R^{-1}, A^+ = R\Lambda^+ R^{-1}$$

Input parameter `ixy` determines if it's in x or y direction.
In latter case splitting is done using B instead of A.
This is all that's required for dimensional splitting.

Wave propagation algorithms in 2D

Clawpack requires:

Normal Riemann solver `rpn2.f`
Solves 1d Riemann problem $q_t + Aq_x = 0$
Decomposes $\Delta Q = Q_{ij} - Q_{i-1,j}$ into $\mathcal{A}^+\Delta Q$ and $\mathcal{A}^-\Delta Q$.
For $q_t + Aq_x + Bq_y = 0$, split using eigenvalues, vectors:

$$A = R\Lambda R^{-1} \implies A^- = R\Lambda^- R^{-1}, A^+ = R\Lambda^+ R^{-1}$$

Input parameter `ixy` determines if it's in x or y direction.
In latter case splitting is done using B instead of A.
This is all that's required for dimensional splitting.

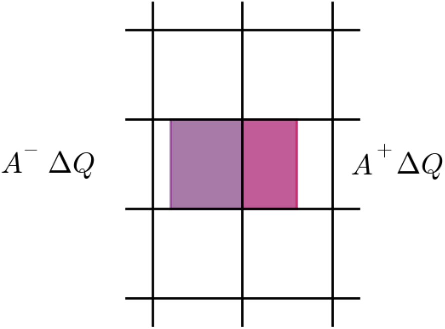
Transverse Riemann solver `rpt2.f`
Decomposes $\mathcal{A}^+\Delta Q$ into $\mathcal{B}^-\mathcal{A}^+\Delta Q$ and $\mathcal{B}^+\mathcal{A}^+\Delta Q$ by splitting this vector into eigenvectors of B.

(Or splits vector into eigenvectors of A if `ixy=2`.)

Wave propagation algorithm for $q_t + Aq_x + Bq_y = 0$

Decompose $A = A^+ + A^-$ and $B = B^+ + B^-$.

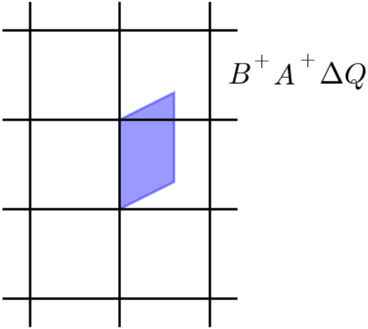
For $\Delta Q = Q_{ij} - Q_{i-1,j}$:



Wave propagation algorithm for $q_t + Aq_x + Bq_y = 0$

Decompose $A = A^+ + A^-$ and $B = B^+ + B^-$.

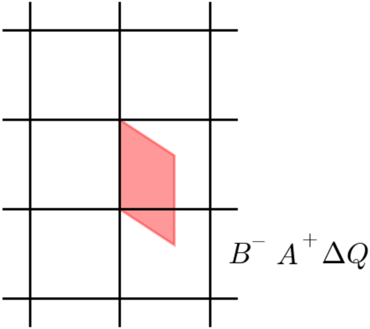
For $\Delta Q = Q_{ij} - Q_{i-1,j}$:



Wave propagation algorithm for $q_t + Aq_x + Bq_y = 0$

Decompose $A = A^+ + A^-$ and $B = B^+ + B^-$.

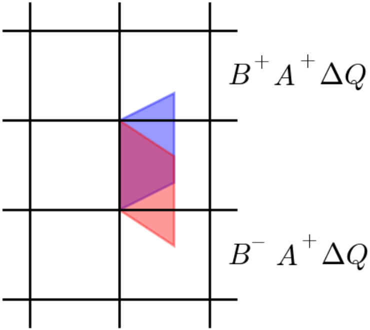
For $\Delta Q = Q_{ij} - Q_{i-1,j}$:



Wave propagation algorithm for $q_t + Aq_x + Bq_y = 0$

Decompose $A = A^+ + A^-$ and $B = B^+ + B^-$.

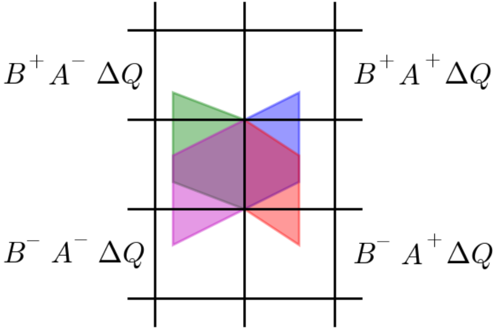
For $\Delta Q = Q_{ij} - Q_{i-1,j}$:



Wave propagation algorithm for $q_t + Aq_x + Bq_y = 0$

Decompose $A = A^+ + A^-$ and $B = B^+ + B^-$.

For $\Delta Q = Q_{ij} - Q_{i-1,j}$:



Transverse solver for 2D Acoustics

Solving Riemann problem in x gives waves and fluctuations
$\mathcal{A}^-\Delta Q_{i-1/2,j},\ \mathcal{A}^+\Delta Q_{i-1/2,j}$.

For $\mathcal{B}^-\mathcal{A}^+\Delta Q_{i-1/2,j}$ we want downward-going part of $\mathcal{A}^+\Delta Q_{i-1/2,j}$,
(partly transmitted an partly reflected at y-interface)

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$$\mathcal{A}^+\Delta Q_{i-1/2,j} = \beta^1 \begin{bmatrix} -Z_{i,j-1} \\ 0 \\ 1 \end{bmatrix} + \beta^2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \beta^3 \begin{bmatrix} Z_{ij} \\ 0 \\ 1 \end{bmatrix},$$

with speeds $-c_{i,j-1},\ 0,\ c_{ij}$ respectively.

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with speeds $-c_{i,j-1},\ 0,\ c_{ij}$ respectively.

Only use downward-going part:

$$\beta^1 = \left(-(\mathcal{A}^+\Delta Q_{i-1/2,j})^1 + (\mathcal{A}^+\Delta Q_{i-1/2,j})^3 Z_{ij}\right) / (Z_{i,j-1} + Z_{ij}),$$

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$$\mathcal{B}^-\mathcal{A}^+\Delta Q_{i-1/2,j} = -c_{i,j-1}\beta^1 \begin{bmatrix} -Z_{i,j-1} \\ 0 \\ 1 \end{bmatrix}$$

Transverse solver for 2D Acoustics

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$$\mathcal{A}^+\Delta Q_{i-1/2,j} = \beta^1 \begin{bmatrix} -Z_{ij} \\ 0 \\ 1 \end{bmatrix} + \beta^2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \beta^3 \begin{bmatrix} Z_{i,j+1} \\ 0 \\ 1 \end{bmatrix},$$

with speeds $-c_{ij},\ 0,\ c_{i,j+1}$ respectively.

Transverse solver for 2D Acoustics

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with speeds $-c_{ij},\ 0,\ c_{i,j+1}$ respectively.

Only use upward-going part:

$$\beta^3 = \left((\mathcal{A}^+\Delta Q_{i-1/2,j})^1 + (\mathcal{A}^+\Delta Q_{i-1/2,j})^3 Z_{i,j+1}\right) / (Z_{ij} + Z_{i,j+1})$$

Transverse solver for 2D Acoustics

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with speeds $-c_{ij},\ 0,\ c_{i,j+1}$ respectively.

Only use upward-going part:

$$\beta^3 = \left((\mathcal{A}^+\Delta Q_{i-1/2,j})^1 + (\mathcal{A}^+\Delta Q_{i-1/2,j})^3 Z_{i,j+1}\right) / (Z_{ij} + Z_{i,j+1})$$

$$\mathcal{B}^+\mathcal{A}^+\Delta Q_{i-1/2,j} = c_{i,j+1}\beta^3 \begin{bmatrix} Z_{i,j+1} \\ 0 \\ 1 \end{bmatrix}$$

Cell averaging material parameters

To solve a variable coefficient problem on a grid,
need to average material parameters onto grid cell.

For acoustics with $\rho(x,y),\ K(x,y)$, on Cartesian grid:

Can use mean value of density:

$$\rho_{ij} = \frac{1}{\Delta x \Delta y} \iint \rho(x,y)\,dx,dy$$

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But need to use harmonic average of bulk modulus:

$$K_{ij} = \left(\frac{1}{\Delta x \Delta y} \iint \frac{1}{K(x,y)}\, dx, dy \right)^{-1}$$

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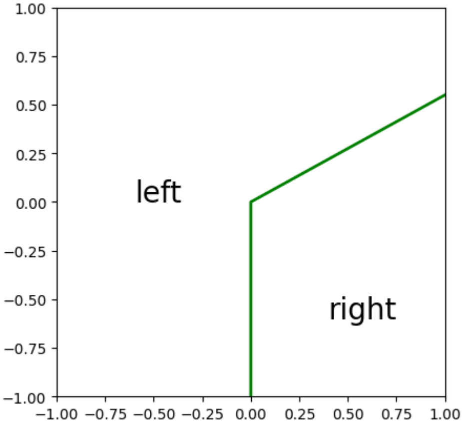
But need to use harmonic average of bulk modulus:

$$K_{ij} = \left(\frac{1}{\Delta x \Delta y} \iint \frac{1}{K(x,y)}\, dx, dy \right)^{-1}$$

Then $\quad c_{ij} = \sqrt{K_{ij}/\rho_{ij}}, \quad Z_{ij} = \sqrt{K_{ij}\rho_{ij}}$

Acoustic wave hitting an interface in 2D

Example from Figure 21.1:

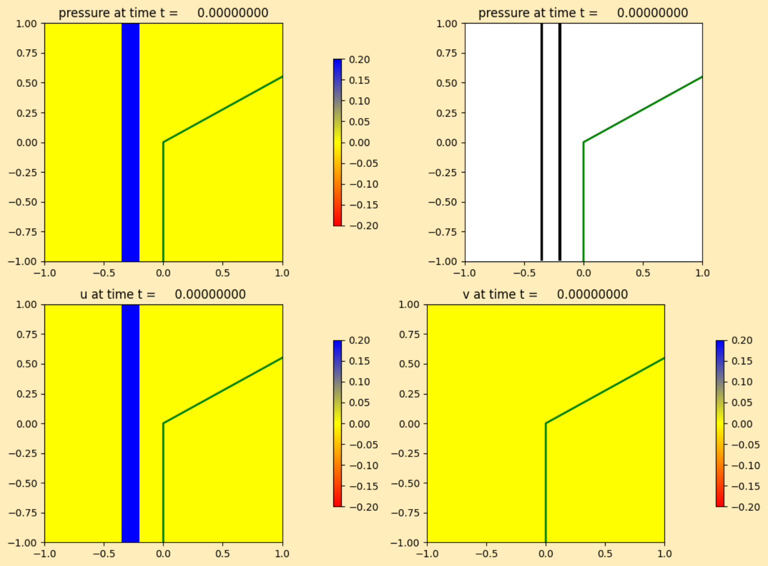


$$\begin{aligned} \rho_\ell &= 1 & \rho_r &= 1 \\ K_\ell &= 1 & K_r &= 0.25 \\ c_\ell &= 1 & c_r &= 0.5 \\ Z_\ell &= 1 & Z_r &= 0.5 \end{aligned}$$

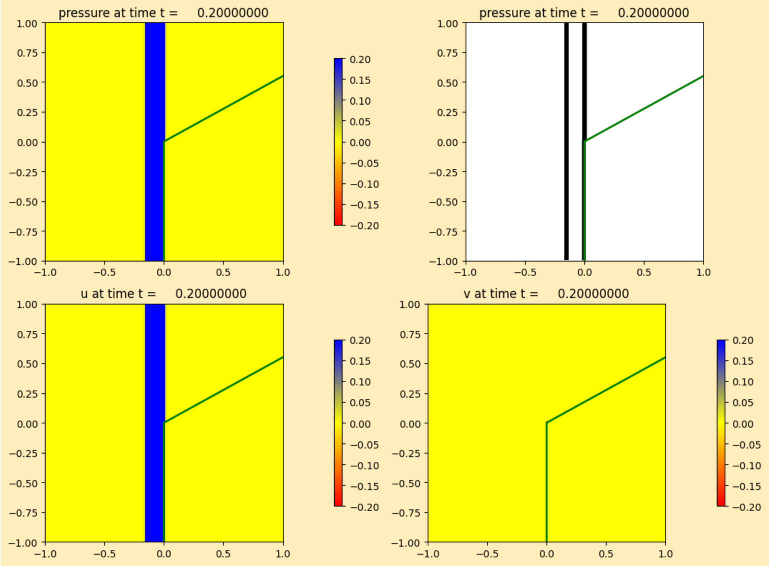
$$\begin{aligned} C_T &= \frac{2Z_r}{Z_\ell + Z_r} \\ &= 2/3 \end{aligned}$$

$$\begin{aligned} C_R &= \frac{Z_r - Z_\ell}{Z_\ell + Z_r} \\ &= -1/3 \end{aligned}$$

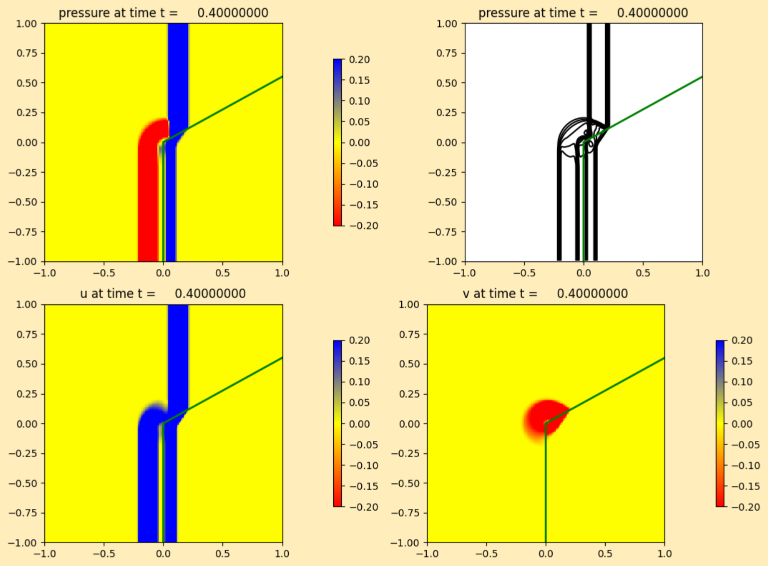
Acoustic wave hitting an interface in 2D



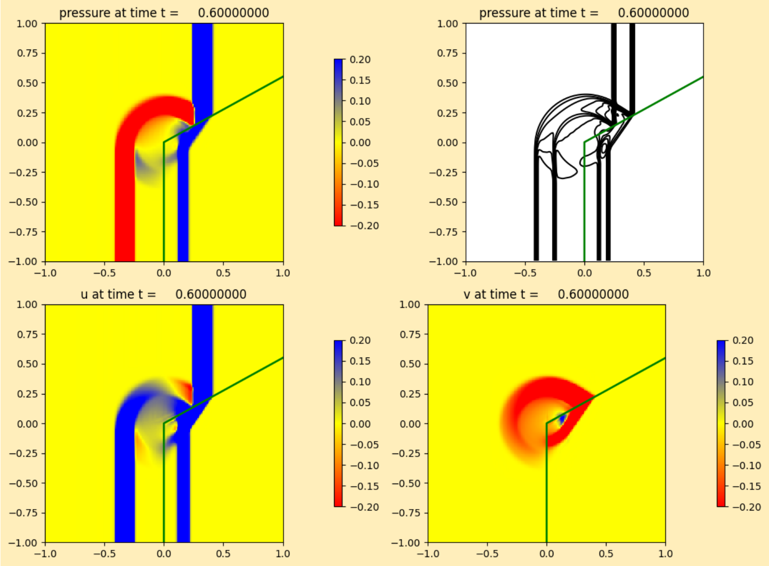
Acoustic wave hitting an interface in 2D



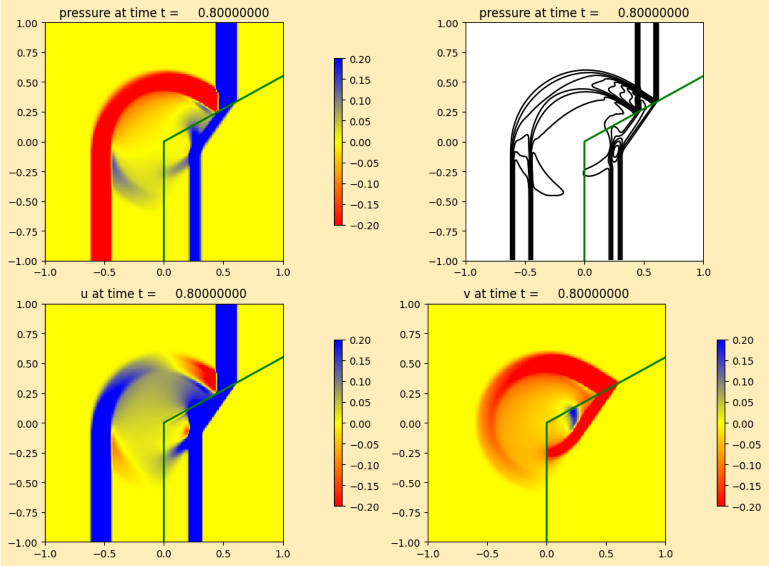
Acoustic wave hitting an interface in 2D



Acoustic wave hitting an interface in 2D

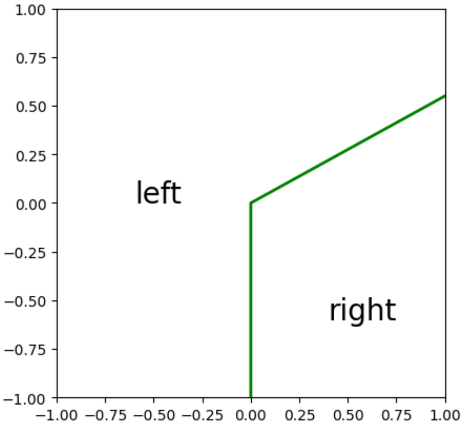


Acoustic wave hitting an interface in 2D



Acoustic wave hitting an interface in 2D

With nearly-incompressible material on right ($\approx$ solid wall)



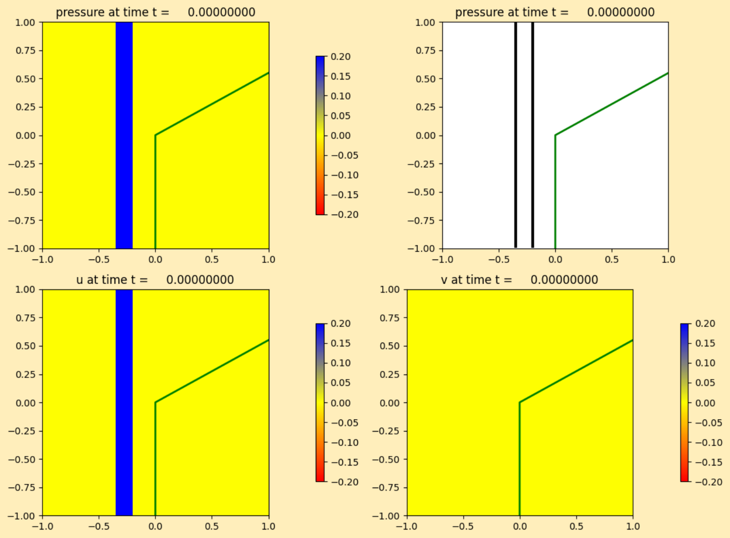
$$\begin{aligned} \rho_\ell &= 1 & \rho_r &= 10^4 \\ K_\ell &= 1 & K_r &= 10^{-8} \\ c_\ell &= 1 & c_r &= 10^{-6} \\ Z_\ell &= 1 & Z_r &= 0.01 \end{aligned}$$

$$\begin{aligned} C_T &= \tfrac{2Z_r}{Z_\ell+Z_r} \\ &\approx 0.02 \end{aligned}$$

$$\begin{aligned} C_R &= \tfrac{Z_r-Z_\ell}{Z_\ell+Z_r} \\ &\approx -0.98 \end{aligned}$$

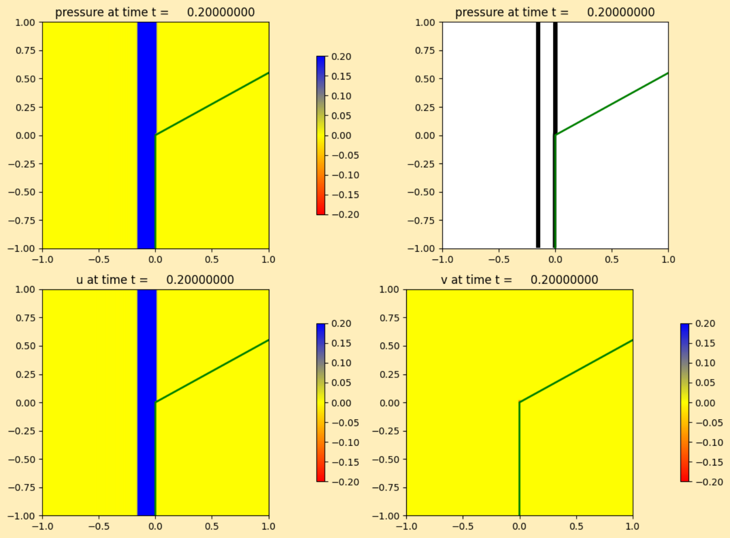
Acoustic wave hitting an interface in 2D

With nearly-incompressible material on right ($\approx$ solid wall)



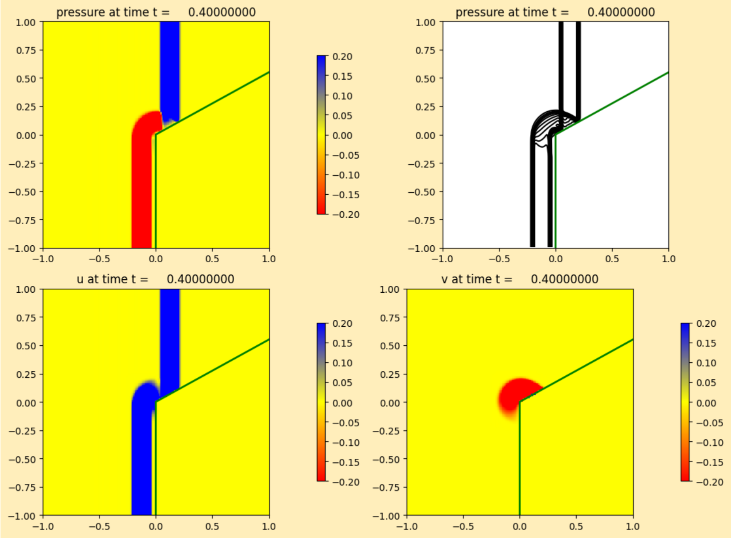
Acoustic wave hitting an interface in 2D

With nearly-incompressible material on right ($\approx$ solid wall)



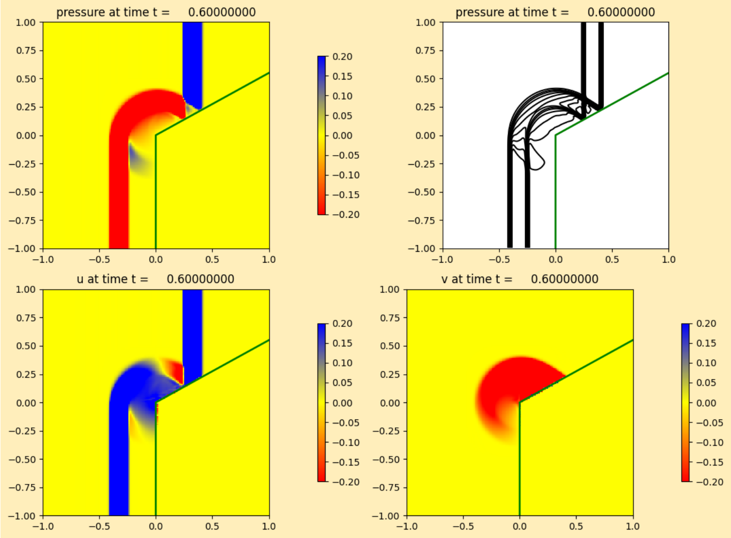
Acoustic wave hitting an interface in 2D

With nearly-incompressible material on right (≈ solid wall)



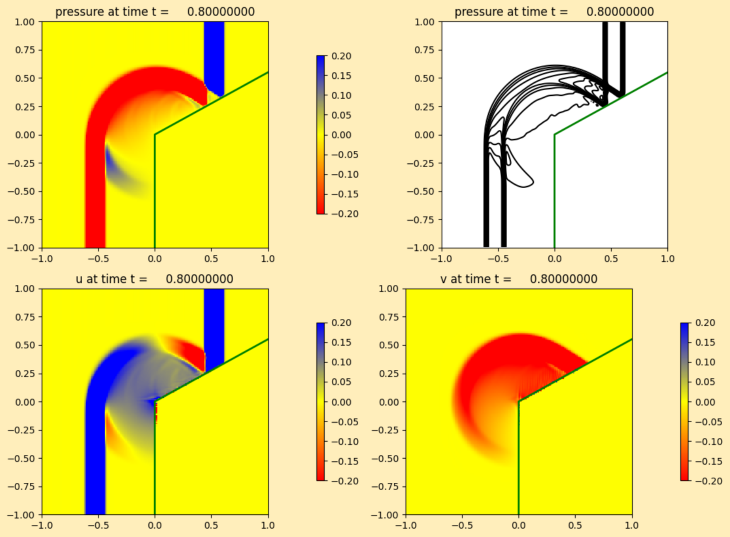
Acoustic wave hitting an interface in 2D

With nearly-incompressible material on right ($\approx$ solid wall)

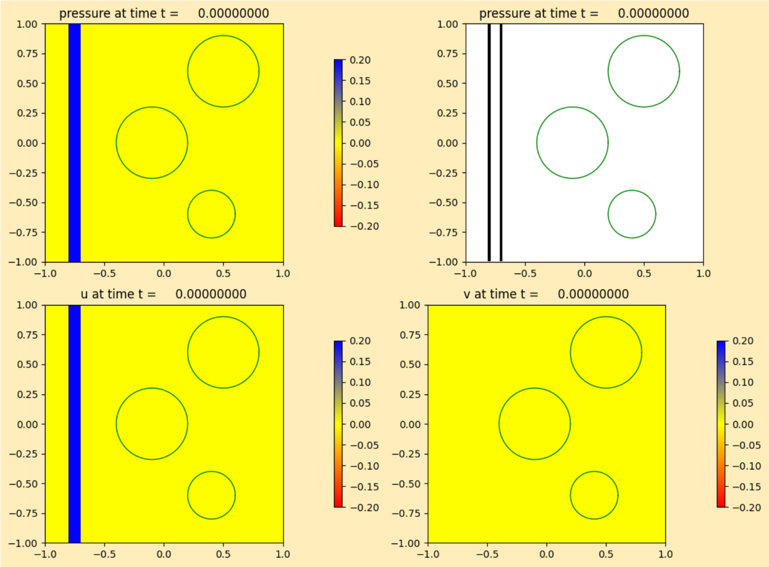


Acoustic wave hitting an interface in 2D

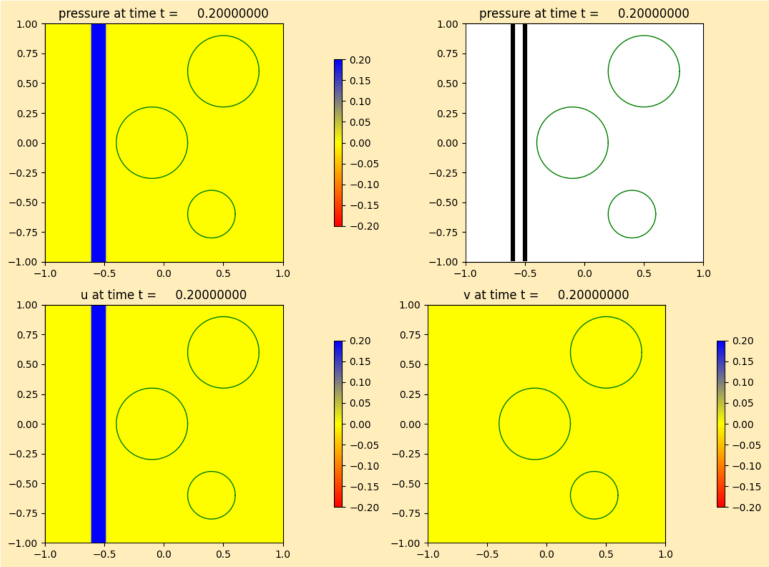
With nearly-incompressible material on right ($\approx$ solid wall)



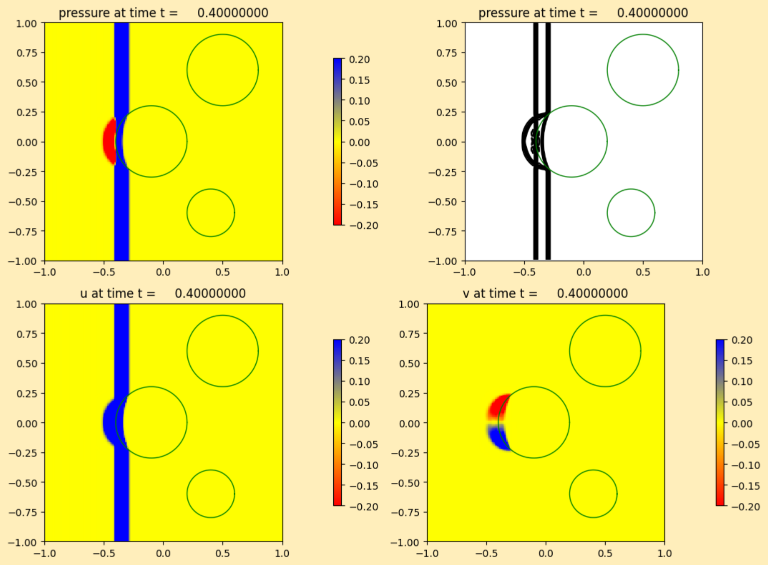
Acoustic wave hitting circular inclusions



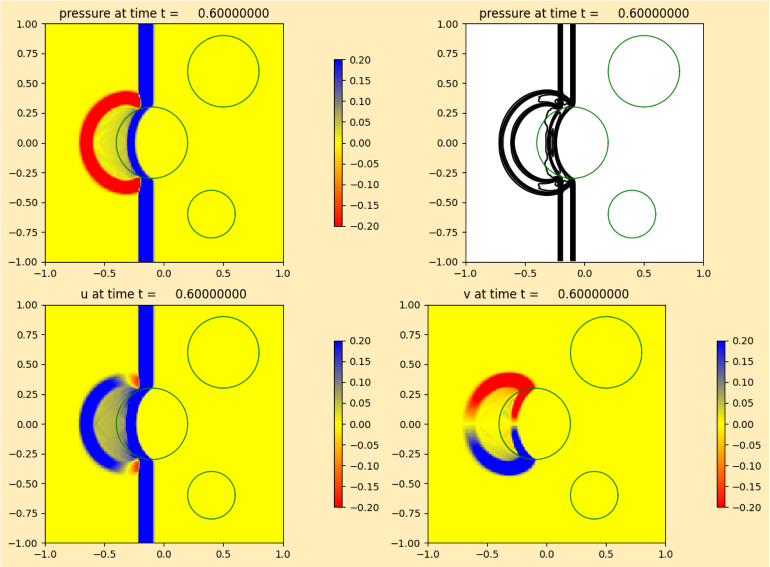
Acoustic wave hitting circular inclusions



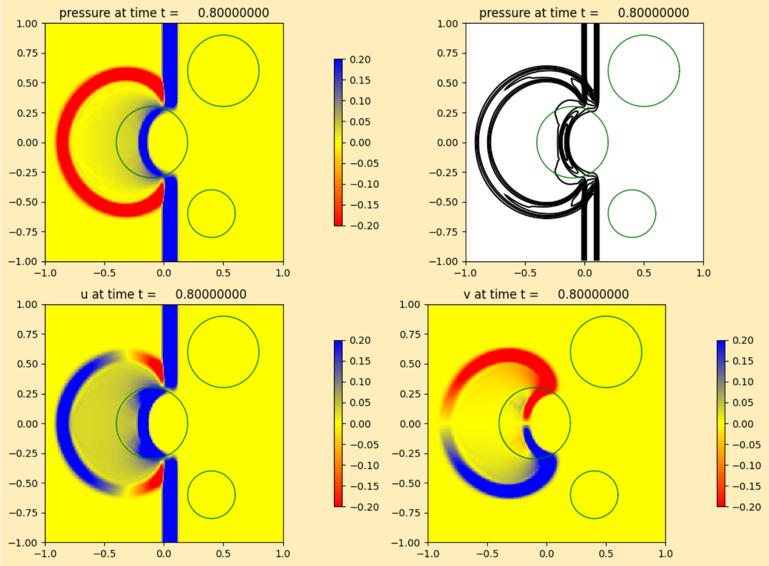
Acoustic wave hitting circular inclusions



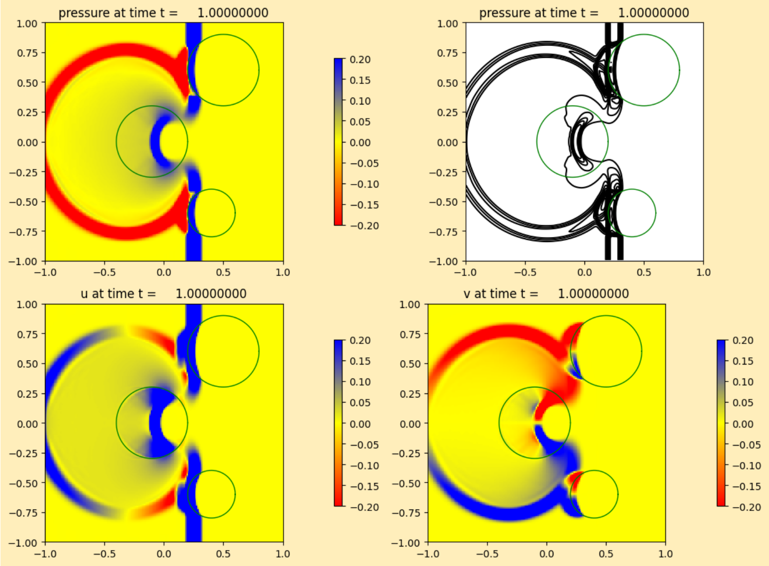
Acoustic wave hitting circular inclusions



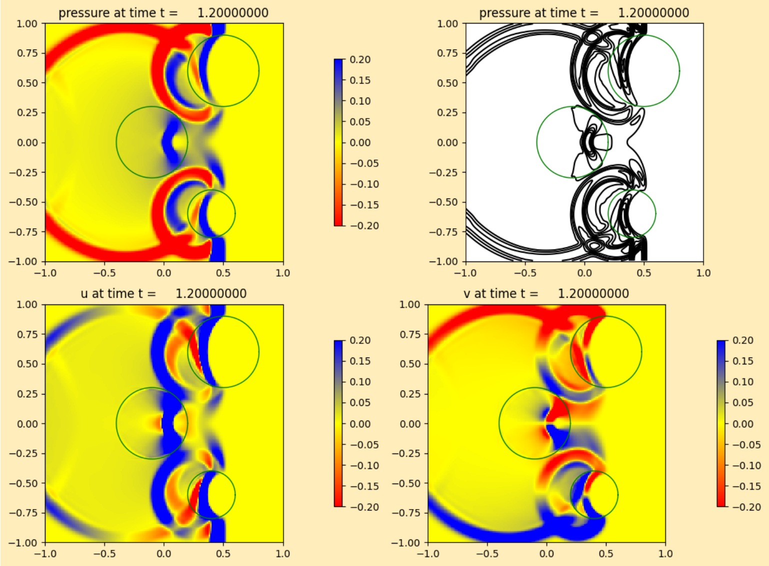
Acoustic wave hitting circular inclusions



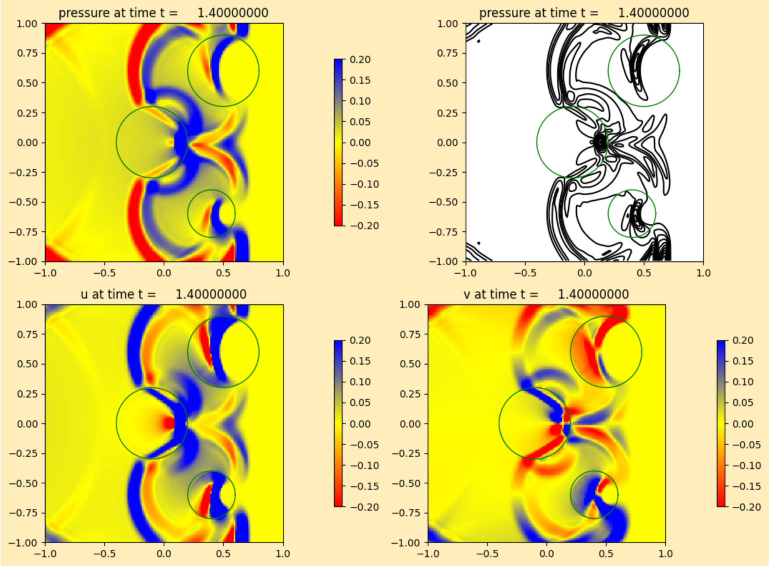
Acoustic wave hitting circular inclusions



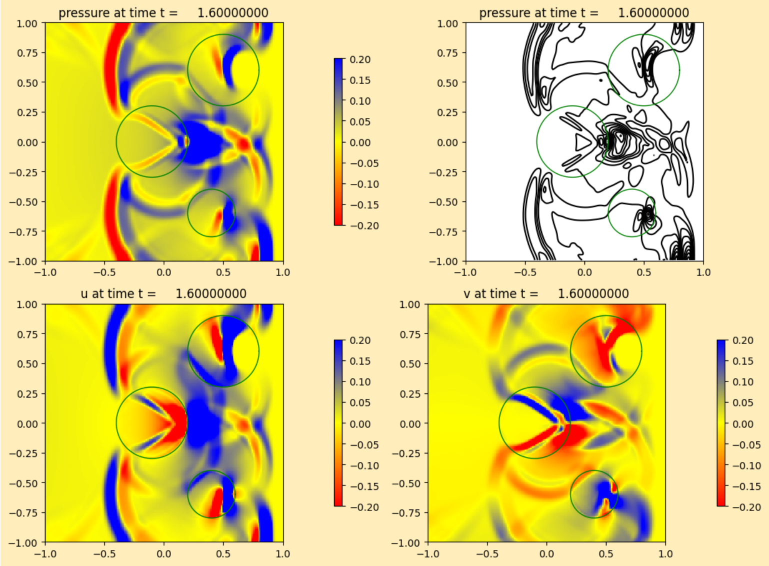
Acoustic wave hitting circular inclusions



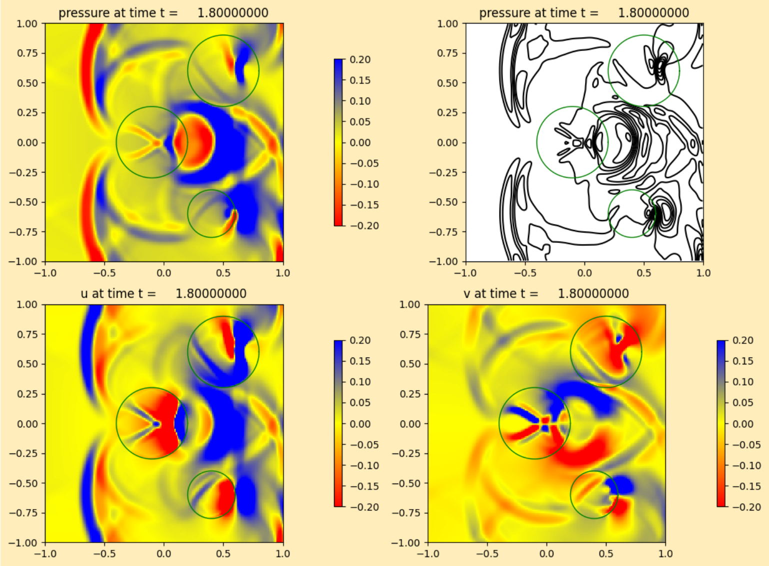
Acoustic wave hitting circular inclusions



Acoustic wave hitting circular inclusions



Acoustic wave hitting circular inclusions



Acoustic wave hitting circular inclusions