Finite Volume Methods for Hyperbolic Problems

Variable Coefficient Advection

- Quasi-1D pipe
- Units in one space dimension
- Conservative form: $q_t + (u(x)q)_x = 0$
- Advective form: $q_t + u(x)q_x = 0$ (color equation)

Variable-coefficient advection

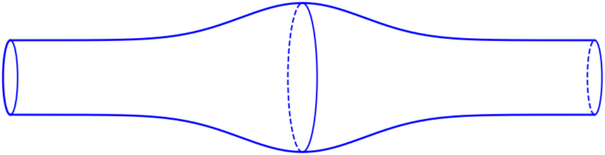
Incompressible flow in 1D pipe with constant cross section

$\Longrightarrow$ $u \equiv$constant in space.

Variable-coefficient advection

Incompressible flow in 1D pipe with constant cross section
$\implies u \equiv$constant in space.

If cross-sectional area $\kappa(x)$ varies, then so does $u(x)$.



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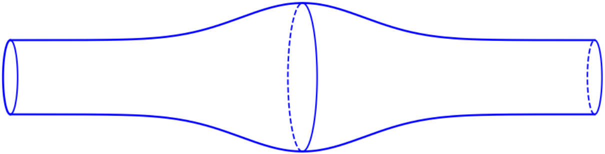

Incompressible $\implies$ flux of fluid must be constant, so

$$\kappa(x)u(x) \equiv U = \text{constant.}$$

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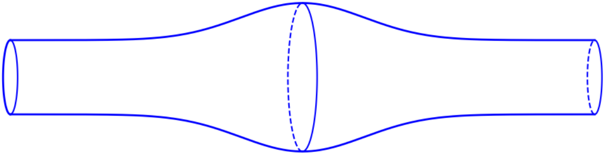

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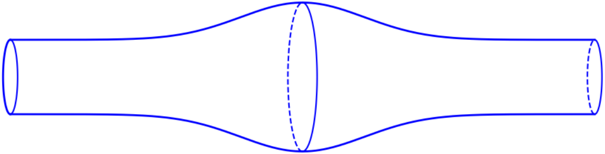
PDE for concentration of a passive tracer advected with flow?

Variable-coefficient advection



Incompressible $\implies$ flux of fluid must be constant, so

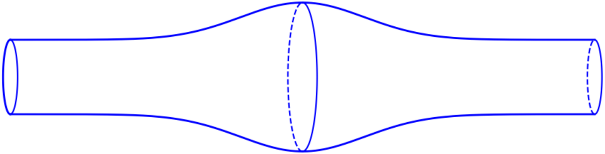
$$\kappa(x)u(x) \equiv U \implies u(x) = U/\kappa(x).$$

Concentration of passive tracer: $q(x,t)$

If units of q are mass / unit length, then q is conserved quantity with flux uq, and we obtain the conservation law

$$q_t(x,t) + (u(x)q(x,t))_x = 0.$$

Variable-coefficient advection



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If units of q are mass / unit length, then q is conserved quantity with flux uq, and we obtain the conservation law

$$q_t(x,t) + (u(x)q(x,t))_x = 0.$$

However, if q is in units of mass / unit volume, then:

$$q_t(x,t) + u(x)q_x(x,t) = 0. \quad \text{(color equation)}$$

Variable-coefficient advection

Derivation of color equation:

Incompressible $\Longrightarrow$ flux of fluid must be constant, so

$$\kappa(x)u(x) \equiv U \implies u(x) = U/\kappa(x).$$

If $q(x,t)$ in units of mass/volume, the mass/length is $\kappa(x)q(x,t)$.
This is now the conserved quantity.

Variable-coefficient advection

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Its flux is $u(x)\kappa(x)q(x,t) = Uq(x,t)$.

Variable-coefficient advection

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Conservation law is:

$$(\kappa(x)q(x,t))_t + (Uq(x,t))_x = 0,$$

$$\kappa(x)q_t(x,t) + Uq_x(x,t)) = 0,$$

$$q_t(x,t) + u(x)q_x(x,t)) = 0.$$

Variable-coefficient advection

Color equation:

$$q_t(x,t) + u(x)q_x(x,t) = 0.$$

Can be rewriten as a balance law
(conservation law plus source term):

$$q_t(x,t) + (u(x)q(x,t))_x = u'(x)q(x,t)$$

Will revisit different forms when studying numerical methods.