Finite Volume Methods for Hyperbolic Problems

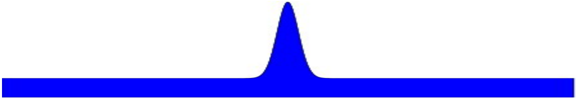
Nonlinear Scalar PDEs – Traffic flow

- Traffic flow — car following models
- Traffic flow — conservation law
- Shock formation
- Rankine-Hugoniot jump conditions
- Riemann problems

Shock formation

For nonlinear problems wave speed generally depends on q.

Waves can steepen up and form shocks
$\Longrightarrow$ even smooth data can lead to discontinuous solutions.



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Note:

- System of two equations gives rise to 2 waves.
- Each wave behaves like solution of nonlinear scalar equation.

Not quite... no linear superposition. Nonlinear interaction!

Shocks in traffic flow

Car following model

$X_j(t) =$ location of jth car at time t on one-lane road.

$$\frac{dX_j(t)}{dt} = V_j(t).$$

Velocity $V_j(t)$ of jth car varies with j and t.

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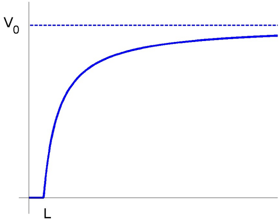
$$V_j(t) = v\big(X_{j+1}(t) - X_j(t)\big)$$

for some function $v(s)$ that defines speed as a function of separation s.

Simulations: `http://www.traffic-simulation.de/`
Select ring road and watch for shock to develop.

Function $v(s)$ (speed as function of separation)



$$v(s) = \begin{cases} u_{\max}\left(1 - \frac{L}{s}\right) & \text{if } s \ge L, \\ 0 & \text{if } s \le L. \end{cases}$$

where:

$L =$ car length

$u_{\max} =$ maximum velocity

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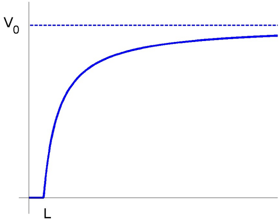

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Local density: $0 < L/s \leq 1$ $\quad (s = L \implies$ bumper-to-bumper)

Continuum model

Switch to density function:

Let $q(x,t) =$ density of cars, normalized so:

Units for x: carlengths, so $x = 10$ is 10 carlengths from $x = 0$.

Units for q: cars per carlength, so $0 \le q \le 1$.

Total number of cars in interval $x_1 \le x \le x_2$ at time t is

$$\int_{x_1}^{x_2} q(x,t)\,dx$$

Flux function for traffic

$q(x,t) =$ density, $u(x,t) =$ velocity $= U(q(x,t))$.

flux: $f(q) = uq$ Conservation law: $q_t + f(q)_x = 0$.

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$$f(q) = u_{\max} q \implies q_t + u_{\max} q_x = 0 \quad \text{(advection)}$$

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Velocity varying with density:

$$V(s) = u_{\max}(1 - L/s) \implies U(q) = u_{\max}(1-q),$$

$$f(q) = u_{\max} q(1-q) \qquad \text{(quadratic nonlinearity)}$$

Characteristics for a scalar problem

$q_t + f(q)_x = 0 \implies q_t + f'(q)q_x = 0$ (if solution is smooth).

Characteristic curves satisfy $X'(t) = f'(q(X(t),t)), \;\; X(0) = x_0$.

How does solution vary along this curve?

$$\begin{aligned}\frac{d}{dt}q(X(t),t) &= q_x(X(t),t)X'(t) + q_t(X(t),t)\\ &= q_x(X(t),t)f'(q(X(t),t)) + q_t(X(t),t)\\ &= 0\end{aligned}$$

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So solution is constant on characteristic
as long as solution stays smooth.

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$q(X(t),t) =$ constant $\implies X'(t)$ is constant on characteristic,
so characteristics are straight lines!

Nonlinear Burgers' equation

Conservation form: $u_t + \left(\frac{1}{2}u^2\right)_x = 0, \qquad f(u) = \frac{1}{2}u^2$.

Quasi-linear form: $\qquad u_t + uu_x = 0.$

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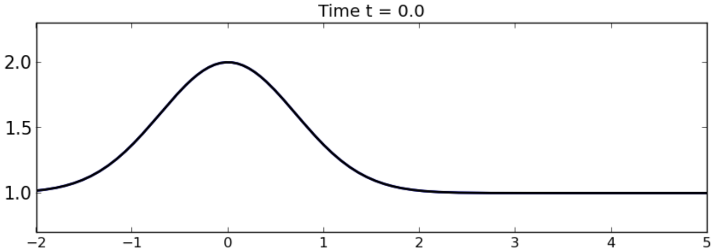
Like an advection equation with u advected with speed u.

True solution: u is constant along characteristic with speed $f'(u) = u$ until the wave "breaks" (shock forms).

Burgers' equation

Quasi-linear form: $u_t + uu_x = 0$

The solution is constant on characteristics so each value advects at constant speed equal to the value...



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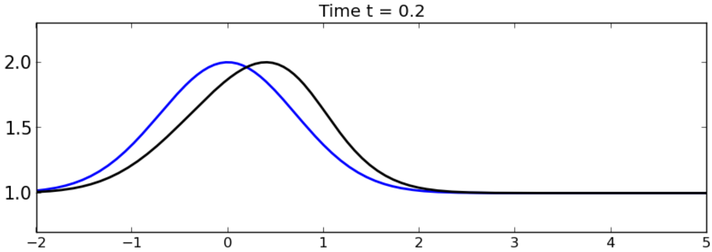

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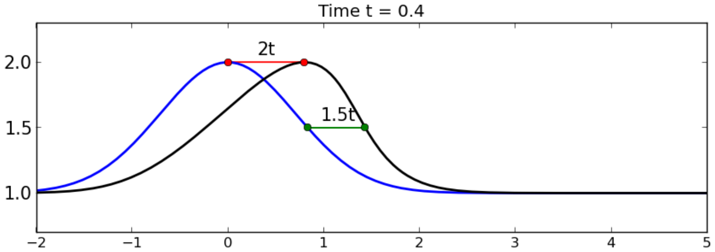

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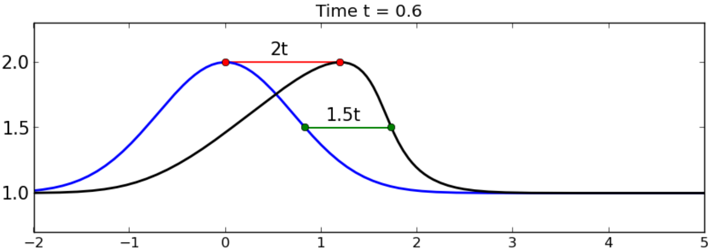

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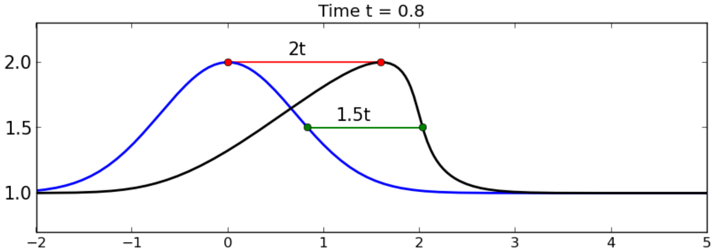

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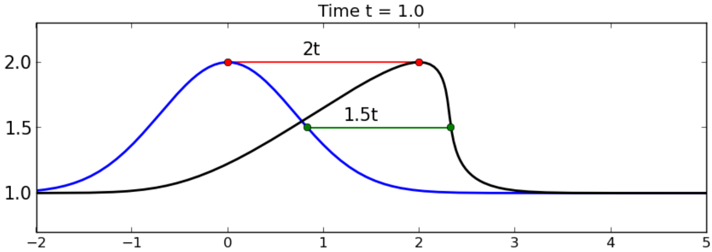

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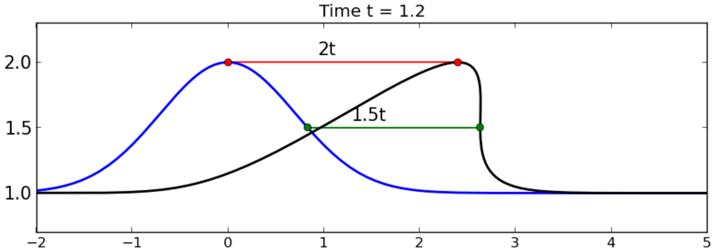

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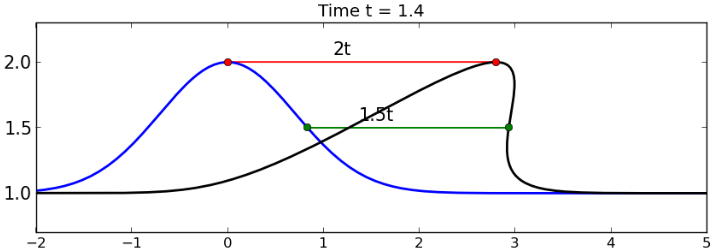

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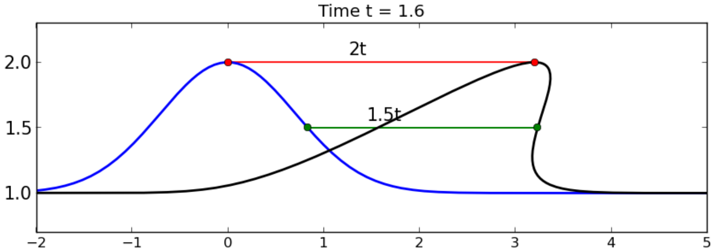

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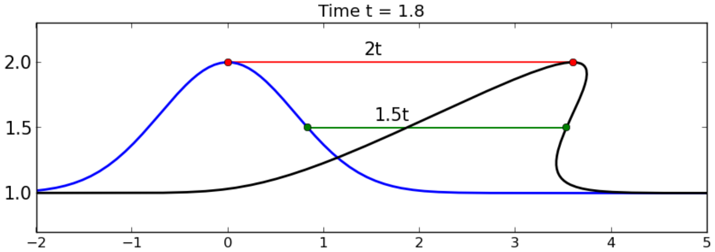

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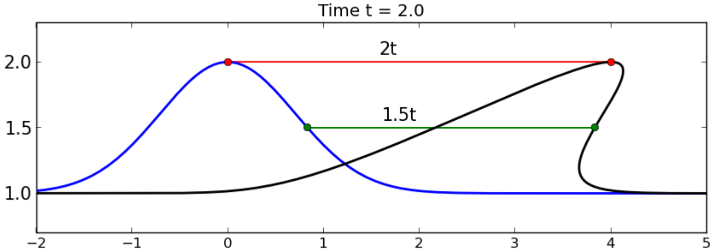


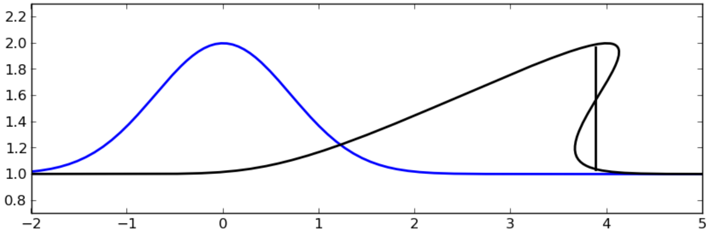
Burgers' equation

Triple valued solution is not physically possible for density.

Equal-area rule:

The area "under" the curve is conserved with time,

We must insert a shock so the two areas cut off are equal.



Vanishing Viscosity solution

Viscous Burgers' equation: $u_t + \left(\frac{1}{2}u^2\right)_x = \epsilon u_{xx}$.

This parabolic equation has a smooth C^∞ solution for all $t > 0$ for any initial data.

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Why try to solve hyperbolic equation?

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Why try to solve hyperbolic equation?

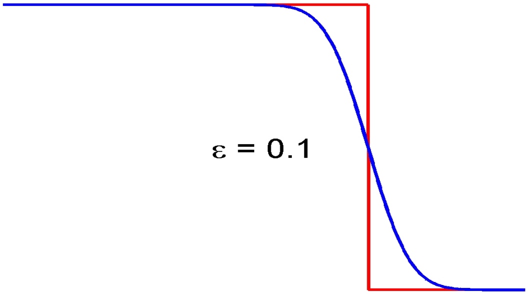
- Solving parabolic equation requires implicit method,
- Often correct value of physical "viscosity" is very small, shock profile that cannot be resolved on the desired grid
  $\Longrightarrow$ smoothness of exact solution doesn't help!

Discontinuous solutions

Vanishing Viscosity solution: The Riemann solution $q(x,t)$ is the limit as $\epsilon \to 0$ of the solution $q^\epsilon(x,t)$ of the parabolic advection-diffusion equation

$$q_t + uq_x = \epsilon q_{xx}.$$

For any $\epsilon > 0$ this has a classical smooth solution:



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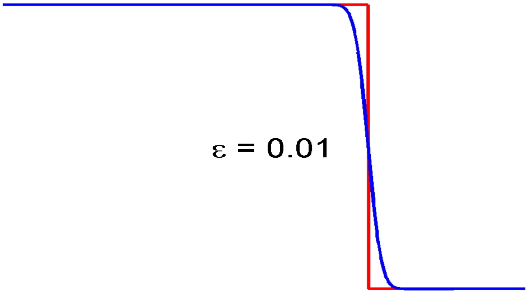

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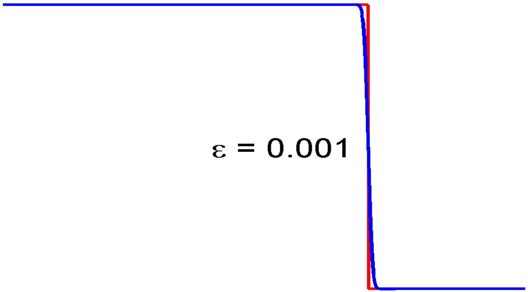

Weak solutions to $q_t + f(q)_x = 0$

$q(x,t)$ is a weak solution if it satisfies the integral form of the conservation law over all rectangles in space-time,

$$\int_{x_1}^{x_2} q(x,t_2)\,dx - \int_{x_1}^{x_2} q(x,t_1)\,dx \\ = \int_{t_1}^{t_2} f(q(x_1,t))\,dt - \int_{t_1}^{t_2} f(q(x_2,t))\,dt$$

Obtained by integrating

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = f(q(x_1,t)) - f(q(x_2,t))$$

from t_n to t_{n+1}.

Weak solutions to $q_t + f(q)_x = 0$

Alternatively, multiply PDE by smooth test function $\phi(x,t)$, with compact support ($\phi(x,t) \equiv 0$ for $|x|$ and t sufficiently large), and then integrate over rectangle,

$$\int_0^\infty \int_{-\infty}^\infty \big(q_t + f(q)_x\big)\phi(x,t)\,dx\,dt$$

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Then we can integrate by parts to get

$$\int_0^\infty \int_{-\infty}^\infty (q\phi_t + f(q)\phi_x)\,dx\,dt = -\int_{-\infty}^\infty q(x,0)\phi(x,0)\,dx.$$

$q(x,t)$ is a weak solution if this holds for all such ϕ.

Weak solutions to $q_t + f(q)_x = 0$

A function $q(x,t)$ that is piecewise smooth with jump discontinuities is a weak solution only if:

- The PDE is satisfied where q is smooth,
- The jump discontinuities all satisfy the Rankine-Hugoniot conditions.

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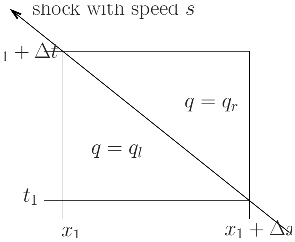
Note: The weak solution may not be unique!

Shock speed with states q_l and q_r at instant t_1



Then

$$\int_{x_1}^{x_1+\Delta x} q(x, t_1 + \Delta t)\,dx - \int_{x_1}^{x_1+\Delta x} q(x, t_1)\,dx$$
$$= \int_{t_1}^{t_1+\Delta t} f(q(x_1, t))\,dt - \int_{t_1}^{t_1+\Delta t} f(q(x_1 + \Delta x, t))\,dt.$$

Since q is essentially constant along each edge, this becomes

$$\Delta x\, q_\ell - \Delta x\, q_r = \Delta t f(q_\ell) - \Delta t f(q_r) + \mathcal{O}(\Delta t^2),$$

Taking the limit as $\Delta t \to 0$ gives

$$s(q_r - q_\ell) = f(q_r) - f(q_\ell).$$

Rankine-Hugoniot jump condition

$$s(q_r - q_\ell) = f(q_r) - f(q_\ell).$$

This must hold for any discontinuity propagating with speed s, even for systems of conservation laws.

For scalar problem, any jump allowed with speed:

$$s = \frac{f(q_r) - f(q_l)}{q_r - q_l}.$$

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For systems, $q_r - q_l$ and $f(q_r) - f(q_l)$ are vectors, s scalar,

R-H condition: $f(q_r) - f(q_l)$ must be scalar multiple of $q_r - q_l$.

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For linear system, $f(q) = Aq$, this says

$$s(q_r - q_l) = A(q_r - q_l),$$

Jump must be an eigenvector, speed s the eigenvalue.

Figure 11.1 — Shock formation in traffic

Discrete cars: Continuum model: $f'(q) = u_{\max}(1 - 2q)$

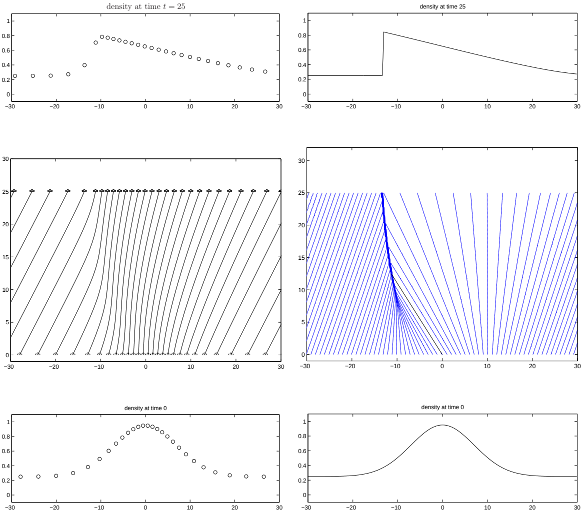


Figure 11.1 — Shock formation

(a) particle paths (car trajectories) $u(x,t) = u_{\max}(1 - q(x,t))$

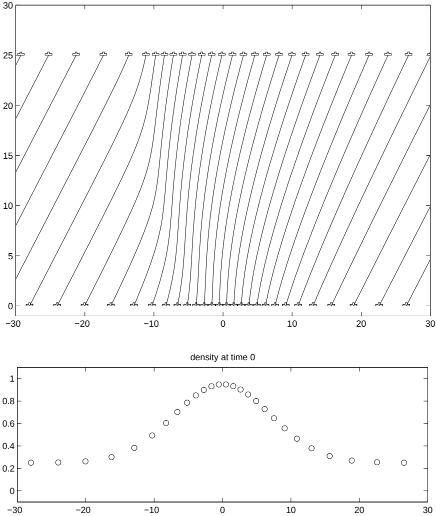
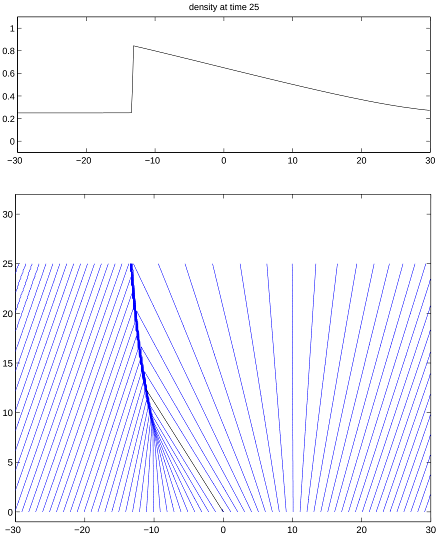


Figure 11.1 — Shock formation

(b) characteristics: $f'(q) = u_{\text{max}}(1 - 2q)$



Riemann problem for traffic flow

Initial data of the form

$$q(x,0) = \begin{cases} q_\ell & \text{if } x < 0 \\ q_r & \text{if } x \geq 0 \end{cases}$$

$$U(q) = u_{\max}(1-q), \quad f(q) = qU(q), \quad 0 \leq q \leq 1$$

Case 1: $q_\ell < q_r$, so $U(q_\ell) > U(q_r)$.

Fast moving cars approaching traffic jam
Expect shock wave.

Case 2: $q_\ell > q_r$, so $U(q_\ell) < U(q_r)$.

Slow moving cars can accelerate
Expect rarefaction wave.

Figure 11.2 — Traffic jam shock wave

Cars approaching red light $(q_\ell < 1, \;\; q_r = 1)$

Shock speed:

$$s = \frac{f(q_r) - f(q_\ell)}{q_r - q_\ell} = \frac{-2u_{\max} q_\ell}{1 - q_\ell} < 0 \quad \text{(for this data, could be } > 0\text{)}$$

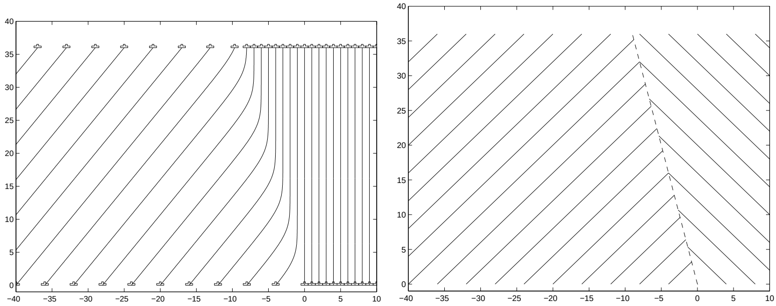


Figure 11.3 — Rarefaction wave

Cars accelerating at green light $(q_\ell = 1, \ \ q_r = 0)$

Characteristic speed $f'(q) = u_{\max}(1 - 2q)$

varies from $f'(q_\ell) = -u_{\max}$ to $f'(q_r) = u_{\max}$.