Finite Volume Methods for Hyperbolic Problems

# Multidimensional Hyperbolic Problems

- Derivation of conservation law
- Hyperbolicity
- Advection
- Gas dynamics and acoustics
- Shear waves

# Derivation of conservation law

$$\frac{d}{dt}\iint_{\Omega} q(x,y,t)\,dx\,dy = \text{net flux across } \partial\Omega.$$

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$\vec{n}(s) = (n^x(s), n^y(s))$ outward-pointing unit normal $(x(s), y(s))$.

Flux at $(x(s), y(s))$ in the direction $\vec{n}(s)$:

$$\vec{n}(s)\cdot\vec{f}(q(x(s),y(s))) = f(q)n^x(s) + g(q)n^y(s),$$

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True for any $\Omega \implies \quad q_t + \vec{\nabla}\cdot\vec{f}(q) = 0.$ (PDE form)

# First order hyperbolic PDE in 2 space dimensions

General conservation law: $q_t + f(q)_x + g(q)_y = 0$

Quasi-linear form: $q_t + f'(q)q_x + g'(q)q_y = 0$

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where $q \in \mathbb{R}^m,\ f(q) = Aq,\ g(q) = Bq$ and $A, B \in \mathbb{R}^{m\times m}$.

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Then plane wave propagating in any direction satisfies 1D hyperbolic equation.

# Plane wave solutions

Suppose

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Then:

$$\begin{aligned} q_x(x,y,t) &= \cos\theta\, \breve{q}_\xi(\xi,t) \\ q_y(x,y,t) &= \sin\theta\, \breve{q}_\xi(\xi,t) \end{aligned}$$

so

$$q_t + Aq_x + Bq_y = \breve{q}_t + (A\cos\theta + B\sin\theta)\breve{q}_\xi$$

and the 2d problem reduces to the 1d hyperbolic equation

$$\breve{q}_t(\xi,t) + (A\cos\theta + B\sin\theta)\breve{q}_\xi(\xi,t) = 0.$$

# Advection in 2 dimensions

Constant coefficient: $q_t + uq_x + vq_y = 0$

In this case solution for arbitrary initial data is easy:

$$q(x, y, t) = q(x - ut,\ y - vt,\ 0).$$

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Variable coefficient:

Conservation form: $q_t + (u(x, y, t)q)_x + (v(x, y, t)q)_y = 0$

Advective form (color eqn): $q_t + u(x, y, t)q_x + v(x, y, t)q_y = 0$

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Equivalent only if flow is divergence-free (incompressible):

$$\nabla \cdot \vec{u} = u_x(x, y, t) + v_y(x, y, t) = 0 \qquad \forall t \geq 0.$$

# Gas dynamics in 2D

$\rho(x, y, t) =$ mass density
$\rho(x, y, t)u(x, y, t) = x$-momentum density
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If pressure $= P(\rho)$, e.g. isothermal or isentropic:

$$\begin{aligned} \rho_t + (\rho u)_x + (\rho v)_y &= 0 \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y &= 0 \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y &= 0 \end{aligned}$$

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For any $\theta$, the matrix $f'(q)\cos\theta + g'(q)\sin\theta$ has eigenvalues

$$\breve{u} - c,\ \breve{u},\ \breve{u} + c$$

where $c = \sqrt{P'(\rho)}$ and $\breve{u} = u\cos\theta + v\sin\theta$.

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Full Euler equations: 1 more equation for Energy

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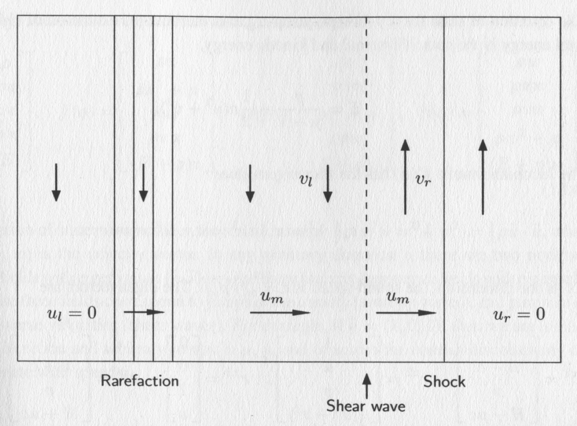
$$\breve{u} - c, \ \breve{u}, \ \breve{u} + c \qquad \text{Euler: another wave with } \lambda = \breve{u}$$

where $c = \sqrt{P'(\rho)}$ and $\breve{u} = u\cos\theta + v\sin\theta$.

# Solution of plane wave Riemann problem in 2D




Jump in $v$ from $v_\ell$ to $v_r$ propagates with the contact discontinuity

# Acoustics in 2 dimensions

Linearize about $u = 0,\ v = 0$ and $p =$ perturbation in pressure:

$$\begin{aligned} p_t + K_0(u_x + v_y) &= 0 \\ \rho_0 u_t + p_x &= 0 \\ \rho_0 v_t + p_y &= 0 \end{aligned}$$

Note: pressure responds to compression or expansion and so $p_t$ is proportional to divergence of velocity.

Second and third equations are $F = ma$.

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Gives hyperbolic system $q_t + Aq_x + Bq_y = 0$ with

$$q = \begin{bmatrix} p \\ u \\ v \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & K_0 & 0 \\ 1/\rho_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & K_0 \\ 0 & 0 & 0 \\ 1/\rho_0 & 0 & 0 \end{bmatrix}.$$

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Plane waves:

$$A\cos\theta + B\sin\theta = \begin{bmatrix} 0 & K_0\cos\theta & K_0\sin\theta \\ \cos\theta/\rho_0 & 0 & 0 \\ \sin\theta/\rho_0 & 0 & 0 \end{bmatrix}.$$

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Eigenvalues: $\lambda^1 = -c_0, \;\; \lambda^2 = 0, \;\; \lambda^3 = +c_0$

where $c_0 = \sqrt{K_0/\rho_0}$ is independent of angle $\theta$.

Isotropic: sound propagates at same speed in any direction.

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Note: Zero wave speed for “shear wave” with variation only in velocity in direction $(-\sin\theta, \; \cos\theta)$.

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In this case, decouples into scalar advection equation for each component of $w$:

$$w_t^p + \lambda^p w_x^p + \mu^p w_y^p = 0 \implies w^p(x, y, t) = w^p(x - \lambda^p t,\ y - \mu^p t,\ 0).$$

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This is not true for most coupled systems, e.g. acoustics.

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$$A = \begin{bmatrix} 0 & K_0 & 0 \\ 1/\rho_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad R^x = \begin{bmatrix} -Z_0 & 0 & Z_0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

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$$B = \begin{bmatrix} 0 & 0 & K_0 \\ 0 & 0 & 0 \\ 1/\rho_0 & 0 & 0 \end{bmatrix} \qquad R^y = \begin{bmatrix} -Z_0 & 0 & Z_0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Solving $q_t + Bq_y = 0$ gives pressure waves in $(p, v)$.