

Finite Volume Methods for Hyperbolic Problems

Approximate Riemann Solvers

- HLL method
- Linearized Jacobian approach
- Roe solvers
- Shallow water example
- HLLE method and positivity
- Harten-Hyman entropy fix

Riemann Problems and Jupyter Solutions

Theory and Approximate Solvers for Hyperbolic PDEs

David I. Ketcheson, RJL, and Mauricio del Razo

General information and links to book, Github, Binder, etc.:

[**bookstore.siam.org/fa16/bonus**](https://bookstore.siam.org/fa16/bonus)

View static version of notebooks at:

[**www.clawpack.org/riemann_book/html/Index.html**](http://www.clawpack.org/riemann_book/html/Index.html)

In particular see: [**Shallow_water_approximate.ipynb**](#)

Approximate Riemann Solvers

For flux-differencing methods: Compute approximation to flux at interface between cells.

Obtain high resolution via higher-order time stepping with flux limiter.

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Can then apply high-resolution wave limiters.

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For wave-propagation algorithm: Approximate true Riemann solution by set of waves consisting of finite jumps propagating at constant speeds.

Can then apply high-resolution wave limiters.

May require entropy fix if a wave should be transonic rarefaction.

Wave propagation methods

- Solving Riemann problem gives waves $\mathcal{W}_{i-1/2}^p$,

$$Q_i - Q_{i-1} = \sum_p \mathcal{W}_{i-1/2}^p$$

and speeds $s_{i-1/2}^p$. (Usually approximate solver used.)

- These waves update neighboring cell averages depending on sign of s^p (Godunov's method) via fluctuations.

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- Waves also give (characteristic) decomposition of slopes:

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- Apply limiter to each wave to obtain $\widetilde{\mathcal{W}}_{i-1/2}^p$.
- Use limited waves in second-order correction terms.

Approximate Riemann Solvers

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by a set of jump discontinuities:

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Local linearization: Replace $q_t + f(q)_x = 0$ by

$$q_t + \hat{A}q_x = 0, \quad \text{where } \hat{A} = \hat{A}(q_l, q_r) \approx f'(q_{ave}).$$

Eigenvectors give waves. **Roe solver** \implies conservative

HLL Solver

Harten – Lax – van Leer (1983): Given $Q_\ell, Q_r \in \mathbb{R}^m$ for $m \geq 2$,
Use only 2 waves with a single intermediate state Q^* .

$s^1 \approx$ minimum characteristic speed

$s^2 \approx$ maximum characteristic speed

$$\mathcal{W}^1 = Q^* - Q_\ell, \quad \mathcal{W}^2 = Q_r - Q^*$$

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Conservation implies unique value for middle state Q^* :

$$s^1 \mathcal{W}^1 + s^2 \mathcal{W}^2 = f(Q_r) - f(Q_\ell)$$

$$\implies Q^* = \frac{f(Q_r) - f(Q_\ell) - s^2 Q_r + s^1 Q_\ell}{s^1 - s^2}.$$

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Choice of speeds:

- Max and min of expected speeds over entire problem,
- Max and min of eigenvalues of $f'(Q_\ell)$ and $f'(Q_r)$.

HLL Solver for Shallow Water Equations

$$\begin{aligned}h_t + (hu)_x &= 0 \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x &= 0\end{aligned}$$

Choose e.g.

$$\begin{aligned}s^1 &= u_\ell - \sqrt{gh_\ell}, \\ s^2 &= u_r + \sqrt{gh_r}\end{aligned}$$

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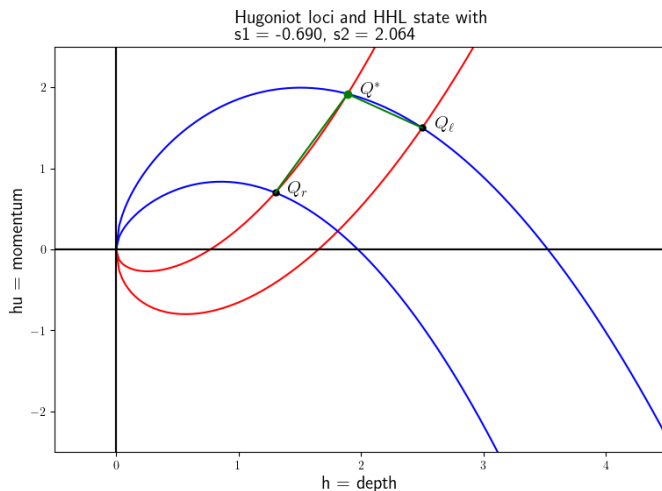
$$\begin{aligned}s^1 &= u_\ell - \sqrt{gh_\ell}, \\ s^2 &= u_r + \sqrt{gh_r}\end{aligned}$$

Then

$$\begin{aligned}Q^* &= \frac{f(Q_r) - f(Q_\ell) - s^2Q_r + s^1Q_\ell}{s^1 - s^2} \\ &= \frac{1}{s^1 - s^2} \left[\begin{array}{c} h_ru_r - h_\ell u_\ell - s^2h_r + s^1h_\ell \\ (h_ru_r^2 + \frac{1}{2}gh_r^2) - (h_\ell u_\ell^2 + \frac{1}{2}gh_\ell^2) - s^2h_ru_r + s^1h_\ell u_\ell \end{array} \right]\end{aligned}$$

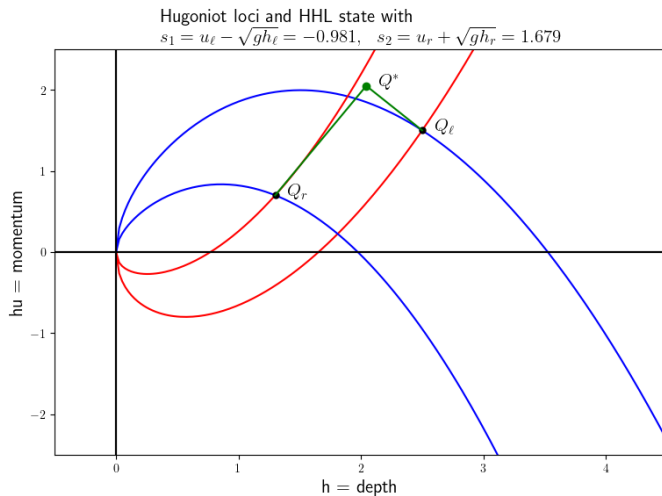
HLL solver for shallow water

If we use the shock speeds from the exact two-shock solution, looks perfect:



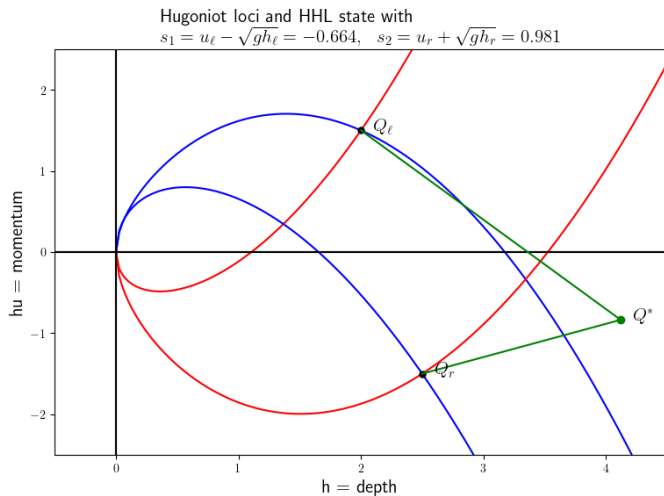
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Using $s_1 = \lambda^1(q_\ell)$ and $s_2 = \lambda^2(q_r)$:



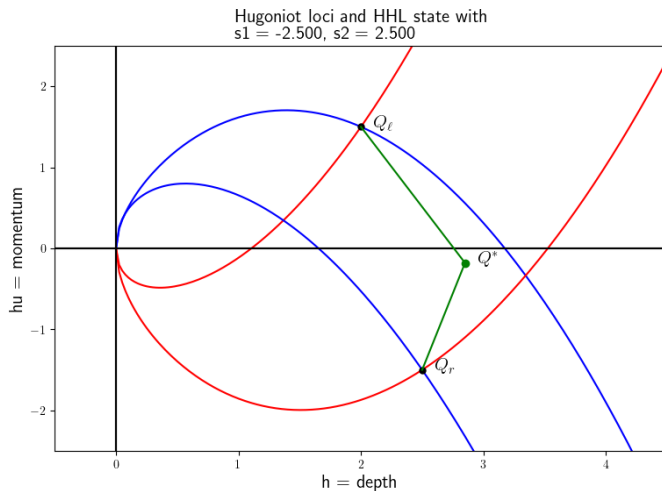
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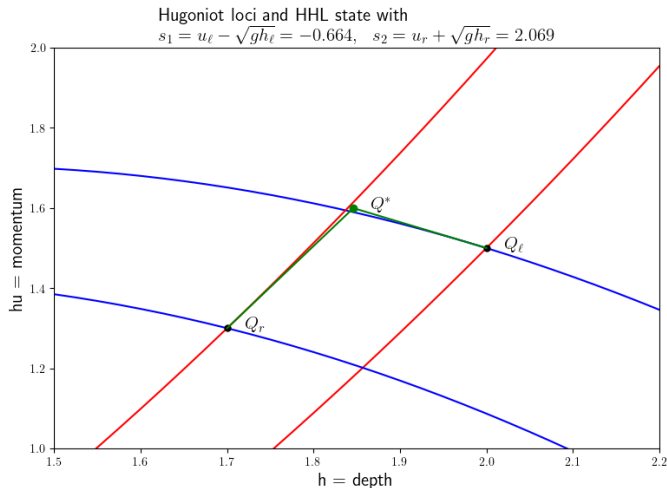
HLL solver for shallow water

Using different choice of s_1, s_2 :



HLL solver for shallow water

If $\Delta Q = Q_r - Q_\ell$ is small, then eigenvalues nearly constant.
For smooth flow, HLL is very accurate (for $m = 2$).



Approximate Riemann Solvers — Local Linearization

Approximate true Riemann solution by set of waves consisting of finite jumps propagating at constant speeds.

Local linearization:

Replace $q_t + f(q)_x = 0$ by

$$q_t + \hat{A}q_x = 0,$$

where $\hat{A} = \hat{A}(q_l, q_r) \approx f'(q_{ave})$.

Then decompose

$$q_r - q_l = \alpha^1 \hat{r}^1 + \dots + \alpha^m \hat{r}^m$$

to obtain waves $\mathcal{W}^p = \alpha^p \hat{r}^p$ with speeds $s^p = \hat{\lambda}^p$.

Approximate Riemann Solvers

How to use?

One approach: determine Q^* = state along $x/t = 0$,

$$Q^* = Q_{i-1} + \sum_{p: s^p < 0} \mathcal{W}^p, \quad F_{i-1/2} = f(Q^*),$$

$$\mathcal{A}^- \Delta Q = F_{i-1/2} - f(Q_{i-1}), \quad \mathcal{A}^+ \Delta Q = f(Q_i) - F_{i-1/2}.$$

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Or, sometimes can use:

$$\mathcal{A}^- \Delta Q = \sum_{p:s^p < 0} s^p \mathcal{W}^p, \quad \mathcal{A}^+ \Delta Q = \sum_{p:s^p > 0} s^p \mathcal{W}^p.$$

Conservative **only if** $\mathcal{A}^- \Delta Q + \mathcal{A}^+ \Delta Q = f(Q_i) - f(Q_{i-1})$.

This holds for Roe solver.

Roe Solver

Given q_ℓ, q_r , solve $q_t + \hat{A}q_x = 0$ where \hat{A} chosen to satisfy

$$\hat{A}(q_r - q_\ell) = f(q_r) - f(q_\ell).$$

Then:

- Good approximation for weak waves (smooth flow)

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- Single shock captured exactly:

$$f(q_r) - f(q_\ell) = s(q_r - q_\ell) \implies q_r - q_\ell \text{ is an eigenvector of } \hat{A}$$

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- Wave-propagation algorithm is conservative since

$$\begin{aligned}\mathcal{A}^- \Delta Q_{i-1/2} &= \sum (s_{i-1/2}^p)^- \mathcal{W}_{i-1/2}^p, \\ \mathcal{A}^+ \Delta Q_{i+1/2} &= \sum (s_{i+1/2}^p)^+ \mathcal{W}_{i+1/2}^p, \quad \implies\end{aligned}$$

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$$\begin{aligned}\mathcal{A}^- \Delta Q_{i-1/2} + \mathcal{A}^+ \Delta Q_{i-1/2} &= \sum s_{i-1/2}^p \mathcal{W}_{i-1/2}^p = \hat{A} \sum \mathcal{W}_{i-1/2}^p \\ &= \hat{A}(q_r - q_\ell) = f(q_r) - f(q_\ell).\end{aligned}$$

Shallow water equations

$h(x, t)$ = depth

$u(x, t)$ = velocity (depth averaged, varies only with x)

Conservation of mass and momentum hu gives system of two equations.

mass flux = hu ,

momentum flux = $(hu)u + p$ where p = hydrostatic pressure

$$\begin{aligned}h_t + (hu)_x &= 0 \\(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x &= 0\end{aligned}$$

Jacobian matrix:

$$f'(q) = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix}, \quad \lambda = u \pm \sqrt{gh}.$$

Roe solver for Shallow Water

Given h_ℓ, u_ℓ, h_r, u_r , define

$$\bar{h} = \frac{h_\ell + h_r}{2}, \quad \hat{u} = \frac{\sqrt{h_\ell}u_\ell + \sqrt{h_r}u_r}{\sqrt{h_\ell} + \sqrt{h_r}}$$

Then

\hat{A} = Jacobian matrix evaluated at this average state

satisfies

$$\hat{A}(q_r - q_\ell) = f(q_r) - f(q_\ell).$$

- Roe condition is satisfied,
- Isolated shock modeled well,
- Wave propagation algorithm is conservative,
- High resolution methods obtained using corrections with limited waves.

Roe solver for Shallow Water

Given h_l, u_l, h_r, u_r , define

$$\bar{h} = \frac{h_l + h_r}{2}, \quad \hat{u} = \frac{\sqrt{h_l}u_l + \sqrt{h_r}u_r}{\sqrt{h_l} + \sqrt{h_r}}$$

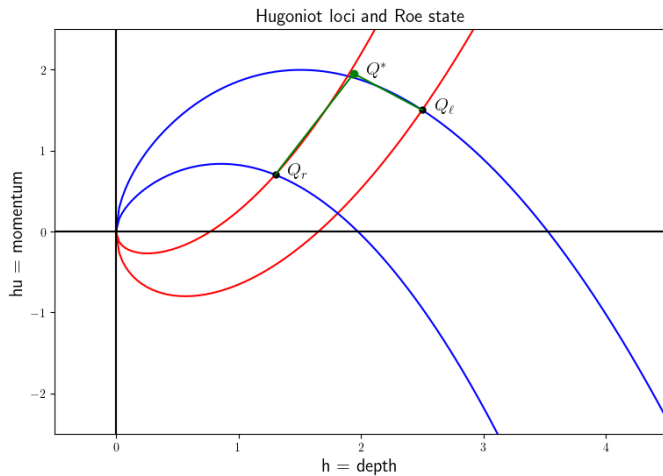
Eigenvalues of $\hat{A} = f'(\hat{q})$ are:

$$\hat{\lambda}^1 = \hat{u} - \hat{c}, \quad \hat{\lambda}^2 = \hat{u} + \hat{c}, \quad \hat{c} = \sqrt{g\bar{h}}.$$

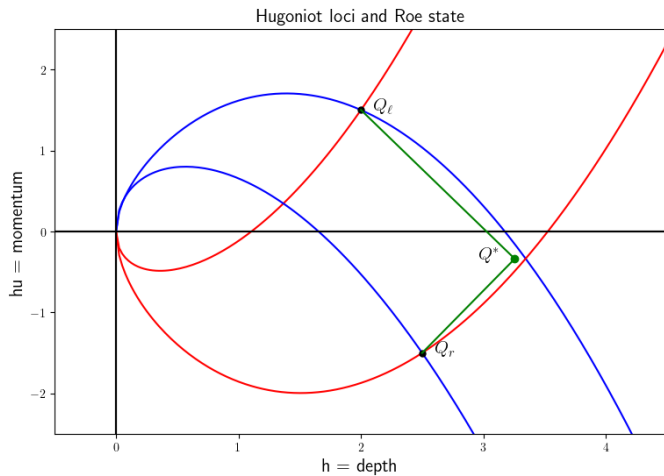
Eigenvectors:

$$\hat{r}^1 = \begin{bmatrix} 1 \\ \hat{u} - \hat{c} \end{bmatrix}, \quad \hat{r}^2 = \begin{bmatrix} 1 \\ \hat{u} + \hat{c} \end{bmatrix}.$$

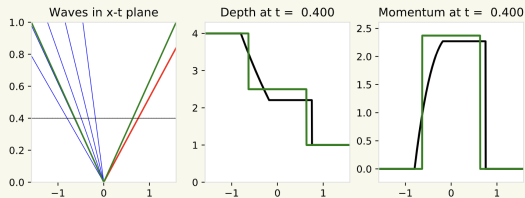
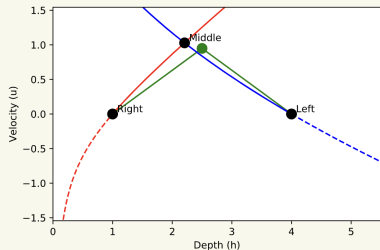
Roe solver for shallow water



Roe solver for shallow water



Dam break problem with Roe solver

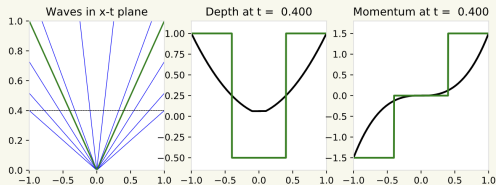


Note that rarefaction replaced by jump.

Example from **RpJs**

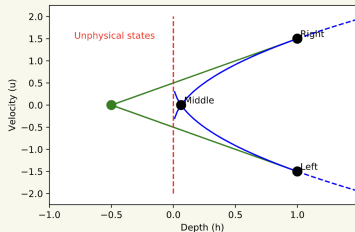
Widgets can be used to experiment.

Nonphysical solution with Roe solver



```
In [12]: sw.phase_plane_plot(q_l,q_r,g=1.,y_axis='u',  
                             approx_states=states,hmin = -1)  
plt.plot([0,0], [-2,2], 'r--')  
plt.text(-0.8,1.5,'Unphysical states',color='red')
```

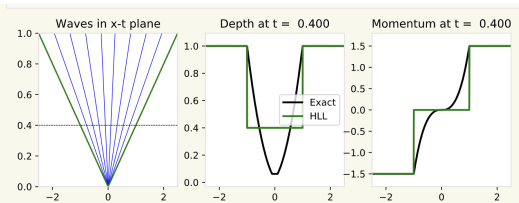
```
Out[12]: Text(-0.8,1.5,'Unphysical states')
```



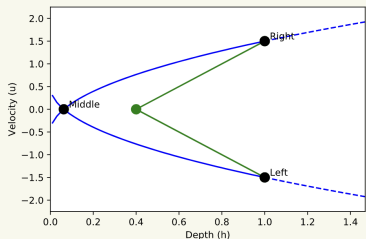
For data that gives near dry state in Q_m , Roe solver may give negative depth.

Example from **RpJs**

HLLC preserves positivity



```
In [25]: sw.phase_plane_plot(q_l,q_r,g=1.,y_axis='u',  
                             approx_states='hllc')
```



For data that gives near dry state in Q_m , Roe solver may give negative depth.

Choosing s_1, s_2 as characteristic speeds in HLL does much better in this case.

Example from **RpJs**

Einfeldt: Choice of speeds for gas dynamics (or shallow water) that **guarantees positivity**.

Based on characteristic speeds and Roe averages:

$$s_{i-1/2}^1 = \min_p (\min(\lambda_{i-1}^p, \hat{\lambda}_{i-1/2}^p)),$$
$$s_{i-1/2}^2 = \max_p (\max(\lambda_i^p, \hat{\lambda}_{i-1/2}^p)).$$

where

λ_i^p is the p th eigenvalue of the Jacobian $f'(Q_i)$,

$\hat{\lambda}_{i-1/2}^p$ is the p th eigenvalue using Roe average $f'(\hat{Q}_{i-1/2})$

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Can also show that:

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Can also show that:

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then Roe speed is used \implies **exact solution in this case**.
- **No entropy fix needed**.
(More diffusive than Roe solver.)

Harten-Hyman entropy fix

For any wave splitting $Q_i - Q_{i-1} = \sum \mathcal{W}^p$, with speeds $\hat{\lambda}^p$.

Define

$$q_\ell^k = Q_{i-1} + \sum_{p=1}^{k-1} \mathcal{W}^p, \quad q_r^k = q_\ell^k + \mathcal{W}^k$$

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$$q_\ell^k = Q_{i-1} + \sum_{p=1}^{k-1} \mathcal{W}^p, \quad q_r^k = q_\ell^k + \mathcal{W}^k$$

If $\lambda_\ell^k \equiv \lambda^k(q_\ell^k) < 0 < \lambda^k(q_r^k) \equiv \lambda_r^k$ then replace \mathcal{W}^k by

$$\begin{aligned} \mathcal{W}_\ell^k &= \beta \mathcal{W}^k, & \text{speed} &= \lambda_\ell^k < 0, \\ \mathcal{W}_r^k &= (1 - \beta) \mathcal{W}^k, & \text{speed} &= \lambda_r^k > 0. \end{aligned}$$

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Conservation requires:

$$\lambda_\ell^k \mathcal{W}_\ell^k + \lambda_r^k \mathcal{W}_r^k = \hat{\lambda}^k \mathcal{W}^k, \quad \implies \beta = \frac{\lambda_r^k - \hat{\lambda}^k}{\lambda_r^k - \lambda_\ell^k}$$

Harten-Hyman entropy fix

In wave propagation algorithm, leave \mathcal{W}^k alone for high-resolution correction terms (with limiters).

Similar to entropy fix for scalar problem:

Only need to modify the fluctuations in the “Godunov update”

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Similar to entropy fix for scalar problem:

Only need to modify the fluctuations in the “Godunov update”

$$\mathcal{A}^- \Delta Q = \sum_{p=1}^m (\lambda^p)^- \mathcal{W}^p, \quad \mathcal{A}^+ \Delta Q = \sum_{p=1}^m (\lambda^p)^+ \mathcal{W}^p,$$

Usually $(\lambda^p)^- = \min(\lambda^p, 0)$, $(\lambda^p)^+ = \max(\lambda^p, 0)$.

Modify for field k :

$$(\lambda^k)^- = \beta \lambda_\ell^k < 0, \quad (\lambda^k)^+ = (1 - \beta) \lambda_r^k > 0,$$

so that

$$(\lambda^k)^- \mathcal{W}^k = \lambda_\ell^k \beta \mathcal{W}^k \quad (\lambda^k)^+ \mathcal{W}^k = \lambda_r^k (1 - \beta) \mathcal{W}^k$$