Finite Volume Methods for Hyperbolic Problems

Acoustics in Heterogeneous Media

- One space dimension
- Reflection and transmission at interfaces
- Non-conservative form, Riemann problems
- Two space dimensions
- Transverse Riemann solver
- Some examples

One-dimensional Elasticity

Compressional waves similar to acoustic waves in gas.

Notation:

$$X(x,t) = \text{ location of particle indexed by } x \text{ in the reference (undeformed) configuration}$$

$$X(x,0) = x \;\text{ if initially undeformed}$$

$$\epsilon(x,t) = X_x(x,t) - 1 = \text{ strain}$$

$$u(x,t) = \text{ velocity of particle indexed by } x$$

$$\sigma(\epsilon) = \text{ stress–strain relation}$$

$$\rho = \text{ density}$$

Linear elasticity

Hyperbolic conservation law:

$$\begin{array}{ll} \epsilon_t - u_x = 0 & \text{since } \epsilon_t = X_{xt} = X_{tx} = u_x \\ \rho u_t - \sigma_x = 0 & \text{conservation of momentum, } F = ma \end{array}$$

Linear stress-strain relation (Hooke's law):

$$\sigma(\epsilon) = K\epsilon$$

where K is the bulk modulus of compressibility.

Then

$$\begin{array}{l} \sigma_t - K u_x = 0 \\ u_t - (1/\rho)\sigma_x = 0 \end{array} \qquad A = \begin{bmatrix} 0 & -K \\ -1/\rho & 0 \end{bmatrix}$$

Eigenvalues: $\lambda = \pm\sqrt{K/\rho}$ as in acoustics.

(Equivalent to acoustics with $\sigma = -p$)

Elasticity in heterogeneous material

Suppose $\rho(x),\ \sigma(\epsilon, x)$ vary with x

Conservative form:

$$\begin{aligned} &\epsilon_t - u_x = 0 \\ &(\rho(x)u)_t - \sigma(\epsilon, x)_x = 0 \end{aligned}$$

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Linear stress-strain relation (Hooke's law):

$$\sigma(\epsilon, x) = K(x)\epsilon$$

Non-conservative variable-coefficient linear system:

$$\begin{aligned} &\sigma_t - K(x)u_x = 0 \\ &u_t - (1/\rho(x))\sigma_x = 0 \end{aligned} \qquad A = \begin{bmatrix} 0 & -K(x) \\ -1/\rho(x) & 0 \end{bmatrix}$$

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Variable coefficient acoustics: $p = -\sigma$

Wave propagation in heterogeneous medium

Multiply system

$$q_t + A(x)q_x = 0$$

by $R^{-1}(x)$ on left to obtain

$$R^{-1}(x)q_t + R^{-1}(x)A(x)R(x)\,R^{-1}(x)q_x = 0$$

or

$$(R^{-1}(x)q)_t + \Lambda(x)\ \left[(R^{-1}(x)q)_x - R_x^{-1}(x)q\right] = 0$$

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Let $w(x,t) = R^{-1}(x)q(x,t)$ (characteristic variable).

There is a coupling term on the right: Note typo in (9.51)

$$w_t + \Lambda(x)\,w_x = \Lambda(x)R_x^{-1}(x)R(x)w$$

If the eigenvectors vary with x (i.e. if $R_x \neq 0$)
then waves in other families are generated (e.g. reflections)

Wave propagation in heterogeneous medium

Linear system $q_t + A(x)q_x = 0$. For acoustics:

$$A = \begin{bmatrix} 0 & K(x) \\ 1/\rho(x) & 0 \end{bmatrix} \qquad q = \begin{bmatrix} p \\ u \end{bmatrix}.$$

eigenvalues: $\lambda^1 = -c(x), \qquad \lambda^2 = +c(x),$

where $c(x) = \sqrt{K(x)/\rho(x)} =$ local speed of sound.

eigenvectors: $r^1(x) = \begin{bmatrix} -Z(x) \\ 1 \end{bmatrix}, \qquad r^2(x) = \begin{bmatrix} Z(x) \\ 1 \end{bmatrix}$

where $Z(x) = \rho c = \sqrt{\rho K} =$ impedance.

Transmission and reflection coefficients

Consider an interface between two materials with constant properties in each.

$$\rho_\ell,\ K_\ell \implies c_\ell = \sqrt{\rho_\ell/K_\ell},\ Z_\ell = \sqrt{\rho_\ell K_\ell}$$

$$\rho_r\ K_r \implies c_r = \sqrt{\rho_r/K_r},\ Z_r = \sqrt{\rho_r K_r}$$

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If impedance $Z_\ell = Z_r$ then $r_\ell^p = r_r^p$ and waves are transmitted through interface with no generation of other waves

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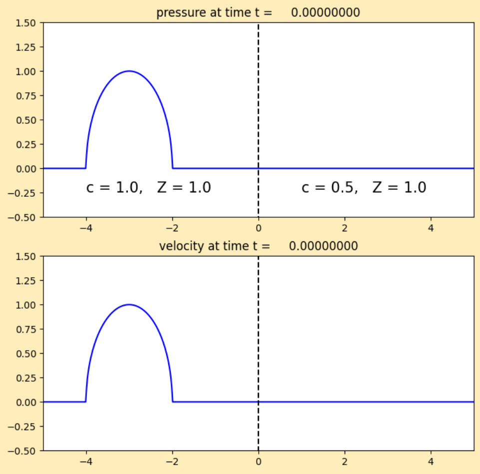
More generally, wave is partly transmitted and partly reflected,

$$C_T = \frac{2Z_r}{Z_\ell + Z_r}, \qquad C_R = \frac{Z_r - Z_\ell}{Z_\ell + Z_r}.$$

Right-going simple wave with $Z_\ell = Z_r$



Note p and u are not conserved, but they are always continuous.

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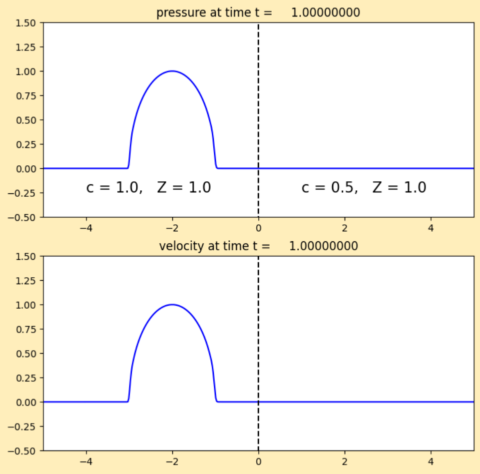

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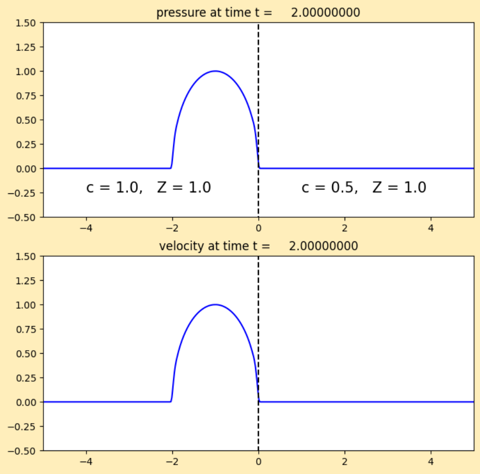

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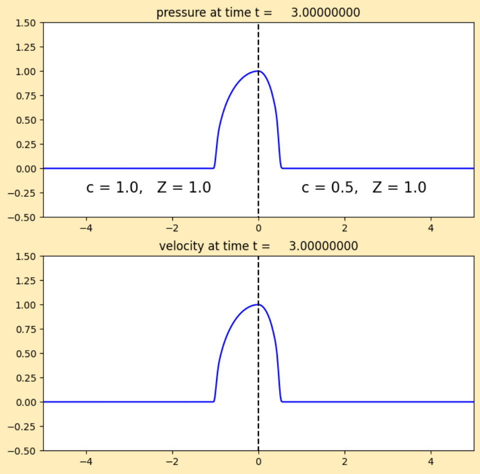

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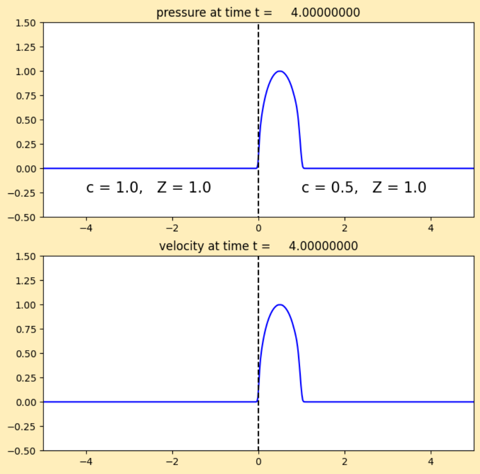

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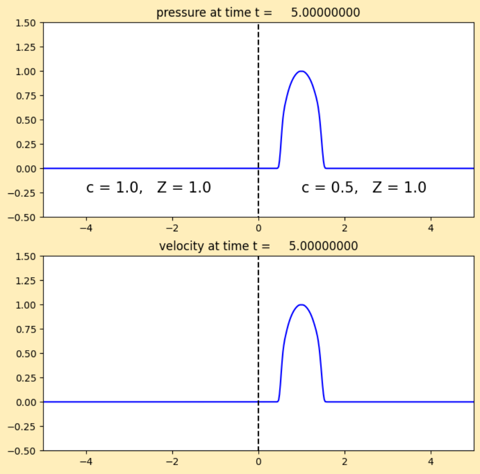

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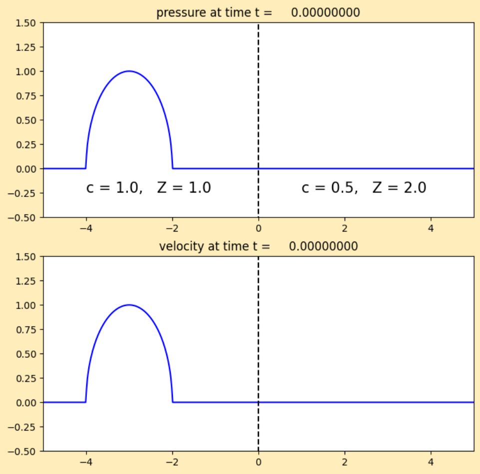
Transmitted/reflected wave with $Z_\ell \neq Z_r$



$$C_T = \frac{2Z_r}{Z_\ell + Z_r} = \frac{4}{3}$$

$$C_R = \frac{Z_r - Z_\ell}{Z_\ell + Z_r} = \frac{1}{3}$$

Note that p and u remain continuous at the interface.

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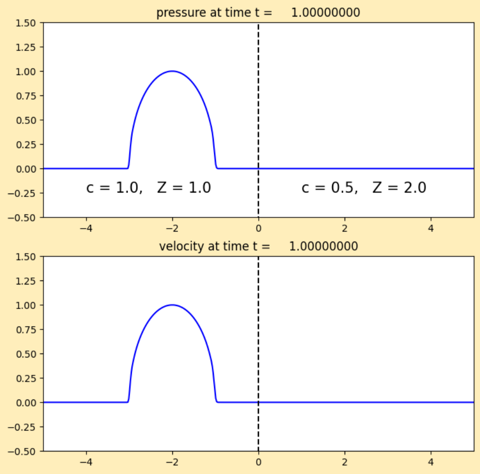

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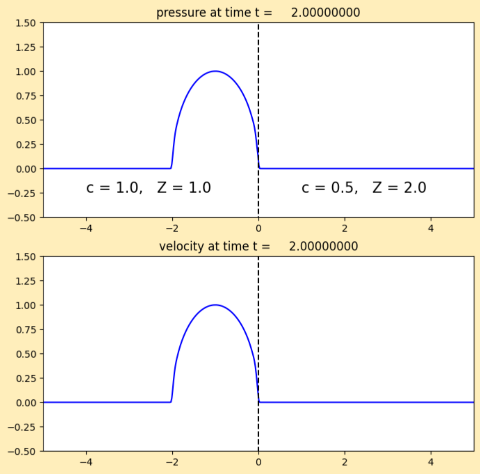

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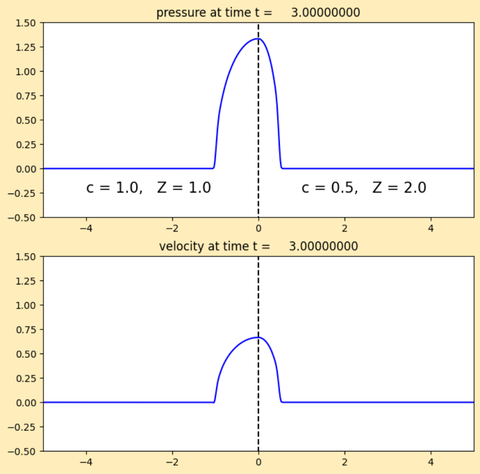

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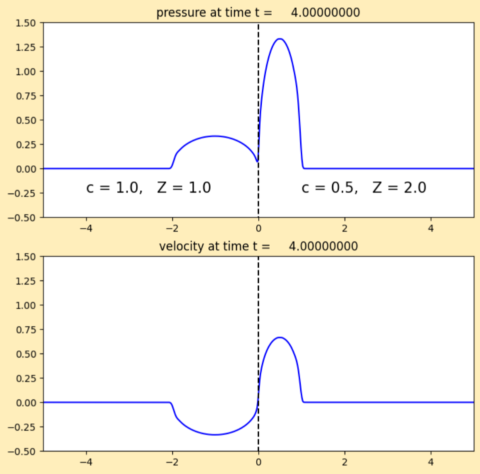

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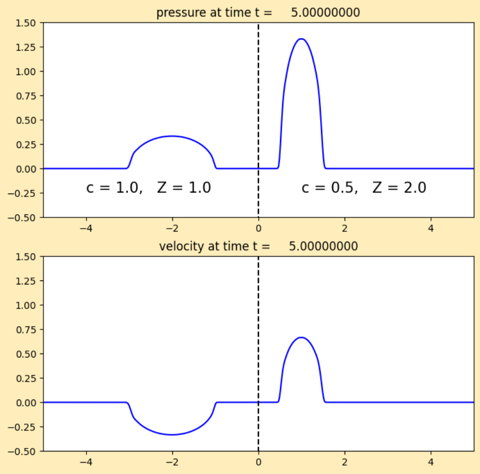

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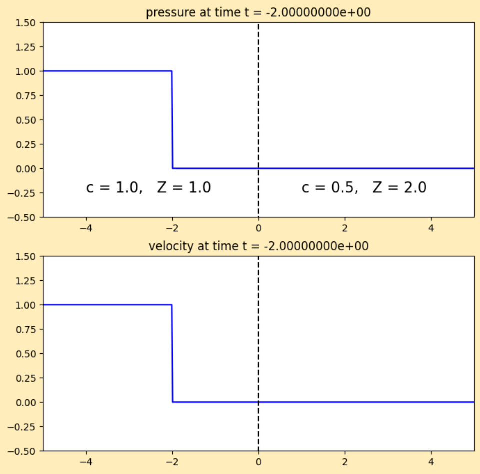

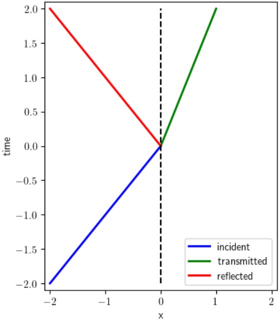

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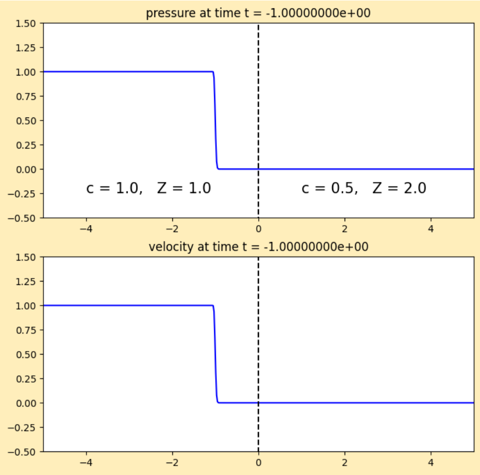

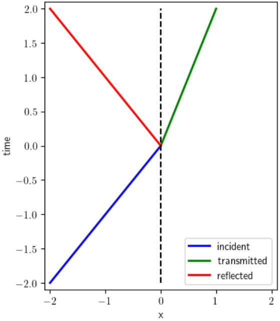

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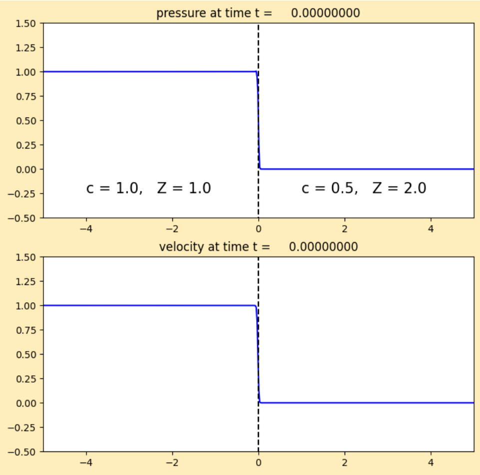

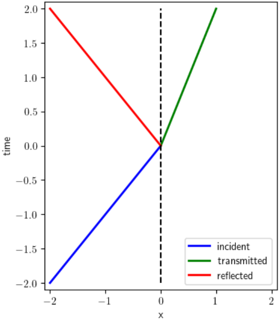

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Looks like Riemann problem data at $t = 0$

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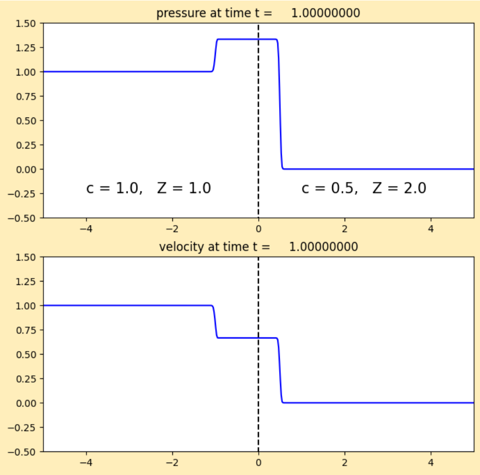

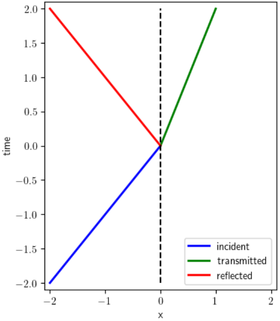

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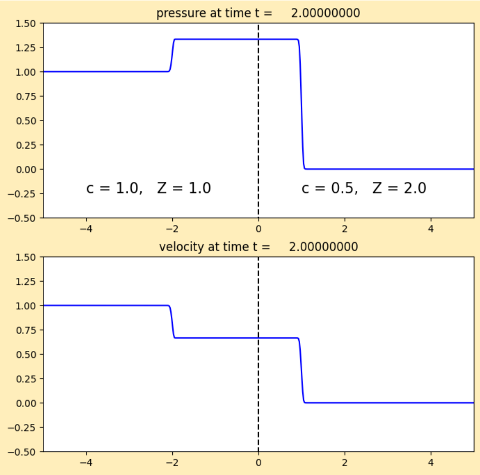

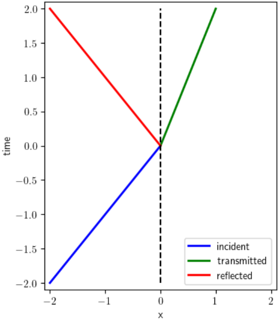

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Riemann problem for heterogeneous medium

Jump discontinuity in $q(x, 0)$ and in $K(x)$ and $\rho(x)$.

Decompose jump in q as linear combination of eigenvectors:

- left-going waves: eigenvectors for material on left,
- right-going waves: eigenvectors for material on right.

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$$R(x) = \begin{bmatrix} -Z(x) & Z(x) \\ 1 & 1 \end{bmatrix}, \qquad R^{-1}(x) = \frac{1}{2Z(x)} \begin{bmatrix} -1 & Z(x) \\ 1 & Z(x) \end{bmatrix}.$$

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Riemann solution: decompose

$$q_r - q_l = \alpha^1 \begin{bmatrix} -Z_l \\ 1 \end{bmatrix} + \alpha^2 \begin{bmatrix} Z_r \\ 1 \end{bmatrix} = \mathcal{W}^1 + \mathcal{W}^2$$

The waves propagate with speeds $s^1 = -c_l$ and $s^2 = c_r$.

Wave propagation in heterogeneous medium

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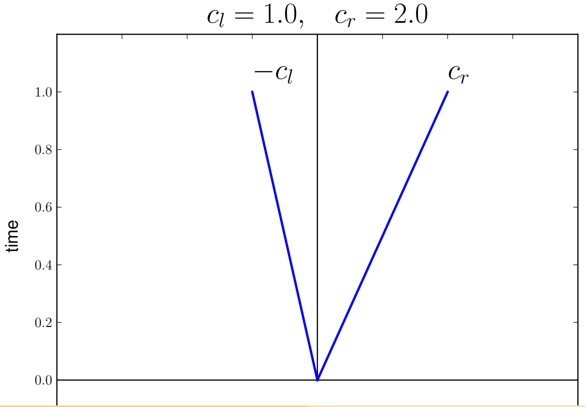

Riemann problem for interface

$$q_r - q_\ell = \alpha^1 \begin{bmatrix} -Z_\ell \\ 1 \end{bmatrix} + \alpha^2 \begin{bmatrix} Z_r \\ 1 \end{bmatrix}.$$

gives the linear system

$$R_{\ell r}\, \alpha = q_r - q_\ell,$$

where

$$R_{\ell r} = \begin{bmatrix} -Z_\ell & Z_r \\ 1 & 1 \end{bmatrix} \qquad \Longrightarrow \; R_{\ell r}^{-1} = \frac{1}{Z_\ell + Z_r} \begin{bmatrix} -1 & Z_r \\ 1 & Z_\ell \end{bmatrix}$$

So

$$\begin{bmatrix} \alpha^1 \\ \alpha^2 \end{bmatrix} = \frac{1}{Z_\ell + Z_r} \begin{bmatrix} -1 & Z_r \\ 1 & Z_\ell \end{bmatrix} \begin{bmatrix} p_r - p_\ell \\ u_r - u_\ell \end{bmatrix}.$$

2-wave hitting interface as a Riemann problem

Incident wave:

$$q_r - q_\ell = \beta r_\ell^2 = \beta \begin{bmatrix} Z_\ell \\ 1 \end{bmatrix},$$

then Riemann solution gives

$$\begin{aligned}
\alpha &= R_{lr}^{-1}(q_r - q_\ell) \\
&= \frac{\beta}{Z_\ell + Z_r} \begin{bmatrix} -1 & Z_r \\ 1 & Z_\ell \end{bmatrix} \begin{bmatrix} Z_\ell \\ 1 \end{bmatrix} \\
&= \frac{\beta}{Z_\ell + Z_r} \begin{bmatrix} Z_r - Z_\ell \\ 2Z_\ell \end{bmatrix}.
\end{aligned}$$

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$$\Longrightarrow \alpha^1 = \left(\frac{Z_r - Z_\ell}{Z_\ell + Z_r}\right)\beta \quad \text{and} \quad \alpha^2 = \left(\frac{2Z_\ell}{Z_\ell + Z_r}\right)\beta$$

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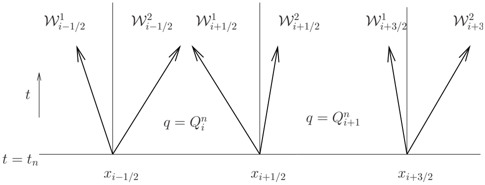
Pressure jump in reflected wave: $c_R \beta Z_\ell$

Pressure jump in transmitted wave: $c_T \beta Z_\ell$

Godunov's method — variable coefficient acoustics



$$
\begin{aligned}
Q_i - Q_{i-1} &= \begin{bmatrix} p_i - p_{i-1} \\ u_i - u_{i-1} \end{bmatrix} \\
&= \alpha^1_{i-1/2} \begin{bmatrix} -\rho_{i-1} c_{i-1} \\ 1 \end{bmatrix} + \alpha^2_{i-1/2} \begin{bmatrix} \rho_i c_i \\ 1 \end{bmatrix} \\
&= \alpha^1_{i-1/2} r^1_{i-1} + \alpha^2_{i-1/2} r^2_i \\
&= \mathcal{W}^1_{i-1/2} + \mathcal{W}^2_{i-1/2}
\end{aligned}
$$

2D Acoustics in Heterogeneous Media

$$q_t + A(x,y)q_x + B(x,y)q_y = 0,$$

$$q = \begin{bmatrix} p \\ u \\ v \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & K(x,y) & 0 \\ 1/\rho(x,y) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & K(x,y) \\ 0 & 0 & 0 \\ 1/\rho(x,y) & 0 & 0 \end{bmatrix}.$$

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Riemann problem in x:

$$\mathcal{W}^1 = \alpha^1 \begin{bmatrix} -Z_{i-1,j} \\ 1 \\ 0 \end{bmatrix}, \qquad \mathcal{W}^2 = \alpha^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathcal{W}^3 = \alpha^3 \begin{bmatrix} Z_{ij} \\ 1 \\ 0 \end{bmatrix},$$

$$\begin{aligned} \alpha^1 &= (-\Delta Q^1 + Z_{ij}\Delta Q^2)/(Z_{i-1,j} + Z_{ij}), \\ \alpha^2 &= \Delta Q^3, \\ \alpha^3 &= (\Delta Q^1 + Z_{i-1,j}\Delta Q^2)/(Z_{i-1,j} + Z_{ij}). \end{aligned}$$

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$$\begin{aligned} \alpha^1 &= (-\Delta Q^1 + Z_{ij}\Delta Q^2)/(Z_{i-1,j} + Z_{ij}), \\ \alpha^2 &= \Delta Q^3, \\ \alpha^3 &= (\Delta Q^1 + Z_{i-1,j}\Delta Q^2)/(Z_{i-1,j} + Z_{ij}). \end{aligned}$$

Wave speeds: $s^1 = -c_{i-1,j}, \quad s^2 = 0, \quad s^3 = c_{ij}$

Only need to propagate and apply limiters to $\mathcal{W}^1,\ \mathcal{W}^3$.

Wave propagation algorithms in 2D

Clawpack requires:

Normal Riemann solver `rpn2.f`
Solves 1d Riemann problem $q_t + Aq_x = 0$
Decomposes $\Delta Q = Q_{ij} - Q_{i-1,j}$ into $\mathcal{A}^+\Delta Q$ and $\mathcal{A}^-\Delta Q$.
For $q_t + Aq_x + Bq_y = 0$, split using eigenvalues, vectors:

$$A = R\Lambda R^{-1} \implies A^- = R\Lambda^- R^{-1}, A^+ = R\Lambda^+ R^{-1}$$

Input parameter `ixy` determines if it's in x or y direction.
In latter case splitting is done using B instead of A.
This is all that's required for dimensional splitting.

Wave propagation algorithms in 2D

Clawpack requires:

Normal Riemann solver `rpn2.f`
Solves 1d Riemann problem $q_t + Aq_x = 0$
Decomposes $\Delta Q = Q_{ij} - Q_{i-1,j}$ into $\mathcal{A}^+\Delta Q$ and $\mathcal{A}^-\Delta Q$.
For $q_t + Aq_x + Bq_y = 0$, split using eigenvalues, vectors:

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Input parameter `ixy` determines if it's in x or y direction.
In latter case splitting is done using B instead of A.
This is all that's required for dimensional splitting.

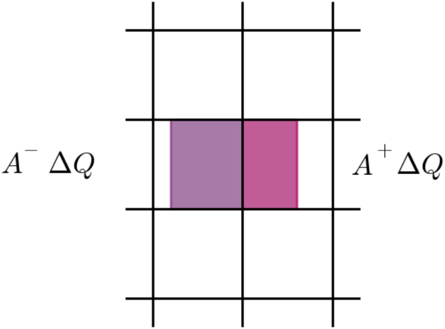
Transverse Riemann solver `rpt2.f`
Decomposes $\mathcal{A}^+\Delta Q$ into $\mathcal{B}^-\mathcal{A}^+\Delta Q$ and $\mathcal{B}^+\mathcal{A}^+\Delta Q$ by splitting this vector into eigenvectors of B.

(Or splits vector into eigenvectors of A if `ixy=2`.)

Wave propagation algorithm for $q_t + Aq_x + Bq_y = 0$

Decompose $A = A^+ + A^-$ and $B = B^+ + B^-$.

For $\Delta Q = Q_{ij} - Q_{i-1,j}$:



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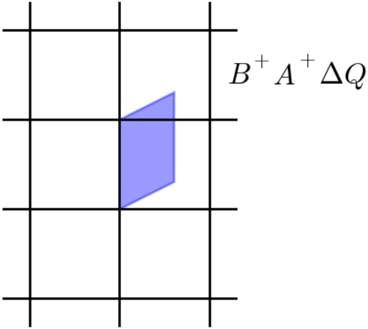
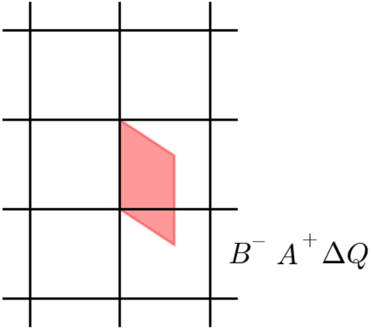


Wave propagation algorithm for $q_t + Aq_x + Bq_y = 0$

Decompose $A = A^+ + A^-$ and $B = B^+ + B^-$.

For $\Delta Q = Q_{ij} - Q_{i-1,j}$:



Wave propagation algorithm for $q_t + Aq_x + Bq_y = 0$

Decompose $A = A^+ + A^-$ and $B = B^+ + B^-$.

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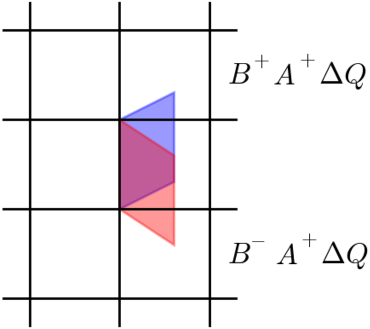
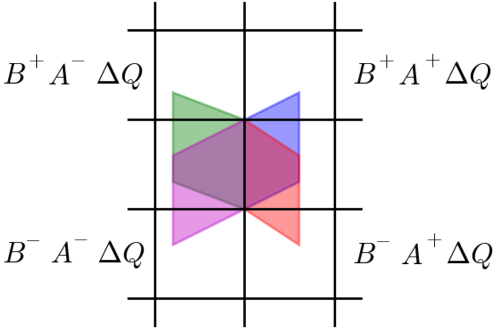


Wave propagation algorithm for $q_t + Aq_x + Bq_y = 0$

Decompose $A = A^+ + A^-$ and $B = B^+ + B^-$.

For $\Delta Q = Q_{ij} - Q_{i-1,j}$:



Transverse solver for 2D Acoustics

Solving Riemann problem in x gives waves and fluctuations
$\mathcal{A}^-\Delta Q_{i-1/2,j}$, $\mathcal{A}^+\Delta Q_{i-1/2,j}$.

For $\mathcal{B}^-\mathcal{A}^+\Delta Q_{i-1/2,j}$ we want downward-going part of $\mathcal{A}^+\Delta Q_{i-1/2,j}$,
(partly transmitted an partly reflected at y-interface)

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$$\mathcal{A}^+\Delta Q_{i-1/2,j} = \beta^1 \begin{bmatrix} -Z_{i,j-1} \\ 0 \\ 1 \end{bmatrix} + \beta^2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \beta^3 \begin{bmatrix} Z_{ij} \\ 0 \\ 1 \end{bmatrix},$$

with speeds $-c_{i,j-1},\ 0,\ c_{ij}$ respectively.

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with speeds $-c_{i,j-1},\ 0,\ c_{ij}$ respectively.

Only use downward-going part:

$$\beta^1 = \left(-(\mathcal{A}^+\Delta Q_{i-1/2,j})^1 + (\mathcal{A}^+\Delta Q_{i-1/2,j})^3 Z_{ij}\right) / (Z_{i,j-1} + Z_{ij}),$$

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$$\mathcal{B}^-\mathcal{A}^+\Delta Q_{i-1/2,j} = -c_{i,j-1}\beta^1 \begin{bmatrix} -Z_{i,j-1} \\ 0 \\ 1 \end{bmatrix}$$

Transverse solver for 2D Acoustics

Solving Riemann problem in x gives waves and fluctuations $\mathcal{A}^-\Delta Q_{i-1/2,j},\ \mathcal{A}^+\Delta Q_{i-1/2,j}$.

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Transverse solver for 2D Acoustics

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$$\mathcal{A}^+\Delta Q_{i-1/2,j} = \beta^1 \begin{bmatrix} -Z_{ij} \\ 0 \\ 1 \end{bmatrix} + \beta^2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \beta^3 \begin{bmatrix} Z_{i,j+1} \\ 0 \\ 1 \end{bmatrix},$$

with speeds $-c_{ij},\ 0,\ c_{i,j+1}$ respectively.

Transverse solver for 2D Acoustics

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with speeds $-c_{ij},\ 0,\ c_{i,j+1}$ respectively.

Only use upward-going part:

$$\beta^3 = \left((\mathcal{A}^+\Delta Q_{i-1/2,j})^1 + (\mathcal{A}^+\Delta Q_{i-1/2,j})^3 Z_{i,j+1}\right) / (Z_{ij} + Z_{i,j+1})$$

Transverse solver for 2D Acoustics

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with speeds $-c_{ij},\ 0,\ c_{i,j+1}$ respectively.

Only use upward-going part:

$$\beta^3 = \left((\mathcal{A}^+\Delta Q_{i-1/2,j})^1 + (\mathcal{A}^+\Delta Q_{i-1/2,j})^3 Z_{i,j+1}\right) / (Z_{ij} + Z_{i,j+1})$$

$$\mathcal{B}^+\mathcal{A}^+\Delta Q_{i-1/2,j} = c_{i,j+1}\beta^3 \begin{bmatrix} Z_{i,j+1} \\ 0 \\ 1 \end{bmatrix}$$

Cell averaging material parameters

To solve a variable coefficient problem on a grid,
need to average material parameters onto grid cell.

For acoustics with $\rho(x, y),\ K(x, y)$, on Cartesian grid:

Can use mean value of density:

$$\rho_{ij} = \frac{1}{\Delta x \Delta y} \iint \rho(x, y)\, dx, dy$$

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But need to use harmonic average of bulk modulus:

$$K_{ij} = \left(\frac{1}{\Delta x \Delta y} \iint \frac{1}{K(x,y)}\, dx, dy \right)^{-1}$$

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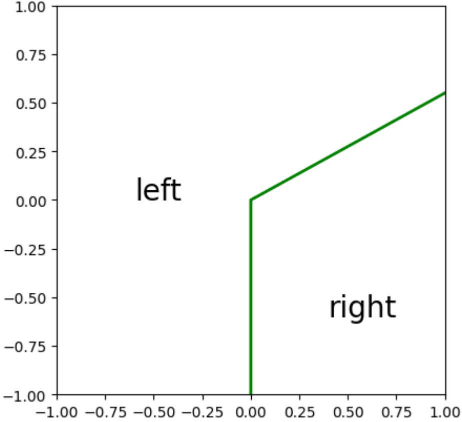
But need to use harmonic average of bulk modulus:

$$K_{ij} = \left(\frac{1}{\Delta x \Delta y} \iint \frac{1}{K(x, y)}\, dx, dy \right)^{-1}$$

Then $\quad c_{ij} = \sqrt{K_{ij}/\rho_{ij}}, \quad Z_{ij} = \sqrt{K_{ij}\rho_{ij}}$

Acoustic wave hitting an interface in 2D

Example from Figure 21.1:

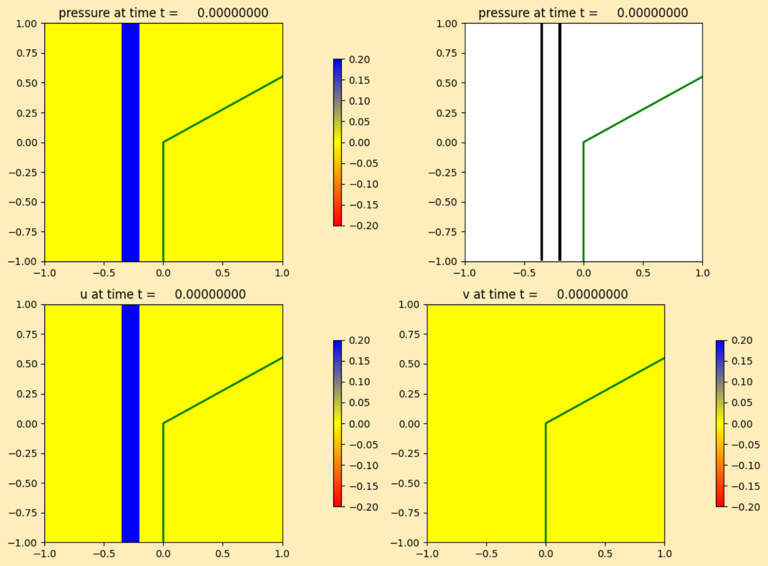


$$\begin{aligned} \rho_\ell &= 1 & \rho_r &= 1 \\ K_\ell &= 1 & K_r &= 0.25 \\ c_\ell &= 1 & c_r &= 0.5 \\ Z_\ell &= 1 & Z_r &= 0.5 \end{aligned}$$

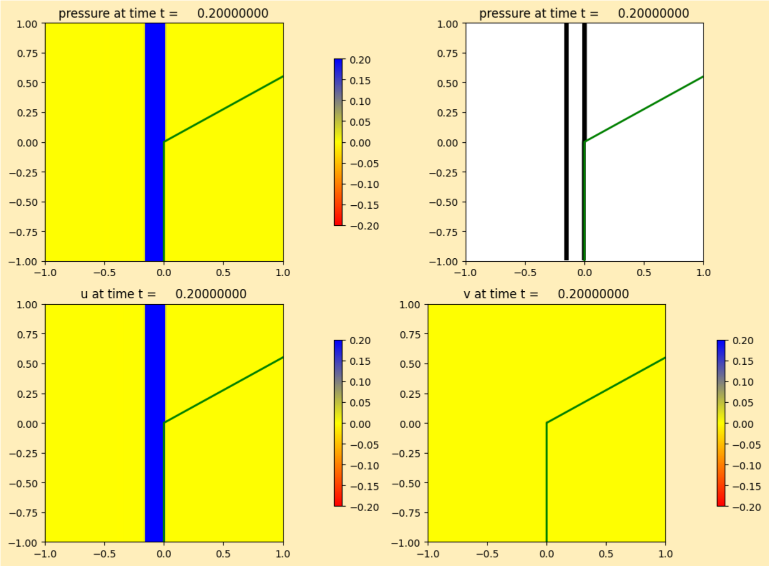
$$\begin{aligned} C_T &= \tfrac{2Z_r}{Z_\ell+Z_r} \\ &= 2/3 \end{aligned}$$

$$\begin{aligned} C_R &= \tfrac{Z_r-Z_\ell}{Z_\ell+Z_r} \\ &= -1/3 \end{aligned}$$

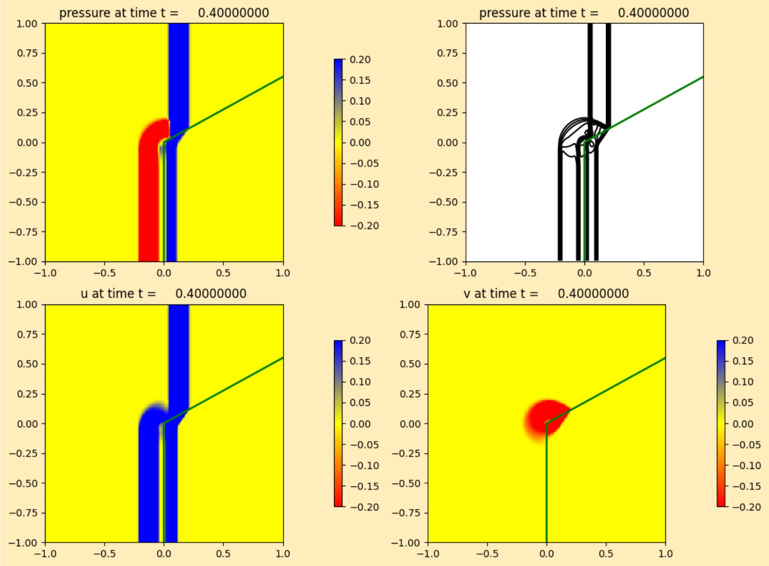
Acoustic wave hitting an interface in 2D



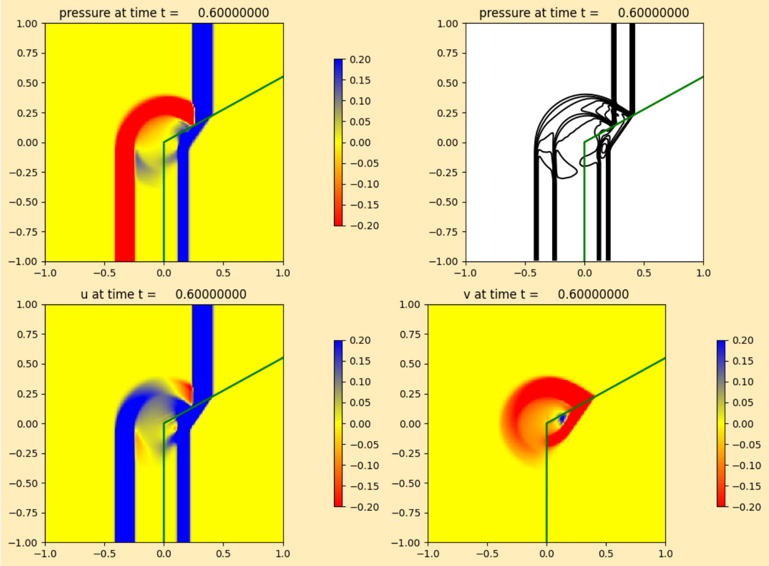
Acoustic wave hitting an interface in 2D



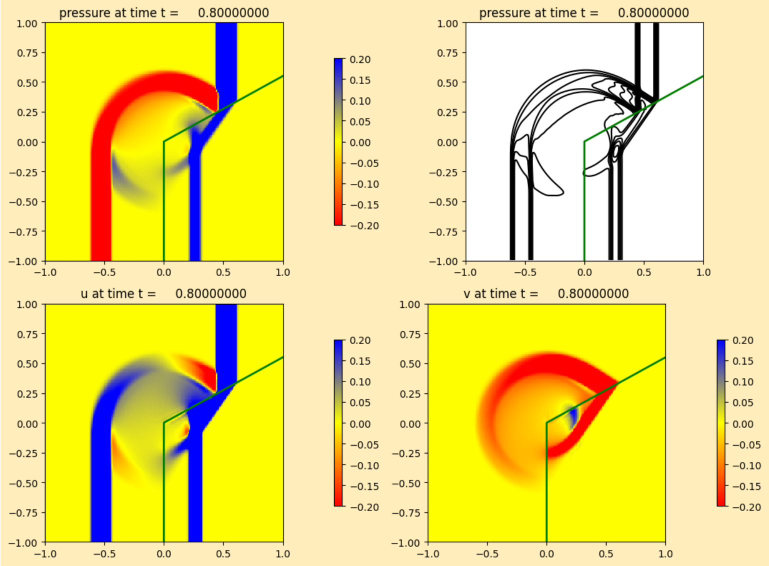
Acoustic wave hitting an interface in 2D



Acoustic wave hitting an interface in 2D

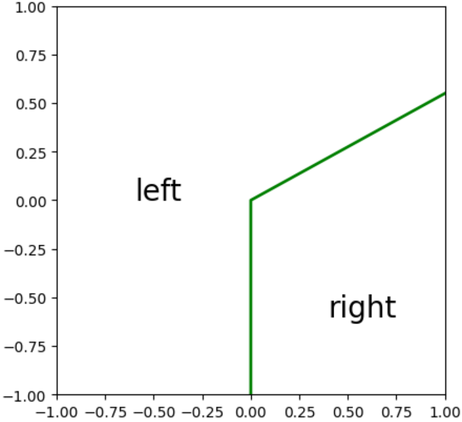


Acoustic wave hitting an interface in 2D



Acoustic wave hitting an interface in 2D

With nearly-incompressible material on right ($\approx$ solid wall)



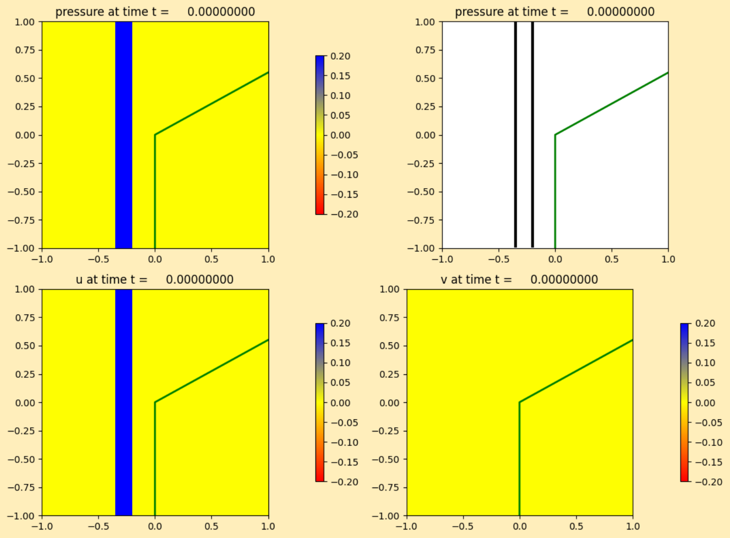
$$\begin{aligned} \rho_\ell &= 1 & \rho_r &= 10^4 \\ K_\ell &= 1 & K_r &= 10^{-8} \\ c_\ell &= 1 & c_r &= 10^{-6} \\ Z_\ell &= 1 & Z_r &= 0.01 \end{aligned}$$

$$\begin{aligned} C_T &= \tfrac{2Z_r}{Z_\ell+Z_r} \\ &\approx 0.02 \end{aligned}$$

$$\begin{aligned} C_R &= \tfrac{Z_r-Z_\ell}{Z_\ell+Z_r} \\ &\approx -0.98 \end{aligned}$$

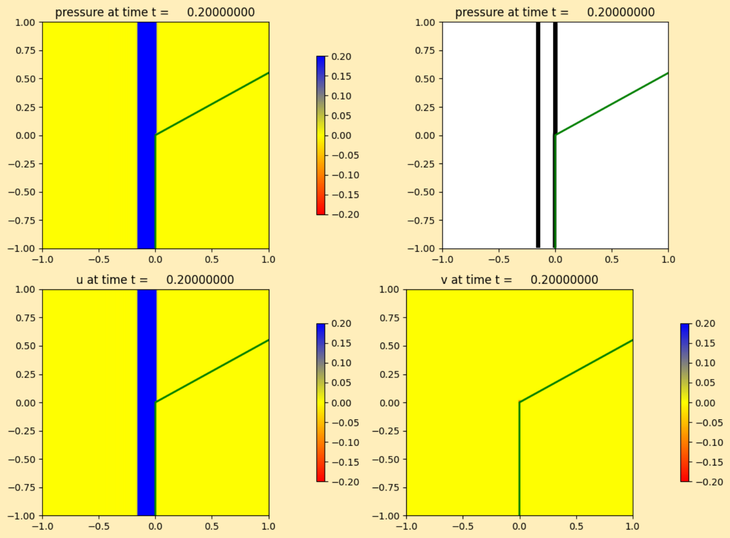
Acoustic wave hitting an interface in 2D

With nearly-incompressible material on right ($\approx$ solid wall)



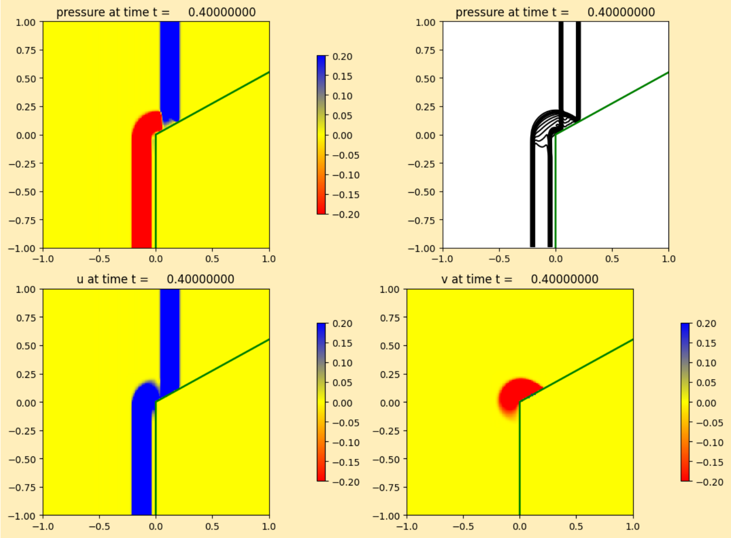
Acoustic wave hitting an interface in 2D

With nearly-incompressible material on right (≈ solid wall)



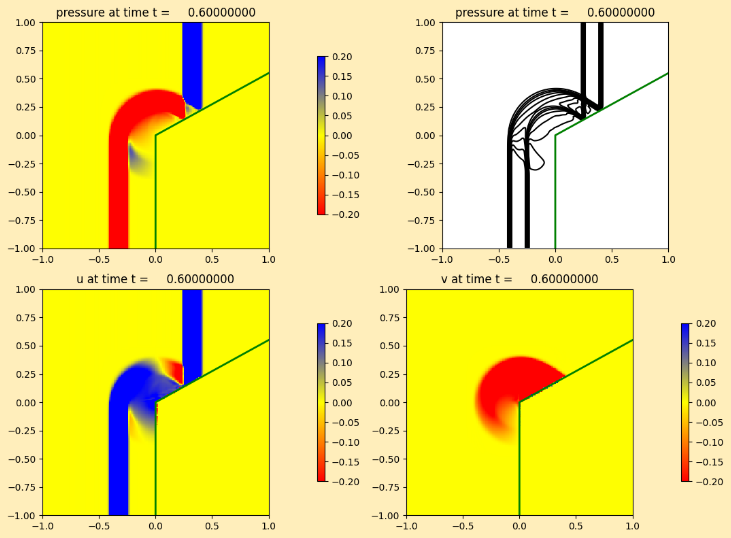
Acoustic wave hitting an interface in 2D

With nearly-incompressible material on right ($\approx$ solid wall)



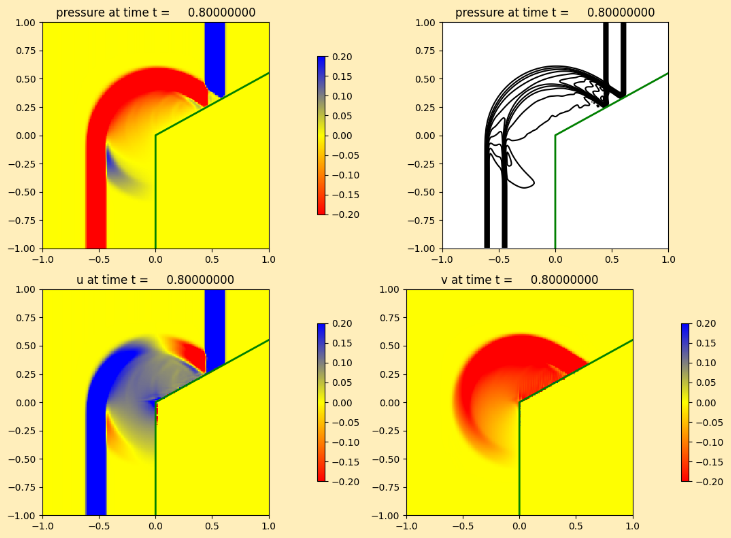
Acoustic wave hitting an interface in 2D

With nearly-incompressible material on right ($\approx$ solid wall)

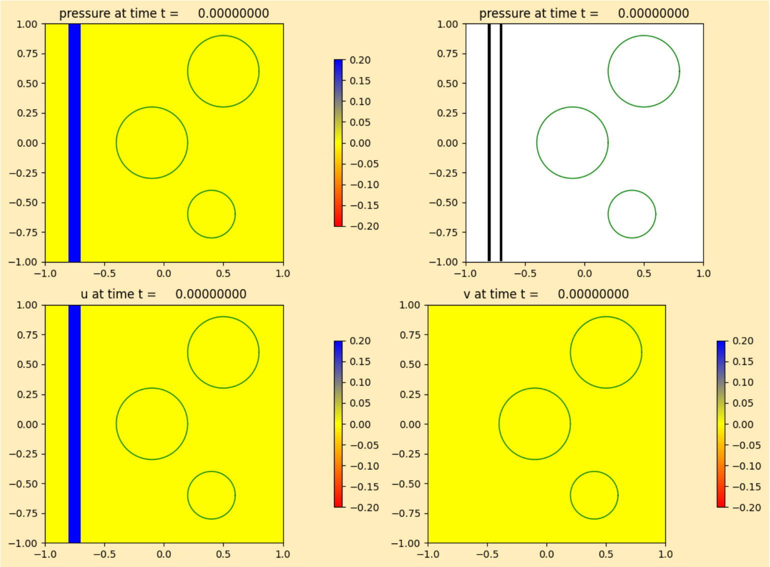


Acoustic wave hitting an interface in 2D

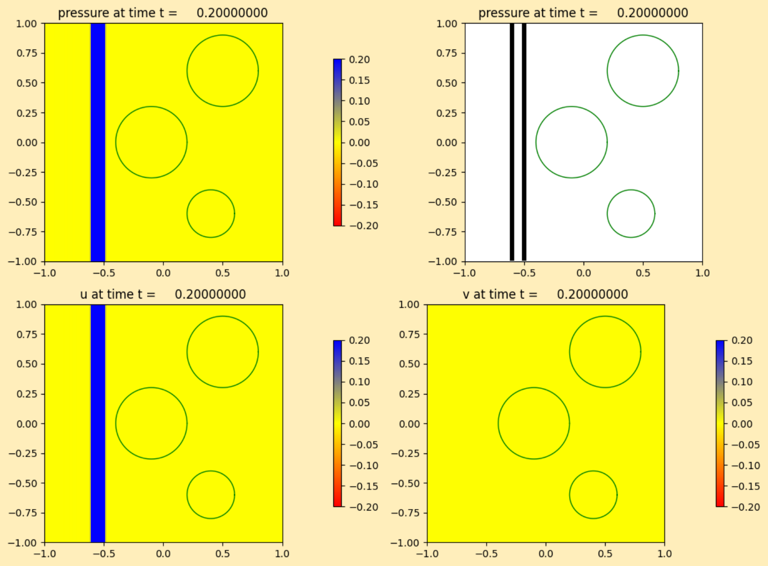
With nearly-incompressible material on right ($\approx$ solid wall)



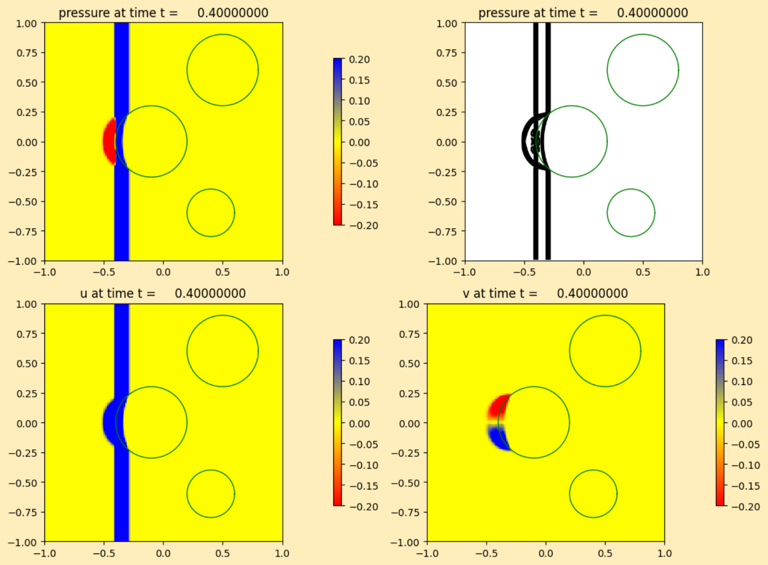
Acoustic wave hitting circular inclusions



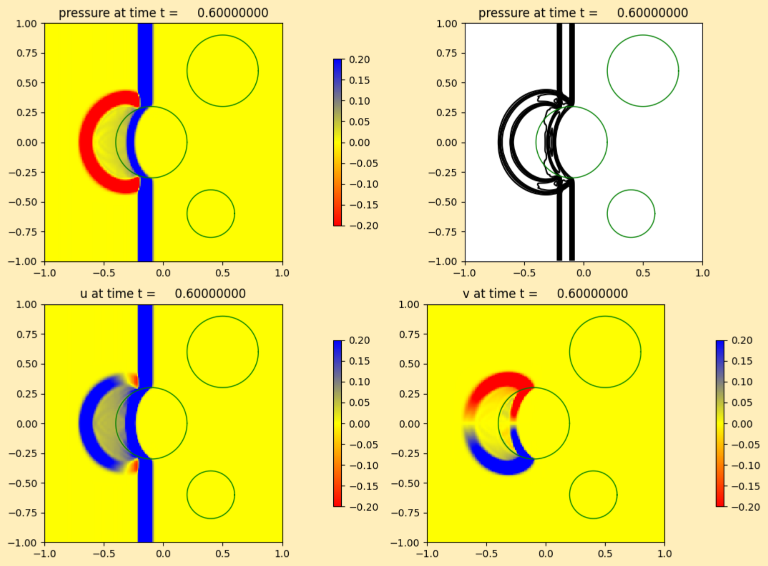
Acoustic wave hitting circular inclusions



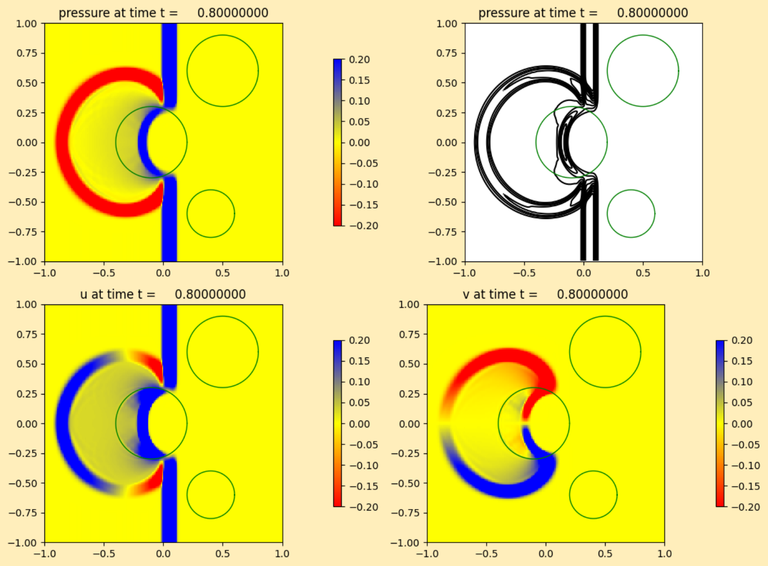
Acoustic wave hitting circular inclusions



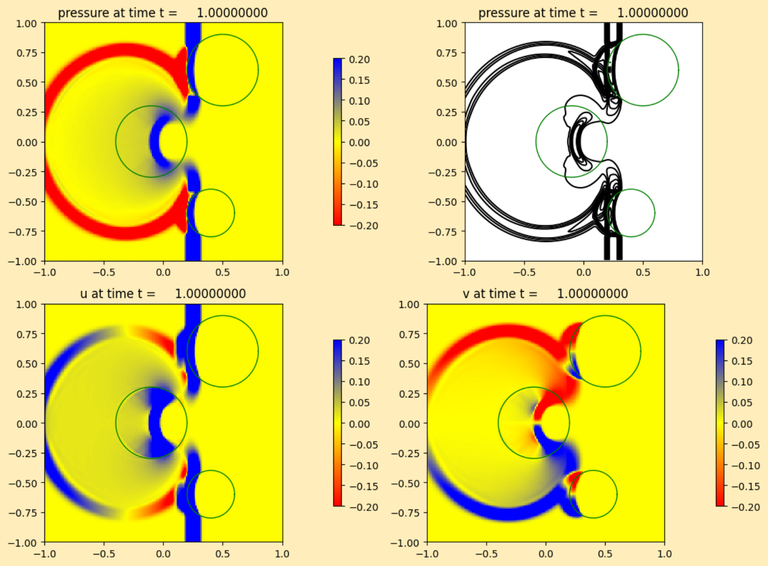
Acoustic wave hitting circular inclusions



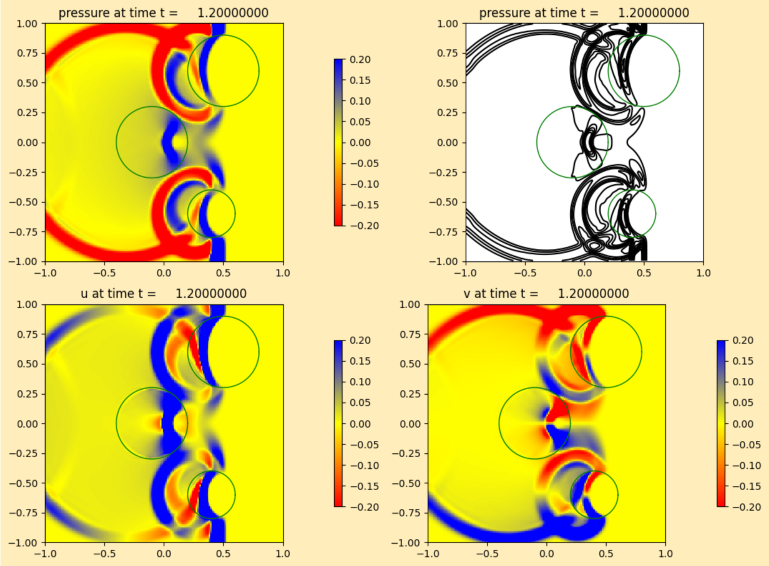
Acoustic wave hitting circular inclusions



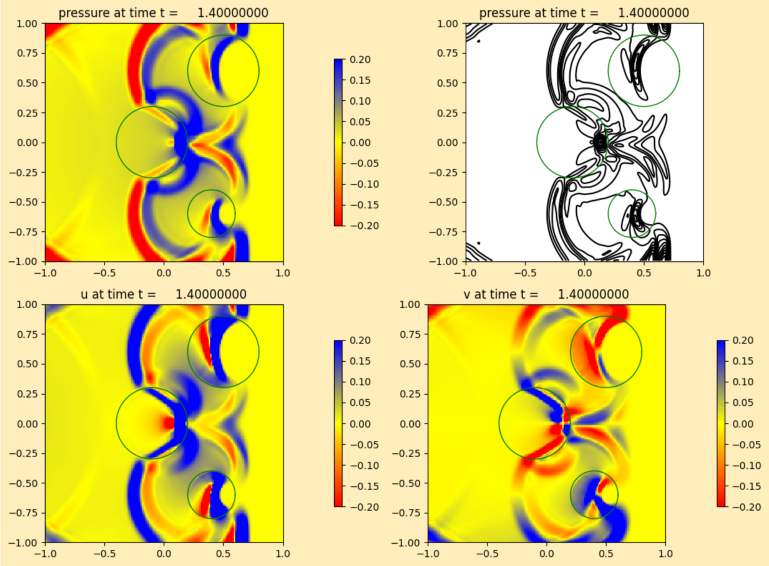
Acoustic wave hitting circular inclusions



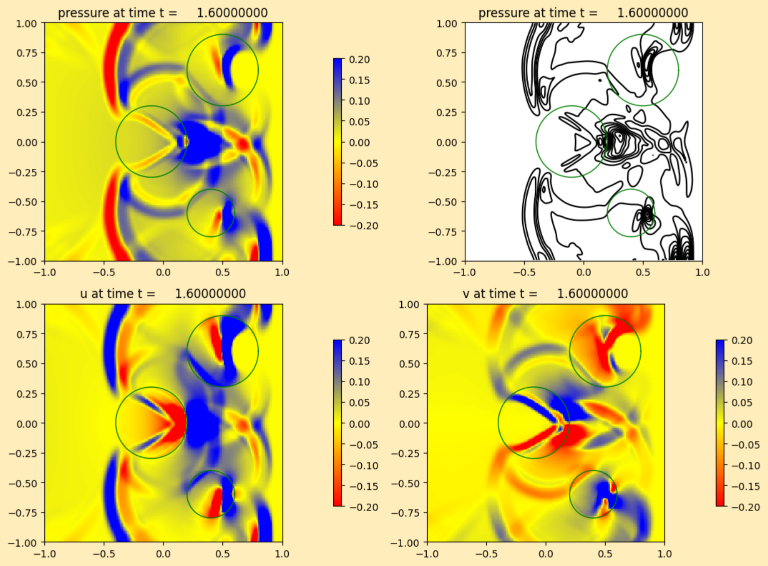
Acoustic wave hitting circular inclusions



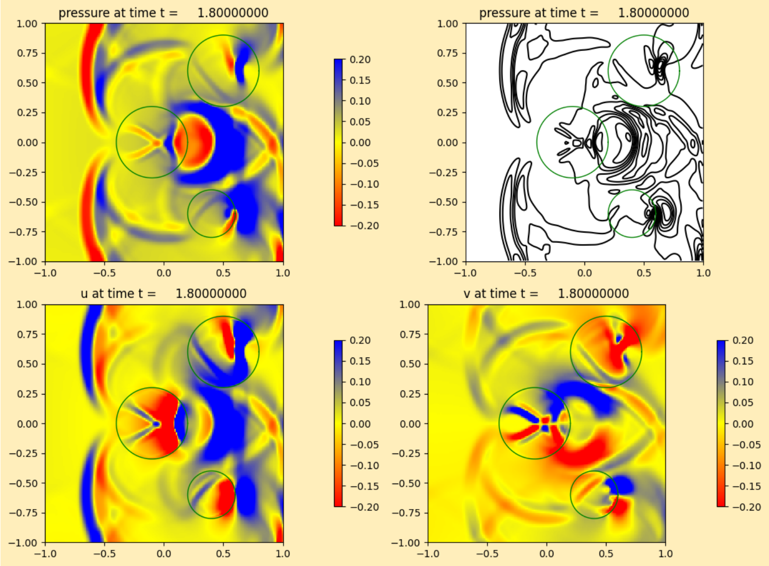
Acoustic wave hitting circular inclusions



Acoustic wave hitting circular inclusions



Acoustic wave hitting circular inclusions



Acoustic wave hitting circular inclusions