Finite Volume Methods for Hyperbolic Problems

Nonlinear Systems

Rarefaction Waves and Integral Curves

- Integral curves
- Genuine nonlinearity and rarefaction waves
- General Riemann solution for shallow water
- Riemann invariants
- Linear degeneracy and contact discontinuities

Shallow water equations

$$h_t + (hu)_x = 0 \implies h_t + \mu_x = 0$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0 \implies \mu_t + \phi(h,\mu)_x = 0$$

where $\mu = hu$ and $\phi = hu^2 + \frac{1}{2}gh^2 = \mu^2/h + \frac{1}{2}gh^2$.

Jacobian matrix:

$$f'(q) = \begin{bmatrix} \partial\mu/\partial h & \partial\mu/\partial\mu \\ \partial\phi/\partial h & \partial\phi/\partial\mu \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix},$$

Eigenvalues:

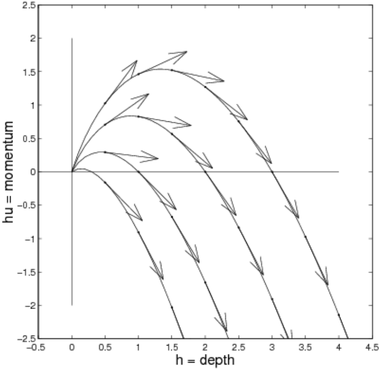
$$\lambda^1 = u - \sqrt{gh}, \qquad \lambda^2 = u + \sqrt{gh}.$$

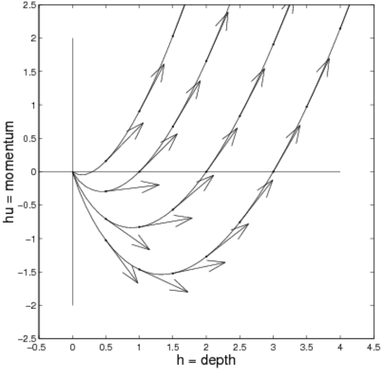
Eigenvectors:

$$r^1 = \begin{bmatrix} 1 \\ u - \sqrt{gh} \end{bmatrix}, \qquad r^2 = \begin{bmatrix} 1 \\ u + \sqrt{gh} \end{bmatrix}.$$

Integral curves of r^p

Curves in phase plane that are tangent to $r^p(q)$ at each q.



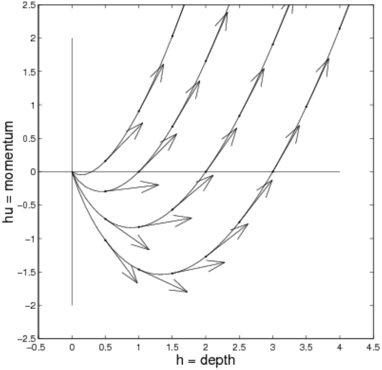


$\tilde{q}(\xi)$: curve through phase space parameterized by $\xi \in \mathbb{R}$.

Satisfying $\tilde{q}'(\xi) = \alpha(\xi) r^p(\tilde{q}(\xi))$ for some scalar $\alpha(\xi)$.

In a simple wave, the values $q(x,t)$ always lie along a single integral curve in some particular pth family.

As initial data, can choose arbitrary smooth $h(x,0)$,
but then $u(x,0)$ is determined.



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Integral curve parameterized by $\tilde{q}(\xi)$.

So $q(x,t) = \tilde{q}(\xi(x,t))$ for some $\xi(x,t)$.

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Not any $\xi(x,t)$ works. When is the PDE satisfied?

Assuming smooth, require $q_t + f'(q)q_x = 0$:

$$q_t(x,t) = \tilde{q}'(\xi(x,t))\xi_t(x,t)$$
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So $q_t + f'(q)q_x = 0 \implies$

$$[\xi_t(x,t) + \lambda^p(\tilde{q}(\xi(x,t)))\,\xi_x(x,t)]\,\tilde{q}'(\xi(x,t)) = 0.$$

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This is a scalar equation and $\tilde{q}(\xi(x,t))$ is constant along characteristic curves $X'(t) = \lambda^p(\tilde{q}(\xi(x,t)))$ as long as the solution stays smooth.

Converging characteristics $\implies$ shock formation.

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Once a shock forms, no longer a simple wave in general (waves in other families can be generated).

Centered rarefaction waves

Similarity solution with piecewise constant initial data:

$$q(x,t) = \begin{cases} q_\ell & \text{if } x/t \le \lambda^p(q_\ell) \\ \tilde{q}(x/t) & \text{if } \lambda^p(q_\ell) \le x/t \le \lambda^p(q_r) \\ q_r & \text{if } x/t \ge \lambda^p(q_r), \end{cases}$$

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Special case of simple wave with $\xi(x,t) = x/t$.
Then $\xi_t(x,t) + \lambda^p(\tilde{q}(\xi(x,t)))\,\xi_x(x,t) = 0$ becomes

$$-\frac{x}{t^2} + \lambda^p(\tilde{q}(x/t))\frac{1}{t} = 0 \qquad \Longrightarrow \; \lambda^p(\tilde{q}(x/t)) = x/t$$

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So we need to solve $\lambda^p(\tilde{q}(\xi)) = \xi$ for $\tilde{q}(\xi)$.

Generalizes the equation $f'(\tilde{q}(\xi)) = \xi$ for scalar PDE.

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Required so that characteristics spread out as time advances.

Also want $\lambda^p(q)$ monotonically increasing from q_ℓ to q_r.

Genuine nonlinearity: generalization of convexity for scalar flux.

Genuine nonlinearity

For scalar problem $q_t + f(q)_x = 0$, want $f''(q) \neq 0 \ \ \forall q$ of interest.

This implies that $f'(q)$ is monotonically increasing or decreasing between q_l and q_r.

Shock if decreasing,   Rarefaction if increasing.

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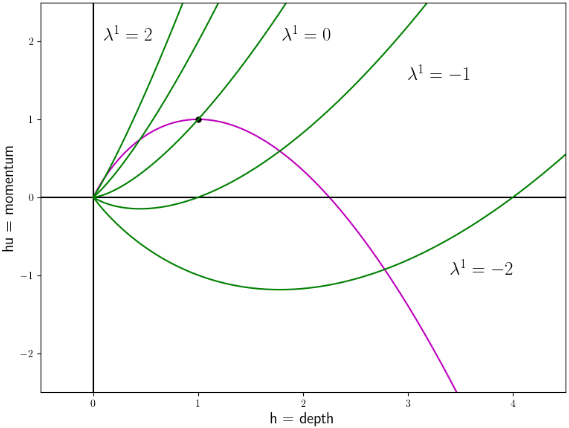
This requires: $\nabla\lambda^p(q) \cdot r^p(q) \neq 0$ for all q in region of interest.

since

$$\frac{d}{d\xi}\lambda^p(\tilde{q}(\xi)) = \nabla\lambda^p(\tilde{q}(\xi)) \cdot \tilde{q}'(\xi).$$

Integral curve for one particular q_*

Green curves are contours of $\lambda^1 = u - \sqrt{gh}$



Note: Increases monotonically in one direction along integral curve.

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Genuine nonlinearity of shallow water equations

1-waves: Requires $\nabla\lambda^1(q)\cdot r^1(q) \neq 0$.

$$\lambda^1 = u - \sqrt{gh} = q^2/q^1 - \sqrt{gq^1},$$

$$\nabla\lambda^1 = \begin{bmatrix} -q^2/(q^1)^2 - \frac{1}{2}\sqrt{g/q^1} \\ 1/q^1 \end{bmatrix} = \begin{bmatrix} -u/h - \frac{1}{2}\sqrt{g/h} \\ 1/h \end{bmatrix}$$

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and hence

$$\nabla\lambda^1 \cdot r^1 = -\frac{3}{2}\sqrt{g/q^1} = -\frac{3}{2}\sqrt{g/h}$$
$$< 0 \quad \text{for all} \;\; h > 0.$$

1-waves: integral curves of r^1

$\tilde{q}(\xi)$: curve through phase space parameterized by $\xi \in \mathbb{R}$.

Satisfies $\tilde{q}'(\xi) = \alpha(\xi) r^1(\tilde{q}(\xi))$ for some scalar $\alpha(\xi)$.

Choose $\alpha(\xi) \equiv 1$ and obtain

$$\left[\begin{array}{c} (\tilde{q}^1)' \\ (\tilde{q}^2)' \end{array} \right] = \tilde{q}'(\xi) = r^1(\tilde{q}(\xi)) = \left[\begin{array}{c} 1 \\ \tilde{q}^2/\tilde{q}^1 - \sqrt{g\tilde{q}^1} \end{array} \right]$$

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Require $\tilde{q}^2(h_*) = h_* u_* \implies$

$$\tilde{q}^2(\xi) = \xi u_* + 2\xi\left(\sqrt{gh_*} - \sqrt{g\xi}\right).$$

1-wave integral curves of r^p

So

$$\tilde{q}^1(\xi) = \xi,$$
$$\tilde{q}^2(\xi) = \xi u_* + 2\xi \left(\sqrt{gh_*} - \sqrt{g\xi}\right).$$

and hence integral curve through $(h_*,\ h_* u_*)$ satisfies

$$hu = hu_* + 2h\left(\sqrt{gh_*} - \sqrt{gh}\right) \quad \text{for } 0 < h < \infty.$$

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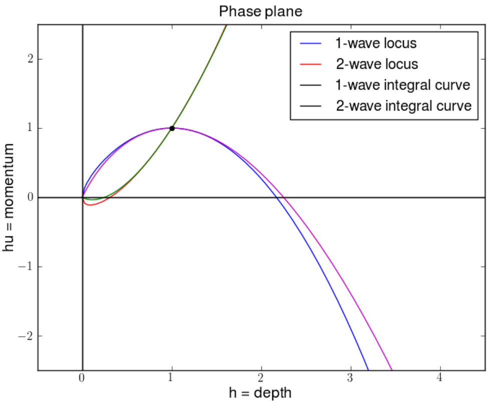
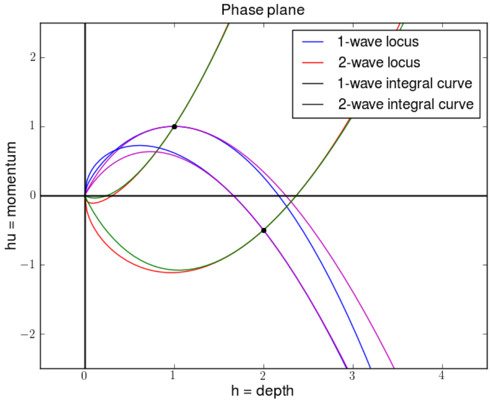
Similarly, 2-wave integral curve through $(h_*,\ h_*u_*)$ satisfies

$$hu = hu_* - 2h\left(\sqrt{gh_*} - \sqrt{gh}\right).$$

Integral curves of r^p versus Hugoniot loci



Solving the shallow water Riemann problem

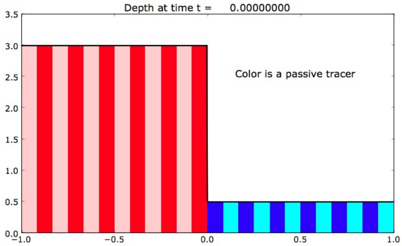


Solution to Riemann problem depends on which state is q_l, q_r.

Also need to choose correct curve from each state.

Dam break problem for shallow water equations

$$h_t + (hu)_x = 0$$
$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$



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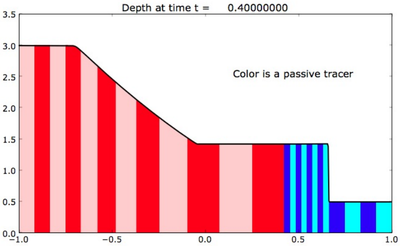

Solving the dam break Riemann problem

$h_\ell > h_r$ and $u_\ell = u_r = 0 \implies$ 1-rarefaction and 2-shock

So the intermediate state q_m lies on:

1-wave integral curve through q_ℓ, and on

2-wave Hugoniot locus through q_r.

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$$u_m = u_l + 2\left(\sqrt{gh_l} - \sqrt{gh_m}\right)$$

and

$$u_m = u_r + (h_m - h_r)\sqrt{\frac{g}{2}\left(\frac{1}{h_m} + \frac{1}{h_r}\right)}$$

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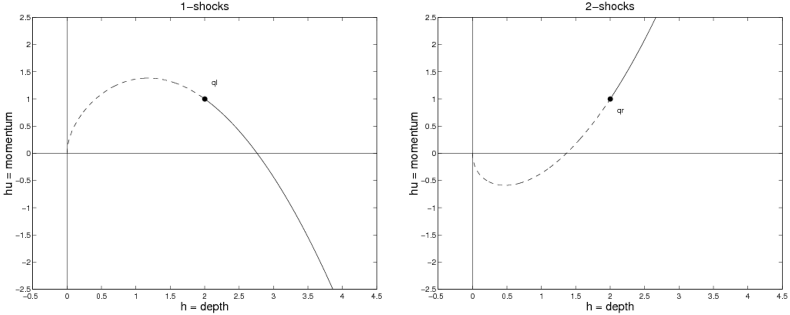
Equate to obtain a single nonlinear equation for h_m:

$$u_l + 2\left(\sqrt{gh_l} - \sqrt{gh_m}\right) = u_r + (h_m - h_r)\sqrt{\frac{g}{2}\left(\frac{1}{h_m} + \frac{1}{h_r}\right)}$$

Hugoniot locus for shallow water

States that can be connected to the given state by a 1-wave or 2-wave satisfying the R-H conditions:



Solid portion: states that can be connected by shock satisfying entropy condition.

Dashed portion: states that can be connected with R-H condition satisfied but not the physically correct solution.

Solving the general Riemann problem

For general data $q_\ell,\ q_r$, the shallow water Riemann solution could have a shock or rarefaction in each family.

Use the fact that across a shock we always expect deeper water "behind" the shock to define 1-wave curve through q_ℓ:

$$\phi_\ell(h) = \begin{cases} u_\ell + 2\left(\sqrt{gh_\ell} - \sqrt{gh}\right) & \text{if } h < h_\ell \\ u_\ell - (h - h_\ell)\sqrt{\frac{g}{2}\left(\frac{1}{h} + \frac{1}{h_\ell}\right)} & \text{if } h \geq h_\ell \end{cases}$$

and 2-wave curve through q_r:

$$\phi_r(h) = \begin{cases} u_r - 2\left(\sqrt{gh_r} - \sqrt{gh}\right) & \text{if } h < h_r \\ u_r + (h - h_r)\sqrt{\frac{g}{2}\left(\frac{1}{h} + \frac{1}{h_r}\right)} & \text{if } h \geq h_r \end{cases}$$

Then determine h_m by using a numerical root finder on

$$\phi(h) = \phi_\ell(h) - \phi_r(h).$$

Riemann invariants

Along a 1-wave integral curve,

$$u = u_* + 2\left(\sqrt{gh_*} - \sqrt{gh}\right)$$

and hence

$$u + 2\sqrt{gh} = u_* + 2\sqrt{gh_*}.$$

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So at every point on the integral curve through (h_*, h_*u_*)

$$w^1(q) = u + 2\sqrt{gh}$$

has the constant value $w^1(q) \equiv w^1(q_*) = u_* + 2\sqrt{gh_*}$.

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The function $w^1(q)$ is a 1-Riemann invariant for this system.

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$$w^1(q) = u + 2\sqrt{gh}$$

has the constant value $w^1(q) \equiv w^1(q_*) = u_* + 2\sqrt{gh_*}$
at every point on any integral curve of $r^1(q)$.

The integral curves are contour lines of $w^1(q)$.

Riemann invariants

1-Riemann invariants:

$$w^1(q) = u + 2\sqrt{gh}$$

has the constant value $w^1(q) \equiv w^1(q_*) = u_* + 2\sqrt{gh_*}$
at every point on any integral curve of $r^1(q)$.

The integral curves are contour lines of $w^1(q)$.

2-Riemann invariants:

$$w^2(q) = u - 2\sqrt{gh}$$

has the constant value $w^2(q) \equiv w^2(q_*) = u_* - 2\sqrt{gh_*}$
at every point on any integral curve of $r^2(q)$.

Linearly degenerate fields

Scalar advection: $q_t + uq_x = 0$ with $u =$ constant.

Characteristics $X(t) = x_0 + ut$ are parallel.

Discontinuity propagates along a characteristic curve.

Characteristics on either side are parallel so not a shock!

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For a system the analogous property arises if

$$\nabla\lambda^p(q) \cdot r^p(q) \equiv 0$$

holds for all q, in which case

$$\frac{d}{d\xi}\lambda^p(\tilde{q}(\xi)) = \nabla\lambda^p(\tilde{q}(\xi)) \cdot \tilde{q}'(\xi) \equiv 0.$$

So λ^p is constant along each integral curve.

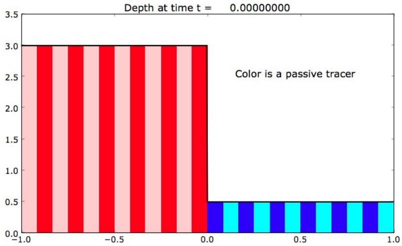
Then pth field is said to be linearly degenerate.

The Riemann problem

Dam break problem for shallow water equations

$$h_t + (hu)_x = 0$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$



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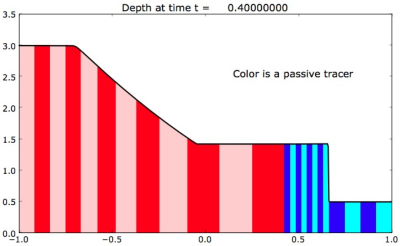

Shallow water with passive tracer

Let $\phi(x,t)$ be tracer concentration and add equation

$$\phi_t + u\phi_x = 0 \implies (h\phi)_t + (uh\phi)_x = 0 \quad \text{(since } h_t + (hu)_x = 0\text{)}.$$

Gives:

$$q = \begin{bmatrix} h \\ hu \\ h\phi \end{bmatrix} = \begin{bmatrix} q^1 \\ q^2 \\ q^3 \end{bmatrix}, \quad f(q) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ uh\phi \end{bmatrix} = \begin{bmatrix} q^2 \\ (q^2)/q^1 + \frac{1}{2}g(q^1)^2 \\ q^2q^3/q^1 \end{bmatrix}.$$

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Jacobian:

$$f'(q) = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + gh & 2u & 0 \\ -u\phi & \phi & u \end{bmatrix}.$$

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$$f'(q) = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + gh & 2u & 0 \\ -u\phi & \phi & u \end{bmatrix}.$$

$$\lambda^1 = u - \sqrt{gh}, \qquad \lambda^2 = u, \qquad \lambda^3 = u + \sqrt{gh},$$

$$r^1 = \begin{bmatrix} 1 \\ u - \sqrt{gh} \\ \phi \end{bmatrix}, \quad r^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad r^3 = \begin{bmatrix} 1 \\ u + \sqrt{gh} \\ \phi \end{bmatrix}.$$

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$$\lambda^2 = u = (hu)/h \implies \nabla\lambda^2 = \begin{bmatrix} -u/h \\ 1/h \\ 0 \end{bmatrix} \implies \lambda^2 \cdot r^2 \equiv 0.$$

So 2nd field is linearly degenerate.

(Fields 1 and 3 are genuinely nonlinear.)