Finite Volume Methods for Hyperbolic Problems

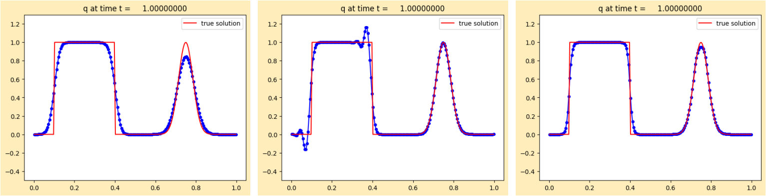
High-Resolution TVD Methods

- Godunov: wave-propagation and REA algorithms
- Extension of REA to piecewise linear
- Relation to Lax-Wendroff, Beam-Warming
- Limiters and minmod
- Monotonicity and Total Variation

Advection tests with periodic BCs

Compare Upwind, Lax-Wendroff, minmod...

With 200 cells:



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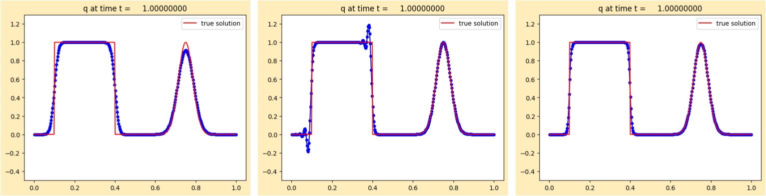

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  Clawpack methods: at best second-order accuracy
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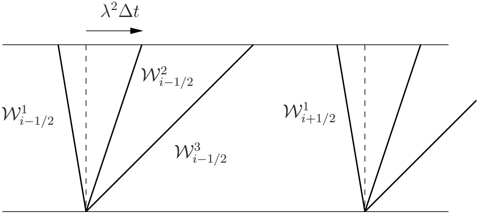
- Godunov-type methods — based on Riemann solvers
  Wave-propagation algorithms with "limiters"

Wave-propagation viewpoint

For linear system $q_t + Aq_x = 0$, the Riemann solution consists of waves $\mathcal{W}^p$ propagating at constant speed λ^p.



$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha^p_{i-1/2} r^p \equiv \sum_{p=1}^{m} \mathcal{W}^p_{i-1/2}.$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\lambda^2 \mathcal{W}^2_{i-1/2} + \lambda^3 \mathcal{W}^3_{i-1/2} + \lambda^1 \mathcal{W}^1_{i+1/2}\right].$$

First-order REA Algorithm

1. Reconstruct a piecewise constant function $\tilde{q}^n(x, t_n)$ defined for all x, from the cell averages Q_i^n.

$$\tilde{q}^n(x, t_n) = Q_i^n \quad \text{for all } x \in \mathcal{C}_i.$$

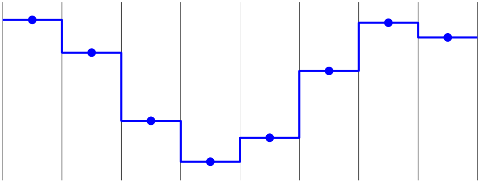
2. Evolve the hyperbolic equation exactly (or approximately) with this initial data to obtain $\tilde{q}^n(x, t_{n+1})$ a time Δt later.

3. Average this function over each grid cell to obtain new cell averages

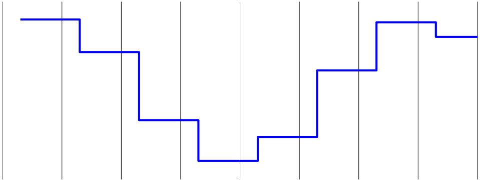
$$Q_i^{n+1} = \frac{1}{\Delta x} \int_{\mathcal{C}_i} \tilde{q}^n(x, t_{n+1})\, dx.$$

First-order REA Algorithm

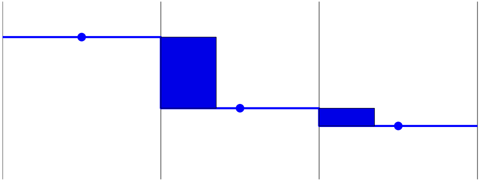
Cell averages and piecewise constant reconstruction:



After evolution:



Cell update



The cell average is modified by

$$\frac{u\Delta t \cdot (Q_{i-1}^n - Q_i^n)}{\Delta x}$$

So we obtain the upwind method

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n).$$

Second-order REA Algorithm

1. Reconstruct a piecewise linear function $\tilde{q}^n(x, t_n)$ defined for all x, from the cell averages Q_i^n.

$$\tilde{q}^n(x, t_n) = Q_i^n + \sigma_i^n (x - x_i) \quad \text{for all } x \in \mathcal{C}_i.$$

2. Evolve the hyperbolic equation exactly (or approximately) with this initial data to obtain $\tilde{q}^n(x, t_{n+1})$ a time Δt later.

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Note: Conservative for any choice of slopes σ_i^n.

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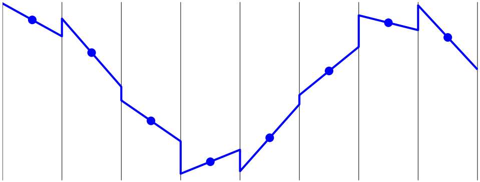
Cell averages and piecewise linear reconstruction:



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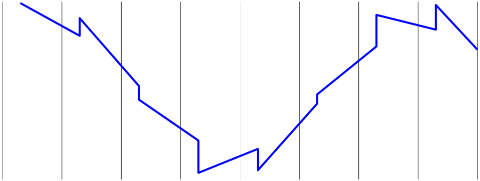

Choice of slopes

$$\tilde{Q}^n(x,t_n) = Q_i^n + \sigma_i^n(x - x_i) \qquad \text{for } x_{i-1/2} \le x < x_{i+1/2}.$$

Applying REA algorithm gives:

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) - \frac{1}{2}\frac{u\Delta t}{\Delta x}\left(\Delta x - u\Delta t\right)\left(\sigma_i^n - \sigma_{i-1}^n\right)$$

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Choice of slopes:

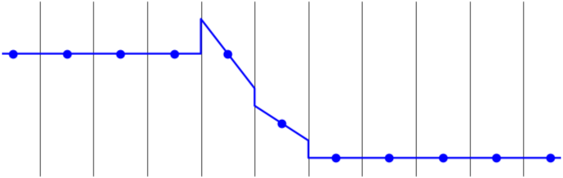
Centered slope: $\sigma_i^n = \dfrac{Q_{i+1}^n - Q_{i-1}^n}{2\Delta x}$ (Fromm)

Upwind slope: $\sigma_i^n = \dfrac{Q_i^n - Q_{i-1}^n}{\Delta x}$ (Beam-Warming)

Downwind slope: $\sigma_i^n = \dfrac{Q_{i+1}^n - Q_i^n}{\Delta x}$ (Lax-Wendroff)

Slopes can create oscillations

Step function data with Lax-Wendroff slope:



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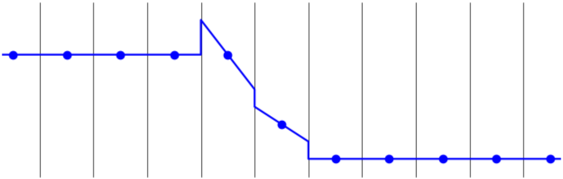
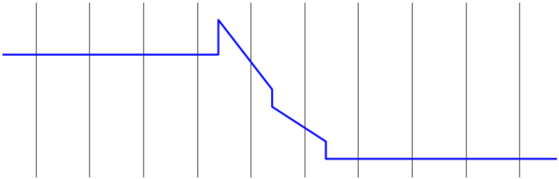


Evolving solution and averaging can result in overshoot:



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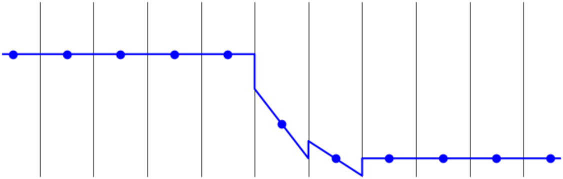
Step function data with Beam-Warming slope:



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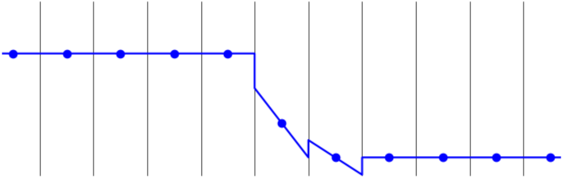

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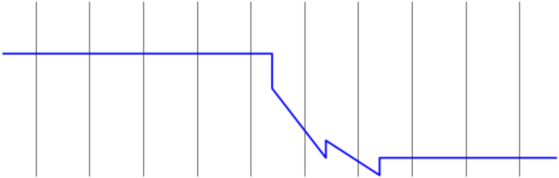


High-resolution methods

Want to use slope where solution is smooth for “second-order” accuracy.

Where solution is not smooth, adding slope corrections gives oscillations.

Limit the slope based on the behavior of the solution, e.g.,

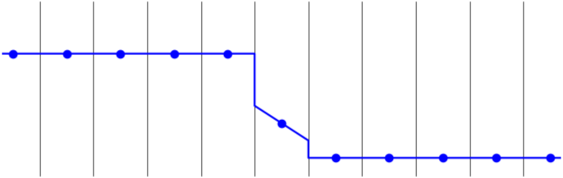
$$\sigma_i^n = \text{minmod}\left(\left(\frac{Q_i^n - Q_{i-1}^n}{\Delta x}\right),\ \left(\frac{Q_{i+1}^n - Q_i^n}{\Delta x}\right)\right)$$

where

$$\text{minmod}(a,b) = \begin{cases} a & \text{if } |a| < |b| \text{ and } ab > 0 \\ b & \text{if } |b| < |a| \text{ and } ab > 0 \\ 0 & \text{if } ab \le 0. \end{cases}$$

Limiters can eliminate oscillations

Step function data with minmod slope:



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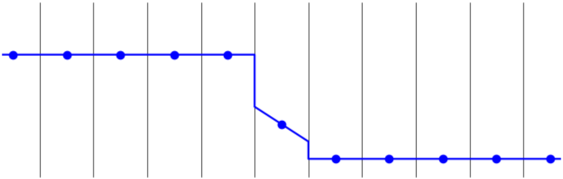
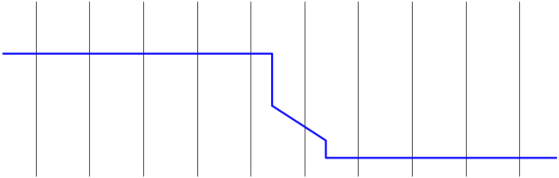


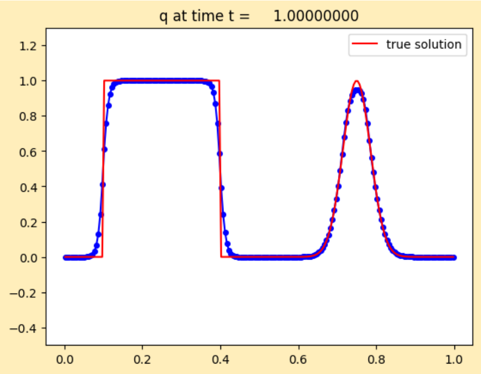
Evolving solution and averaging maintains monotonicity:



Advection tests

$q_t + q_x = 0$ with periodic BCs
Solution at $t = 1$ should agree with initial data.

Minmod solution with 200 cells:



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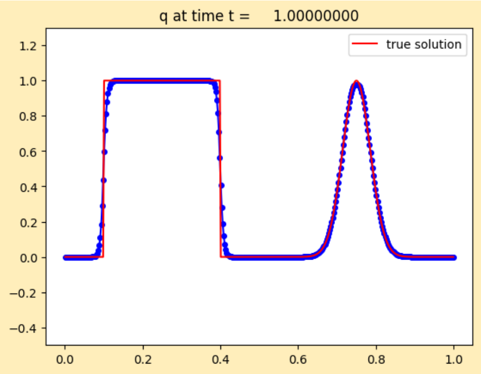

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Compare Lax-Wendroff, Beam-Warming, minmod...

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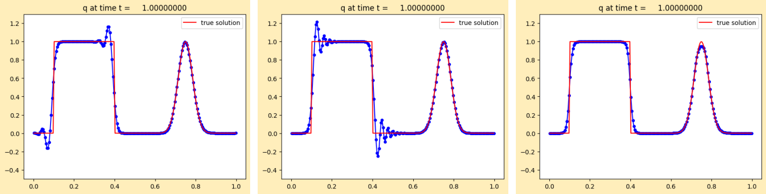

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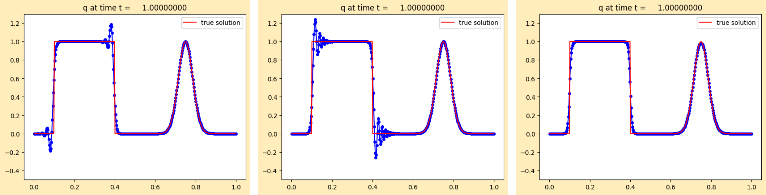
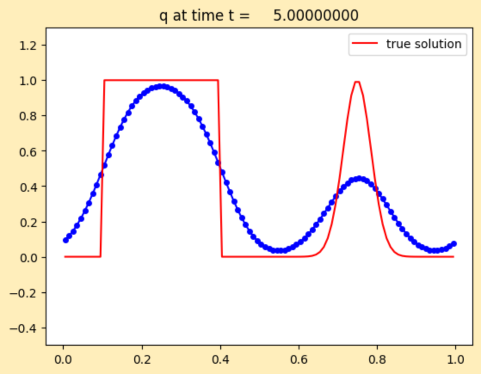


Advection tests

$q_t + q_x = 0$ with periodic BCs
Solution at $t = 1, 2, 3, 4, 5, \ldots$ should agree with initial data.

Upwind solution with 100 cells at $t = 5$:



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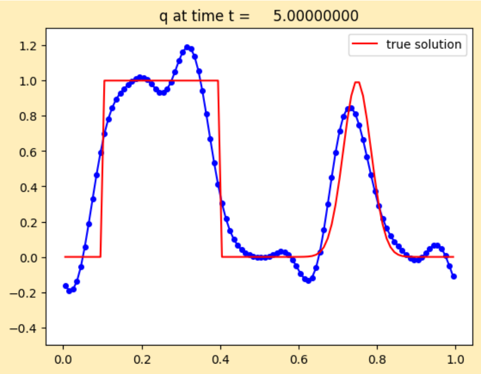

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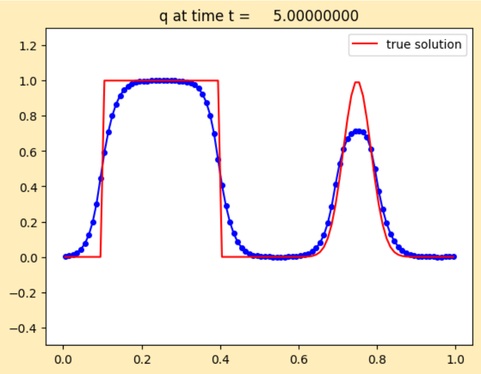
Minmod limiter solution with 100 cells at $t = 5$:



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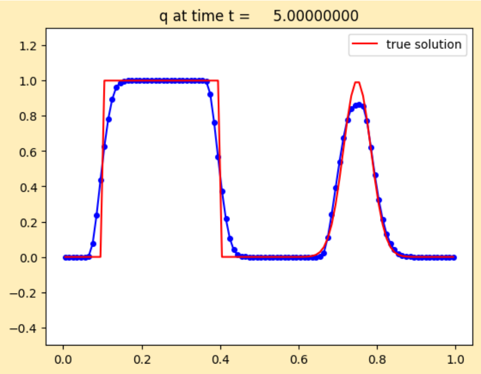
Monotonized Central limiter solution with 100 cells at $t = 5$:



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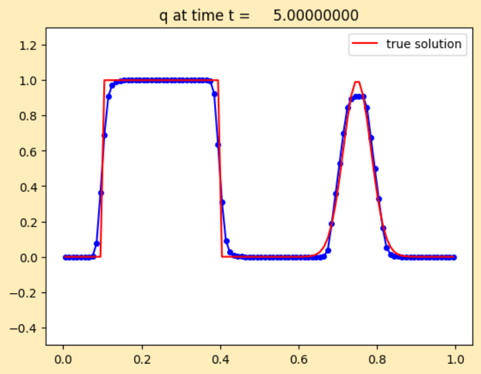
Superbee limiter solution with 100 cells at $t = 5$:



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Monotonicity Preserving methods

A scalar method is said to be monotonicity preserving if:

Given any data Q_i^n that satisfies

$$Q_{i-1}^n \geq Q_i^n \quad \text{for all } i.$$

Taking one time step preserves this property:

$$Q_{i-1}^{n+1} \geq Q_i^{n+1} \quad \text{for all } i.$$

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And similarly if $\geq$ replaced by $\leq$.

In particular:
An isolated discontinuity propagates without any oscillations.

TVD Methods

Total variation:

$$TV(Q) = \sum_i |Q_i - Q_{i-1}|$$

For a function, $TV(q) = \int |q_x(x)|\, dx$.

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A method is Total Variation Diminishing (TVD) if

$$TV(Q^{n+1}) \leq TV(Q^n).$$

Gives a form of stability useful for proving convergence, also for nonlinear scalar conservation laws.

TVD implies monotonicity preserving

Any TVD method for a scalar PDE is monotonicity preserving.

Prove the contrapositive:

Suppose

$$Q_{i-1}^n \geq Q_i^n \quad \text{for all } i$$

but after one step we do not have $Q_{i-1}^{n+1} \geq Q_i^{n+1}$ for all i.

Then the total variation of the solution must have increased.

Deriving methods that are TVD

Since TV is a global property, how do we derive methods that we can prove are TVD for any data?

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Since TV is a global property, how do we derive methods that we can prove are TVD for any data?

Use these facts (for scalar conservation law):

- Exact solution is TVD
- If we average $q(x,t)$ over grid cells to compute Q_i, then $TV(Q_i) \leq TV(q(\cdot,t))$.

$$TV(Q) = \sum_i |Q_i - Q_{i-1}|, \qquad TV(q) = \int |q_x(x)|\, dx$$

TVD REA Algorithm

1. Reconstruct a piecewise linear function $\tilde{q}^n(x, t_n)$ defined for all x, from the cell averages Q_i^n.

$$\tilde{q}^n(x, t_n) = Q_i^n + \sigma_i^n(x - x_i) \quad \text{for all } x \in \mathcal{C}_i$$

   with the property that $TV(\tilde{q}^n) \le TV(Q^n)$.

2. Evolve the hyperbolic equation exactly (or approximately) with this initial data to obtain $\tilde{q}^n(x, t_{n+1})$ a time k later.

3. Average this function over each grid cell to obtain new cell averages

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Note: Steps 2 and 3 are always TVD.

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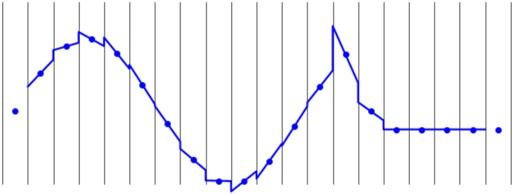
So $\quad TV(Q^{n+1}) \leq TV(\tilde{q}^n(\cdot, t_{n+1})) \leq TV(\tilde{q}^n(\cdot, t_n)) \leq TV(Q^n)$

Reconstruction step

Lax-Wendroff slopes do not give TVD reconstruction:



Minmod slopes do give TVD reconstruction: