University of Oklahoma
Department of Mathematics

Homework #7

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Course: *Abstract Linear Algebra* – Professor: *Dr. Gregory Muller*
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Problem 1

Let $T : \mathbb{P}_2(\mathbb{F}) \to \mathbb{P}_2(\mathbb{F})$ be the linear map defined by

$$T(p(x)) = p(1)x^2 + p(2)x + p(-1)$$

a) Find the trace, determinant, and characteristic polynomial of T.
b) Find the eigenvalues of T.
c) For each eigenvalue, find an eigenvector of T.

Answer. For any $\mathbb{F}$-linear vector space V and endomorphism $T : V \to V$, the trace and determinant of T can be deduced from the second and last coefficients of the characteristic polynomial $p_T \in \mathbb{P}(\mathbb{F})$ of T. The characteristic polynomial p_T may be calculated from the matrix A_T of T w.r.t. the basis $\mathcal{B} = (x^2, x, 1)$ of $\mathbb{P}_2(\mathbb{F})$.

$$\begin{aligned} x^2 &\mapsto x^2 + 4x + 1 \\ x &\mapsto x^2 + 2x + -1 \\ 1 &\mapsto x^2 + x + 1 \end{aligned} \qquad \Longrightarrow \qquad A_T = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

The characteristic polynomial p_T is defined as the determinant,

$$\begin{aligned} \det(\lambda Id_v - A_T) &= (\lambda-1)(\lambda-2)(\lambda-1) + (-1) + (4) - ((\lambda-2) + 4(\lambda-1) + (-1)(\lambda-1)) \\ &= (\lambda-1)(\lambda-2)(\lambda-1) - 3(\lambda-1) - (\lambda-2) + 3 \\ &= (\lambda^3 - 4\lambda^2 + 5\lambda - 2) - (3\lambda - 3) - (\lambda - 2) + 3 \\ &= \lambda^3 - 4\lambda^2 + \lambda + 6 \end{aligned}$$

a) So $\mathrm{tr}(T) = -(-4) = 4$ and $\det(T) = (-1)^3(6) = -6$, and

$$p_T(\lambda) = (\lambda+1)(\lambda-2)(\lambda-3)$$

b) The eigenvalues of T are characterized by the roots of p_T, $\{-1, 2, 3\}$

c) Using elimination to compute bases for the kernels of the linear transformation defined as multiplication by $\lambda Id - A_T$, the vectors in $\mathbb{P}_2(\mathbb{F})$ associated with each eigenvalue of T is spanned by the following vectors

$$\lambda = 3 : -x^2 - 3x + 1 \qquad \lambda = 2 : -x^2 - 5x + 4 \qquad \lambda = -1 : -x^2 + x + 1$$

Problem 2

Let $R : \mathbb{C}^3 \to \mathbb{C}^3$ be the linear map which acts like multiplication by

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

a) Find the trace, determinant, and characteristic polynomial of R.
b) Find the eigenvalues of R.
c) For each eigenvalue, find an eigenvector of R.

Answer. .
a) The second-highest and lowest degree coefficients of the characteristic polynomial p_R indicate the trace and determinant of R. The characteristic polynomial of R is defined as the determinant,

$$\det(\lambda Id_v - [R]_{\mathcal{B}\mathcal{B}}) = (\lambda - 1)^3 + 2^3 = \lambda^3 - 3\lambda^2 + 3\lambda - 9$$

So $\text{tr}(R) = 3$ and $\det(R) = 9$, and

$$p_R(\lambda) = \lambda^3 - 3\lambda^2 + 3\lambda - 9 = (\lambda - 3)(\lambda - i\sqrt{3})(\lambda + i\sqrt{3})$$

b) The eigenvalues of R are the roots of $p_R(\lambda) : \{3, i\sqrt{3}, -i\sqrt{3}\}$.
c) Using elimination to compute bases for the kernels of the linear transformation defined as multiplication by $\lambda Id - A_T$, the vectors in $\mathbb{P}_2(\mathbb{F})$ associated with each eigenvalue of T is spanned by the following vectors

$$\begin{aligned} \lambda = 3 &: (1, 1, 1) \\ \lambda = i\sqrt{3} &: (-1 - i\sqrt{3}, -1 + i\sqrt{3}, 2) \\ \lambda = -i\sqrt{3} &: (-1 + i\sqrt{3}, -1 - i\sqrt{3}, 2) \end{aligned}$$

Problem 3

Let $\mathbf{dim}(V) < \infty$ and let $P : V \to V$ be an **idempotent** linear map; that is, $P^2 = P$. Assuming we know that P is not the zero map nor the identity map, find the minimal polynomial of P.

Answer. Let $\mathbf{dim}(V) = n$. Since P is idempotent, $P = P^2$, so $P(P - Id) = 0$.

$$\det P(P - Id) = 0$$

the Cayley-Hamilton Theorem implies p_P has the property,

$$p_P(P) = p_P(P^2) = 0$$

Since any matrix of a linear map on V must have exactly n entries along the diagonal,

$$\begin{aligned} p_P(P) &= P^n + a_{n-1}P^{n-1} + \cdots + a_2P^2 + a_1P^1 + a_0P^0 \\ &= P^n - \operatorname{tr}(P)P^{n-1} + \cdots + a_1P^1 + (-1)^n \det(P)Id \\ &= P - \operatorname{tr}(P)P + \cdots + a_1P + (-1)^n \det(P)Id \\ &= P(-\operatorname{tr}(P)Id + \cdots + a_1) + (-1)^n \det(P)Id \end{aligned}$$

Problem 4

Let $T : V \to V$ be an invertible linear map with $\mathbf{dim}(V) < \infty$.

a) Show that the characteristic polynomial of T^{-1} is

$$\frac{1}{p_T(0)} x^{\mathbf{dim}(V)} p_T(x^{-1})$$

b) Show that there is a polynomial $p(x)$ with $p(T) = T^{-1}$.

Answer. T invertible $\implies$ $\ker(T) = 0$ $\implies$ $Tv = 0$ only if $v = 0$ $\implies$ 0 is not an eigenvalue of T $\implies$ 0 is not a root of p_T $\implies$ $1/p_T(0)$ exists.

Problem 5

Goal: show that the eigenvalues of a linear transformation are always roots of the minimal polynomial.

Let $T : V \to V$ be a linear map with $\mathbf{dim}(V) < \infty$.

a) Let v be an eigenvector of T with eigenvalue λ. Show that, for every polynomial $p(x)$,

$$p(T)v = p(\lambda)v$$

b) Show that every eigenvalue is a root of the minimal polynomial of T.

c) Show that, if T has $\mathbf{dim}(V)$-many distinct eigenvalues $\lambda_1, \ldots, \lambda_{\mathbf{dim}(V)}$, then the characteristic polynomial of T equals the minimal polynomial of T.

Answer. .
a) If v is an eigenvector of T with eigenvalue λ. Then

$$\begin{aligned} p(T)v &= (a_m T^m + a_{m-1} T^{m-1} + \cdots + a_1 T^1 + a_0 T^0)v \\ &= a_m T^m v + a_{m-1} T^{m-1} v + \cdots + a_1 T^1 v + a_0 T^0 v \\ &= a_m \lambda^m v + a_{m-1} \lambda^{m-1} v + \cdots + a_1 \lambda^1 v + a_0 \lambda^0 v \\ &= (a_m \lambda^m + a_{m-1} \lambda^{m-1} + \cdots + a_1 \lambda^1 + a_0 \lambda^0)v \\ &= p(\lambda)v \end{aligned}$$

b) Let μ_T denote the minimal polynomial of T, λ_i denote the i'th eigenvalue of T, and $v_i \in V$ be the non-zero eigenvector associated with λ_i.

Since $\mu(T)v_i = \mu(\lambda_i)v_i = 0$, we have two cases $\mu(\lambda_i) = 0$ or $v_i = 0$, but we assumed v_i is a non-zero eigenvector, so λ_i must be a root of μ. Since i is arbitrary, all eigenvalues of T are roots of the minimal polynomial of T.

c) Let μ_T denote the minimal polynomial of T. Since the set of eigenvalues of $T, \lambda_1, \lambda_2, \ldots, \lambda_{\mathbf{dim}(V)}$ are the of roots of μ_T and p_T, we can factorize μ_T into the map defined by

$$\mu_T(z) = q(z)(z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_{\mathbf{dim}(V)})$$

and p_T into the map defined by

$$p_T(z) = r(z)(z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_{\mathbf{dim}(V)})$$

Since $\mathbf{deg}(\mu_T) \leq \mathbf{deg}(p_T) = \mathbf{dim}(V)$ and μ_T and p_T is monic, the above factors $q(z) = r(z) = (-1)^{\mathbf{dim}(V)}$. Therefore, when T has $\mathbf{dim}(V)$ distinct eigenvalues, $\mu_T = p_T$.

Since the roots of the minimal polynomial are eigenvalues of T. any polynomial of the form $g(T) = 0$ cannot have any roots other than the roots of p_T, Then λ is an eigenvalue of T with an associated eigenvector $v \neq 0$