University of Oklahoma
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Homework #6

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Course: *Abstract Linear Algebra* – Professor: *Dr. Gregory Muller*
Due date: *Oct 3, 2021*

10.A: 2

Suppose A and B are square matrices of the same size and $AB = I$. Prove that $BA = I$.

Answer. If A, B are square matrices and $AB = I$, then the linear transformation T_A is surjective, which implies T_A is injective (because A is a square matrix). Since T_A is injective and surjective, T_A is isomorphic and therefore A^{-1} exists. Since A^{-1} exists,

$$\begin{aligned} I &= A^{-1}A \\ &= A^{-1}IA \\ &= A^{-1}(AB)A \\ &= (A^{-1}A)BA \\ &= I(BA) \\ &= BA \end{aligned}$$

10A: 3

Suppose $T \in \mathcal{L}(V)$ has the same matrix with respect to every basis of V. Prove that T is a scalar multiple of the identity operator.

Answer. The following lemmas, (1) and (2), yield the following implications, demonstrating the above statement.

1. If there exists a non-zero vector in V that is not an eigenvector of T, then there are two bases for V such that the associated matrices of T are not equal.

2. If every non-zero vector in V is an eigenvector of T, then T is a scalar multiple of the identity map.

$$\begin{aligned} \forall \text{ bases } \mathcal{B}_1, \mathcal{B}_2 \text{ of } V, [T]_{\mathcal{B}_1,\mathcal{B}_1} = [T]_{\mathcal{B}_2,\mathcal{B}_2} &\overset{(1)}{\Longrightarrow} \forall v \in V, \exists \lambda \in \mathbb{F}, \text{ such that } Tv = \lambda v \\ &\overset{(2)}{\Longrightarrow} T \text{ is a scalar multiple of the identity map.} \end{aligned}$$

Therefore, any linear endomorphism T that has the same matrix with respect to every basis of V is a scalar multiple of the identity operator on V. ■

Proof of Lemma (1)
Define the linear endomorphism T on V by the equation $Tv_k = \lambda_k v_k$, where each v_k belongs to an ordered basis $\mathcal{B}_1 = (v_1, \ldots, v_n)$ of V, $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$, and $\lambda_i \neq \lambda_j$. Since each $v_k \in \mathcal{B}_1$ is an eigenvector of T, the matrix associated with T for the basis $\mathcal{B}_1$ is the diagonal matrix,

$$[T]_{\mathcal{B}_1,\mathcal{B}_1} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

By extension, $\mathcal{B}_2 = (v_1, \ldots, v_{n-1}, v_1 + \cdots + v_n)$ is a basis of V. Since $\nexists \lambda \in \mathbb{F}$ where, $Tv_k = \lambda v_k$ for all $v_k \in \mathcal{B}_1$, $(v_1 + \cdots + v_n)$ is not an eigenvector of T and the matrix $[T]_{\mathcal{B}_2,\mathcal{B}_2}$ cannot be diagonal. Since, there exists a non-zero vector in V that is not an eigenvector of T, $[T]_{\mathcal{B}_1,\mathcal{B}_1} \neq [T]_{\mathcal{B}_2,\mathcal{B}_2}$.

Proof of Lemma (2)
If, as assumed, any vector $v \in V$ is an eigenvector, then there exists a scalar map $\phi : V \to \mathbb{F}$, such that

$$Tv = (\phi v)v \tag{1}$$

where ϕv is the eigenvalue cooresponding to v. If it is shown that ϕ maps all $v \in V$ to the same element in $\mathbb{F}$, then it is demonstrated that T is a scalar multiple of the identity map on V.

If $u, v \in V$ are related by the equation, $u = \beta v$ for some $\beta \in \mathbb{F}$, we have,

$$\phi(u)u = \phi(\beta v)\beta v = T(\beta v) = \beta T(v) = \beta\phi(v)v = \phi(v)\beta v \tag{2}$$

When $(\beta v) \neq 0$, (2) shows that

$$\phi(\beta v) = \phi(v) \tag{3}$$

i.e. ϕ maps every member of the one-dimensional subspace generated by v to $\phi(v)$. When u, v are linearly independent,

$$T(u + v) = \phi(u + v)(u + v) = \phi(u + v)u + \phi(u + v)v \tag{4}$$

but additive property of T also requires that,

$$T(u + v) = T(u) + T(v) = \phi(u)u + \phi(v)v \tag{5}$$

combining (4) and (5),

$$(\phi(u + v) - \phi(u))u + (\phi(u + v) - \phi(v))v = 0 \tag{6}$$

equation (6) tells us $\phi(u + v) = \phi(u) = \phi(v)$ because we assumed that u, v are linearly independent.

Since u, v are an arbitrary pair of linearly independent vectors, the result produced by (6) combined with (3) shows that $\phi(v)$ is a constant for all $v \in V$; taking $\lambda = \phi(v)$ for any $v \in V$ then yields the complete solution. ■

5.A: 2

Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that null(S) is invariant under T.

Answer. Let $v \in \text{null}(S)$. Then

$$\begin{aligned} S(Tv) &= T(Sv) = T(0) = 0 \\ &\implies \forall v \in \text{null}(S), Tv \in \text{null}(S) \\ &\implies T(\text{null}(S)) \subseteq \text{null}(S) \end{aligned}$$

Therefore null(S) is invariant under T.

5.A: 7

Suppose $T \in \mathcal{L}(\mathbb{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find the eigenvalues of T.

Answer. Suppose λ is an eigenvalue of T. Then

$$\begin{aligned} \lambda x &= -3y \\ \lambda y &= x \\ 0 &= (\lambda x + 3y) + (\lambda y - x) \\ (x - 3y) &= \lambda(x + y) \\ \lambda &= \frac{(x - 3y)}{(x + y)} \end{aligned}$$

Hence

$$\lambda(1, 0) = 1(1, 0) = (1, 0)$$

but

$$T(1, 0) = (0, 1) \neq \lambda(1, 0)$$

Therefore, no $\lambda \in \mathbb{R}$ is an eigenvalue of T

5.A: 11

Define $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $Tp = p'$. Find all eigenvalues and eigenvectors of T.

Answer. $\mathcal{P}(\mathbb{R}) \subsetneq \mathcal{C}^\infty$ and the only function $f \in C^\infty$, such that $Tf = f' = cf$, is the function defined by $f(x) = e^{cx}$ for all $x \in \mathbb{R}$, but $e^{cx} \notin \mathcal{P}(\mathbb{R})$. Therefore, no element in $\mathcal{P}(\mathbb{R})$ is an eigenvector for T and there is no eigenvalue for T in $\mathcal{P}(\mathbb{R})$.

5.A: 12

Define $T \in \mathcal{L}(\mathcal{P}_4(\mathbb{R}))$ by

$$(Tp)(x) = xp'(x)$$

for all $x \in \mathbb{R}$. Find all eigenvalues and eigenvectors of T.

Answer. Note that

$$\begin{aligned} Tx^4 &= x(4x^3) = 4x^4 & Tx^3 &= x(3x^2) = 3x^3 \\ Tx^2 &= x(2x) = 2x^2 & Tx &= x(1) = 1 \\ T(1) &= x(0) = 0 \end{aligned}$$

Hence the span of each vector in the standard basis of $\mathcal{P}_4(\mathbb{R})$ forms a distinct invariant subspace under T, with eigenvalue equal to the exact degree of the spanning polynomial.