UNIVERSITY OF OKLAHOMA
DEPARTMENT OF MATHEMATICS

Execise Set #1

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Course: *Intro to Abstract Algebra (MATH 4323)* – Professor: *Dr. Roi Docampo*
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Question 1

Suppose that

$$\begin{aligned} A &= \{x : x \in \mathbb{N} \text{ and x is even}\} \\ B &= \{x : x \in \mathbb{N} \text{ and x is prime}\} \\ C &= \{x : x \in \mathbb{N} \text{ and x is a multiple of 5}\} \end{aligned}$$

Answer. Describe each of the following sets:
a) $A \cap B = \{2\}$ because 2 is the only even prime number
b) $B \cap C = \{5\}$ because a multiple of 5 cannot be prime
c) $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16, 17, 18, 19, \ldots\}$, the set of all naturals minus composite odd numbers
d) $A \cap (B \cup C)$ includes 2 and the even multiples of 5, because 2 is the only even prime.

Question 2

Suppose that $A = \{a, b, c\}, B = \{1, 2, 3\}, C = \{x\}$, and $D = \emptyset$.

Answer. List all of the elements in each of the following sets
a) $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3)\}$
b) $B \times A = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$
c) $A \times B \times C = \{(a, 1, x), (a, 2, x), (a, 3, x), (b, 1, x), (b, 2, x), (b, 3, x), (c, 1, x), (c, 2, x), (c, 3, x)\}$
d) $A \times D = \emptyset$

Question 8

Prove $A \subset B$ if and only if $A \cap B = A$

Answer. Take any two sets B and $A \subset B$.

$$A \subset B \iff \forall a \in A, a \in B \iff A \cap B = A$$

where the first iff holds by definition; the second iff holds because the set A contains every element in A, B contains at least every element in A, and their intersection contains no more elements than the collection defined by A.■

Question 10

Prove $A \cup B = (A \cap B) \cup (A \setminus B) \cup (B \setminus A)$

Answer. Take any two sets $A, B \subset U$. By De Morgan's Laws and set opertation rules of distributivity,

$$\begin{aligned} A \cup B &= \{x : x \in A\} \cup \{x : x \in B\} \\ &= (A \cap U) \cup (B \cap U) \\ &= (A \cap (B \cup B')) \cup (B \cap (A \cup A')) \\ &= ((A \cap B) \cup (A \cap B')) \cup ((B \cap A) \cup (B \cap A')) \\ &= ((A \setminus B) \cup (A \setminus B')) \cup ((B \setminus A) \cup (B \setminus A')) \\ &= ((A \setminus B) \cup (A \cap B)) \cup ((B \setminus A) \cup (B \cap A)) \\ &= (A \cap B) \cup (B \cap A) \cup (A \setminus B) \cup (B \setminus A) \\ &= (A \cap B) \cup (A \setminus B) \cup (B \setminus A) \end{aligned}$$

as required. ■

Question 11

Prove $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Answer. Take any three sets A, B, C.

$$\begin{aligned} (A \cup B) \times C &= \{(x, c) : x \in (A \cup B), c \in C\} \\ &= \{(x, c) : x \in A, c \in C\} \cup \{(x, c) : x \in B, c \in C\} \\ &= (A \times C) \cup (B \times C) \end{aligned}$$

as required. ■

Question 14

Prove $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

Answer. Take any three sets A, B, C. By De Morgan's Laws and set opertation rules of distributivity,

$$\begin{aligned} A \setminus (B \cup C) &= A \cap (B \cup C)' \\ &= A \cap A \cap (B' \cap C') \\ &= A \cap B' \cap A \cap C' \\ &= (A \setminus B) \cap (A \setminus C) \end{aligned}$$

as required. ■

Question 16

Prove $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$

Answer. Take any two sets $A, B \subset U$. By De Morgan's Laws and set opertation rules of distributivity,

$$\begin{aligned}
(A \setminus B) \cup (B \setminus A) &= (A \cap B') \cup (B \cap A') \\
&= ((A \cap B') \cup B) \cap ((A \cap B') \cup A') \\
&= ((A \cup B) \cap (B \cup B')) \cap ((A \cup A') \cap (B' \cup A')) \\
&= ((A \cup B) \cap U) \cap (U \cap (B' \cup A')) \\
&= (A \cup B) \cap (B' \cup A') \\
&= (A \cup B) \cap (A \cap B)' \\
&= (A \cup B) \setminus (A \cap B)
\end{aligned}$$

as required. ■

Question 18

Determine which of the following functions are one-to-one and which are onto. If the function is not onto, determine its range.

Answer. Functions:
a) $f : \mathbb{R} \to \mathbb{R} : x \mapsto e^x$
This function is one-to-one but not onto. Its range is $\mathbb{R}_{>0}$
b) $f : \mathbb{Z} \to \mathbb{Z} : n \mapsto n^2 + 3$
This function is not one-to-one or onto. Its range is $\mathbb{Z}_{\geq 3}$
c) $f : \mathbb{R} \to \mathbb{R} : x \mapsto \sin(x)$
This function is not one-to-one or onto. Its range is $[-1, 1]$.
d) $f : \mathbb{Z} \to \mathbb{Z} : n \mapsto n^2$
This function is not one to one or onto. Its range is $\{0\} \cup \mathbb{N}$.

Question 19

Let $f : A \to B$ and $g : B \to C$ be invertable mappings; that is, mappings such that f^{-1} and g^{-1} exist. Show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Answer. Take any $a \in A$, and suppose $a \overset{f}{\mapsto} b \overset{g}{\mapsto} c$. Therefore $(g \circ f)(a) = c$. Then

$$a = f^{-1}(b) = f^{-1}(g^{-1}(c)) = (f^{-1} \circ g^{-1})(c)$$

Hence $(g \circ f)^{-1} = (f^{-1} \circ g^{-1})$, which was to be demonstrated. ■

Question 20

> Define a function $f : \mathbb{N} \to \mathbb{N}$ that is one-to-one but not onto. Define a function $f : \mathbb{N} \to \mathbb{N}$ that is not one-to-one but onto.

Answer. Suppose $f : \mathbb{N} \to \mathbb{N}$.
a) The function defined by $f : x \mapsto x + 1$ is one-to-one but not onto.
b) The function defined by $f = (1,1) \cup \{(x, x-1) : x \in \mathbb{N} \text{ and } x \geq 2\}$ is onto but not one-to-one.

Question 22b

> Let $f : A \to B$ and $g : B \to C$ be maps. If $g \circ f$ is onto, show that g is onto.

Answer. If $f : A \to B$, and $g \circ f$ is onto, then,

$$f(A) \subset B \implies g(f(A)) \subset g(B) \subset C \implies C \subset g(B) \subset C$$

where the first $\implies$ holds because the preimage of $g(B)$ contains the preimage of $g(f(A))$, and the second $\implies$ holds because $(g \circ f)(A) = g(f(A)) = C$, by definition of onto. This demonstrates $g(B) = C$. Therefore, the mapping g is onto. ■

Question 22c

> Let $f : A \to B$ and $g : B \to C$ be maps. If $g \circ f$ is one-to-one, show that f is one-to-one.

Answer. Take any one-to-one composition, $g \circ f$. Since $g \circ f$ is one-to-one, it is invertible (the inverse was shown earlier to be $f^{-1} \circ g^{-1}$). Therefore

$$\begin{aligned}
(g \circ f)(a) &= (g \circ f)(b) \\
g^{-1}(g \circ f)(a) &= g^{-1}(g \circ f)(b) \\
f(a) &= f(b) \\
f^{-1}f(a) &= f^{-1}f(b) \\
a &= b
\end{aligned}$$

where we see that $f(a) = f(b) \implies a = b$. Hence, f is injective. ■

Question 24

Let $f : X \to Y$ be a map with $A_1, A_2 \subset X$ and $B_1, B_2 \subset Y$.

Answer. Prove the following statements
a)
$$f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$$
Let $f : X \to Y$ be a map with $A_1, A_2 \subset X$. Then,

$$\begin{aligned} b \in f(A_1 \cup A_2) &\iff \exists a \text{ such that } b = f(a), \text{ where } a \in A_1 \cup A_2 \\ &\iff a \in A_1 \text{ or } a \in A_2 \\ &\iff b \in f(A_1) \text{ or } b \in f(A_2) \\ &\iff b \in f(A_1) \cup f(A_2) \end{aligned}$$

where the first $\Leftrightarrow$ restates the asssumption; the second $\Leftrightarrow$ holds by definition; the third $\Leftrightarrow$ holds because the preimage of b belongs to A_1 or A_2; and the final $\Leftrightarrow$ reiterates the preceding statement. Since $b \in f(A_1 \cup A_2) \Leftrightarrow b \in f(A_1) \cup f(A_2)$, the equality $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$ is demonstrated. ■

b)
$$f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$$
Consider any $b \in f(A_1 \cap A_2)$.

$$\begin{aligned} \exists a \in A_1, A_2 \text{ such that } b = f(a) &\implies b \in f(A_1) \text{ and } b \in f(A_2) \\ &\implies b \in f(A_1) \cap f(A_2) \end{aligned}$$

where the first $\implies$ holds because the preimage of b must exist in both A_1 and A_2, and the second $\implies$ holds by definition. Since $b \in f(A_1 \cap A_2) \implies b \in f(A_1) \cap f(A_2)$, the relation $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$ is demonstrated. ■

For an example that disproves the equality, consider the mapping $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ for all $x \in \mathbb{R}$. Define the subsets,

$$\begin{aligned} \mathbb{R} \supset A_1 &= [-1, 0] \\ \mathbb{R} \supset A_2 &= [0, 1] \end{aligned}$$

Notice,

$$f(A_1 \cap A_2) = f(\{0\}) = \{0\} \neq [0, 1] = [0, 1] \cap [0, 1] = f(A_1) \cap f(A_2)$$

Hence
$$f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2).$$

c)
$$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$$
Let the mapping $g : Y \to X$ be defined by the equation $g(b) = f^{-1}(b)$ for all $b \in B$. In part a),
$$g(B_1 \cup B_2) = g(B_1) \cup g(B_2)$$
was demonstrated for any arbitrary mapping. Hence,
$$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2),$$
as required. ■

d)
$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$$
Notice, in part b),
$$f^{-1}(B_1 \cap B_2) \subset f^{-1}(B_1) \cap f^{-1}(B_2)$$
was demonstrated for any arbitrary mapping.

The right-hand set will now be shown to contain the left-hand set. Suppose $f(A_1) = B_1$ and $f(A_2) = B_2$. Then for all $b \in B_1 \cap B_2$, $\exists a \in f^{-1}(B_1 \cap B_2)$ such that $b = f(a)$. Hence $(f^{-1}(B_1) \cap f^{-1}(B_2)) \subset f^{-1}(B_1 \cap B_2)$, as required. ■

e)
$$f^{-1}(Y \setminus B_1) = X \setminus f^{-1}(B_1)$$
In part d), it was shown that
$$f^{-1}(Y \setminus B_1) = f^{-1}(Y \cap B_1') = f^{-1}(Y) \cap f^{-1}(B_1')$$
Since f is invertible, f is bijective, hence $f^{-1}(Y) = X$. Since $f^{-1}(B_1')$ contains all the elements that do not map to B_1, $f^{-1}(B_1') = f^{-1}(B_1)'$, which is every element that does not map to B_1. Hence
$$f^{-1}(Y \setminus B_1) = f^{-1}(Y) \cap f^{-1}(B_1') = X \cap f^{-1}(B_1)' = X \setminus f^{-1}(B_1)$$
as required. ■