# Abstract Algebra
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## 2 The Integers

### 2.1 Induction

**Definition: First Principle of Mathematical Induction**

Let $S(n)$ be a statement about integers for $n \in \mathbb{N}$ and suppose $S(n_0)$ is true for some integer $n_0$. If for all integers $k$ with $k \geq n_0$, $S(k)$ implies that $S(k+1)$ is true, then $S(n)$ is true for all integers $n$ greater than or equal to $n_0$.

**Definition: Second Principle of Mathematical Induction**

Let $S(n)$ be a statement about integers for $n \in \mathbb{N}$ and suppose $S(n_0)$ is true for some integer $n_0$. If $S(n_0), S(n_0+1), \ldots, S(k)$ imply that $S(k+1)$ for $k \geq n_0$, then the statement $S(n)$ is true for all integers $n \geq n_0$.

**Definition: Principle of Well-Ordering**

Every non-empty subset of the natural numbers contains a least element.

**Theorem:**

The Principle of Mathematical Induction implies that 1 is the least natural number

Proof: $\square$

**Theorem:**

The Principle of Mathematical Induction implies the Principle of Well-Ordering. That is, every nonempty subset of $\mathbb{N}$ contains a least element.

Proof: $\square$

## 2.2 The Division Algorithm

An application of the Principle of Well-Ordering that is often-used is the division algorithm.

**Theorem: Division Algorithm**

Let $a$ and $b$ be integers, with $b \geq 0$. Then there exists unique integers $q$ and $r$ such that

$$a = bq + r$$

where $0 \leq r < b$.

Proof: *existence of $q$ and $r$*. Consider the set,

$$R = \{a - bx : x \in \mathbb{Z} \wedge a - bx \geq 0\}$$

If $0 \in R$, then $b|a$, and we can let $q = a/b$ and $r = 0$. If $0 \notin R$, then the WOP guarentees the existence of a smallest element in a set $R$ iff $R \subseteq \mathbb{N}$ and $R \neq \emptyset$. Since each element $x \in R$ satisfies $x \in \mathbb{Z}$ and $x \geq 0$ and $0 \notin R$, the first condition of the WOP is satisfied, $R \subseteq \mathbb{N}$.
To show that $R \neq \emptyset$, consider the two cases:

**Case 1:** $a \geq 0$. Then it is clear that $a \in R$, by letting $x = 0$.
**Case 2:** $a < 0$. Then if $x = 2a$, $a - bx = a - b(2a) = a(1 - 2b)$, we have the product of a negative integer $a$ and a negative integer $(1 - 2b)$ when $b \geq 1$, therefore $a - bx \geq 0$. $\square$