University of Oklahoma
Department of Mathematics

Homework #7

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Course: *Abstract Linear Algebra* – Professor: *Dr. Gregory Muller*
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Problem 1

Let $T : \mathbb{P}_2(\mathbb{F}) \to \mathbb{P}_2(\mathbb{F})$ be the linear map defined by

$$T(p(x)) = p(1)x^2 + p(2)x + p(-1)$$

a) Find the trace, determinant, and characteristic polynomial of T.
b) Find the eigenvalues of T.
c) For each eigenvalue, find an eigenvector of T.

Answer. .
Because the trace, determinant, and characteristic polynomial of T is invariant under any choice of basis, the following matrix can be constructed for T for the basis $\mathcal{B} = (x^2, x, 1)$ of $\mathbb{P}_2(\mathbb{F})$.

$$\begin{aligned} x^2 &\mapsto x^2 + 4x + 1 \\ x &\mapsto x^2 + 2x + -1 \\ 1 &\mapsto x^2 + x + 1 \end{aligned} \qquad \Longrightarrow \qquad \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

a) The second-highest and lowest degree coefficients of the characteristic polynomial p_T indicate the trace and determinant of T. The characteristic polynomial of T is defined as the determinant,

$$\det(\lambda Id_v - [T]_{\mathcal{B}\mathcal{B}}) = (\lambda - 1)(\lambda - 2)(\lambda - 1) +$$

So $\operatorname{tr}(R) = 3$ and $\det(R) = 9$, and

$$p_R(\lambda) = \lambda^3 - 3\lambda^2 + 3\lambda - 9 = (\lambda - 3)(\lambda - i\sqrt{3})(\lambda + i\sqrt{3})$$

Problem 2

Let $R : \mathbb{C}^3 \to \mathbb{C}^3$ be the linear map which acts like multiplication by

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

a) Find the trace, determinant, and characteristic polynomial of R.
b) Find the eigenvalues of R.
c) For each eigenvalue, find an eigenvector of R.

Answer. .
a) The second-highest and lowest degree coefficients of the characteristic polynomial p_R indicate the trace and determinant of R. The characteristic polynomial of R is defined as the determinant,

$$\det(\lambda Id_v - [R]_{\mathcal{B}\mathcal{B}}) = (\lambda - 1)^3 + 2^3 = \lambda^3 - 3\lambda^2 + 3\lambda - 9$$

So $\mathrm{tr}(R) = 3$ and $\det(R) = 9$, and

$$p_R(\lambda) = \lambda^3 - 3\lambda^2 + 3\lambda - 9 = (\lambda - 3)(\lambda - i\sqrt{3})(\lambda + i\sqrt{3})$$

b) The eigenvalues of R are the roots of $p_R(\lambda) : \{3, i\sqrt{3}, -i\sqrt{3}\}$.
c) Solving for each λ, we get,

$$\begin{aligned} 3\text{-Eigenspace} &= \mathbf{span}((1, 1, 1)) \\ i\sqrt{3}\text{-Eigenspace} &= \mathbf{span}((-1 - i\sqrt{3}, -1 + i\sqrt{3}, 2)) \\ -i\sqrt{3}\text{-Eigenspace} &= \mathbf{span}((-1 + i\sqrt{3}, -1 - i\sqrt{3}, 2)) \end{aligned}$$

Problem 3

Let $\mathbf{dim}(V) < \infty$ and let $P : V \to V$ be an **idempotent** linear map; that is, $P^2 = P$. Assuming we know that P is not the zero map nor the identity map, find the minimal polynomial of P.

Answer. .

Problem 4

Let $T : V \to V$ be an invertible linear map with $\mathbf{dim}(V) < \infty$.

a) Show that the characteristic polynomial of T^{-1} is

$$\frac{1}{p_T(0)} x^{\mathbf{dim}(V)} p_T(x^{-1})$$

b) Show that there is a polynomial $p(x)$ with $p(T) = T^{-1}$.

Answer. .

Problem 5

Goal: show that the eigenvalues of a linear transformation are always roots of the minimal polynomial.

Let $T : V \to V$ be a linear map with $\mathbf{dim}(V) < \infty$.

a) Let v be an eigenvector of T with eigenvalue λ. Show that, for every polynomial $p(x)$,

$$p(T)v = p(\lambda)v$$

b) Show that every eigenvalue is a root of the minimal polynomial of T.

c) Show that, if T has $\mathbf{dim}(V)$-many distinct eigenvalues $\lambda_1, \ldots, \lambda_{\mathbf{dim}(V)}$, then the characteristic polynomial of T equals the minimal polynomial of T.

Answer. .
a) If v is an eigenvector of T with eigenvalue λ. Then

$$\begin{aligned}
p(T)v &= (a_m T^m + a_{m-1} T^{m-1} + \cdots + a_1 T^1 + a_0 T^0)v \\
&= a_m T^m v + a_{m-1} T^{m-1} v + \cdots + a_1 T^1 v + a_0 T^0 v \\
&= a_m \lambda^m v + a_{m-1} \lambda^{m-1} v + \cdots + a_1 \lambda^1 v + a_0 \lambda^0 v \\
&= (a_m \lambda^m + a_{m-1} \lambda^{m-1} + \cdots + a_1 \lambda^1 + a_0 \lambda^0)v \\
&= p(\lambda)v
\end{aligned}$$

b) Suppose λ is a root of $p_T(x)$. Then λ is an eigenvalue of T with an associated eigenvector $v \neq 0$.