4 Eigentheory and Diagnolization

In this chapter we develop the tools that will help us **understand the structure of vector space endomorphisms (linear operators)**, that is, maps from a vector space $V \to V$ that preserve the operations of addition and scalar multiplication over some field $\mathbb{F}$ that endow structure on V.

By focusing on certain subspaces, called **invariant subspaces** of the endomorphism, we will show that these so called **"eigenspaces"** allow us to create a simpler characterization of endomorphisms on V.

Definition: Linear Endomorphisms

In the study of linear maps on vector spaces, an **endomorphism** is a structure preserving map from a vector space to itself, or **homomorphism** from a vector space to itself. For the rest of the class, we will focus on linear maps from a vector space to itself

$$T : V \to V$$

These are sometimes called **linear endomorphisms** of V.

Suppose T is a linear endomorphism on V. Notice that if we have a direct sum decomposition of V

$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_m$$

where each U_j is a proper subspace of V, then to understand the behavior of T, we need only to understand the behavior of **the maps of T restricted to each subspace U_j; each $T|_{U_j}$**. The reason we can characterize all of T by focusing on each $T|_{U_j}$ is because any element of V is described by one unique linear combination of elements in U_j.

Note: Notational Shortcuts

- $\mathbb{F}$ denotes $\mathbb{R}$ or $\mathbb{C}$
- V and W denote vector spaces over $\mathbb{F}$

4.1 Recall:

Definition: Change of Basis Matrix

For any two bases $\mathcal{B}_1, \mathcal{B}_2$ of V, the **change of basis matrix** from $\mathcal{B}_1$ to $\mathcal{B}_2$ is the matrix $[Id_V]_{\mathcal{B}_2,\mathcal{B}_1}$

Definition: Conjugation of matrices

Let A, B be matrices. The **conjugation of A by B** is the matrix BAB^{-1}. Therefore, the matrix of T w.r.t. $\mathcal{B}_2$ is the conjugation of the matrix of T w.r.t. $\mathcal{B}_1$ by the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$.

Definition: Similarity

Two square matrices are **similar** if one is the conjugation of the other by some invertible matrix.

Definition: Determinant

If $T : V \to V$ is linear and $\dim V < \infty$, the **determinant** of T is $\det([T]_{\mathcal{B},\mathcal{B}})$, where $\mathcal{B}$ is any choice of basis for V.

4.2 Invariant Subspaces

Notice the above description of T being characterized by m restriction maps $T|_{U_j}$ on proper subspaces of V requires T to be an endomorphism on each U_j. The notion of a subspace that is its own codomain under T is sufficiently important to deserve a name.

Definition: Invariant Subspace

Suppose T is a linear endomorphism on V. A subspace U of V is called invariant under T if $u \in U$ implies $Tu \in U$. That is, U is said to be an **invariant subspace of** V if

$$T(U) \subseteq U$$

In other words, U is invariant under T if $T|_U$ is an endomorphism on U.

Take any $v \in V$ with $v \neq 0$ and let U equal the set of all scalar multiples of v. Because U has dimension 1, if U is invariant under an endomorphism T, then T acts as multiplication on U, and hence there is a scalar $\lambda \in \mathbb{F}$ such that

$$Tv = \lambda v \tag{1}$$

Equation 1 is important enough that the vectors v and scalars λ satisfying it are given special names.

Definition: Eigenvalue

Suppose T is a linear endomorphism on V. A number $\lambda \in \mathbb{F}$ is called an **eigenvalue** of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.

Definition: Eigenvector

Suppose T is an endomorphism on V and $\lambda \in \mathbb{F}$ is an eigenvalue of T. A vector $v \in V$ is called an **eigenvector** of T cooresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

Theorem: Equivalent Conditions to be an Eigenvalue

Suppose V is finite-dimensional, T is an endomorphism on V, and $\lambda \in \mathbb{F}$. Then the following are equivalent.

- λ is an eigenvalue of T with v the cooresponding eigenvector;
- $(T - \lambda I_V)$ is not injective and $v \in \ker(T - \lambda I_V)$;
- $(T - \lambda I_V)$ is not surjective;
- $(T - \lambda I_V)$ is not invertable;

Therefore, the eigenvectors of T with eigenvalue λ are the non-zero vectors in $\ker(T - \lambda I_V)$

Proof:

$\square$

The eigenvectors associated with a particular eigenvalue is important enough that the set of all vectors associated with a particular eigenvalues is given a special name.

Definition: Eigenspace

Let T be an endomorphism on V and $\lambda \in \mathbb{F}$. The **λ-eigenspace of T** is

$$\ker(T - \lambda I_V)$$

4.3 Characteristic Polynomials