1 Finite-Dimensional Vector Spaces

Linear algebra focueses on finite-dimensional vector spaces, which are introduced in this chapter.

1.1 Span and Linear Independence

Adding up scalar multiples of vectors in a list gives what is called a **linear combination** of the list.

Linear Combinations and Span

Definition: Linear Combination

A **linear combination** of a list $v_1, \ldots, v_m$ of vectors in $\mathbf{V}$ is a vector of the form

$$a_1 v_1 + \cdots + a_m v_m$$

where $a_1, \ldots, a_m \in \mathbf{F}$.

Definition: Span

The set of all linear combinations of a list of vector $v_1, \ldots, v_m \in V$ is called the **span** of $v_1, \ldots, v_m$, denoted $\mathbf{span}(v_1, \ldots, v_m)$. In other words

$$\mathbf{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_1, \ldots, a_m \in \mathbf{F}\}.$$

The span of the empty list () is defined to be zero.

Some mathematicians use the term **linear span**, which means the same thing as span.

Theorem: Span is the smallest containing subspace

The span of a list of vectors in V is the smallest subspace of V containing all the vectors in the list.

Proof: $\square$

Definition: Spans

If $\textbf{span}(v_1, \ldots, v_m)$ equals V, then we say $v_1, \ldots, v_m$ **spans** V.

Example: Natual Basis

Suppose $n \in \mathbf{N}$. Show that

$$(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$$

spans $\mathbf{F^n}$. Here the j^{th} vector in the list above is the n-tuple with 1 in the j^{th} slot and 0 in all the other slots. Suppose $(x_1, \ldots, x_n) \in \mathbf{F^n}$. Then

$$(x_1, \ldots, x_n) = (x_1, 0, \ldots, 0) + \cdots + (0, \ldots, 0, x_n) = x_1(1, 0, \ldots, 0) + \cdots + x_n(0, \ldots, 0, 1)$$

Thus $(x_1, \ldots, x_n) \in \textbf{span}((1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1))$, as desired.

Now we can make one of the key definitions in linear algebra. Recall that every list, by definition, has a finite length.

Definition: Finite Dimensional Vector Space

A vector space is called **finite-dimensional** if some list of vectors in it spans the space.

The above example shows that $\mathbf{F^n}$ is a finite-dimensional vector space for every positive integer n.

Definition: Polynomial, $\mathcal{P}(\mathbf{F})$

A function $p : \mathbf{F} \to \mathbf{F}$ is called a **polynomial** with coefficients in $\mathbf{F}$ if there exists $a_0, a_1, \ldots, a_m \in \mathbf{F}$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

for all $z \in \mathbf{F}$.

- $\mathcal{P}(\mathbf{F})$ is the set of all polynomials with coefficients in $\mathbf{F}$.

With the usual operations of addition and scalar multiplication, $\mathcal{P}(\mathbf{F})$ is a vector space over $\mathbf{F}$. As such, $\mathcal{P}(\mathbf{F})$ is a subspace of $\mathbf{F}^{\mathbf{F}}$.

It will later be shown that the coefficients of the polynomial uniquely determine the polynomial.

Definition: degree of a polynomial, deg p

- A polynomial $p \in \mathcal{P}(\mathbf{F})$ is said to have ***degree*** m if there exists scalars $a_0, a_1, \ldots, a_m \in \mathbf{F}$ with $a_m \neq 0$ such that

  $$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

  for all $z \in \mathbf{F}$. If p has degree m, we write deg $p = m$.
- The polynomial that is identically 0 is said to have degree $-\infty$.

We use the convention that $-\infty < m$, which means that the polynomial 0 is in $\mathcal{P}_m(\mathbf{F})$.

Definition: $\mathcal{P}_m(\mathbf{F})$

For any $m \in \mathbb{N}$, $\mathcal{P}_m(\mathbf{F})$ denotes the set of all polynomials with coefficients in $\mathbf{F}$ and degree at most m.

Example: Finite-dimensional vector space

$\mathcal{P}_m(\mathbf{F})$ is a finite dimensional vector space for each non-negative integer m.

Definition: Infinite-dimensional vector space

A vector space is called **infinite-dimensional** if it is not finite dimensional.

Example: Infinite-dimensional vector space

$\mathcal{P}(\mathbf{F})$ is infinite dimensional.

Linear Independence

Suppose $v_1, \ldots, v_m \in V$ and $v \in \textbf{span}(v_1, \ldots, v_m)$. By the definition of span, there exists $a_1, \ldots, a_m \in \mathbf{F}$ such that

$$v = a_1 v_1 + \cdots + a_m v_m.$$

Consider the question of whether the choice of scalars in the expression is unique. Suppose $c_1, \ldots, c_n \in \mathbf{F}$ is another set of scalars such that

$$v = c_1 v_1 + \cdots + c_m v_m.$$

Subtracting the last two equations, we have

$$0 = (a_1 - c_1)v_1 + \cdots + (a_m - c_m)v_m.$$

If the only way to write 0 as a linear combination of $v_1, \ldots, v_m$ is by taking 0 for each scalar then, $a_j = c_j$ (the choice of scalars for each vector in $\textbf{span}(v_1, \ldots, v_m)$ must be unique). This situation is so important that we give it a special name, linear independence.

Definition: Linearly Independent

- A list $v_1, \ldots, v_n$ of vectors in V is called **linearly independent** if the only choice of $a_1, \ldots, a_n \in \mathbf{F}$ that makes $a_1 v_1 + \cdots + a_n v_n$ equal to 0 is $a_1 = \cdots = a_n = 0$.
- The empty list () is also declared to be linearly independent.

Definition: Linearly dependent

- A list $v_1, \ldots, v_n$ of vectors in V is called **linearly dependent** if it is not linearly independent.

Theorem: Linear Dependence Lemma

Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then there exists $j \leq m$ such that the following hold:

- $v_j \in \textbf{span}(v_1, \ldots, v_{j-1})$;
- If the j^{th} term is removed from $v_1, \ldots, v_m$, the span of the remaining list equals $\textbf{span}(v_1, \ldots, v_m)$.

Proof: $\square$

Theorem: Length of a linearly independent list $\leq$ length of spanning list

In a finite-dimensional vector space, the length of every linearly indepenedent list of vectors is less than or equal to the length of every spanning list of vectors.

Proof: $\square$

Theorem: Finite dimensional subspaces

Every subspace of a finite-dimensional vector space is finite-dimensional

Proof: $\square$

1.2 Bases

The following section derives crucial theorems built on concepts of linear independence and span.

Definition: Basis

A **basis** of V is a list of vectors in V that is linearly independent and spans V.