1 Generalized Eigenvectors

1.1 Complexification

Recall that a matrix T is diagnolizable if and only if **the sum of the geometric multiplicities equals the dimension of V.** Recall that if there are the maximum number of distinct eigenvalues, then T is diagonalizable.

Theorem: Enough roots of the characteristic polynomial

If T is diagonalizable; that is,

$$\sum_{\lambda} \text{geom.multi. of } \lambda = \dim(V)$$

then

$$\sum_{\lambda} \text{alg.multi. of } \lambda = \dim(V)$$

that is, the characteristic polynomial p_T factors as the product of linear terms.

Proof:

$\square$

If we consider the vector spaces over $\mathbb{C}$, we know that any characteristic polynomial factors as the product of linear terms by the Fundamental Theorem of Algebra

Theorem: Fundamental Theorem of Algebra

Every nonzero complex polynomial $p \in \mathcal{P}(\mathbb{C})$ can be factored, in essentially a unique way, as a product of a constant and linear terms, in the form

$$p(x) = \prod a(x - \lambda)^{\mu(\lambda)}$$

where a is the leading coefficient of $p(x)$ and $\mu(\lambda)$ denotes the algebraic multiplicity of λ.

Proof:

$\square$

These two facts can be combined to show the following

Theorem: Consequences for multiplicity

If $T : V \to V$ is $\mathbb{C}$-linear and $\dim_{\mathbb{C}}(V) < \infty$, then

$$\sum_{\lambda} \text{alg.multi. of } \lambda = \dim_{\mathbb{C}}(V)$$

also T is diagonalizable iff for each eigenvalue λ of T,

$$\text{alg.multi. of } \lambda = \text{geom.multi. of } \lambda$$

Proof:

$\square$