Linear Algebra Done Right
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1 Vector Spaces

Linear algebra is the study of linear maps on finite-dimensional vector spaces. Vector spaces are defined in this chapter, and their basic properties are developed. Vector spaces are a generalization of the description of a plane using two coordinates, as published by Descartes in 1637.

1.1 $\mathbf{R^n}$, $\mathbf{C^n}$, and $\mathbf{F^n}$

Definition: Complex Number

A **complex number** is an ordered pair $(a, b) \in \mathbf{R}^2$, denoted $a + bi$.

- The set of all complex numbers is denoted by $\mathbf{C}$:

$$\mathbf{C} = \{a + bi : a, b \in \mathbf{R}\}$$

- **Addition and multiplication** on $\mathbf{C}$ are defined by

$$(a + bi) + (c + di) \equiv (a + c) + (b + d)i$$

$$(a + bi) \cdot (c + di) \equiv (ac - bd) + (ad + bc)i$$

The following properies are proven using the familiar properites of real numbers and the definition of complex addition and multiplication.

Property: Properties of Complex Numbers

- **Commutativity :** $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbf{C}$
- **Associativity :** $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ and $(\alpha \cdot \beta) \cdot \lambda = \alpha \cdot (\beta \cdot \lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$
- **Identities :** $\lambda + e_+ \equiv \lambda + 0 \equiv \lambda$ and $\lambda \cdot e_. \equiv \lambda \cdot 1 \equiv \lambda$ for all $\lambda \in \mathbf{C}$
- **Additive Inverse :** $\forall \alpha \in \mathbf{C}, \exists \beta \in \mathbf{C}$, such that $\alpha + \beta = e_+ = 0$
- **Multiplicative Inverse :** $\forall \alpha \in \mathbf{C}, \exists \beta \in \mathbf{C}$, such that $\alpha \cdot \beta = e_. = 1$
- **Distributive Property** $\lambda \cdot (\alpha + \beta) = \lambda \cdot \alpha + \lambda \cdot \beta$, for all $\alpha, \beta, \lambda \in \mathbf{C}$

Definition: Constructed Operations on C: Subtraction and Division

Let $\alpha, \beta \in \mathbf{C}$.

- Let $-\alpha$ denote the additive inverse of α. Thus, $-\alpha$ is the unique element of $\mathbf{C}$ such that $\alpha + (-\alpha) = 0$.
- Subtraction on $\mathbf{C}$ is defined by

$$\beta - \alpha = \beta + (-\alpha)$$

- Let $(1/\alpha)$ denote the multiplicative inverse of α. Thus, $(1/\alpha)$ is the unique element of $\mathbf{C}$ such that $\alpha \cdot (1/\alpha) = 1$.
- Division on $\mathbf{C}$ is defined by

$$\beta/\alpha = \beta \cdot (1/\alpha)$$

Throughout these notes, $\mathbf{F}$ stands for either $\mathbf{R}$ or $\mathbf{C}$. The letter $\mathbf{F}$ is used because $\mathbf{R}$ and $\mathbf{C}$ are examples of the algebraic structure known as a **field**.