UNIVERSITY OF OKLAHOMA
DEPARTMENT OF MATHEMATICS

# Homework #7

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Course: *Abstract Linear Algebra* – Professor: *Dr. Gregory Muller*
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## Problem 1

Let $T : \mathbb{P}_2(\mathbb{F}) \to \mathbb{P}_2(\mathbb{F})$ be the linear map defined by

$$T(p(x)) = p(1)x^2 + p(2)x + p(-1)$$

a) Find the trace, determinant, and characteristic polynomial of $T$.
b) Find the eigenvalues of $T$.
c) For each eigenvalue, find an eigenvector of $T$.

**Answer.** For any $\mathbb{F}$-linear vector space $V$ and endomorphism $T : V \to V$, the trace and determinant of $T$ can be deduced from the second and last coefficients of the characteristic polynomial $p_T \in \mathbb{P}(\mathbb{F})$ of $T$. The characteristic polynomial $p_T$ may be calculated from the matrix $A_T$ of $T$ w.r.t. the basis $\mathcal{B} = (x^2, x, 1)$ of $\mathbb{P}_2(\mathbb{F})$.

$$
\begin{aligned}
x^2 &\mapsto x^2 + 4x + 1 \\
x &\mapsto x^2 + 2x + -1 \\
1 &\mapsto x^2 + x + 1
\end{aligned}
\qquad \Longrightarrow \qquad
A_T = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}
$$

The characteristic polynomial $p_T$ is defined as the determinant,

$$
\begin{aligned}
\det(\lambda Id_v - A_T) &= (\lambda - 1)(\lambda - 2)(\lambda - 1) + (-1) + (4) - ((\lambda - 2) + 4(\lambda - 1) + (-1)(\lambda - 1)) \\
&= (\lambda - 1)(\lambda - 2)(\lambda - 1) - 3(\lambda - 1) - (\lambda - 2) + 3 \\
&= (\lambda^3 - 4\lambda^2 + 5\lambda - 2) - (3\lambda - 3) - (\lambda - 2) + 3 \\
&= \lambda^3 - 4\lambda^2 + \lambda + 6
\end{aligned}
$$

a) So $\mathrm{tr}(T) = -(-4) = 4$ and $\det(T) = (-1)^3(6) = -6$, and

$$p_T(\lambda) = (\lambda + 1)(\lambda - 2)(\lambda - 3)$$

b) The eigenvalues of $T$ are characterized by the roots of $p_T$, $\{-1, 2, 3\}$

c) Using elimination to compute bases for the kernels of the linear transformation defined as multiplication by $\lambda Id - A_T$, the vectors in $\mathbb{P}_2(\mathbb{F})$ associated with each eigenvalue of $T$ is spanned by the following vectors

$$\lambda = 3 : -x^2 - 3x + 1 \qquad \lambda = 2 : -x^2 - 5x + 4 \qquad \lambda = -1 : -x^2 + x + 1$$

## Problem 2

Let $R : \mathbb{C}^3 \to \mathbb{C}^3$ be the linear map which acts like multiplication by

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

a) Find the trace, determinant, and characteristic polynomial of $R$.
b) Find the eigenvalues of $R$.
c) For each eigenvalue, find an eigenvector of $R$.

**Answer.** .
a) The second-highest and lowest degree coefficients of the characteristic polynomial $p_R$ indicate the trace and determinant of $R$. The characteristic polynomial of $R$ is defined as the determinant,

$$\det(\lambda Id_v - [R]_{\mathcal{B}\mathcal{B}}) = (\lambda - 1)^3 + 2^3 = \lambda^3 - 3\lambda^2 + 3\lambda - 9$$

So $\operatorname{tr}(R) = 3$ and $\det(R) = 9$, and

$$p_R(\lambda) = \lambda^3 - 3\lambda^2 + 3\lambda - 9 = (\lambda - 3)(\lambda - i\sqrt{3})(\lambda + i\sqrt{3})$$

b) The eigenvalues of $R$ are the roots of $p_R(\lambda) : \{3, i\sqrt{3}, -i\sqrt{3}\}$.
c) Using elimination to compute bases for the kernels of the linear transformation defined as multiplication by $\lambda Id - A_T$, the vectors in $\mathbb{P}_2(\mathbb{F})$ associated with each eigenvalue of $T$ is spanned by the following vectors

$$\lambda = 3 : (1, 1, 1)$$
$$\lambda = i\sqrt{3} : (-1 - i\sqrt{3}, -1 + i\sqrt{3}, 2)$$
$$\lambda = -i\sqrt{3} : (-1 + i\sqrt{3}, -1 - i\sqrt{3}, 2)$$

## Problem 3

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Let $\mathbf{dim}(V) < \infty$ and let $P : V \to V$ be an **idempotent** linear map; that is, $P^2 = P$. Assuming we know that $P$ is not the zero map nor the identity map, find the minimal polynomial of $P$.

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**Answer.** Let $\mathbf{dim}(V) = n$. Since $P$ is idempotent, $P = P^2$, so $P(P - Id) = 0$.

$$\det P(P - Id) = 0$$

the Cayley-Hamilton Theorem implies $p_P$ has the property,

$$p_P(P) = p_P(P^2) = 0$$

Since any matrix of a linear map on $V$ must have exactly $n$ entries along the diagonal,

$$
\begin{aligned}
p_P(P) &= P^n + a_{n-1}P^{n-1} + \cdots + a_2 P^2 + a_1 P^1 + a_0 P^0 \\
&= P^n - \mathrm{tr}(P)P^{n-1} + \cdots + a_1 P^1 + (-1)^n \det(P)Id \\
&= P - \mathrm{tr}(P)P + \cdots + a_1 P + (-1)^n \det(P)Id \\
&= P(-\mathrm{tr}(P)Id + \cdots + a_1) + (-1)^n \det(P)Id
\end{aligned}
$$

## Problem 4

Let $T : V \to V$ be an invertible linear map with $\mathbf{dim}(V) < \infty$.

a) Show that the characteristic polynomial of $T^{-1}$ is

$$\frac{1}{p_T(0)} x^{\mathbf{dim}(V)} p_T(x^{-1})$$

b) Show that there is a polynomial $p(x)$ with $p(T) = T^{-1}$.

**Answer.** $T$ invertible $\implies$ $\ker(T) = 0$ $\implies$ $Tv = 0$ only if $v = 0$ $\implies$ $0$ is not an eigenvalue of $T$ $\implies$ $0$ is not a root of $p_T$ $\implies$ $1/p_T(0)$ exists.

## Problem 5

Goal: show that the eigenvalues of a linear transformation are always roots of the minimal polynomial.

Let $T : V \to V$ be a linear map with $\mathbf{dim}(V) < \infty$.

a) Let $v$ be an eigenvector of $T$ with eigenvalue $\lambda$. Show that, for every polynomial $p(x)$,
$$p(T)v = p(\lambda)v$$

b) Show that every eigenvalue is a root of the minimal polynomial of $T$.

c) Show that, if $T$ has $\mathbf{dim}(V)$-many distinct eigenvalues $\lambda_1, \ldots, \lambda_{\mathbf{dim}(V)}$, then the characteristic polynomial of $T$ equals the minimal polynomial of $T$.

**Answer.** .
a) If $v$ is an eigenvector of $T$ with eigenvalue $\lambda$. Then

$$
\begin{aligned}
p(T)v &= (a_m T^m + a_{m-1} T^{m-1} + \cdots + a_1 T^1 + a_0 T^0)v \\
&= a_m T^m v + a_{m-1} T^{m-1} v + \cdots + a_1 T^1 v + a_0 T^0 v \\
&= a_m \lambda^m v + a_{m-1} \lambda^{m-1} v + \cdots + a_1 \lambda^1 v + a_0 \lambda^0 v \\
&= (a_m \lambda^m + a_{m-1} \lambda^{m-1} + \cdots + a_1 \lambda^1 + a_0 \lambda^0)v \\
&= p(\lambda)v
\end{aligned}
$$

b) Let $\mu_T$ denote the minimal polynomial of $T$, $\lambda_i$ denote the $i$'th eigenvalue of $T$, and $v_i \in V$ be the non-zero eigenvector associated with $\lambda_i$.

Since $\mu(T)v_i = \mu(\lambda_i)v_i = 0$, we have two cases $\mu(\lambda_i) = 0$ or $v_i = 0$, but we assumed $v_i$ is a non-zero eigenvector, so $\lambda_i$ must be a root of $\mu$. Since $i$ is arbitrary, all eigenvalues of $T$ are roots of the minimal polynomial of $T$.

c) Let $\mu_T$ denote the minimal polynomial of $T$. Since the set of eigenvalues of $T$, $\lambda_1, \lambda_2, \ldots, \lambda_{\mathbf{dim}(V)}$ are the of roots of $\mu_T$ and $p_T$, we can factorize $\mu_T$ into the map defined by

$$\mu_T(z) = q(z)(z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_{\mathbf{dim}(V)})$$

and $p_T$ into the map defined by

$$p_T(z) = r(z)(z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_{\mathbf{dim}(V)})$$

Since $\mathbf{deg}(\mu_T) \leq \mathbf{deg}(p_T) = \mathbf{dim}(V)$ and $\mu_T$ and $p_T$ is monic, the above factors $q(z) = r(z) = (-1)^{\mathbf{dim}(V)}$. Therefore, when $T$ has $\mathbf{dim}(V)$ distinct eigenvalues, $\mu_T = p_T$.