University of Oklahoma
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Homework #3

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Course: *Abstract Linear Algebra* – Professor: *Dr. Gregory Muller*
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2C: 1

Suppose V is finite-dimensional and U is a subspace of V such that dim U = dim V. Prove that $U = V$.

Answer. Suppose $\mathbf{dim}(U) = \mathbf{dim}(V) = n$ and U is a subspace of V. Then,

$$\begin{aligned} U \text{ is finite dimensional} &\implies \exists \text{ a basis } u_1, \ldots, u_n \text{ of } U \\ &\implies u_1, \ldots, u_n \text{ is linearly independent in } V \end{aligned}$$

Since every linearly independent list of vectors in V of length n is a basis of V, it must be true that $u_1, \ldots, u_n$ is a basis of V. Therefore $U = \mathbf{span}(u_1, \ldots, u_n) = V$. ■

2C: 7

Let
$$U = \{p \in \mathcal{P}_4(\mathbb{F}) \ : p(2) = p(5)\}.$$

Answer. .
(a) Find a basis of U.
Let $p \in U$ be any polynomial such that $p(2) = p(5) = d$. Then $\exists a, b, c, d \in \mathbb{F}$ s.t.,

$$\begin{aligned} p(x) &= (ax^2 + bx + c)(x-2)(x-5) + d && (1) \\ &= ax^2(x-2)(x-5) + bx(x-2)(x-5) + c(x-2)(x-5) + d \cdot 1 && (2) \\ &= ax^4 + (-7a+b)x^3 + (10a - 7b + c)x^2 + (10b+c)x + (10c+d)1 && (3) \end{aligned}$$

Hence (2) describes all elements of the substructure $\mathcal{P}_4(\mathbb{F})$ with a single linear combination. Additionally, by solving for $\mathbf{0}$ using the system of coefficients in (3),

$$p(x) = \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ -7 & 1 & 0 & 0 \\ 10 & -7 & 1 & 0 \\ 0 & 10 & -7 & 0 \\ 0 & 0 & 10 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \right) \cdot \begin{pmatrix} x^4 \\ x^3 \\ x^2 \\ x \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x^4 \\ x^3 \\ x^2 \\ x \\ 1 \end{pmatrix} \qquad (4)$$

it is clear that $p = \mathbf{0} \iff a = b = c = d = 0$. Therefore,

$$\mathcal{B} = \{x^2(x-2)(x-5), x(x-2)(x-5), (x-2)(x-5), 1\}$$

is a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbb{F})$
By solving for $\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix}^T$ using the matrix equation in (4), it is clear that $x \notin \mathbf{span}(\mathcal{B})$. Therefore, appending x to $\mathcal{B}$ gives five linearly independent polynomials in $\mathcal{P}_4(\mathbb{F})$. Consequentally, $\mathcal{B} \cup \{x\}$ is a basis of $\mathcal{P}_4(\mathbb{F})$ because any $\mathbf{dim}(\mathcal{P}_4(\mathbb{F}))$ linearly independent vectors in $\mathcal{P}_4(\mathbb{F})$ generates $\mathcal{P}_4(\mathbb{F})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbb{F})$ such that $\mathcal{P}_4(\mathbb{F}) = U \oplus W$.
Let $W = \mathbf{span}(x)$. According to (b), when $p \in \mathcal{P}(\mathbb{F})$ is the mapping from $x \overset{p}{\mapsto} x$, we have $p \notin U$. By the closure of scalar multiplication in U, we have $kp \notin U$ for all $k \in \mathbb{F}$. Therefore, we know,

$$W \not\subset U \implies W \cap U = \{0\}$$

Part (b) also concludes that

$$\mathbf{span}(\{x^2(x-2)(x-5), x(x-2)(x-5), (x-2)(x-5), 1\} \cup \{x\}) = \mathcal{P}_4(\mathbb{F})$$

By the definition of span, the previous equation implies

$$\mathcal{P}_4(\mathbb{F}) = \mathbf{span}(\{x^2(x-2)(x-5), x(x-2)(x-5), (x-2)(x-5), 1\}) + \mathbf{span}(\{x\}) = U \oplus W$$

2C: 10

Suppose $p_0, p_1, \ldots, p_m \in \mathcal{P}(\mathbb{F})$ are such that each p_j has degree j. Prove that $p_0, p_1, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbb{F})$.

Answer. By strong induction it will be shown that $p_0, p_1, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbb{F})$ where each p_j has degree j.

Base cases: Let $m = 0$. Any $p_0, g_0 \in \mathcal{P}(\mathbb{F})$ where $\mathbf{deg}(p_0) = \mathbf{deg}(g_0) = 0$, say $p_0(x) = c$ and $g_0(x) = c'$, can be expressed in terms of each other by choosing c'/c as a scaling factor for p_0, since

$$\frac{c'}{c} p_0(x) = \frac{c'}{c} c = c' = g_0(x)$$

Hence any $g \in \mathcal{P}_0(\mathbb{F})$ can be expressed as a unique linear combination of a degree 0 polynomial.

Let $m = 1$. For any $a'x + b' \in \mathcal{P}_1(\mathbb{F})$, we have

$$\begin{aligned} a'x + b' &= y_1(ax+b) + y_0(c) \\ \iff y_1 = \frac{a'}{a}, y_0 &= (b' - y_1 b) = (b' - \frac{a'}{a} b) \end{aligned}$$

demonstrating that any $g \in \mathcal{P}_1(\mathbb{F})$ can be expressed as a unique linear combination of a degree 1 polynomial and degree 0 polynomial. Hence p_1, p_0 is a basis of $\mathcal{P}_1(\mathbb{F})$.

Let $m = 2$. For any $a'x^2 + b'x + c' \in \mathcal{P}_2(\mathbb{F})$, we have

$$\begin{aligned} a'x^2 + b'x + c' &= y_2(a_{22}x^2 + a_{21}x + a_{20}) + y_1(a_{11}x + a_{10}) + y_0(a_{00}) \\ \iff y_2 = \frac{a'}{a_{22}}, y_1 = \frac{1}{a_{11}}(b' - \frac{a'a_{21}}{a_{22}}), y_0 &= \frac{1}{a_{00}}(c' - \frac{a'a_{20}}{a_{22}} - \frac{a_{10}}{a_{11}}(b' - \frac{a'a_{21}}{a_{22}})) \end{aligned}$$

demonstrating that any $g \in \mathcal{P}_2(\mathbb{F})$ can be expressed as a unique linear combination of a degree 2 polynomial, degree 1 polynomial, and degree 0 polynomial. Therefore p_2, p_1, p_0 is a basis of $\mathcal{P}_2(\mathbb{F})$.

Induction step: Let $n = k$ be given and suppose $p_0, p_1, \ldots, p_k$ is a basis for $\mathcal{P}_k(\mathbb{F})$. Then any $g \in \mathcal{P}_{k+1}(\mathbb{F})$ can be written as a linear combination of $p_0, p_1, \ldots, p_k$ and a $p_{k+1}(x) = a_{(k+1)0} + a_{(k+1)1}x + \cdots + a_{(k+1)(k+1)}x^{k+1}$ term. More precisely, any g in

$$g(x) = a_0 p_0(x) + a_1 p_1(x) + \cdots + p_{k+1}(x)$$

where $p_{k+1}(x)$ is defined immediately above, but we assumed,

$$a_{(k+1)0} + a_{(k+1)1}x + \cdots + a_{(k+1)k}x^k \in \mathbf{span}(p_0, p_1, \ldots, p_k)$$

Using the process described in the proof of the linear dependence lemma, these redunant terms of p_{k+1} can be removed until either every term in p_{k+1} vanishes or until $g(x)$ can be expressed as a unique linear combination of $p_0, p_1, \ldots, p_k$, and x^{k+1}. More precisely, any $g \in \mathcal{P}_{k+1}(\mathbb{F})$ can be expressed as a unique linear combination of $p_0, p_1, \ldots, p_{k+1}$ such that each p_j has degree j. Therefore, the statement $p_0, p_1, \ldots, p_n$ such that each p_j has degree j is a basis for $\mathcal{P}_n(\mathbb{F})$ holds for $n = k+1$, and the inductive step is complete.

2C: 12

Suppose U and W are both five-dimensional subspaces of $\mathbb{R}^9$. Prove that $U \cap W \neq \{0\}$.

Answer. If U and W are subspaces with $\mathbf{dim}(U) = \mathbf{dim}(W) = 5$, then there exists 5 linearly independet elements $u_1, \ldots, u_5 \in U \subset \mathbb{R}^9$ and 5 linearly independent elements $w_1, \ldots, w_5 \in W \subset \mathbb{R}^9$. Assume $u_j \neq w_j$ for all $j \in \{1, 2, 3, 4, 5\}$.

Because $\mathbb{R}^9$ is finite-dimensional, the cardinality of any linearly independent set in $\mathbb{R}^9$ is less than or equal to the cardinality of a spanning set. Since $e_1, \ldots, e_9 \in \mathbb{R}^9$ is a spanning set of $\mathbb{R}^9$,

$$u_1, \ldots, u_5 \cup w_1, \ldots, w_5$$

cannot possibly be linearly independent. Therefore there exists a linear combination,

$$\mathbf{0} = a_1 u_1 + \cdots + a_5 u_5 + b_1 w_1 + \cdots + b_5 w_5$$

where $a_1, \ldots, a_5, b_1, \ldots, b_5$ are not all $\mathbf{0}$. By a theorem, if U and W are subspaces, then $U \cap W = \{0\}$ if and only if they are linearly independent. Since there exists a non-zero linear combination of elements of U and W equal to $\mathbf{0}$, the sets are not linearly independent. Therefore,

$$U \cap W \neq \{0\}$$