UNIVERSITY OF OKLAHOMA
DEPARTMENT OF MATHEMATICS

Homework #4

Student name: *Clayton Curry*

Course: *Abstract Linear Algebra* – Professor: *Dr. Gregory Muller*
Due date: *Sep 19, 2021*

1C: 24

A function $f : \mathbb{R} \to \mathbb{R}$ is called even if

$$f(-x) = f(x)$$

for all $x \in \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is called odd if

$$f(-x) = -f(x)$$

for all $x \in \mathbb{R}$.
Let U_e denote the set of real-valued even functions on $\mathbb{R}$ and let U_o denote the set of real-valued odd functions on $\mathbb{R}$. Show that

$$\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$$

Answer. If $f \in \mathbb{R}^{\mathbb{R}}$, then

$$\begin{aligned} f(x) &= \frac{f(x)+f(x)}{2} + 0 \\ &= \frac{f(x)+f(x)}{2} + \frac{f(-x)-f(-x)}{2} \\ &= \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} \end{aligned}$$

for all $x \in \mathbb{R}$. Notice,

$$\frac{f(x)+f(-x)}{2} \in U_e$$

and

$$\frac{f(x)-f(-x)}{2} \in U_o$$

which implies that f can be uniquely writted as the sum of an even function and an odd function. Since any $f \in \mathbb{R}^{\mathbb{R}}$ can be expressed as a unique sum of an even function and odd function, $\mathbb{R}^{\mathbb{R}}$ is a direct sum of U_o and U_e. ■

2C: 11

Suppose that U and W are subspaces of $\mathbb{R}^8$ such that $\mathbf{dim}(U) = 3$, $\mathbf{dim}(W) = 5$ and $U + W = \mathbb{R}^8$. Prove that $\mathbb{R}^8 = U \oplus W$.

Answer. Since $\mathbf{dim}(U) = 3$ and $\mathbf{dim}(W) = 5$, there exists three linearly independent spanning vectors u_1, u_2, u_3 in U and five linearly independent spanning vectors w_1, w_2 w_3, w_4, w_5 in W. Since any spanning list of vectors in a subspace with length equal to the dimension of the subspace is a basis of the subspace, $u_1, u_2, u_3, w_1, w_2, w_3, w_4, w_5$ are necessarily a basis of $U + W$ and by extension linearly independent in $U + W$. Since $u_1, u_2, u_3, w_1, w_2, w_3, w_4, w_5$ are linearly independent, the only way to write $\mathbf{0}$ as a linear combination of u's and w's is by taking each coefficient to $\mathbf{0}$. Another theorem states that the sum of subspaces is a direct sum if and only if a linear combination equal to zero is the trivial one. Therefore, it then follows that $\mathbb{R}^8 = U \oplus W$.

3A: 4

Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_m$ is a list of vectors in V such that $Tv_1, \ldots, Tv_m$ is linearly independent in W. Prove that $v_1, \ldots, v_m$ is linearly independent.

Answer. Suppose $c_1, \ldots, c_n \in \mathbb{F}$ where at least one $c_j \neq 0$ such that

$$\mathbf{0} = c_1 v_1 + \cdots + c_n v_n$$

If $Tv_1, \ldots, Tv_m$ is linearly independent in W, we have

$$\mathbf{0} = T(c_1 v_1 + \cdots + c_n v_n) = c_1 T v_1 + \cdots + c_n T v_n$$

where at least one $c_j \neq 0$, which is a contratiction. More precisely, by assuming some set of independent vectors in W under some $T \in \mathcal{L}(V, W)$ is dependent in V, we have reached a contradiction. Therefore any set of independent vectors in W under some $T \in \mathcal{L}(V, W)$ must also be independent in V. ■

3A: 7

Show that every linear map from a 1-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\mathbf{dim}(V) = 1$ and $T \in \mathcal{L}(V, V,)$ then there exists $\lambda \in \mathbb{F}$ such that $Tv = \lambda v$ for all $v \in V$.

Answer. If $T \in \mathcal{L}(V, V)$, then T maps elements from $\{kv \in V : k \in \mathbb{F}, v \in V, v \neq 0\}$ back to itself, because any non-zero $v \in V$ is a basis of V. By selecting the appropriate $\lambda \in \mathbb{F}$ such that $v \overset{T}{\mapsto} \lambda v$, then for any $kv \in V$, we have $T(kv) = kT(v) = k(\lambda v) = (k\lambda)v = (\lambda k)v = \lambda(kv)$. Since any element kv maps to $\lambda(kv)$ under T, the transformation T is indistinguishable from multiplication by λ.

3A: 14

Suppose V is finite-dimensional with $\mathbf{dim}(V) = 2$. Prove that there exist $S, T \in \mathcal{L}(V, V)$ such that $ST \neq TS$.

Answer. Let $V = \mathbb{R}^2$, and define the maps T and S by the following equations,

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad S\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

for all $\begin{pmatrix} a & b \end{pmatrix}^T \in \mathbb{R}^2$. Notice ST is defined by

$$ST\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}\left(\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}\right) = \left(\begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\right)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

and

$$TS\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\left(\begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}\right) = \left(\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}\right)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

but

$$\begin{pmatrix} 3 & 4 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 6 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

for some $\begin{pmatrix} a & b \end{pmatrix}^T \in \mathbb{R}^2$. Hence this example provides a case of S, T with $ST \neq TS$. ■