1 Linear Maps

Note: Notational Shortcuts

- $\mathbf{F}$ denotes $\mathbf{R}$ or $\mathbf{C}$
- $\mathbf{V}$ and $\mathbf{W}$ denote vector spaces over $\mathbf{F}$

1.1 The Vector Space of Linear Maps

Definition: Linear Map

A **linear map** from V to W is a function $T : V \to W$ with the following properies:

- **Additivity:** $T(u + v) = Tu + Tv$ for all $u, v \in V$;
- **Homogeneity:** $T(\lambda v) = \lambda Tv$ for all $\lambda \in \mathbf{F}$ and $v \in V$.

Some mathematicians use the term **linear transformation** or **vector space homomorphism**, which means the same thing as linear map.

Definition: $\mathcal{L}(V, W)$

The set of all linear maps from $\mathbf{V}$ to $\mathbf{W}$ is denoted $\mathcal{L}(\mathbf{V}, \mathbf{W})$.

It is easy to verify that each of the functions defined below is indeed a linear map.

Example: Linear Maps

Zero
In addition to other uses, the symbol 0 can be used to denote the function that takes each element of the vector space to zero. To be specific, $0 \in \mathcal{L}(V, W)$ is defined by

$$0v = 0$$

Identity
The **identity map** denoted I can be used to denote the function that takes each element of the vector space to itself. To be specific, $I \in \mathcal{L}(V, V)$ is defined by

$$Iv = v$$

Differentiation
Let $f, g \in \mathbf{R}^{\mathbf{R}}$ be once-differentiable functions in $\mathbf{R}$. Notice

$$\frac{d}{dx}(f+g)(x) = \frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) = (\frac{d}{dx}f + \frac{d}{dx}g)(x).$$

$$\frac{d}{dx}(\lambda f)(x) = \frac{d}{dx}(\lambda f(x)) = \lambda\frac{d}{dx}f(x)$$

Integration
Let $f, g \in \mathbf{R}^{\mathbf{R}}$ be once-integrable functions over $(a, b) \subseteq \mathbf{R}$. Notice

$$\int_a^b (f+g)(x)dx = \int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b (\lambda f)(x)dx = \int_a^b (\lambda f(x))dx = \lambda\int_a^b f(x)dx$$

The previous two assertions, that differentiation and integration are linear, are other ways of stating the basic results about differentiation and integration: the derivative or integral of a sum is the sum of the derivatives or integrals; the derivative or integral of a constant times a function is the constant multiple of the derivative or integral of the function.

Example: Linear Maps

Multiplication by x^2
Define $T \in \mathcal{L}(\mathcal{P}(\mathbf{F}), \mathcal{P}(\mathbf{F}),)$ by,

$$(Tp)(x) = x^2 p(x) \quad \text{for } x \in \mathbf{F}.$$

Backward shift
Recall that $\mathbf{F}^\infty$ denotes the vector space of all sequences of elements in $\mathbf{F}$. Define $T \in \mathcal{L}(\mathbf{F}^\infty, \mathbf{F}^\infty)$ by

$$T(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots)$$

From $\mathbf{R}^3$ to $\mathbf{R}^2$
Define $T \in \mathcal{L}(\mathbf{R}^3, \mathbf{R}^2)$ by

$$T(x, y, z) = (2x - y + 3z, 7x + 5y - 6z)$$

The uniqueness part of the next result means that a linear map is completely determined by its values on a basis.

Theorem: Linear maps and basis of domain

Suppose $v_1, \ldots, v_n$ is a basis of $\mathbf{V}$ and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T : \mathbf{V} \to \mathbf{W}$ such that

$$Tv_j = w_j$$

for each $j = 1, \ldots, n$.

Proof:
First we show existence, then uniqueness. Define $T : \mathbf{V} \to \mathbf{W}$ by

$$T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n$$

where $c_1, \ldots, c_n$ are arbitrary elements of $\mathbf{F}$. Since $v_1, \ldots, v_n$ is a basis, T is well-defined and is therefore a function from V to W. For each $j \in \{1, \ldots, n\}$, taking $c_j = 1$ and the other c's equal to 0 in the equation shows $Tv_j = w_j$. The map can be easily verified to be a vector space homomorphism. Thus, the desired map exists.

To prove uniqueness, supppose $S, T \in \mathcal{L}(\mathbf{V}, \mathbf{W})$ and that $\forall j \in \{1, \ldots, n\}, Sv_j = Tv_j = w_j$. Then $\forall a_1, \ldots, a_n \in \mathbf{F}$,

$$\begin{aligned}
S(a_1 v_1 + \cdots + a_n v_n) &= a_1 S v_1 + \cdots a_n S v_n \\
&= a_1 w_1 + \cdots a_n w_n \\
&= a_1 T v_1 + \cdots a_n T v_n \\
&= T(a_1 v_1 + \cdots + a_n v_n) \\
&\implies S = T \text{ for all elements in } \mathbf{span}(v_1, \ldots, v_n).
\end{aligned}$$

But $\mathbf{V} = \mathbf{span}(c_1, \ldots, c_n)$, which implies $S = T5pt\forall v \in \mathbf{V} \implies S = T \implies T$ is unique. $\square$

Algebraic Operations on $\mathcal{L}(\mathbf{V}, \mathbf{W})$

Definition: Addition and Scalar Multiplication on $\mathcal{L}(\mathbf{V}, \mathbf{W})$

Suppose $S, T \in \mathcal{L}(\mathbf{V}, \mathbf{W})$ and $\lambda \in \mathbf{F}$. The **sum** $S + T$ and the **product** λT are the maps from $\mathbf{V}$ to $\mathbf{W}$ defined by:

$$(S+T)(v) = Sv + Tv \quad \text{and} \quad (\lambda T)(v) = \lambda(Tv)$$

for all $v \in \mathbf{V}$.

Theorem: $\mathcal{L}(\mathbf{V}, \mathbf{W})$ is a vector space

With the operations of addition and scalar multiplication as defined above, $\mathcal{L}(\mathbf{V}, \mathbf{W})$ is a vector space.

Proof:

□

Definition: Product of Linear Maps

If $T \in \mathcal{L}(\mathbf{U}, \mathbf{V})$ and $S \in \mathcal{L}(\mathbf{V}, \mathbf{W})$, then the **product** $ST : \mathbf{U} \to \mathbf{W}$ is defined by

$$(ST)(u) = S(Tu)$$

for $u \in \mathbf{U}$.

Theorem: The product of linear maps is a linear map

If $T \in \mathcal{L}(\mathbf{U}, \mathbf{V})$ and $S \in \mathcal{L}(\mathbf{V}, \mathbf{W})$, then $ST \in \mathcal{L}(\mathbf{U}, \mathbf{W})$.

Proof:

Suppose $u, v \in \mathbf{U}$, $T \in \mathcal{L}(\mathbf{U}, \mathbf{V})$, and $S \in \mathcal{L}(\mathbf{V}, \mathbf{W})$

$$\begin{aligned} ST(u+v) &= S(Tu + Tv) \\ &= S(Tu) + S(Tv) \\ &= ST(u) + ST(v) \end{aligned} \qquad \begin{aligned} ST(\lambda u) &= S(\lambda Tu) \\ &= \lambda S(Tu) \\ &= \lambda ST(u) \end{aligned}$$

□

Property: Algebraic properties of products of linear maps

- **Associativity:**
$$(T_1 T_2) T_3 = T_1 (T_2 T_3)$$
whenever T_1, T_2, and T_3 are linear maps such that products make sense.
- **Identity:**
$$TI = IT = T$$
whenever $T \in \mathcal{L}(V, W)$ and the left I is the identity map on V and the right I is the identity map on W.
- **Distributitvity:**
$$(S_1 + S_2)T = S_1 T + S_2 T \quad \text{and} \quad S(T_1 + T_2) = ST_1 + ST_2$$
whenever $S, S_1, S_2 \in \mathcal{L}(V, W)$ and $T, T_1, T_2 \in \mathcal{L}(U, V)$.

Theorem: Linear maps take 0 to 0

Suppose $T \in \mathcal{L}(\mathbf{V}, \mathbf{W})$. Then $T(\mathbf{0}) = \mathbf{0}$.

Proof:

Let $v \in \mathbf{V}$. Then

$$T(\mathbf{0}) = T(0v) = 0T(v) = \mathbf{0}$$

as desired $\square$

1.2 Null Spaces and Ranges

Null Spaces and Injectivity

In this section, we will learn about two subspaces intimitely connected with each linear map. The first subspace exists in the domain; the second exists in the codomain.

Definition: Null Space

For $T \in \mathcal{L}(\mathbf{V}, \mathbf{W})$, the **null space** of T, denoted null T, is the subset of $\mathbf{V}$ consisting of all the vectors $v \in \mathbf{V}$ that T maps to $0 \in \mathbf{W}$:

$$\text{null } T = \{v \in \mathbf{V} : Tv = 0 \in \mathbf{W}\}$$

Some mathematicians use the term **kernel** instead of null space.

Theorem: The null space is a subspace

Suppose $T \in \mathcal{L}(\mathbf{V}, \mathbf{W})$. Then null T is a subspace of $\mathbf{V}$.

Proof:

Suppose $u, v \in$ null T.

1. The additive identity $\mathbf{0} \in$ null T because $T(\mathbf{0}) = \mathbf{0}$.
2. $T(u + v) = Tu + Tv = \mathbf{0} + \mathbf{0} = \mathbf{0} \implies u + v \in \text{null } T$
3. $T(\lambda v) = \lambda Tv = \lambda \mathbf{0} = \mathbf{0} \implies \lambda v \in \text{null } T$

Hence, null T is a subspace of $\mathbf{V}$. $\square$

Definition: Injective, one-to-one

A function $T : \mathbf{V} \to \mathbf{W}$ is called injective if $Tu = Tv$ implies $u = v$.

Theorem: Injectivity is equivalent to null space equals $\{0\}$

Let $T \in \mathcal{L}(\mathbf{V}, \mathbf{W})$. Then T is injective if and only if null $T = \{\mathbf{0}\}$

Proof:

T is injective $\implies$ only one vector in $\mathbf{V}$ is sent to $\mathbf{0}$ in $\mathbf{W} \implies$ null $T = \{\mathbf{0}\}$.

null $T = \{\mathbf{0}\}$ and $Tv = Tu \implies \mathbf{0} = Tv - Tu = T(v - u) \implies v - u \in \text{null } T \implies v = u$. $\square$

Range and Surjectivity

Definition: Range

For a function $T : \mathbf{V} \to \mathbf{W}$, the range of T is the subset of $\mathbf{W}$ consisting of those vectors that are of the form Tv for some $v \in \mathbf{V}$:

$$\text{range } T = \{Tv : v \in \mathbf{V}\}$$

Theorem: The range is a subspace

Suppose $T \in \mathcal{L}(\mathbf{V}, \mathbf{W})$. Then range T is a subspace of $\mathbf{W}$.

Proof:

Suppose $u, w \in$ range T and $T(v_1) = u$ and $T(v_2) = w$.

1. The additive identity $\mathbf{0} \in$ range T because $T(\mathbf{0}) = \mathbf{0}$.
2. $u + w = Tv_1 + Tv_2 = T(v_1 + v_2) \implies u + v \in$ range T
3. $\lambda u = \lambda Tv_1 = T(\lambda v_1) \implies \lambda u \in$ range T

□

Definition: Surgective, onto

A function $T : V \to W$ is called **surjective** or **onto** if its range is equal to W.

Fundamental Theorem of Linear Maps

The next result is so important that it gets a dramatic name.

Theorem: Fundamental Theorem of Linear Maps

Suppose $\mathbf{V}$ is finite-dimensional and $T \in \mathcal{L}(\mathbf{V}, \mathbf{W})$. Then range T is finite-dimensional and

$$\dim \mathbf{V} = \dim \operatorname{null} T + \dim \operatorname{range} T$$

Proof:

Because null T is a subspace of $\mathbf{V}$, null T has a basis $v_1, \ldots, v_m$. Because $v_1, \ldots, v_m$ is linearly independent in $\mathbf{V}$, the list can be extended to a basis of $v_1, \ldots, v_m, w_1, \ldots, w_n$ of $\mathbf{V}$. Therefore dim $V = m + n =$ dim null $T + n$, and the above proposition holds if it can be demonstrated that dim range $T = n$.

Notice that when T acts on any $v \in \mathbf{V}$, we have

$$\begin{aligned} Tv &= T(c_1 v_1 + \cdots + c_m v_m + d_1 w_1 + \cdots + d_n w_n) \\ &= c_1 Tv_1 + \cdots + c_m Tv_m + d_1 Tw_1 + \cdots + d_n Tw_n \\ &= \mathbf{0} + \cdots + \mathbf{0} + d_1 Tw_1 + \cdots + d_n Tw_n \\ &= d_1 Tw_1 + \cdots + d_n Tw_n \\ &\implies Tw_1, \ldots, Tw_n \text{ spans range } T \end{aligned}$$

If we select a $v \in \mathbf{V}$ such that $Tv = \mathbf{0}$, then

$$\begin{aligned} Tv = d_1 Tw_1 + \cdots + d_n Tw_n = \mathbf{0} &\implies v \in \text{null } T \\ &\implies v = c_1 v_1 + \cdots + c_m v_m \\ &\implies d_1 = \cdots = d_m = 0 \\ &\implies Tw_1, \cdots, Tw_n \text{ is linearly independent} \\ &\implies Tw_1, \cdots, Tw_n \text{ is a basis for range } T \end{aligned}$$

Because $Tw_1, \cdots, Tw_n$ is a basis, it follows that dim range $T = n$, as desired. By this logic, we conclude that if $\mathbf{V}$is finite-dimensional and $T \in \mathcal{L}(\mathbf{V}, \mathbf{W})$ then,

$$\dim \mathbf{V} = \dim \operatorname{null} T + \dim \operatorname{range} T$$

□

Now we can show that no injective linear map exists from a finite-dimensional vector space to a "smaller" (is measured in dimension).

Theorem: A map to a smaller dimensional subspace is not injective

Suppose $\mathbf{V}$ and $\mathbf{W}$ are finite-dimensional vector spaces such that dim $\mathbf{V} >$ dim $\mathbf{W}$. Then no linear map from $\mathbf{V}$ to $\mathbf{W}$ is injective.

Proof:
Take any $T \in \mathcal{L}(\mathbf{V}, \mathbf{W})$ and suppose dim $\mathbf{V} >$ dim $\mathbf{W}$. By the Fundamental Theorem of Linear Maps,

$$\begin{aligned} \text{dim null } T &= \text{dim } \mathbf{V} - \text{dim range } T \\ \text{dim null } T &\geq \text{dim } \mathbf{V} - \text{dim } \mathbf{W} \\ \text{dim null } T &> 0 \end{aligned}$$

This means null T contains vectors other than $\mathbf{0}$. Therefore, T is not injective. □

Theorem: A map to a larger dimensional subspace is not surjective

Suppose V and W are finite-dimensional vector spaces such that dim $V <$ dim W. Then no linear map from V to W is surjective.

Proof:
Take any $T \in \mathcal{L}(\mathbf{V}, \mathbf{W})$ and suppose dim $\mathbf{V} <$ dim $\mathbf{W}$. By the Fundamental Theorem of Linear Maps,

$$\begin{aligned} \text{dim range } T &= \text{dim } \mathbf{V} - \text{dim null } T \\ &\leq \text{dim } \mathbf{V} \\ &< \text{dim } \mathbf{W} \end{aligned}$$

Since dim range $T <$ dim $\mathbf{W}$, range T does not span $\mathbf{W}$. Therefore, T is not surjective. □

Theorem: Homogenous system of linear equations

A homogenous system of linear equations with more variables than equations has non-zero solutions.

Proof:

□

Theorem: Inhomogeneous system of linear equations

An inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms.

Proof:

□

1.3 Matrices

Suppose $T \in \mathcal{L}(\mathbf{V}, \mathbf{W})$ and $v_1, \ldots, v_n$ is a basis of $\mathbf{V}$, then the values of $Tv_1, \ldots, Tv_n$ determine the action of T on any arbitrary vectors in V. Matrices became popular for recording the values of the Tv_j's in terms of a basis of W.

Definition: Coefficient Vectors

If $\mathcal{B} = (v_1, \ldots, v_n)$ is an ordered basis for $\mathbf{V}$, then for any $u \in \mathbf{V}$, where

$$u = a_1 v_1 + \cdots + a_n v_n$$

the **coefficient vector** of $u \in V$ is defined to be the element of $\mathbf{F^n}$,

$$[u]_{\mathcal{B}} := (a_1 n_1 + \cdots + a_) n_)$$

where the entries of $[u]_{\mathcal{B}}$ correspond to the coefficients in the unique linear combination of $_{\mathcal{B}}$ equal to u.

Definition: Matrix, $A_{j,k}$

Let m and n denote positive integers. An m-by-n **matrix** A is a rectangular array of elements of $\mathbf{F}$ with m rows and n columns:

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$$

The notation $A_{j,k}$ denotes the entry in row j, column k of A. In other words, the first index refers to the row number and the second index refers to the column number.

Now we come to the key definition in this section.

Definition: Matrix of a Linear Map, $\mathcal{M}(T)$

Suppose $T \in \mathcal{L}(\mathbf{V}, \mathbf{W})$ and $v_1, \ldots, v_n$ is a basis of $\mathbf{V}$and $w_1, \ldots, w_m$ is a basis of $\mathbf{W}$. The **matrix of** T with respect to these bases is the m-by-n matrix $\mathcal{M}(T)$ whose entries $A_{j,k}$ are defined by,

$$Tv_k = A_{1,k} w_1 + \cdots + A_{m,k} w_m \quad \text{for } k = 1, \cdots, n.$$

If the bases are not clear from the context, then the notation $\mathcal{M}(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m))$ is used.

To remember how $\mathcal{M}(T)$ is constructed from T, you might write across the top of the matrix the basis vectors $v_1, \ldots, v_n$ for the domain and along the left the basis vectors $w_1, \ldots, w_m$ for the vector space into which T maps as follows:

Example: Matrix encoding $(v_1, \dots, v_n) \mapsto (w_1, \dots, w_m)$

Only the k^{th} column is shown. Thus the second index of each displayed entry of the matrix is k.

$$\mathcal{M}(T) = \begin{array}{c} \\ w_1 \\ \vdots \\ w_m \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & \cdots & v_k & \cdots & v_n \end{array} \\ \begin{pmatrix} & & A_{1,k} & & \\ & & \vdots & & \\ & & A_{m,k} & & \end{pmatrix} \end{array}$$

The above picture should remind you that Tv_k **can be computed from $\mathcal{M}(T)$ by multiplying each entry in the $\mathbf{k^{th}}$ column by the corresponding $\mathbf{w_j}$ then adding up the resulting vectors.**

If T is a linear map from $\mathbf{F^n}$ to $\mathbf{F^m}$, then unless stated otherwise, assume the bases in question are the standard ones (where the k^{th} basis vector is 1 in the k^{th} slot and 0 in all the other slots).

Example: Matrix of $T \in \mathcal{L}(\mathbf{F^n}, \mathbf{F^m})$

If $T \in \mathcal{L}(\mathbf{F^n}, \mathbf{F^m})$ and you think of elements of $\mathbf{F^m}$ as columns of numbers, then you can think of the k^{th} column of $\mathcal{M}(T)$ as T applied to the k^{th} standard basis vector of $\mathbf{F^n}$.

Addition and Scalar Multiplication of Matrices

Suppose $S, T \in \mathcal{L}(\mathbf{V}, \mathbf{W})$. For the rest of the section, assume $\mathbf{V}$and $\mathbf{W}$are finite-dimensional and have a basis chosen for them.

We are interested in whether $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$ and whether $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$?

Definition: Matrix Addition

The sum of two matrices of the same size is the matrix obtained by the corresponding entries of two matrices.

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} + \begin{pmatrix} B_{1,1} & \cdots & B_{1,n} \\ \vdots & & \vdots \\ B_{m,1} & \cdots & B_{m,n} \end{pmatrix} = \begin{pmatrix} A_{1,1} + B_{1,1} & \cdots & A_{1,n} + B_{1,n} \\ \vdots & & \vdots \\ A_{m,1} + B_{m,1} & \cdots & A_{m,n} + B_{m,n} \end{pmatrix}$$

In the following result, the assumption is that the same bases are used for $\mathcal{M}(S)$, $\mathcal{M}(T)$, and $\mathcal{M}(S+T)$.

Theorem: The Matrix of the Sum of Linear Maps

Suppose $S, T \in \mathcal{L}(\mathbf{V}, \mathbf{W})$. Then $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$.

Proof:

$\square$

Still assuming that the same bases are used, is the matrix of a scalar times a linear map the same as the scalar times the matrix of the linear map? Again we must provide a definition for scalar multiplication.

1.4 Isomorphisms

Taking coefficient vectors as a map.

Definition: Coefficient vectors as a map

Let $\mathbf{V}$ be a finite-dimensional vector space with $\dim \mathbf{V} = n$. Sending an element of $\mathbf{V}$ to its coefficient vector gives a map

$$C_{\mathcal{B}} : \mathbf{V} \to \mathbf{F^n}$$

Property: Properties of the coefficient map

- Coefficient vectors are unique; that is

$$[u_1]_{\mathcal{B}} = [u_2]_{\mathcal{B}} \implies u_1 = u_2$$

- Every vector in $\mathbf{F^n}$ is a coefficient vector, that is

$$\forall w \in \mathbf{F^n}, \exists u \in \mathbf{V} \text{ with } [u]_m athcal B = w$$

Definition: The inverse to $C_{\mathcal{B}}$

Let $C_{\mathcal{B}}^{-1} : \mathbf{F^n} \to V$ be the linear map defined by

$$C_{\mathcal{B}}^{-1} : (c_1, \ldots, c_n) \mapsto c_1 v_1 + \cdots + c_n v_n$$

This definition can be generalized as follows

Definition: Inverse map

The **inverse** to a linear map $T : V \to W$ is a linear map

$$T^{-1} : W \to V$$

such that $TT^{-1} = Id_W$ and $T^{-1}T = Id_V$.

The next definition captures the idea of two vector spaces that are essentially the same, excep for the names fo the elements of the vector space.

Definition: Isomorphism

A linear map with an inverse is called an **isomorphism**, or equivalently an **invertible** linear map.

The point of isomorphism is that they allow us to translate problems from one vector space to another.

Theorem: Dimension shows whether vector spaces are isomorphic

Two finite-dimensional vector spaces over $\mathbf{F}$ are isomorphic if and only if they have the same dimension

Proof:

$\square$

Theorem: $\mathcal{L}(V, W,)$ and $\mathbf{F^{m,n}}$ are isomorphic

Suppose $v_1, \ldots, v_n$ is a basis of $\mathbf{V}$ and $w_1, \ldots, w_m$ is a basis of $\mathbf{W}$. Then $\mathcal{M}()$ is an isomorphism between $\mathcal{L}(V, W,)$ and $\mathbf{F^{m,n}}$.

Proof:

$\square$