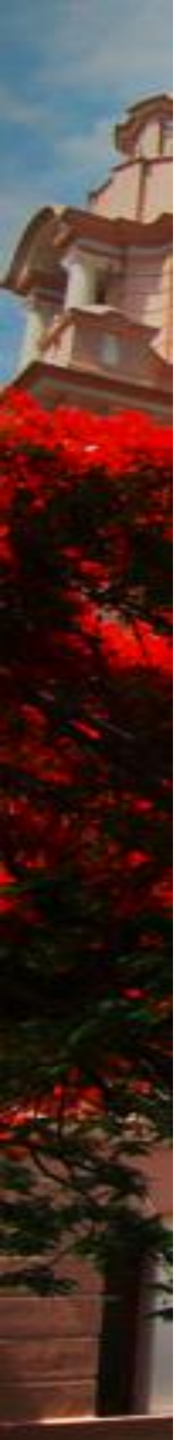
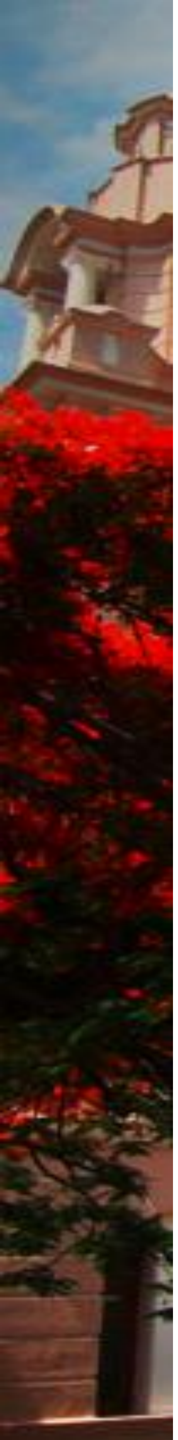
MSBA7003 Decision Analytics

06 Mathematical Programming II



Agenda

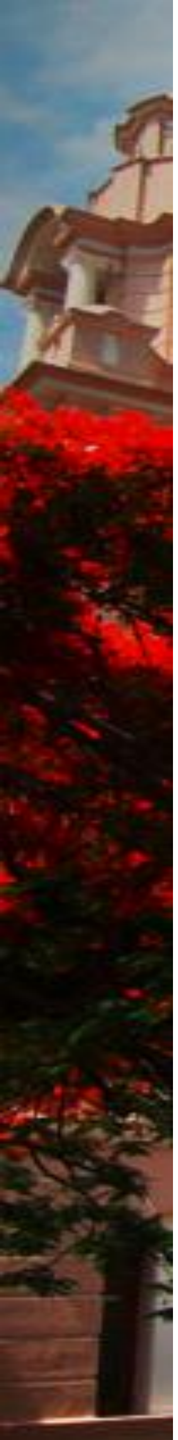
- Linear Programming Applications
  - Worker scheduling
  - Portfolio selection
  - Transportation planning
  - Worst case maximization
- Mixed Integer Programming
  - Knapsack problem
  - Assignment problem
  - Supply chain planning



Worker Scheduling

- A post office requires different numbers of full-time employees on different days of the week. Union rules states that each full-time employee must work five consecutive days and then receive two days off. The post office wants to meet its daily requirements using only full-time employees, while minimizing the total number of full-time employees on its payroll.

Day of Week	Minimum Number of Employees Required
Monday	17
Tuesday	13
Wednesday	15
Thursday	19
Friday	14
Saturday	16
Sunday	11

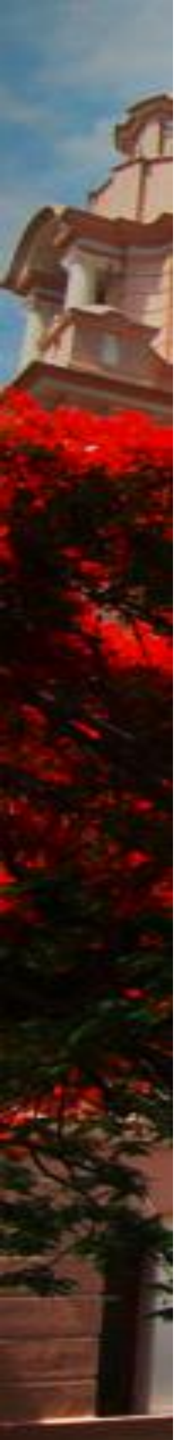


Worker Scheduling

- Decision variables:
  - X_1: the number of employees whose first working day is Monday
  - X_2: the number of employees whose first working day is Tuesday
  - X_3: the number of employees whose first working day is Wednesday
  - X_4: the number of employees whose first working day is Thursday
  - X_5: the number of employees whose first working day is Friday
  - X_6: the number of employees whose first working day is Saturday
  - X_7: the number of employees whose first working day is Sunday

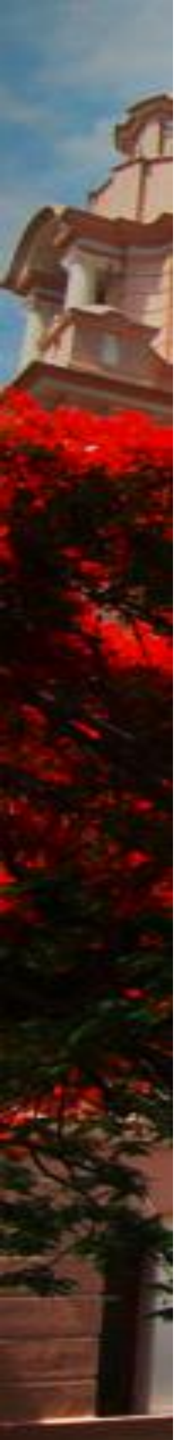
When defining decision variables, try not to introduce addition constraints!

- Objective:
  - Minimize: $X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7$



Worker Scheduling

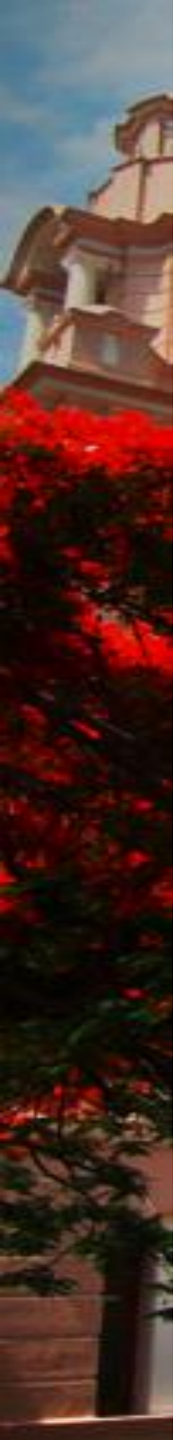
- Constraints:
- Monday constraint: $X_1 + X_4 + X_5 + X_6 + X_7 \geq 17$
- Tuesday constraint: $X_1 + X_2 + X_5 + X_6 + X_7 \geq 13$
- Wednesday constraint: $X_1 + X_2 + X_3 + X_6 + X_7 \geq 15$
- Thursday constraint: $X_1 + X_2 + X_3 + X_4 + X_7 \geq 19$
- Friday constraint: $X_1 + X_2 + X_3 + X_4 + X_5 \geq 14$
- Saturday constraint: $X_2 + X_3 + X_4 + X_5 + X_6 \geq 16$
- Sunday constraint: $X_3 + X_4 + X_5 + X_6 + X_7 \geq 11$
- Non-negativity: $X_1, X_2, X_3, X_4, X_5, X_6, X_7 \geq 0$



Worker Scheduling

Worker Scheduling Model											
Decision Variables		Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday	Total No. of FT Employees		
No. of FT Employees:		6.333333333	3.333333333	2	7.333333333	0	3.333333333	0			22.33333
Constraints											RHS
	Monday	1			1	1	1	1	17	>=	17
	Tuesday	1	1			1	1	1	13	>=	13
	Wednesday	1	1	1			1	1	15	>=	15
	Thursday	1	1	1	1			1	19	>=	19
	Friday	1	1	1	1	1			19	>=	14
	Saturday		1	1	1	1	1		16	>=	16
	Sunday			1	1	1	1	1	12.66667	>=	11

- Interpretation of non-integer solutions
  - Average value
  - Change the unit of measurement
- To get integer solutions
  - Add integer constraints
  - Round the solutions to the nearest integers



Portfolio Selection

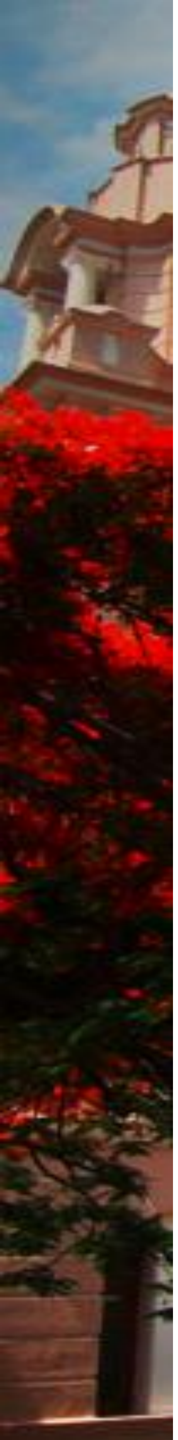
- The Heinlein and Krampf Brokerage firm is instructed by a client to invest $250,000 in five possible options, with guidelines:
  - Municipal bonds should constitute at least 20% of the investment
  - At least 40% of the investment should be placed in a combination of electronic firms, aerospace firms, and drug manufacturers
  - No more than 50% of the amount invested in municipal bonds should be placed in a high-risk, high-yield nursing home stock
- The goal is to maximize the projected return.

Investment	Los Angeles municipal bonds	Thompson Electronics, Inc.	United Aerospace Corp.	Palmer Drugs	Happy Days Nursing Homes
Projected Return (%)	5.3	6.8	4.9	8.4	11.8

Portfolio Selection

	A	B	C	D	E	F	G	H	I
1	Investment	LA Municipal Bonds	Thompson Electronics, Inc.	United Aerospace Corp.	Palmer Drugs	Happy Days Nursing Homes			
2	Projected Return (%)	5.3	6.8	4.9	8.4	11.8			
3	Amount	50000	0	0	175000	25000			
4									
5	Total Return	20300							
6									
7	Constraints								
8		1	1	1	1	1	250000	<=	250000
9		1					50000	>=	50000
10			1	1	1		175000	>=	100000
11						1	25000	<=	25000

Question: How to set cells I9 and I10?



Portfolio Selection: Sensitivity Report

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B3	Amount LA Municipal Bonds	50000	0	0.053	0.014	0.406
C3	Amount Thompson Electronics, Inc.	0	-0.016	0.068	0.016	1E+30
D3	Amount United Aerospace Corp.	0	-0.035	0.049	0.035	1E+30
E3	Amount Palmer Drugs	175000	0	0.084	0.034	0.009333333
F3	Amount Happy Days Nursing Homes	25000	0	0.118	0.028	0.034

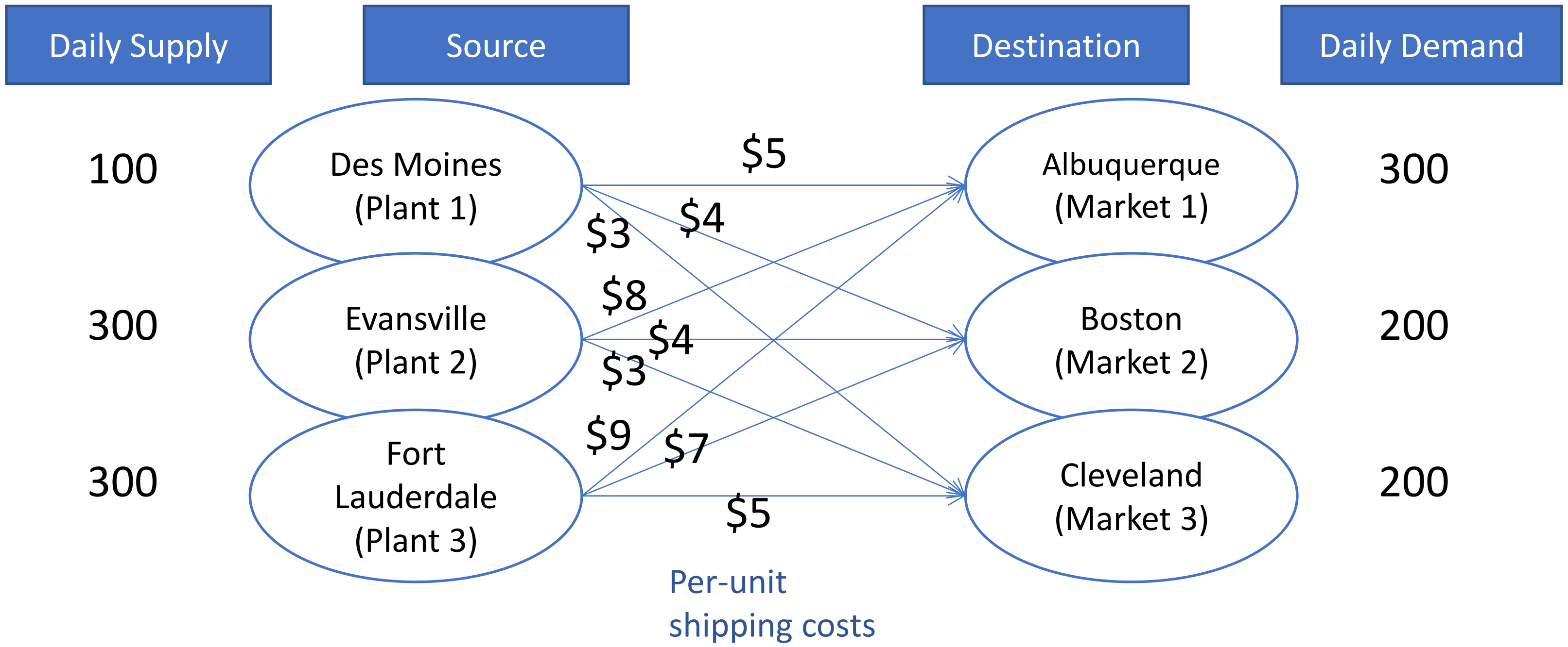
Constraints

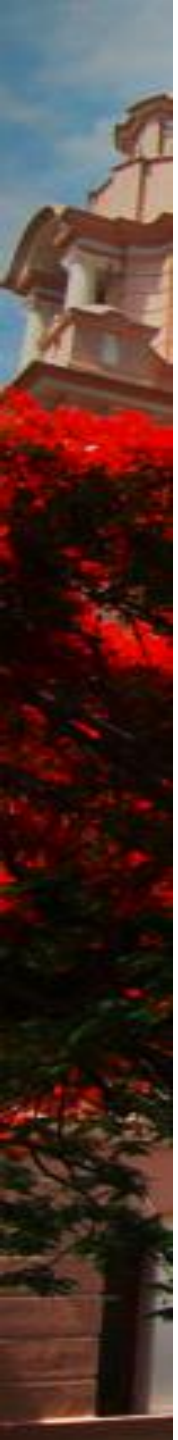
Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
G8	Total Amount Constraint	250000	0.0812	250000	1E+30	250000
G9	Municipal Bond Constraint	50000	-0.014	0	50000	50000
G10	E.A.D. Constraint	175000	0	0	75000	1E+30
G11	Nursing Home Constraint	25000	0.034	0	75000	25000

Question: What is the marginal return rate?

Transportation Problem

- The Executive Furniture Corporation is faced with the following transportation problem and is trying to minimize the daily transportation cost. How to optimize the shipping plan, while the demand must be satisfied?





Transportation Problem

	A	B	C	D	E	F
1	**Executive Furniture Corporation**					
2						
3	Source		Des Moines	Evansville	Fort Lauderdale	Demand Sum
4	Destination	Albuquerque	100	0	200	300
5		Boston	0	200	0	200
6		Cleveland	0	100	100	200
7		Supply Sum	100	300	300	
8						
9	Model Parameters		Des Moines	Evansville	Fort Lauderdale	Demand
10		Albuquerque	$ 5.00	$ 8.00	$ 9.00	300
11		Boston	$ 4.00	$ 4.00	$ 7.00	200
12		Cleveland	$ 3.00	$ 3.00	$ 5.00	200
13		Supply	100	300	300	
14						
15	Total Cost	$ 3,900.00				

In-Class Exercise

- The following table shows the sensitivity analysis of the transportation problem. If you can expand the supply by 100, which plant would you choose?

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
C7	Supply Sum Des Moines	100	-4	100	200	0
D7	Supply Sum Evansville	300	-2	300	100	0
E7	Supply Sum Fort Lauderdale	300	0	300	1E+30	0
F4	Albuquerque Demand Sum	300	9	300	0	200
F5	Boston Demand Sum	200	6	200	0	100
F6	Cleveland Demand Sum	200	5	200	0	100

Dealing with Nonlinear Functions

- Sometimes, nonlinear objective functions and constraint functions can be reformulated as linear ones in an LP.

Maximize $\min\{X, Y\}$

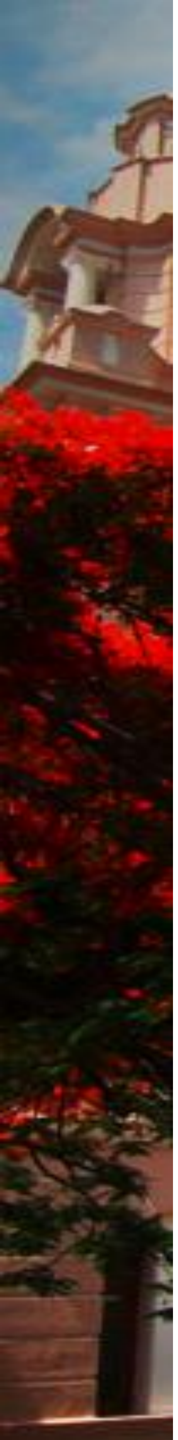
↓

Maximize Z
Subject to: $Z \leq X$ and $Z \leq Y$

Maximize $2X + 3Y$
Subject to: $2X + Y \leq \min\{X, Y - X\}$

↓

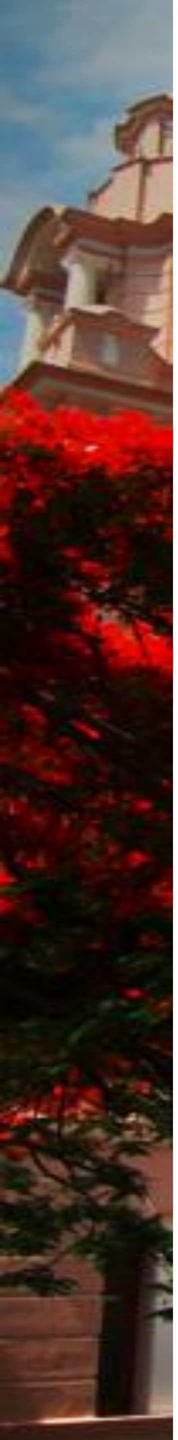
Maximize $2X + 3Y$
Subject to:
$2X + Y \leq X$ and $2X + Y \leq Y - X$



Transportation Problem Revisited

- Reconsider the Transportation Problem.
- Suppose each source can supply more than its capacity by outsourcing at price $p_i = 3$, and demand can be unsatisfied with a revenue loss of $r_j = 10$ per unit. What is the optimal outsourcing and transportation plan? Formulate the LP.

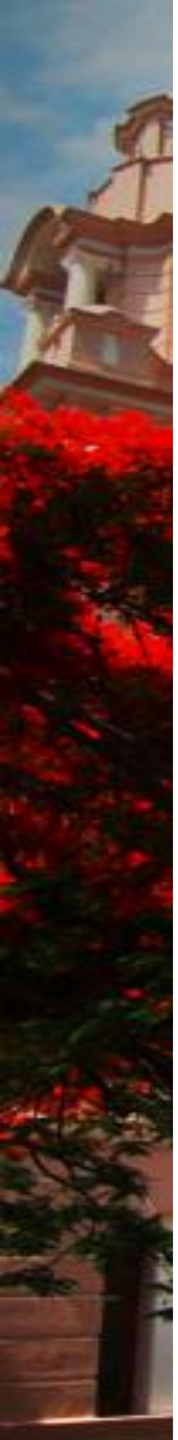
- $\min_{x_{ij} \geq 0} \quad \sum_{i,j} c_{ij} \cdot x_{ij} + \sum_i p_i \cdot \max\{0, \sum_j x_{ij} - s_i\} + \sum_j r_j \cdot \max\{0, d_j - \sum_i x_{ij}\}$
- Introduce $z_i \geq 0$ and $w_j \geq 0$ such that
  - $z_i \geq 0$ and $z_i \geq \sum_j x_{ij} - s_i$
  - $w_j \geq 0$ and $w_j \geq d_j - \sum_i x_{ij}$
  - Objective $= \sum_{i,j} c_{ij} \cdot x_{ij} + \sum_i p_i \cdot z_i + \sum_j r_j \cdot w_j$.



Worst Case Maximization

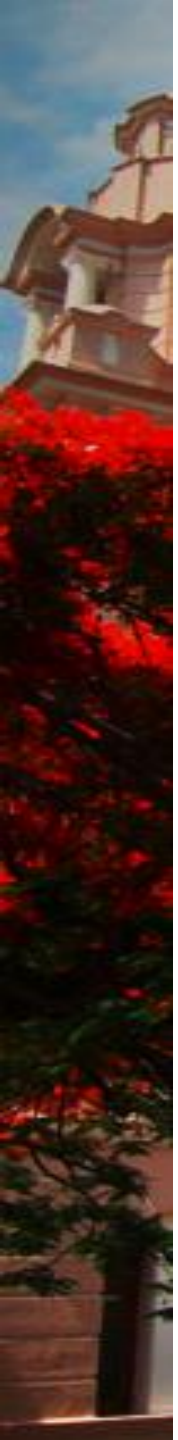
- The Hong Kong Family Office is instructed by a client to invest $250,000 among the following five asset classes. The market has three possible scenarios, and the asset classes will have different returns.
- How to allocate the money such that worst case return is maximized?

Investment Return (%)	Class 1	Class 2	Class 3	Class 4	Class 5
Scenario 1	5	7	5	8	10
Scenario 2	5	6	3	12	15
Scenario 3	5	8	8	4	-5



Worst Case Maximization

- **Maximize** t
- **s.t.** $t \leq 5x_1 + 7x_2 + 5x_3 + 8x_4 + 10x_5$
- $t \leq 5x_1 + 6x_2 + 3x_3 + 12x_4 + 15x_5$
- $t \leq 5x_1 + 8x_2 + 8x_3 + 4x_4 - 5x_5$
- $x_1 + x_2 + x_3 + x_4 + x_5 \leq 250,000$
- $x_1, x_2, x_3, x_4, x_5 \geq 0$



Modeling with 0-1 Variables

- 0-1 (binary) variables are very useful in practical problems.
    - Making a selection among a set of choices
    - Discrete (either-or) choices with fixed costs
    - Dependent selections

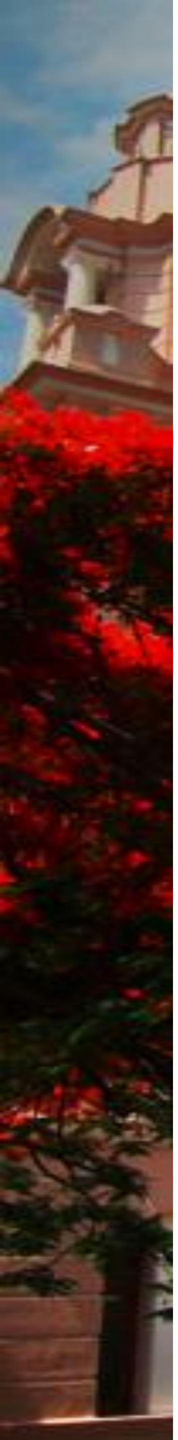
- An LP with 0-1 variables is called a Mixed Integer Program.

Modeling with 0-1 Variables

- **Making a selection**
- Quemo Chemical Company is considering three possible improvement projects:

Project	NPV	Required Investment	
		Year 1	Year 2
(1) Catalytic Converter	$25,000	$8,000	$7,000
(2) Software Upgrade	$18,000	$6,000	$4,000
(3) Warehouse Expansion	$32,000	$12,000	$8,000
	Available Funds	$20,000	$16,000

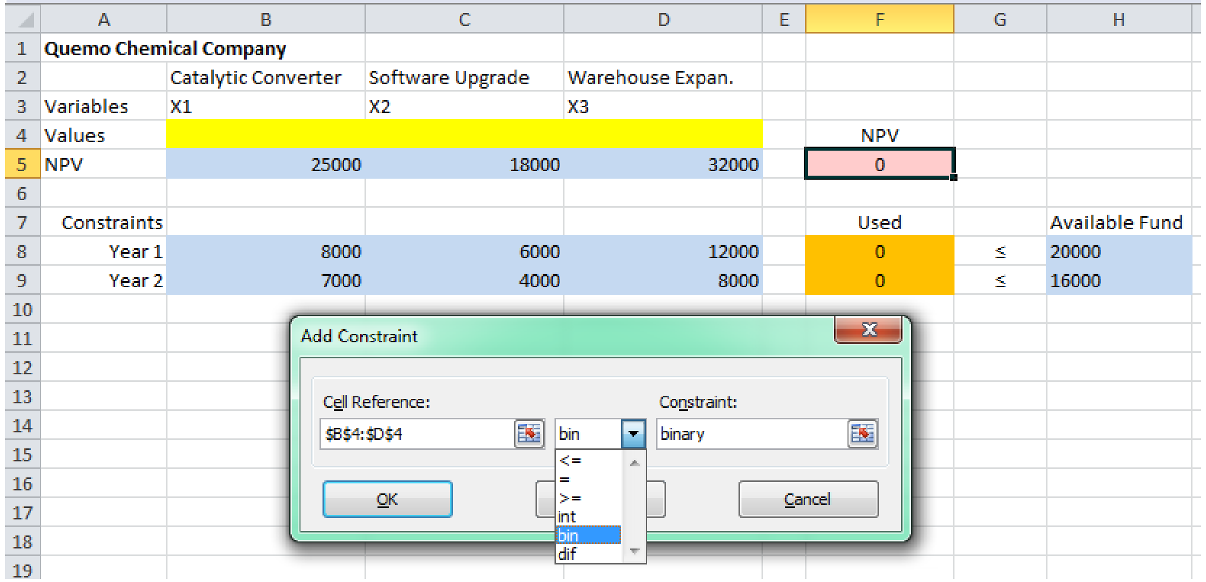
- Which project(s) to undertake to maximize NPV?



Modeling with 0-1 Variables

- Define the decision variables as

$$X_i = \begin{cases} 1 \text{ if project } (i) \text{ is funded} \\ 0 \text{ otherwise} \end{cases}$$



	A	B	C	D	E	F	G	H
1	Quemo Chemical Company							
2		Catalytic Converter	Software Upgrade	Warehouse Expan.				
3	Variables	X1	X2	X3				
4	Values					NPV		
5	NPV	25000	18000	32000		0		
6								
7	Constraints					Used		Available Fund
8	Year 1	8000	6000	12000		0	≤	20000
9	Year 2	7000	4000	8000		0	≤	16000

- The mathematical statement of the problem:

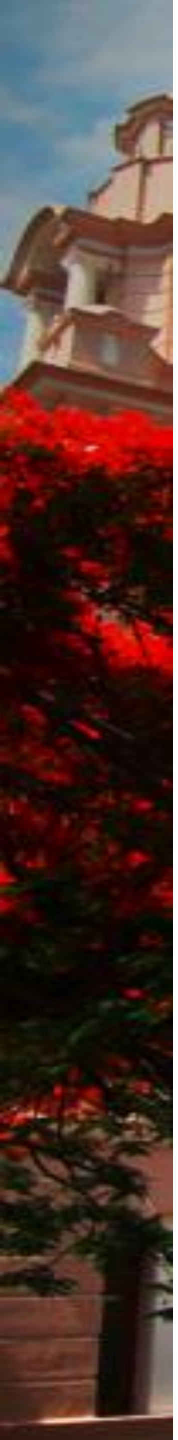
$$\text{Max NPV} = 25{,}000X_1 + 18{,}000X_2 + 32{,}000X_3$$

Subject to

$$8{,}000X_1 + 6{,}000X_2 + 12{,}000X_3 \le 20{,}000$$

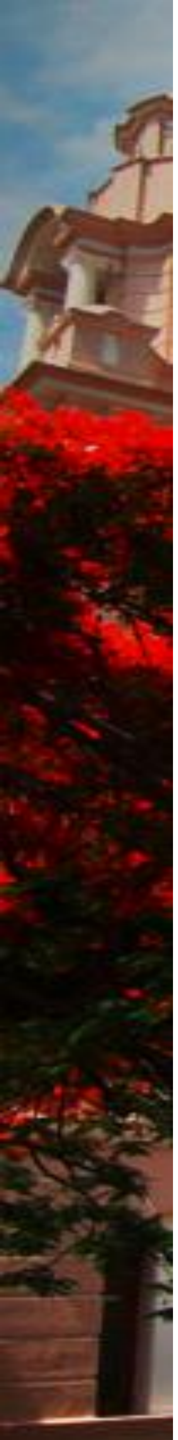
$$7{,}000X_1 + 4{,}000X_2 + \ \ 8{,}000X_3 \le 16{,}000$$

$$X_1, X_2, X_3 \in \{0,1\}$$



Modeling with 0-1 Variables

- **Modeling dependent selections**
- Suppose that the catalytic converter could be purchased only if the software was upgraded.
- Add a constraint: $X_1 \leq X_2$.
- What if the two projects must be undertaken together?
- Add the constraint: $X_1 = X_2$.

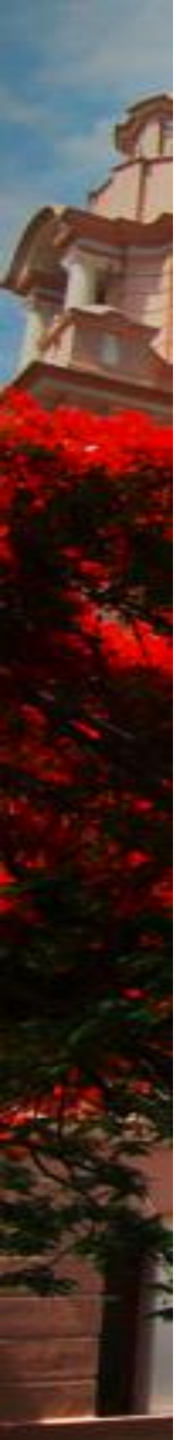


Modeling with 0-1 Variables

- **Modeling fixed and variable costs**

- Suppose there is a fourth option: a marketing program to build the brand name. To start the program, the company has to hire a marketing team, which costs $10,000 in year 1. The company can then decide the amount of money to invest in year 2. The NPV should be 1.5 times the year-2 investment minus the hiring cost.

- Add variables X_4 and M_4.

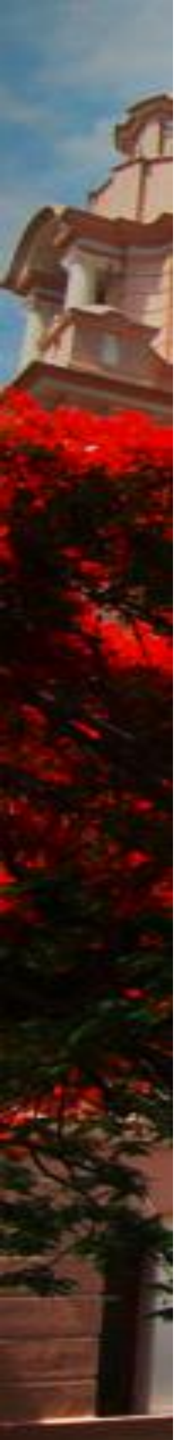


Modeling with 0-1 Variables

- The new problem can be modeled as follows:

- Max $\text{NPV} = 25{,}000X_1 + 18{,}000X_2 + 32{,}000X_3 + 1.5M_4 - 10{,}000X_4$
- New constraints:
  - $8{,}000X_1 + 6{,}000X_2 + 12{,}000X_3 + 10{,}000X_4 \leq 20{,}000$
  - $7{,}000X_1 + 4{,}000X_2 + 8{,}000X_3 + M_4 \leq 16{,}000$
  - $M_4 \leq 16{,}000X_4$
  - $M_4 \geq 0$ and $X_1, X_2, X_3, X_4 \in \{0,1\}$

This constraint is to ensure $M_4 = 0$ when $X_4 = 0$. The control limit can be any positive number greater than 16,000.

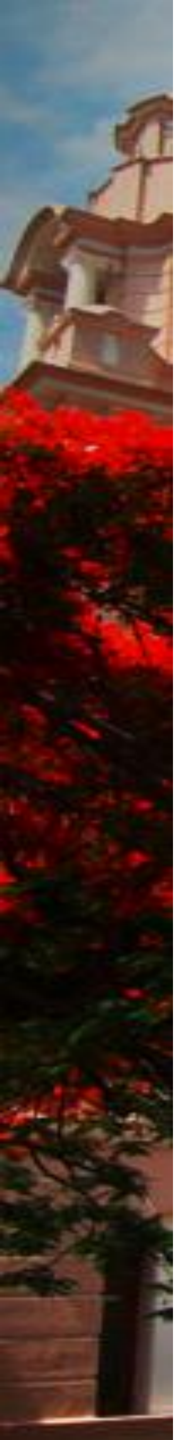


In-class Exercises

- How to model “at most one can be selected between catalytic converter and warehouse expansion?”

- How to model “catalytic converter could be purchased only when either the software upgrading or the marketing program was undertaken but not both?”

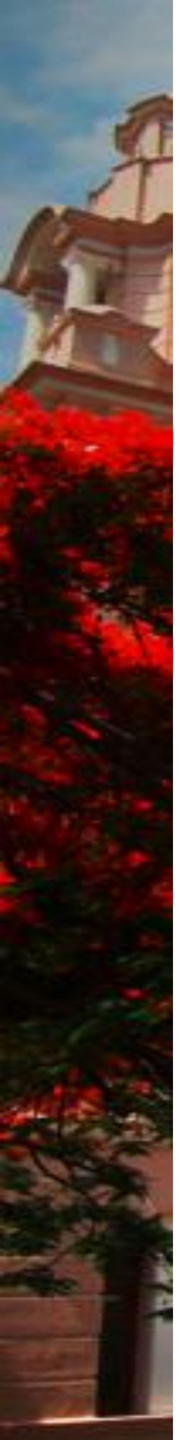
- How to formulate the model if, for the marketing program, an investment could be made in year 1 in addition to the $10,000 needed to hire marketing team. The NPV is 1.5 times the total marketing investment in two years, minus the hiring cost.



Truck Loading (Knapsack Problem)

- Goodman Shipping Co. is deciding which items to load on a truck so as to maximize the total value shipped.
- The truck has a capacity of 10,000 pounds and the following items are awaiting shipment.

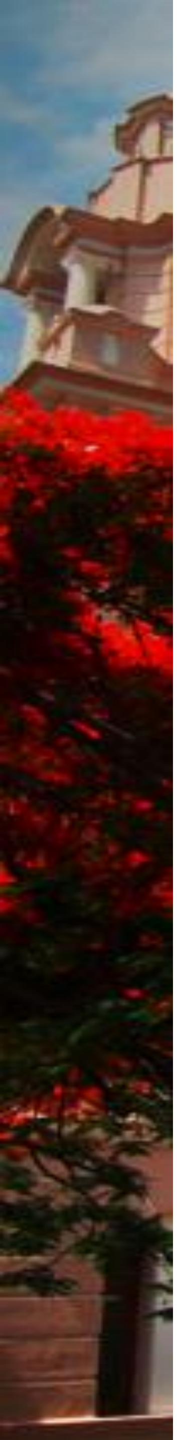
ITEM	VALUE ($)	WEIGHT (lbs)
1	22,500	7,500
2	24,000	7,500
3	8,000	3,000
4	9,500	3,500
5	11,500	4,000
6	9,750	3,500



Truck Loading (Knapsack Problem)

- All the decision variables are binary.

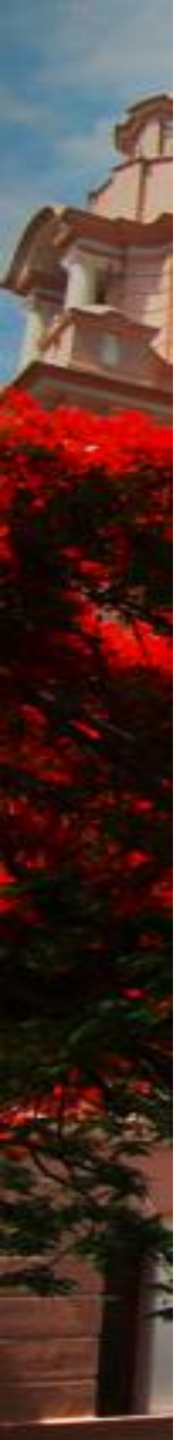
	A	B	C	D	E	F	G	H	I	J	K
1	**Goodman Shipping Co.**										
2											
3	Item	1	2	3	4	5	6				
4	Variable	X1	X2	X3	X4	X5	X6				
5	Decision	0	0	1	1	0	1		Total Value		
6	Value	22500	24000	8000	9500	11500	9750		27250		
7									Total Weight		Capacity
8	Weight	7500	7500	3000	3500	4000	3500		10000	≤	10000



Truck Loading (Assignment Problem)

- If Goodman Shipping Co. must ship all the items and they can use more than one truck in their fleet. What is the loading plan that minimizes the unused capacity.

Truck No.	1	2	3	4	5	6
Capacity (lbs)	10,000	5,000	12,000	8,000	4,500	4,000



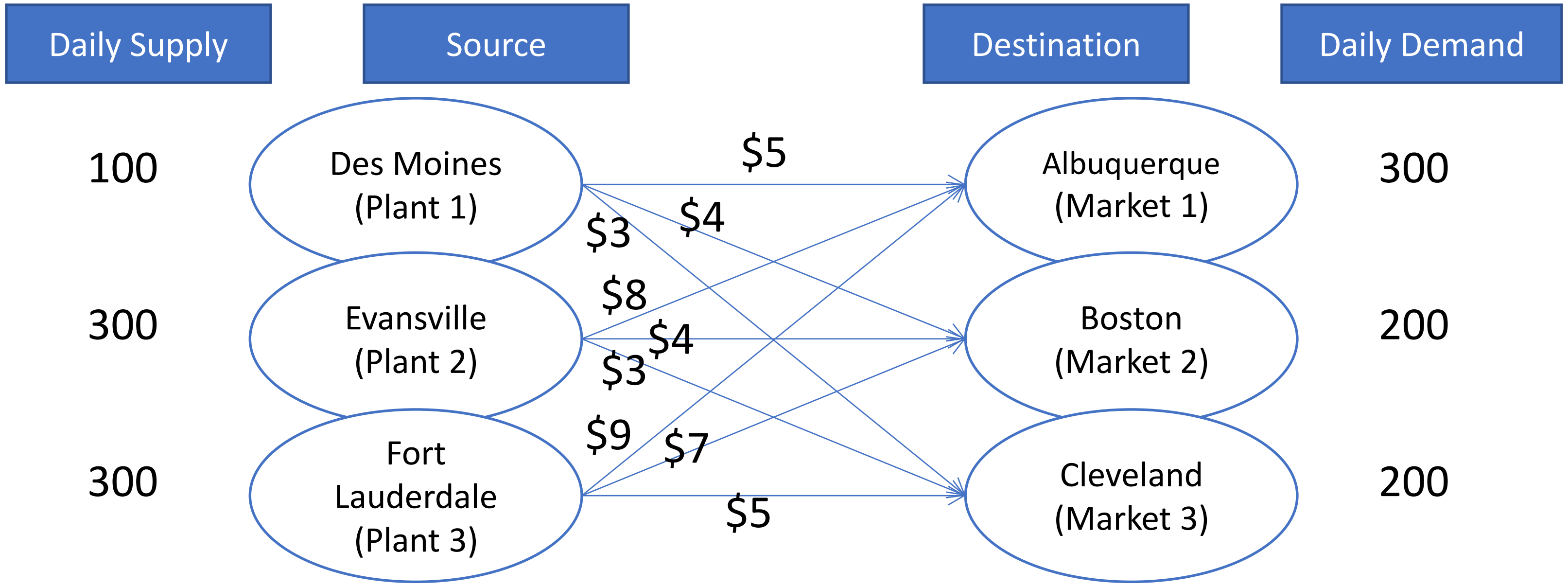
Truck Loading (Assignment Problem)

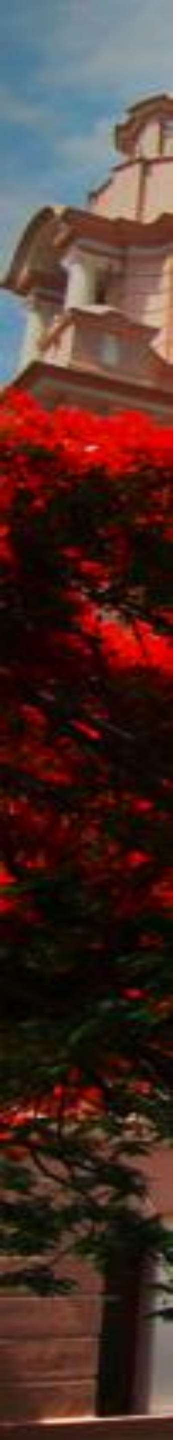
- All the decision variables are binary.

Goodman Shipping Co.													
Item	1	2	3	4	5	6		Total Wasted Capacity					
Weight	7500	7500	3000	3500	4000	3500		1000				Available	Spare
Truck								Capacity	Used?	Load		Capacity	Capacity
1	0	0	1	1	0	1		10000	1	10000	≤	10000	0
2	0	0	0	0	0	0		5000	0	0	≤	0	0
3	1	0	0	0	1	0		12000	1	11500	≤	12000	500
4	0	1	0	0	0	0		8000	1	7500	≤	8000	500
5	0	0	0	0	0	0		4500	0	0	≤	0	0
6	0	0	0	0	0	0		4000	0	0	≤	0	0
Shipped?	1	1	1	1	1	1							
	=	=	=	=	=	=							
	1	1	1	1	1	1							

Supply Chain Planning

- The Executive Furniture Corporation is faced with the following supply chain planning problem. There are three possible locations to operate a plant, and there are three possible markets.

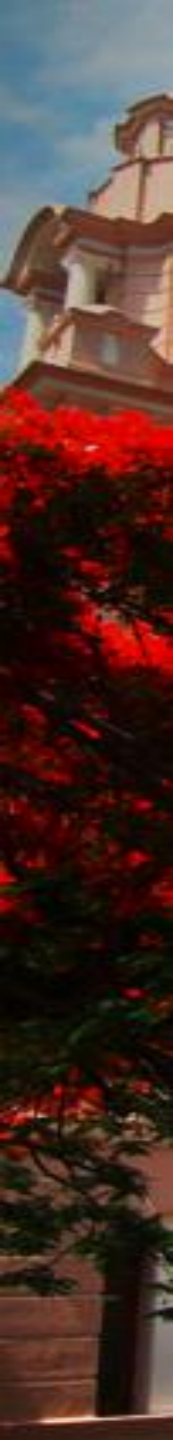




Supply Chain Planning

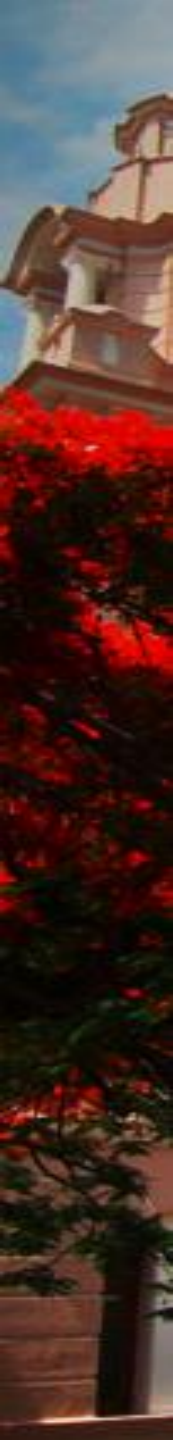
- Suppose there is a daily fixed cost of running each plant and the selling prices at the three destinations differ.
- Which plants and destinations should be chosen to maximize profit?

Source	Des Moines	Evansville	Fort Lauderdale
Daily Fixed Cost	$1,000	$3,000	$5,000
Destination	Albuquerque	Boston	Cleveland
Selling Price	$18	$30	$25



Supply Chain Planning

Executive Furniture Corporation (Supply Chain Planning)							
Using Source?	Des Moines	Evansville	Fort Lauderdale				Total Profit
Decision	1	1	0	(binary)			$ 5,600.00
Fixed Cost	$ 1,000.00	$ 3,000.00	$ 5,000.00				=
Cover Market?	Albuquerque	Boston	Cleveland				Total Revenue
Decision	0	1	1	(binary)			$ 11,000.00
Selling Price	$ 18.00	$ 30.00	$ 25.00				-
							Total Cost
Model Parameters		Des Moines	Evansville	Fort Lauderdale	Demand		$ 5,400.00
	Albuquerque	$ 5.00	$ 8.00	$ 9.00	300		
	Boston	$ 4.00	$ 4.00	$ 7.00	200		
	Cleveland	$ 3.00	$ 3.00	$ 5.00	200		
	Supply	100	300	300			
Source		Des Moines	Evansville	Fort Lauderdale	Demand Sum		Planned Demand
Destination	Albuquerque	0	0	0	0	=	0
	Boston	100	100	0	200	=	200
	Cleveland	0	200	0	200	=	200
	Supply Sum	100	300	0			
		≤	≤	≤			
	Planned Supply	100	300	0			



Quiz

- As the factory HR, you are hiring two workers to work on a two-stage production line. Each stage requires one worker only. Stage A precedes stage B, so it is required that the processing rate of stage B should not be lower than that of stage A. There are three candidates: Jack, Ken, and Logan. Their job processing rates along with their required wage rates are listed in the table below. To ensure the line throughput rate is no less than M units per hour, how can the hiring plan minimize the wage cost for the factory? Formulate this problem as a mix integer program.

		Jack	Ken	Logan
Processing Rate	Stage A	r_{Aj}	r_{Ak}	r_{Al}
	Stage B	r_{Bj}	r_{Bk}	r_{Bl}
Required wage rate		w_j	w_k	w_l

- True/False: If we use binary variable x_{sn} $(n = j, k, l; s = A, B)$ to indicate whether to assign candidate n to stage s, we need constraint $x_{Bj} \cdot r_{Bj} + x_{Bk} \cdot r_{Bk} + x_{Bl} \cdot r_{Bl} \geq M$ to ensure the line throughput rate is no less than M.