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A governing equation for fluid flow in rough fractures

Shemin Ge

Department of Geological Sciences, University of Colorado, Boulder

Abstract. Fluid flow in a rock fracture bounded by two irregular surfaces is complex even under a laminar flow regime. The major factor causing deviation of predicted fracture flow behavior from the ideal parallel plate theory is the nature of nonparallel and nonsmooth geometry of fracture surfaces. Important questions on the validity of the cubic law and the Reynolds equation for complicated fracture geometries have been studied by many researchers. The general conclusion from these efforts is that the cubic law is valid provided that an appropriate average aperture can be defined. Many average apertures have been proposed, and for some cases, some work better than others. Nonetheless, to date, these efforts have not converged to form a unified definition on the fracture aperture needed in the cubic law, which stimulates the current effort to develop a general governing equation for fracture flow from a fundamental consideration. In this study, a governing equation stemming from the principle of mass conservation and the assumption that the cubic law holds locally is derived for incompressible laminar fluid flow in irregular fractures under steady state conditions. The equation is formulated in both local and global coordinates and explicitly incorporates two vectorial variables of fracture geometry: true aperture and tortuosity. Under the assumption of small variations in both tortuosity and aperture, the governing equation can be reduced to the Reynolds equation. Two examples are provided to show the importance and generality of the new governing equation in both local and global coordinate systems. In a simple fracture with two nonsmooth and nonparallel surfaces, the error in permeability estimation can be induced using the Reynolds equation with the apparent aperture and can reach 10% for a 25° inclination between the fracture surfaces. In a fracture with sinusoidal surfaces, the traditional method can cause significant errors in both permeability and pressure calculation.

Introduction

Fluid flow in fractures of natural rocks has been a subject of active research for the last 2 decades. Because of the various long-term geologic processes, rock fractures at all scales commonly exist in the Earth's upper crust and, consequently, greatly control groundwater movement, petroleum migration, and mineral transport. Understanding the controlling mechanisms of fluid flow in a single fracture and network systems has been the focus in recent research. For a single fracture consisting of two perfectly parallel surfaces the most important finding, perhaps, is the well-known "cubic law." The cubic law states that the volume rate of fluid flow across a section in such a fracture is proportional to the applied pressure gradient and the cube of the separation distance. The important implication of the cubic law is that fluid flow may be fully characterized by the separation distance, called "aperture," although the velocity varies across that distance. How well the cubic law predicts flow in natural rock fractures is, however, a question. Because virtually all natural rock fractures are nonsmooth and nonparallel, the validity of the cubic law has been a natural extension as a research subject and has been challenged by many researchers from both experimental and theoretical aspects. Using the experimental study of *Iwai* [1976] on fracture permeability change in response to compressional and tensional stress loading as a basis, *Witherspoon et al.* [1980] examined the validity of the cubic law. They found that permeability can be uniquely defined by fracture aperture and is not dependent on stress history and rock type used in their investigations (compressional and extensional loading on granite, basalt, and marble samples). However, fracture permeability is sensitive to applied normal stress and fracture roughness [*Tsang and Witherspoon*, 1981; 1983]. *Neuzil and Tracy* [1981] conducted a theoretical study on the validity of the cubic law for flow through rough fractures. Their major conclusions are that a single-valued aperture can not sufficiently characterize flow rates in rough fractures and other physical and theoretical considerations such as the frequency distribution of apertures need to be incorporated into the cubic law. Using the theoretical analysis and previous experiments as a basis, *Walsh* [1981] demonstrated that the fracture permeability also depends on the effective pressure and is linearly proportional to the cube of the natural logarithm of the effective pressure. *Brown* [1987] conducted numerical experiments solving the Reynolds equation in a three-dimensional fracture and found that fracture roughness of natural rocks can cause a deviation from the cubic law prediction ranging from 10 to 50%. The Reynolds equation was also employed by *Zimmerman et al.* [1991] to establish the analytical expressions of permeability dependency on the fracture roughness and mean aperture for sinusoidal surface fractures. Their theoretical results indicate that in order for the Reynolds equation to be valid, the fracture surfaces must be smooth over the length of the order of 1 standard deviation of the fracture aperture, but associated errors could be considerable if the fracture surfaces are very rough.

Most of the previous studies have targeted the search for an appropriate definition of an average aperture in order to fit the

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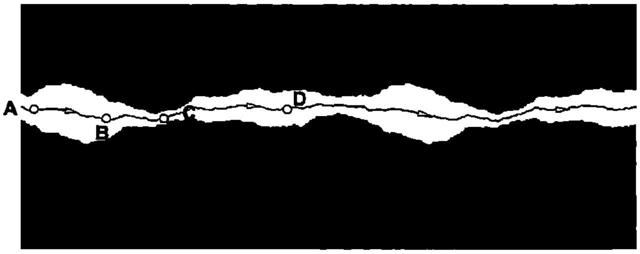

Figure 1. An illustration of the complex nature of fracture geometry and fluid flow. The fracture is generated artificially using the similar fractal dimension of natural rock fractures.

cubic law. The general conclusion from these efforts is that the cubic law is valid provided that an appropriate average aperture can be defined. Many definitions of an average aperture have been proposed, and for some cases, some work better than others. Nonetheless, to date, these efforts have not converged to form a unified definition of the fracture aperture for the cubic law.

As mentioned earlier, the advantage of using the cubic law is its simplicity and capability to reveal the physics of fracture flow. Theoretically speaking, solving the Navier-Stokes equations under a set of complicated fracture surfaces will provide details on pressure and flow velocity distributions in fractures. However, a great number of practical constraints limit the direct application of the Navier-Stokes equations. Three coupled, nonlinear partial differential equations have to be solved in a three-dimensional domain. The explicit treatment of fracture surfaces as boundary conditions further complicates the approach and, in most cases, does not lend to simple analytical and numerical treatments. In contrast, under certain geometric and kinematic constraints the Navier-Stokes equations can be reduced to the much simpler Reynolds equation [*Pinkus and Sternlicht*, 1961; *Zimmerman et al.*, 1991]. The Reynolds equation can also be derived from other physical considerations. The approach employed by *Walsh* [1981] and *Brown* [1989] to arrive at the Reynolds equation provides a simple and attractive alternative. The advantage of the approach lies in the direct incorporation of simple surface feature (plates) and the locally held cubic law. However, because of the simplification of the fracture geometry in arriving at the Reynolds equation, the effect of roughness and tortuosity on the validity of the cubic law can only be examined after the solution to the Reynolds equation using some sort of statistical averaging schemes [e.g., *Tsang and Witherspoon*, 1983; *Brown*, 1987; *Zimmerman et al.*, 1991].

Characteristics of Fracture Flow

I first examine qualitatively how fluid flows through a rock fracture (see Figure 1). As fluid flows through the two-dimensional fracture, fluid particles experiencing the disturbance due to the fracture surfaces take tortuous pathways. The degree of tortuosity can be evidenced at two scales: microscopic scale and fracture-length scale. The microscopic effect can be observed by the nonsmooth path line (see between points A and B) and is caused by the small variations of fracture surfaces over that interval. Although the vertical locations of points A and B could be the same, the length of the actual particle path line is different from the straight distance between the two points. This deviation, noticeable at the fracture-length scale, can be characterized by the concept of tortuosity (see *Walsh and Brace* [1984], and later discussion). An important feature of tortuosity is that it varies along the flow path. For example, the average tortuosity between A and B is different from that between B and C, or C and D.

Another phenomenon is that the average flow velocity varies significantly as a fluid particle travels through the fracture. Using the principle of mass conservation and assuming that the fluid is incompressible, we know that the total flux across any vertical section should be the same. Also, by the cubic law the particle velocity from point C to point D could decrease several orders of magnitude, depending on the actual degree of fracture aperture enlargement. To examine the validity of the cubic law, researchers have proposed and studied many definitions of fracture apertures. The following are a few examples: apparent aperture shown in Figure 2 between points b and e mean aperture or mechanical aperture [*Brown*, 1989, Figure 3] and hydraulic radius [*Walsh and Brace*, 1984, equation (12)]. These concepts are examined in more detail in the following.

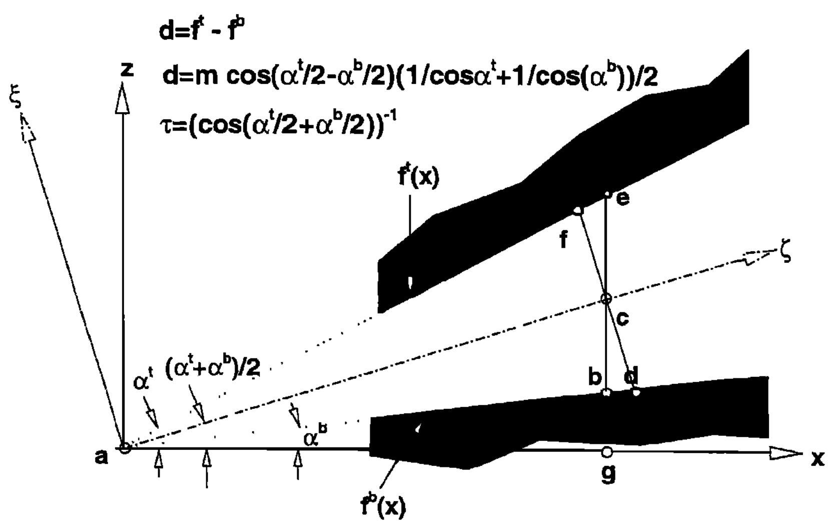

Figure 2. Conceptual illustrations of tortuosity, apparent hydraulic aperture, and true hydraulic aperture.

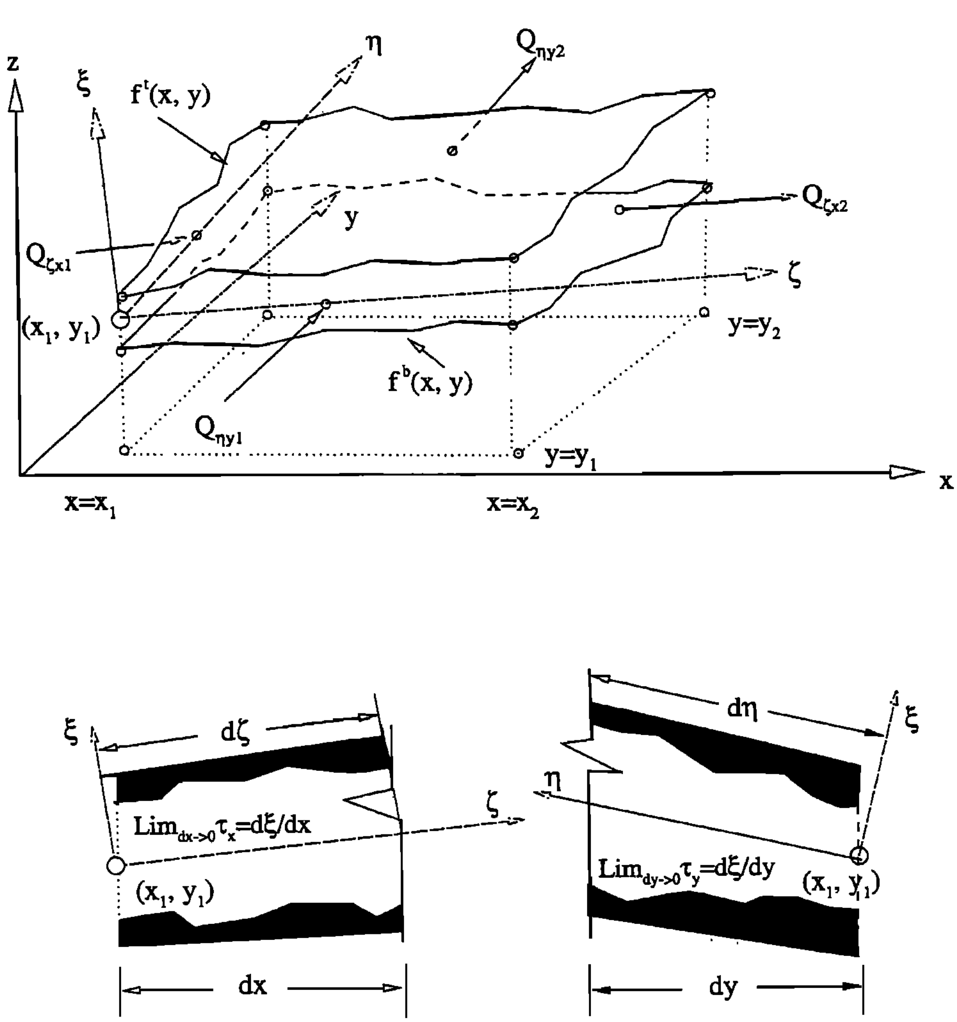

Figure 3. A description of an infinitesimal volume of a fracture bounded by two irregular surfaces.

Fracture Geometry

Assuming the top surface $f^t(x, y)$ and bottom surface $f^b(x, y)$ (Figures 2 and 3) being single-valued functions and at least piecewise continuous, we have the following geometric relations at a point (x_1, y_1):

$$\alpha_x^t = \tan^{-1}\left(\frac{\partial f^t(x_1, y_1)}{\partial x}\right), \qquad \alpha_y^t = \tan^{-1}\left(\frac{\partial f^t(x_1, y_1)}{\partial y}\right)$$
$$\alpha_x^b = \tan^{-1}\left(\frac{\partial f^b(x_1, y_1)}{\partial x}\right), \qquad \alpha_y^b = \tan^{-1}\left(\frac{\partial f^b(x_1, y_1)}{\partial y}\right) \tag{1}$$

where α_x^t and α_y^t are angles of the top surface in the x and y direction at the point (x_1, y_1), respectively, and α_x^b and α_y^b are the angles of the bottom surface at the point (x_1, y_1).

Fracture Apertures

Two kinds of fracture apertures are discussed here: the apparent aperture and the true aperture. By definition, the apparent aperture d at the coordinate x is the distance between the top and bottom surfaces (see Figure 2) in the global coordinates x-z. A local coordinate system is chosen in the direction of the average velocity, which is approximated as the average of the angle of the top and bottom surfaces at location x. For piecewise continuous surfaces, the angles exist everywhere except at corners. Although the angle at a corner point cannot be defined uniquely, an assumption is made that a generalized angle exists at the corner point and is approximated as the arithmetic average of the left and right angles about the point. Point c (Figure 2) is defined as the intersection of the apparent aperture and the local coordinate ζ axis, whose velocity is defined as the average velocity for the cubic law. For the same material point c the apparent aperture varies as the coordinate system rotates. Therefore the apparent aperture is not unique.

The true aperture for point c can be found uniquely as follows. Take a straight line and rotate around point c at the cross section x to the point where the straight line is perpendicular to the local coordinate ζ, the cross section between these two points is proposed as the one for the cubic law and the distance between the two contact points is defined as the true aperture m (distance between points d and f in Figure 2). Therefore the true aperture at a given location is unique. Conducting the same experiment along the x direction through the entire fracture, the trajectory of point c can be defined as the path line $f^m(x)$ or surface $f^m(x, y)$ in a three-dimensional fracture where the true apertures and directions are used to represent the locally held cubic law. Although many researchers might have recognized that the apparent aperture does not represent the hydraulic aperture [e.g., *Walsh and Brace*, 1984] well, most previous studies neglected this fact and used the apparent aperture in solving the Reynolds equation or estimating fracture permeability. Under the global coordinates x-z the Reynolds equation can be written as

$$\frac{d}{dx}\left(d_a^3\frac{dp}{dx}\right)=0 \tag{2}$$

where p is fluid pressure. If the fracture surfaces are nearly parallel and planar, and the coordinate system is aligned nearly parallel to the fracture plane, the difference between the apparent and true apertures could be small. As illustrated in Figure 1, these assumptions do not hold for most natural fractures. Hence the difference could be considerable and the errors associated with aperture estimation could be large. Other physical considerations also support this notion. Darcy's law implies that the flux vector is proportional to the cross-sectional area (the true aperture here) perpendicular to the flux. Therefore the aperture used in the local cubic law should be the true aperture. Furthermore, the cubic law requires that velocity distribution across a section be symmetrical with respect to the axis passing through the average velocity point in the flow direction.

When two fracture surfaces are parallel to each other (see Figure 2), the true aperture can be simply expressed as $m = d_a/\tau$, where τ is the tortuosity. This relation has been used in previous studies [e.g., *Walsh and Brace*, 1984; *Brown*, 1987]. However, it does not hold when the two fracture surfaces at the cross section x are not parallel to each other (see Figure 2). Under general geometric conditions the true aperture in the x direction can be defined as

$$m_x = 2d_a\frac{\cos\alpha_x^t\cos\alpha_x^b}{\cos\dfrac{\alpha_x^t-\alpha_x^b}{2}\left(\cos\alpha_x^t+\cos\alpha_x^b\right)}=d_ac_x \tag{3}$$

Similarly, the true aperture in the y direction can be defined as

$$m_y = 2d_a\frac{\cos\alpha_y^t\cos\alpha_y^b}{\cos\dfrac{\alpha_y^t-\alpha_y^b}{2}\left(\cos\alpha_y^t+\cos\alpha_y^b\right)}=d_ac_y \tag{4}$$

Therefore, for a three-dimensional fracture bounded by two rough surfaces the true aperture is considered as a two-component vectorial field.

Fracture Tortuosities

Local tortuosity is defined by *Walsh and Brace* [1984] as the actual path length relative to the apparent path length. Tortuosity can be calculated from the center path line (or surface) as simply the inverse of cosine of the local flow direction, which may be calculated as the arithmetic average of the cosines of the inclination angles of the fracture surfaces (see Figure 2). Therefore it is a vectorial field in a three-dimensional fracture. For natural rock fractures, tortuosity values range from 1 (parallel plates) to 2 (60 deg inclination). Because the Reynolds equation does not account for tortuosity, the viscous force due to tortuosity is totally ignored. Consequently, this oversight can cause errors in fracture permeability estimation and in the accuracy of the cubic law. In an effort to examine the cubic law under a three-dimensional fracture condition, *Brown* [1987] used tortuosity to correct the calculated flow rate. The tortuosity was approximated from the deviation of flow direction away from the applied pressure gradient in the fracture plane direction and therefore was recognized as a scalar quantity. For fluid flow in a three-dimensional rough fracture, two components of tortuosity exist at each point. This is because the deviation of a fluid particle away from the applied pressure is due to the variation of fracture geometry in both x and y directions. The tortuosity vector at a point is defined as

$$\begin{aligned}\tau_x &= \frac{\partial\zeta}{\partial x}=\sqrt{1+(f_x^m)^2}\\ \tau_y &= \frac{\partial\eta}{\partial x}=\sqrt{1+(f_y^m)^2}\end{aligned} \tag{5}$$

where the subscripts in the function f^m represent the derivatives with respect to the x and y directions.

Governing Equation

A governing equation explicitly accounting for velocity and tortuosity variations is desirable and can be established by incorporating two independent geometry variables: true aperture and tortuosity.

Besides the assumptions for the explicit incorporation of the aperture and tortuosity, the additional assumptions involved in arriving at a new governing equation are similar to those provided by *Pinkus and Sternlicht* [1961] in deriving the generalized Reynolds equation. They are summarized as follows:

1. The aperture of the fracture is relatively small compared to the fracture dimensions in x and y directions. This allows us to ignore the rotational flow components and consider only the transversal velocities, namely q_x and q_y.
2. The fluid is incompressible and flow is laminar. This implies that isothermal and nonturbulent conditions hold everywhere in the fracture.
3. No external forces act on the fracture.
4. No slip occurs at the fracture surfaces.
5. The cubic law holds locally at any point in the fracture. This implies that the pressure is constant across the aperture.
6. Compared to the two velocity gradients $\partial q_x/\partial x$ and $\partial q_y/\partial y$, all other velocity gradients are considered negligible. This assumption permits us to consider only the above derivatives in the elementary volume balance.

The physical implications of these assumptions are straightforward are easily understood except the last one. The validity of the last assumption for numerical simulation of fracture flow has been explored by *Brown* [1987], who concludes that the ratio of the standard deviation of surface height to the spacing between simulation points has to be small, say less than 0.1.

Mass Balance

On the basis of assumptions (1)–(6), the total fluxes at each of the following mutual perpendicular faces of an elementary volume are (Figure 3)

$$\begin{aligned}
x=x_1 \qquad & Q_{\zeta x1}=\frac{q_\zeta}{\tau_x}\left[d_a+\frac{1}{2}\frac{\partial d_a}{\partial y}dy\right]dy\\
x=x_2 \qquad & Q_{\zeta x2}=\frac{q_\zeta+\dfrac{\partial q_\zeta}{\partial x}dx}{\tau_x}\left[d_a+\frac{\partial d_a}{\partial x}dx+\frac{1}{2}\frac{\partial d_a}{\partial y}dy\right]dy\\
y=y_1 \qquad & Q_{\eta y1}=\frac{q_\eta}{\tau_y}\left[d_a+\frac{1}{2}\frac{\partial d_a}{\partial x}dx\right]dx
\end{aligned} \tag{6}$$

$$y = y_2 \qquad Q_{\eta y2} = \frac{q_\eta + \frac{\partial q_\eta}{\partial y} dy}{\tau_y} \left[d_a + \frac{\partial d_a}{\partial y} dy + \frac{1}{2} \frac{\partial d_a}{\partial x} dx \right] dx$$

For steady state and incompressible fluid flow, mass conservation requires that

$$Q_{\zeta x2} - Q_{\zeta x1} + Q_{\eta y2} - Q_{\eta y1} = 0 \tag{7}$$

Substituting (6) into (7) and taking limits at the point (x, y) $dx = dy \rightarrow 0$ yields

$$q_\zeta \frac{\partial d_a}{\partial x} \frac{1}{\tau_x d_a} + q_\eta \frac{\partial d_a}{\partial y} \frac{1}{\tau_y d_a} + \frac{1}{\tau_x} \frac{\partial q_\zeta}{\partial x} + \frac{1}{\tau_y} \frac{\partial q_\eta}{\partial y} = 0 \tag{8}$$

Equation in Local Coordinates

The formulation of equations governing fluid flow has primarily been done under the global coordinates in fluid mechanics. Use of the local coordinates is generally avoided, because they usually move with the path lines of fluid particles, and cannot be easily determined a priori. For fluid flow in a relatively narrow channel of rock fractures the path line of a fluid particle is strongly constrained by the fracture geometry. Therefore the problems sometimes may be simplified greatly under the local coordinates.

Using the chain rule and (5), we have

$$\begin{aligned} \frac{\partial d_a}{\partial x} &= \tau_x \frac{\partial d_a}{\partial \zeta} \\ \frac{\partial d_a}{\partial y} &= \tau_y \frac{\partial d_a}{\partial \eta} \end{aligned} \tag{9}$$

The averaged Darcy velocities and their gradients across the sections at $x = x_1$ along ζ direction and at $y = y_1$ along the η direction can be written [after *Brace et al.*, 1968]

$$\begin{aligned} q_\zeta &= -\frac{m_x^2}{12\mu} \frac{\partial p}{\partial \zeta}, \qquad \frac{\partial q_\zeta}{\partial x} = \frac{\partial q_\zeta}{\partial \zeta} \frac{\partial \zeta}{\partial x} = \frac{\partial q_\zeta}{\partial \zeta} \tau_x \\ q_\eta &= -\frac{m_y^2}{12\mu} \frac{\partial p}{\partial \eta}, \qquad \frac{\partial q_\eta}{\partial y} = \frac{\partial q_\eta}{\partial \eta} \frac{\partial \eta}{\partial y} = \frac{\partial q_\eta}{\partial \eta} \tau_y \end{aligned} \tag{10}$$

Substituting (9) and (10) into (8), I arrive at the following form of the governing equation:

$$3m_x \frac{\partial p}{\partial \zeta} \frac{\partial m_x}{\partial \zeta} + 3m_y \frac{\partial p}{\partial \eta} \frac{\partial m_y}{\partial \eta} + m_x^2 \frac{\partial^2 p}{\partial \zeta^2} + m_y^2 \frac{\partial^2 p}{\partial \eta^2} - \frac{m_x^2}{c_x} \frac{\partial p}{\partial \zeta} \frac{\partial c_x}{\partial \zeta} - \frac{m_y^2}{c_y} \frac{\partial p}{\partial \eta} \frac{\partial c_y}{\partial \eta} = 0 \tag{11}$$

Several observations can be made from (11). Although I start the derivation with tortuosity and the true hydraulic aperture, tortuosity is eliminated and only the true aperture is retained in the governing equation. Because the coordinates move with the center path line (or surface), no deviation of flow trajectory is involved. It can be easily shown that for a fracture of two parallel plates with the local coordinates chosen parallel to the fracture plane direction, (11) is reduced to the well-known Reynolds equation.

Equation in Global Coordinates

The averaged Darcy velocities in both ζ and η directions along the fracture plane can be projected to the x and y coordinates by

$$\begin{aligned} q_\zeta &= -\frac{m_x^2}{12\mu} \frac{1}{\tau_x} \frac{\partial p}{\partial x} \\ q_\eta &= -\frac{m_y^2}{12\mu} \frac{1}{\tau_y} \frac{\partial p}{\partial y} \end{aligned} \tag{12}$$

Taking the derivatives of (12) with respect to the coordinates x and y yields

$$\begin{aligned} \frac{\partial q_\zeta}{\partial x} &= -\frac{1}{12\mu} \frac{\partial}{\partial x} \left(\frac{m_x^2}{\tau_x} \frac{\partial p}{\partial x} \right) \\ \frac{\partial q_\eta}{\partial y} &= -\frac{1}{12\mu} \frac{\partial}{\partial y} \left(\frac{m_y^2}{\tau_y} \frac{\partial p}{\partial y} \right) \end{aligned} \tag{13}$$

Using (12), (13), and (9), we can rearrange (8) in the global coordinates as

$$\frac{d_a c_x^2}{\tau_x^2} \frac{\partial p}{\partial x} \frac{\partial d_a}{\partial x} + \frac{d_a c_y^2}{\tau_y^2} \frac{\partial p}{\partial y} \frac{\partial d_a}{\partial y} + \frac{1}{\tau_x} \frac{\partial}{\partial x} \left(\frac{d_a^2 c_x^2}{\tau_x} \frac{\partial p}{\partial x} \right) + \frac{1}{\tau_y} \frac{\partial}{\partial y} \left(\frac{d_a^2 c_y^2}{\tau_y} \frac{\partial p}{\partial y} \right) = 0 \tag{14}$$

If the tortuosity variation is ignored (i.e., $\tau_x = \tau_y = 1.0$), (14) can be reduced to the Reynolds equation.

Quantitative Illustrations

Fluid Flow in a Nonparallel and Planar Fracture

To illustrate the importance and generality of the new governing equations, I examine the following two examples. The first is a fracture with nonparallel and planar surfaces as illustrated in Figure 4a. Tortuosity exists because the center flow line is not parallel to the global coordinate x. However, because the center line can be easily identified, tortuosity can be avoided in solving the governing equation by using the local coordinate ζ (i.e., a particle flowing through a fracture does not deviate from its path line). For the fracture illustrated in Figure 4a, the fracture surfaces can be described mathematically as

$$\begin{aligned} f^t(x) &= ax + \frac{b}{2} \qquad 0 \le x \le 1 \\ f^b(x) &= -\frac{b}{2} \qquad 0 \le x \le 1 \\ \alpha_x^t &= \tan^{-1}(a) = \alpha \qquad 0 \le x \le 1 \\ \alpha_x^b &= 0 \qquad 0 \le x \le 1 \end{aligned} \tag{15}$$

where a is the slope of the top surface and b is the minimum apparent aperture. By definition, we have

$$\begin{aligned} d_a &= ax + b \\ d_m &= \frac{a}{2} + b \\ m_x &= d_a c_x = d_a \frac{2 \cos \alpha}{\cos \frac{\alpha}{2} (1 + \cos \alpha)} \end{aligned} \tag{16}$$

Under the geometry defined in Figure 4a the governing (11) becomes

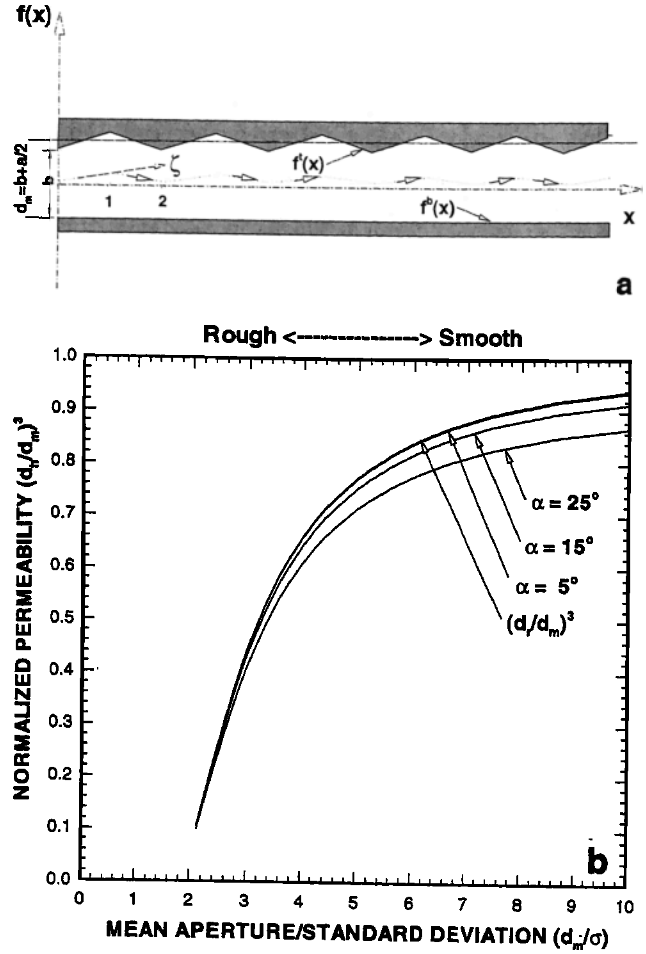

Figure 4. Fluid flow in a nonparallel planar fracture: (a) geometry and (b) normalized permeability as a function of the standard deviation of the roughness and the inclination angle of the upper surface.

$$3m_x \frac{dp}{d\zeta}\frac{dm_x}{d\zeta} + m_x^2 \frac{d^2p}{d\zeta^2} = 0 \tag{17}$$

or

$$\frac{d}{d\zeta}\left(m_x^3 \frac{dp}{d\zeta}\right) = 0 \tag{18}$$

Following the approach commonly used in fluid mechanics and in studying fluid flow in fractures [e.g., *Wilson and Witherspoon*, 1974; *Zimmerman et al.*, 1991], integrating the above equation, and imposing a measured or known total flux rate Q_ζ yields

$$dp = -Q_\zeta \frac{12\mu d\zeta}{m_x^3} \tag{19}$$

Integrating (19) and imposing the boundary conditions of $p = p_1|_{\zeta=\zeta_1}$ and $p = p_2|_{\zeta=\zeta_2}$ gives

$$\frac{p_2 - p_1}{\zeta_2 - \zeta_1} = \frac{12\mu Q_\zeta}{\zeta_2 - \zeta_1}\int_{\zeta_1}^{\zeta_2} \frac{d\zeta}{m_x^3} \tag{20}$$

For the cubic law to hold macroscopically we need

$$Q_\zeta = -\frac{\langle d_h\rangle^3}{12\mu}\frac{p_2 - p_1}{\zeta_2 - \zeta_1} \tag{21}$$

Comparing the above two equations, we arrive at

$$\langle d_h\rangle^3 = \left[\frac{1}{\zeta_2 - \zeta_1}\int_{\zeta_1}^{\zeta_2} \frac{d\zeta}{m_x^3}\right]^{-1} \tag{22}$$

From (16) we know that $m = d_a c_x$, which can be substituted into the above equation to yield

$$\langle d_h\rangle^3 = c_x^3\left[\frac{1}{\zeta_2 - \zeta_1}\int_{\zeta_1}^{\zeta_2} \frac{d\zeta}{d_a^3}\right]^{-1} = c_x^3\left[\frac{1}{x_2 - x_1}\int_{x_1}^{x_2} \frac{dx}{d_a^3}\right]^{-1} \tag{23}$$

Zimmerman et al. [1991] show that the Reynolds equation can lead to the following expression for the hydraulic aperture $\langle d_r\rangle$:

$$\langle d_r\rangle^3 = \left[\frac{1}{x_2 - x_1}\int_{x_1}^{x_2} \frac{dx}{d_a^3}\right]^{-1} \tag{24}$$

Comparing the above two equations, we can see that $\langle d_h\rangle = c_x\langle d_r\rangle$. Using (16) and (23) and conducting the integration for x from 0 to 1, we can arrive at

$$\langle d_r\rangle^3 = \left[\int_0^1 \frac{dx}{d_a^3}\right]^{-1} = \frac{2b^2(a+b)^2}{a+2b} \tag{25}$$

The standard deviation σ, under the current geometry, can be estimated as

$$\sigma = \left[\int_0^1 (d_a - d_m)^2\, dx\right]^{1/2}$$

$$= \left[\int_0^1 a^2\left(x - \frac{1}{2}\right)^2 dx\right]^{1/2} = \frac{\sqrt{3}}{6}a \tag{26}$$

Using (16) and (24), the normalized permeability for the solution from the Reynolds equation (2) is

$$\left(\frac{\langle d_r\rangle}{\langle d_m\rangle}\right)^3 = \frac{16\left(\frac{a}{b}+1\right)^2}{\left(\frac{a}{b}+2\right)^4} \tag{27}$$

The normalized permeability for the solution from (18) is

$$\left(\frac{\langle d_h\rangle}{\langle d_m\rangle}\right)^3 = \frac{\left(\frac{a}{b}+1\right)^2}{\left(\frac{a}{2b}+1\right)^4}\left[\frac{2\cos\alpha}{\cos\frac{\alpha}{2}(1+\cos\alpha)}\right]^3 \tag{28}$$

The relative roughness is usually calculated as d_m/σ; that is,

$$\frac{d_m}{\sigma} = 2\sqrt{3}\left(\frac{b}{a}+\frac{1}{2}\right) \tag{29}$$

Figure 4b shows the comparison of the calculated normalized permeabilities. As the inclination angle and the fracture roughness increase, the absolute error in permeability increases. Therefore, using the Reynolds equation with the apparent aperture will result in an overestimation of the permeability [*Brown et al.*, 1995]. The relative error due to the use of the apparent aperture can also be estimated as

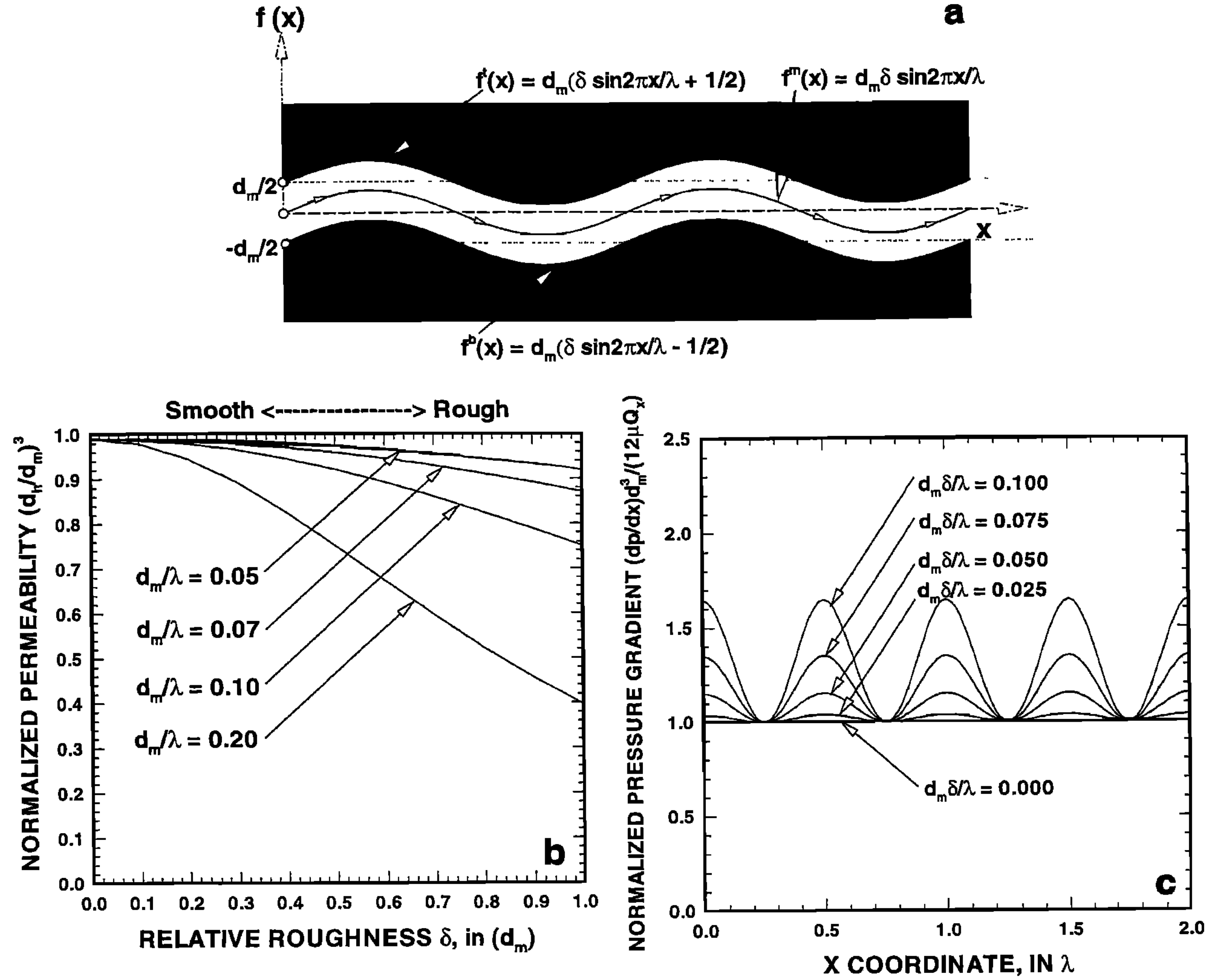

Figure 5. Fluid flow in a sinusoidal fracture: (a) geometry, (b) normalized permeability as a function of relative roughness and mean aperture, and (c) pressure gradient as a function of x coordinate and roughness.

$$\frac{\langle d_h\rangle^3 - \langle d_a\rangle^3}{\langle d_h\rangle^3} = 1 - \frac{1}{c_x^3} = 1 - \frac{\cos^3 \frac{\alpha}{2}\,(1 + \cos\alpha)^3}{8\cos^3\alpha} \tag{30}$$

As illustrated in Figure 4b, the error in the permeability estimation is small when the surfaces are nearly parallel, but increases quickly as the angle increases. For example, the error is about 1.2% when α equals 5°, and increases to 10% when α equals 25°.

Several observations can be made. In this case, using the Reynolds equation with the apparent aperture will cause overestimation of the fracture permeability. Although the local coordinate formulation is generally avoided in fluid mechanics, it is sometimes an attractive approach for fluid flow in fractures. This is because fluid path lines (or surface) are confined in two relatively narrow rough surfaces and the center line assumption for the cubic law is reasonably accurate. Although tortuosity exists macroscopically, it can be avoided in the calculation if the local coordinate formulation is employed. Nonetheless, two separate effects can be easily interpreted in (16): the first bracket can be regarded as the effect of roughness and the second as being due to the nonparallel arrangement of the fracture surfaces.

Fluid Flow in a Parallel and Nonplanar Fracture

The second example shown here is fluid flow through a fracture with sinusoidal fracture surfaces as illustrated in Figure 5a. The fracture surfaces are separated by a mean aperture d_m, and the variation of the aperture is characterized by a sinusoidal profile with the magnitude $d_m\delta$ and wavelength λ. Under these geometric conditions the governing (14) can be simplified as

$$\frac{d}{dx}\left(\frac{m_x^2}{\tau_x}\frac{dp}{dx}\right) = 0 \tag{31}$$

The solution of the above equation using appropriate boundary conditions (see Appendix A) leads to the estimation of the permeability in terms of the mean permeability as

$$\frac{\langle d_h\rangle^3}{\langle d_m\rangle^3} = \left[\frac{1}{\lambda}\int_0^\lambda \left[1 + \left(\frac{d_m\delta 2\pi}{\lambda}\right)^2 \cos^2\frac{2\pi x}{\lambda}\right]^{3/2} dx\right]^{-1} \tag{32}$$

Figure 5b shows the sensitivity of the normalized permeability to the relative aperture variation (d_m/λ) and the roughness δ. The calculated permeability decreases as the relative aperture variation (d_m/λ) and the roughness δ increase, whereas in the traditional method the normalized permeability keeps as a constant 1 due to the fact that the apparent aperture is a constant everywhere in the fracture.

Figure 5c shows the pressure gradient profiles along the x direction. In the current theory, pressure gradient varies periodically in the fracture. The magnitude of the pressure gradient increases as the fracture roughness increases. In contrast, the traditional method predicts a constant pressure gradient due to the fact that the apparent aperture is a constant everywhere in the fracture.

Although both the Reynolds equation and the new governing (14) conserve the total fluid flux rate throughout the fracture domain, the latter shows a nonlinear dynamic variation in pressure as the fluid flows through the fracture, whereas the pressure solution of the Reynolds equation using the apparent aperture is linearly distributed. The relative error in the permeability calculation using the Reynolds equation with the apparent aperture increases as the relative aperture variation increases. The error in this case is caused by neglecting the variations in the fluid flow tortuosity and the true aperture.

Conclusions

I have shown that a more general governing equation for fluid flow in a single fracture bounded by two rough surfaces can be arrived at and can better predict fluid flow behavior. The governing equation is derived based on the rigorous consideration of the true aperture and tortuosity variations. Both of these variables are redefined and generalized as vectorial fields in order to capture the rough and tortuous nature of fracture geometry. The governing equation is formulated in both local and global coordinates for convenience and can be reduced to the Reynolds equation if the variations in both tortuosity and true aperture are small. Two examples, one under the local coordinate and the other under the global coordinate, are provided to illustrate the importance and generality of the theoretical results. The two analytical solutions of the given examples can be used for future model verification and theoretical study. Results from these examples show that errors in permeability estimation and pressure solution due to neglecting changes in both tortuosity and aperture are considerable under natural fracture conditions. Therefore the presented governing equation may provide a rigorous basis for validating the cubic law and a potentially useful tool for predicting fluid flow and transport in natural fractures bounded by two rough surfaces.

Appendix A: Solution for Fluid Flow in a Sinusoidal Fracture

For the fracture illustrated in Figure 5a the fracture surfaces can be described mathematically as

$$f^t(x) = d_m\left(\delta \sin\frac{2\pi x}{\lambda} + \frac{1}{2}\right)$$
$$f^b(x) = d_m\left(\delta \sin\frac{2\pi x}{\lambda} - \frac{1}{2}\right) \tag{A1}$$

where δ is the magnitude of the aperture variation, d_m is mean aperture, and λ is the wavelength of the aperture variation. The center path line $f^m(x)$ can be approximated as

$$f^m(x) = d_m\delta \sin\frac{2\pi x}{\lambda} \tag{A2}$$

By definition, we have

$$d_a = d_m$$
$$\tau_x = \left[1 + \left(\frac{2\pi d_m\delta}{\lambda}\right)^2 \cos^2\frac{2\pi x}{\lambda}\right]^{1/2} \tag{A3}$$
$$m_x = d_a c_x = \frac{d_m}{\tau_x}$$

Under the two-dimensional and sinusoidal surface conditions, (14) becomes

$$\frac{d}{dx}\left(\frac{m_x^2}{\tau_x}\frac{dp}{dx}\right) = 0 \tag{A4}$$

Integrating the above equation and imposing a known total boundary flux Q_x yields

$$m_x^3\frac{dp}{dx} = -12\mu Q_x \tag{A5}$$

Integrating (A5) and imposing the boundary conditions of $p = p_1|_{x=0}$ and $p = p_2|_{x=\lambda}$ gives

$$\frac{p_2 - p_1}{\lambda} = \frac{12\mu Q_x}{\lambda}\int_0^\lambda \frac{dx}{m_x^3} \tag{A6}$$

For the cubic law to hold macroscopically, we need

$$Q_x = \frac{\langle d_h\rangle^3}{12\mu}\frac{p_2 - p_1}{\lambda} \tag{A7}$$

Comparing the above two equations, we arrive at

$$\langle d_h\rangle^3 = \left[\frac{1}{\lambda}\int_0^\lambda \frac{dx}{m_x^3}\right]^{-1} \tag{A8}$$

Imposing the geometry condition (A3) to the above equation gives

$$\frac{\langle d_h\rangle^3}{\langle d_m\rangle^3} = \left[\frac{1}{\lambda}\int_0^\lambda \tau_x^3\, dx\right]^{-1}$$
$$= \left[\frac{1}{\lambda}\int_0^\lambda \left[1 + \left(\frac{d_m\delta 2\pi}{\lambda}\right)^2 \cos^2\frac{2\pi x}{\lambda}\right]^{3/2} dx\right]^{-1} \tag{A9}$$

The variation in pressure gradient can be calculated using (A5) as

$$\frac{\frac{dp}{dx}d_m^3}{12\mu Q_x} = \left[1 + \left(\frac{2\pi d_m\delta}{\lambda}\right)^2 \cos^2\frac{2\pi x}{\lambda}\right]^{3/2} \tag{A10}$$

Notation

c_x, c_y fracture surface geometry coefficients in x and y directions, respectively, dimensionless.

d_a apparent aperture, distance between top and bottom surfaces for a given x, m.

d_m mean apparent aperture, m.

$\langle d_h \rangle$ averaged true aperture, m.
$\langle d_m \rangle$ averaged mean aperture, m.
$\langle d_r \rangle$ averaged aperture, m.
m true aperture vector, m.
m_x, m_y true aperture components in x and y directions, m.
x, y, z global coordinates.
p pressure, N m^{-2}.
q_ζ averaged Darcy velocity along channel center surface in ζ direction m s^{-1}.
q_η averaged Darcy velocity along channel center surface in η direction m s^{-1}.
q_x averaged Darcy velocity along x direction, m s^{-1}.
q_y averaged Darcy velocity along y direction, m s^{-1}.
$Q_{\zeta x1}$ total flux across face $x = x_1$ along ζ direction, m^3 s^{-1}.
$Q_{\zeta x2}$ total flux across face $x = x_2$ along ζ direction, m^3 s^{-1}.
$Q_{\eta y1}$ total flux across face $y = y_1$ along η direction, m^3 s^{-1}.
$Q_{\eta y2}$ total flux across face $y = y_2$ along η direction, m^3 s^{-1}.
α_x inclination angle of fracture surface in x direction, deg.
α_y inclination angle of fracture surface in y direction, deg.
δ magnitude coefficient of aperture variation, dimensionless.
μ fluid viscosity, kg m^{-1} s^{-1}.
τ tortuosity vector, dimensionless.
τ_x, τ_y tortuosity components in x and y directions, dimensionless.
σ standard deviation of fracture roughness, m.
ζ local coordinate in fracture along fluid flow direction, m.
η local coordinate in fracture along fluid flow direction, m.
ξ local coordinate in fracture perpendicular to fluid flow direction, m.
λ wavelength of aperture variation, m.

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S. Ge, Department of Geological Sciences, University of Colorado, Campus Box 250, Boulder, CO 80309-0250.

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