Solving the heat equation for a slice of pie using Green's functions

ECH 259

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Abstract

The heat equation with homogeneous Dirichlet boundary conditions was solved for a slice of pie cooling on a slab of ice cream. The method was finding a Green's function for a source term, then modifying it to solve the original problem. Evaluation of the series solution showed that it took 22 minutes to cool the pie from 375°F to 85°F. The reference was the notes [1].

1 Introduction

Consider a piece of pie with the following dimensions: radius: 5", angle 40°, and thickness: 1.5".
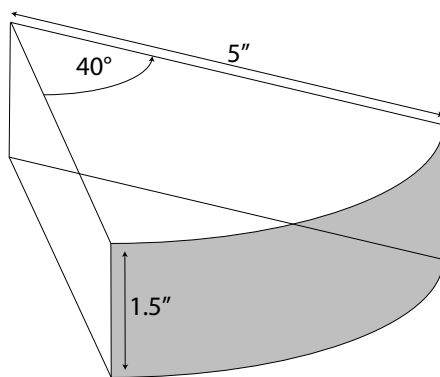
Assume the pie came out of the oven with a homogeneous initial temperature 375°F, and rests on a slab of ice cream with constant temperature 32°F. The room temperature is 70°F. The objective is to determine how long it will take the pie to reach an average internal temperature of 85°F. Assume the temperature of the outer surface of the pie is equal to the temperature of the room. The first step is to solve the problem analytically. To do this, the problem will be modified, but in a way so that a re-useable Green's function can be generated. Then, the Green's function will be used to solve the original problem.

2 Analytic solution

The heat equation is

$$\frac{\partial T}{\partial t} = \alpha \,\triangle T$$

Where
T = temperature, $°C$
t = time, s
α = thermal diffusivity = $\frac{k}{\rho c_p}$, $\frac{m^2}{s}$
k = thermal conductivity, $\frac{J}{m°C}$
ρ = density, $\frac{kg}{m^3}$
c_p = specific heat capacity, $\frac{J}{kg°C}$

The problem was changed into the international system of units. Using this change, the objective was to solve for time at which the average internal temperature of the pie reaches 29.4 °C. The thermal diffusivity of custard was measured by Betta, et. al (2009), and found to be **$1.34\text{x}10^{-7} \pm 0.03\frac{m^2}{s}$** [2]. This was used for the thermal diffusivity of the pie.

In cylindrical coordinates, the heat equation is

$$\frac{\partial T}{\partial t} = \alpha \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right]$$

Subject to the boundary conditions
$T(0, r, \theta, z) = 190.6$
$T(t, a, \theta, z) = 21.1$
$T(t, r, \theta, h) = 21.1$
$T(t, r, \theta, 0) = 0.0$
$T(t, r, 2\pi/9, z) = 21.1$
$T(t, r, 0, z) = 21.1$

The first step is to get rid of all of the room temperature inhomogeneities using a change of variables.

$$u = T - 21.1$$

Now, the boundary conditions, in $°C$, are
$u(0, r, \theta, z) = 169.5$
$u(t, a, \theta, z) = 0$
$u(t, r, \theta, h) = 0$
$u(t, r, \theta, 0) = -21.1$
$u(t, r, 2\pi/9, z) = 0$
$u(t, r, 0, z) = 0$

There are two inhomogeneities remaining: the initial condition and the pie-ice cream boundary. These will be dealt with by changing the problem again to include a source term made up of Dirac

delta functions located at the coordinates of the inhomogeneous boundary conditions. This allows one to integrate over the domain and only recover the inhomogeneities where they are relevant.

$$\frac{\partial g_{src}}{\partial t} = \alpha \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g_{src}}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 g_{src}}{\partial \theta^2} + \frac{\partial^2 g_{src}}{\partial z^2} \right] + \frac{1}{r}\delta(r - r_0)\delta(\theta - \theta_0)\delta(z - z_0)$$

The next step is to take the Laplace transform of everything.

$$\mathcal{L}\left\{ g_{src}(t, r, \theta, z) \right\} = G_{src}(s, r, \theta, z)$$

The heat equation becomes

$$sG_{src} = \alpha \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G_{src}}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 G_{src}}{\partial \theta^2} + \frac{\partial^2 G_{src}}{\partial z^2} \right] + \int_0^\infty \frac{1}{r}\delta(r - r_0)\delta(\theta - \theta_0)\delta(z - z_0)e^{-st}dt$$

The Dirac deltas are zero everywhere, except for when $t = t_0$, thus

$$sG_{src} = \alpha \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G_{src}}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 G_{src}}{\partial \theta^2} + \frac{\partial^2 G_{src}}{\partial z^2} \right] + \frac{1}{r}\delta(r - r_0)\delta(\theta - \theta_0)\delta(z - z_0)e^{-st_0}$$

At this point, the boundary conditions and initial condition are all homogeneous, because they have been accounted for by the source term.
$G_{src}(0, r, \theta, z) = 0$
$G_{src}(t, a, \theta, z) = 0$
$G_{src}(t, r, \theta, h) = 0$
$G_{src}(t, r, \theta, 0) = 0$
$G_{src}(t, r, 2\pi/9, z) = 0$
$G_{src}(t, r, 0, z) = 0$

Remark 1 (Solving the wrong problem). The boundary conditions are all homogeneous because of the delta functions in the source term. Thus, this is not the same as the original problem. However, by solving this new problem a reuseable Green's function can be obtained, which will be used to solve the original problem by integrating it over the inhomogeneities.

2.1   Finding the re-useable Green's function

Now, the term $\frac{\partial^2 G_{src}}{\partial z^2}$ can be recognized as a Sturm-Liouville operator. It is expanded using a sine series.

$$G_{src}(s, r, \theta, z) = \sum_{n=1}^{\infty} a_n(s, r, \theta)sin\left(\frac{n\pi z}{H} \right)$$

a_n periodic in z

Substituting the definition for u into the differential equation yields

$$\frac{s}{\alpha}\left[\sum_{n=1}^{\infty}a_n(s,r,\theta)sin\left(\frac{n\pi z}{H}\right)\right] = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left[\sum_{n=1}^{\infty}a_n(s,r,\theta)sin\left(\frac{n\pi z}{H}\right)\right]\right)$$

$$+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left[\sum_{n=1}^{\infty}a_n(s,r,\theta)sin\left(\frac{n\pi z}{H}\right)\right] + \frac{\partial^2}{\partial z^2}\left[\sum_{n=1}^{\infty}a_n(s,r,\theta)sin\left(\frac{n\pi z}{H}\right)\right]$$

$$+\frac{1}{r\alpha}\delta(r-r_0)\delta(\theta-\theta_0)\delta(z-z_0)e^{-st_0}$$

Evaluate the z derivative. Then, integrate everything over the domain of z, and multiply by a particular eigenfunction, $sin(\frac{n'\pi z}{H})$

$$\int_0^H \frac{s}{\alpha}\left[\sum_{n=1}^{\infty}a_n(s,r,\theta)sin\left(\frac{n\pi z}{H}\right)sin\left(\frac{n'\pi z}{H}\right)\right]dz$$

$$=\int_0^H \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left[\sum_{n=1}^{\infty}a_n(s,r,\theta)sin\left(\frac{n\pi z}{H}\right)\right]\right)sin\left(\frac{n'\pi z}{H}\right)dz$$

$$+\int_0^H \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left[\sum_{n=1}^{\infty}a_n(s,r,\theta)sin\left(\frac{n\pi z}{H}\right)\right]sin\left(\frac{n'\pi z}{H}\right)dz$$

$$-\int_0^H \left[\sum_{n=1}^{\infty}a_n(s,r,\theta)sin\left(\frac{n\pi z}{H}\right)\left(\frac{n^2\pi^2}{H^2}\right)\right]sin\left(\frac{n'\pi z}{H}\right)dz$$

$$+\int_0^H \frac{1}{r\alpha}\delta(r-r_0)\delta(\theta-\theta_0)\delta(z-z_0)e^{-st_0}sin\left(\frac{n'\pi z}{H}\right)dz$$

Rearrange, recognizing that the coefficients a_n cannot be pulled outside any derivative, but the sums of the sine functions can be. This is because everything in the equation is a linear operator, and the sines are not a function of r or θ, the variables the remaining derivatives are taken with respect to.

$$\int_0^H \frac{s}{\alpha}\sum_{n=1}^{\infty}a_n(s,r,\theta)sin\left(\frac{n\pi z}{H}\right)sin\left(\frac{n'\pi z}{H}\right)dz$$

$$=\int_0^H \sum_{n=1}^{\infty}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}a_n(s,r,\theta)\right)sin\left(\frac{n\pi z}{H}\right)sin\left(\frac{n'\pi z}{H}\right)dz$$

$$+\int_0^H \sum_{n=1}^{\infty}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left(a_n(s,r,\theta)\right)sin\left(\frac{n\pi z}{H}\right)sin\left(\frac{n'\pi z}{H}\right)dz$$

$$-\int_0^H \sum_{n=1}^{\infty}a_n(s,r,\theta)sin\left(\frac{n\pi z}{H}\right)\left(\frac{n^2\pi^2}{H^2}\right)sin\left(\frac{n'\pi z}{H}\right)dz$$

$$+\frac{1}{r\alpha}\delta(r-r_0)\delta(\theta-\theta_0)e^{-st_0}\int_0^H \delta(z-z_0)sin\left(\frac{n'\pi z}{H}\right)dz$$

Recognize that the infinite sums are zero for all instances where $n \neq n'$. Thus, the infinite sum over a function of n times a function of n' works like a Kronecker delta, $\delta_{nn'}$, collapsing the indices. This also changes the index of the sine function in the source term. Also recognize that when integrated

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over the domain, the z Dirac delta in the source term returns any function of z in the same integral, but evaluated at the inhomogeneity, z_0.

$$\int_0^H \frac{s}{\alpha} a_n(s, r, \theta) \sin^2\left(\frac{n\pi z}{H}\right) dz$$

$$= \int_0^H \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} a_n(s, r, \theta)\right) \sin^2\left(\frac{n\pi z}{H}\right) dz$$

$$+ \int_0^H \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}\left(a_n(s, r, \theta)\right) \sin^2\left(\frac{n\pi z}{H}\right) dz$$

$$- \int_0^H a_n(s, r, \theta) \sin^2\left(\frac{n\pi z}{H}\right)\left(\frac{n^2\pi^2}{H^2}\right) dz$$

$$+ \frac{1}{r\alpha} \delta(r - r_0)\delta(\theta - \theta_0) e^{-st_0} \sin\left(\frac{n\pi z_0}{H}\right)$$

Now, the integral of $\sin^2\left(\frac{n\pi z}{H}\right)$ must be done. This is done easily using a trig identity substitution, and even more easily using Wolfram alpha. However, trig identities are hard to remember but are easily obtained using Demoive's theorem.

Remark 2 (Demoive's theorem).

$$\text{Start with the Euler formula } e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$\text{Recognize that } \left(e^{i\theta}\right)^2 = e^{2i\theta}$$

$$\text{Thus } \left(\cos(\theta) + i\sin(\theta)\right)^2 = \cos(2\theta) + i\sin(2\theta)$$

$$\cos^2(\theta) + 2i\sin(\theta)\cos(\theta) - \sin^2(\theta) = \cos(2\theta) + i\sin(2\theta)$$

The above is an equation with real and imaginary parts, which can be equated separately.

$$Re : \cos^2(\theta) - \sin^2(\theta) = \cos(2\theta) \qquad\qquad Im : 2i\sin(\theta)\cos(\theta) = i\sin(2\theta)$$

$$\text{Use the left one to get a trig identity for } \sin^2$$

$$\sin^2(\theta) = -\cos(2\theta) + \cos^2(\theta)$$

$$\sin^2(\theta) + \sin^2(\theta) = -\cos(2\theta) + \cos^2(\theta) + \sin^2(\theta)$$

$$2\sin^2(\theta) = -\cos(2\theta) + 1$$

$$\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$$

Returning to the main problem, use the identity to solve the integral.

$$\int_0^H \sin^2\left(\frac{n\pi z}{H}\right) dz = \int_0^H \frac{1}{2}\left(1 - \cos\left(\frac{2n\pi z}{H}\right)\right) dz$$

$$= \frac{1}{2}\left[x - \sin\left(\frac{2n\pi z}{H}\right)\frac{H}{2n\pi}\right]\Bigg|_0^H$$

$$= \frac{1}{2}\left[x - \sin\left(\frac{2n\pi z}{H}\right)\frac{H}{2n\pi}\right] = \frac{H}{2}$$

Now put the solution to the integral into the four places where it was found in the last statement of the main problem.

$$\frac{s}{\alpha}a_n(s,r,\theta)\frac{H}{2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}a_n(s,r,\theta)\right)\frac{H}{2}$$

$$+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left(a_n(s,r,\theta)\right)\frac{H}{2} - a_n(s,r,\theta)\left(\frac{n^2\pi^2}{H^2}\right)\frac{H}{2}$$

$$+\frac{1}{r\alpha}\delta(r-r_0)\delta(\theta-\theta_0)e^{-st_0}sin\left(\frac{n\pi z_0}{H}\right)dz$$

Collect terms.

$$a_n(s,r,\theta)\left(\frac{s}{\alpha}+\frac{n^2\pi^2}{H^2}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}a_n(s,r,\theta)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left(a_n(s,r,\theta)\right)$$

$$+\frac{2}{Hr\alpha}\delta(r-r_0)\delta(\theta-\theta_0)e^{-st_0}sin\left(\frac{n\pi z_0}{H}\right)$$

The z expansion is finished. The next step is to conduct a similar expansion on the $\frac{\partial^2}{\partial\theta^2}$ Sturm-Liouville operator. The same steps are used. First, propose that

$$a_n(s,r,\theta) = \sum_{m=1}^{\infty} b_{nm}(s,r)sin\left(\frac{m\pi\theta}{\psi}\right)$$

Where
$\psi = 40° = \frac{2\pi}{9}$
b_{nm} zero at $\theta = 0$ and at $\theta = \psi$

As before, substitute the sine expansion into the differential equation.

$$\left[\sum_{m=1}^{\infty} b_{nm}(s,r)sin\left(\frac{m\pi\theta}{\psi}\right)\right]\left(\frac{s}{\alpha}+\frac{n^2\pi^2}{H^2}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left[\sum_{m=1}^{\infty} b_{nm}(s,r)sin\left(\frac{m\pi\theta}{\psi}\right)\right]\right)$$

$$+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left(\left[\sum_{m=1}^{\infty} b_{nm}(s,r)sin\left(\frac{m\pi\theta}{\psi}\right)\right]\right) + \frac{2}{Hr\alpha}\delta(r-r_0)\delta(\theta-\theta_0)e^{-st_0}sin\left(\frac{n\pi z_0}{H}\right)$$

Take the second derivative with respect to θ, Multiply by a particular eigenfunction and integrate over the domain. Make sure to treat the term with the r derivative and the following term together, as later it is necessary to put it in the form of a Bessel function.

$$\left(\frac{s}{\alpha}+\frac{n^2\pi^2}{H^2}\right)\int_0^\psi \sum_{m=1}^{\infty} b_{nm}(s,r)sin\left(\frac{m\pi\theta}{\psi}\right)sin\left(\frac{m'\pi\theta}{\psi}\right)d\theta$$

$$= \int_0^\psi \sum_{m=1}^{\infty}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}b_{nm}(s,r)sin\left(\frac{m\pi\theta}{\psi}\right)\right) - \frac{m^2\pi^2}{r^2\psi^2}\left(b_{nm}(s,r)sin\left(\frac{m\pi\theta}{\psi}\right)\right)\right]sin\left(\frac{m'\pi\theta}{\psi}\right)d\theta$$

$$+\int_0^\psi \frac{2}{Hr\alpha}\delta(r-r_0)\delta(\theta-\theta_0)e^{-st_0}sin\left(\frac{n\pi z_0}{H}\right)sin\left(\frac{m'\pi\theta}{\psi}\right)d\theta$$

As before, the sums act as $\delta_{mm'}$.

$$\left(\frac{s}{\alpha} + \frac{n^2\pi^2}{H^2}\right)\int_0^\psi b_{nm}(s,r)sin^2\left(\frac{m\pi\theta}{\psi}\right)d\theta$$

$$= \int_0^\psi \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}b_{nm}(s,r)sin^2\left(\frac{m\pi\theta}{\psi}\right)\right) - \frac{m^2\pi^2}{r^2\psi^2}\left(b_{nm}(s,r)sin^2\left(\frac{m\pi\theta}{\psi}\right)\right)\right]d\theta$$

$$+ \frac{2}{Hr\alpha}\delta(r-r_0)e^{-st_0}sin\left(\frac{n\pi z_0}{H}\right)\int_0^\psi \delta(\theta-\theta_0)sin\left(\frac{m\pi\theta}{\psi}\right)d\theta$$

Note that $\int_0^\psi sin^2\left(\frac{m\pi\theta}{\psi}\right)d\theta = \pi/9$. Integrating the delta function over the θ interval yields the function of θ evaluated at θ_0.

$$\left(\frac{s}{\alpha} + \frac{n^2\pi^2}{H^2}\right)b_{nm}(s,r)\frac{\pi}{9}$$

$$= \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}b_{nm}(s,r)\right) - \frac{m^2\pi^2}{r^2\psi^2}\left(b_{nm}(s,r)\right)\right]\frac{\pi}{9}$$

$$+ \frac{2}{Hr\alpha}\delta(r-r_0)e^{-st_0}sin\left(\frac{n\pi z_0}{H}\right)sin\left(\frac{m\pi\theta_0}{\psi}\right)$$

Simplifying

$$\left(\frac{s}{\alpha} + \frac{n^2\pi^2}{H^2}\right)b_{nm}(s,r)$$

$$= \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}b_{nm}(s,r)\right) - \frac{m^2\pi^2}{r^2\psi^2}\left(b_{nm}(s,r)\right)\right]$$

$$+ \frac{18}{\pi Hr\alpha}\delta(r-r_0)e^{-st_0}sin\left(\frac{n\pi z_0}{H}\right)sin\left(\frac{m\pi\theta_0}{\psi}\right)$$

The expansions for z and θ are now complete. Recognize that the middle term is in the form of a Bessel differential equation, which can be represented in the following form:

$$\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}J_\nu(kr)\right) - \frac{m^2}{r}\left(J_\nu(kr)\right)\right] = -k^2 r J_\nu(kr)$$

Where
k are the eigenvalues of the Bessel function
ν is the order of the Bessel function

In this specific case, $\nu = \frac{m\pi}{\psi}$. Thus, a Bessel series expansion is proposed

$$b_{nm}(s,r) = \sum_{l=1}^\infty c_{nml}(s)J_{\frac{m\pi}{\psi}}\left(k_{ml}r\right)$$

Where
$J_{\frac{m\pi}{\psi}}$ is the $(\frac{m\pi}{\psi})^{th}$ order Bessel function of the first kind
$k_{ml}a$ is the lth zero of $J_{\frac{m\pi}{\psi}}$

As before, substitute into the differential equation.

$$\left(\frac{s}{\alpha} + \frac{n^2\pi^2}{H^2}\right)\sum_{l=1}^{\infty} c_{nml}(s)J_{\frac{m\pi}{\psi}}(k_{ml}r)$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\sum_{l=1}^{\infty} c_{nml}(s)J_{\frac{m\pi}{\psi}}(k_{ml}r)\right) - \frac{m^2\pi^2}{r\psi^2}\left(\sum_{l=1}^{\infty} c_{nml}(s)J_{\frac{m\pi}{\psi}}(k_{ml}r)\right)\right]$$

$$+ \frac{18}{\pi Hr\alpha}\delta(r-r_0)e^{-st_0}sin\left(\frac{n\pi z_0}{H}\right)sin\left(\frac{m\pi\theta_0}{\psi}\right)$$

Rearranging,

$$\left(\frac{s}{\alpha} + \frac{n^2\pi^2}{H^2}\right)\sum_{l=1}^{\infty} c_{nml}(s)J_{\frac{m\pi}{\psi}}(k_{ml}r)$$

$$= \sum_{l=1}^{\infty} c_{nml}(s)\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}J_{\frac{m\pi}{\psi}}(k_{ml}r)\right) - \frac{m^2\pi^2}{r\psi^2}\left(J_{\frac{m\pi}{\psi}}(k_{ml}r)\right)\right]$$

$$+ \frac{18}{\pi Hr\alpha}\delta(r-r_0)e^{-st_0}sin\left(\frac{n\pi z_0}{H}\right)sin\left(\frac{m\pi\theta_0}{\psi}\right)$$

Once again, recognize that the middle term is in the form of a Bessel differential equation, which can be represented by $-k_{ml}^2 r J_{\frac{m\pi}{\psi}}(k_{ml}r)$. This can be proved by directly substituting the series expansion and evaluating the r derivatives of the Bessel function. Multiply everything by r and then substitute.

$$\left(\frac{s}{\alpha} + \frac{n^2\pi^2}{H^2}\right)\sum_{l=1}^{\infty} c_{nml}(s)J_{\frac{m\pi}{\psi}}(k_{ml}r)\,r$$

$$= -\sum_{l=1}^{\infty} c_{nml}(s)k_{ml}^2 J_{\frac{m\pi}{\psi}}(k_{ml}r)\,r$$

$$+ \frac{18}{\pi Hr\alpha}\delta(r-r_0)e^{-st_0}sin\left(\frac{n\pi z_0}{H}\right)sin\left(\frac{m\pi\theta_0}{\psi}\right)r$$

Don't cancel the common factor of r, as later we will want to use it as part of a convenient definition for Sturm-Liouville orthogonality of Bessel functions. Next, multiply by a particular eigenfunction, $J_{\frac{m\pi}{\psi}}(k_{ml'}r)$ and integrate over the domain of r.

$$\int_0^a \left(\frac{s}{\alpha} + \frac{n^2\pi^2}{H^2}\right)\sum_{l=1}^{\infty} c_{nml}(s)J_{\frac{m\pi}{\psi}}(k_{ml}r)\,J_{\frac{m\pi}{\psi}}(k_{ml'}r)\,rdr$$

$$= -\int_0^a \sum_{l=1}^{\infty} c_{nml}(s)k_{ml}^2 J_{\frac{m\pi}{\psi}}(k_{ml}r)\,J_{\frac{m\pi}{\psi}}(k_{ml'}r)\,rdr$$

$$+ \int_0^a \frac{18}{\pi Hr\alpha}\delta(r-r_0)e^{-st_0}sin\left(\frac{n\pi z_0}{H}\right)sin\left(\frac{m\pi\theta_0}{\psi}\right)J_{\frac{m\pi}{\psi}}(k_{ml'}r)\,rdr$$

Rearranging,

$$\sum_{l=1}^{\infty} c_{nml}(s)\left(\frac{s}{\alpha} + \frac{n^2\pi^2}{H^2}\right) \int_0^a J_{m\pi/\psi}\left(k_{ml}r\right) J_{m\pi/\psi}\left(k_{ml'}r\right) r dr$$

$$= \sum_{l=1}^{\infty} c_{nml}(s)k_{ml}^2 - \int_0^a J_{m\pi/\psi}\left(k_{ml}r\right) J_{m\pi/\psi}\left(k_{ml'}r\right) r dr$$

$$+ sin\left(\frac{n\pi z_0}{H}\right) sin\left(\frac{m\pi\theta_0}{\psi}\right) e^{-st_0} \frac{18}{\pi H\alpha} \int_0^a \frac{\delta(r-r_0)}{r} J_{m\pi/\psi}\left(k_{ml'}r\right) r dr$$

Now cancel the factor of r that is in both the numerator and denominator of the source term. Recognize that Sturm-Liouville orthogonality can be applied. This relevant property is

$$\int_0^a J_\nu\left(k_{ml}r\right) J_\nu\left(k_{ml'}r\right) r dr = \begin{cases} \frac{1}{2}\left[aJ_{\nu+1}(k_{ml}a)\right]^2 & l = l' \\ 0 & l \neq l' \end{cases}$$

Also, recognize that the delta function in the source term returns the Bessel function, but evaluated at r_0. Applying orthogonality and the delta function property

$$c_{nml}(s)\left(\frac{s}{\alpha} + \frac{n^2\pi^2}{H^2}\right)\frac{1}{2}\left[aJ_{\frac{m\pi}{\psi}+1}(k_{ml}a)\right]^2$$

$$= -c_{nml}(s)k_{ml}^2 \frac{1}{2}\left[aJ_{\frac{m\pi}{\psi}+1}(k_{ml}a)\right]^2$$

$$+ sin\left(\frac{n\pi z_0}{H}\right) sin\left(\frac{m\pi\theta_0}{\psi}\right) e^{-st_0} \frac{18}{\pi H\alpha} J_{\frac{m\pi}{\psi}}\left(k_{ml}r_0\right)$$

Rearranging,

$$c_{nml}(s)\left(\frac{s}{\alpha} + \frac{n^2\pi^2}{H^2}\right)$$

$$= -c_{nml}(s)k_{ml}^2 + sin\left(\frac{n\pi z_0}{H}\right) sin\left(\frac{m\pi\theta_0}{\psi}\right) e^{-st_0} \frac{36}{\left[aJ_{\frac{m\pi}{\psi}+1}(k_{ml}a)\right]^2 \pi H\alpha} J_{\frac{m\pi}{\psi}}\left(k_{ml}r_0\right)$$

Simplifying,

$$c_{nml}(s)\left(\frac{s}{\alpha} + \frac{n^2\pi^2}{H^2} + k_{ml}^2\right) = sin\left(\frac{n\pi z_0}{H}\right) sin\left(\frac{m\pi\theta_0}{\psi}\right) e^{-st_0} \frac{36}{a^2 J_{\frac{m\pi}{\psi}+1}^2(k_{ml}a)\pi H\alpha} J_{\frac{m\pi}{\psi}}\left(k_{ml}r_0\right)$$

Rearranging again,

$$c_{nml}(s)\left(s + \frac{\alpha n^2\pi^2}{H^2} + \alpha k_{ml}^2\right) = e^{-st_0} sin\left(\frac{n\pi z_0}{H}\right) sin\left(\frac{m\pi\theta_0}{\psi}\right) \frac{36}{a^2 J_{\frac{m\pi}{\psi}+1}^2(k_{ml}a)\pi H} J_{\frac{m\pi}{\psi}}\left(k_{ml}r_0\right)$$

Solving for c_{nml},

$$c_{nml}(s) = \frac{e^{-st_0}}{\left(s + \frac{\alpha n^2\pi^2}{H^2} + \alpha k_{ml}^2\right)} sin\left(\frac{n\pi z_0}{H}\right) sin\left(\frac{m\pi\theta_0}{\psi}\right) \frac{36}{a^2 J_{\frac{m\pi}{\psi}+1}^2(k_{ml}a)\pi H} J_{\frac{m\pi}{\psi}}\left(k_{ml}r_0\right)$$

Now that c_{nml} has been found, the solution can be put together from the expanded series. First, show b_{nm}.

$$b_{nm}(s,r) = \sum_{l=1}^{\infty} J_{\frac{m\pi}{\psi}}(k_{ml}r) \frac{36}{\pi H a^2 J_{\frac{m\pi}{\psi}+1}^2 (k_{ml}r_0)(k_{ml}a)} sin\left(\frac{n\pi z_0}{H}\right) sin\left(\frac{m\pi\theta_0}{\psi}\right) J_{\frac{m\pi}{\psi}}(k_{ml}r_0)$$

$$\frac{e^{-st_0}}{s + \alpha\frac{n^2\pi^2}{H^2} + \alpha k_{ml}^2}$$

Second, put together a_n.

$$a_n(s,r,\theta) = \sum_{m=1}^{\infty}\sum_{l=1}^{\infty} sin\left(\frac{m\pi\theta}{\psi}\right) J_{\frac{m\pi}{\psi}}(k_{ml}r) \frac{36}{\pi H a^2 J_{\frac{m\pi}{\psi}+1}^2 (k_{ml}a)} sin\left(\frac{n\pi z_0}{H}\right)$$

$$sin\left(\frac{m\pi\theta_0}{\psi}\right) J_{\frac{m\pi}{\psi}}(k_{ml}r_0) \frac{e^{-st_0}}{s + \alpha\frac{n^2\pi^2}{H^2} + \alpha k_{ml}^2}$$

Third, put together G_{src}.

$$G_{src}(s,r,\theta,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} sin\left(\frac{n\pi z}{H}\right) sin\left(\frac{m\pi\theta}{\psi}\right) J_{\frac{m\pi}{\psi}}(k_{ml}r) \frac{36}{\pi H a^2 J_{\frac{m\pi}{\psi}+1}^2 (k_{ml}a)}$$

$$sin\left(\frac{n\pi z_0}{H}\right) sin\left(\frac{m\pi\theta_0}{\psi}\right) J_{\frac{m\pi}{\psi}}(k_{ml}r_0) \frac{e^{-st_0}}{s + \alpha\frac{n^2\pi^2}{H^2} + \alpha k_{ml}^2}$$

Putting like functions next to each other,

$$G_{src}(s,r,\theta,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} sin\left(\frac{n\pi z}{H}\right) sin\left(\frac{n\pi z_0}{H}\right) sin\left(\frac{m\pi\theta}{\psi}\right) sin\left(\frac{m\pi\theta_0}{\psi}\right)$$

$$J_{\frac{m\pi}{\psi}}(k_{ml}r) J_{\frac{m\pi}{\psi}}(k_{ml}r_0) \frac{36}{\pi H a^2 J_{\frac{m\pi}{\psi}+1}^2 (k_{ml}a)} \frac{e^{-st_0}}{s + \alpha\frac{n^2\pi^2}{H^2} + \alpha k_{ml}^2}$$

Recall that G_{src} was the Laplace transform of g_{src}. This must be undone by inverting the previous statement. At this point, the green's function for a source term has been obtained.

$$g_{src}(t,r,\theta,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} sin\left(\frac{n\pi z}{H}\right) sin\left(\frac{n\pi z_0}{H}\right) sin\left(\frac{m\pi\theta}{\psi}\right) sin\left(\frac{m\pi\theta_0}{\psi}\right)$$

$$J_{\frac{m\pi}{\psi}}(k_{ml}r) J_{\frac{m\pi}{\psi}}(k_{ml}r_0) \frac{36}{\pi H a^2 J_{\frac{m\pi}{\psi}+1}^2 (k_{ml}a)} e^{-\alpha(t-t_0)\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)} H(t-t_0)$$

Where
H is the Heaviside function

In the next section, this function will be used to solve the original problem.

Remark 3 (Mellin inversion). The Mellin formula was used to invert the Laplace transform using the residue theorem and a contour in the complex plane. The first step is to rearrange the function to be inverted, giving

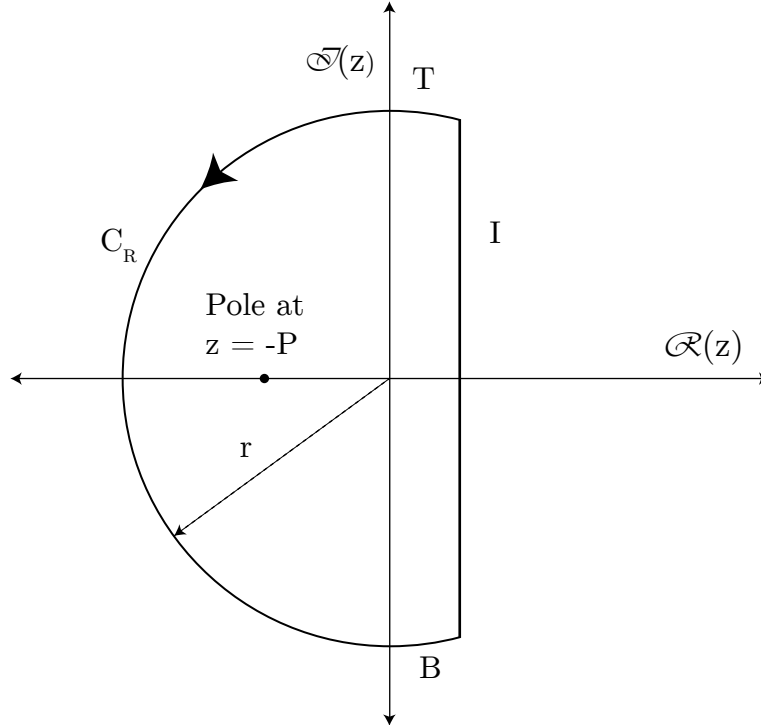$$G_{src}(s, r, \theta, z) = \frac{Qe^{-st_0}}{P + s}$$

Where
$Q = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} sin\left(\frac{n\pi z}{H}\right) sin\left(\frac{n\pi z_0}{H}\right) sin\left(\frac{m\pi\theta}{\psi}\right) sin\left(\frac{m\pi\theta_0}{\psi}\right)$
$J_{\frac{m\pi}{\psi}}(k_{ml}r)J_{\frac{m\pi}{\psi}}(k_{ml}r_0)\frac{36}{\pi Ha^2 J_{\frac{m\pi}{\psi}+1}^2(k_{ml}a)}$

$P = \alpha\frac{n^2\pi^2}{H^2} + \alpha k_{ml}^2$

Next, invoke the Mellin formula:

$$f(t) = \mathcal{L}^{-1}F(s) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{Qe^{-st_0}}{P + s}e^{st}ds$$

The next step is to draw a contour in the input space of the function.

The integral that is desired is a vertical line from $-\infty$ to ∞ at an x value of γ, which is chosen so that all of the poles of the function lie to the left. The only pole has multiplicity 1 and is located at $s = -P$. To use the residue theorem, a closed contour must be used. Close left, yielding three more segments that must be evaluated, T, C_R, and B. The residue theorem gives

$$\oint = 2\pi i \sum res = I + T + C_R + B$$

Analyze T and B, using the change of variables $z = x + ir$ or $z = x - ir$. Argue that both approach a horizontal line as r approaches ∞. Both go to zero.

$$T = \lim_{r\to\infty} Q \int_\gamma^0 \frac{e^{-st_0}e^{st}}{P+s}dx = \lim_{r\to\infty} Q \int_\gamma^0 \frac{e^{-(x+ir)t_0}e^{(x+ir)t}}{P+(x+ir)}dx = 0$$

$$B = \lim_{r\to\infty} Q \int_0^\gamma \frac{e^{-st_0}e^{st}}{P+s}dx = \lim_{r\to\infty} Q \int_0^\gamma \frac{e^{-(x-ir)t_0}e^{(x-ir)t}}{P+(x-ir)}dx = 0$$

Analyze C_R using a different change of variables: $z = re^{i\theta}$.

$$C_R = \lim_{r\to\infty} Q \int_{\pi/2-\epsilon}^{3\pi/2+\epsilon} \frac{e^{-re^{i\theta}t_0}e^{re^{i\theta}t}}{P+re^{i\theta}}rie^{i\theta}d\theta = 0$$

Since the arc and line segments are zero, the integral that was desired, I, can be found by

$$2\pi i \sum res = I$$

The residue at $s = -P$ can be found using a simple formula that applies to functions that can be represented by $F = \frac{X}{Y}$, and have $X(s) \neq 0$ and $Y(s) = 0$.

$$res(a) = \frac{X(s)}{Y'(s)} = \frac{Qe^{--Pt_0}e^{-Pt}}{1} = \frac{Qe^{P(t_0-t)}}{1} = Qe^{-P(t-t_0)}$$

Using the residue theorem

$$2\pi i \sum res = 2\pi i res(a) = 2\pi i Qe^{-P(t-t_0)}$$

Returning to the Mellin theorem

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{Qe^{-st_0}}{P+s}e^{st}ds = \frac{1}{2\pi i}2\pi i Qe^{-P(t-t0)}$$

In order to enforce that t must be greater than t_0, add the Heaviside step function.

$$f(t) = Qe^{-P(t-t0)}H(t-t_0)$$

Substituting back the definitions of Q and P yields the inverted statement shown previously.

$$f(t) = \sum_{n=1}^\infty \sum_{m=1}^\infty \sum_{l=1}^\infty \sin\left(\frac{n\pi z}{H}\right)\sin\left(\frac{n\pi z_0}{H}\right)\sin\left(\frac{m\pi\theta}{\psi}\right)\sin\left(\frac{m\pi\theta_0}{\psi}\right)J_{\frac{m\pi}{\psi}}(k_{ml}r)J_{\frac{m\pi}{\psi}}(k_{ml}r_0)$$

$$\frac{36}{\pi Ha^2 J^2_{\frac{m\pi}{\psi}+1}(k_{ml}a)}e^{\left(-\alpha\frac{n^2\pi^2}{H^2}+\alpha k^2_{ml}\right)(t-t_0)}H(t-t_0)$$

2.2 Using the Green's function for the particular problem

The green's function must respect the boundary conditions of the original problem. After the initial change of variables, all boundary conditions were homogeneous except for the pie-ice cream boundary and initial temperature. For the boundary condition, $-\alpha$ times the normal gradient of the Green's function with respect to the inhomogeneous variable, z_0, and evaluated at the location of the inhomogeneity, $z_0 = 0$, will satisfy the BC. This integral must also contain the value of the inhomogeneous condition, and must then be integrated over the other variables. Let this part of the solution be u_1.

$$u_1(t, r, \theta) = \int_0^a \int_0^\psi \int_0^{t\infty} -\alpha \frac{\partial g_{src}}{\partial z_0}\bigg|_{z_0} u_{BC}(r_0, \theta_0, t_0) dt_0 r_0 d\theta_0 dr_0$$

However, the problem also had an inhomogeneous initial condition. As luck would have it, the Green's function for an initial condition is the Green's function for the source term, evaluated at $t_0 = 0$, multiplied by the value of the initial condition, and integrated over the domain. Call this part of the solution u_2.

$$u_2(r, \theta, z) = \int_0^a \int_0^\psi \int_0^H g_{src}\bigg|_{t_0=0} u_{IC}(r_0, \theta_0, z_0) dz_0 r_0 d\theta_0 dr_0$$

Because of superposition, the two solutions can be added together to obtain a solution for the overall temperature profile.

$$u(t, r, \theta, z) = u_1(t, r, \theta) + u_2(r, \theta, z)$$

First, evaluate u_1. Recognize that the boundary condition temperature is constant with respect to all of the variables. Also recognize that since the normal gradient points out of the object, at the bottom of the pie where this boundary condition is relevant this is in the $-z$ direction, canceling the negative signs.

$$u_1(t, r, \theta) = u_{BC} \int_0^a \int_0^\psi \int_0^t \alpha \sum_{n=1}^\infty \sum_{m=1}^\infty \sum_{l=1}^\infty \sin\left(\frac{n\pi z}{H}\right) \cos\left(\frac{n\pi(0)}{H}\right) \frac{n\pi}{H} \sin\left(\frac{m\pi\theta}{\psi}\right) \sin\left(\frac{m\pi\theta_0}{\psi}\right)$$

$$J_{\frac{m\pi}{\psi}}(k_{ml}r) J_{\frac{m\pi}{\psi}}(k_{ml}r_0) \frac{36}{\pi H a^2 J_{\frac{m\pi}{\psi}+1}^2(k_{ml}a)} e^{-\alpha(t-t_0)\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)} H(t-t_0) dt_0 r_0 d\theta_0 dr_0$$

Integrating the heaviside function changes the limits of the time integral based on the point where it turns on.

$$u_1(t, r, \theta) = u_{BC} \int_0^a \int_0^\psi \int_{t_0}^t \sum_{n=1}^\infty \sum_{m=1}^\infty \sum_{l=1}^\infty \sin\left(\frac{n\pi z}{H}\right) \sin\left(\frac{m\pi\theta}{\psi}\right) \sin\left(\frac{m\pi\theta_0}{\psi}\right)$$

$$J_{\frac{m\pi}{\psi}}(k_{ml}r) J_{\frac{m\pi}{\psi}}(k_{ml}r_0) \frac{36\alpha n}{H^2 a^2 J_{\frac{m\pi}{\psi}+1}^2(k_{ml}a)} e^{-\alpha(t-t_0)\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)} dt_0 r_0 d\theta_0 dr_0$$

Next, the exponential function of time must be integrated.

$$u_1(t, r, \theta) = u_{BC} \frac{36\alpha}{H^2 a^2} \int_0^a \int_0^\psi \sum_{n=1}^\infty \sum_{m=1}^\infty \sum_{l=1}^\infty \sin\left(\frac{n\pi z}{H}\right) \sin\left(\frac{m\pi\theta}{\psi}\right) \sin\left(\frac{m\pi\theta_0}{\psi}\right)$$

$$J_{\frac{m\pi}{\psi}}(k_{ml}r) J_{\frac{m\pi}{\psi}}(k_{ml}r_0) \frac{n}{J_{\frac{m\pi}{\psi}+1}^2(k_{ml}a)} \left[\frac{e^{-\alpha(t-t_0)\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}}{\alpha\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}\right]_{t_0=0}^t r_0 d\theta_0 dr_0$$

Simplifying,

$$u_1(t,r,\theta) = u_{BC}\frac{36\alpha}{H^2a^2}\int_0^a\int_0^\psi\sum_{n=1}^\infty\sum_{m=1}^\infty\sum_{l=1}^\infty sin\left(\frac{n\pi z}{H}\right)sin\left(\frac{m\pi\theta}{\psi}\right)sin\left(\frac{m\pi\theta_0}{\psi}\right)$$

$$J_{\frac{m\pi}{\psi}}(k_{ml}r)J_{\frac{m\pi}{\psi}}(k_{ml}r_0)\frac{n}{J_{\frac{m\pi}{\psi}+1}^2(k_{ml}a)}\left(\frac{1-e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}}{\alpha\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}\right)r_0 d\theta_0 dr_0$$

Evaluate the integral with respect to θ_0.

$$u_1(t,r,\theta) = u_{BC}\frac{36\alpha}{H^2a^2}\int_0^a\sum_{n=1}^\infty\sum_{m=1}^\infty\sum_{l=1}^\infty sin\left(\frac{n\pi z}{H}\right)sin\left(\frac{m\pi\theta}{\psi}\right)J_{\frac{m\pi}{\psi}}(k_{ml}r)J_{\frac{m\pi}{\psi}}(k_{ml}r_0)$$

$$\frac{n}{J_{\frac{m\pi}{\psi}+1}^2(k_{ml}a)}\left(\frac{1-e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}}{\alpha\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}\right)\left[-cos\left(\frac{m\pi\theta_0}{\psi}\right)\frac{\psi}{m\pi}\right]\Bigg|_0^\psi r_0 dr_0$$

Note that

$$-cos\left(\frac{m\pi\theta_0}{\psi}\right)\frac{\psi}{m\pi}\Bigg|_0^\psi = \frac{-\psi}{m\pi}\left[cos\left(m\pi\right)-cos\left(0\right)\right] = \frac{-\psi}{\pi}\frac{\left((-1)^m-1\right)}{m}$$

Simplifying,

$$u_1(t,r,\theta) = u_{BC}\frac{-36\psi\alpha}{\pi H^2a^2}\int_0^a\sum_{n=1}^\infty\sum_{m=1}^\infty\sum_{l=1}^\infty\frac{n\left((-1)^m-1\right)}{m}sin\left(\frac{n\pi z}{H}\right)sin\left(\frac{m\pi\theta}{\psi}\right)J_{\frac{m\pi}{\psi}}(k_{ml}r)J_{\frac{m\pi}{\psi}}(k_{ml}r_0)$$

$$\frac{1}{J_{\frac{m\pi}{\psi}+1}^2(k_{ml}a)}\left(\frac{1-e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}}{\alpha\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}\right)r_0 dr_0$$

Rearranging,

$$u_1(t,r,\theta) = u_{BC}\frac{-36\psi\alpha}{\pi H^2a^2}\sum_{n=1}^\infty\sum_{m=1}^\infty\sum_{l=1}^\infty\frac{n\left((-1)^m-1\right)}{m}sin\left(\frac{n\pi z}{H}\right)sin\left(\frac{m\pi\theta}{\psi}\right)J_{\frac{m\pi}{\psi}}(k_{ml}r)\frac{1}{J_{\frac{m\pi}{\psi}+1}^2(k_{ml}a)}$$

$$\left(\frac{1-e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}}{\alpha\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}\right)\int_0^a J_{\frac{m\pi}{\psi}}(k_{ml}r_0)r_0 dr_0$$

Recognize that the integral of the Bessel function in r_0 cannot be expressed in closed form with easy methods. Thus, it is replaced with a symbol, ξ_{ml}. These integrals will be evaluated numerically and tabulated. When evaluating the solution, the necessary amount will be called by querying the table.

$$u_1(t,r,\theta) = u_{BC}\frac{-36\psi\alpha}{\pi H^2a^2}\sum_{n=1}^\infty\sum_{m=1}^\infty\sum_{l=1}^\infty\frac{n\left((-1)^m-1\right)}{m}sin\left(\frac{n\pi z}{H}\right)sin\left(\frac{m\pi\theta}{\psi}\right)J_{\frac{m\pi}{\psi}}(k_{ml}r)\frac{1}{J_{\frac{m\pi}{\psi}+1}^2(k_{ml}a)}$$

$$\left(\frac{1-e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}}{\alpha\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}\right)\xi_{ml}$$

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The function that respects the BC has been found. The next step is to evaluate the function that respects the initial condition.

$$u_2(r, \theta, z) = \int_0^a \int_0^\psi \int_0^H \sum_{n=1}^\infty \sum_{m=1}^\infty \sum_{l=1}^\infty \sin\left(\frac{n\pi z}{H}\right) \sin\left(\frac{n\pi z_0}{H}\right) \sin\left(\frac{m\pi \theta}{\psi}\right) \sin\left(\frac{m\pi \theta_0}{\psi}\right)$$

$$J_{\frac{m\pi}{\psi}}(k_{ml} r) J_{\frac{m\pi}{\psi}}(k_{ml} r_0) \frac{36}{\pi H a^2 J^2_{\frac{m\pi}{\psi}+1}(k_{ml} a)} e^{-\alpha(t-(0))\left(\frac{n^2\pi^2}{H^2}+k^2_{ml}\right)} H(t-(0)) u_{IC}(r_0, \theta_0, z_0) dz_0 r_0 d\theta_0 dr_0$$

For positive t, the Heaviside function is equal to 1. As it was for the boundary condition, the initial condition is not a function of the independent variables so it can be taken out of the integrals. Evaluate the integral in z_0.

$$u_2(r, \theta, z) = u_{IC} \frac{36}{\pi H a^2} \int_0^a \int_0^\psi \sum_{n=1}^\infty \sum_{m=1}^\infty \sum_{l=1}^\infty \sin\left(\frac{n\pi z}{H}\right) \sin\left(\frac{m\pi \theta}{\psi}\right) \sin\left(\frac{m\pi \theta_0}{\psi}\right)$$

$$J_{\frac{m\pi}{\psi}}(k_{ml} r) J_{\frac{m\pi}{\psi}}(k_{ml} r_0) \frac{1}{J^2_{\frac{m\pi}{\psi}+1}(k_{ml} a)} e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k^2_{ml}\right)} \left[-\cos\left(\frac{n\pi z_0}{H}\right)\frac{H}{n\pi}\right]\Bigg|_0^H r_0 d\theta_0 dr_0$$

Simplifying,

$$u_2(r, \theta, z) = u_{IC} \frac{-36}{\pi^2 a^2} \int_0^a \int_0^\psi \sum_{n=1}^\infty \sum_{m=1}^\infty \sum_{l=1}^\infty \frac{((-1)^n - 1)}{n} \sin\left(\frac{n\pi z}{H}\right) \sin\left(\frac{m\pi \theta}{\psi}\right) \sin\left(\frac{m\pi \theta_0}{\psi}\right)$$

$$J_{\frac{m\pi}{\psi}}(k_{ml} r) J_{\frac{m\pi}{\psi}}(k_{ml} r_0) \frac{1}{J^2_{\frac{m\pi}{\psi}+1}(k_{ml} a)} e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k^2_{ml}\right)} r_0 d\theta_0 dr_0$$

Evaluating the integral in θ_0,

$$u_2(r, \theta, z) = u_{IC} \frac{-36}{\pi^2 a^2} \int_0^a \sum_{n=1}^\infty \sum_{m=1}^\infty \sum_{l=1}^\infty \frac{((-1)^n - 1)}{n} \sin\left(\frac{n\pi z}{H}\right) \sin\left(\frac{m\pi \theta}{\psi}\right)$$

$$J_{\frac{m\pi}{\psi}}(k_{ml} r) J_{\frac{m\pi}{\psi}}(k_{ml} r_0) \frac{1}{J^2_{\frac{m\pi}{\psi}+1}(k_{ml} a)} e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k^2_{ml}\right)} \left[-\cos\left(\frac{m\pi \theta_0}{\psi}\right)\frac{\psi}{m\pi}\right]\Bigg|_0^\psi r_0 dr_0$$

Simplifying,

$$u_2(r, \theta, z) = u_{IC} \frac{36\psi}{\pi^3 a^2} \int_0^a \sum_{n=1}^\infty \sum_{m=1}^\infty \sum_{l=1}^\infty \frac{((-1)^n - 1)}{n} \frac{((-1)^m - 1)}{m} \sin\left(\frac{n\pi z}{H}\right) \sin\left(\frac{m\pi \theta}{\psi}\right)$$

$$J_{\frac{m\pi}{\psi}}(k_{ml} r) J_{\frac{m\pi}{\psi}}(k_{ml} r_0) \frac{1}{J^2_{\frac{m\pi}{\psi}+1}(k_{ml} a)} e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k^2_{ml}\right)} r_0 dr_0$$

Rearrange to show clearly that the integral of the Bessel function over r_0 gives the same issue as before.

$$u_2(r, \theta, z) = u_{IC} \frac{36\psi}{\pi^3 a^2} \sum_{n=1}^\infty \sum_{m=1}^\infty \sum_{l=1}^\infty \frac{((-1)^n - 1)}{n} \frac{((-1)^m - 1)}{m} \sin\left(\frac{n\pi z}{H}\right) \sin\left(\frac{m\pi \theta}{\psi}\right)$$

$$J_{\frac{m\pi}{\psi}}(k_{ml} r) \frac{1}{J^2_{\frac{m\pi}{\psi}+1}(k_{ml} a)} e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k^2_{ml}\right)} \int_0^a J_{\frac{m\pi}{\psi}}(k_{ml} r_0) r_0 dr_0$$

Once again, replace it with ξ_{ml}.

$$u_2(r, \theta, z) = u_{IC} \frac{36\psi}{\pi^3 a^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \frac{((-1)^n - 1)}{n} \frac{((-1)^m - 1)}{m} \sin\left(\frac{n\pi z}{H}\right) \sin\left(\frac{m\pi\theta}{\psi}\right)$$

$$J_{\frac{m\pi}{\psi}}(k_{ml}r) \frac{1}{J_{\frac{m\pi}{\psi}+1}^2(k_{ml}a)} e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)} \xi_{ml}$$

Now, the overall solution can be put together $u(t, r, \theta, z) = u_1(t, r, \theta) + u_2(r, \theta, z)$.

$$u(t, r, \theta, z) = u_{BC} \frac{-36\psi\alpha}{\pi H^2 a^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \frac{n\left((-1)^m - 1\right)}{m} \sin\left(\frac{n\pi z}{H}\right) \sin\left(\frac{m\pi\theta}{\psi}\right)$$

$$J_{\frac{m\pi}{\psi}}(k_{ml}r) \frac{1}{J_{\frac{m\pi}{\psi}+1}^2(k_{ml}a)} \left(\frac{1 - e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}}{\alpha\left(\frac{n^2\pi^2}{H^2} + k_{ml}^2\right)}\right) \xi_{ml}$$

$$+u_{IC} \frac{36\psi}{\pi^3 a^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \frac{((-1)^n - 1)}{n} \frac{((-1)^m - 1)}{m} \sin\left(\frac{n\pi z}{H}\right) \sin\left(\frac{m\pi\theta}{\psi}\right)$$

$$J_{\frac{m\pi}{\psi}}(k_{ml}r) \frac{1}{J_{\frac{m\pi}{\psi}+1}^2(k_{ml}a)} e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)} \xi_{ml}$$

Some simplification yields a statement of the analytic solution.

$$u(t, r, \theta, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \sin\left(\frac{n\pi z}{H}\right) \sin\left(\frac{m\pi\theta}{\psi}\right) J_{\frac{m\pi}{\psi}}(k_{ml}r) \frac{36\psi\xi_{ml}}{\pi a^2 J_{\frac{m\pi}{\psi}+1}^2(k_{ml}a)} \frac{((-1)^m - 1)}{m}$$

$$\left[\frac{-u_{BC}n\alpha}{H^2}\left(\frac{1 - e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}}{\alpha\left(\frac{n^2\pi^2}{H^2} + k_{ml}^2\right)}\right) + \frac{u_{IC}}{\pi^2} \frac{((-1)^n - 1)}{n} e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}\right]$$

3 Evaluating the average temperature

The next step is to evaluate the average value of temperature on a volume basis, $T_{ave}(t)$. This will be done using

$$T_{ave}(t) = \frac{u_{vol}(t)}{V_{pie}} = \frac{\int_0^a \int_0^\psi \int_0^H u(t,r,\theta,z)dzrd\theta dr}{V_{pie}} + 21.1$$

Start by working on the numerator. Substituting the analytic solution into the volume integral,

$$u_{vol}(t) = \int_0^a \int_0^\psi \int_0^H \sum_{n=1}^\infty \sum_{m=1}^\infty \sum_{l=1}^\infty sin\left(\frac{n\pi z}{H}\right) sin\left(\frac{m\pi\theta}{\psi}\right) J_{\frac{m\pi}{\psi}}(k_{ml}r)$$

$$\frac{\xi_{ml}}{J^2_{\frac{m\pi}{\psi}+1}(k_{ml}a)}\left[u_{BC}\frac{-36\psi\alpha}{\pi H^2 a^2}\frac{n\left((-1)^m-1\right)}{m}\left(\frac{1-e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k^2_{ml}\right)}}{\alpha\left(\frac{n^2\pi^2}{H^2}+k^2_{ml}\right)}\right)\right.$$

$$\left.+u_{IC}\frac{36\psi}{\pi^3 a^2}\frac{\left((-1)^n-1\right)}{n}\frac{\left((-1)^m-1\right)}{m}e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k^2_{ml}\right)}\right]dzrd\theta dr$$

Rearranging,

$$u_{vol}(t) = \sum_{n=1}^\infty \sum_{m=1}^\infty \sum_{l=1}^\infty \frac{\xi_{ml}}{J^2_{\frac{m\pi}{\psi}+1}(k_{ml}a)}\left[u_{BC}\frac{-36\psi\alpha}{\pi H^2 a^2}\frac{n\left((-1)^m-1\right)}{m}\left(\frac{1-e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k^2_{ml}\right)}}{\alpha\left(\frac{n^2\pi^2}{H^2}+k^2_{ml}\right)}\right)\right.$$

$$\left.+u_{IC}\frac{36\psi}{\pi^3 a^2}\frac{\left((-1)^n-1\right)}{n}\frac{\left((-1)^m-1\right)}{m}e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k^2_{ml}\right)}\right]$$

$$\int_0^a \int_0^\psi \int_0^H sin\left(\frac{n\pi z}{H}\right) sin\left(\frac{m\pi\theta}{\psi}\right) J_{\frac{m\pi}{\psi}}(k_{ml}r)dzrd\theta dr$$

Evaluating the integral over z.

$$u_{vol}(t) = \sum_{n=1}^\infty \sum_{m=1}^\infty \sum_{l=1}^\infty \frac{\xi_{ml}}{J^2_{\frac{m\pi}{\psi}+1}(k_{ml}a)}\left[u_{BC}\frac{-36\psi\alpha}{\pi H^2 a^2}\frac{n\left((-1)^m-1\right)}{m}\left(\frac{1-e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k^2_{ml}\right)}}{\alpha\left(\frac{n^2\pi^2}{H^2}+k^2_{ml}\right)}\right)\right.$$

$$\left.+u_{IC}\frac{36\psi}{\pi^3 a^2}\frac{\left((-1)^n-1\right)}{n}\frac{\left((-1)^m-1\right)}{m}e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k^2_{ml}\right)}\right]$$

$$\int_0^a \int_0^\psi sin\left(\frac{m\pi\theta}{\psi}\right) J_{\frac{m\pi}{\psi}}(k_{ml}r)\left[-cos\left(\frac{n\pi z}{H}\right)\frac{H}{n\pi}\right]\Big|_0^H rd\theta dr$$

Simplifying,

$$u_{vol}(t) = \sum_{n=1}^\infty \sum_{m=1}^\infty \sum_{l=1}^\infty \frac{\xi_{ml}}{J^2_{\frac{m\pi}{\psi}+1}(k_{ml}a)}\left[u_{BC}\frac{-36\psi\alpha}{\pi H^2 a^2}\frac{n\left((-1)^m-1\right)}{m}\left(\frac{1-e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k^2_{ml}\right)}}{\alpha\left(\frac{n^2\pi^2}{H^2}+k^2_{ml}\right)}\right)\right.$$

$$\left.+u_{IC}\frac{36\psi}{\pi^3 a^2}\frac{\left((-1)^n-1\right)}{n}\frac{\left((-1)^m-1\right)}{m}e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k^2_{ml}\right)}\right]$$

$$\frac{-H}{\pi}\frac{\left((-1)^n-1\right)}{n}\int_0^a \int_0^\psi sin\left(\frac{m\pi\theta}{\psi}\right) J_{\frac{m\pi}{\psi}}(k_{ml}r)rd\theta dr$$

Evaluating the integral over θ,

$$u_{vol}(t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} \frac{\xi_{ml}}{J_{\frac{m\pi}{\psi}+1}^2\left(k_{ml}a\right)}\left[u_{BC}\frac{-36\psi\alpha}{\pi H^2 a^2}\frac{n\left((-1)^m-1\right)}{m}\left(\frac{1-e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}}{\alpha\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}\right)\right.$$

$$\left.+u_{IC}\frac{36\psi}{\pi^3 a^2}\frac{\left((-1)^n-1\right)}{n}\frac{\left((-1)^m-1\right)}{m}e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}\right]$$

$$\frac{-H}{\pi}\frac{\left((-1)^n-1\right)}{n}\int_0^a J_{\frac{m\pi}{\psi}}(k_{ml}r)\left[-\cos\left(\frac{m\pi\theta}{\psi}\right)\right]\frac{\psi}{m\pi}\Bigg|_0^{\psi} d\theta r dr$$

Simplifying,

$$u_{vol}(t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} \frac{\xi_{ml}}{J_{\frac{m\pi}{\psi}+1}^2\left(k_{ml}a\right)}\left[u_{BC}\frac{-36\psi\alpha}{\pi H^2 a^2}\frac{n\left((-1)^m-1\right)}{m}\left(\frac{1-e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}}{\alpha\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}\right)\right.$$

$$\left.+u_{IC}\frac{36\psi}{\pi^3 a^2}\frac{\left((-1)^n-1\right)}{n}\frac{\left((-1)^m-1\right)}{m}e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}\right]$$

$$\frac{-H}{\pi}\frac{\left((-1)^n-1\right)}{n}\frac{-\psi}{\pi}\frac{\left((-1)^m-1\right)}{m}\int_0^a J_{\frac{m\pi}{\psi}}(k_{ml}r)r dr$$

Once again, the interesting integral of the Bessel function has been obtained. It is in the same form as before, so once again it is replaced using ξ_{ml}, giving

$$u_{vol}(t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} \frac{\xi_{ml}^2}{J_{\frac{m\pi}{\psi}+1}^2\left(k_{ml}a\right)}\frac{H\psi}{\pi^2}\frac{\left((-1)^n-1\right)}{n}\frac{\left((-1)^m-1\right)}{m}$$

$$\left[u_{BC}\frac{-36\psi\alpha}{\pi H^2 a^2}\frac{n\left((-1)^m-1\right)}{m}\left(\frac{1-e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}}{\alpha\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}\right)+u_{IC}\frac{36\psi}{\pi^3 a^2}\frac{\left((-1)^n-1\right)}{n}\frac{\left((-1)^m-1\right)}{m}e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}\right]$$

Simplifying,

$$u_{vol}(t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} \frac{36H\psi^2}{\pi^3 a^2}\frac{\xi_{ml}^2}{J_{\frac{m\pi}{\psi}+1}^2\left(k_{ml}a\right)}\frac{\left((-1)^n-1\right)}{n}\left(\frac{\left((-1)^m-1\right)}{m}\right)^2$$

$$\left[u_{BC}\frac{-\alpha n}{H^2}\left(\frac{1-e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}}{\alpha\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}\right)+u_{IC}\frac{1}{\pi^2}\frac{\left((-1)^n-1\right)}{n}e^{-\alpha t\left(\frac{n^2\pi^2}{H^2}+k_{ml}^2\right)}\right]$$

This is the volume integral of the dependent variable. However, to find the volume average, the

nominal volume of the pie must be evaluated. This is done using

$$V_{pie} = \int_0^a \int_0^\psi \int_0^H dz r d\theta dr$$

$$V_{pie} = \int_0^a \int_0^\psi \left[z \Big|_0^H \right] r d\theta dr$$

$$V_{pie} = \int_0^a \int_0^\psi H r d\theta dr$$

$$V_{pie} = H \int_0^a \left[\theta \Big|_0^\psi \right] r dr$$

$$V_{pie} = H\psi \int_0^a r dr$$

$$V_{pie} = H\psi \left[r^2/2 \Big|_0^a \right]$$

$$V_{pie} = H\psi a^2/2$$

Now, the original shift can be undone to obtain temperature, T, from the dependent variable, u. Recall that $T = u + 21.1$. On the same step, the volume-basis average temperature is obtained by dividing the volume integral of the temperature by the volume of the pie.

$$T_{ave}(t) = \frac{u_{vol}(t)}{V_{pie}} + 21.1$$

The analytic solution for the average internal temperature is thus

$$T_{ave}(t) = 21.1 + \sum_{n=1}^\infty \sum_{m=1}^\infty \sum_{l=1}^\infty \frac{36H\psi^2}{\pi^3 a^2} \frac{\xi_{ml}^2}{J_{\frac{m\pi}{\psi}+1}^2(k_{ml}a)} \frac{((-1)^n - 1)}{n} \left(\frac{((-1)^m - 1)}{m} \right)^2$$

$$\left[u_{BC} \frac{-\alpha n}{H^2} \left(\frac{1 - e^{-\alpha t \left(\frac{n^2\pi^2}{H^2} + k_{ml}^2 \right)}}{\alpha \left(\frac{n^2\pi^2}{H^2} + k_{ml}^2 \right)} \right) + u_{IC} \frac{1}{\pi^2} \frac{((-1)^n - 1)}{n} e^{-\alpha t \left(\frac{n^2\pi^2}{H^2} + k_{ml}^2 \right)} \right] / \left(H\psi a^2/2 \right)$$

Where
$H = $ pie height $= 0.0381$m
$\psi = $ pie angle $= 40° = 2\pi/9$
$a = $ pie radius $= 0.127$ m
$\alpha = $ thermal diffusivity $= 1.34 \times 10^{-7} \pm 0.03 \frac{m^2}{s}$ [2]
$u_{BC} = $ constant temperature on the pie-ice cream boundary $= -21.1$°C
$u_{IC} = $ initial temperature of the pie $= 169.6$°C
$J_{\frac{m\pi}{\psi}} = $ Bessel function of the first kind, of order $\frac{m\pi}{\psi}$
$k_{ml} = l^{th}$ zero of Bessel function of the first kind, of order m
$\xi_{ml} = \int_0^a J_{\frac{m\pi}{\psi}}(k_{ml}r) r dr = $ Bessel function of the first kind, of order $\frac{m\pi}{\psi}$, integrated from 0 to a

The next step is to evaluate the analytic solution.

4 Implementing the solution

The objective was to find the time for T_{ave} to reach $29.4°C$, or $85°F$. The solution was evaluated using MATLAB. Since I have a date later, I don't have time to include all infinity of the terms in the series solution. Thus, this step is where the solution becomes a numerical approximation.

```matlab
%% ECH259_clay_swackhamer_project_solution
%2017-12-12
%Notes: Runs on click.
%Evaluates solution and makes a quick plot
%more plots are in
%ECH259_clay_swackhamer_plot_code

tic, clc, clear, close all
fontname = 'CMU Serif'; %to match tex
set(0,'defaultaxesfontname',fontname);
set(0,'defaulttextfontname',fontname);
map = lines; %colors

%% Constants
H = 0.0381;               %height, m
psi = 2*pi/9;             %angle, rad
a = 0.127;                %radius, m
Tbc = -21.1;              %boundary temp, C
Tic = 169.5;              %initial temp, C
alpha = 1.34e-7;          %diffusivity, m^2/s
Vpie = (H*psi*a^2/2);     %pie volume, m^3

%% Numerically evaluate zeros of the bessel function
tableSize = 50; %can't keep more terms than this in the series
k = zeros(tableSize, tableSize);   %preallocate
for m = 1:1:tableSize
    roots = besselzero(m*pi/psi, tableSize, 1); %(o, nzeros, J or Y)
    k(m,:) = roots; %(order, zeros)
    %credit for besselzero is [Greg von Winckel, 2005,
    %https://www.mathworks.com/matlabcentral/fileexchange/
    %6794-bessel-function-zeros?focused=5058085&tab=function]
end
k = k/a; %eigenvalues are k, roots are kml*a

%% Numerically evaluate zeta integrals
%These are the Bessel integrals that can't be done easily by hand
zeta = zeros(tableSize,tableSize);
for m = 1:1:tableSize
    nu = m*pi/psi; %order of bessel
    for el = 1:1:tableSize
        bfun = @(x)besselj(nu,x*k(m,el)).*x;
        zeta(m,el) = integral(bfun,0,a); %order, integral
```

```matlab
43          end
44  end
45
46  %% Evaluate temperature, keeping different numbers of terms
47  %Set up and preallocate
48  t85 = nan(1,50);
49  NtermsRecord = nan(1,50);
50  j = 1;
51  Nterms = 1; %terms to keep in solution
52  t = linspace(0, 3600, 60); %time, full hour on one min intervals
53
54  while Nterms < 9
55      T = zeros(length(t),1); %preallocate temp for SN
56      for i = 1:1:length(t)
57          u = 0; %clear u after each time point
58          for n = 1:1:Nterms
59              for m =1:1:Nterms
60                  for el = 1:1:Nterms
61
62                      coef = (36*H*psi^2)/(pi^3*a^2) ...
63                          * (zeta(m,el))^2/(besselj((m*pi/psi+1), ...
64                          k(m,el)*a))^2 ...
65                          * ((-1)^n-1)/n * (((-1)^m-1)/m)^2;
66
67                      bc = Tbc*-n/H^2 * (1-exp(-alpha*t(i) ...
68                          *(n^2*pi^2/H^2+k(m,el)^2))) ...
69                          /((n^2*pi^2/H^2+k(m,el)^2));
70
71                      ic = Tic/pi^2 * ((-1)^n-1)/n ...
72                          *exp(-alpha*t(i)*(n^2*pi^2/H^2+k(m,el)^2));
73
74                      u = u + coef*(bc+ic);
75                  end
76              end
77          end
78          T(i) = u/Vpie + 21.1;        %volume ave temp in celsius
79      end
80
81      %Find t85
82      Tf = T*9/5+32;                   %convert to F
83      [~, t85ind] = min(abs(Tf-85)); %find t85 index
84      t85(j) = t(t85ind)/60;         %find t85 value
85
86      %Find Richardson error
87      NtermsRecord(j) = Nterms; %number of terms
88
89      j = j + 1;
```
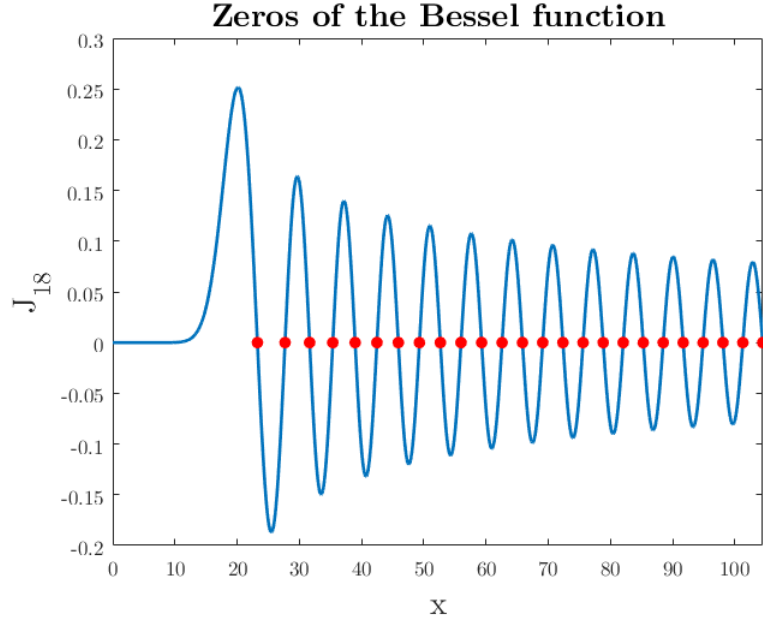
```matlab
        Nterms = Nterms *2;
end

%Clean up results
t85 = t85(~isnan(t85))
NtermsRecord = NtermsRecord(~isnan(NtermsRecord))

%% Quick plot for last series
plot(t, Tf)
title('Quick check: Temp (F) vs Time (s)')
toc
```

5 Results

5.1 Bessel function zeros

Zeros of the Bessel functions were found using Greg von Winckel's `besselzero.m` [3], which uses Halley's method. For details on Halley's method please see [4]. An example plot is shown below, with the first 25 zeros of the Bessel function of the first kind, of order 18. These zeros are the values $k_{ml}a$ in the series solution.


5.2 Bessel function integrals

The integrals that were labeled ξ_{ml} were evaluated numerically using function `integrate` in MAT-LAB. A table was generated and saved, and entries were called when building the solution. The code shown in this report is designed to run everything on one click, so the table is left in the workspace. The first nine ξ_{ml} integrals are visualized below. Numbers in the title are m, l, and represent the following integral: $\int_0^a J_{\frac{m\pi}{\psi}}\left(k_{ml}r\right) r dr$.
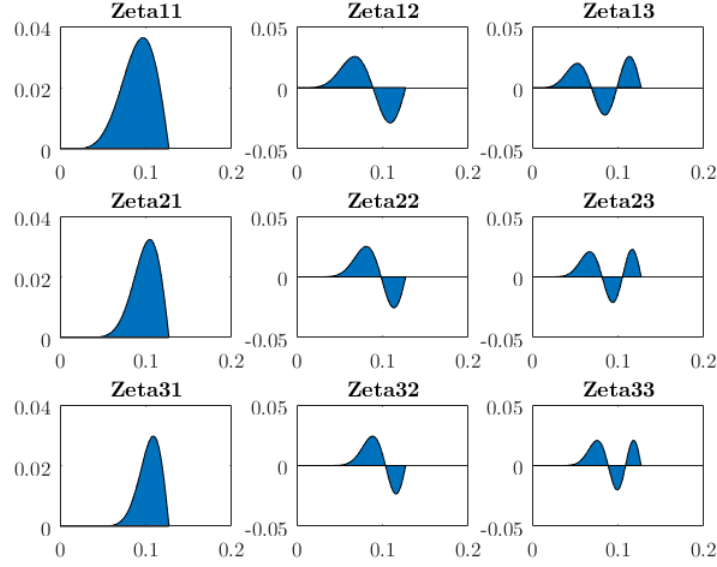
5.3 Cooling time

The time to cool down was evaluated by finding the temperature value closest to 85°F, and then taking the corresponding time value. All cooling profiles were evaluated on 1000 evenly spaced time steps between time 0 and one hour. This temperature profile has an exponentially decaying characteristic, starting at the initial, oven temperature and eventually settling to an value between the ambient room temperature and the ice cream temperature.
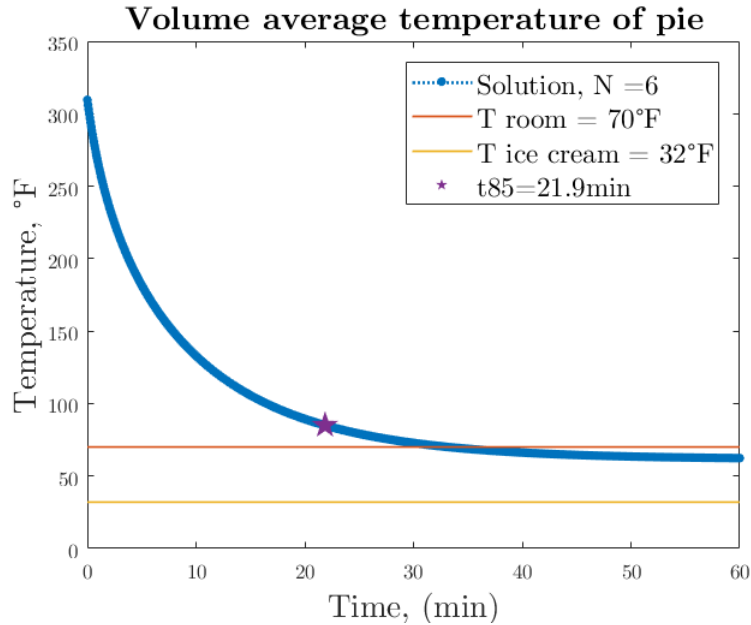
A plot of cooling times for solutions with increasing number of terms is shown below. Results suggest that the series is converging, and that there are diminishing returns to keeping more terms as N becomes large.

Cooling time depends on number of terms
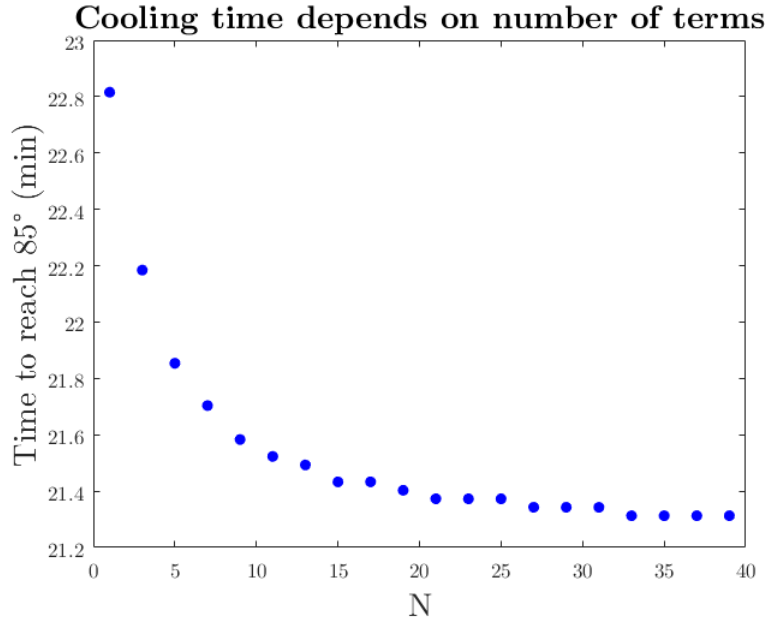
5.4 Convergence analysis

The error introduced by truncating the infinite series solution was estimated using the Richardson approach. This involves assuming a model for the difference between the true value of an infinite sum and its truncation with varying numbers of terms. The error for a series with N terms is defined as

$$E_N = t85_N - t85_\infty$$

Where
E_N = error for a truncated series with N terms
$t85_N$ = time to reach 85°, as calculated using the series with N terms
$t85_\infty$ = true time to reach 85°

The model is given by

$$E_N = CN^P$$

Where
C, P are constants

Estimates for C and P can be obtained from evaluating specific series.

$$t85_N - t85_\infty = C(N)^P$$
$$t85_{2N} - t85_\infty = C(2N)^P$$

Subtracting these equations yields

$$t85_{2N} - t85_N = C(2N)^P - C(N)^P = CN^P(2^P - 1)$$

25

Repeating the steps for two another series yields

$$t85_{2N} - t85_\infty = C(2N)^P$$
$$t85_{4N} - t85_\infty = C(4N)^P$$

$$t85_{4N} - t85_{2N} = C(4N)^P - C(2N)^P = CN^P 2^P(2^P - 1)$$

By dividing the results obtained above, an estimate for P can be obtained.

$$\frac{t85_{4N} - t85_{2N}}{t85_{2N} - t85_N} = \frac{CN^P 2^P(2^P - 1)}{CN^P(2^P - 1)} = 2^P$$
$$ln(2^P) = ln\left(\frac{t85_{4N} - t85_{2N}}{t85_{2N} - t85_N}\right)$$
$$Pln(2) = ln\left(\frac{t85_{4N} - t85_{2N}}{t85_{2N} - t85_N}\right)$$
$$P = \frac{ln\left(\frac{t85_{4N} - t85_{2N}}{t85_{2N} - t85_N}\right)}{ln(2)}$$

The constant C can be found using

$$C = \frac{t85_{4N} - t85_{2N}}{2^P(2^P - 1)}$$

For $N = 2$, these constants were calculated from the results of the series with 10, 20, and 40 terms and found to be $P = -1.3$, and $C = -75.0$. Using these values, the Richardson error of the best solution that was evaluated (40 terms), was calculated using

$$E_{4N} = C(4N)^P = -75.0(4 * 10)^{-0.1.3} = -0.57min.$$

The L2 Richardson error, $L2$ was also calculated for initial temperature, $T0$ using

$$L2_N = T0_{2N} - T0_N$$

The log of this error was then plotted verses the log of the number of terms in the series solution. The expected behavior is a linear decrease.

Initial Temperature L2 Error

$$\ln(err) = -1.7345*\ln(N) + 5.1351$$
$$R^2 = 0.9993$$
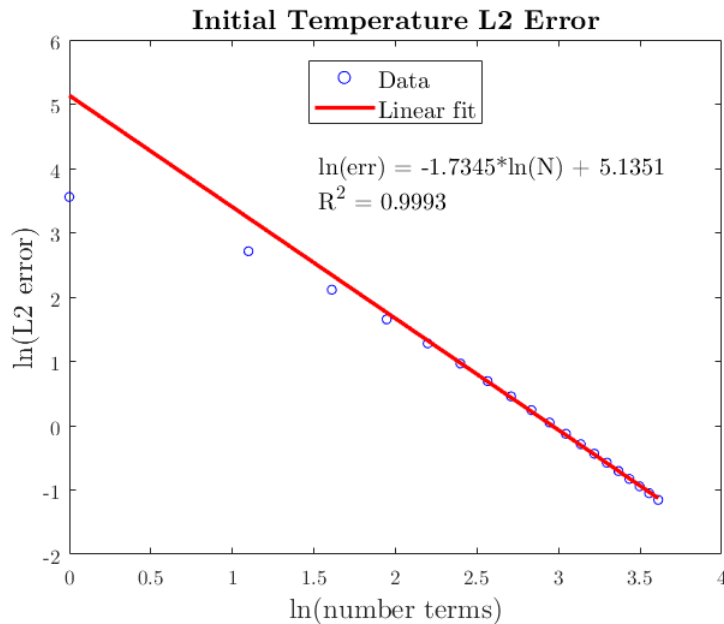
Using the equation for the linear fit, an estimate for the number of terms required in order to achieve an error in initial temperature less than a certain value, ϵ, can be determined. For $\epsilon = 1\text{x}10^{-4}$ it was estimated that $3.9\text{x}10^3$ terms would be required. The largest solution that was actually built had only 40 terms, so achieving this target could require increased computer resources.

References

[1] Greg Miller. *ECH 259 Course Notes*. UC Davis, 2017.

[2] Giampaolo Betta, Massimiliano Rinaldi, Roberto Massini. *A quick method for thermal diffusivity estimation: Application to several foods*. Journal of Food Engineering, 91(1): 34âĂŞ-41, 2009.

[3] Greg von Winckel *Bessel Function Zeros: Computes the first k zeros of the Bessel Function of the 1st and 2nd Kinds*. MathWorks File Exchange: https://www.mathworks.com/matlabcentral/fileexchange/6794-bessel-function-zeros?focused=5058085&tab=function, 2005.

[4] Scavo, T.R. and Thoo, J.B. *On the Geometry of Halley's Method*. Amer. Math. Monthly, 102: 417–426, 1995.

6 Appendix: code for plotting

```
%% ECH_259_plot_code
%Notes: does not run on click
%Needs results of "ECH_259_clay_swackhamer_project_solution"

%% Visualize L2 error
%Whatever values are stored in Lvals will be analyzed
%To run this code for L2 norm of initial temperature,
```

```matlab
 8  %create a smaller mesh, for instance  t = [0, 1];
 9
10  Lvals = t85;
11  for i = 1:1:length(Lvals)-1
12      L2(i) = ((Lvals(i+1) - Lvals(i))^2)^(1/2);
13  end
14  logN = log(NtermsRecord(1:length(L2)))';
15  logErr = log(L2)';
16
17  %Plot data and create fit
18  figure, scatter(logN, logErr, 20, 'bo'); box on;
19  [f, gof] = fit(logN(5:end), logErr(5:end), 'poly1');
20
21  %Evaluate fit for plotting
22  fitFun = @(dummyVar) f.p1*logN + f.p2;
23  fitVals = fitFun(logN);
24
25  hold on
26  plot(logN, fitVals, 'r-', 'LineWidth', 2)
27  title('Initial Temperature L2 Error', 'FontSize', 14)
28  xlabel('ln(number terms)', 'FontSize', 14)
29  ylabel('ln(L2 error)', 'FontSize', 14)
30  leg = legend('Data', 'Linear fit');
31  leg.FontSize = 12;
32  leg.Location = 'north';
33  p = leg.Position;
34
35  %Add annotation with equation and rsquare
36  dim = [p(1), 0.7*(p(2)), p(3), p(3)];
37  message1 = horzcat('ln(err) = ', num2str(round(f.p1,4)),...
38      '*ln(N) + ', num2str(round(f.p2, 4)));
39  message2 = horzcat('R^2 = ', num2str(round(gof.rsquare,4)));
40
41  str ={message1,message2};
42  annotation('textbox', dim, 'String', str, 'FitBoxToText',...
43      'on', 'BackgroundColor','white',...
44      'FontSize', 12, 'LineStyle', 'none');
45
46  %Estimate how many terms we would need to actually achieve
47  %error less than epsilon
48  epsilon = log(1e-4);
49  slope = f.p1;
50  intercept = f.p2;
51  estimateN = exp((epsilon - intercept)/slope); %recall x=ln(N)
52  %}
53
54  %% Visualize t85 vs N
```

```matlab
figure
scatter(NtermsRecord, t85, 30, 'bo', 'filled'); box on
title('Cooling time depends on number of terms', 'FontSize', 16)
xlabel('N', 'FontSize', 16)
ylabel('Time to reach 85ř (min)', 'FontSize', 16)


%% Visualize Temperature profile
    %
figure;  hold on
plot(t, Tf, 'o', 'MarkerFaceColor', map(1,:),...
    'LineStyle', ':', 'LineWidth', 1.5, 'MarkerSize', 3);
x70 = [min(t), max(t)];
y70 = [70, 70];
plot(x70, y70, 'LineWidth', 1, 'LineStyle', '--');
x32 = [min(t), max(t)];
y32 = [32, 32];
plot(x32, y32, 'LineWidth', 1);
scatter(t(t85ind), 85, 200, 'p', 'filled')
leg = legend(strcat('Solution, N = ', num2str(Nterms)),...
    'T room = 70řF', 'T ice cream = 32řF', strcat('t85=',...
    num2str(round(t85(end),1)), 'min'));
leg.Location = 'northeast';
leg.FontSize = 14;
xlabel('Time, (min)', 'FontSize', 16)
ylabel('Temperature, řF', 'FontSize', 16)
title('Volume average temperature of pie', 'FontSize', 16)
```