ADDITIONAL FILE-1

Related Proofs, Tables and Graphs

Appendix 1

Deriving the conditions for optimization:

$$p\alpha k S^{\alpha-1} I^{\beta} P^{\gamma} N^{\delta} = w_1 \tag{1}$$
$$p\beta k S^{\alpha} I^{\beta-1} P^{\gamma} N^{\delta} = w_2 \tag{2}$$
$$p\gamma k S^{\alpha} I^{\beta} P^{\gamma-1} N^{\delta} = w_3 \tag{3}$$
$$p\delta k S^{\alpha} I^{\beta} P^{\gamma} N^{\delta-1} = w_4 \tag{4}$$

Multiplying these equations with S, I, P and N, respectively-

$$p\alpha k S^{\alpha} I^{\beta} P^{\gamma} N^{\delta} = w_1 S \Rightarrow p\alpha y = w_1 S \tag{5}$$
$$p\beta k S^{\alpha} I^{\beta} P^{\gamma} N^{\delta} = w_2 I \Rightarrow p\beta y = w_2 I \tag{6}$$
$$p\gamma k S^{\alpha} I^{\beta} P^{\gamma} N^{\delta} = w_3 P \Rightarrow p\gamma y = w_3 P \tag{7}$$
$$p\delta k S^{\alpha} I^{\beta} P^{\gamma} N^{\delta} = w_4 N \Rightarrow p\delta y = w_4 N \tag{8}$$

Dividing equations (6), (7) and (8) by (5) following equations are obtained:

$$I = \frac{\beta}{\alpha}\frac{w_1}{w_2} S \tag{9}$$
$$P = \frac{\gamma}{\alpha}\frac{w_1}{w_3} S \tag{10}$$
$$N = \frac{\delta}{\alpha}\frac{w_1}{w_4} S \tag{11}$$

Using values of I, P and N, we get

$$\begin{aligned} & p\alpha k S^{\alpha-1} I^{\beta} P^{\gamma} N^{\delta} = w_1 \\ \Rightarrow & p\alpha k S^{\alpha-1}\left(\frac{\beta}{\alpha}\frac{w_1}{w_2}S\right)^{\beta}\left(\frac{\gamma}{\alpha}\frac{w_1}{w_3}S\right)^{\gamma}\left(\frac{\delta}{\alpha}\frac{w_1}{w_4}S\right)^{\delta} = w_1 \\ \Rightarrow & pkS^{\alpha+\beta+\gamma+\delta-1}\beta^{\beta}\gamma^{\gamma}\delta^{\delta}w_1^{\beta+\gamma+\delta-1}w_2^{-\beta}w_3^{-\gamma}w_4^{-\delta} = 1 \end{aligned} \tag{12}$$

$$\begin{aligned} S = & \Big(pk\alpha^{1-(\beta+\gamma+\delta)}\beta^{\beta}\gamma^{\gamma}\delta^{\delta} \\ & w_1^{\beta+\gamma+\delta-1}w_2^{-\beta}w_3^{-\gamma}w_4^{-\delta}\Big)^{\frac{1}{1-(\alpha+\beta+\gamma+\delta)}} \end{aligned} \tag{13}$$

Performing similar calculations the following values of I, P and N are obtained:

$$\begin{aligned} I = & \Big(pk\alpha^{\alpha}\beta^{1-(\alpha+\gamma+\delta)}\gamma^{\gamma}\delta^{\delta} \\ & w_1^{-\alpha}w_2^{\alpha+\gamma+\delta-1}w_3^{-\gamma}w_4^{-\delta}\Big)^{\frac{1}{1-(\alpha+\beta+\gamma+\delta)}} \end{aligned} \tag{14}$$

$$\begin{aligned} P = & \Big(pk\alpha^{\alpha}\beta^{\beta}\gamma^{1-(\alpha+\beta+\delta)}\delta^{\delta} \\ & w_1^{-\alpha}w_2^{-\beta}w_3^{\alpha+\beta+\delta-1}w_4^{-\delta}\Big)^{\frac{1}{1-(\alpha+\beta+\gamma+\delta)}} \end{aligned} \tag{15}$$

$$\begin{aligned} N = & \Big(pk\alpha^{\alpha}\beta^{\beta}\gamma^{\gamma}\delta^{1-(\alpha+\beta+\gamma)} \\ & w_1^{-\alpha}w_2^{-\beta}w_3^{-\gamma}w_4^{\alpha+\beta+\gamma-1}\Big)^{\frac{1}{1-(\alpha+\beta+\gamma+\delta)}} \end{aligned} \tag{16}$$

These values of S, I, P and N are the profit maximizing data center's demand for inputs, as a function of the prices of all the inputs, and of the price of output. Using values of S, I, P and N, we get

$$y = \left(kp^{\alpha+\beta+\gamma+\delta}\alpha^{\alpha}\beta^{\beta}\gamma^{\gamma}\delta^{\delta}w_1^{-\alpha}w_2^{-\beta}w_3^{-\gamma}w_4^{-\delta}\right)^{\frac{1}{1-(\alpha+\beta+\gamma+\delta)}} \tag{17}$$

Appendix 2

Consider the following production function:

$$y = \prod_{i=1}^{n} kx_i^{\alpha_i}$$

To prove:

$$\sum_{i=1}^{n} \alpha_i < 1$$

Consider the profit function:

$$\pi_n = \prod_{i=1}^{n} kx_i^{\alpha_i} - \sum_{i=1}^{n} w_i x_i$$

w_i:Unit cost of inputs
Profit maximization is achieved when: $p\frac{\partial f}{\partial x_i} = w_i$. Deriving the condition for optimization:

$$pk\frac{\alpha_1}{x_1}\prod_{i=1}^{n} x_i^{\alpha_i} = w_1 \tag{18}$$
$$pk\frac{\alpha_2}{x_2}\prod_{i=1}^{n} x_i^{\alpha_i} = w_2 \tag{19}$$

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$$pk\frac{\alpha_n}{x_n}\prod_{i=1}^{n} x_i^{\alpha_i} = w_n \tag{20}$$

Multiplying these equations with x_i, respectively-

$$p\alpha_1\prod_{i=1}^{n} kx_i^{\alpha_i} = w_1 x_1 \Rightarrow p\alpha_1 y = w_1 x_1 \tag{21}$$
$$p\alpha_2\prod_{i=1}^{n} kx_i^{\alpha_i} = w_2 x_2 \Rightarrow p\alpha_2 y = w_2 x_2 \tag{22}$$

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$$p\alpha_n\prod_{i=1}^{n} kx_i^{\alpha_i} = w_n x_n \Rightarrow p\alpha_n y = w_n x_n \tag{23}$$

Dividing equations (20) to (23) by (19), following equations are obtained:

$$x_2 = \frac{\alpha_2}{\alpha_1}\frac{w_1}{w_2}x_1$$
$$x_3 = \frac{\alpha_3}{\alpha_1}\frac{w_1}{w_3}x_1$$
$$.$$
$$.$$
$$x_{n-1} = \frac{\alpha_{n-1}}{\alpha_1}\frac{w_1}{w_{n-1}}x_1$$
$$x_n = \frac{\alpha_n}{\alpha_1}\frac{w_1}{w_n}x_1$$

Substituting these values of x_i in equation (13),

$$pk\frac{\alpha_1}{x_1}\prod_{i=1}^{n}x_i^{\alpha_i} = w_1$$

$$pk\alpha_1 x_1^{\alpha_1-1}\left(\frac{\alpha_2}{\alpha_1}\frac{w_1}{w_2}x_1\right)^{\alpha_2}\left(\frac{\alpha_3}{\alpha_1}\frac{w_1}{w_3}x_1\right)^{\alpha_3}... \left(\frac{\alpha_{n-1}}{\alpha_1}\frac{w_1}{w_{n-1}}x_1\right)^{\alpha_{n-1}}\left(\frac{\alpha_n}{\alpha_1}\frac{w_1}{w_n}x_1\right)^{\alpha_n} = w_1 \tag{24}$$

$$pk{x_1}^{(\alpha_1+\alpha_2+....+\alpha_n)-1}\alpha_1^{1-(\alpha_2+\alpha_3+....+\alpha_n)}\alpha_2^{\alpha_2}... \alpha_n^{\alpha_n}w_1^{-1+(\alpha_2+\alpha_3+....+\alpha_n)}w_2^{-\alpha_2}....w_n^{-\alpha_n} = 1 \tag{25}$$

$$x_1 = \Big(pk\alpha_1^{1-(\alpha_2+\alpha_3+....+\alpha_n)}\alpha_2^{\alpha_2}...\alpha_n^{\alpha_n} w_1^{-1+(\alpha_2+\alpha_3+....+\alpha_n)}w_2^{-\alpha_2}.... w_n^{-\alpha_n}\Big)^{\frac{1}{1-(\alpha_1+\alpha_2+....+\alpha_n)}} \tag{26}$$

Performing similar calculations following values of $x_i, (i >= 2)$ are obtained,

$$x_2 = \Big(pk\alpha_2^{1-(\alpha_1+\alpha_3+....+\alpha_n)}\alpha_1^{\alpha_1}...\alpha_n^{\alpha_n} w_2^{-1+(\alpha_1+\alpha_3+....+\alpha_n)}w_1^{-\alpha_1}.... w_n^{-\alpha_n}\Big)^{\frac{1}{1-(\alpha_1+\alpha_2+....+\alpha_n)}} \tag{27}$$

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$$x_n = \Big(pk\alpha_n^{1-(\alpha_1+\alpha_2+....+\alpha_{n-1})}\alpha_1^{\alpha_1}....\alpha_{n-1}^{\alpha_{n-1}} w_n^{-1+(\alpha_1+\alpha_2+....+\alpha_n)}w_2^{-\alpha_2}.... w_{n-1}^{-\alpha_{n-1}}\Big)^{\frac{1}{1-(\alpha_1+\alpha_2+....+\alpha_n)}} \tag{28}$$

Substituting values of x_i in production function,

$$y = \Big(kp^{(\alpha_1+\alpha_2+....+\alpha_n)}\alpha_1^{\alpha_1} \alpha_2^{\alpha_2}....\alpha_n^{\alpha_n}w_1^{-\alpha_1}w_2^{-\alpha_2}.... w_n^{-\alpha_n}\Big)^{\frac{1}{1-(\alpha_1+\alpha_2+....+\alpha_n)}} \tag{29}$$

y increases in price of its output and decreases in price of its inputs iff:

$$1-\sum_{i=1}^{n}\alpha_i > 0$$
$$\sum_{i=1}^{n}\alpha_i < 1$$

Therefore decreasing returns to scale.

Appendix 3

Matlab code which has been applied throughout the data set for optimal elasticity and maximum revenue calculation.

Code for Increasing return to scale:

```
A = [1 1;-1 -1;-1 0; 0 -1];
b = [1.9;-1.1;-0.1;-0.1];
x0 = [0.4; 0.1];
[x, fval] = fmincon(@cobbfun, x0, A, b);
function f = cobbfun(x)
%Cobb-Douglas function with k=1
%f is a representation of Cobb-Douglas function.
% x(1),x(2) is representing elasticity constant of new server spending and power/cooling cost respectively.
f = -62^x(1) * 5^x(2);
end
```

Code for Constant return to scale:

```
A = [ -1 0; 0 -1];
b = [-0.1;-0.1];
Aeq = [1 1];
beq = [1]
x0 = [0.4; 0.1];
[x, fval] = fmincon(@cobbfun, x0, A, b, Aeq, beq);
function f = cobbfun(x)
%Cobb-Douglas function with k=1
% f is a representation of Cobb-Douglas function.
%x(1),x(2) is representing elasticity constant of new server spending and power/cooling cost respectively.
f = -62^x(1) * 5^x(2);
end
```

decreasing returns to scale:

```
A = [1 1; -1 0; 0 -1];
b = [0.9;-0.1;-0.1];
x0 = [0.4; 0.1];
[x, fval] = fmincon(@cobbfun, x0, A, b);
function f = cobbfun(x)
%Cobb-Douglas function with k=1
%f is a representation of Cobb-Douglas function.
%x(1),x(2) is representing elasticity constant of new server spending and power/cooling cost respectively.
f = -62^x(1) * 5^x(2);
end
```

Code for least square and quadratic programming

```
CRS = 1; DRS = 2; IRS = 3;
dataType = IRS;
threeParameters = 1;

   y = log(Revenue);
parameter1 = log(Server);
parameter1 =reshape(parameter1,length(parameter1),1);
parameter2 = log(Power);
parameter2 = reshape(parameter2,length(parameter2),1);

   if(threeParameters)
A = [ones(numRows,1) parameter1 parameter2];
x_leastSquares = A \y;
alpha_leastSquares = x_leastSquares(2);
beta_leastSquares = x_leastSquares(3);

   else
x_leastSquares = A \y;
alpha_leastSquares = x_leastSquares(1);
beta_leastSquares = x_leastSquares(2);

   end

   if(dataType == CRS)
alphaBetaUpperBound = 1;

   if(threeParameters)
C = [0, -1, 0; 0, 0, -1];
b = [0, 0];

   Ceq = [0 1 1];
beq = [1];

   lb = [-Inf,0,0]';
ub = [Inf,alphaBetaUpperBound,alphaBetaUpperBound]';
else

   C = [-1, 0; 0, -1];

   b = [0, 0];

   Ceq = [1 1];

   beq = [1];

   lb = [0,0];
ub = [alphaBetaUpperBound,alphaBetaUpperBound];

   end
end

   if(dataType == DRS)

   alphaBetaUpperBound = 0.995;
if(threeParameters)
C = [0, -1, 0; 0, 0, -1;0 1 1];
b = [0, 0, alphaBetaUpperBound]';

   Ceq = [];

   beq = [];

   lb = [-Inf,0,0]';

   ub = [Inf,alphaBetaUpperBound,alphaBetaUpperBound]';
else
C = [-1, 0; 0, -1;1 1];
b = [0, 0, alphaBetaUpperBound]';

   Ceq = [];

   beq = [];

   lb = [0,0];
ub = [alphaBetaUpperBound,alphaBetaUpperBound];
end
end

   if(dataType == IRS)
alphaBetaUpperBound = 1.005;
if(threeParameters)

   C = [0, -1, 0; 0, 0, -1;-0 -1 -1];

   b = [0, 0, -alphaBetaUpperBound];

   Ceq = [];
beq = [];

   lb = [-Inf,0,0];
ub = [Inf,alphaBetaUpperBound,alphaBetaUpperBound];
else C = [-1, 0; 0, -1;-1 -1];

   b = [0, 0, -alphaBetaUpperBound];

   Ceq = [];

   beq = [];

   lb = [0,0];
ub = [alphaBetaUpperBound,alphaBetaUpperBound];
end
end

   H = 2*A'*A;
   f = -2*y'*A;
   if(threeParameters)
alpha_constOpt = x_constOpt(2);
beta_constOpt = x_constOpt(3);
else
alpha_constOpt = x_constOpt(1);
beta_constOpt = x_constOpt(2);
end
```

Appendix 4

A c^2 function $f : U \subset R^n \to R$ defined on a convex open set U is concave if and only if the Hessian matrix $D^2 f(x)$ is negative semi-definite for all $x \in U$. A matrix H is negative semi-definite if and only if it's $2^n - 1$ principal minors alternate in sign so that odd order minors are less than equal to 0 and even order minors are greater than equal to 0. Cobb-Douglas function for 2 inputs is:

$$f(x, y) = cx^a y^b$$

Its Hessian is

$$\begin{bmatrix} a(a-1)cx^{a-2}y^b & abcx^{a-1}y^{b-1} \\ abcx^{a-1}y^{b-1} & b(b-1)cx^a y^{b-2} \end{bmatrix}$$

$$\begin{aligned} \Delta_1 &= a(a-1)cx^{a-2}y^b \\ \Delta_1 &= b(b-1)cx^a y^{b-2} \\ \Delta_2 &= abc^2 x^{2a-2} y^{2b-2}(1-(a+b)) \end{aligned}$$

Condition for a function to be concave,

$$\begin{aligned} \Delta_1 &\leq 0 \\ \Delta_2 &\geq 0 \end{aligned}$$

For decreasing and constant returns to scale: $a + b \leq 1$ Therefore,

$$\begin{aligned} & a \leq 1,\ b < 1 \\ & \Rightarrow (a-1) \leq 0 \\ & \Rightarrow \Delta_1 \leq 0 \end{aligned}$$

$$\begin{aligned} & (1-(a+b)) \geq 0 \\ & \Rightarrow \Delta_2 \geq 0 \end{aligned}$$

Both conditions for concave function are satisfied by decreasing and constant returns to scale. Therefore, the graph obtained for decreasing and constant returns to scale is concave, while for increasing returns the graph is neither concave nor convex.

A Note on the animation representing optimal revenue:

Since we have 24 graphs so, we'll ahve(IRS+DRS+CRS) 24 frames; therefore the video duration can't be for more than 1 second.
100 frames would lead to 6 second video in Matlab sample code:

```
clear; close all;
x=linspace(0,1,100);

[X,Y] = meshgrid(x,x);
N=100; for i = 1:N
Z = sin(2*pi*(X-i/N)).*sin(2*pi*(Y-i/N));

surf(X,Y,Z)

M(i)=getframe(gcf);
end
```

Output the movie as an avi file:
movie2avi(M,'WaveMovie.avi');

https://youtu.be/Uqyqgvkn3Yw

	Data Center	Annual Labor Cost	Electric power cost	Amortization cost	Property and Sales Tax	Heating and Air conditioning	Corporate Travel Cost	Elasticity() Elasticity()	Maximum Revenue
Oakland, Ca	7,000,000	500,000	9,000,000	800,000	200,000	100,000	0.1000	0.8000	1832053
Boston, Ma	7,000,000	800,000	7,000,000	1,500,000	300,000	100,000	0.8000	0.1000	1824394
Newark, Nj	7,000,000	250,000	8,000,000	2,000,000	300,000	100,000	0.8000	0.1000	1840014
San Francisco, Ca	7,500,000	500,000	13,000,000	1,000,000	250,000	100,000	0.1000	0.8000	2472845
New York, Ny	7,500,000	700,000	12,000,000	5,000,000	600,000	200,000	0.8000	0.1000	2549741
Rolla,Mo	6,000,000	150,000	4,500,000	500,000	200,000	100,000	0.8000	0.1000	1390875
Winston-Salem,Nc	6,000,000	150,000	5,000,000	500,000	200,000	100,000	0.8000	0.1000	1299628
Bloomington,Ind	6,000,000	150,000	4,200,000	500,000	200,000	100,000	0.8000	0.1000	1286556
Huntsville,Ala	6,000,000	150,000	4,000,000	500,000	200,000	100,000	0.8000	0.1000	1320025
Sious Falls,Sd	6,000,000	175,000	4,000,000	500,000	200,000	100,000	0.8000	0.1000	1223993

TABLE I: Data Center Comparison Cost for DRS

	Data Center	Annual Labor Cost	Electric power cost	Amortization cost	Property and Sales Tax	Heating and Air conditioning	Corporate Travel Cost	Elasticity() Elasticity()	Maximum Revenue
Oakland, Ca	7,000,000	500,000	9,000,000	800,000	200,000	100,000	0.1000	0.9000	105773.7
Boston, Ma	7,000,000	800,000	7,000,000	1,500,000	300,000	100,000	0.9000	0.1000	9077514.2
Newark, Nj	7,000,000	250,000	8,000,000	2,000,000	300,000	100,000	0.9000	0.1000.	9150300.7
San Francisco, Ca	7,500,000	500,000	13,000,000	1,000,000	250,000	100,000	0.1000	0.9000	12747309.8
New York, Ny	7,500,000	700,000	12,000,000	5,000,000	600,000	200,000	0.9000	0.1000	13138736.1
Rolla,Mo	6,000,000	150,000	4,500,000	500,000	200,000	100,000	0.9000	0.1000	6726637.4
Winston-Salem,Nc	6,000,000	150,000	5,000,000	500,000	200,000	100,000	0.9000	0.1000	6224388.4
Bloomington,Ind	6,000,000	150,000	4,200,000	500,000	200,000	100,000	0.9000	0.1000	6161779.2
Huntsville,Ala	6,000,000	150,000	4,000,000	500,000	200,000	100,000	0.9000	0.1000	6351326.0
Sious Falls,Sd	6,000,000	175,000	4,000,000	500,000	200,000	100,000	0.9000	0.1000	6372811.4

TABLE II: Data Center Comparison Cost for CRS

	Data Center	Annual Labor Cost	Electric power cost	Amortization cost	Property and Sales Tax	Heating and Air conditioning	Corporate Travel Cost	Elasticity() Elasticity()	Maximum Revenue
Oakland, Ca	7,000,000	500,000	9,000,000	800,000	200,000	100,000	0.1000	0.9000	1.6854889e+13
Boston, Ma	7,000,000	800,000	7,000,000	1,500,000	300,000	100,000	0.9000	0.1000	1.6966865e+13
Newark, Nj	7,000,000	250,000	8,000,000	2,000,000	300,000	100,000	1.8000	0.1000.	1.7020132e+13
San Francisco, Ca	7,500,000	500,000	13,000,000	1,000,000	250,000	100,000	0.1000	1.8000	3.2765201e+13
New York, Ny	7,500,000	700,000	12,000,000	5,000,000	600,000	200,000	1.8000	0.1000	3.3656591e+13
Rolla,Mo	6,000,000	150,000	4,500,000	500,000	200,000	100,000	1.8000	0.1000	9.7361276e+12
Winston-Salem,Nc	6,000,000	150,000	5,000,000	500,000	200,000	100,000	1.8000	0.1000	8.2526426e+12
Bloomington,Ind	6,000,000	150,000	4,200,000	500,000	200,000	100,000	1.8000	0.1000	8.1696319e+12
Huntsville,Ala	6,000,000	150,000	4,000,000	500,000	200,000	100,000	1.8000	0.1000	8.778168e+12
Sious Falls,Sd	6,000,000	175,000	4,000,000	500,000	200,000	100,000	1.8000	0.1000	8.8376585e+12

TABLE III: Data Center Comparison Cost for IRS

Year	Server Cost	Energy Cost	Infrastructure	Annual I&E	Elasticity()	Elasticity()	Maximum Revenue
1992	1400	50	200	220	0.8000	0.1000	640.3720
1995	1400	75	250	300	0.8000	0.1000	678.2507
2000	1400	200	500	1200	0.8000	0.1000	866.1950
2005	1400	1000	1500	2800	0.1000	0.8000	1622.0591
2010	1400	1600	1800	3400	0.1000	0.8000	2040.2857

TABLE IV: Maximum revenue and optimal elasticity constants for DRS

Year	Server Cost	Energy Cost	Infrastructure	Annual I&E	Elasticity()	Elasticity()	Maximum Revenue
1992	1400	50	200	220	0.9000	0.1000	1339.1966
1995	1400	75	250	300	0.9000	0.1000	1422.74
2000	1400	200	500	1200	0.9000	0.1000	1842.8544
2005	1400	1000	1500	2800	0.1000	0.9000	3698.66
2010	1400	1600	1800	3400	0.1000	0.9000	4781.76

TABLE V: The revenue data, elasticities and different cost for CRS

Year	Server Cost	Energy Cost	Infrastructure	Annual I&E	Elasticity()	Elasticity()	Maximum Revenue
1992	1400	50	200	220	1.8000	0.1000	1024595.2
1995	1400	75	250	300	1.8000	0.1000	1119113.7
2000	1400	200	500	1200	1.8000	0.1000	1645770.5
2005	1400	1000	1500	2800	0.1000	1.8000	6163824.6
2010	1400	1600	1800	3400	0.1000	1.8000	10201428.8

TABLE VI: The revenue data, elasticities and different cost for IRS