ECEN303 - Sections 200 and 501

Programming Assignment

Due: Thursday, May 6, 2021 (9:00 PM)

This assignment is optional and for extra credit only (a maximum of 10 points). The number of points you receive is based on my judgment of your performance as reflected by your answers. If your final grade was X (out of 100) without this assignment and you receive Y points for this assignment, your final grade will be X+Y (out of 100).

This assignment has three parts. In the first part, we generate samples from an exponential random variable. In the second part, we generate samples from a Poisson random variable. Finally, in the last part, we tackle a real-world problem using our codes for the previous two parts.

Part I: In class we have discussed how to generate samples from an exponential random variable using samples from a continuous uniform random variable between 0 and 1.

1. Write a code using the programming language of your interest (preferably, Python, C++, or MATLAB) that takes a positive scalar λ as input, and returns **one** sample of an exponential random variable X with parameter λ (i.e., $X \sim \text{Exp}(\lambda)$). You can only use the RAND function (or its equivalent) that generates a random number in the interval $[0, 1]$.

   To report: The script of your code.

2. Write a code that takes a positive integer n and a positive scalar λ as input, and generates n independent samples of $X \sim \text{Exp}(\lambda)$. You can use the function you have defined for problem 1 as part of the code for this problem.

   To report: The script of your code.

3. Write a code that takes n independent samples of a random variable X as input, and returns an approximation of the CDF of X. You can use the following procedure. Suppose that the n samples of X are $x_1, \dots, x_n$, and we wish to find an approximation $\widehat{F}_X(x)$ of $F_X(x)$ for any $x \in \{x_{\min}, x_{\min} + \Delta, x_{\min} + 2\Delta, \dots, x_{\max}\}$. For any given x, you can approximate $F_X(x)$ by $\widehat{F}_X(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{x_i \le x\}}$, where $\mathbb{1}_{\{x_i \le x\}} = 1$ if $x_i \le x$, and $\mathbb{1}_{\{x_i \le x\}} = 0$ otherwise.

   To report: The script of your code.

4. Using your code for problem 2, generate two sets of samples of $X \sim \text{Exp}(\frac{1}{2})$, one for $n = 100$ and the other one for $n = 5000$; and for each set of samples, compute an approximation of the CDF of X using your code for problem 3 (when $x_{\min} = 0, x_{\max} = 15$, and $\Delta = 0.01$).

   To report: The plot of your approximation of the CDF of X for both $n = 100$ and $n = 5000$ together with the (exact) CDF of X (all three on the same plot).

5. Using the samples generated for both $n = 100$ and $n = 5000$, compute the sample mean $\bar{x}$ and the sample variance s^2, where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$.

   To report: The sample mean and the sample variance for both $n = 100$ and $n = 5000$. Compare these estimated values with the mean and variance of $X \sim \text{Exp}(\frac{1}{2})$.

Part II: In class we have seen a few definitions for the Poisson random variable. In what follows, we will see another definition for the Poisson random variable. This definition is particularly of interest for generating samples from a Poisson random variable M with parameter μ (i.e., $M \sim \mathrm{Po}(\mu)$) using independent samples from $Y \sim \mathrm{Exp}(1)$. Consider an infinite number of independent samples of $Y \sim \mathrm{Exp}(1)$, say $y_1, y_2, y_3, \ldots$. Let $m \geq 0$ be the largest integer such that $\sum_{i=0}^{m} y_i \leq \mu$, where $y_0 = 0$. Then, the integer m is a sample of $M \sim \mathrm{Po}(\mu)$. This is one of the fundamental results in the theory of *Poisson processes*.

6. Write a code that takes a positive scalar μ, and generates **one** sample of a Poisson random variable M with parameter μ. Use the function you wrote for problem 1 in part I.

   To report: The script of your code.

7. Using your code for problem 6, generate 5000 independent samples of $M \sim \mathrm{Po}(5)$, and compute the sample mean and the sample variance using the generated samples.

   To report: The sample mean and the sample variance. Compare these values with the mean and the variance of $M \sim \mathrm{Po}(5)$.

Part III: Each month, Jane goes to the bank N times (to make a deposit or withdrawal), where $N \sim \mathrm{Po}(5)$. Every time, she finds M customers ahead of her, where $M \sim \mathrm{Po}(2)$. Assume that N and M are independent. The service time (in minutes) of each customer ahead of Jane, if present, is denoted by X, where $X \sim \mathrm{Exp}(\frac{1}{2})$. The service time of different customers are assumed to be independent from each other. Assume that X is also independent of N and M. Let T be the random variable representing Jane's total waiting time at the bank during a month. We wish to estimate the probability of some events associated with T as well as the mean and the variance of T. To do so, we will use the commonly-used *Monte-Carlo simulation* technique described below.

8. Write a code to generate 1,000,000 independent samples of T. You can use the functions you wrote for the problems in parts I and II.

   To report: The script of your code.

9. Compute an estimate of the probability of each of the following events, by computing the fractions of times (out of 1,000,000 generated samples) that the event of interest has occurred:

   (i) The event that T is no more than 20 minutes.

   (ii) The event that T is more than 20 minutes *given that* Jane goes to the bank at least 5 times during a month.

   (iii) The event that T is more than 20 minutes *given that* each time there is at least 1 customer ahead of Jane.

   (iv) The event that T is more than 20 minutes *given that* Jane goes to the bank at least 5 times during a month and each time there is at least 1 customer ahead of her.

   To report: The estimated probabilities for events (i), (ii), (iii) and (iv).

10. Compute the sample mean and the sample variance of T.

    To report: The sample mean and the sample variance. Can you guess the exact values of the mean and the variance of T based on the estimated values? What are your guesses?