Boolean Algebra

GBW ACSL - Contest #3

Clayton O'Neill — clayton@oneill.net

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What is Boolean Algebra?

- Boolean algebra is algebra using only two numbers: 0 and 1
- Closely related to Digital Electronics in contest #4
- It uses the logic operations we've already learned: AND, OR, XOR, NOT
- NOT is represented with a line over the value: $\overline{A}$
- AND is represented by multiplication (A * B) or AB
- XOR is represented with $\oplus$: $A \oplus B$
- OR is represented by addition: A + B
- Order of operations: NOT first then AND, XOR and OR.
- 0 is considered FALSE, 1 is considered TRUE

Truth Tables for Boolean Operations

A	B	$\overline{A}$	A + B	A * B	$A \oplus B$
0	0	1	0	0	0
0	1	1	1	0	1
1	0	0	1	0	1
1	1	0	1	1	0

Remember:

- $+$ is short for OR
- * is the short way of saying AND
- $\oplus$ is short for Exclusive-OR (XOR)
- $\overline{A}$ is short for NOT A

Don't confuse $+$ and * in boolean algebra with normal math. Remember that $1 + 1 = 1$ in boolean algebra because it's another way of writing 1 OR $1 = 1$. You need to memorize truth tables to solve these.

Translating Boolean Algebra to Logic

Here are a few examples of how you can mentally translate boolean algebra expressions into logic statements you may be more used to.

A+B = A OR B

AB = A * B = A AND B

$A \oplus B$ = A XOR B

$A\overline{B} = A * \overline{B}$ = A AND (NOT B)

$(\overline{A + B})BC = (\overline{A + B}) * B * C$ = (NOT(A OR B)) AND B AND C

Boolean Algebra Identities: The Rules

Boolean algebra has rules you can apply to solve or simplify equations. You will need to memorize these and practice using them. The numbers of the identities shown here are from the ACSL materials.

1. Communicative Property — Order of OR (+) and AND (*) doesn't matter:

   $$A + B = B + A$$

   $$A * B = B * A$$

2. Associative Property — Parentheses don't matter with same operation:

   $$A + (B + C) = (A + B) + C$$

   $$A * (B * C) = (A * B) * C$$

3. Distributive Property:

   $$A * (B + C) = A * B + A * C$$

De Morgan's Law[1]

4. Negated OR of two variables is the same as AND of the negative of each variable.

   $$\overline{A + B} = \overline{A} * \overline{B}$$

5. Negated AND of two variables is the same as OR of the negative of each variable.

   $$\overline{A * B} = \overline{A} + \overline{B}$$

[1] They are named after Augustus De Morgan, a 19th-century British mathematician

The Basics - OR

6. Value ORed with zero(FALSE) is itself.

$$A + 0 = A$$

7. Value ORed with one(TRUE) is always one(TRUE).

$$A + 1 = 1$$

8. Value ORed with it's opposite is always one(TRUE).

$$A + \overline{A} = 1$$

9. Value ORed with itself is one(TRUE).

$$A + A = 1$$

The Basics - AND

6. Value ANDed with zero(FALSE) is always zero(FALSE).

$$A * 0 = 0$$

7. Value ANDed with one(TRUE) is itself.

$$A * 1 = A$$

8. Value ANDed with it's opposite is always zero(FALSE).

$$A * \overline{A} = 0$$

9. Value ANDed with itself is itself.

$$A * A = A$$

Identities: Part V

11. $A + \overline{A} * B = A + B$
12. $(A + B) * (A + C) = A + B * C$
13. $(A + B) * (C + D) = A * C + A * D + B * C + B * D$
    This is an expanded form of the more general Distributive Property in #3
14. $A * (A + B) = A$
    If A is true then it doesn't matter if B is true or false, and if A is false, then the first A will force the AND to be false.
15. $A \oplus B = A * \overline{B} + \overline{A} * B$
16. $\overline{A \oplus B} = \overline{A} \oplus B = A \oplus \overline{B}$

What do you need to know?

There are mainly two types of problems on the ACSL practice tests:

- Simplify Expression
  These require you to use the identities to make the problem as simple as possible. You will need to memorize the identities and know how to apply them.
- Simplify Expression and Solve
  These are the same as above, but require you to also find all the combination of values for the variables that can make the expression true. These can be easier, since you can stop simplifying when you feel the problem can be solved with a truth table. This means you can get away not knowing the identities quite as well for these problems.

These aren't the the only problem types, just the most common on past tests.

Simplify This Expression Example I

1. Simplify:

   $$(\overline{A} * B + A * B)(\overline{A * B})$$

2. Apply Distributive Property (#3) to move B outside of first parentheses:

   $$B * (\overline{A} + A) * (\overline{AB})$$

3. Apply De Morgan's Law (#5) to the second parentheses:

   $$B * (\overline{A} + A) * (\overline{A} + \overline{B})$$

4. Apply identity #8 to simplify first parentheses:

   $$B * 1 * (\overline{A} + \overline{B})$$

Simplify This Expression Example II

5. Apply identity #7 to remove 1:

   $$B * (\overline{A} + \overline{B})$$

6. Apply Distributive Property (#3) to spread B:

   $$B * \overline{A} + B * \overline{B}$$

7. Apply identity #8 to convert second half to zero:

   $$B * \overline{A} + 0$$

8. Apply identity #6 to get rid of zero:

   $$B * \overline{A}$$

Simplify and Solve Example I

1. Simplify first:

$$AB(\overline{A} + C) + \overline{A}(B + C)$$

2. Apply Distributive identity (#3) to both sides:

$$AB\overline{A} + ABC + \overline{A}B + \overline{A}C$$

3. Apply identity #8 to convert $A\overline{A}$ to zero:

$$0 * B + ABC + \overline{A}B + \overline{A}C$$

4. Apply identity #6 to remove $0 * B$:

$$ABC + \overline{A}B + \overline{A}C$$

At this point no further simplification is possible, so we must figure out what values for A, B and C will make the expression true. We want this truth table to contain each of the parts of the expression that we'll solve for:

Solving the Equation I

To figure out what values will make the expression true, we make a truth table that shows every value for A, B and C.

A	B	C
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Solving the Equation II

Then we want to add columns for each of the parts of the expression, and the expression itself, and then solve each part:

A	B	C	ABC	$\overline{A}B$	$\overline{A}C$	$ABC + \overline{A}B + \overline{A}C$
0	0	0	0	0	0	0
0	0	1	0	0	1	1
0	1	0	0	1	0	1
0	1	1	0	1	1	1
1	0	0	0	0	0	0
1	0	1	0	0	0	0
1	1	0	0	0	0	0
1	1	1	1	0	0	1

The rows in gray are the ones the expression is true for. ACSL expects you to provide them in alphabetic order as a sequence of numbers. For example: (0, 0, 1), (0, 1, 0), (0, 1, 1), and (1, 1 1).

Tips and Tricks

- Be careful about NOT:

  $$\overline{A + B} \neq \overline{A} + \overline{B}$$

  $$\overline{A * B} \neq \overline{A} * \overline{B}$$

- When simplifying look for opportunities to combine negated terms to eliminate them. ($B * \overline{B} = 0$).
- Logical reasoning can save you a lot of time. For $A(BC + AD)$ to be true, A must always be true since it is ANDed with the $(BC + AD)$. If asked to identify the true values for A, B and C for that expression, you can solve for just B and C since you know A must be true.
- In the ACSL examples the the Communicative Property is frequently used to sort the expressions. For example: $A\overline{B}C$ instead of $CA\overline{B}$.
- Memorize the identities: http://www.cram.com/flashcards/boolean-algebra-acsl-6988254