SPRING 2022

Probability and Statistics
FINAL EXAM

Notes: (1) this is an open-book/notes exam; (2) there are 8 questions with a total of 100 points; (3) suggestion: have a look at all problems and first solve the problems you feel easiest; (4) good luck!

1. **[10 pts]** Xiaoming has two coins, α and β. Coin α has $P(H) = 1/3$ and Coin β has $P(H) = 1/4$. Suppose Xiaoming picks one coin randomly and toss it for 4 times.

   (a) **[6 pts]** What is the probability of observing at most 2 heads?

   (b) **[4 pts]** If you ask Xiaoming, "did you have at most 2 heads observed?" He says, "yes". What is the probability that the coin that Xiaoming picked is Coin β?

2. **[10 pts]** The pdf of a continuous random variable X is an even function, i.e., $f_X(-x) = f_X(x)$ for all $x \in \mathbb{R}$. Suppose $\mathrm{Var}(X) = \sigma^2 < \infty$. Define $Y = |X|$.

   (a) **[5 pts]** Please find $Cov(X, Y)$.

   (b) **[5 pts]** Are X and Y independent? Please provide your reasoning with detailed steps.

3. **[12 pts]** Suppose a random sample $\{X_i\}_{i=1}^n \sim i.i.d. N(0, \sigma^2)$. Define the following three estimators

$$S_n^2 = (n-1)^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2,$$

$$\hat{\sigma}_n^2 = n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2,$$

and

$$\tilde{\sigma}_n^2 = n^{-1} \sum_{i=1}^n X_i^2.$$

where $\bar{X}_n = n^{-1}\sum_{i=1}^{n} X_i$.

(a) [**6 pts**] Check whether S_n^2, $\hat{\sigma}_n^2$, and $\tilde{\sigma}_n^2$ are unbiased estimators for σ^2 respectively. Give your reasoning.

(b) [**6 pts**] Which estimator is the most efficient? Please rank them using the MSE criterion. Give your reasoning.

4. [**10 pts**] Suppose $\{X_i\}_{i=1}^n$ is an i.i.d. random sample with the following pmf

$$f_X(x) = \begin{cases} e^{-\lambda}\frac{\lambda^x}{x!} & \text{for } x = 0, 1, 2, ... \\ 0 & \text{otherwise.} \end{cases}$$

$\lambda > 0$ is an unknown parameter. Define the following estimator

$$\hat{\theta} = \frac{1}{n^2}\left(\sum_{i=1}^{n} X_i\right)\left(\sum_{i=1}^{n} X_i - 1\right).$$

Is $\hat{\theta}$ an unbiased estimator for λ^2? If so, please prove it. If not, please find the bias of $\hat{\theta}$ for λ^2.

5. [**10 pts**] Suppose a random sample $\{X_i\}_{i=1}^n \sim i.i.d.(\mu, \sigma^2)$. Define the following estimator

$$\hat{\mu} = \frac{2}{n(n+1)}\sum_{j=1}^{n} jX_j.$$

Please show that $\hat{\mu}$ is a consistent estimator for μ.

6. [**18 pts**] Suppose $\{X\}_{i=1}^n$ is an i.i.d. random sample with the following marginal pdf

$$f_X(x) = \begin{cases} \frac{c}{\theta}e^{-x/\theta} & \text{for } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) [**4 pts**] Please find the value of c that makes the above pdf valid.

   (b) [**7 pts**] Please show that $2\sum_{i=1}^{n} X_i/\theta \sim \chi^2(2n)$;

   (c) [**7 pts**] Suppose an observable sample is obtained with sample size n =15 and sample mean $\bar{x}_n = 5010$. Please find a 95% confidence interval for θ. Provide your reasoning in detail.

HINT: The MGF for a $\chi^2(n)$ RV is $M_X(t) = (1 - 2t)^{-n/2}$.

7. **[20 pts]** Suppose a product's weight follows normal distribution. Now we independently draw a sample from it and the weights are

$$481, 473, 471, 464, 500, 439, 461, 466, 485, 460.$$

(a) [5 pts] What is the 99% confidence interval estimate for the average weight of this product ? Give your reasoning.

(b) [5 pts] What is the 95% confidence interval estimate for the variance of weight for this product ? Give your reasoning.

(c) [5 pts] What is the interpretation of the confidence interval estimates in Parts (a) and (b)?

(d) [5 pts] Suppose one would like to test if the average weight is below 450. Please conduct a test at the 5% significance level and a conclusion by explicitly writing down hypotheses and test statistic.

8. **[10 pts]** Consider the following regression equation

$$Y_i = X_i'\beta + \varepsilon_i,$$

where Y_i is a one-dimensional random variable, X_i is a p-dimensional vector and β is an unknown p dimensional vector, and ε_i is an unobservable random variable such that it follows i.i.d. $N(0,1)$. Furthermore, ε_i is independent of X_i. Given a dataset $\{y_i, x_i'\}_{i=1}^n$. Please find the MLE for β.

[**HINT**: the joint density of X_t, Y_t can be written as $f_{X_t,Y_t}(x_t, y_t) = f_{Y_t|X_t}(y_t|x_t)f_{X_t}(x_t)$. Since $f_{X_t}(x_t)$ does not depend on β, then objective function in MLE only depends on the conditional density $f_{Y_t}(y_t|x_t)$.]