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1. (a) If Xiaoming picks Coin α, we denote it as A. Under this case, X means the number of heads.

$$P(X=0) = C_4^0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^4 = \frac{16}{81}$$

$$P(X=1) = C_4^1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^3 = \frac{32}{81}$$

$$P(X=2) = C_4^2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2 = \frac{24}{81}.$$

$$\therefore P(X\le 2) = \frac{16}{81}+\frac{32}{81}+\frac{24}{81} = \frac{72}{81}$$

If Xiaoming picks Coin β, we define Y as the number of heads and the case as B.

$$P(Y=0) = C_4^0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^4 = \frac{81}{256}$$

$$P(Y=1) = C_4^1\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^3 = \frac{108}{256}.$$

$$P(Y=2) = C_4^2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2 = \frac{54}{256}.$$

$$\therefore P(Y\le 2) = \frac{81}{256}+\frac{108}{256}+\frac{54}{256} = \frac{243}{256}.$$

Since Xiaoming picks one coin randomly, we define Z as the number of heads.

$$P(Z\le 2) = P(Z\le 2\mid A)P(A) + P(Z\le 2\mid B)P(B)$$

$$= \frac{72}{81}\times\frac{1}{2} + \frac{243}{256}\times\frac{1}{2} = \frac{4235}{4608}.$$

(b) We denote the case that the coin Xiaoming picked is Coin β as H.

So $$P(H\mid Z\le 2) = \frac{P(H, Z\le 2)}{P(Z\le 2)}$$

$$= \frac{\frac{243}{256}\times\frac{1}{2}}{\frac{4235}{4608}} = \frac{2187}{4235}.$$

2.(a) $Cov(X,Y) = E(X)E(Y) - E(XY)$

$= E(X)E(|X|) - E(X|X|)$

since X is an even function, $x f_X(x)$ is an odd function.

$E(X) = \int_{-\infty}^{+\infty} x f_X(x)\,dx = 0.$

and we know that $x|x| f_X(x)$ is also an odd functio

$E(XY) = \int_{-\infty}^{+\infty} x|x| f_X(x)\,dx = 0.$

So $Cov(X,Y) = 0$.

(b) X and Y are not independent.

Because when $x \geq 0$, $Y = X$; $X < 0$, $Y = -X$

So Y and X have a strict function relationship. We can not say that X and Y are independent.

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3.(a) $\cancel{E(S_n^2) = E\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2\right)}$

we define a as the mean, so:

$$E(S_n^2) = E\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2\right)$$

$$\because \sum_{i=1}^{n}(X_i-\bar{X})^2 = \sum_{i=1}^{n}\left[(X_i-a)-(\bar{X}-a)\right]^2$$

$$= \sum_{i=1}^{n}(X_i-a)^2 - 2(\bar{X}-a)\sum_{i=1}^{n}(X_i-a) + n(\bar{X}-a)^2$$

notice that $\sum_{i=1}^{n}(X_i-a) = n(\bar{X}-a)$.

so $\sum_{i=1}^{n}(X_i-\bar{X})^2 = \sum_{i=1}^{n}(X_i-a)^2 - n(\bar{X}-a)^2$.

$\because$ $a = E(X_i) = E(\bar{X})$.

$\therefore$ $E(X_i-a)^2 = Var(X_i) = \sigma^2 \quad (i=1,\cdots,n)$.

$E(\bar{X}-a)^2 = Var(\bar{X}) = \frac{\sum_{i=1}^{n} Var(X_i)}{n^2} = \frac{\sigma^2}{n}$.

so $E(S_n^2) = \frac{1}{n-1}E\left(\sum_{i=1}^{n}(X_i-\bar{X})^2\right) = \frac{1}{n-1}\left(n\sigma^2 - n\cdot\frac{\sigma^2}{n}\right) = \sigma^2$.

S_n^2 is an unbiased estimators for σ^2.

$\because \hat{\sigma}_n^2 = \frac{n-1}{n} S_n^2$

$\therefore$ $E(\hat{\sigma}_n^2) = E\left(\frac{n-1}{n}S_n^2\right) = \frac{n-1}{n}\sigma^2 \neq \sigma^2$

$\hat{\sigma}_n^2$ is not an unbiased estimators for σ^2.

$$E(\tilde{\sigma}_n^2) = n^{-1}E\left(\sum_{i=1}^{n}X_i^2\right) = \frac{1}{n}\sum_{i=1}^{n}E(X_i^2)$$

And $\{X_i\}_{i=1}^{n} \sim$ i.i.d. $N(0,\sigma^2)$.

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So $E(X_i^2)=\int_{-\infty}^{+\infty} x^2 f_X(x)\,dx = 2$.

$E(\tilde{\sigma}_n^2) = \frac{1}{n}\sum_{i=1}^{n} E(X_i^2) = 2$.

so when $\sigma^2=2$, $\tilde{\sigma}_n^2$ is an unbiased estimator for σ^2;

$\sigma^2\neq 2$, $\tilde{\sigma}_n^2$ is not an unbiased estimator for σ^2.

(b) Since $MSE(\hat{\theta}) = [E(\hat{\theta})-\theta]^2 + Var(\hat{\theta})$.

we can find that:

$$MSE(S_n^2)=Var(S_n^2)=\frac{2\sigma^4}{n-1}.$$

$$MSE(\hat{\sigma}_n^2) = [E(\hat{\sigma}_n^2)-\sigma^2]^2+Var(\hat{\sigma}_n^2)$$

$$= \frac{1}{n^2}\sigma^4 + Var(\hat{\sigma}_n^2)$$

$$= \frac{1}{n^2}\sigma^4+\frac{2(n-1)}{n^2}\sigma^4 = \frac{2n-1}{n^2}\sigma^4$$

$$MSE(\tilde{\sigma}_n^2) = (2-\sigma^2)^2 + Var(\tilde{\sigma}_n^2)$$

when $n\to+\infty$, $MSE(\tilde{\sigma}_n^2) \geq (2-\sigma^2)^2 \nrightarrow 0$

and $\frac{2}{n-1} > \frac{2n-1}{n^2}$, $MSE(S_n^2) > MSE(\hat{\sigma}_n^2)$

$MSE(S_n^2)\to 0$, $MSE(\hat{\sigma}_n^2)\to 0$.

so $MSE(\hat{\sigma}_n^2) < MSE(S_n^2) < MSE(\tilde{\sigma}_n^2)$

so $\hat{\sigma}_n^2$ is the most efficient, S_n^2 is the second

and $\tilde{\sigma}_n^2$ is the third.

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4. $E(\hat{\theta}) = E\left[\frac{1}{n^2}\left(\sum_{i=1}^{n} X_i\right)\left(\sum_{i=1}^{n} X_i - 1\right)\right]$

$= \frac{1}{n^2} E\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] - \frac{1}{n^2} \sum_{i=1}^{n} E(X_i)$

we know that $\{X_i\}_{i=1}^{n}$ is an i.i.d. possion distribution

$E(X_i) = \sum x f_X(x) = \lambda$. we that

and $\sum_{i=1}^{n} X_i \sim Possion(n\lambda)$, define $X = \sum_{i=1}^{n} X_i$

$E\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] = E(X^2) = Var(X) + [E(X)]^2$.

$= n\lambda + n^2\lambda^2$.

So $E(\hat{\theta}) = \frac{1}{n^2} E\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] - \frac{1}{n^2} \sum_{i=1}^{n} E(X_i)$

$= \frac{1}{n^2}(n\lambda + n^2\lambda^2) - \frac{1}{n^2} \cdot n\lambda = \lambda^2$.

So $\hat{\theta}$ is an unbiased estimator for λ^2.

5. If we want to show $\hat{\mu}$ is a consistent estimator for μ, we just need to prove that $\hat{\theta} \xrightarrow{P} \theta$.

$\forall \varepsilon > 0$, $|\hat{\mu} - \mu| = \left| \frac{2}{n(n+1)} \sum_{j=1}^{n} j X_j - \mu \right|$.

$= \left| \frac{2\sigma}{n(n+1)} \sum_{j=1}^{n} j \frac{(X_j - \mu)}{\sigma} \right|$.

Since $\{X_i\}_{i=1}^{n} \sim i.i.d (\mu, \sigma^2)$, we know that according to LLN, given $\varepsilon > 0, \delta > 0$

we can find $N_i \in \mathbb{Z}^+$, $n > N_i$, $P(|X_i - \mu| \geq \varepsilon) < \delta$

So we choose $N = \max\{N_1, N_2, \cdots, N_n\}$,

when $n > N$, $|\hat{\mu} - \mu| = \left| \frac{2}{n(n+1)} \sum_{j=1}^{n} j(X_j - \mu) \right|$.

$\leq \frac{2}{n(n+1)} \sum_{j=1}^{n} j \, |X_j - \mu|$

$P(|\hat{\mu} - \mu| > \varepsilon) \leq P\left(\frac{2}{n(n+1)} \sum_{j=1}^{n} j |X_j - \mu| > \varepsilon \right)$.

$\leq \frac{2}{n(n+1)} \sum_{j=1}^{n} j P(|X_j - \mu| > \varepsilon) < \frac{2}{n(n+1)} \sum_{j=1}^{n} j \delta = \delta$

$\therefore \hat{\theta} \xrightarrow{P} \theta$.

μ is a consistent estimator for μ.

6. (a) Since $\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$.

So $\int_{-\infty}^{0} f_X(x)\,dx + \int_0^{+\infty} \frac{C}{\theta} e^{-\frac{x}{\theta}}\,dx$

$= \frac{C}{\theta}\int_0^{+\infty} e^{-\frac{x}{\theta}}\,dx = C = 1 \Rightarrow C = 1$.

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(b). We can find the MGF of $\frac{2X_i}{\theta}$ as follows:

$Y = \frac{2X_i}{\theta}$

We can know that

$P(Y \le y)$

$= P\left(\frac{2X_i}{\theta} \le y\right)$

$= P\left(X_i \le \frac{\theta y}{2}\right)$

$= \int_0^{\frac{\theta y}{2}} \frac{1}{\theta} e^{-\frac{x}{\theta}}\,dx$

$\Rightarrow f_Y(y) = \frac{1}{\theta} e^{-\frac{y}{2}} \cdot \frac{\theta}{2}$

$= \frac{1}{2} e^{-\frac{y}{2}}$

$M_Y(t) = \int_{-\infty}^{+\infty} e^{yt} f_Y(y)\,dy = \int_0^{+\infty} e^{yt} \cdot e^{-\frac{y}{2}} \cdot \frac{1}{2}\,dy$

$= \int_0^{+\infty} e^{y(t-\frac{1}{2})} \frac{1}{2}\,dy$

$= \int_0^{+\infty} \frac{1}{2} e^{-y(\frac{1}{2}-t)}\,dy = \frac{1}{1-2t}$

So if $A \sim \chi^2_{(2n)}$,

$(1-2t)^{-n} = E(e^{At}) = E\left(e^{\frac{2}{\theta}\sum_{i=1}^{n} X_i t}\right)$

$\because \{X_i\}_{i=1}^{n}$ are i.i.d. $\therefore E\left(e^{\frac{2}{\theta}\sum_{i=1}^{n} X_i t}\right) = \prod_{i=1}^{n} E\, e^{\frac{2}{\theta} X_i t}$

and according to the caculation,

$E\left(e^{\frac{2}{\theta} X_i t}\right) = (1-2t)^{-1}$

So we know that $2\sum_{i=1}^{n} \frac{X_i}{\theta} \sim \chi^2_{(2n)}$

(c). We can know that the 95% CI is:

$\frac{2n\bar{x}}{\theta} \sim \chi^2_{(2n)}$, $\bar{x} = 5010$, $n = 15$.

So the interval is: $\left[\frac{\chi^2_{2n}(1-\frac{\alpha}{2})}{2n\bar{x}}, \frac{\chi^2_{2n}(\frac{\alpha}{2})}{2n\bar{x}}\right]$

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7. We can find that:

$\mu = \frac{1}{10}\sum_{i=1}^{10} x_i = 470$.

$S^2 = \frac{1}{10-1}\sum_{i=1}^{n}(x_i-\bar{x})^2 = \frac{2450}{9}$.

(a) We can suppose that $T = \frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}} \sim N(0,1)$

$\Rightarrow$ 99% CI: $\alpha = 0.01$.

$\left[\mu - u_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}},\ \mu + u_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}\right]$

CI: [467.5, 472.5]

(b) $\alpha = 0.025$.

CI: $\left[\mu - u_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}},\ \mu + u_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}\right]$

CI: [467.4, 472.6]

(c). It means that ~~the probability that~~ if we choose another product randomly, the probability that it may be in this interval.

(d) $H_0: \mu < 450$

$H_1: \mu \geq 450$. $\alpha = 0.95$

if $T = \frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}} \geq u_\alpha$, we reject H_0.

$T < u_\alpha$, we do not reject H_0.

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8. We can find that.

$$\hat{Y}_i = \alpha_0 + \alpha_1 (X_i - \bar{X})$$

$$Q(\alpha_0, \alpha_1) = \sum_{i=1}^{n} [Y_i - \alpha_0 - \alpha_1 (X_i - \bar{X})]^2.$$

We use the fact that.

$$\frac{\partial Q}{\partial \alpha_0} = -2 \sum_{i=1}^{n} [Y_i - \alpha_0 - \alpha_1 (X_i - \bar{X})] = 0.$$

$$\frac{\partial Q}{\partial \alpha_1} = -2 \sum_{i=1}^{n} (X_i - \bar{X}) [Y_i - \alpha_0 - \alpha_1 (X_i - \bar{X})] = 0.$$

$$\Rightarrow \hat{\beta}_0 = \bar{Y}.$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} (X_i - \bar{X}) Y_i}{\sum_{i=1}^{n} (X_i - \bar{X})^2}.$$

So the β is $\dfrac{\sum_{i=1}^{n} (X_i - \bar{X}) Y_i}{\sum_{i=1}^{n} (X_i - \bar{X})^2}.$