SPRING 2021

Probability and Statistics
MIDTERM EXAM

Notes: (1) this is a closed book/notes exam; (2) there are 8 questions, with a total of 100 points; (3) suggestion: have a look at all problems and first solve the problems you feel easiest; (4) good luck!

1. **[10 pts]** Suppose there is a Ping-Pang tournament which lasts for 3 days. The tournament organizers prepare a box of 12 brand new ping-pang balls for all matches. In each day, only 3 ping-pang balls are picked randomly from the box. At the end of that day, they will be put back to the box. What is the probability that at the last day of the tournament, the 3 ping-pang balls that are picked from the box are unused?

2. **[10 pts]** A spam filter usually identifies spam emails by looking at the words contained in an email. Suppose that (i) 50% emails are spam; (ii) 1% of spam emails contain the word "lottery"; and (iii) 0.001% of non-spam emails contain the word "lottery". Suppose that an email is received that contains the word "lottery". What is the probability that the email is spam?

3. **[15 pts]** Let X follow an exponential distribution with parameter $\lambda = 2$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $Y = 2 + 3X$.

   (a) Find $P(X > 2)$;

   (b) Find $E(Y)$ and $\text{var}(Y)$;

   (c) Find $P(X > 2|Y < 11)$.

4. **[10 pts]** Suppose X is a discrete random variable, and its support is the nonnegative integers, i.e., $\{0, 1, 2, ...\}$. Please verify whether the following statement is correct:

$$E(X) = \sum_{j=0}^{\infty} P(X > j).$$

If you think it is correct, please prove it. If not, please explain or provide a counter-example.

5. **[15 pts]** Suppose a continuous random variable X has the following pdf

$$f_X(x) = \begin{cases} c\frac{\beta\alpha^\beta}{x^{\beta+1}} & \alpha < x < \infty \\ 0 & otherwise, \end{cases}$$

where c is an unknown constant, and $\alpha > 0$ and $\beta > 0$.

   (a) Find the value of constant c that makes $f_X(x)$ a valid pdf.

   (b) Find the mean $E(X)$ and the variance $\text{var}(X)$.

   (c) Based on the result in (b), do you need any additional restrictions on α or β?

6. **[15 pts]** Suppose discrete random variables X and Y have a joint pmf that assigns equal probability to each pair of points in their support G, where $G = \{(x, y)|x, y \in \mathbb{Z}, x^2 + |y| \leq 2\}$.

   (a) Find the joint pmf of (X, Y);

   (b) Find the marginal pmf of X;

   (c) Find the conditional pmf of Y given $X = x$. What is the name of this type of distribution?

7. **[15 pts]** Two continuous random variables X and Y have the joint probability density function

$$f_{XY}(x, y) = \begin{cases} k & \text{for } 0 < x^2 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the value of k.

   (b) Find the conditional density function $f_{Y|X}(y|x)$ of Y given $X = x$.

   (c) Are X and Y independent?

8. **[10 pts]** The function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad -\infty < x < \infty,$$

where μ and $\sigma^2 > 0$ are constants, is the probability density function of the normal distribution. Find the value of integral

$$\int_0^\infty x^2 e^{-2x^2} dx.$$

Give your reasoning.