Statistical Properties of the Asymmetric Power ARCH Process

by

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Abstract. The asymmetric power ARCH model is a recent addition to time series models that may be used for predicting volatility. Its performance is compared with that of standard models of conditional heteroskedasticity such as GARCH. This has previously been done empirically. In this paper the same issue is studied theoretically using unconditional fractional moments for the A-PARCH model that are derived for the purpose. The role of the heteroskedasticity parameter of the A-PARCH process is highlighted and compared with corresponding empirical results involving autocorrelation functions of power-transformed absolute-valued return series.

Key Words. GARCH, heteroskedasticity, financial time series, nonlinearity, S&P 500, volatility

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1 Introduction

The temporal properties of power-transformed returns, r_t, from speculative assets have recently received considerable attention. Clive Granger, whose interest in commodity and stock markets can be traced back to Elliott and Granger (1967) and Granger and Morgenstern (1970), has discussed them in a series of papers: see Ding, Granger and Engle (1993), henceforth DGE, Granger and Ding (1995a, b; 1996) and Ding and Granger (1996). In DGE the authors reminded the reader of the fact which Taylor (1986, pp. 52-56) first observed, namely, that $|r_t|$ is positively autocorrelated even at long lags. DGE found that $|r_t|^d$, $d > 0$, is even long-memory in that the autocorrelations remain not only positive but rather high at very long lags. Moreover, the authors observed that the property is strongest for $d = 1$ and near it. They noted that this may argue against ARCH specifications that are based on squared returns such as the classical GARCH model of Bollerslev (1986) and Taylor (1986, pp. 78-79). Consequently, DGE introduced a new GARCH model with a separate heteroskedasticity parameter δ. This model, the Asymmetric Power ARCH (A-PARCH) model, contained as special cases a host of previous GARCH models. DGE demonstrated its potential by fitting the model with normal errors to the long S&P 500 daily stock return series from 3 January 1928 to 30 April 1991, 17054 observations in all.

In this paper we take another look at statistical properties of the A-PARCH process. As the motivation for the model lies in observed autocorrelation functions of $|r_t|^d$, see DGE, we mainly focus on the corresponding autocorrelation function of the A-PARCH process. In particular, the relationship between d and the corresponding model parameter which will be called δ is of interest. It turns out not to be as straightforward as one might think. We demonstrate how the heteroskedasticity parameter and an asymmetry parameter also present in the A-PARCH model add to the exibility of the speci cation. In order to proceed we need certain (fractional) moments of the A-PARCH process and having obtained those we have the autocorrelation function necessary for our discussion.

The paper is organized as follows. The fractional moments and the autocorrelation function of the A-PARCH process are presented in Section 2. A transformation of observations implicit in the A-PARCH speci cation is the topic of Section 3. Section 4 considers the autocorrelation function of $|r_t|\,\delta$ as a function of δ. Section 5 discusses possible advantages of the A-PARCH speci cation in the light of the empirical example in DGE. Section 6 concludes.

2 Moments and the autocorrelation function

Consider the general A-PARCH(p, q, δ) model

$$\varepsilon_t = z_t h_t \tag{1}$$

where $\{z_t\}$ is a sequence of independent identically distributed random variables with a symmetric density with zero mean and a nite 2δ-th unconditional absolute moment. Furthermore,

$$\begin{aligned} h_t\delta &= \alpha_0 + \sum_{j=1}^{p} \alpha_j(|\varepsilon_{t-j}| - \phi_j\varepsilon_{t-j})\delta + \sum_{j=1}^{q} \beta_j h_{t-j}\delta \\ &= \alpha_0 + \sum_{j=1}^{p} c_{\delta,t-j} h_{t-j}\delta \end{aligned} \tag{2}$$

where $c_{\delta,t-j} = \alpha_j(|z_{t-j}| - \phi_j z_{t-j})\delta + \beta_j$, $j = 1, ..., p$. Expression (2) for $\delta = 2$ and $\phi_j = 0$, $j = 1, ..., p$, appeared in He and Teräsvirta (1997a). Following their arguments it is possible to derive the autocorrelation function of $|\varepsilon_t|\,\delta$ when $\{\varepsilon_t\}$ is generated by (1) and (2). The result is not explicitly stated here because we shall concentrate on the rst-order case

$(p = q = 1)$. This implies that (2) becomes

$$h_t\delta = \alpha_0 + c_{\delta,t-1}h_{t-1}\delta \tag{3}$$

where $c_{\delta t} = \alpha_1(|z_t| - \phi z_t)\delta + \beta_1$ and $\phi = \phi_1$, and $\{c_{\delta t}\}$ is a sequence of iid observations. Letting $\gamma\delta = \mathsf{E}c_{\delta t}$ and $\gamma_{2\delta} = \mathsf{E}c_{\delta t}^2$ we have

Theorem 1 . *For the A-PARCH$(1, 1, \delta)$ model (1) with (3), a necessary and sufficient condition for the existence of the 2δ-th absolute moment $\mu_{2\delta} = \mathsf{E}\left|\varepsilon_t\right|^{2\delta}$ is*

$$\gamma_{2\delta} < 1. \tag{4}$$

If (4) holds, then

$$\mu_{2\delta} = \alpha_0^2\nu_{2\delta}(1 + \gamma\delta)/\{(1 - \gamma\delta)(1 - \gamma_{2\delta})\} \tag{5}$$

and the moment coefficient

$$\kappa_{2\delta} = \mu_{2\delta}/\mu\delta^2 = \kappa_{2\delta}(z_t)(1 - \gamma\delta^2)/(1 - \gamma_{2\delta}) \tag{6}$$

where $\nu\psi = \mathsf{E}\left|z_t\right|\psi$ and $\kappa_{2\delta}(z_t) = \nu_{2\delta}/\nu\delta^2$. Furthermore, the autocorrelation function $\rho_n(\delta) = \rho(\left|\varepsilon_t\right|\delta, \left|\varepsilon_{t-n}\right|\delta)$, $n \geq 1$, of $\{\left|\varepsilon_t\right|\delta\}$ has the form

$$\rho_n(\delta) = \frac{\nu\delta\gamma\delta^{n-1}[\overline{\gamma}\delta(1 - \gamma\delta^2) - \nu\delta\gamma\delta(1 - \gamma_{2\delta})]}{\nu_{2\delta}(1 - \gamma\delta^2) - \nu\delta^2(1 - \gamma_{2\delta})} \tag{7}$$

where $\overline{\gamma}\delta = \mathsf{E}(\left|z_t\right|\delta c_{\delta t})$.

In particular, when $\delta = 2$, the kurtosis of $\{\varepsilon_t^2\}$ equals $\kappa_4 = \kappa_4(z_t)(1 - \gamma_2^2)/(1 - \gamma_4)$ and $\rho_n(2) = \kappa_4(z_t)\gamma_2^{n-1}(\overline{\gamma}_2 - \gamma_2\kappa_4^{-1})/(1 - \kappa_4^{-1})$, $n \geq 1$, where $\kappa_4(z_t)$ is the kurtosis of $\{z_t\}$; see He and Teräsvirta (1997b) for a discussion.

The results follow from the general GARCH(p, q) case He and Teräsvirta (1997a) considered, but they may also be derived by substituting $h_t\delta$ and $c_{\delta t}$ for their counterparts in the proofs of Theorems 1 and 4 of He and Teräsvirta (1997b). DGE also discussed $\gamma\delta$, and Ding and Granger (1996) derived the (conditional) autocorrelation function of $|\varepsilon_t|\,\delta$ under the assumption $\nu\delta = 1$. As DGE pointed out, many well-known GARCH models are special cases of the A-PARCH model. Theorem 1 applies to those as well, but they are considered in more detail in He and Teräsvirta (1997b).

As stated in the beginning, we shall examine the role of the power parameter δ in the A-PARCH model. The model was introduced to characterize processes $\{\varepsilon_t\}$ in which $\rho(|\varepsilon_t|, |\varepsilon_{t-n}|)$ decays slowly as a function of n. From (7) we have

$$\rho_n(\delta) = \rho_1(\delta)\gamma_\delta^{n-1}, \ n \geq 1. \tag{8}$$

The decay of the autocorrelation function is exponential, as Ding and Granger (1996) already noted, and $\gamma\delta$ controls this decay. For a closer look at the maximum value of the rst-order autocorrelation (7) note that, due to the symmetry of the density of z_t, we can express $\gamma\delta$, $\overline{\gamma}\delta$ and $\gamma_{2\delta}$, respectively, as

$$\gamma\delta = (\alpha_1/2)\phi\delta\nu\delta + \beta_1 \tag{9}$$

$$\overline{\gamma}\delta = (\alpha_1/2)\phi\delta\nu_{2\delta} + \beta_1\nu\delta \tag{10}$$

$$\gamma_{2\delta} = (\alpha_1^2/2)\phi_{2\delta}\nu_{2\delta} + \alpha_1\beta_1\phi\delta\nu\delta + \beta_1^2 \tag{11}$$

where $\phi\delta = (1+\phi)\delta + (1-\phi)\delta$ and $\phi_{2\delta} = (1+\phi)^{2\delta} + (1-\phi)^{2\delta}$. Using (9), expressions (10) and (11) may be rewritten as

$$\overline{\gamma}\delta = \gamma\delta\nu\delta + (\alpha_1/2)\phi\delta(\nu_{2\delta} - \nu\delta^2) \tag{12}$$

$$\gamma_{2\delta} = \gamma\delta^2 + (\alpha_1^2/2)\phi_{2\delta}\nu_{2\delta} - (\alpha_1^2/4)\phi\delta^2\nu\delta^2. \tag{13}$$

Inserting (12) and (13) into (7) yields

$$\rho_1(\delta) = \frac{(\alpha_1/2)\nu\delta\phi\delta(\nu_{2\delta} - \nu\delta^2)(1-\gamma\delta^2) + (\alpha_1^2/2)\nu\delta^2\gamma\delta(\phi_{2\delta}\nu_{2\delta} - (1/2)\phi\delta^2\nu\delta^2)}{(\nu_{2\delta} - \nu\delta^2)(1-\gamma\delta^2) + (\alpha_1^2/2)\nu\delta^2(\phi_{2\delta}\nu_{2\delta} - (1/2)\phi\delta^2\nu\delta^2)}. \tag{14}$$

From (14) it is seen that for given values of α_1, β_1 and ϕ, $\rho_1(\delta)$ is a function of δ expressed in terms of $\phi\delta$, $\phi_{2\delta}$, $\nu\delta$ and $\nu_{2\delta}$. Assuming $\phi = 0$, $\rho_1(\delta)$ reduces to

$$\rho_1(\delta) = \frac{\alpha_1\nu\delta(1-\gamma\delta^2) + \alpha_1^2\gamma\delta\nu\delta^2}{(1-\gamma\delta^2) + \alpha_1^2\nu\delta^2}. \tag{15}$$

On the other hand, setting $\beta_1 = 0$ in (14) yields the rst-order autocorrelation of squares in the A-PARCH$(1,\delta)$ model:

$$\begin{aligned}\rho_1(\delta) &= \frac{(\alpha_1/2)\nu\delta\phi\delta(\nu_{2\delta}-\nu\delta^2) + (\alpha_1^3/4)\nu\delta^3\phi\delta\nu_{2\delta}(\phi_{2\delta}-(1/2)\phi\delta^2)}{(\nu_{2\delta}-\nu\delta^2) + (\alpha_1^2/2)\nu\delta^2\nu_{2\delta}(\phi_{2\delta}-(1/2)\phi\delta^2)} \\ &= (\alpha_1/2)\phi\delta\nu\delta. \end{aligned} \tag{16}$$

Further giving up asymmetry by setting $\phi = 0$ in (16) leads to

$$\rho_1(\delta) = \gamma\delta = \alpha_1\nu\delta, \tag{17}$$

which is analogous to a familiar expression from an ARCH(1) model.

3 The power parameter

The unconditional moment (5) is a fractional moment. Obtaining integer moments for the A-PARCH process analytically seems difficult. Nonetheless, the fractional moments are useful in considering the role of the power parameter δ in the A-PARCH(1,1) model. Transform ε_t as follows:

$$\xi_t = \mathrm{sgn}(\varepsilon_t)\,|\varepsilon_t|^{\delta/2}. \tag{18}$$

Assuming $\phi = 0$ in (3) so that no asymmetry is present then Theorem 1 yields the fourth moment of ξ_t. The kurtosis of ξ_t is

$$\kappa_4(\xi_t) = \mu_{2\delta}/\mu\delta^2 = \kappa_4^*(z_t)(1-\gamma\delta^2)/(1-\gamma_{2\delta}) \tag{19}$$

where $\kappa_4^*(z_t) = \nu_{2\delta}/\nu\delta^2$ is the kurtosis of $\mathrm{sgn}(z_t)|z_t|^{\delta/2}$ and $\gamma\delta = \alpha_1\nu\delta + \beta_1$. Furthermore, (7) is the autocorrelation function of $\{\xi_t^2\}$. In fact, (1) with (3) where $\phi = 0$ is the ordinary

GARCH(1,1) model for $\{\xi_t\}$. Note, however, that the distribution of z_t also changes although the density retains its symmetry about zero. There does not exist a corresponding transformation for $\phi \neq 0$. Nonetheless, if we reparameterize the A-PARCH(1,1) model by replacing the term $(|\varepsilon_{t-1}| - \phi\varepsilon_{t-1})$, $\phi < 1$, by $(\varepsilon_{t-1} - \phi\,|\varepsilon_{t-1}|)$ we may de ne

$$\xi_t^* = \operatorname{sgn}(\varepsilon_t - \phi\,|\varepsilon_{t-1}|)\,|\varepsilon_t - \phi\,|\varepsilon_{t-1}||^{\delta/2}\,. \tag{20}$$

When $\{\varepsilon_t\}$ has the modi ed A-PARCH(1,1) representation the sequence $\{\xi_t^*\}$ obeys a symmetric GARCH(1,1) model. Its kurtosis and the autocorrelation function of $(\xi_t^*)^2$ are given by (20) and (7) for $\phi \neq 0$.

This may give interesting insight in what the contribution of the extra parameters δ and ϕ is as regards the kurtosis and the autocorrelation function of the squares. To illustrate, consider rst real data. We choose the long daily return series of the S&P 500 stock index of 17054 observations from 3 January 1928 to 30 April 1991 compiled by William Schwert and analyzed in DGE, Granger and Ding (1995a, b), and Ding and Granger (1996); see also Rydén, Teräsvirta and Åsbrink (1997). The series is transformed according to (18) and (20). Figure 1 shows that the rst-order autocorrelation of the squared observations is a concave function of δ. Value $\delta = 2$ corresponds to the identity transformation. The solid line represents the case $\phi = 0$. The maximum of the autocorrelation function occurs in the vicinity of $\delta = 1$ which agrees with what DGE and Granger and Ding (1995a) observed. The dashed-dotted line is the rst-order autocorrelation for the case $\phi = -0.3$. Introducing asymmetry in the transformation seems to reduce the rst-order autocorrelation and the value of the autocorrelation-maximizing δ. Figure 2 shows the relationship between the kurtosis and the rst-order autocorrelation of the transformed variables. The kurtosis is an increasing function of δ. While the kurtosis decreases the autocorrelation as a concave function of the kurtosis rst increases and then starts to decline. Although the kurtosis may be reduced to that of the normal distribution the higher-order dependence in the original series does not vanish with the transformation. Some skewness also remains. The asymmetry introduced through ϕ tends to increase kurtosis and, at the same time, reduce the autocorrelation (the dashed-dotted line in Figure 2). Given the problem of modelling series displaying high kurtosis and low rst-order autocorrelation of squared observations with GARCH models with normal errors (see, for example, He and Teräsvirta, 1997b) this is an interesting observation. Of course, ϕ also affects the skewness of the transformed observations. A value such as $\phi = -0.1$ already transforms the originally negatively skewed series to be positively skewed for a wide range of values of δ (small values are an exception).

Next we turn to the theoretical A-PARCH model and the role of δ and ϕ there.

4 Convexity of the theoretical autocorrelation function

In this section we consider $\rho_1(\delta)$ as a function of δ. We begin by de ning $\rho_1(\delta)$ on a set of values of δ such that $\rho_1(\delta)$ exists. Let $I = (0, l)$, $l > 0$, be an open interval such that $\delta_0 \in I$ if and only if $\gamma_{2\delta_0} < 1$. Then there exists $I\delta = [\delta_a, \delta_b]$, which is a closed interval, such that $I\delta \subset I$. Obviously, for any $\delta \in I\delta$ we have $\gamma_{2\delta} < 1$.

We now state our main result:

Property 4.1 *Assume that $\nu\delta$ is a convex function of δ on $I\delta$ with a nite second derivative for any $\delta \in I\delta$. Then $\rho_1(\delta)$ given in (15) is a convex function of δ on $I\delta$. Furthermore, if $\frac{\partial \nu\delta}{\partial \delta}\big|_{\delta=\delta_0} = 0$ and $\delta_0 \in I\delta$, then $\frac{\partial \rho_1(\delta)}{\partial \delta}\Big|_{\delta=\delta_0} = 0$.*

Proof. See Appendix.

Remark 1. The convexity assumption for $\nu\delta$ is not very restrictive. In fact, what we have assumed for z_t in Section 2 is sufficient for $\nu\delta$ to be a convex function of δ on $I\delta$. To see this, suppose that a random variable X has a continuous density $f(x)$ on $(-\infty, +\infty)$ such that $f(x)$ is nonnegative everywhere. Then $\nu\delta'' = \mathsf{E}(|X|\,\delta \ln^2 |X|) > 0$ on $I\delta$ so that $\nu\delta$ is a convex function of δ on $I\delta$. Moreover, $\nu\delta' = \mathsf{E}(|X|\,\delta \ln |X|)$ implies that there exists a unique δ_0 such that $\nu'_{\delta_0} = 0$.

Moreover the asymmetry terms $\phi\delta$, $\phi_{2\delta}$ and $\phi\delta^2$ in (14) are convex functions of δ on the interval (0,1) and they reach their minimum values on the interval (0, 0.5]. The minimizing value is a function of ϕ.

Remark 2. If $\phi = 0$, then it follows from Property 4.1 that the rst-order autocorrelation $\rho_1(\delta)$ reaches its minimum value at $\delta_0 \in I\delta$, in which $\nu\delta$ also has its minimum value. For given parameter values β_1 and α_1, $\underset{\delta \in I\delta}{min}\, \{\rho_1(\delta)\}$ thus depends on the distribution of z_t.

From (15) we see that $\rho_1(\delta)$ only depends on the δ-th absolute moment $\nu\delta$ if we do not allow asymmetry. If $\phi \neq 0$ then it follows from (14) that $\rho_1(\delta)$ also depends on $\nu_{2\delta}$, $\phi\delta$ and $\phi_{2\delta}$. This complicates the analysis. Nevertheless, assume as before that $\nu\delta$ with a second derivative for any $\delta \in I\delta$ is a convex function of δ on $I\delta$. This implies that $\nu_{2\delta}$ is also a convex function of δ on $I\delta$. Moreover, $\phi\delta'' = (1 + \phi)\delta \ln^2(1 + \phi) + (1 - \phi)\delta \ln^2(1 - \phi) \geq 0$ indicates that $\phi\delta$ is a convex function of δ on $I\delta$. Therefore, there exists a subset $I_{\delta^*} \subseteq I\delta$ such that $\nu\delta'$, $\nu'_{2\delta}$, $\phi\delta'$, $\phi'_{2\delta} < 0$ for any $\delta \in I_{\delta^*}$. Then there exists an unique $\delta_0 \in I\delta$ such that $u'v - uv' = 0$ at $\delta_0 \in I\delta$, where $\rho_1(\delta) = u/v$ in (14). We thus have $\rho'_1(\delta_0) = 0$. Proving $\rho''_1(\delta_0) > 0$ seems complicated and we content ourselves with

Conjecture 4.2 *Assume that $\nu\delta$ with a second derivative for any $\delta \in I\delta$ is a convex function of δ on $I\delta$. Then, for $\phi \neq 0$, $\rho_1(\delta)$ is a convex function of δ on $I\delta$. Furthermore, if $\frac{\partial \nu\delta}{\partial \delta}\big|_{\delta=\delta_0} = 0$ and $\delta_0 \in I\delta$, then $\frac{\partial \rho_1(\delta)}{\partial \delta}\Big|_{\delta=\delta_0} \neq 0$.*

Remark 3. When $\phi \neq 0$, $\rho_1(\delta)$ is still a convex function of δ on $I\delta$. However, the minimizing $\delta_0 \in I\delta$ such that $\nu'_{\delta_0} = 0$ is not the same as the $\delta_* \in I\delta$ for which $\rho'_1(\delta_*) = 0$.

The inequality $\delta_* \neq \delta_0$ is due to the effects of the asymmetric terms $\phi\delta$ and $\phi_{2\delta}$ in (14).

We already noticed that the asymmetry parameter ϕ in (14) affects the location of the minimum value of $\rho_1(\delta)$ on the interval $I\delta$. We are also able to assess the effect of ϕ on the value of the autocorrelation function. Let $\rho_1^*(\delta)$ denote the rst order autocorrelation $\rho_1(\delta)$ in (14) corresponding to $\phi \neq 0$ whereas $\rho_1^\circ(\delta) = \rho_1(\delta)$ in (15) where $\phi = 0$. We have

Property 4.3. *Consider the A-PARCH model (1) with (3) and assume that* $\rho_1^*(\delta)$ *and* $\rho_1^\circ(\delta)$ *exist for some* δ*. Then*

$$\rho_1^*(\delta) \quad > \quad \rho_1^\circ(\delta), \textit{when } \delta \geq 1 \tag{21}$$

$$\rho_1^*(\delta) \quad > \quad \rho_1^\circ(\delta), \textit{when } 0 < \delta < 1 \textit{and } \gamma\delta \textit{is sufficiently large.} \tag{22}$$

Proof. See Appendix.

The situation is illustrated in Figure 3. The solid concave curve is the rst-order autocorrelation of $|\varepsilon_t|\,\delta$ estimated from the S&P 500 series for various values of δ. The solid convex one is the theoretical autocorrelation obtained from (15) by setting $\alpha_1 = 0.091$ and $\beta_1 = 0.9$. The dashed-dotted curves are the counterparts of the solid curves after introducing skewness by setting $\phi = -0.3$. It is seen how the nonzero asymmetry parameter has the effect of shifting the autocorrelation-minimizing value of δ to the left. Furthermore, the value of the theoretical autocorrelation function increases with $|\phi|$. The point where the theoretical and the empirical autocorrelations are equal is also shifted to the left. One may expect the estimated value of δ to lie in the neighbourhood of the value at which the two curves intersect.

As Remark 2 indicates, the minimizing value of $\rho(|\varepsilon_t|\,\delta,\ |\varepsilon_{t-1}|\,\delta)$ may also be shifted away from $\delta_{min} = 0.87$ where it is located under the assumption of normality by assuming another distribution for the error term. It can be shown that if the error term is t-distributed then $\delta_{min} < 0.87$. In fact, the minimizing value is an increasing function of degrees of freedom in the t-distribution. On the other hand, the t-distribution can be generalized further into a generalized t (GT) distribution with the density

$$f(x) = \frac{p}{2\sigma q^{1/p} B(1/p, q)(1 + \frac{|x|^p}{q\sigma^p})^{q+1/p}}, \ |x| < \infty, \ \sigma, p, q > 0 \tag{23}$$

where B is the beta-function, see, for example, McDonald and Newey (1988). Bollerslev, Engle and Nelson (1994) employed the GT distribution as the error distribution when modelling U.S. stock index returns. Suitable choices of the two shape parameters p and q shift the minimizing value to the right. For instance, $p = 1.5$ and $q = 200$ yield $\delta_{min} = 1.07$. For comparison, $p = 2$ for the t-distribution.

5 An empirical example

The above theory considers the role of δ and ϕ in the A-PARCH model under the *ceteris paribus* assumption that the other parameters remain constant. If a standard GARCH model is augmented by these parameters, the estimates of the other parameters almost certainly change. DGE considered the contribution of δ and ϕ to the GARCH speci cation by comparing the values of the estimated log-likelihood for three different models tted to the above-mentioned S&P 500 daily return series of 17054 observations. The models were the AVGARCH(1,1) model, the standard GARCH(1,1) model, and the A-PARCH(1,1,δ) model. We carry out a similar comparison by studying estimated autocorrelation functions. Estimation results from DGE are reproduced in Table 1. The last row of Table 1 contains the estimated left-hand side of the existence condition of the absolute moment of order 2δ. It is seen that according to the results from the standard GARCH model, the fourth unconditional moment does not exist for the S&P 500 series. On the other hand, according to the A-PARCH model, the absolute moment of order 2.86 exists. In line with this result, the estimated AVGARCH model signals the existence of the second moment of $\{\varepsilon_t\}$.

Table 2 shows the autocorrelation functions of absolute values of $|\varepsilon_t|\ \delta$ for $\delta = 2$, 1.43, 1. They are estimated directly from the data (D) or from the estimated models (M) applying (14) and (15). The autocorrelation function from the model is not available for the standard GARCH model because the fourth moment does not exist. We are thus not able to compute any autocorrelations of squared observations from the model. Yet, the left-hand side of the existence condition for the fourth moment is a monotonically increasing function of both α_1 and β_1, and the autocorrelation function converges to an asymptote as the fourth moment vanishes. In order to obtain an approximation to an asymptotic autocorrelation function of squares we decrease the estimates of these parameters in turn such that the fourth moment just exists ($\gamma_4 = 0.9999$) for the two obtained parameter combinations. This yields approximations to two extreme asymptotic autocorrelation functions that, however, lie close to each other; the rest may be obtained by reducing both parameters in size simultaneously until $\gamma_4 < 1$. It is seen from Table 2 that the discrepancy between the asymptotic autocorrelation functions and the autocorrelations estimated directly from the data is large. For the AVGARCH(1,1) model, there is also a large discrepancy between the autocorrelation function obtained from the model and the one estimated from the data. The autocorrelations from the model are very high compared to those estimated directly from the series. The gap between the two is much smaller for the power ARCH model. Interestingly enough, the model-based autocorrelations seem to decay at a slower rate up to 50 lags than the direct ones although the decay rate of the model-based autocorrelations is just exponential. Nevertheless, the extra parameters in the A-PARCH(1,1) model seem to have considerably improved the correspondence between stylized facts in the data on the one hand and the model on the other.

6 Conclusions

The results of this paper indicate how the power or heteroskedasticity (DGE) parameter adds to the flexibility of the GARCH family when it comes to characterizing stylized facts in the observed series. The considerations are restricted to autocorrelations of powers of absolute values whereas the asymmetric response to news handled by a specific asymmetry parameter is not a topic here. It will be discussed in future work. The paper merely demonstrates the effect of a nonzero asymmetry parameter on the autocorrelation function of powers of absolute-value transformed observations. It is also shown how the choice of error distribution would further enhance the flexibility of the A-PARCH parametrization. As for the heteroskedasticity parameter, its role turns out to be different from what intuition based on empirical considerations might suggest.

The results on moments and the autocorrelation function generalize to higher-order A-PARCH processes by applying the results in He and Teräsvirta (1997a) to the present situation. As such a generalization involves tedious notation and because the (1,1) case is by far the most popular one in practice, we do not discuss higher-order A-PARCH processes in this paper. They are needed if the first few autocorrelations in the observed autocorrelation function of absolute values or squares do not fit into the general exponential decay pattern of the autocorrelations. This possibility has recently been discussed in He and Teräsvirta (1997c) in the context of the standard GARCH process (see also Nelson and Cao, 1992), and their results pertain to the present situation.

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Appendix. Proofs.

Proof of Property 4.1.

Let $\rho_1(\delta) = u/v$ where $u = \alpha_1(1-\beta_1^2)\nu\delta - \alpha_1^2\beta_1\nu\delta^2$ and $v = (1-\beta_1^2) - 2\alpha_1\beta_1\nu\delta$ from (16). It follows that $\rho_1(\delta)$ is continuous and twice differentiable on $I\delta$. The rst derivative $\rho_1'(\delta) = (u'v - uv')/v^2$, where u' and v' are the rst order derivatives with respect to δ.

First we show that there exists an unique $\delta_0 \in I\delta$ such that $\rho_1'(\delta_0) = 0$. Set

$$\begin{aligned} & u'v - uv' \\ = \;& \alpha_1(1-\beta_1^2)^2\nu\delta' \; - 2\alpha_1^2\beta_1(1-\beta_1^2)\nu\delta\nu\delta' \; + 2\alpha_1^3\beta_1^2\nu\delta^2\nu\delta' \; = 0 \end{aligned} \tag{A.1}$$

where $\nu\delta'$ is a rst derivative of $\nu\delta$ with respect to δ. From (A.1) it follows that either (i) $\nu\delta' = 0$, or (ii) $2\alpha_1^2\beta_1^2\nu\delta^2 \; - 2\alpha_1\beta_1(1-\beta_1^2)\nu\delta \; - 2(1-\beta_1^2)^2 \; = 0$. If (ii) holds, then $\nu\delta = (1-\beta_1^2)(1\pm\sqrt{3})/2\alpha_1\beta_1$. Consider $\nu_d = (1-\beta_1^2)(1-\sqrt{3})/2\alpha_1\beta_1$. This implies $\nu\delta < 0$, which is not possible. Alternatively, the solution $\nu\delta = (1-\beta_1^2)(1+\sqrt{3})/2\alpha_1\beta_1$. Then $2\alpha_1\beta_1\nu\delta < 1-\beta_1^2$, which violates the moment condition $\gamma_{2\delta} < 1$. Thus $\nu_{\delta_0}' = 0$. Assume there exists $\delta_0 \in I\delta$ such that $\nu_{\delta_0}' = 0$. Then $\rho_1'(\delta_0) = 0$.

Second, we show that the second derivative $\rho_1''(\delta_0) > 0$. Write $\rho_1''(\delta) \; = [(u''v \; - uv'') \; - 2(u'v \; - uv')v']/v^3$. It suffices to show that $u''v - uv'' > 0$ at $\delta = \delta_0$. From (4) we have $1-\beta_1^2 > \delta_1^2\nu\delta + 2\alpha_1\beta_1\nu\delta$ so that

$$\begin{aligned} & u''v - uv''|_{\delta=\delta_0} \\ = \;& \alpha_1\nu_{\delta_0}''[(1-\beta_1^2)^2 + 2\alpha_1\beta_1(1-\beta_1^2)\nu_{\delta_0} - 6\alpha_1^2\beta_1^2\nu_{\delta_0}^2] \\ > \;& \alpha_1\nu_{\delta_0}''[\alpha_1^4\nu_{2\delta_0}^2 + 6\alpha_1^3\beta_1\nu_{\delta_0}\nu_{2\delta_0} + 2\alpha_1^2\beta_1^2\nu_{\delta_0}^2] \\ > \;& 0. \; \dashv \end{aligned} \tag{A.2}$$

Proof of Property 4.3.

Suppose rst that $\delta > 1$ and compare (14) with (15). In this case, $\phi\delta > 2$, and $(\phi_{2\delta}\nu_{2\delta} - \frac{1}{2}\phi\delta^2\nu\delta^2)/(\nu_{2\delta} - \nu\delta^2) > 2$ implies that $\rho_1^*(\delta) > \rho_1^\circ(\delta)$. When $\delta = 1$, $\phi\delta = 2$ and $(\phi_{2\delta}\nu_{2\delta} - \frac{1}{2}\phi\delta^2\nu\delta^2)/(\nu_{2\delta} - \nu\delta^2) > 2$. This proves (21). When $0 < \delta < 1$, extra conditions are needed for $\rho_1^*(\delta) > \rho_1^\circ(\delta)$. The moment $\gamma\delta$ should be sufficiently large so that the numerator on the right side of (14) is dominated by $\gamma\delta$. This is because $1 < \phi\delta < 2$ and $(\phi_{2\delta}\nu_{2\delta} - \frac{1}{2}\phi\delta^2\nu\delta^2)/(\nu_{2\delta} - \nu\delta^2) > 2$. (For example, from $\gamma\delta \geq 0.8$ it already follows that $\rho_1^*(\delta) > \rho_1^\circ(\delta)$. In practice, (22) tends to hold for $0 < \delta < 1$ because the estimated $\widehat{\gamma}\delta$ is usually quite close to unity.) $\dashv$

Table 1. GARCH speci cations estimated for the S&P 500 daily stock return series, 3 January 1928 to 30 April 1991. Source: DGE (value of the t-statistic in parentheses)

Parameter estimate	GARCH	A-PARCH	AVGARCH
	$\delta = 2$	$\widehat{\delta} = 1.43$	$\delta = 1$
$\widehat{\alpha}_0$	8.0×10^{-7} (12.5)	1.4×10^{-6} (4.5)	9.6×10^{-6} (12.6)
$\widehat{\alpha}_1$	0.091 (50.7)	0.083 (32.4)	0.104 (67)
$\widehat{\beta}_1$	0.906 (43.4)	0.920 (474)	0.913 (517)
$\widehat{\phi}$		−0.373 (−20.7)	
$\widehat{\gamma}_{2\delta}^1$	1.0106	0.99526	0.99591

[1] $\widehat{\gamma}_{2\delta}$ = left-hand side of the 2δ moment condition. A value less than unity indicates the existence of the 2δ-th moment of $\{|\varepsilon_t|\}$.

Table 2. Autocorrelation functions of $\{|\varepsilon_t|\ \delta\}$, $\delta = 2, 1.43, 1$, *estimated from the linearly ltered S&P 500 daily stock return series, January 3, 1928 to April 30, 1991, and computed from estimated models.*

Model	D/M[1]	Lag									
		1	2	3	4	5	10	20	30	40	50
GARCH	D	0.252	0.243	0.182	0.153	0.200	0.115	0.095	0.094	0.067	0.069
	M(a)	0.390	0.387	0.383	0.380	0.377	0.362	0.332	0.306	0.281	0.258
	M(b)	0.387	0.384	0.381	0.378	0.376	0.362	0.335	0.311	0.288	0.267
A-PARCH	D	0.337	0.328	0.294	0.291	0.287	0.212	0.196	0.181	0.154	0.152
	M	0.421	0.418	0.415	0.412	0.409	0.395	0.368	0.343	0.320	0.298
AVGARCH	D	0.343	0.338	0.330	0.314	0.316	0.258	0.249	0.230	0.207	0.200
	M	0.506	0.504	0.502	0.500	0.498	0.488	0.469	0.450	0.433	0.416

[1]D= Autocorrelations estimated from the S&P 500 daily return series

M= Autocorrelations computed from the estimated model

M(a)=Autocorrelations computed from the estimated model by reducing the estimate of α_1 until the stationarity condition is satis ed ($\widehat{\gamma}_4 = 0.9999$)

M(b)=Autocorrelations computed from the estimated model by reducing the estimate of β_1 until the stationarity condition is satis ed ($\widehat{\gamma}_4 = 0.9999$)

Fig. 1. The rst-order autocorrelations for the S&P 500 stock returns transformed according to (18) (solid curve) and (20) with $\phi = -0.3$ *(dashed-dotted curve) as a function of the transformation parameter* δ

Fig. 2. The kurtosis of the S&P 500 stock returns transformed according to (18) (solid curve) and (20) with $\phi = -0.3$ *(dashed-dotted curve) as a function of the rst-order autocorrelation*

Fig. 3. The rst-order autocorrelations for the power-transformed S&P 500 series for $\phi = 0$ *(solid concave curve) and* $\phi = -0.3$ *(dashed -dotted concave curve) and the corresponding theoretical autocorrelations from an A-PARCH(1,1) model with* $\alpha_1 = 0.091$, $\beta_1 = 0.9$, *and* $\phi = 0$ *(solid convex curve) and* $\phi = -0.3$ *(dashed-dotted convex curve)*