PHYS 3033/3053 Assignment 6

Due: 6 Nov 2015 at begin of lecture at 3:00 pm

Problem 1.

(a) Find the force on a square loop placed as shown in Fig. 1(a), near an infinite straight wire. Both the loop and the wire carry a steady current I.

(b) Find the force on the triangular loop in Fig. 1(b).

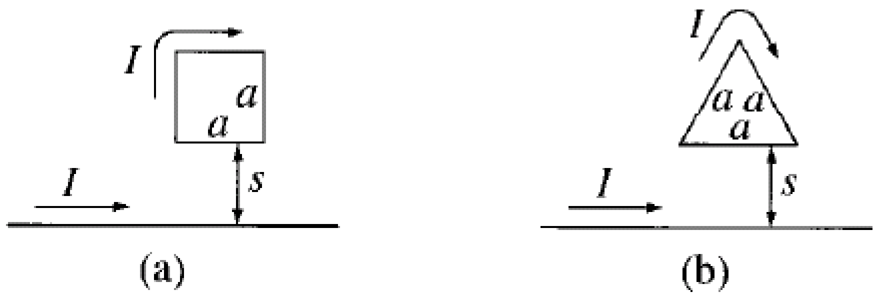


Fig. 1

Solution 1

(a) The forces on the two sides cancel. At the bottom, $B = \dfrac{\mu_0 I}{2\pi s} \Rightarrow F = \left(\dfrac{\mu_0 I}{2\pi s}\right) Ia$ (up).

At the top, $B = \dfrac{\mu_0 I}{2\pi(s+a)} \Rightarrow F = \left(\dfrac{\mu_0 I}{2\pi(s+a)}\right) Ia$ (down).

The net force is $\dfrac{\mu_0 I^2 a^2}{2\pi s(s+a)}$ (up).

(b) The force on the bottom is the same as before, $\dfrac{\mu_0 I^2 a}{2\pi s}$ (up).

On the left side,

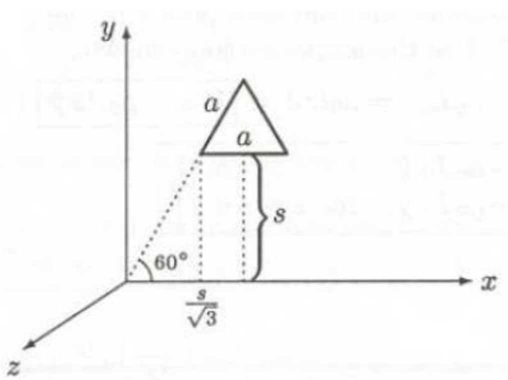
$$d\mathbf{F} = I(d\mathbf{l} \times \mathbf{B}) = I(dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dx\,\hat{\mathbf{z}}) \times \left(\frac{\mu_0 I}{2\pi y}\hat{\mathbf{z}}\right) = \frac{\mu_0 I^2}{2\pi y}(-dx\,\hat{\mathbf{y}} + dy\,\hat{\mathbf{x}}).$$

But the x component cancels the corresponding term from the right side, and

$$F_y = -\frac{\mu_0 I^2}{2\pi}\int_{s/\sqrt{3}}^{s/\sqrt{3}+a/2} \frac{1}{y}dx = -\frac{\mu_0 I^2}{2\sqrt{3}\pi}\ln\left(1+\frac{\sqrt{3}a}{2s}\right), \text{ where } y = \sqrt{3}x.$$

The force on the right side is the same, so the net force on the triangle is

$$\frac{\mu_0 I^2}{2\pi}\left[1 - \frac{2}{\sqrt{3}}\ln\left(1+\frac{\sqrt{3}a}{2s}\right)\right].$$



Problem 2.

A long cylindrical conductor of radius *R* has a cylindrical cavity of radius *R*’ cut out of its volume; the axis of the cavity is parallel to the central axis but displaced a distance *h* to one side. The conductor carries a current *I*, uniformly distributed over its remaining volume. Find the magnetic field in the cavity by Ampere’s law.

Solution 2

If there is no cavity, the magnetic field in the conductor is

$$\oint \mathbf{B}_1 \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

$$2\pi s B_1 = \mu_0 \pi s^2 J$$

$$\therefore \quad \mathbf{B}_1 = \frac{1}{2}\mu_0 s J \hat{\phi}$$

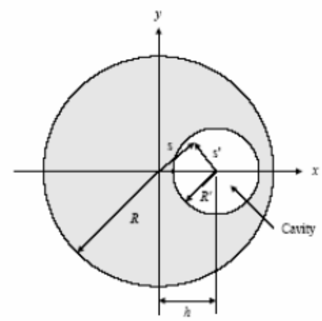


Figure 1

Now, the cavity can be treated as with a current $J\hat{\mathbf{z}}$ and current $-J\hat{\mathbf{z}}$. The former together with the current in the rest of the conductor gives the field above. The latter give a field

$$\mathbf{B}_2 = -\frac{1}{2}\mu_0 s' J \hat{\phi}'$$

where s' is the distance from the axis of the cavity (Figure 1).
The net magnetic field is the sum of the above two field by superposition principle.

$$\begin{aligned}
\mathbf{B} &= \mathbf{B}_1 + \mathbf{B}_2 \\
&= \frac{1}{2}\mu_0 J \left(s\hat{\phi} - s'\hat{\phi}' \right) \\
&= \frac{1}{2}\mu_0 J \left\{ s\left(\hat{\mathbf{z}} \times \hat{\mathbf{s}}\right) - s'\left(\hat{\mathbf{z}} \times \hat{\mathbf{s}}'\right) \right\} \\
&= \frac{1}{2}\mu_0 J\hat{\mathbf{z}} \times \left(\mathbf{s} - \mathbf{s}'\right) \\
&= \frac{1}{2}\mu_0 J\hat{\mathbf{z}} \times h\hat{\mathbf{x}}
\end{aligned}$$

$$\boxed{\mathbf{B} = \frac{1}{2}\mu_0 J h \hat{\mathbf{y}}} \text{ (inside cavity)}$$

where $J = \dfrac{I}{\pi\left(R^2 - R'^2\right)}$. So, the magnetic field inside the cavity is uniform.

Problem 3.

Consider a very large flat slab of thickness *D*. Use Cartesian coordinates so that the slab is parallel to the *xy* plane, with its thickness extending from *z* = - *D*/2 to *z* = *D*/2. The slab carries a uniform volume current density **J** flowing in the *x* direction. Find the **B** field at a distance *z* from the *xy* plane by Ampere's law.

Solution 3

By Biot-Savart law, the field should have no x component.
Rotation by π about the z axis reverses the direction of current and hence the field. But this should not change the z component of the field. Therefore, the z

component must be zero.

Translational symmetry along any direction on the plane implies the field must be independent of x, y.

Hence $\quad \mathbf{B} = B(z)\hat{\mathbf{y}}$.

Besides, rotation by π about the x-axis leaves the current unchanged. Therefore the field must also be the same. This implies the field at the same distance above and below the xy plane must have the same magnitude but pointing in opposite directions:

$$B(-z) = -B(z)$$

Consider a rectangular unit-length Amperian loop extending equal distances above and below the plane, with its surface pointing in the x direction.

Ampere's law implies

$$-B(z) + B(-z) = \mu_0 I_{\text{enc}} \Rightarrow B(z) = -\frac{\mu_0 I_{\text{enc}}}{2}$$

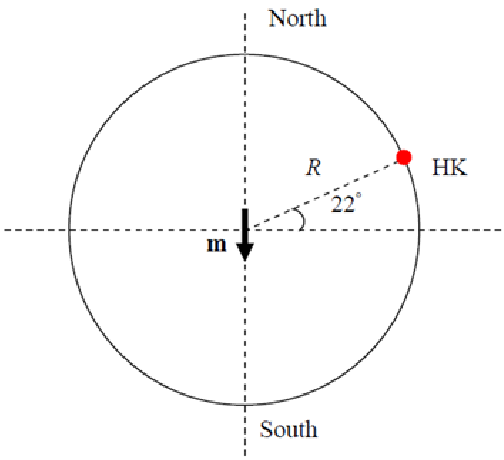
For $0 \le z \le D/2$, $I_{\text{enc}} = J \times 2z$ $\rightarrow$ $B(z) = -\mu_0 Jz$.

For $z > D/2$, $I_{\text{enc}} = J \times D$ $\rightarrow$ $B(z) = -\dfrac{\mu_0 JD}{2}$.

$$\text{Hence,}\quad \mathbf{B} = \begin{cases} -\dfrac{\mu_0 JD}{2}\hat{\mathbf{y}} & \text{for } z > D/2 \\ -\mu_0 Jz\,\hat{\mathbf{y}} & \text{for } D/2 \ge z \ge -D/2 \\ \dfrac{\mu_0 JD}{2}\hat{\mathbf{y}} & \text{for } z < -D/2 \end{cases}$$

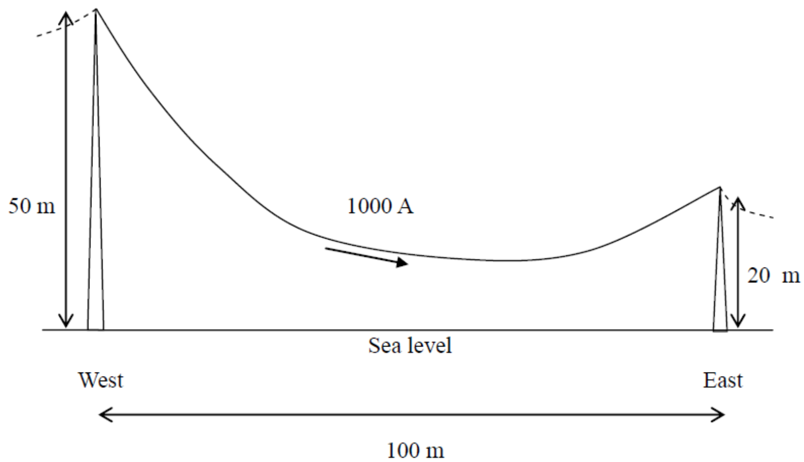
Problem 4.

The Earth's magnetic field is approximately a dipole field. One can imagine there is a pure magnetic dipole **m** at the center of the Earth. For simplicity, we ignore the fact that the magnetic poles do not coincide with the geographic poles, and assume that **m** lies on the rotation axis of the Earth, pointing from north to south, as shown in the figure below.



(a) Assume that $m = 8\times10^{22}$ Am^2, find the Earth's magnetic field at Hong Kong, of which the latitude is assumed to be 22° N exactly. We shall also assume that the Earth is a perfect sphere with a radius of R = 6400 km exactly. Use spherical coordinates with **m** at the origin and pointing at the negative z direction. Express your answer in terms of $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$, and use the unit of gauss, where 1 gauss $= 10^{-4}$ tesla.

(b) A wire between two towers in Hong Kong carries a constant current of 1000 A. The length of the wire is 120 m. The horizontal distance of the two towers is 100 m, and the heights of the two towers are 50 m and 20 m. The higher tower is at the west of the lower one, as shown in the figure below.



Find the magnitude of the total magnetic force acting on the wire due to the Earth's magnetic field.

Solution 4

(a)

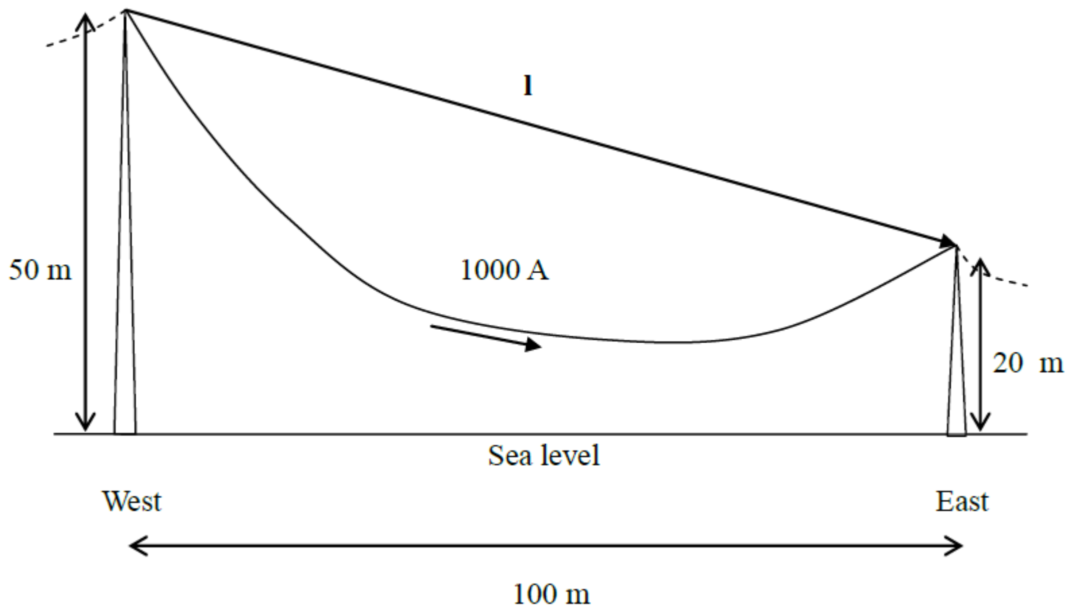
$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}\right] \\
&= \frac{\mu_0}{4\pi}\frac{1}{R^3}\left[3\left((-m\hat{\mathbf{z}})\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}-(-m\hat{\mathbf{z}})\right] \\
&= \frac{\mu_0}{4\pi}\frac{m}{R^3}\left[-3\cos 68^\circ\,\hat{\mathbf{r}}+\left(\cos 68^\circ\,\hat{\mathbf{r}}-\sin 68^\circ\,\hat{\boldsymbol{\theta}}\right)\right] \\
&= \frac{\mu_0}{4\pi}\frac{m}{R^3}\left[-2\cos 68^\circ\,\hat{\mathbf{r}}-\sin 68^\circ\,\hat{\boldsymbol{\theta}}\right] \\
&\approx -0.23\hat{\mathbf{r}}-0.28\hat{\boldsymbol{\theta}}
\end{aligned}$$

in gauss. (3)

(b) The magnetic field can be assumed uniform locally. Hence the force is

$$\mathbf{F} = \int I d\mathbf{l} \times \mathbf{B} = I\left(\int d\mathbf{l}\right) \times \mathbf{B} = I\mathbf{l} \times \mathbf{B},$$

where $\mathbf{l} = \int d\mathbf{l}$ is the vector pointing from the top of the higher tower to that of the lower tower, as shown in the figure below. (3)



The Earth's magnetic field is $\mathbf{B} \approx -0.23\hat{\mathbf{r}} - 0.28\hat{\boldsymbol{\theta}}$ gauss.
Here $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are the vertically upward and south direction, respectively. (2)

The east direction is $\hat{\boldsymbol{\phi}}$. Hence $\mathbf{l} = -30\hat{\mathbf{r}} + 100\hat{\boldsymbol{\phi}}$ in meter. (2)

Therefore,

$$\begin{aligned}\mathbf{F} &= 1000\left(-30\hat{\mathbf{r}} + 100\hat{\boldsymbol{\phi}}\right) \times \left(-0.23\hat{\mathbf{r}} - 0.28\hat{\boldsymbol{\theta}}\right) \times 10^{-4} \\ &= 0.1\left(8.4\hat{\boldsymbol{\phi}} - 23\hat{\boldsymbol{\theta}} + 28\hat{\mathbf{r}}\right) \qquad (2) \\ &= 2.8\hat{\mathbf{r}} - 2.3\hat{\boldsymbol{\theta}} + 0.84\hat{\boldsymbol{\phi}}\end{aligned}$$

The magnitude of the force is $F = \sqrt{2.8^2 + 2.3^2 + 0.84^2} = 3.7\,\mathrm{N}$. (1)