PHYS 3033/3053 Assignment 5

Due: 23 Oct 2015 at begin of lecture at 3:00 pm

Problem 1
When a neutral dielectric material is being polarized, charges move a bit, but the *total charge* in the material remains zero. This fact should be reflected in the bound charges σ_b and ρ_b. Given that $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$ and $\rho_b = -\nabla\cdot\mathbf{P}$, prove that the total bound charge vanishes.

Solution 1:
Let the volume occupied by the neutral dielectric to be *V* and its boundary surface be *A*, then the total bound charge is:

$$Q_b = \int_V \rho_b(\mathbf{r}')d\tau' + \int_A \sigma_b(\mathbf{r}')da'$$

Now let $\hat{\mathbf{n}}$ be the unit normal vector of a surface area $d\mathbf{a}$ of *A*, then,

$$Q_b = \int_V \rho_b(\mathbf{r}')d\tau' + \int_A \sigma_b(\mathbf{r}')da' = -\int_V \nabla\cdot\mathbf{P}(\mathbf{r}')d\tau' + \int_A \mathbf{P}(\mathbf{r}')\cdot\hat{\mathbf{n}}da' = -\int_A \mathbf{P}(\mathbf{r}')\cdot d\mathbf{a}' + \int_A \mathbf{P}(\mathbf{r}')\cdot d\mathbf{a}' = 0$$

Problem 2
A thick spherical shell (inner radius *a*, outer radius *b*, see Figure 1) is made of the dielectric material with a "frozen-in" polarization

$$\mathbf{P}(\mathbf{r}) = \frac{k}{r^2}\hat{\mathbf{r}}$$

where *k* is a constant and *r* is the distance from the center (figure below). (There is no *free* charge in the problem.) Find the electric field in all three regions by two different methods:

(a) Locate the bound charge, and use Gauss's law to calculate the field it produces

(b) Find **D**, and then get **E** from **D**. [Notice that the second method is much faster, and avoids any explicit reference to the bound charges.]

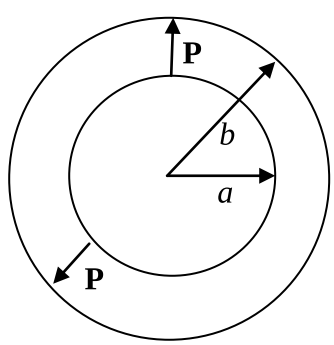

Figure 1

Solution 2

Method 1:

It is unnecessary to calculate the bound charges here. It is enough to know there is ρ_b between $a<r<b$ and σ_b at $r=a$ and $r=b$. But we may as well calculate them out.

$$\rho_b = -\nabla\cdot\mathbf{P} = -\nabla\cdot\left(\frac{k}{r^2}\hat{\mathbf{r}}\right) = 0 \text{ for } a<r<b$$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \frac{k}{b^2}\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \frac{k}{b^2} \text{ at } r=b \text{ and } \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \frac{k}{a^2}\hat{\mathbf{r}}\cdot(-\hat{\mathbf{r}}) = -\frac{k}{a^2} \text{ at } r=a$$

We consider a Gaussian Surface with different r of the three regions.
For $r<a$, no charge can be enclosed by our Gaussian Surface, so

$$\mathbf{E}_{r<a}(\mathbf{r}) = 0$$

Without calculating the exact bound charge

For $a\le r<b$, we have both ρ_b and σ_b at $r=a$ being enclosed, with spherical symmetry,

$$\mathbf{E}_{a\le r<b}(\mathbf{r}) = \frac{\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}\int_{a\le r'<b}\left(-\nabla'\cdot\mathbf{P}\right)d\tau' = \frac{-\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}\int_{r'=r}\mathbf{P}\cdot d\mathbf{a}' = \frac{-\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}4\pi r^2\mathbf{P}(r)\cdot\hat{\mathbf{r}}$$

And so

$$\mathbf{E}_{a\le r<b}(\mathbf{r}) = \frac{-\hat{\mathbf{r}}}{\varepsilon_0 r^2}\left[r^2\frac{k}{r^2}\hat{\mathbf{r}}\right]\cdot\hat{\mathbf{r}} = -\frac{k}{\varepsilon_0 r^2}\hat{\mathbf{r}}$$

For $r\ge b$, $\mathbf{P}$ outside is zero and so the surface integral vanishes, then

$$\mathbf{E}_{b\le r}(\mathbf{r}) = 0$$

With calculation of the bound charges

For $r<b$

$$\mathbf{E}_{a\le r<b}(\mathbf{r}) = \frac{\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}\left(\int_{a\le r'<b}\rho_b d\tau' + \int_{r'=a}\sigma_b da'\right) = \frac{-\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}\left(0-\frac{k}{a^2}4\pi a^2\right) = -\frac{k}{\varepsilon_0 r^2}\hat{\mathbf{r}}$$

For $r>b$, we need to include the surface bound charge, i.e. $\mathbf{E}_{a\le r<b}(\mathbf{r}) = \dfrac{\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}Q_{total}$

$$Q_{total} = \int_{sphere} \rho_b d\tau' + \int_{r'=a} \sigma_b da' + \int_{r'=b} \sigma_b da' = 0 - 4\pi k + 4\pi k = 0$$

So: $\mathbf{E}_{b \le r}(\mathbf{r}) = 0$

Method 2:

With **D**, $\int_A \mathbf{D} \cdot d\mathbf{a} = 0$ for all A as there is no free charge and with spherical symmetry, $\mathbf{D} = 0$ everywhere. So

For $r < a$, $\mathbf{D} = \varepsilon_0 \mathbf{E} = 0 \Rightarrow \mathbf{E} = 0$

For $a \le r < b$, $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = 0 \Rightarrow \mathbf{E} = -\dfrac{\mathbf{P}}{\varepsilon_0} = -\dfrac{k}{\varepsilon_0 r^2}\hat{\mathbf{r}}$

For $r \ge b$, $\mathbf{D} = \varepsilon_0 \mathbf{E} = 0 \Rightarrow \mathbf{E} = 0$

Problem 3

Consider a very large slab with thickness t, located in the region $0 < z < t$, as shown in Figure 2. The slab is uniformly polarized with polarization **P** making an angle of 45° with the positive z axis, as shown in the figure. Find the electric field inside the slab at a point (x, y, z), where $0 < z < t$.

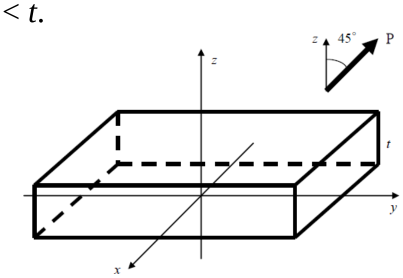

Figure 2

Solution 3:

The system is equivalent to some volume bound charge and surface bound charge.

The volume bound charge density is $\rho_b = -\nabla \cdot \mathbf{P} = 0$ since $\mathbf{P}$ is uniform inside the slab and there is no polarization outside.

The surface bound charge density on the top surface is

$$\sigma_b^+ = \mathbf{P} \cdot \hat{\mathbf{z}} = P \cos 45^\circ = P / \sqrt{2} .$$

The surface bound charge density on the bottom surface is

$$\sigma_b^- = \mathbf{P} \cdot (-\hat{\mathbf{z}}) = -P \cos 45^\circ = -P / \sqrt{2} .$$

Hence inside the slab, the field is

$$\mathbf{E} = \frac{\sigma_b^+}{2\varepsilon_0}(-\hat{\mathbf{z}}) + \frac{\sigma_b^-}{2\varepsilon_0}(\hat{\mathbf{z}}) = \frac{\sigma_b^- - \sigma_b^+}{2\varepsilon_0}\hat{\mathbf{z}} = -\frac{P}{\sqrt{2}\varepsilon_0}\hat{\mathbf{z}} .$$