PHYS 3033 Assignment 4

Due: 2 Oct 2015 at begin of lecture at 3:00 pm

Problem 1.

Show that in general the average potential over a spherical surface of radius R is

$$V_{ave} = V_{center} + \frac{Q_{enc}}{4\pi\varepsilon_0 R},$$

where V_{center} is the potential at the center due to all the *external* charges, and Q_{enc} is the total enclosed charge.

Solution 1

Consider a point charge q inside the sphere. The argument is exactly the same as Sect. 3.1.4 of the textbook and the lecture notes, in which the point charge q is outside, except that since $z < R$, $\sqrt{z^2 + R^2 - 2zR} = R - z$, instead of $z - R$. Hence

$$V_{ave} = \frac{q}{4\pi\varepsilon_0}\frac{1}{2zR}\left[(z+R)-(R-z)\right] = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}.$$

By superposition principle, if there are more than one charges inside the sphere, the average potential due to interior charges is $\frac{1}{4\pi\varepsilon_0}\frac{Q_{enc}}{R}$, and the average due to exterior charges is V_{center}, so $V_{ave} = V_{center} + \frac{Q_{enc}}{4\pi\varepsilon_0 R}$.

Problem 2.

In one sentence, justify **Earnshaw's theorem**: *A charged particle cannot be held in a stable equilibrium by electrostatic forces alone*.

Solution 2.

A stable equilibrium is a point of local minimum in the potential energy. Here the potential energy is qV. But we know that Laplace's equation allows no local minima for V.

Problem 3.

Two infinite grounded metal plates lie parallel to the xz plane, one at $y = 0$, the other at $y = a$. The left end, at $x = 0$, is closed off with two infinite strips insulated from each other and from the two infinite plates. One of the strips is from y = 0 to y = a/2 and is held at a constant potential $-V_0$, and the other, from $y = a/2$ to $y = a$, is at potential V_0.

(a) Find the potential inside this "slot."

(b) Determine the surface charge density $\sigma(y)$ on the two strips at $x = 0$.

Solution 3

(a) By using separation of variables, let $V(x, y) = X(x)Y(y)$, then

$$\nabla^2 V = 0 \quad \Rightarrow \quad \frac{X''}{X} + \frac{Y''}{Y} \quad \Rightarrow \quad \begin{cases} X'' = k^2 X \\ Y'' = -k^2 Y \end{cases}$$

In order to make $V = 0$ at $y = 0$ and $y = a$, Y cannot be exponential or linear, then k should be real and positive.

The solutions of X and Y are

$$\begin{cases} X(x) = A_k \exp(kx) + B_k \exp(-kx) \\ Y(y) = C_k \cos ky + D_k \sin ky \end{cases}.$$

To make $V = 0$ at both $y = 0$ and $y = a$,

$$Y(0) = Y(a) = 0 \quad \Rightarrow \quad \begin{cases} C_k = 0 \\ \sin ka = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} C_k = 0 \\ ka = n\pi, \quad n = 1, 2, 3, \ldots \end{cases}$$

For position very far away from the strips, the potential should tend to zero:
$\lim_{x \to \infty} V = \lim_{x \to \infty} X = 0 \quad \Rightarrow \quad A_k = 0$.

Hence the general solution is

$$V = \sum_{n=1}^{\infty} B_n \exp\left(-\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$$

where D_n is absorbed into B_n.

At $x = 0$,

$$V(0,y)=\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi y}{a}\right)$$

Since $V(0,y)=\begin{cases} -V_0 & \text{for} \quad 0<y<a/2 \\ V_0 & \text{for} \quad a/2<y<a \end{cases}$,

$$\begin{aligned}
B_n &= \frac{2}{a}\int_0^a V(0,y')\sin\left(\frac{n\pi y'}{a}\right)dy' = \frac{2}{a}\left(-\int_0^{a/2} V_0 \sin\left(\frac{n\pi y'}{a}\right)dy' + \int_{a/2}^{a} V_0 \sin\left(\frac{n\pi y'}{a}\right)dy'\right) \\
&= \frac{2V_0}{a}\left(\left[\frac{a}{n\pi}\cos\left(\frac{n\pi y'}{a}\right)\right]_0^{a/2} - \left[\frac{a}{n\pi}\cos\left(\frac{n\pi y'}{a}\right)\right]_{a/2}^{a}\right) = \frac{2V_0}{n\pi}\left(-1+2\cos\frac{n\pi}{2}-\cos n\pi\right) \\
&= \begin{cases} -\dfrac{8V_0}{n\pi} & \text{for} \quad n=2,6,10,\ldots \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

The potential inside the slot is

$$\begin{aligned}
V &= -\frac{8V_0}{\pi}\sum_{n=2,6,10,\ldots}^{\infty} \frac{1}{n}\exp\left(-\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right) \\
&= -\frac{4V_0}{\pi}\sum_{p=0}^{\infty}\frac{1}{2p+1}\exp\left(-\frac{(4p+2)\pi x}{a}\right)\sin\left(\frac{(4p+2)\pi y}{a}\right)
\end{aligned}$$

(b) The electric field is

$$\begin{aligned}
\mathbf{E} = -\nabla V &= \frac{4V_0}{\pi}\left(\frac{\partial}{\partial x}\hat{\mathbf{x}}+\frac{\partial}{\partial y}\hat{\mathbf{y}}\right)\left(\sum_{p=0}^{\infty}\frac{1}{2p+1}\exp\left(-\frac{(4p+2)\pi x}{a}\right)\sin\left(\frac{(4p+2)\pi y}{a}\right)\right) \\
&= -\frac{8V_0}{a}\sum_{p=0}^{\infty}\exp\left(-\frac{(4p+2)\pi x}{a}\right)\left(\sin\left(\frac{(4p+2)\pi y}{a}\right)\hat{\mathbf{x}}-\cos\left(\frac{(4p+2)\pi y}{a}\right)\hat{\mathbf{y}}\right)
\end{aligned}$$

At $x=0$, the surface charge density at the two strips is

$$\sigma(y)=\varepsilon_0 E_x = -\frac{8\varepsilon_0 V_0}{a}\sum_{p=0}^{\infty}\sin\left(\frac{(4p+2)\pi y}{a}\right)$$

Problem 4.

A rectangular pipe, running parallel to the *z*-axis (from $-\infty$ to ∞), has three grounded metal sides, at $x = 0$, $x = a$, and $y = 0$. The fourth side, at $y = b$, is maintained at a specified potential $V_0(x)$.

(a) Develop a general formula for the potential within the pipe.

(b) Find the potential explicitly, for the case $V_0(x) = V_0$ (a constant).

Solution 4

(a) Let $V(x, y) = X(x)Y(y)$. Then by using separation of variables,

$$\nabla^2 V = 0 \quad \Rightarrow \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad \Rightarrow \quad Y\frac{d^2 X}{dx^2} + X\frac{d^2 Y}{dy^2} = 0 \quad \Rightarrow \quad \frac{X''}{X} + \frac{Y''}{Y} = 0$$

which implies $\frac{X''}{X}$ and $\frac{Y''}{Y}$ should both be constants, and the sum of the two constants should be zero. Therefore,

$$\begin{cases} X'' = \lambda X \\ Y'' = -\lambda Y \end{cases}.$$

In addition, since both $x = 0$ and $x = a$ are grounded, then *X* must be sinusoidal. Therefore, $\lambda < 0$.

Let $\lambda = -k^2$, where *k* is real. The solutions of *X* and *Y* are

$$\begin{cases} X(x) = A_k \cos ky + B_k \sin ky \\ Y(y) = C_k \exp(kx) + D_k \exp(-kx) \end{cases}.$$

To make $V = 0$ at both $x = 0$ and $x = a$,

$$X(0) = X(a) = 0 \quad \Rightarrow \quad \begin{cases} A_k = 0 \\ \sin ka = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} A_k = 0 \\ ka = n\pi, \quad n = 1, 2, 3, \ldots \end{cases}$$

To make $V = 0$ at $y = 0$,

$$Y(0) = 0 \quad \Rightarrow \quad C_k + D_k = 0$$

Hence the general solution is

$$V=\sum_{n=1}^{\infty} C_n\left[\exp\left(\frac{n\pi y}{a}\right)-\exp\left(\frac{n\pi y}{a}\right)\right]B_n\sin\left(\frac{n\pi x}{a}\right)$$
$$=\sum_{n=1}^{\infty} 2B_nC_n\sinh\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi x}{a}\right)$$
$$=\sum_{n=1}^{\infty} C_n\sinh\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi x}{a}\right)$$

where $2B_n$ is absorbed into C_n.

Since $V=V_0(x)$ at $y=b$,

$$V(x,b)=V_0(x)=\sum_{n=1}^{\infty} C_n\sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi x}{a}\right)$$
$$\Rightarrow\quad \int_0^a V_0(x)\sin\left(\frac{m\pi x}{a}\right)dx=\sum_{n=1}^{\infty} C_n\sinh\left(\frac{n\pi b}{a}\right)\int_0^a\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)dx$$
$$=\frac{a}{2}\sum_{n=1}^{\infty} C_n\sinh\left(\frac{n\pi b}{a}\right)\delta_{mn}=C_m\frac{a}{2}\sinh\left(\frac{m\pi b}{a}\right)$$
$$\Rightarrow\quad C_m=\frac{2}{a\sinh\left(\frac{m\pi b}{a}\right)}\int_0^a V_0(x)\sin\left(\frac{m\pi x}{a}\right)dx$$

The potential within the pipe is

$$V=\frac{2}{a}\sum_{n=1}^{\infty}\left[\int_0^a V_0(x')\sin\left(\frac{n\pi x'}{a}\right)dx'\right]\frac{\sinh\left(\frac{n\pi y}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}\sin\left(\frac{n\pi x}{a}\right)$$

(b) If $V_0(x)=V_0$,

$$\int_0^a V_0(x')\sin\left(\frac{n\pi x'}{a}\right)dx'=V_0\left[-\frac{a}{n\pi}\cos\left(\frac{n\pi x'}{a}\right)\right]_0^a=\begin{cases}\dfrac{2aV_0}{n\pi} & \text{for } n=1,3,5,\dots\\ 0 & \text{for } n=2,4,6,\dots\end{cases}$$

$$\therefore \quad V = \frac{2}{a} \sum_{n=1,3,5,\ldots}^{\infty} \frac{2aV_0}{n\pi} \frac{\sinh\left(\frac{n\pi y}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi x}{a}\right)$$

$$= \frac{4V_0}{\pi} \sum_{\text{odd } n}^{\infty} \frac{1}{n \sinh\left(\frac{n\pi b}{a}\right)} \sinh\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi x}{a}\right)$$

Problem 5

An amount of charge Q has been deposited on an isolated conducting sphere of radius R, and the sphere has been placed in a uniform electric field $\mathbf{E_0}$ in the z direction. What is the potential outside the sphere?

Solution 5

The general solution is

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta).$$

Let us choose $V(R,\theta) = 0$. Hence

$$\sum_{l=0}^{\infty} \left(A_l R^l + \frac{B_l}{R^{l+1}} \right) P_l(\cos\theta) = 0 \quad \Rightarrow \quad B_l = -A_l R^{2l+1}$$

Therefore,

Since we know that when $r \to \infty$, the potential cannot grow faster than r, as a term which varies as r^2 corresponds to an electric field which varies as r, and so on. Hence,

$$A_l = 0 \quad \text{for } l = 2,\ 3,\ 4,\ \ldots$$

Therefore,

$$V(r,\theta) = A_0 \left(1 - \frac{R}{r} \right) + A_1 \left(r - \frac{R^3}{r^2} \right) \cos\theta$$

The term that varies as $1/r$ is non-vanishing, due to the non-zero net charge carried by the conductor. At large distance, the field due to the surface charges on the conducting sphere is dominated by the monopole term, which is a Coulomb potential that varies as $1/r$. In addition, knowing that the net charge is Q, we know that this field should be given by

$$\frac{Q}{4\pi\varepsilon_0}\frac{1}{r}.$$

In other words,

$$A_0 = -\frac{Q}{4\pi\varepsilon_0 R}.$$

We know that the asymptotic behavior of the potential due to the uniform applied field should be

$$-E_0 r\cos\theta,$$

Hence $A_1 = -E_0$.

In conclusion,

$$V(r,\theta) = -\frac{Q}{4\pi\varepsilon_0 R}\left(1-\frac{R}{r}\right) - E_0\left(r-\frac{R^3}{r^2}\right)\cos\theta.$$