Assignment 1 Solution

1. (a) Direction: perpendicular to the contour line and pointing in the direction of increasing H.
   Relative magnitude: Gradient at **a** should have greater magnitude than that at **b**.

   (b) Print the figure on A4 paper. Use a ruler to estimate the distance for the height to increase by, say, 100m. Then use the scale 1cm ←→ 1km to estimate the real distance. $|\nabla H|$ equals increase in H divided by horizontal distance.

   (c) Use a protractor to measure the angle between the gradient at **b** and the north direction. Let it be θ. Then the increase in height required is approximately given by $|\nabla H| dl \cos\theta = 0.05 \times 100 \times \cos\theta$.

2. $$\oint \mathbf{F} \cdot d\mathbf{l} = \iint_{\text{Rectangle}} (\nabla \times \mathbf{F}) \cdot d\mathbf{a}$$

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & xy & 0 \end{vmatrix} \\
&= 0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + (y - 2y)\hat{\mathbf{z}} \\
&= -y\hat{\mathbf{z}}
\end{aligned}$$

$$\begin{aligned}
\oint \mathbf{F} \cdot d\mathbf{l} &= \iint_{\text{Area enclosed}} (\nabla \times \mathbf{F}) \cdot d\mathbf{a} \\
&= \iint_{\text{Area enclosed}} -y\hat{\mathbf{z}} \cdot \hat{\mathbf{z}} da \\
&= \int_0^1 \int_{-1}^1 -y dx dy \\
&= \int_0^1 -2y dy \\
&= -1
\end{aligned}$$

3. (a) Parameterize the path by $\mathbf{r}(t) = \cos(\pi t)\hat{\mathbf{x}} + \sin(\pi t)\hat{\mathbf{y}}$ where $0 \le t \le 1$.

Then $\mathbf{F}(t) = -2\sin(\pi t)\hat{\mathbf{x}}$ and $d\mathbf{l} = -\pi\sin(\pi t)dt\hat{\mathbf{x}} + \pi\cos(\pi t)dt\hat{\mathbf{y}}$.

$$\begin{aligned}
\int \mathbf{F}\cdot d\mathbf{l} &= \int_0^1 (-2\sin(\pi t)\hat{\mathbf{x}}\cdot\left(-\pi\sin(\pi t)dt\hat{\mathbf{x}} + \pi\cos(\pi t)dt\hat{\mathbf{y}}\right) \\
&= \int_0^1 2\pi\sin^2(\pi t)\,dt \\
&= \pi
\end{aligned}$$

(b) Consider the closed loop line integral along the upper half of the unit circle in the counter-clockwise direction, followed by the straight path from (-1,0) to (1,0) on the x axis. However, since $\mathbf{F} = \mathbf{0}$ on the x axis, the closed loop line integral is the same as the line integral evaluated in part (a).

$$\oint \mathbf{F}\cdot d\mathbf{l} = \iint\limits_{\text{Area enclosed}} (\nabla\times\mathbf{F})\cdot d\mathbf{a}$$

$$\begin{aligned}
\nabla\times\mathbf{F} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -2y & 0 & 0 \end{vmatrix} \\
&= 2\hat{\mathbf{z}}
\end{aligned}$$

$$\begin{aligned}
\oint \mathbf{F}\cdot d\mathbf{l} &= \iint\limits_{\text{Area enclosed}} (\nabla\times\mathbf{F})\cdot d\mathbf{a} \\
&= \iint\limits_{\text{Area enclosed}} 2\hat{\mathbf{z}}\cdot\hat{\mathbf{z}}da \\
&= \iint\limits_{\text{Area enclosed}} 2da \\
&= 2\times\frac{\pi}{2} \\
&= \pi
\end{aligned}$$

4. (a)

$$
\begin{aligned}
\mathbf{F} &= xz\hat{\mathbf{x}} + yz\hat{\mathbf{y}} - z^2\hat{\mathbf{z}} \\
&= (r\sin\theta\cos\phi)(r\cos\theta)\left(\sin\theta\cos\phi\hat{\mathbf{r}} + \cos\theta\cos\phi\hat{\boldsymbol{\theta}} - \sin\phi\hat{\boldsymbol{\phi}}\right) \\
&\quad + (r\sin\theta\sin\phi)(r\cos\theta)\left(\sin\theta\sin\phi\hat{\mathbf{r}} + \cos\theta\sin\phi\hat{\boldsymbol{\theta}} + \cos\phi\hat{\boldsymbol{\phi}}\right) \\
&\quad - (r\cos\theta)^2\left(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}\right) \\
&= r^2\left[\left(\sin^2\theta\cos\theta\cos^2\phi + \sin^2\theta\cos\theta\sin^2\phi - \cos^3\theta\right)\hat{\mathbf{r}}\right. \\
&\quad + \left(\sin\theta\cos^2\theta\cos^2\phi + \sin\theta\cos^2\theta\sin^2\phi + \cos^2\theta\sin\theta\right)\hat{\boldsymbol{\theta}} \\
&\quad \left. - (\sin\theta\cos\theta\sin\phi\cos\phi - \sin\theta\cos\theta\sin\phi\cos\phi)\hat{\boldsymbol{\phi}}\right] \\
&= r^2\left[\cos\theta\left(\sin^2\theta - \cos^2\theta\right)\hat{\mathbf{r}} + 2\sin\theta\cos^2\theta\hat{\boldsymbol{\theta}}\right]
\end{aligned}
$$

(b)

$$
\begin{aligned}
\nabla\cdot\mathbf{F} &= \nabla\cdot\left[r^2\cos\theta\left(\sin^2\theta - \cos^2\theta\right)\hat{\mathbf{r}} + 2r^2\sin\theta\cos^2\theta\hat{\boldsymbol{\theta}}\right] \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2r^2\cos\theta\left(\sin^2\theta - \cos^2\theta\right)\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\times 2r^2\sin\theta\cos^2\theta\right) \\
&= \frac{4r^3}{r^2}\cos\theta[\sin^2\theta - \cos^2\theta] + \frac{r^2}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{2}\sin^2 2\theta\right) \\
&= -4r\cos\theta\cos 2\theta + \frac{r}{2\sin\theta}4\sin 2\theta\cos 2\theta \\
&= -4r\cos\theta\cos 2\theta + 4r\cos\theta\cos 2\theta \\
&= 0
\end{aligned}
$$

5. (a) Let $\mathbf{r} = (x, y, z)$ and $\mathbf{r}' = (x', y', z')$. Then

$$
r = |\mathbf{r} - \mathbf{r}'| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}\ .
$$

Hence,

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{1}{r}\right) &= \frac{\partial}{\partial x}\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \\
&= -\frac{1}{2}\frac{2(x-x')}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2}} \\
&= -\frac{x-x'}{r^3}
\end{aligned}
$$

Similarly, $\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = -\frac{y-y'}{r^3}$, $\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -\frac{z-z'}{r^3}$

Therefore,

$$
\begin{aligned}
\nabla\left(\frac{1}{r}\right) &= \hat{\mathbf{x}}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{\mathbf{y}}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{\mathbf{z}}\frac{\partial}{\partial z}\left(\frac{1}{r}\right) \\
&= \hat{\mathbf{x}}\left(-\frac{x-x'}{r^3}\right) + \hat{\mathbf{y}}\left(-\frac{y-y'}{r^3}\right) + \hat{\mathbf{z}}\left(-\frac{z-z'}{r^3}\right) \\
&= -\frac{1}{r^3}\left[\left(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}\right) - \left(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} + z'\hat{\mathbf{z}}\right)\right] \\
&= -\frac{1}{r^3}\left(\mathbf{r} - \mathbf{r}'\right) \\
&= -\frac{\mathbf{r}}{r^3} \\
&= -\frac{\hat{\mathbf{r}}}{r^2}
\end{aligned}
$$

(b) $\nabla^2 \frac{1}{r} = \nabla \cdot \left(\nabla \frac{1}{r}\right) = \nabla \cdot \left(-\frac{\hat{\mathbf{r}}}{r^2}\right) = -\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right)$

Because $\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) = \nabla \cdot \left(\frac{\mathbf{r}}{|\mathbf{r}|^3}\right) = 4\pi\delta^3(\mathbf{r})$.

Hence $\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) = \nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right) = \nabla \cdot \left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right) = 4\pi\delta^3(\mathbf{r}-\mathbf{r}') = 4\pi\delta^3(\mathbf{r})$.

So $\nabla^2 \frac{1}{r} = -4\pi\delta^3(\mathbf{r})$.