Assignment 7 Solution

Problem 1.

Define the surface and contour of the square be A and C respectively, then the force acting on the loop is

$$\mathbf{F} = \oint_C I d\mathbf{l} \times \mathbf{B}(\mathbf{r})$$
$$= I\left[\int_0^\varepsilon dy\hat{\mathbf{y}} \times \mathbf{B}(0, y, 0) + \int_0^\varepsilon dz\hat{\mathbf{z}} \times \mathbf{B}(0, \varepsilon, z) + \int_\varepsilon^0 dy\hat{\mathbf{y}} \times \mathbf{B}(0, y, \varepsilon) + \int_\varepsilon^0 dz\hat{\mathbf{z}} \times \mathbf{B}(0, 0, z)\right]$$
$$= I\left[\int_0^\varepsilon dy\hat{\mathbf{y}} \times \mathbf{B}(0, y, 0) - \int_0^\varepsilon dy\hat{\mathbf{y}} \times \mathbf{B}(0, y, \varepsilon) - \int_0^\varepsilon dz\hat{\mathbf{z}} \times \mathbf{B}(0, 0, z) + \int_0^\varepsilon dz\hat{\mathbf{z}} \times \mathbf{B}(0, \varepsilon, z)\right]$$

Since ε is small, one can expand $\mathbf{B}(0, y, \varepsilon)$ about (0, y, 0):

$$\mathbf{B}(0, y, \varepsilon) = \mathbf{B}(0, y, 0) + \varepsilon \left.\frac{\partial \mathbf{B}}{\partial z}\right|_{(0,y,0)}.$$

Similarly, $$\mathbf{B}(0, \varepsilon, z) = \mathbf{B}(0, 0, z) + \varepsilon \left.\frac{\partial \mathbf{B}}{\partial y}\right|_{(0,0,z)}.$$

Hence,

$$\mathbf{F} \approx I\left\{\int_0^\varepsilon dy\hat{\mathbf{y}} \times \mathbf{B}(0, y, 0) - \int_0^\varepsilon dy\hat{\mathbf{y}} \times \left[\mathbf{B}(0, y, 0) + \varepsilon \left.\frac{\partial \mathbf{B}}{\partial z}\right|_{(0,y,0)}\right]\right.$$
$$\left. + \int_0^\varepsilon dz\hat{\mathbf{z}} \times \left[\mathbf{B}(0, 0, z) + \varepsilon \left.\frac{\partial \mathbf{B}}{\partial y}\right|_{(0,0,z)}\right] - \int_0^\varepsilon dz\hat{\mathbf{z}} \times \mathbf{B}(0, 0, z)\right\}$$
$$= I\varepsilon\left(\int_0^\varepsilon dz\hat{\mathbf{z}} \times \left.\frac{\partial \mathbf{B}}{\partial y}\right|_{(0,0,z)} - \int_0^\varepsilon dy\hat{\mathbf{y}} \times \left.\frac{\partial \mathbf{B}}{\partial z}\right|_{(0,y,0)}\right)$$

Again, since z in $\left.\frac{\partial \mathbf{B}}{\partial y}\right|_{(0,0,z)}$ inside the integral varies from 0 to ε, which is small,

one can expand the derivative about (0, 0, 0):

$$\left.\frac{\partial}{\partial y}\mathbf{B}\right|_{(0,0,z)} = \left.\frac{\partial}{\partial y}\mathbf{B}\right|_{(0,0,0)} + z\left.\frac{\partial}{\partial z}\frac{\partial}{\partial y}\mathbf{B}\right|_{(0,0,0)}$$

Similarly, $\left.\frac{\partial}{\partial z}\mathbf{B}\right|_{(0,y,0)} = \left.\frac{\partial}{\partial z}\mathbf{B}\right|_{(0,0,0)} + y\left.\frac{\partial}{\partial y}\frac{\partial}{\partial z}\mathbf{B}\right|_{(0,0,0)}$.

Hence,

$$\mathbf{F} \approx I\varepsilon\left\{\int_0^\varepsilon dz\hat{\mathbf{z}}\times\left[\left.\frac{\partial\mathbf{B}}{\partial y}\right|_{(0,0,0)} + z\left.\frac{\partial^2\mathbf{B}}{\partial z\partial y}\right|_{(0,0,0)}\right] - \int_0^\varepsilon dy\hat{\mathbf{y}}\times\left[\left.\frac{\partial\mathbf{B}}{\partial z}\right|_{(0,0,0)} + y\left.\frac{\partial^2\mathbf{B}}{\partial y\partial z}\right|_{(0,0,0)}\right]\right\}$$

Since $\left.\frac{\partial\mathbf{B}}{\partial y}\right|_{(0,0,0)}$ is a constant vector,

$$\int_0^\varepsilon dz\hat{\mathbf{z}}\times\left.\frac{\partial\mathbf{B}}{\partial y}\right|_{(0,0,0)} = \left(\int_0^\varepsilon dz\hat{\mathbf{z}}\right)\times\left.\frac{\partial\mathbf{B}}{\partial y}\right|_{(0,0,0)} = \varepsilon\hat{\mathbf{z}}\times\left.\frac{\partial\mathbf{B}}{\partial y}\right|_{(0,0,0)}$$

Similarly, $\int_0^\varepsilon dy\hat{\mathbf{y}}\times\left.\frac{\partial\mathbf{B}}{\partial z}\right|_{(0,0,0)} = \varepsilon\hat{\mathbf{y}}\times\left.\frac{\partial\mathbf{B}}{\partial z}\right|_{(0,0,0)}$

Because $\left.\frac{\partial^2\mathbf{B}}{\partial z\partial y}\right|_{(0,0,0)}$ and $\left.\frac{\partial^2\mathbf{B}}{\partial y\partial z}\right|_{(0,0,0)}$ are also constant vectors, so

$$\int_0^\varepsilon dz\hat{\mathbf{z}}\times z\left.\frac{\partial^2\mathbf{B}}{\partial z\partial y}\right|_{(0,0,0)} = \int_0^\varepsilon zdz\hat{\mathbf{z}}\times\left.\frac{\partial^2\mathbf{B}}{\partial z\partial y}\right|_{(0,0,0)} = \left(\int_0^\varepsilon zdz\hat{\mathbf{z}}\right)\times\left.\frac{\partial^2\mathbf{B}}{\partial z\partial y}\right|_{(0,0,0)} = \frac{\varepsilon^2}{2}\hat{\mathbf{z}}\times\left.\frac{\partial^2\mathbf{B}}{\partial z\partial y}\right|_{(0,0,0)}$$

Similarly, $\int_0^\varepsilon dy\hat{\mathbf{y}}\times y\left.\frac{\partial^2\mathbf{B}}{\partial y\partial z}\right|_{(0,0,0)} = \frac{\varepsilon^2}{2}\hat{\mathbf{y}}\times\left.\frac{\partial^2\mathbf{B}}{\partial y\partial z}\right|_{(0,0,0)}$.

Therefore, $\mathbf{F} \approx I\varepsilon^2\left[\left(\hat{\mathbf{z}}\times\frac{\partial\mathbf{B}}{\partial y} - \hat{\mathbf{y}}\times\frac{\partial\mathbf{B}}{\partial z}\right) + \frac{1}{2}\varepsilon\left(\hat{\mathbf{z}}\times\frac{\partial^2\mathbf{B}}{\partial z\partial y} - \hat{\mathbf{y}}\times\frac{\partial^2\mathbf{B}}{\partial y\partial z}\right)\right]_{(0,0,0)}$

To leading order in ε, the term involving second derivatives can be ignored. In the first term,

$$\hat{\mathbf{z}}\times\frac{\partial\mathbf{B}}{\partial y}\bigg|_{(0,0,0)} = \hat{\mathbf{z}}\times\left(\hat{\mathbf{x}}\frac{\partial B_x}{\partial y}\bigg|_{(0,0,0)} + \hat{\mathbf{y}}\frac{\partial B_y}{\partial y}\bigg|_{(0,0,0)} + \hat{\mathbf{z}}\frac{\partial B_z}{\partial y}\bigg|_{(0,0,0)}\right)$$

$$= \hat{\mathbf{y}}\frac{\partial B_x}{\partial y}\bigg|_{(0,0,0)} - \hat{\mathbf{x}}\frac{\partial B_y}{\partial y}\bigg|_{(0,0,0)}$$

Similarly, $\hat{\mathbf{y}}\times\frac{\partial\mathbf{B}}{\partial z}\bigg|_{(0,0,0)} = -\hat{\mathbf{z}}\frac{\partial B_x}{\partial z}\bigg|_{(0,0,0)} + \hat{\mathbf{x}}\frac{\partial B_z}{\partial z}\bigg|_{(0,0,0)}$.

So

$$\left(\hat{\mathbf{z}}\times\frac{\partial\mathbf{B}}{\partial y} - \hat{\mathbf{y}}\times\frac{\partial\mathbf{B}}{\partial z}\right)_{(0,0,0)} = \hat{\mathbf{y}}\frac{\partial B_x}{\partial y}\bigg|_{(0,0,0)} - \hat{\mathbf{x}}\frac{\partial B_y}{\partial y}\bigg|_{(0,0,0)} + \hat{\mathbf{z}}\frac{\partial B_x}{\partial z}\bigg|_{(0,0,0)} - \hat{\mathbf{x}}\frac{\partial B_z}{\partial z}\bigg|_{(0,0,0)}$$

$$= \hat{\mathbf{y}}\frac{\partial B_x}{\partial y}\bigg|_{(0,0,0)} + \hat{\mathbf{z}}\frac{\partial B_x}{\partial z}\bigg|_{(0,0,0)} - \hat{\mathbf{x}}\left(\frac{\partial B_y}{\partial y}\bigg|_{(0,0,0)} + \frac{\partial B_z}{\partial z}\bigg|_{(0,0,0)}\right)$$

$$= \hat{\mathbf{x}}\frac{\partial B_x}{\partial x}\bigg|_{(0,0,0)} + \hat{\mathbf{y}}\frac{\partial B_x}{\partial y}\bigg|_{(0,0,0)} + \hat{\mathbf{z}}\frac{\partial B_x}{\partial z}\bigg|_{(0,0,0)}$$

$$- \hat{\mathbf{x}}\left(\frac{\partial B_x}{\partial x}\bigg|_{(0,0,0)} + \frac{\partial B_y}{\partial y}\bigg|_{(0,0,0)} + \frac{\partial B_z}{\partial z}\bigg|_{(0,0,0)}\right)$$

$$= \nabla(B_x)_{(0,0,0)} - \hat{\mathbf{x}}(\nabla\cdot\mathbf{B})_{(0,0,0)}$$

$$= \nabla(\hat{\mathbf{x}}\cdot\mathbf{B})_{(0,0,0)}$$

The force is therefore, $\mathbf{F} = I\varepsilon^2\nabla(\hat{\mathbf{x}}\cdot\mathbf{B})_{(0,0,0)} = \nabla\left(I\varepsilon^2\hat{\mathbf{x}}\cdot\mathbf{B}\right)_{(0,0,0)} = \nabla(\mathbf{m}\cdot\mathbf{B})_{(0,0,0)}$

Problem 2.

Choose coordinates such that both dipoles point towards $+x$ direction. Put $\mathbf{m}_1$ and $\mathbf{m}_2$ at $(x, y, z) = (0, 0, 0)$ and $(r, 0, 0)$ respectively, the force acting by $\mathbf{m}_1$ on $\mathbf{m}_2$ is

$$
\begin{aligned}
\mathbf{F}_{12} &= \nabla\left(\mathbf{m}_2 \cdot \mathbf{B}_1\right) \\
&= \nabla\left(\mathbf{m}_2 \cdot \frac{\mu_0}{4\pi r^3}\left(\frac{3}{r^2}\left(\mathbf{m}_1 \cdot \mathbf{r}_{12}\right)\mathbf{r}_{12} - \mathbf{m}_1\right)\right) \\
&= \frac{\mu_0}{4\pi}\nabla\left(m_2\hat{\mathbf{x}} \cdot \frac{1}{r^3}\left(\frac{3}{r^2}\left(m_1 x\right)\left(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}\right) - m_1\hat{\mathbf{x}}\right)\right) \\
&= \frac{\mu_0}{4\pi}\nabla\left(m_1 m_2\left(\frac{3x^2}{r^5} - \frac{1}{r^3}\right)\right) \\
&= \frac{\mu_0 m_1 m_2}{4\pi}\nabla\frac{2x^2 - y^2 - z^2}{r^5} \\
&= \frac{\mu_0 m_1 m_2}{4\pi}\left[\frac{1}{r^5}\nabla\left(2x^2 - y^2 - z^2\right) + \left(2x^2 - y^2 - z^2\right)\nabla\frac{1}{r^5}\right] \\
&= \frac{\mu_0 m_1 m_2}{4\pi}\left[\frac{1}{r^5}\left(4x\hat{\mathbf{x}} - 2y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}}\right) + \left(2x^2 - y^2 - z^2\right)\frac{-5\mathbf{r}}{r^7}\right]
\end{aligned}
$$

At the position of $\mathbf{m}_2$, $y = z = 0$, $x = r$,

$$
\mathbf{F}_{12} = \frac{\mu_0 m_1 m_2}{4\pi}\left[\frac{1}{r^5}\left(4r\hat{\mathbf{x}}\right) + \left(2r^2\right)\frac{-5r\hat{\mathbf{x}}}{r^7}\right] = \frac{\mu_0 m_1 m_2}{4\pi}\left[\frac{4}{r^4}\hat{\mathbf{x}} - \frac{10}{r^4}\hat{\mathbf{x}}\right] = -\frac{3\mu_0 m_1 m_2}{2\pi r^4}\hat{\mathbf{x}}
$$

Similarly, Put $\mathbf{m}_1$ and $\mathbf{m}_2$ at $(x, y, z) = (-r, 0, 0)$ and $(0, 0, 0)$ respectively, the force acting by $\mathbf{m}_2$ on $\mathbf{m}_1$ is

$$
\begin{aligned}
\mathbf{F}_{21} &= \nabla\left(\mathbf{m}_1 \cdot \mathbf{B}_2\right) \\
&= \nabla\left(\mathbf{m}_1 \cdot \frac{\mu_0}{4\pi r^3}\left(\frac{3}{r^2}\left(\mathbf{m}_2 \cdot \mathbf{r}_{21}\right)\mathbf{r}_{21} - \mathbf{m}_2\right)\right) \\
&= \frac{\mu_0}{4\pi}\nabla\left(m_1\hat{\mathbf{x}} \cdot \frac{1}{r^3}\left(\frac{3}{r^2}\left(m_2 x\right)\left(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}\right) - m_2\hat{\mathbf{x}}\right)\right) \\
&= \frac{\mu_0 m_1 m_2}{4\pi}\left[\frac{1}{r^5}\left(4x\hat{\mathbf{x}} - 2y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}}\right) + \left(2x^2 - y^2 - z^2\right)\frac{-5\mathbf{r}}{r^7}\right]
\end{aligned}
$$

At the position of $\mathbf{m}_2$, $y = z = 0$, $x = -r$,

$$
\mathbf{F}_{21} = \frac{\mu_0 m_1 m_2}{4\pi}\left[\frac{1}{r^5}\left(-4r\hat{\mathbf{x}}\right) + \left(2r^2\right)\frac{5r\hat{\mathbf{x}}}{r^7}\right] = \frac{3\mu_0 m_1 m_2}{2\pi r^4}\hat{\mathbf{x}}
$$

$\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ are action and reaction pair. The force between the dipoles is attractive.

Problem 3.

Let $\mathbf{m}_1$ and $\mathbf{m}_2$ be the magnetic dipole moments of the circular and square loops respectively. Choose coordinates such that points $\mathbf{m}_1$ towards +*z* direction and $\mathbf{m}_2$ points towards +*x* direction respectively. Then $\mathbf{m}_1 = I\pi a^2\hat{\mathbf{z}}, \mathbf{m}_2 = Ib^2\hat{\mathbf{x}}$.

(a) The torque acting by $\mathbf{m}_1$ on $\mathbf{m}_2$ is

$$\begin{aligned}
\mathbf{N}_{12} &= \mathbf{m}_2 \times \mathbf{B}_1 = \mathbf{m}_2 \times \frac{\mu_0}{4\pi r^3}\left(3\left(\mathbf{m}_1 \cdot \hat{\mathbf{r}}_{12}\right)\hat{\mathbf{r}}_{12} - \vec{\mathbf{m}}_1\right) \\
&= Ib^2\hat{\mathbf{x}} \times \frac{\mu_0}{4\pi r^3}\left(3\left(I\pi a^2\hat{\mathbf{z}} \cdot \hat{\mathbf{x}}\right)\hat{\mathbf{x}} - I\pi a^2\hat{\mathbf{z}}\right) \\
&= Ib^2\hat{\mathbf{x}} \times -\frac{\mu_0 I\pi a^2}{4\pi r^3}\hat{\mathbf{z}} = \frac{\mu_0 I^2 a^2 b^2}{4r^3}\hat{\mathbf{y}}
\end{aligned}$$

The torque acting by $\mathbf{m}_2$ on $\mathbf{m}_1$ is

$$\begin{aligned}
N_{21} &= \mathbf{m}_1 \times \mathbf{B}_2 = \mathbf{m}_1 \times \frac{\mu_0}{4\pi r^3}\left(3\left(\mathbf{m}_2 \cdot \hat{\mathbf{r}}_{21}\right)\hat{\mathbf{r}}_{21} - \mathbf{m}_2\right) \\
&= I\pi a^2\hat{\mathbf{z}} \times \frac{\mu_0}{4\pi r^3}\left(3\left(Ib^2\hat{\mathbf{x}} \cdot \left(-\hat{\mathbf{x}}\right)\right)\left(-\hat{\mathbf{x}}\right) - Ib^2\hat{\mathbf{x}}\right) \\
&= I\pi a^2\hat{\mathbf{z}} \times \frac{2\mu_0 Ib^2}{4\pi r^3}\hat{\mathbf{x}} = \frac{\mu_0 I^2 a^2 b^2}{2r^3}\hat{\mathbf{y}}
\end{aligned}$$

(b) Assume $\mathbf{m}_1$ is fixed in +*z* direction and $\mathbf{m}_2$ is free to rotate on the *x*-*z* plane. The interaction energy between the two dipoles is

$$\begin{aligned}
U &= -\mathbf{m}_2 \cdot \mathbf{B}_1 = -\mathbf{m}_2 \cdot \frac{\mu_0}{4\pi r^3}\left(3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_1\right) \\
&= -\frac{\mu_0}{4\pi r^3}\left(3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_2 \cdot \hat{\mathbf{r}}) - \mathbf{m}_1 \cdot \mathbf{m}_2\right) \\
&= -\frac{\mu_0}{4\pi r^3}\left(3(m_1\hat{\mathbf{z}} \cdot \hat{\mathbf{x}})(\mathbf{m}_2 \cdot \hat{\mathbf{x}}) - \mathbf{m}_1 \cdot \mathbf{m}_2\right) \\
&= -\frac{\mu_0 m_1 m_2}{4\pi r^3}\left(0 - \cos\theta\right) = \frac{\mu_0 m_1 m_2}{4\pi r^3}\cos\theta
\end{aligned}$$

where θ is the angle between $\mathbf{m}_1$ and $\mathbf{m}_2$. Obviously U is a minimum when $\theta = \pi$. Hence the square loop has its equilibrium orientation when it points towards –*z* direction.

Problem 4.

Volume bound current density $\mathbf{J}_b = \nabla \times \mathbf{M} = \frac{1}{s}\frac{\partial}{\partial s}\left(sks^2\right)\hat{\mathbf{z}} = \frac{1}{s}\frac{\partial}{\partial s}\left(ks^3\right)\hat{\mathbf{z}} = 3ks\hat{\mathbf{z}}$

Surface bound current density $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = ks^2\hat{\phi} \times \hat{\mathbf{s}} = -ks^2\hat{\mathbf{z}} = -kR^2\hat{\mathbf{z}}$

Consider a circular Ampere loop inside the cylinder with radius $s < R$ and it has the same axis as the cylinder. By Ampere's law,

$$\begin{aligned}
&\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \\
\Rightarrow \quad & 2\pi sB = \mu_0 \int_S \mathbf{J}_b \cdot d\mathbf{a} = \mu_0 \int_0^{2\pi}\int_0^s 3ks\hat{\mathbf{z}} \cdot \hat{\mathbf{z}} s ds d\phi \\
&\qquad = 6\mu_0 \pi k \int_0^s s^2 ds = 6\mu_0 \pi k \left[\frac{s^3}{3}\right]_0^s = 2\mu_0 \pi k s^3 \\
\Rightarrow \quad & \mathbf{B} = \mu_0 k s^2 \hat{\phi}
\end{aligned}$$

Consider another circular Ampere loop surrounding the cylinder with radius $s > R$ and it has the same axis as the cylinder. By Ampere's law,

$$\begin{aligned}
&\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \\
\Rightarrow \quad & 2\pi sB = \mu_0 \left(\int_S \mathbf{J}_b \cdot d\mathbf{a} + \int_C \mathbf{K}_b \cdot d\mathbf{l}\right) = \mu_0 \left(\int_0^{2\pi}\int_0^R 3ks\hat{\mathbf{z}} \cdot \hat{\mathbf{z}} s ds d\phi - 2\pi RkR^2\right) \\
&\qquad = \mu_0 \left(6\pi k \int_0^R s^2 ds - 2\pi kR^3\right) = \mu_0 \left(2\pi kR^3 - 2\pi kR^3\right) = 0 \\
\Rightarrow \quad & \mathbf{B} = \mathbf{0}
\end{aligned}$$

Therefore

$$\mathbf{B} = \begin{cases} \mu_0 k s^2 \hat{\phi} & \text{for } r < R \\ \mathbf{0} & \text{for } r > R \end{cases}$$