PHYS 3033 Assignment 3

Problem 1.

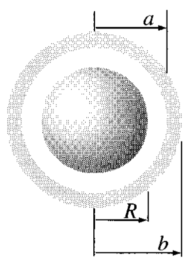
A sphere of radius R carries a charge density $\rho(r) = kr^2$ (where k is a constant). Find the energy of the configuration using

(a) $W = \frac{\varepsilon_0}{2} \int_{\text{all space}} E^2 d\tau$,

(b) $W = \frac{1}{2} \int \rho V d\tau$.

Problem 2.

A metal sphere of radius R, carrying charge q, is surrounded by a thick concentric metal shell (inner radius a, outer radius b, as in the figure below). The shell carries no net charge.



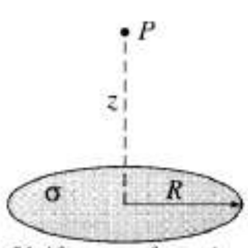
(a) Find the surface charge density σ at R, at a, and at b.

(b) Find the potential at the center, using infinity as the reference point.

(c) Now the outer surface is touched to a grounding wire, which lowers its potential to zero (same as at infinity). How do your answers to (a) and (b) change?

Problem 3.

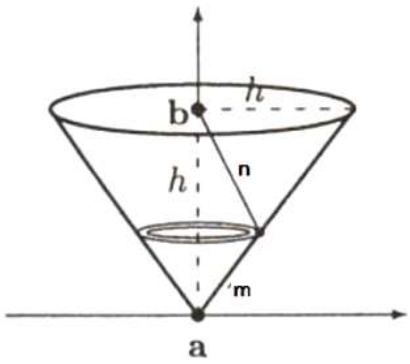
Using $W = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma(r')}{r} da'$, find the potential at a distance z above the center of the charge distribution in the figure shown below. Compute the z-component of the $\mathbf{E}$ field by $\mathbf{E} = -\nabla V$.



Uniform surface charge

Problem 4.

A conical surface (an empty ice-cream cone) carries a uniform surface charge σ. The height of the cone is h and the radius on the top is also h. Find the potential difference between points **a** (the vertex) and **b** (the center of the top).



<u>Solutions:</u>

1) a)

Using Gauss's Law,

For $r < R$,

$$\oint \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\varepsilon_0}Q_{enc} \Rightarrow E\left(4\pi r^2\right) = \frac{1}{\varepsilon_0}\int_0^{\pi}\int_0^{2\pi}\int_0^{r}\rho\left(r'\right)r'^2\sin\theta dr'd\theta d\phi = \frac{4\pi k}{\varepsilon_0}\frac{r^5}{5}$$

$$\Rightarrow \mathbf{E} = \frac{kr^3}{5\varepsilon_0}\hat{\mathbf{r}}$$

For $r > R$,

$$\oint \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\varepsilon_0}Q_{enc} \Rightarrow E\left(4\pi r^2\right) = \frac{1}{\varepsilon_0}\int_0^{\pi}\int_0^{2\pi}\int_0^{R}\rho\left(r'\right)r'^2\sin\theta dr'd\theta d\phi = \frac{4\pi k}{\varepsilon_0}\frac{R^5}{5}$$

$$\Rightarrow \mathbf{E} = \frac{kR^5}{5\varepsilon_0 r^2}\hat{\mathbf{r}}$$

So,

$$W = \frac{\varepsilon_0}{2}\int_{all\ space} E^2 d\tau = \frac{\varepsilon_0}{2}\int_0^R \left(\frac{kr^3}{5\varepsilon_0}\right)^2 4\pi r^2 dr + \frac{\varepsilon_0}{2}\int_R^\infty \left(\frac{kR^5}{5\varepsilon_0 r^2}\right)^2 4\pi r^2 dr$$

$$= 2\pi\varepsilon_0 \left(\frac{k}{5\varepsilon_0}\right)^2 \left\{\int_0^R r^8 dr + R^{10}\int_R^\infty \frac{1}{r^2} dr\right\} = \frac{2\pi k^2}{25\varepsilon_0}\left(\frac{R^9}{9} + R^9\right) = \frac{4\pi k^2 R^9}{45\varepsilon_0} \ .$$

(b) The potential at $r < R$ is

$$V(r) = -\int_\infty^r \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^R \left(\frac{kR^5}{5\varepsilon_0 r^2}\right) dr - \int_R^r \left(\frac{kr^3}{5\varepsilon_0}\right) dr = -\frac{k}{5\varepsilon_0}\left(-R^4 + \frac{r^4}{4} - \frac{R^4}{4}\right)$$

$$= \frac{k}{4\varepsilon_0}\left(R^4 - \frac{r^4}{5}\right)$$

Therefore,

$$W = \frac{1}{2}\int \rho V d\tau = \frac{1}{2}\int_0^R kr^2 \left[\frac{k}{4\varepsilon_0}\left(R^4 - \frac{r^4}{5}\right)\right] 4\pi r^2 dr = \frac{2\pi k^2}{4\varepsilon_0}\int_0^R \left(R^4 r^4 - \frac{r^8}{5}\right) dr$$

$$= \frac{\pi k^2}{2\varepsilon_0}\left[R^4 \times \frac{R^5}{5} - \frac{R^9}{45}\right] = \frac{4\pi k^2 R^9}{45\varepsilon_0}$$

2)

(a) By symmetry, the charge q should be uniformly distributed on the surface of the sphere. Hence $\sigma_R = \frac{q}{4\pi R^2}$.

Consider a spherical Gaussian surface inside the conducting shell just inside the inner surface. Because the E field inside conductor is zero, therefore by Gauss's law, the total amount of charge enclosed by this Gaussian surface should be zero. Hence, the inner surface must carry a total amount of charge of $-q$. By symmetry, these charges must be uniformly distributed, so $\sigma_a = -\frac{q}{4\pi a^2}$.

Because the shell carries no net charge, and there can be no charge inside a conductor, the outer surface must carry an amount of charge of q. Again, by symmetry, the distribution must be uniform, and so $\sigma_R = \frac{q}{4\pi b^2}$.

(b)

$$V(0) = -\int_{\infty}^{0} \mathbf{E}\cdot d\mathbf{l} = -\int_{\infty}^{b}\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)dr - \int_{b}^{a}(0)\,dr - \int_{a}^{R}\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)dr - \int_{R}^{0}(0)\,dr$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{b}+\frac{1}{R}-\frac{1}{a}\right)$$

(c) Now, $\sigma_b = 0$, then

$$V(0) = -\int_{\infty}^{0} \mathbf{E}\cdot d\mathbf{l} = -\int_{\infty}^{a}(0)\,dr - \int_{a}^{R}\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)dr - \int_{R}^{0}(0)\,dr = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{R}-\frac{q}{a}\right)$$

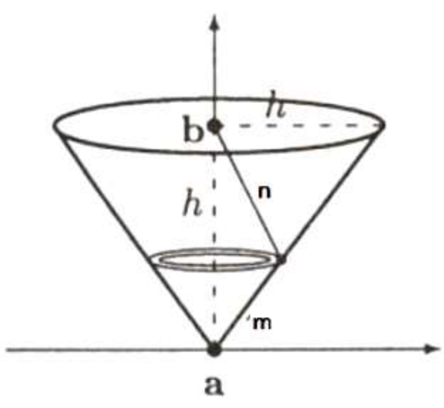
3)

The potential on the axis is $V = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_0^R \frac{\sigma 2\pi r dr}{\sqrt{r^2+z^2}} = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2+z^2}-z\right)$.

The z-component of the electric field along the axis is

$$E_z = -(\nabla V)_z = -\frac{\partial V}{\partial z} = -\frac{\sigma}{2\varepsilon_0}\left(\frac{1}{2}\frac{1}{\sqrt{R^2+z^2}}2z-1\right) = \frac{\sigma}{2\varepsilon_0}\left(1-\frac{1}{\sqrt{R^2+z^2}}z\right)$$

4)



$$V(\mathbf{a}) = \frac{1}{4\pi\varepsilon_0}\int_0^{\sqrt{2}h}\frac{\sigma 2\pi r}{m}dm, \quad (where\ r = m/\sqrt{2})$$

$$= \frac{\sigma}{2\varepsilon_0}\frac{1}{\sqrt{2}}\left(\sqrt{2}h\right) = \frac{\sigma h}{2\varepsilon_0}$$

:

$$V(\mathbf{b}) = \frac{1}{4\pi\varepsilon_0}\int_0^{\sqrt{2}h} \frac{\sigma 2\pi r}{n} dm$$

$$= \frac{1}{4\pi\varepsilon_0}\int_0^{\sqrt{2}h} \frac{\sigma 2\pi r}{\sqrt{h^2+m^2-\sqrt{2}hm}} dm = \frac{\sigma}{\sqrt{2}\varepsilon_0}\int_0^{h} \frac{r}{\sqrt{h^2+2r^2-2rh}} dr$$

$$= \frac{\sigma}{2\varepsilon_0}\int_0^{h} \frac{r}{\sqrt{(r-h/2)^2+h^2/4}} dr = \frac{\sigma}{2\varepsilon_0}\int_0^{h} \frac{(r-h/2)+h/2}{\sqrt{(r-h/2)^2+h^2/4}} dr$$

$$= \frac{\sigma}{2\varepsilon_0}\left(\int_{-h/2}^{h/2} \frac{u}{\sqrt{u^2+h^2/4}} du + \frac{h}{2}\int_{-h/2}^{h/2} \frac{du}{\sqrt{u^2+h^2/4}}\right) \qquad \text{where } u = r - h/2$$

$$= \frac{\sigma}{2\varepsilon_0}\left(0 + \frac{h}{2}\int_{-\pi/4}^{\pi/4} \frac{h\sec^2\theta d\theta/2}{h\sqrt{1+\tan^2\theta}/2}\right) \qquad \text{where } u = h\tan\theta/2$$

$$= \frac{\sigma h}{4\varepsilon_0}\int_{-\pi/4}^{\pi/4} \sec\theta d\theta = \frac{\sigma h}{4\varepsilon_0}\int_{-\pi/4}^{\pi/4} \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta+\tan\theta} d\theta$$

$$= \frac{\sigma h}{4\varepsilon_0}\int_{-\pi/4}^{\pi/4} \frac{d(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta} = \frac{\sigma h}{4\varepsilon_0}\ln\left|\sec\theta+\tan\theta\right|_{-\pi/4}^{\pi/4}$$

$$= \frac{\sigma h}{4\varepsilon_0}\left(\ln\left|\sqrt{2}+1\right| - \ln\left|\sqrt{2}-1\right|\right) = \frac{\sigma h}{4\varepsilon_0}\ln\frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{\sigma h}{4\varepsilon_0}\ln\left(1+\sqrt{2}\right)^2$$

$$= \frac{\sigma h}{2\varepsilon_0}\ln\left(1+\sqrt{2}\right)$$

So, $V(\mathbf{a}) - V(\mathbf{b}) = \frac{\sigma h}{2\varepsilon_0}\left[1-\ln\left(1+\sqrt{2}\right)\right]$.