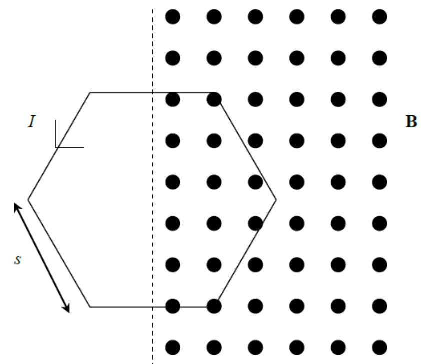
PHYS 3033/3053 Assignment 8

Due: 27 Nov 2015 at begin of lecture at 3:00 pm

Problem 1.
A wire of the shape of a regular hexagon with side length s and carrying uniform current I flowing in the counterclockwise direction has its right half inside a uniform magnetic field **B** pointing outwards, as shown in the figure below:



(a) Find the force experienced by the loop.

(b) The current loop has its own magnetic field. Find the magnitude and direction of this field when observed at a point P at a large distance d ($d >> s$) **on the plane of the loop**. The distance is so large that one can use the multipole expansion and keep only the dipole term.

Solution 1:

(a) Let **L** be the vector joining the two points of intersection of the wire with the boundary of B field, pointing from the lower one to the upper one. Then the length of **L** is $L = \sqrt{3}s$. The force is

$$F = \left|\int I d\mathbf{l} \times \mathbf{B}\right| = |I\mathbf{L} \times \mathbf{B}| = ILB = \sqrt{3}sIB$$

Direction is to the right.

(b) The area of the hexagon is $a = \frac{3\sqrt{3}}{2}s^2$.

$m = Ia = \frac{3\sqrt{3}}{2}Is^2$

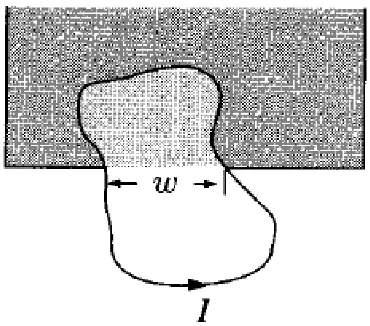
Direction of **m** is out of the page.

Hence $\mathbf{m} \cdot \hat{\mathbf{r}} = 0$.

The dipole field is $\mathbf{B}_{\text{dip}} = -\frac{\mu_0}{4\pi}\frac{\mathbf{m}}{r^3}$

Problem 2.

A plane wire loop of irregular shape is situated so that part of it is in a uniform magnetic field $\mathbf{B}$ (in Fig. 4 the field occupies that shaded region, and points perpendicular to the plane of the loop). The loop carries a current I . Show that the net magnetic force on the loop is $F = IBw$, where w is the chord subtended. Generalize this result to the case where the magnetic field region itself has an irregular shape. What is the direction of the force?



Solution 2.

From $\mathbf{F} = I\int (d\mathbf{l} \times \mathbf{B})$. But $\mathbf{B}$ is constant, in this case, so it comes outside the integral

$\mathbf{F} = I(\int d\mathbf{l}) \times \mathbf{B}$, and $\int d\mathbf{l} = \mathbf{w}$, the vector displacement from the point at which the wire first enters the field to the point where it leaves. Since $\mathbf{w}$ and $\mathbf{B}$ are perpendicular, $F = IBw$, and $\mathbf{F}$ is perpendicular to $\mathbf{w}$.

Problem 3.

For a configuration of charges and currents confined within a volume $\mathcal{V}$, show that

$$\int_{\mathcal{V}} \mathbf{J} d\tau = d\mathbf{p} / dt,$$

Where $\mathbf{p}$ is the total dipole moment. [*Hint:* evaluate $\int_{\mathcal{V}} \nabla \cdot (x\mathbf{J}) d\tau$.]

Solution 3.

$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}\int_v \rho \mathbf{r} d\tau = \int \frac{\partial \rho}{\partial t} \mathbf{r} d\tau = -\int (\nabla \cdot \mathbf{J}) \mathbf{r} d\tau$ (by the continuity equation).

Apply product rule $\nabla \cdot (x\mathbf{J}) = x(\nabla \cdot \mathbf{J}) + \mathbf{J} \cdot (\nabla x)$.

However $\nabla x = \hat{\mathbf{x}}$, so $\nabla \cdot (x\mathbf{J}) = x(\nabla \cdot \mathbf{J}) + J_x$.

Thus $\int_v (\nabla \cdot \mathbf{J}) x d\tau = \int_v \nabla \cdot (x\mathbf{J}) d\tau - \int_v J_x d\tau$.

The first term is $\int_s x\mathbf{J}\cdot d\mathbf{a}$ (by the divergence theorem), and since $\mathbf{J}$ is entirely inside V, it is zero on the surface S.

Therefore $\int_v (\nabla\cdot\mathbf{J})x\,d\tau = -\int_v J_x\,d\tau$, or, combining this with the y and z components,

$\int_v (\nabla\cdot\mathbf{J})\mathbf{r}\,d\tau = -\int_v \mathbf{J}\,d\tau$.

Referring back to the first line, $\dfrac{d\mathbf{p}}{dt} = \int \mathbf{J}\,d\tau$.