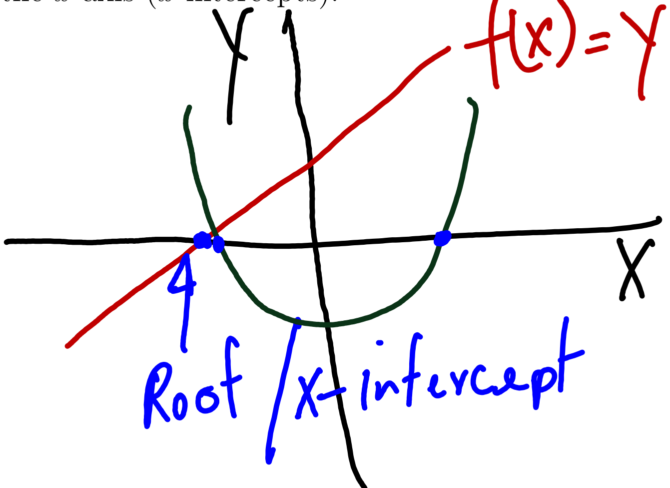
Newton's Method: Finding the root(s) of a function

We can use linear and quadratic approximations to estimate function values. Now we want to approximate the location of *roots* of a function.

* **Root** of a function is a value x at which $f(x) = 0$, or the x values at which the graph of f intersects the x-axis (x-intercepts).



Put $y = 0$, and solve for x.

Recall that for simple functions such as a linear function or a quadratic function, we can find the exact value of the roots by using simple formulas.

Example 1. Find the roots of the following functions.

a) $f(x) = 5 - 3x = 0 \Rightarrow 3x = 5 \Rightarrow x = 5/3$.

b) $f(x) = 2x^2 + x - 1 = 0$

$ax^2 + bx + c = 0 \Rightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$
\begin{aligned}
& 2x^2 + x - 1 \\
&= 2x^2 + 2x - x - 1 \\
&= 2x(x+1) - 1(x+1) \\
&= (x+1)(2x-1) = 0 \\
\Rightarrow\ & x = -1 \text{ and } x = 1/2.
\end{aligned}
$$

Now what about roots of $f(x) = x^5 - 5x + 3$?

In this case, we approximate the root by a numeric technique which is called the **Newton's Method**.

What is Newton's method? It involves finding successive tangent lines to the graph of f, following a certain algorithm until we get close enough to the root.

A function f is given and assume r is a root of f that we wish to approximate, we also assume f is differentiable in an interval containing r.

Suppose an initial approximation $\boxed{\text{of } r \text{ is}}$ also given, call that $\boxed{x_0.}$ — best guess.

⇝ **Algorithm**:

$x^5 - 5x + 3 = 0$

$$X_{n+1} = X_n \boxed{- \frac{f(X_n)}{f'(X_n)}}$$

— Correction factor.

X_{n+1} → improved estimate

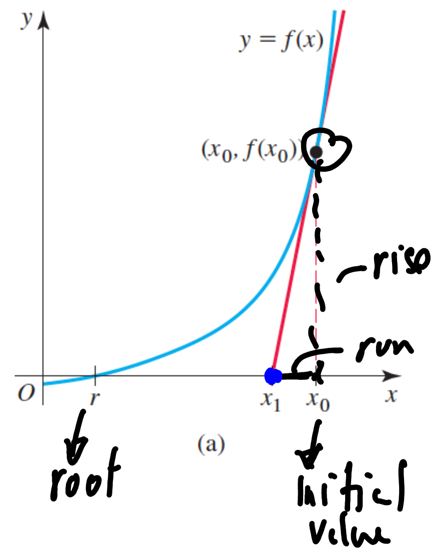
X_n → old estimate

$f'(X_n)$ → First derivative of f evaluated at X_n.

$$|X_{n+1} - X_n| < \varepsilon$$

where $\varepsilon > 0$.

1) Draw a tangent line to the curve of f at the point $(x_0, f(x_0))$:



(a)

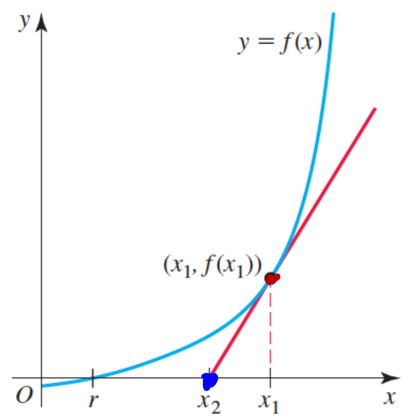
2) Continue the tangent line such that it intersects the x-axis at a point, call that point x_1. x_1 is our new approximation:

$$m = \text{slope} = f'(x_0) = \frac{\text{rise}}{\text{run}}$$

$$y - f(x_0) = f'(x_0)(x - x_0)$$

$$\Rightarrow 0 - f(x_0) = f'(x_0)(x_1 - x_0)$$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$



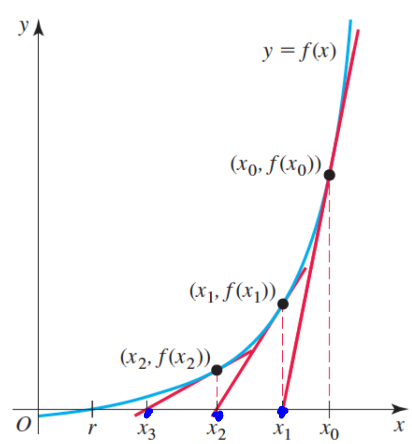
(b)

3) Now repeat steps 1 and 2 for x_1:
Draw a tangent line to the curve of f at the point $(x_1, f(x_1))$, continue the tangent line such that it intersects the x-axis at x_2 and repeat the steps.

$$y - f(x_1) = f'(x_1)(x - x_1)$$

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

$$\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$



Repeating the algorithm for each point, we will obtain a sequence of points, $\{x_0, x_1, x_2, x_3, ...\}$, that ideally get closer and closer to the root r.

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

This pattern continues: If a previous approximation is known, say x_n, then the new approximation is calculated by the following formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

for $n = 0, 1, 2, \cdots$

Example 2. Back to the example, let's approximate the root of $f(x) = x^5 - 5x + 3$ with initial starting values $x_0 = \{-2, 0, 2\}$.

$$f'(x) = 5x^{5-1} - 5 = 5x^4 - 5$$

$$x_{n+1} = x_n - \frac{x_n^5 - 5x_n + 3}{5x_n^4 - 5}$$

for $n = 0, 1, 2, 3, \cdots$

Remark 1. Newton's method is an example of a repetitive loop calculation called an *iteration*. It is mainly done by calculators and computers and it is included in many scientific computing software.

Remark 2. When to stop?

There are different ways to decide when to terminate the iterations. Either the number of iterations are given, or the number of agreeing digits between two successive approximations are given, for instance continue the iterations until two successive approximations agree to 4 digits. The effectiveness of the algorithm is to get close enough to the root (small error) as quickly as possible (not many iterations).

Exercise. Use Newton's method to estimate the value for $\sqrt{10}$. Stop calculating approximations when two successive approximations agree to four digits to the right of the decimal point.

$$x = \sqrt{10}$$

$$\Rightarrow x^2 = (\sqrt{10})^2 = 10$$

$$\Rightarrow x^2 - 10 = f(x) = y.$$

$$f'(x) = 2x.$$

$$x_{n+1} = x_n - \frac{x_n^2 - 10}{2x_n}$$

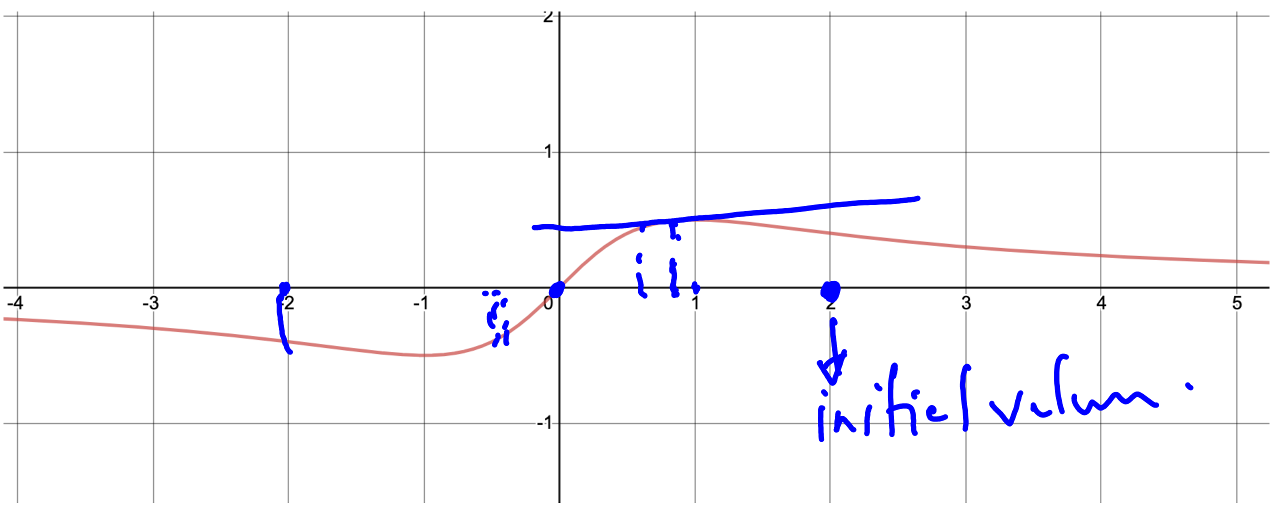
for $n = 0, 1, 2, 3, \cdots$

Remark 3. Newton's method is not always working. The location of the initial approximation is important.

Consider the function $f(x) = \dfrac{x}{x^2+1}$:

$$X_{n+1} = X_n - \boxed{\frac{f(X_n)}{f'(X_n)}} = \frac{a}{0}$$

We know that the root of this function is at $x = 0$,



Now let's apply Newton's method with two initial approximations $x_0 = 2$ and $x_0 = \frac{1}{\sqrt{3}}$: