Modifying An Existing Energy-Economic Oil Production Model to Reflect Logistic Labor Population

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1 Introduction

Since the 1950's differential equations have been used to model fossil fuel extraction. One of the most influential of these models, the Hubbert model, has been the subject of lengthy debate in the years following. The controversy surrounding Hubbert's model stems from the simplistic and in many cases unrealistic assumptions that are made by the model. Fully understanding the assumptions made and properly assessing the model's limitations is key to utilizing Hubbert's model for prediction.

What sets the crude extraction models of the 1950's apart from models used today is the inclusion of economic theory. The model presented by Hubbert was purely a petroleum engineer's account of fossil fuel extraction, that is it lacked any form of economic intuition. The assumptions made in these newer extraction models still carry some theoretical shortcomings and currently stand to be improved. Over the course of this paper I will present Hubbert's influential model and outline the progression into a newer oil extraction model which incorporates economic theory. Once comfortable with the evolution of Hubbert's oil extraction model into the present oil extraction model, I will explore the effect of improving assumptions current academics have made about labour population (Berg, Hanz, and Milton 2011).

2 Hubbert Model (1956)

During the 1950's, Marion Hubbert, a geoscientist working for the petroleum giant Shell, modeled the rate of fossil-fuel extraction using a first-order differential equation. Being a geoscientist, Hubbert's model was a technical one which relied entirely on geophysical constraints to explain the rate of fossil fuel extraction.[1] Hubbert's model was a good approximation for the traditonal oil extraction techniques of the 1960's; a well would be drilled into the Earth's crust into a reserve of oil in order to create a pressure differential between the surface of the well (low pressure) and the bottom of the reserve (high pressure) pushing the oil upward.[2] Increasing production translates into drilling an additional well next to the first well until the pressure in the reserve is roughly equal to the pressure at the well's surface. Hubbert's model lacked any economic factors such as the price of oil or the maximization/minimization routines traditionally used in economic analysis. Hubbert's first-order differential equation is shown in equation 1.

$$\frac{dQ}{dt} = gQ(R_{\text{Max}} - Q) \tag{1}$$

Parameter	
Q	Cumulative extraction (Barrels)
R_{Max}	Maximum amount of reserves (Barrels)
g	Extraction growth (Dimensionless)

The cumulative extraction parameter (Q), reflects the amount of oil that has been extracted from the reserve. R_{Max}, the maximum amount of reserves, represents the total amount of oil that has yet to be extracted. g, the extraction growth, is a positive constant put in place to appropriately scale cumulative extraction over time.

In deriving equation 1, Hubbert makes the following assumptions: the discovery of oil would grow exponentially in the short term but eventually taper off to a finite maximum; geology is the sole motivator of discovery, depletion, and production; and finally that the ultimate recoverable reserves are static.

Hubbert began with the assumption that the discovery of oil would grow exponentially in the short term, but eventually the growth would slow and approach a threshold amount (Q_{Max}) at which all available oil is found. This discovery curve is labeled Q_{D} in Figure 2. The cumulative production curve (labeled Q_{P} in Figure 2) would resemble the discovery curve Q_{D} but be separated by a lag in time which accounts for market expansion. The amount remaining in the reserve at any point in time is quantified by $\text{Q}_{\text{R}} = \text{Q}_{\text{D}}$-$\text{Q}_{\text{P}}$. The total area under the curve Q_{R} is the ultimate recoverable resource.

[1] Hubbert notes that his model of production relied on intrinsic characteristics about the resource: "...although production rates tend to initially increase, physical limits prevent their continuing to do so." Hubbert (1956)

[2] Bernoulli's Equation, *http://hyperphysics.phy-astr.gsu.edu/hbase/pber.html*

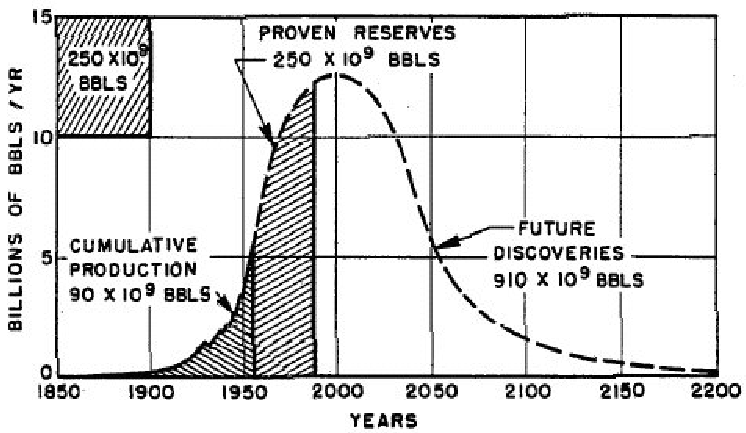

Figure 1: "Ultimate world crude-oil production based upon initial reserves of 250 billion barrels." Hubbert (1956) The vertical axis is measured in billions of barrels extracted/year, reflecting the rate that cumulative extraction changes over time. The horizontal axis is measured in years allowing for the rate of cumulative extraction to change temporally. The area under the curve or $\int \frac{dQ}{dt} dt = \int dQ = Q$, represents the cumulative amount of oil extracted. The maximum amount of oil production based off of Hubbert's model occurs in the year 2000. Oil production experiences a point of undulation in the 1970's in which the rate of oil extraction is increasing at a decreasing rate.

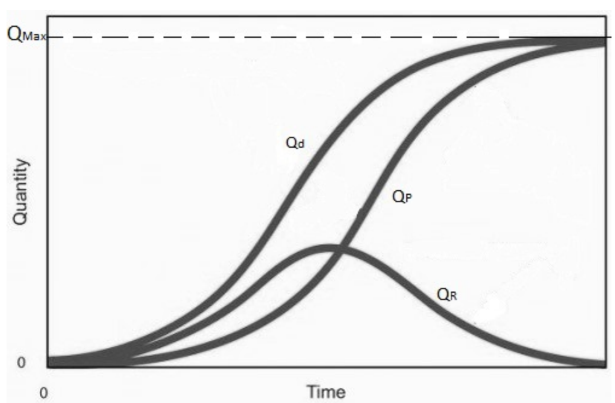

Figure 2: Theoretical resource discovery $\mathrm{Q_D}$, theoretical resource production $\mathrm{Q_P}$, and the amount remaining $\mathrm{Q_R}$

In order to satisfies Hubbert's assumption about cumulative extraction, Q, initially being exponential, forces $Q(t)=Q_0e^{gt}$. The rate of extraction $\frac{dQ}{dt}$, is shown in equation 2,

$$\frac{dQ}{dt} = gQ_0e^{gt} = gQ(t) \tag{2}$$

Hubbert quickly realized that this functional form of cumulative extraction is not theoretically sound; exponential growth implies that as time approaches ∞ the amount of the resource extracted approaches ∞. To correct cumulative extractions climb to infinity Hubbert suggested that the rate of cumulative extraction once a significant portion of the resource has been extracted would be proportional to the fraction of remaining resource $1 - \frac{Q(t)}{R_{\text{Max}}}$.[3] Incorporating this feature into equation 2 we have an expression for the rate of extraction shown in equation 3,

$$\frac{dQ}{dt} = gQ(1 - \frac{Q}{R_{\text{Max}}}) \tag{3}$$

Solving the logistic function (equation 3) for Q(t), Hubbert has an expression for cumulative oil extraction shown in equation 4,

$$Q(t) = \frac{R_{\text{Max}}}{(1 + ae^{-gt})} \tag{4}$$

At the time this was a reason for many to laugh at Hubbert as the classical macroeconomic models assumed natural resources to be infinite.

3 Economic Maximization Models

There are several limitations to modeling oil production using the Hubbert model this is due largely to the simplistic assumptions being made here[4]

[3] A detailed derivation of this can be found in Foundations of Environmental Physics: Understanding Energy Use and Human Impacts Forinash (2010)

[4] Cavallo criticizes some of these assumptions in his article on Hubbert's oil production model, "there is no geophysical, physical, or chemical law that compels cumulative production to follow equation 4. The logistic growth curve has nothing to do with any of the physical and geochemical factors that govern oil flows and well productivity."Cavallo. For example the Hubbert model neglects the influence that the price of oil has on oil extraction. The Hubbert model lacks the maximization strategies typically seen from the agent extracting from the resource. Several economists have tackled this by proposing an economic foundation for the Hubbert model; Reynolds (1999) interprets the Hubbert curve as a cost function which combines both information and depletion effects outlined by Uhler (1976).

3.1 Nonrenewable Resource: Two-Period Model

To construct the economic foundation behind explaining the Hubbert model we first must examine the classical two period model of a non-renewable resource.[5] We start with the assumption that time is defined over two periods, period 0 and period 1. Because the resource is non-renewable we can make the assumption that there is a fixed amount of said resource denoted R_{Max} at the beginning of period 0. We denote the amount extracted in the period t be defined as Q_{t} and assume that each period has its own inverse demand function given by equation 5,

$$P_{\mathrm{t}} = a - bQ_{\mathrm{t}} \tag{5}$$

where P_{t} is the price in period t, with a and b being positive constants. This demand curve

is illustrated in figure 3,

The area under the demand curve shaded in grey represents the gross benefit to society in consuming the quantity Q_{t} in period t.[6]

Net benefit differs from gross benefit by including a cost function. The cost function assumes two things: these extraction costs are completely covered by the firm extracting the resource and marginal cost, c, is a constant. Total extraction cost, C_{t}, can now be expressed by $C_{\mathrm{t}} = cQ_{\mathrm{t}}$. Total net benefit experienced by society, N_{t}, becomes the difference in cost and benefits, $N_{\mathrm{t}} = B_{\mathrm{t}} - C_{\mathrm{t}}$.

Using the expression for net benefits we seek to optimize our extraction based on the resources constraints. In order to optimize extraction we define a welfare function that is in discounted utilitarian form. The two period welfare function can be written as $W = N_0 + \frac{N_1}{1+\rho}$, where ρ is the social utility discount rate.[7] At this point we assume a technical constraint; society wishes to have no resource remaining in the reserve at the end of period 1, or $Q_0 + Q_1 = Q_{\mathrm{Max}}$.

Thus the maximization of welfare, when choosing levels of resource extraction levels Q_0 and Q_1, is defined in equation 6 when choosing levels of resource extraction levels Q_0 and Q_1 subject to the technical constraint outlined above.,

$$Max\ W_{Q_0,Q_1} = N_0 + \frac{N_1}{1+\rho} \tag{6}$$

Subject to

$$Q_0 + Q_1 = Q_{\mathrm{Max}}$$

3.2 Economic Foundation for the Hubbert Model

Let us now consider the following economic foundation for the Hubbert model: an oil producer who is operating in competitive market faces production(technology), cost and profit functions:

Production:

$$Y_{\mathrm{t}} = (I_t)^{\theta} \tag{7}$$

Where Y is oil production, I is an index of all possible inputs used in the production process(labour,capital, energy, etc.), and θ is the level of returns to scale. $\theta > 1$ corresponds to all inputs increasing by a given factor yet output increases more than that given factor(increasing returns to scale), $\theta = 1$ reflects if all inputs increase by a given factor then the output will increase by a given factor(constant returns to scale), and finally if $\theta < 1$ corresponds to all inputs increasing by a given factor yet output increases by less than that given factor (decreasing returns to scale).

Cost:

$$C_t = P_t^{Input} I_t \tag{8}$$

Where C is the cost of production, P_t^{Input} is an vector of prices associated with every input I_t,I_t is an index of all possible inputs used in the production process(labour, capital, energy, etc.).

4 Global Model for Oil Production

In order to accurately describe global oil production one must consider the following, exponential growth and eventual decline in production, the depletable nature of the resource, the amount of capital invested into oil infrastructure, the population of the labour force, and a variety of factors that influence oil production. Due to the scope of this project the factors listed above will be the only ones considered.

Production of Oil in the USA, Norway, and UK has experienced the behavior predicted in Hubbert's model (exponential growth followed by a tapering or terminal decline). Shown in figure 4 is Norway's oil production and percent change in production,

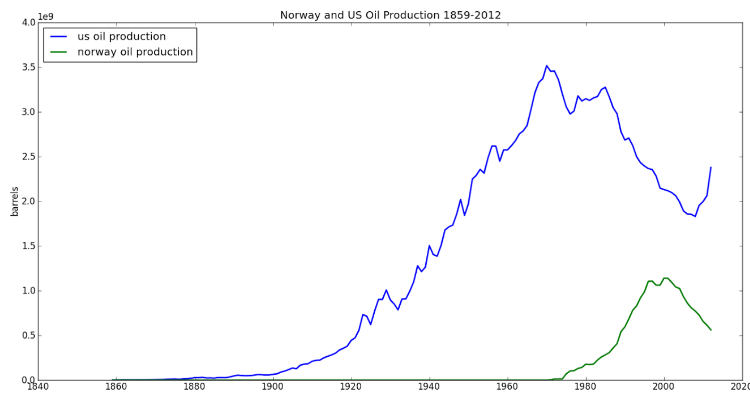

Figure 4: Hubbert's models were able to properly predict the peak in oil produciton for the United States in the 1970's, but was unable to predict the resurgence of oil production that came from discovery of oil in Alaska.

Empirical data for global oil consumption indicates that global consumption of oil starting in the 1950's was steadily increasing with the exception of the

Profit:

$$\prod_t = P_t Y_t - C_t = P_t Y_t - P_t^{Input} Y_t^{\frac{1}{\theta}} \tag{9}$$

Where $\prod$ is revenue minus cost, with the new variable P_t for output price, and t is the time operator.

Oil production is now optimized by maximizing profits subject to the technical constraint set by the resource shown in equation 10.

$$Max_{Y_t} \prod_{t=1}^{n} \frac{\prod_t}{(1+\rho)^t} \quad st. \prod_{t=1}^{n} Y_t \leq Q_{Max} \tag{10}$$

energy glut in the 1980's shown in figure 4. This implies that in order to keep up with oil consumption, oil production must also be increasing at a proportional rate. Due to the finite nature of oil the rate of production must eventually taper off as the amount of available resource becomes depleted.

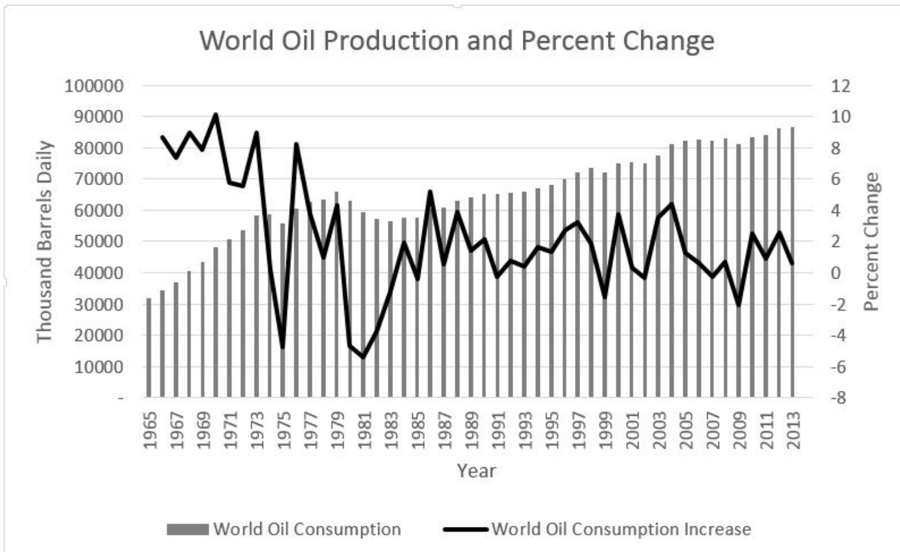

Figure 5: Notice the growth in global oil consumption in 1965 was followed by the sharp decline during the the beginning of the OPEC embargos starting in 1973 to 1974, where the price of oil increased from \$3 a barrel to roughly \$12 a barrel.Petroleum (2013) The next period of decline in oil production occured in the 1980's when the world entered what is known as the energy glut where a surplus of crude oil combined with failing oil demand caused the price of oil to crash. From 1985 to 2013 the percentage increase in World Oil Production has overall been increasing.

5 Oil Production Function

Following the macroeconomic framework established by Robert Solow, oil consumption, can be expressed explicitly in a production function. Economist

Joseph Stiglitz suggests that the production function takes the form of equation 11,

$$Y = Y(K, L, Res) \tag{11}$$

Where Y is defined as output as a function of inputs K (Capital), L (Labour), and Res(Natural Resource). This is a slight modification of the Solow production function in which natural resource is explicitly embedded in the output function Y Solow. The production function will be used later to express the price of oil as an implicit function of E.

5.1 Berg, Hanz, and Milton Model

A key factor in the production function are the relationships shared between inputs. In particular we are concerned with the elasticity of substitution, σ, which is a measure of the curvature of an isoquant for given amounts of input (Capital, Energy). An isoquant is a contour line drawn between a set of points at which a constant level of output is produced while changing the quantities of two or more inputs shown in Figure 6.

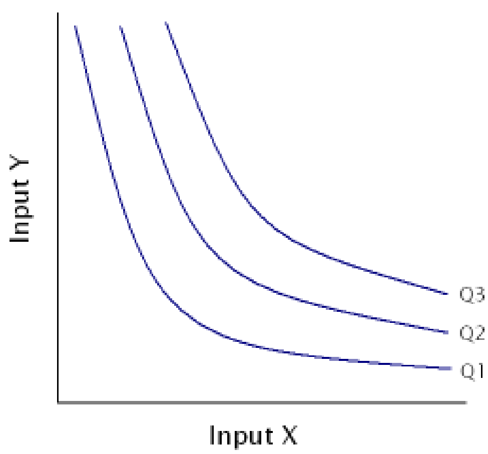

Figure 6: Shown is an isoquant map with various levels of output. Any point on a given curve corresponds to a constant level of output; moving from curve to curve, up and to the right, corresponds to higher levels of constant output ($Q_3 > Q_2 > Q_1$).

The elasticity of substitution is a measure of how easy it is to shift between inputs, this is defined as the % change in factor proportions resulting from a one-percent change in the marginal rate of technical substitution, equation 12.

$$\sigma = \frac{d\ ln\frac{K}{E}}{d\ ln\frac{\partial Y/\partial E}{\partial Y/\partial K}} = \frac{\%\Delta\frac{K}{E}}{\%\Delta\frac{\partial Y/\partial E}{\partial Y/\partial K}} = \frac{\%\Delta\frac{K}{E}}{\%\Delta MRTS} \tag{12}$$

The marginal rate of technical substitution (MRTS), is the amount one has to be decreased when a one-unit increase of another input is increased, so that output remains constant. This can be mathematically expressed as the slope at any point on any of the isoquants in Figure 6.

When conceptualizing elasticity of substitution it is perhaps easiest to think of the relationship between labourers and machinery in a factory. Berg, Hanz, and Milton suggest that an elasticity of substitution $\sigma = \infty$ would represent machinery that is able to do all of the work without the need of a single labourer. This could also could refer to a scenario where labourers are able to do all of the work without the need of machinery. At the other extreme, an elasticity of substitution $\sigma = 0$, describes a situation in which both labour and machinery are required and one cannot be used in place of the other (ex. one scientist and one electron microscope).

Following the work of Berg, Hanz, and Milton, a constant elasticity of substitution (CES), is used in the model implying that elasticity of substitution, σ remains unchanged independent of the values of the inputs, K and E, shown in equation 13.

$$Y = [\beta(BE)^{\rho} + (1-\beta)K^{\rho}]^{\alpha/\rho}(AL)^{1-\alpha} \tag{13}$$

Parameter	
Y	Total output ($)
L	Labour population ($/Mega Barrels)
K	Capital input ($/year)
A	Labour productivity ($)
B	Energy efficiency ($/Mega Barrels)
α	Output elasticity of capital
β	Output elasticty of labour
$\mathrm{g_a}$	Growth rate in labour productivity
$\mathrm{g_b}$	Growth rate in energy efficiency
$\mathrm{g_l}$	Growth rate in labour population

In the production function above labour productivity(A), energy efficiency (B), and labour productivity (L) are all assumed to grow exponentially with time as follows[8],

$$A(t) = A_0 e^{\mathrm{g_a t}} \tag{14}$$

$$B(t) = B_0 e^{\mathrm{g_b t}} \tag{15}$$

and

$$L(t) = L_0 e^{\mathrm{g_l t}} \tag{16}$$

[8]Exponential growth is a reasonable assumption since this closely follows empirical data.

These parameters all have the form of the classical Malthusian Growth Model, that is essentially exponential growth based on a constant rate.[9] Next we apply the production function derived in equation 13 to a profit function for global oil production.

To understand the various parts of equation 13 we must first must consider a simpler version of the Cobb-Douglas production function discussed in section 5.2.

5.2 Cobb-Douglas Production Function

We begin by looking at the functional form of the Cobb-Douglas in equation 17,

$$Y = WK^{\alpha}L^{1-\alpha} \tag{17}$$

Where Y is total production, W is the efficiency parameter, L is labour input, K is capital input, and α and $1-\alpha$ are the output elasticities of capital and labour, respectively. The Cobb-Douglas is widely used to represent the technological relationship between the amounts of physical capital and labour.

Based off of equation 17 we note the following assumptions that are implicit to the Cobb-Douglas: If either capital (K) or labour (L) disappears, then production also vanishes. The marginal productivity of labour is proportional to the amount of production per unit of labour, and finally the marginal productivity of capital is proportional to the amount of production per unit of capital this is shown in equation 18 and 19 respectively(Chiang Wainwright 2005).

$$\frac{\partial Y}{\partial K} = W\alpha K^{\alpha-1}L^{-(\alpha-1)} = W(\frac{K}{L})^{\alpha-1} \tag{18}$$

$$\frac{\partial Y}{\partial L} = WK^{\alpha}(1-\alpha)L^{-\alpha} = W(1-\alpha)(\frac{K}{L})^{\alpha} \tag{19}$$

If we assume each input is to be paid by the amount of its marginal product, the relative share of total product capital accrues is α, and the relative share of total product labour accrues is $1-\alpha$ this is shown in equations 20 and 21 respectively.

$$\frac{K(\partial Y/\partial K)}{Y} = \alpha \tag{20}$$

$$\frac{L(\partial Y/\partial L)}{Y} = 1-\alpha \tag{21}$$

Equations 20 and 21 above indicate that the relative share of that input in the total product.

[9] This model is named after Thomas Robert Malthus, whose influential work on population can be found in *An Essay on the Principle of Population(1798)*

5.3 CES Production Function

Like the Cobb-Douglas production function explored in the previous section, the CES production function expresses total output based upon different bundles of inputs. The functional form of the CES production function is shown in equation 22,

$$Y = W(\beta K^{\rho} + (1-\beta)L^{\rho})^{1/\rho} \tag{22}$$

Where Y is total production, W is the efficiency parameter, L is labour input, K is capital input, and β is the distribution parameter, like the α in the Cobb-Douglas function, this has to do with the relative factor shares in the total product, and $\rho = \frac{\sigma-1}{\sigma}$ (substitution parameter), which shares no counterpart in the Cobb-Douglas production function.

Just as we did with the Cobb-Douglas we now will look at the marginal products of Labour and Capital in equations 23 and 24 respectively,

$$\frac{\partial Y}{\partial L} = \frac{(1-\beta)}{W^{\rho}}(\frac{Y}{L})^{1+\rho} \tag{23}$$

$$\frac{\partial Y}{\partial L} = \frac{\beta}{W^{\rho}}(\frac{Y}{L})^{1+\rho} \tag{24}$$

We can see in equations 23 and 24 the impact of the distribution parameter β and $1-\beta$ similar to equations 20 and 21 (Chiang Wainwright 2005).

5.4 Berg, Hanz, and Milton CES Revisited

With a better understanding of the Cobb-Douglas and CES production functions we can begin to tease out an economic interpretation of Berg, Hanz, and Miltons CES, equation 13. First we consider equation 25,

$$Y_{EC} = [\beta(BE)^{\rho} + (1-\beta)K^{\rho}]^{1/\rho} \tag{25}$$

Here Y_{EC} is merely the energy and capital segment of the entire production function(equation 13). In order discern a more digestible interpretation of equation 25 we must begin a dimensional analysis of each of the individual terms. Energy efficiency (B) is measured in (\$/Mega Barrels), Oil production (E) is measured in (Mega Barrels/Year), Capital Invested (K) is measured in (\$/Year), finally ρ and β are dimensionless parameters. We look at our discussion in section 5.3 as to how the distribution term (β) affects relative share of each input (BE and K) in the total product. Next we look at the relationship between Y_{EC} and labour in equation 26,

$$Y = Y_{EC}^{\alpha}(AL)^{1-\alpha} \tag{26}$$

This is the familiar form of a Cobb-Douglas which we discussed at length in Section 5.2. Here α represents the output elasticity of Y_{EC} and $1-\alpha$ represents the output elasticity of Labour.

Berg, Hanz, and Milton explore the assumption that $\rho = 1$, which in turn forces equation 13 to assume the form of a Cobb-Douglas shown in equation 27,[10]

$$Y = [(BE)^{\beta} K^{1-\beta}]^{\alpha} (AL)^{1-\alpha} \tag{27}$$

6 Global Oil Prices

In order to derive an expression for the price of energy, p_E, we maximize profit. Profit is defined as output minus all inputs shown in equation 28.

$$\prod_{Y} = Y - p_E E - r_K K - wL \tag{28}$$

where p_E represents the price of oil, r_K the rent on capital, and w the wage for labour. Now we can maximize our profit with respect to oil equation 29 as well as maximize profit with respect to capital equation 30,

$$\frac{\partial \prod_Y}{\partial E} = 0 \Longrightarrow p_E = \frac{\partial Y}{\partial E} \tag{29}$$

and

$$\frac{\partial \prod_Y}{\partial K} = 0 \Longrightarrow r_K = \frac{\partial Y}{\partial K} \tag{30}$$

Where Y was previously defined in equation 13. We now have two expressions p_E(E,K) and r_K(E,K) both of which are dependent on capital, K. By dividing expression for p_E by r_K and solving for capital, K, results in K(p_E,r_K,E). Finally if we solve our expression K(p_E,r_K,E) for p_E(r_K,E,K) shown in equation 31,

$$p_E = \beta\left(\frac{\alpha B^{\alpha\beta}(1-\beta)^{1-\beta}}{r_K^{\alpha(1-\beta)}}\right)^{\frac{1}{1-\alpha(1-\beta)}} \left(\frac{AL}{E}\right)^{\frac{1-\alpha}{1-\alpha(1-\beta)}} \tag{31}$$

This expression for price will be used later in section 9, when it is substituted into the oil production function.

[10]The limit of the CES production function is shown in the appendix B.

7 Oil Supply

Based on Hubbert's model, recall equation 1, we note that oil production should be scale linearly with the amount of remaining reserves. Linear scaling takes the form of a geological constraint, (R-Q), imposed by the intrinsic physical properties of the resource. The supply of oil also needs to account for the amount of capital invested in oil related infrastructure, K_E. Berg, Hanz, and Milton uses the following model for oil supply equation 32,

$$E = \frac{dQ}{dt} = CK_E^{\epsilon}(R-Q)^{1-\epsilon}, \quad 0 < \epsilon < 1, \tag{32}$$

where C is an Malthusian growth model for oil extraction efficiency expressed in equation 33,

$$C(t) = C_0 e^{g_c t} \tag{33}$$

which is representative of advancement of technology in oil extraction.[11] ϵ and 1-ϵ are representative of diminishing returns on capital and reserves.[12] We note that oil production does not scale linearly with capital. Like Hubbert's model, equation 1, equation 21 has the geological constraint that suggests as Q $\rightarrow$ R, oil extraction becomes increasingly more difficult, which demands more capital in order to extract the entire reserve. Following the work done by Berg, Hanz, and Milton, $\epsilon = 0.1$ which is in line with the suggestions of A. Pickering Pickering (2007).

Using equation 33, we derive the rate of oil production as a function of the price of energy,p_E. Berg, Hanz, and Milton. defines oil industry profits, $\prod_E$, as shown in equation 34,

$$\prod_E = p_E E - r_E K_E \tag{34}$$

where r_E is the rent on oil capital. Berg, Hanz, and Milton make the assumption that "the oil industry is capital intensive but not labor intensive"Peter Berg and Milton (2011). Because Berg, Hanz, and Milton make this assumption, labour, L, does not appear in equation 34.

Like the routine followed in equation 30, profit is maximized with respect to capital shown in equation 35,

$$\frac{\partial \prod_E}{\partial K_E} = 0 \Longrightarrow \frac{\partial E}{\partial K_E} = \frac{r_E}{p_E} \tag{35}$$

[11] Advancement can take the form of improvements in drilling techniques, such as horizontal drilling and hydraulic fracturing.

[12] This follows the groundwork established earlier in equation 9

Solving $\frac{\partial E}{\partial K_E}$ for K_E and substituting this expression back into equation 32 we now have a new expression for E shown in equation 36,

$$E = \frac{dQ}{dt} = C^{\frac{1}{1-\epsilon}}(\frac{p_E \epsilon}{r_E})^{\frac{\epsilon}{1-\epsilon}}(R - Q) \tag{36}$$

8 Data

Due to the scope of this project a majority of parameter values were taken from Berg, Hanz, and Milton in order to finish in a timely fashion. With the use of equation 13 and the economic and industrial data found in Kemfert (1998), the value of B, the energy efficiency parameter, was determined. By using the value for B in the CES production function, equation 13, and current oil industry estimates for the unknowns; the value for A can be estimated. To obtain the value for C the oil production function, equation 25 is used alongside oil industry estimates for current production, E = 85 mega barrels/day, capital invested K_E = $\$1.4 \times 10^{12}$, and cumulative production, Q=10^{12}barrels(BP, 2008).[13]

Parameter	Description	Value
A	Labour Productivity	$2.98 \times 10^5\$$
B	Energy efficiency	$273 \times 10^6\$/mega\ barrels$
D	Oil extraction efficiency	$1.5 \times 10^{-3}\$^{1-\epsilon} mega\ barrels^{\epsilon-1}$
g_a	Growth rate of labour productivity	.03
g_b	Growth rate of energy efficiency	.015
g_c	Growth rate of oil extraction efficiency	.015
g_l	Labour population growth rate	.01
Q_0	Initial cumulative production	$.2 \times R$
r_E	Rent on oil capital	.1
r_K	Rent on capital	.1
R	Initial reserves	2.2 trillion barrels
α		.3
β		.066
ϵ		.1

Recall section(5) equation 16, where Berg, Hanz, and Milton makes the claim that labour population follows a Malthusian model for growth. Theoretically this is distressing: as $t \to \infty$, labour population, $L(t) \to \infty$, which is an impossible result due to obvious issues with the carrying capacity of Earth. Instead of a Malthusian model for labor population which is categorized by exponential growth, the use of a Verhulst function is used in its place. The Verhulst function,

[13]The units of D presented here is from Berg, Hanz, and Milton however these units are incorrect an issue addressed in appendix D.

better known as the Logistic function, is a family of "S" shaped curves. Logistic functions have the feature of initial exponential growth followed by the function tapering off to a finite value. The proposed model for labour population, L(t), is shown in equation 37,

$$L(t) = \frac{L_{max}}{1 + \kappa e^{-\delta t}} \tag{37}$$

Where L_{Max} is the theoretical maximum for Earth's population, κ and δ are positive constants found by fitting the curve to empirical data. In order to find values for κ and δ I used the data from the World Bank and the US census bureau, specifically, world labour force participation total and world population total shown in figures 7 and 9, respectively,

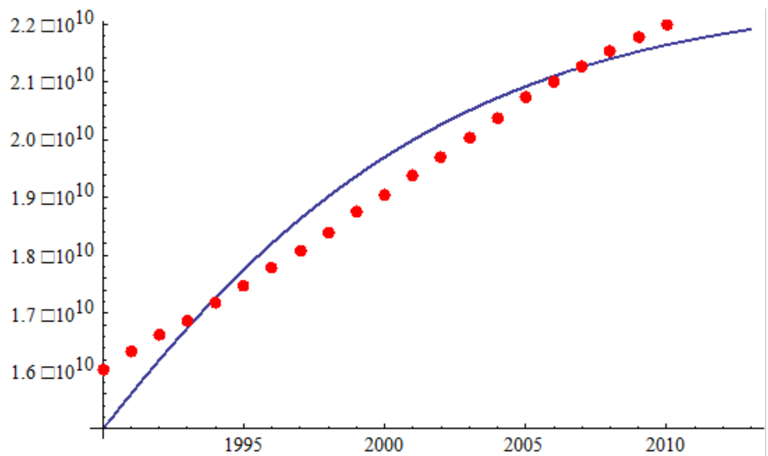

Figure 7: Logistic curve fit of World Labour Force Participation Total appears as the continuous line and empirical data appears as dots. A total of 20 observations. Notice the inflection point of the empirical data occurs during 2002.

Ideally the logistic fit for World Labour Force Participation Total would be used to in place of the Malthusian Growth model for Labour yet fitting a curve to 20 points of data yields a crude fit at best. Instead of using the the logistic fit for World Labour Force Participation Total a logistic fit of World Population Data is used instead. Looking at Figure's 8 and 9, we notice that the inflection points of both fitted curves are similar.

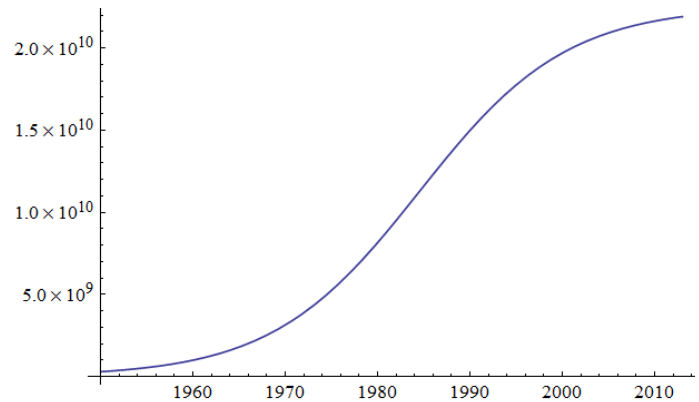

Figure 8: The logistic fit of World Labour Force Participation Total. Notice the inflection point occurs in 1989.

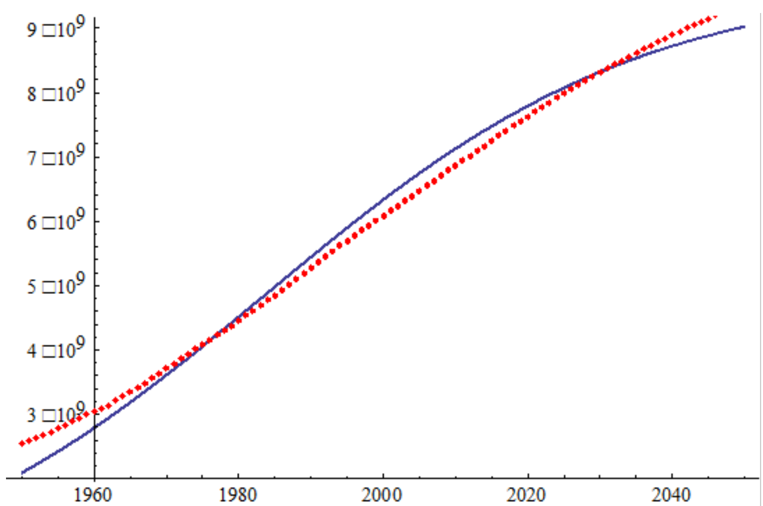

Figure 9: Logistic fit of World Population including estimates for 2050 appears as the continuous line and empirical data appears as dots. A total of 100 observations. Notice the inflection point of the fitted curve occurs in the early 1990's.

9 Results of the Analytical Model

In order to assess any results that the modified Berg, Hanz, and Milton model produces we must make a comparison to the original Berg, Hanz, and Milton model as well as to the Hubbert Model. Unlike the Hubbert model which requires an infinite amount of time to extract all the ultimate reserves (R); my model hits the point Q = R in a finite amount of time; at which oil production (E) is equal to zero. After the peak in production the decline is much steeper than the incline, a feature which was not present in the Hubbert Model.

The final form of global oil production is derived step-by-step in appendix A. Unfortunately I was unsuccessful in using the functional form established over the course of this paper to produce sensible results. Finding a functional form that matches closest to empirical results is shown in appendix C, where a number of variations to the functional form of global oil production are explored. The model with the closest fit to empirical data only captured contributions from labour productivity (A), labour population (L), and the parameters which capture the substitution effects discussed in sections 5.2-5.4. Plotting this trimmed down version global oil production with respect to time yields the curve seen in Figure 10,

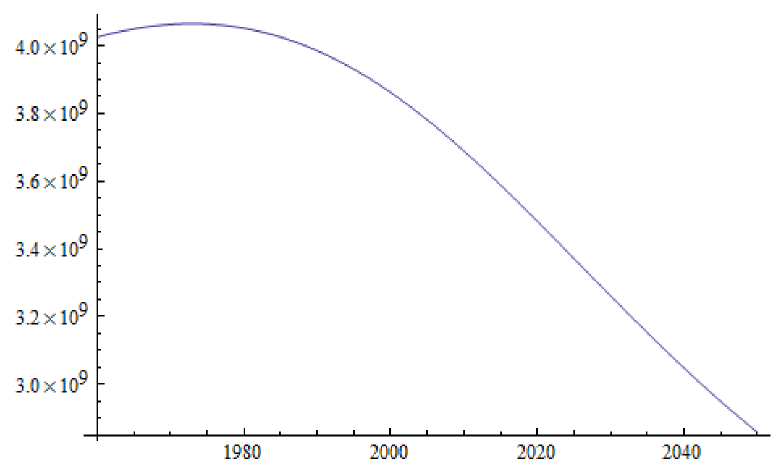

Figure 10: The oil production curve plotted above is equation 28. The vertical axis represents Barrels of Oil/year and the horizontal axis represents years.

Comparing the model in Figure 10 to the model Berg, Hanz, and Milton present in Figure 11, we notice both models feature the Gaussian shape. Berg, Hanz, and Milton's model features a much sharper decline after the peak in production, occurring in the early 2000's whereas my model featured a more gradual decline after the peak in the late 70's. The difference in the rate of declines after the global maximum is likely due to the exclusion of the contributions made to oil production from improvements in oil related technologies D (oil extraction efficiency) and the energy efficiency of oil production B.[14]

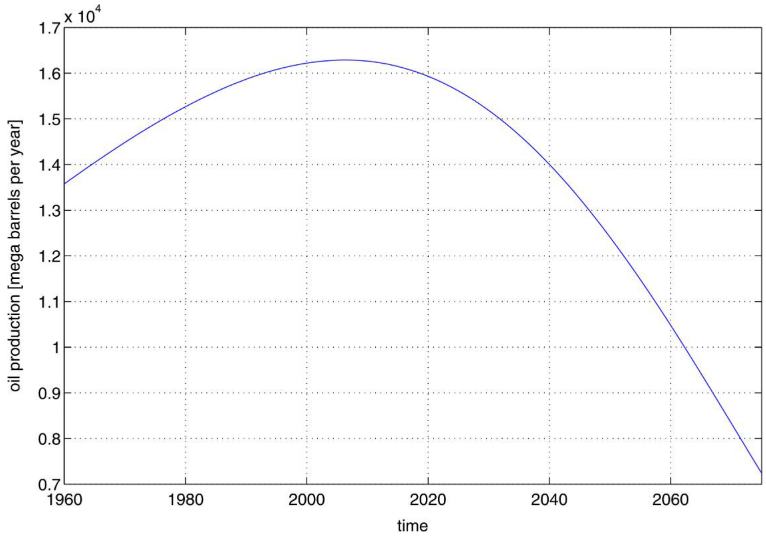

Figure 11: Analytical solution: oil production versus time for = 0. The symmetry of the traditional Hubbert curve is broken, resulting in steep decline rates past the production peak, referred to as the oil peak.

Based on the predictions the model shows global oil production reached an all-time high in 1979 of 4.067×10^9 barrels/year. To assess the accuracy of this model we compare the predicted amount to the observed level of oil production in 1979 shown in the equation 38 is the percent error calculation for the "peak"(OPEC 2014)[15],

$$\frac{2.279 \times 10^{10} - 4.067 \times 10^9}{2.279 \times 10^{10}} \times 100\% = 82.161\% \tag{38}$$

Such a high percentage error is being caused by the predicted result being off by an order of magnitude from the empirical result. The modified Berg, Hanz, and Milton model lacks some of the intricacies seen in global oil production data which is what gives it in certain places such a large percent error. Considering the assumptions made by this final model some percent error is to be expected. The model predicts that in the year 2776 global oil production will hit zero.

[14]Recall that both the technology parameters D and B are specified as Malthusian growth model.

[15]Note that the data from OPEC.org is given in 1000 barrels/day and must be transformed to barrels/year before using it.

Parameter values as well as the derivation and evaluation of the function can be checked in appendix C. The fact remains that using Berg, Hanz, and Milton's methodology, I was unable to recreate their results, this is shown in appendix D, where the Mathematica symbolic evaluation is written exactly as it appears in Berg, Hanz, and Milton's IMA Journal article. Skeptical of these results I performed a dimensional analysis of the routine in order to confirm the integrity of Berg, Hanz, and Milton's results.[16] The results of the dimensional analysis indicate that indeed the units as specified by Berg, Hanz, and Milton were incorrect. While the units themselves have no bearing on the numerical results of the model it could point towards a deeper issue than a mistake in the arithmetic.[17]

10 Conclusion

The foundational work of Royal Dutch Shell geoscientist Marion Hubbert is still the focus of energy economic research among academics. Hubbert's original model was completely dependent on physical constraints and required a number of unrealistic assumptions, yet it accurately predicted the peak in oil in a number of cases. Being completely dependent on physical constraints and ignoring obvious influences to oil production such as oil prices, improvements in drilling technology/techniques, and a growing number of educated labourers definitely damages the theoretical credibility of the original Hubbert model thus the need to expand the theoretical foundation. Several academics have contributed to the theoretical base of the Hubbert model with the inclusion traditional economic maximization routines and economic intuition (Berg, Hanz, and Milton 2011).

Although the results of modifying Berg, Hanz, and Milton's model were largely unimpressive, a big improvement was made on the underlying assumptions about global population. Respecifying the world's population as a logistic function made the Berg, Hanz, and Milton model more theoretically sound thus strengthening the predictive power. Recreation of the oil production model as specified by Berg, Hanz, and Milton was unsuccessful which really begged for a by hand derivation to ensure mathematical mistakes were not made.[18]

A couple of pertinent conclusions can be drawn from this investigation of Hubbert energy economic oil production models, (1) The results for the analytical solution printed in the Berg, Hanz, and Milton IMA Journal article cannot be recreated using the published methodology. Attempts to recreate this result in nonsensical Gaussian shaped curves that feature oil production off by roughly a hundred orders of magnitude and global maxima that are off by roughly 500

[16] The parameters were initialized as only their unit values, B = \$/Mega Barrel per year.

[17] If time allowed for it I would estimate the elasticities for CES production using the micEconCES R package.

[18] In addition to a by hand derivation, a thorough review of how parameter values were obtained would also be helpful.

years. (2) Respecifying world labour population as a logistic function expands upon the theoretical foundation established by Berg, Hanz, and Milton. However big improvements stand to be made here as the lack of historical labour force population data forces the use of global population data as a proxy.

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