EL2450: Hybrid and Embedded Control Systems: Homework 1

[To be handed in **January 28**]

Introduction

The objective of this homework is to understand the basics of digital control including modelling, controller design and implementation. The process to be controlled is a 2-D robotic manipulator, shown in Figure 1.

The state of the arm is the vector of the joint angles $q = [q_1, q_2]^T \in \mathbb{R}^2$, as shown in Figure 1. The links have positive lengths $l_1, l_2 > 0$, moments of inertia I_1, I_2 and their masses m_1, m_2 are assumed to be concentrated in their center. The control inputs are the torques applied at the joints $\tau = [\tau_1, \tau_2]^T \in \mathbb{R}^2$. The inertial frame of reference is $\{O_0\}$. The end-effector's 2-D position and angle with respect to x_0 is $p_E = [p_x, p_y]^T \in \mathbb{R}^2$ (position of frame $\{O_E\}$) and $\theta_z \in [-\pi, \pi]$, respectively, whereas the position and angle of the masses are denoted as $p_{c_1}, p_{c_2} \in \mathbb{R}^2, \theta_{c_1}, \theta_{c_2} \in [-\pi, \pi]$. Moreover, the 3-D generalized velocity of the end-effector is $v_E = [\dot{p}_E^T, \omega_E]^T$, where $\omega_E \in \mathbb{R}$ is its angular velocity with respect to z_0. In the same vein, the generalized velocities fo the link masses are denoted as $v_{c_i} = [\dot{p}_{c_i}^T, \omega_{c_i}]^T, i = 1, 2$. The motors are considered massless.

The values used in this assignment are: $m_1 = m_2 = 1\text{kg}, l_1 = l_2 = 0.75\text{m}, I_1 = I_2 = 1\text{kgm}^2$ and the gravity acceleration $g = 9.81\text{m/s}^2$.

Exercises

For the implementation of the exercises, download the matlab files that you will find on the website.

1. (4p) Write the forward kinematics of the arm links, i.e., express p_{c_i} and θ_{c_i} as functions of q, for $i = 1, 2$.

2. (4p) Write the differential kinematics of the arm links, i.e., write v_{c_i} in the form:

$$v_{c_i} = J_i(q)\dot{q}, i = 1, 2 \tag{1}$$

where $J_i \in \mathbb{R}^{3\times 2}, i = 1, 2$, are the manipulator Jacobians.

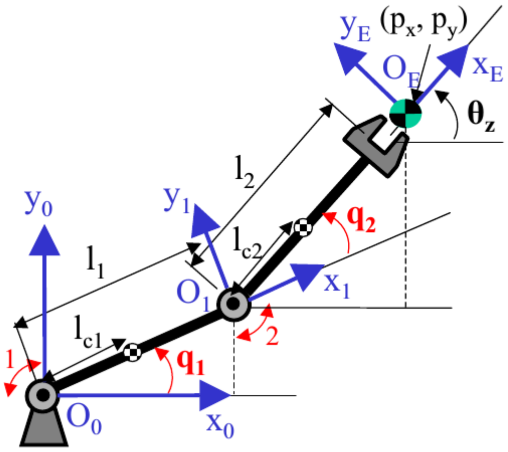

Figure 1: 2-D robotic arm.

3. (6p) Find the Kinetic and Potential Energy of the arm $K = K_1 + K_2$ and $P = P_1 + P_2$, where K_i, P_i are the Kinetic and Potential energy of link $i = 1, 2$. Then use the Lagrange function $L = K - P$ and the Lagrangian formulation:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_i, i = 1, 2$$

to obtain the dynamic model of the arm in the form:

$$B(q)\ddot{q} + N(q, \dot{q})\dot{q} + P_g(q) = \tau, \tag{2}$$

where $B \in \mathbb{R}^{2\times 2}$ is the positive definite inertia matrix, $N \in \mathbb{R}^{2\times 2}$ is the Coriolis matrix and $P_g \in \mathbb{R}^2$ is the gravity vector. Note: The choice for the matrix N is not unique. Choose it such that the matrix $\dot{B} - 2N$ is skew-symmetric, i.e., $\dot{B} - 2N = -(\dot{B} - 2N)^T$. In order to do that, you can use the following formula to compute $N = [n_{ij}]$:

$$n_{ij} = \sum_{k=1}^{2} n_{ijk}\dot{q}_k$$

where

$$n_{ijk} = \frac{1}{2}(\frac{\partial b_{ij}}{\partial q_k} + \frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i}),$$

where b_{ij} is the ij- element of B. Verify the skew-symmetric property of $\dot{B} - 2N$.

4. (4p) Suppose we want the joint angles $q = [q_1, q_2]^T$ to go to a desired pose $q_{\text{des}} = [q_{1,\text{des}}, q_{2,\text{des}}]^T$. Prove that the controller $\tau = P_g - K_v\dot{q} - K_p e$, achieves asymptotic stability (i.e. $\lim_{t\to\infty} e(t) = 0$), where $e = [q_1 - q_{1,\text{des}}, q_2 - q_{2,\text{des}}]^T \in \mathbb{R}^2$ is the corresponding error and K_p, K_v are positive definite diagonal gain matrices. ***Hint***: Use the Lyapunov function $V = \frac{1}{2}e^T K_p e + \frac{1}{2}\dot{q}^T B(q)\dot{q}$ and LaSalle's invariance principle [1, Chapter 4].

5. (3p) Implement the aforementioned controller in matlab for $q_{1,\text{des}} \in [0, \pi], q_{2,\text{des}} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ of your choice. More specifically, fill in the appropriate lines in the block tracking_controller . Then run the file plot_simulink_arm.m to check your controller's performance.

6. (5p) Make a sampled implementation of the controller that you have designed in Task 5. You can achieve this by simply adding two zero-order-hold blocks in the simulink diagram: one to the measurement of q, and one to the measurement of $\dot{q}$. Start from a sampling time $T = 0.05$, then try $T = 0.10, 0.15, \ldots, 0.30$. How is the control performance affected by the sampling time? Is the closed-loop system stable for all the sampling times?

7. (6p) Add a quantization module to your sampled time controller in Task 6. You can achieve this by adding three quantization blocks in your simulink diagram. Two quantization block need to be between the zero-order-hold blocks and the controller. Another quantization block needs to be between the controller and the input to the arm. Start with quantization levels of 1.0 degree for the sensing and 1.0 Nm for the actuation, then try different quantization levels. Try to find the largest quantization levels that preserve the stability of the closed-loop system. How is the control performance affected?

8. (4p) Suppose we now want the joint angles $q = [q_1, q_2]^T$ to track a desired trajectory $q_{\text{des}} = [q_{1,\text{des}}(t), q_{2,\text{des}}(t)]^T$. Prove that the controller $\tau = B\dot{v}_{\text{des}} + Nv_{\text{des}} + P_g - K_p e - K_v e_v$ achieves asymptotic stability, where $e = [q_1 - q_{1,\text{des}}(t), q_2 - q_{2,\text{des}}(t)]^T$, $v_{\text{des}} = \dot{q}_{\text{des}} - e$, $e_v = \dot{q} - v_{\text{des}}$, and K_p, K_v are positive definite matrices.
   Hint: Use the Lyapunov function $V = \frac{1}{2}e^T K_p e + \frac{1}{2}e_v^T B(q) e_v$.

9. (3p) Implement the aforementioned controller in matlab for sinusoidal $q_{1,\text{des}}(t) \in [0, \pi]$, $q_{2,\text{des}}(t) \in [-\frac{\pi}{2}, \frac{\pi}{2}], \forall t \geq 0$ of your choice. More specifically, fill in the appropriate lines in the tracking_controller block. Then, run the file plot_simulink_arm.m to check your controller's performance.

10. (5p) Make a sampled implementation of the controller that you have designed in Task 9. You can achieve this by simply adding two zero-order-hold blocks in the simulink diagram: one to the measurement of q, and one to the measurement of $\dot{q}$. Start from a sampling time $T = 0.05$, then try $T = 0.10, 0.15, \ldots, 0.30$. How is the control performance affected by the sampling time? Is the closed-loop system stable for all the sampling times?

11. (6p) Add a quantization module to your sampled time controller in Task 10. You can achieve this by adding three quantization blocks in your simulink diagram. Two quantization block need to be between the zero-order-hold blocks and the controller. Another quantization block needs to be between the controller and the input to the arm. Start with quantization levels of 1.0 degree for the sensing and 1.0 Nm for the actuation, then try different quantization levels. Try to find the

largest quantization levels that preserve the stability of the closed-loop system. How is the control performance affected?

Questions to the TAs

For this homework, the responsible TA is:

- Chris Verginis (cverginis@kth.se) for tasks 1, 2, 3, 4, 5, 8, 9;
- Antonio Adaldo (adaldo@kth.se) for tasks 6, 7, 10, 11.

Please address possible questions to the appropriate TA.

References

[1] H. K. Khalil, Nonlinear Systems, Prentice Hall, 2002.