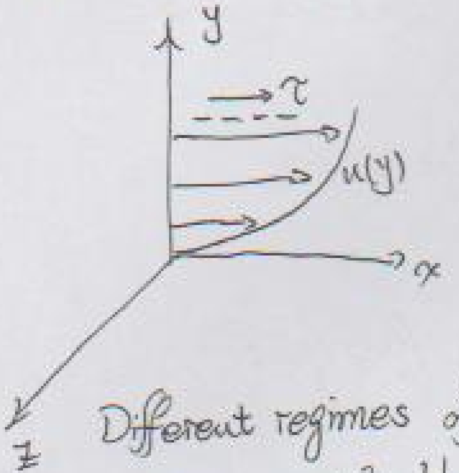
Fluid Mechanics

Navier-Stokes equation: $\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i$; $\frac{\partial u_i}{\partial x_i} = 0$

Definition of fluid: A fluid deforms continuously under the action of a shear force, however small.

Newton's law of motion in fluid for:

1. Open fix control volume (integral form): Momentum principle.
2. Infinitesimal fluid element (differential form): Navier-Stokes equations.



Newtons law of friction: $\tau = \mu \frac{du}{dy} \approx \mu \frac{U}{L}$

Different regimes of flows:

$$\left.\begin{aligned} ma &\approx \rho L^3 \cdot \frac{U}{L/U} \\ \tau L^2 &\approx \mu \frac{U}{L} \cdot L^2 \end{aligned}\right\} \Rightarrow \frac{ma}{\tau L^2} \approx \frac{\rho U L}{\mu} = \text{Reynolds number. } Re$$

Vorticies and vorticity: trailing-line vortices

Vorticity: $\omega_i = \varepsilon_{ijk} \frac{\partial u_j}{\partial x_k}$; $\frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \omega_j \frac{\partial u_i}{\partial x_j} + \nu \nabla^2 \omega_i$

Tubulence separation: Laminar flow separates faster than turbulent flow

Potential flow: $u_i = \frac{\partial \phi}{\partial x_i} \Rightarrow \nabla^2 \phi = 0$ $\frac{\partial \phi}{\partial t} + \frac{1}{2} u_i u_i + \frac{p}{\rho} = C$ + free surface condition

Navier-Stokes equations:

Vector form: $\rho\left(\frac{\partial \bar{u}}{\partial t} + \bar{u} \cdot \nabla \bar{u}\right) = -\nabla p + \mu \nabla^2 \bar{u}$; $\nabla \cdot \bar{u} = 0$

Cartesian tensor form: $\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$, $\frac{\partial u_i}{\partial x_i} = 0$

Velocity: $\bar{u} = (u, v, w) = (u_1, u_2, u_3)$

Dynamic viscosity: $\bar{\mu}$ Density: $\bar{\rho}$ Pressure: p

Lecture 2.

Vector form & Cartesian form of Navier-Stokes equation.

x-component of vector form:

$$\rho\left[\frac{\partial u}{\partial t} + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)u\right] = -\frac{\partial p}{\partial x} + \mu\underbrace{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}_{\nabla^2 u.}$$

$i=1$ for tensor form

$$\rho\left[\frac{\partial u_1}{\partial t} + u_j\frac{\partial}{\partial x_j}u_1\right] = -\frac{\partial p}{\partial x_1} + \mu\frac{\partial^2}{\partial x_j \partial x_j}u_1.$$

Kinematics: Description of fluid particle motion, acceleration and deformation.

Streamlines: tangent to the instantaneous velocity field

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Pathlines: line following the path of a marked fluid particle.

$$\frac{d}{dt}\underset{\sim}{x}_p(t) = \underset{\sim}{u}(\underset{\sim}{x}_p(t), t).$$

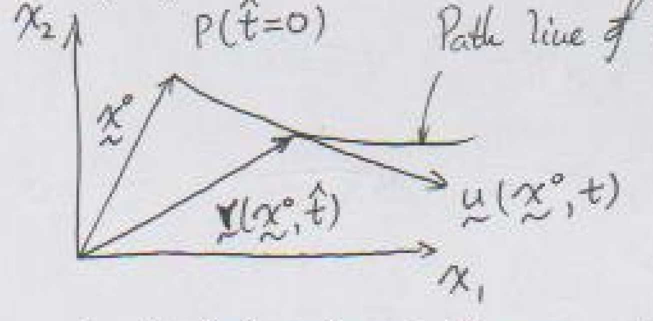
Streaklines: line connecting marked fluid particles that has passed specific point in the flow. (Smoke line).

For steady state flow, these lines are the same.

Timelines: line connecting neighbouring fluid particles at time t that were marked at $t=0$.

Lagrangian and Eulerdian coordinates.

1. Lagrange coordinates (classical mechanics)



Every particle is marked and followed in the flow.

Independent variables:

x_i° – initial position $i=1,2,3$.

$\hat{t}$ – time.

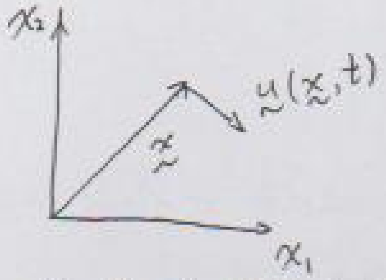
$$r_i = r_i(x_1^\circ, x_2^\circ, x_3^\circ, \hat{t}) = r_i(x_k^\circ, \hat{t})$$

Velocity of particle $u_i = \frac{\partial}{\partial \hat{t}} r_i(x_k^\circ, \hat{t})$

Acceleration of particles: $a_i = \frac{\partial^2 r_i}{\partial \hat{t}^2} = \frac{\partial u_i}{\partial \hat{t}}(x_k^0, \hat{t})$.

When x_k^0 changes we consider new particles. Any flow variable $F(x_k^0, \hat{t})$.

2. Euler coordinates.



Consider fixed point in space. Fluid flows past point $\underset{\sim}{x}$ at time t.

Independent variables: x_i – space coordinate
t – time

Fluid velocity at x_k and time t: $u_i(x_k, t)$. Flow variable $F(x_k, t)$

3. Relation between Lagrangian & Eulerian coordinates.

$\underset{\sim}{x}^0$ defines a Lagrangian particles passing the point.

$$\underset{\sim}{x} = \underset{\sim}{r}(\underset{\sim}{x}^0, \hat{t}) \text{ at the time } t = \hat{t}.$$

The Eulerian velocity is $\underset{\sim}{u}(\underset{\sim}{x}, t) = \frac{\partial}{\partial \hat{t}} \underset{\sim}{r}(\underset{\sim}{x}^0, \hat{t})$ at $t = \hat{t}$.

4. Material time derivative.

$F = F_L(\underset{\sim}{x}^0, \hat{t}) = F_E(\underset{\sim}{r}(\underset{\sim}{x}^0, t), t)$ at $t = \hat{t}$. Rate of change of F for fluid particle $\frac{\partial}{\partial \hat{t}} F_L(\underset{\sim}{x}^0, t)$. Express this in Euler coordinate.

$$\frac{\partial F_L}{\partial \hat{t}} = \frac{\partial F_E}{\partial x_i} \underbrace{\frac{\partial r_i}{\partial \hat{t}}}_{=u_i} + \frac{\partial F_E}{\partial t} \underbrace{\frac{\partial t}{\partial \hat{t}}}_{=1} = \frac{\partial F_E}{\partial t} + u_i \frac{\partial F_E}{\partial x_i}$$

$$\frac{\partial}{\partial \hat{t}} = \frac{D}{Dt} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i} = \frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3} = \frac{\partial}{\partial t} + \underset{\sim}{u} \cdot \nabla.$$

Material/Substantial time derivative.

$$\frac{D}{Dt} F = \frac{\partial F}{\partial t} + \underset{\sim}{u} \cdot \nabla F \longrightarrow \text{Advective rate of change as particle moves through special gradients of } F.$$

Rate of change for material fluid particle

Rate of change at fixed position in space.

Let F be fluid velocity $\underset{\sim}{u}(\underset{\sim}{x}, t)$.

$$\frac{D}{Dt}\underset{\sim}{u} = \frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} \leftarrow \text{Acceleration of material fluid particle.}$$

Let F be position vector $\underset{\sim}{x}$. $\frac{D}{Dt} x_i = \frac{\partial x_i}{\partial t} + u_j \frac{\partial x_i}{\partial x_j} = u_j \delta_{ij} = u_i$.

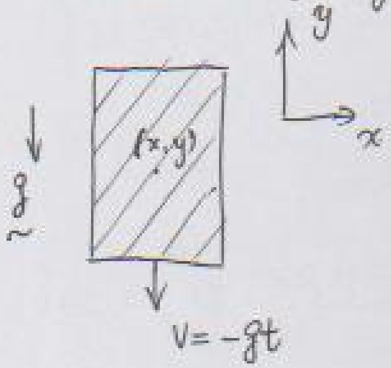
$$\frac{D}{Dt}\underset{\sim}{x} = \frac{\partial}{\partial t}\underset{\sim}{x} + \underset{\sim}{u} \nabla \underset{\sim}{x} = u \frac{\partial}{\partial x}\underset{\sim}{x} + v \frac{\partial}{\partial y}\underset{\sim}{x} + w \frac{\partial}{\partial z}\underset{\sim}{x} = u \underset{\sim}{e_x} + v \underset{\sim}{e_y} + w \underset{\sim}{e_z} = \underset{\sim}{u}.$$

Acceleration of fluid particle in river.

$$a_x = \underbrace{\frac{\partial u}{\partial t}}_{=0} + u\frac{\partial u}{\partial x} + \underset{0}{\underset{\parallel}{v\frac{\partial u}{\partial y}}} + \underset{0}{\underset{\parallel}{w\frac{\partial u}{\partial z}}} = u\frac{\partial u}{\partial x}$$

Acceleration of fluid element $\neq 0$ even if velocity at fixed x does not change change with time.

Acceleration of falling can filled with fluid.

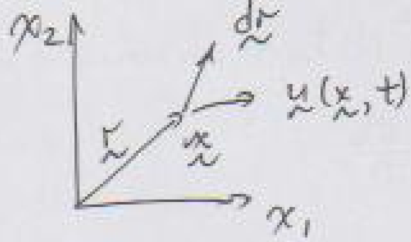


$$a_y = \frac{D}{Dt}v = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} = -g - gt\cdot 0 = -g$$

No gradients of velocity field in bucket.

$\Rightarrow$ advective term is zero.

Lecture 2:



Consider material line element.

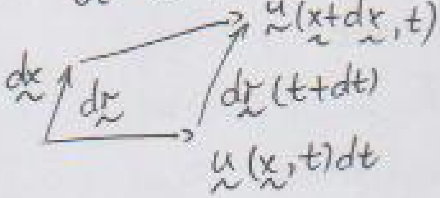
$$d\underset{\sim}{r} = \underset{\sim}{r}(\underset{\sim}{x}^{\circ} + d\underset{\sim}{x}^{\circ}, \hat{t}) - \underset{\sim}{r}(\underset{\sim}{x}^{\circ}, \hat{t}).$$

Translation with velocity

$$\frac{\partial \underset{\sim}{r}(\underset{\sim}{x}^{\circ}, \hat{t})}{\partial \hat{t}} = \underset{\sim}{u}(\underset{\sim}{x},t)\ ;\ \underset{\sim}{x} = \underset{\sim}{r}(\underset{\sim}{x}^{\circ}, \hat{t}),\ t = \hat{t}.$$

The instantaneous rate of change of length and orientation of a material line element is given by the relative motion.

$$\frac{\partial}{\partial \hat{t}}(\underset{\sim}{r}(\underset{\sim}{x}^{\circ} + dx^{\circ}, \hat{t})) = \underset{\sim}{u}(\underset{\sim}{x} + dx, t)\ ;\ d\underset{\sim}{x} = d\underset{\sim}{r}(\underset{\sim}{x}^{\circ}, d\underset{\sim}{x}^{\circ}, \hat{t}),\ t = \hat{t}.$$



$$\frac{\partial}{\partial \hat{t}} d\underset{\sim}{r} = \frac{D}{Dt} d\underset{\sim}{r} = \underset{\sim}{u}(\underset{\sim}{x} + dx, t) - u(\underset{\sim}{x}, t) = d\underset{\sim}{u}$$

Taylor expansion around $\underset{\sim}{x}$ $\Rightarrow$

Relative velocity

$$du_i = u_i(\underset{\sim}{x} + dx, t) - u_i(\underset{\sim}{x}, t)$$

$$= u_i(\underset{\sim}{x},t) + \frac{\partial u_i}{\partial x_1}(\underset{\sim}{x},t)dx_1 + \frac{\partial u_i}{\partial x_2}(\underset{\sim}{x},t)dx_2 + \frac{\partial u_i}{\partial x_3}(\underset{\sim}{x},t)dx_3 - u_i(\underset{\sim}{x},t)$$

$$= \sum_{j=1}^{3} \frac{\partial u_i}{\partial x_j}(\underset{\sim}{x},t)dx_j = \frac{\partial u_i}{\partial x_j}(x,t)dx_j$$

du_i are components of relative velocity of $\underset{\sim}{x}, t$

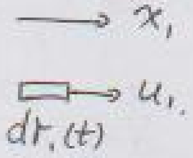
dx_j are components of considered line element.

$\frac{\partial u_i}{\partial x_j}(\underset{\sim}{x}, t)$ the velocity gradient tensor (independent of length & orientation of line element).

$$\frac{\partial u_i}{\partial x_j} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

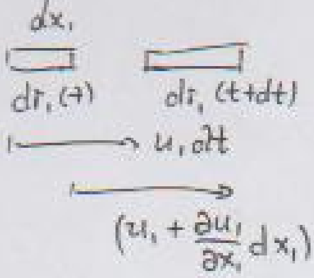
Evaluation of the effect on the relative motion from individual components of $\frac{\partial u_i}{\partial x_j}$

* Line element along x_1-axis :



$$\frac{D}{Dt} dr_1 = \frac{\partial u_1}{\partial x_1} dr_1 + 0 + 0$$

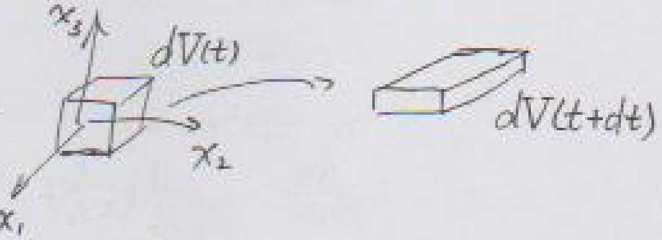
Normal (Linear) strain rate : $\frac{1}{dr_1}\frac{D}{Dt} dr_1 = \frac{\partial u_1}{\partial x_1}$



$$u_1 dt + dr_1(t+dt) = dr_1(t) + (u_1 + \frac{\partial u_1}{\partial x_1} dx_1) dt$$

$$\frac{dr_1(t+dt) - dr_1(t)}{dt} = \frac{D}{Dt} dr_1 = \frac{\partial u_1}{\partial x_1} dx_1$$

* Material volume $dV = dr_1 dr_2 dr_3$:

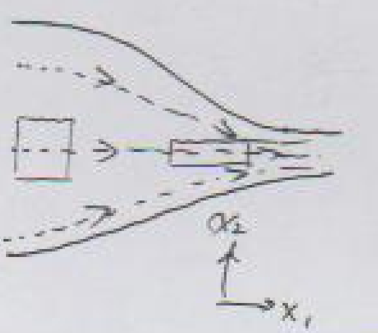


Bulk strain rate $\frac{1}{dV}\frac{D}{Dt} dV = \frac{1}{dr_1 dr_2 dr_3}\frac{D}{Dt}(dr_1 dr_2 dr_3) =$

$$= \frac{1}{dr_1}\frac{D}{Dt} dr_1 + \frac{1}{dr_2}\frac{D}{Dt} dr_2 + \frac{1}{dr_3}\frac{D}{Dt} dr_3 = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u_i}{\partial x_i} = \nabla \cdot \underset{\sim}{u}$$

For incompressible fluid $\frac{D}{Dt} dV = 0 \Leftrightarrow \nabla \cdot \underset{\sim}{u} = 0$.

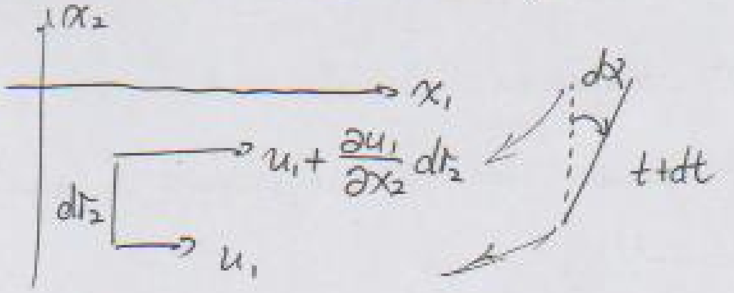
2D ex



Fix control volume

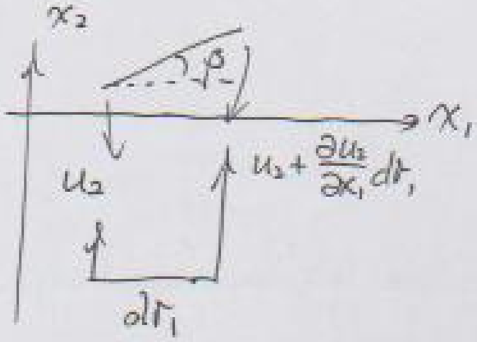
$$\underbrace{\frac{\partial u_1}{\partial x_1}}_{>0} + \underbrace{\frac{\partial u_2}{\partial x_2}}_{<0} = 0.$$

* Line element $\perp$ to the flow



$$\underbrace{\tan(d\alpha)}_{\approx d\alpha} = \frac{dt \cdot \frac{\partial u_1}{\partial x_2} dr_2}{dr_2} \Rightarrow$$

$$\frac{d\alpha}{dt} = \frac{\partial u_1}{\partial x_2}.$$



$$\tan(d\beta) = \frac{dt \frac{\partial u_2}{\partial x_1} dr_1}{dr_1} \Rightarrow \frac{d\beta}{dt} = \frac{\partial u_2}{\partial x_1}$$

Def: Shear strain rate:

$$\frac{d}{dt}\alpha + \frac{d}{dt}\beta = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$$

gives deformation in shape.

If $d\beta = -d\alpha$. $\Rightarrow$ No shape change, just rotation.

Vorticity: $$-\frac{d\alpha}{dt} + \frac{d\beta}{dt} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = \omega_3$$

The average rotation rate $= \frac{1}{2}\omega_3 = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$

Velocity gradient tensor

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\substack{\text{Symmetric part} \\ e_{ij} = e_{ji} \\ \text{Strain rate tensor}}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\substack{\text{Antisymmetric part.} \\ \xi_{ij} = -\xi_{ji} \\ \text{Rotation tensor}}}$$

$$e_{ij} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

$$e_{kk} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \nabla \cdot \underset{\sim}{u}$$

$$\xi_{ij} = \begin{bmatrix} 0 & -\frac{1}{2}\omega_3 & \frac{1}{2}\omega_2 \\ \frac{1}{2}\omega_3 & 0 & -\frac{1}{2}\omega_1 \\ -\frac{1}{2}\omega_2 & \frac{1}{2}\omega_1 & 0 \end{bmatrix} \qquad \begin{aligned} \omega_3 &= \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \\ \omega_1 &= \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \\ \omega_2 &= \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}. \end{aligned}$$

Shorter notation : $\xi_{ij} = -\frac{1}{2}\varepsilon_{ijk}\omega_k$.

Cross product: $(\underset{\sim}{a}\times\underset{\sim}{b})_i = \varepsilon_{ijk}a_j b_k$.

Rotation of 3D line element:

$$\left(\frac{D}{Dt}\,d\underset{\sim}{r}\right)_{rot} = \xi_{ij}dx_j = -\frac{1}{2}\varepsilon_{ijk}\omega_k dx_j = -\frac{1}{2}d\underset{\sim}{x}\times\underset{\sim}{\omega} = \frac{1}{2}\underset{\sim}{\omega}\times d\underset{\sim}{x}$$

i.e. solid body rotation at angular velocity $\frac{1}{2}\underset{\sim}{\omega}$. (c.f. $\dot{r} = \underset{\sim}{\Omega}\times\underset{\sim}{r}$)

One also finds that

$$\underset{\sim}{\omega} = \nabla\times\underset{\sim}{u} = \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ u_1 & u_2 & u_3 \end{vmatrix}.$$

Tensor notation:

$$\omega_i = \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j} = \varepsilon_{ijk}e_{kj} + \varepsilon_{ijk}\xi_{kj}$$

$$-\frac{1}{2}\varepsilon_{ijk}\omega_k = -\frac{1}{2}\varepsilon_{ijk}\varepsilon_{klm}\xi_{ml} = -\frac{1}{2}\xi_{ml}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) =$$

$$= -\frac{1}{2}\xi_{ji} + \frac{1}{2}\xi_{ij} = \frac{1}{2}\xi_{ij} + \frac{1}{2}\xi_{ij} = \xi_{ij}$$

Partition of e_{ij}

- Isotropic part.

$$\overline{\overline{e_{ij}}} = \begin{bmatrix} \frac{1}{3}e_{kk} & 0 & 0 \\ 0 & \frac{1}{3}e_{kk} & 0 \\ 0 & 0 & \frac{1}{3}e_{kk} \end{bmatrix} = \frac{1}{3}e_{kk}\delta_{ij} = \frac{1}{3}\nabla\cdot\underset{\sim}{u}\,\delta_{ij}$$

gives deformation due to volume expansion

- Deviatoric part.

$$\overline{e_{ij}} = e_{ij} - \overline{\overline{e_{ij}}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)$$

gives deformation rate without change of volume.

Traceless $\overline{e_{kk}} = 0$

Summary of relative motion.

$$\frac{D}{Dt}dr_i = du_i = \frac{\partial u_i}{\partial x_k}dx_k = \underbrace{\overline{e_{ik}}\,dx_k}_{(1)} + \underbrace{\overline{\overline{e_{ik}}}\,dx_k}_{(2)} + \underbrace{\xi_{ik}\,dx_k}_{(3)}$$

The terms of du_i are result of:

(1) Pure deformation, without change of volume

(2) Isotropic change of volume

(3) Local solid body rotation.

Ex: Shear flow

$u_1 = \dot{\gamma} x_2$,

$$\frac{\partial u_i}{\partial x_j} = \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{2}\dot{\gamma} & 0 \\ \frac{1}{2}\dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{e_{ij}} + \underbrace{\begin{bmatrix} 0 & \frac{1}{2}\dot{\gamma} & 0 \\ -\frac{1}{2}\dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\xi_{ij}}$$

$$e_{kk} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0 + 0 + 0$$. No rotational deformation.

$\bar{\bar{e}}_{ij} = 0$. Isotropic strain rate tensor.

Relative velocity field

$$du_i = \frac{\partial u_i}{\partial x_k} \cdot dx_k = \frac{1}{2}\begin{bmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & \gamma & 0 \\ -\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

$$= \underbrace{\frac{\gamma}{2}\begin{bmatrix} x_2 \\ x_1 \\ 0 \end{bmatrix}}_{(du_i)_{def}} + \underbrace{\frac{\gamma}{2}\begin{bmatrix} x_2 \\ -x_1 \\ 0 \end{bmatrix}}_{(du_i)_{rot}}$$

Streamlines : $\dfrac{dx_2}{dx_1} = \dfrac{u_2}{u_1}$

Deformation $\dfrac{dx_2}{dx_1} = \dfrac{\frac{1}{2}\gamma x_1}{\frac{1}{2}\gamma x_2} = \dfrac{x_1}{x_2} \Rightarrow x_2 dx_2 = x_1 dx_1$

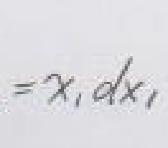
$$\Rightarrow \frac{x_2^2}{2} = \frac{x_1^2}{2} + d$$

Rotation $\dfrac{dx_2}{dx_1} = \dfrac{-\frac{1}{2}\gamma x_1}{\frac{1}{2}\gamma x_2} = -\dfrac{x_1}{x_2} \Rightarrow x_2 dx_2 + x_1 dx_1 = 0$

$$\Rightarrow \frac{x_2^2}{2} + \frac{x_1^2}{2} = 0 .$$

Lecture 3.

Forces on a fluid element

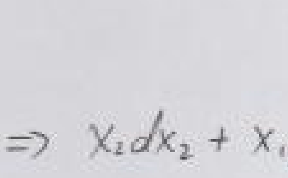


- External body forces (long range interaction)

  Proportional to mass of fluid element

- Surface forces (short range interaction)

  Contact forces cancel.

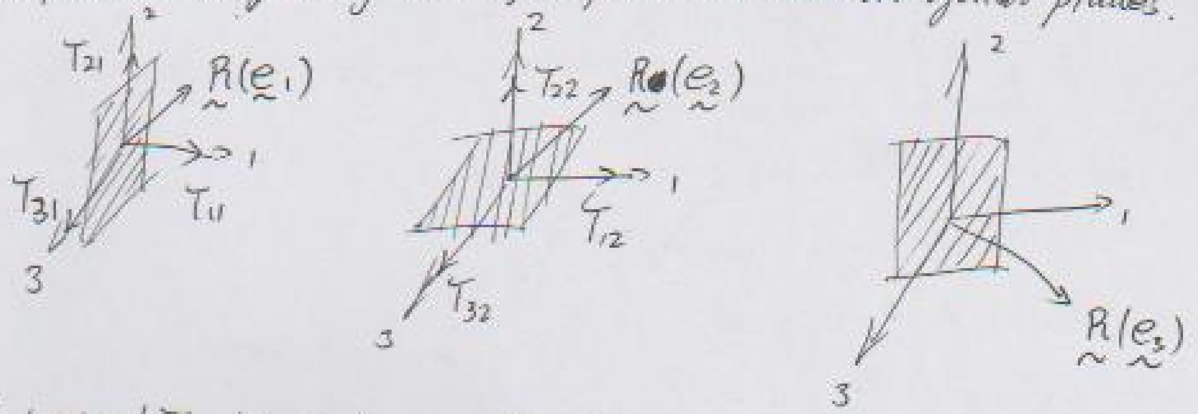


  Proportional to area of fluid element $d\underset{\sim}{F} = \underbrace{\underset{\sim}{R}(\underset{\sim}{n}, \underset{\sim}{x})}_{\text{Stress force (force/unit area)}} dS$

Stress tensor

The stress state is uniquely determined by a tensor field $T_{ij}(\underset{\sim}{x}, t)$

The components are given by the surface forces on three orthogonal planes.



$\underset{\sim}{R}(\underset{\sim}{e}_1) = (T_{11}, T_{12}, T_{13})$; $\underset{\sim}{R}(\underset{\sim}{e}_2) = (T_{12}, T_{22}, T_{32})$

T_{ij} is i-component of stress on surface element with unit normal in j-direction.

$$T_{ij} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

shear stresses

normal stresses

The stress tensor may be used to obtain the stress on a surface with any direction

Let $d\underset{\sim}{S} = \underset{\sim}{n}\,dS$ or $dS_i = n_i\,dS$

Surface forces must balance each other to lowest order on small fluid element.

at dS : $dF_i = R_i\,dS$

at orthogonal surfaces: $-T_{i1}dS_1 - T_{i2}dS_2 - T_{i3}dS_3 = -T_{ij}dS_j = -T_{ij}n_j\,dS$

$\Rightarrow$ Net force $R_i\,dS - T_{ij}n_j\,dS = 0 \Rightarrow R_i = T_{ij}n_j \Leftrightarrow \underset{\sim}{R} = \underset{\sim}{T}\cdot\underset{\sim}{n}$.

Moment balance around fluid particle shows that: $T_{ij} = T_{ji}$.

$\Rightarrow$ T_{ij} has only 6 independent components.

Pressure and viscous stress:

For fluid element at rest only isotropic normal stresses are present.

At rest : $T_{ij} = -p\,\delta_{ij} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$ At deformation : $T_{ij} = -p\delta_{ij} + \tau_{ij}$

p is hydrostatic pressure (directed inward). τ_{ij} is viscous stress tensor, depends on the fluids rate of deformation.

Newtonian fluid.

Viscous stresses proportional to deformation rate of fluid element.

$$\tau_{ij} = \underbrace{2\mu(T)\,\bar{e}_{ij}}_{\text{deviatoric strain rate tensor}} + \underbrace{\mu_B(T)\,\bar{\bar{e}}_{ij}}_{\text{isotropic strain rate tensor}}$$

τ_{ij} does not depend on rotation tensor ξ_{ij}

$\mu(T)$ dynamic viscosity $[kg/ms]$.

$\mu_B(T)$ bulk viscosity $[kg/ms]$

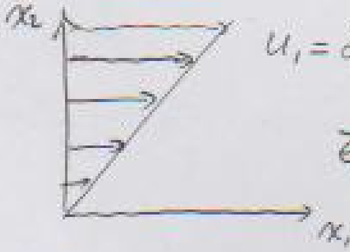
(in air $\mu_B \sim 0{,}6\mu$, in CO_2 $\mu_B \sim 1000\mu$)

$\nu = \mu/\rho$ kinematic viscosity $[m^2/s]$

μ_B often disregarded since usually $\left|\frac{\partial u_n}{\partial x_n}\right| \lll |\bar{e}_{ij}|$

(exception, e.g. strong shock waves).

Ex:



$u_1 = \alpha x_2$, $u_2 = 0$ $\quad \frac{\partial u_n}{\partial x_n} = 0$

$$\bar{e}_{ij} = \begin{bmatrix} 0 & \frac{1}{2}\alpha \\ \frac{1}{2}\alpha & 0 \end{bmatrix}; \quad \bar{\bar{e}}_{ij} = 0; \quad \xi_{ij} = \begin{bmatrix} 0 & \frac{1}{2}\alpha \\ -\frac{1}{2}\alpha & 0 \end{bmatrix}$$

$$T_{ij} = -p\,\delta_{ij} + 2\mu\,\bar{e}_{ij} = \begin{bmatrix} -p & \mu\alpha \\ \mu\alpha & -p \end{bmatrix}$$

Here viscous stresses appear normal to surface

$$R_i = T_{ij} n_j = \begin{bmatrix} -p & \mu\alpha \\ \mu\alpha & -p \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = -\frac{p}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{\mu\alpha}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\underset{\sim}{R}(\underset{\sim}{n}) = (-p + \mu\alpha)\,\underset{\sim}{n}.$$

Conservation of momentum; Newton $\Rightarrow$

$$\frac{D}{Dt}\int_{\underbrace{V(t)}_{\text{material volume}}} \rho u_i \, dV = \int_V \rho f_i \, dV + \oint_S T_{ij} n_j \, dS \Rightarrow \frac{D}{Dt}\int_{V(t)} \rho \underset{\sim}{u} \, dV = \int_V \rho \underset{\sim}{f} \, dV + \oint_S \underset{\sim}{T}\,\underset{\sim}{n} \, dS$$

Transform to fixed volume:

$$\frac{D}{Dt}\int_{\underbrace{V(t)}_{\text{material volume}}} F \, dV = \frac{d}{dt}\int_{\underbrace{V}_{\text{fixed volume}}} F \, dV + \underbrace{\oint_S F(\underset{\sim}{u}\cdot\underset{\sim}{n}) \, dS}_{\text{net outflow of } F \text{ through fixed surface.}}$$

Gauss' theorem: $\iiint_V \frac{\partial}{\partial x_k}(A_{ijk})\, dV = \oint_S A_{ijk} n_k \, dS.$

$$\Downarrow \quad \frac{D}{Dt}\int_{V(t)} F\,dV(t) = \int_V \frac{D}{Dt}F\,dV + \int_V F\frac{D}{Dt}dV = \int_V \left(\frac{\partial F}{\partial t} + \underset{\sim}{u}\cdot\nabla F\right)dV + \int_V F\cdot\nabla\cdot\underset{\sim}{u}\,dV$$

$$= \int_V \left(\frac{\partial F}{\partial t} + \nabla\cdot(F\underset{\sim}{u})\right)dV = \left\{\text{Gauss theorem}\right\} = \frac{d}{dt}\int_V F\,dV + \oiint_S F(\underset{\sim}{u}\cdot\underset{\sim}{n})\,dS$$

Newtons law of motion in fixed open volume. (Momentum theorem)

$$\underbrace{\frac{d}{dt}\int_V \rho\underset{\sim}{u}\,dV}_{\substack{\text{Rate of change}\\ \text{of momentum}\\ \text{in fixed control}\\ \text{volume}}} + \underbrace{\oiint_S \rho\underset{\sim}{u}(\underset{\sim}{u}\cdot\underset{\sim}{n})\,dS}_{\substack{\text{Net outflow of}\\ \text{momentum per}\\ \text{unit time}}} = \underbrace{\int_V \rho\underset{\sim}{f}\,dV}_{\substack{\text{bodies}\\ \text{force}\\ \text{on } V}} + \underbrace{\oiint_S \underset{\sim}{T}\cdot\underset{\sim}{n}\,dS}_{\substack{\text{surface}\\ \text{forces}\\ \text{on } V.}}$$

Gauss' theorem $\Rightarrow$

$$\iiint_V \left\{\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_k}(\rho u_i u_k) - \rho f_i - \frac{\partial}{\partial x_k}(\tau_{ik})\right\} dV = 0$$

Should hold for arbitrary volume $\Rightarrow$ Differential form of Newton law of motion:

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_k}(\rho u_i u_k) = \rho f_i + \frac{\partial}{\partial x_k}\tau_{ik}.$$

$$\underbrace{u_i\left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k)\right)} + \rho\frac{\partial u_i}{\partial t} + \rho u_k\frac{\partial}{\partial x_k}u_i = \rho f_i + \frac{\partial \tau_{ik}}{\partial x_k}$$

$$u_i\left(\underbrace{\frac{\partial \rho}{\partial t} + u_k\frac{\partial \rho}{\partial x_k}} + \rho\frac{\partial u_n}{\partial x_n}\right)$$

$$\frac{D\rho}{Dt} + \rho\frac{1}{dV}\frac{D}{Dt}dV = \frac{1}{dV}\underbrace{\left(\frac{D\rho}{Dt}dV + \rho\frac{D}{Dt}dV\right)}_{\frac{D}{Dt}(\rho\,dV)=0}$$

$\underbrace{\rho\,dV}_{dm}$, fluid mass element.

Reimann Cauchy's equation of motion.

$$\rho\frac{D}{Dt}u_i = \rho f_i + \frac{\partial}{\partial x_k}\tau_{ik}$$

Navier-Stoke's equation are obtained if we insert

$$T_{ij} = -p\delta_{ij} + 2\mu\overline{e_{ij}} + \mu_B\overline{\overline{e_{ij}}}$$

Lecture 4.

Complete set of conservation equations.

Conservation of:

1) Mass : Continuity equation.
2) Momentum : Newton's law of motion.
3) Energy : 1:st law of thermodynamics.

1) $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k) = 0$$ Continuity equation.

2) $$\rho \frac{D}{Dt} u_i = \rho f_i + \frac{\partial}{\partial x_k} \tau_{ik}$$ Cauchy's equation.

In 2) let $\tau_{ik} = -p\delta_{ik} + 2\mu \overline{e_{ik}} + \mu_B \overline{\overline{e_{ik}}}$:

$$\frac{\partial}{\partial x_k}\tau_{ik} = -\frac{\partial p}{\partial x_k}\delta_{ik} + \frac{\partial}{\partial x_k}\left(2\mu(T)\overline{e_{ik}}\right) + \frac{\partial}{\partial x_k}\left(\mu_B(T)\frac{\delta_{ik}}{3}\overline{\overline{e_{ik}}}\right)$$

$$= -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_k}\left(2\mu(T)\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) - \frac{1}{3}\delta_{ik}\frac{\partial u_j}{\partial x_j}\right]\right) + \frac{\partial}{\partial x_i}\left(\mu_B(T)\frac{1}{3}\frac{\partial u_j}{\partial x_j}\right.$$

Insert then we got the Navier-Stoke's equation.

- For incompressible fluid: $\frac{\partial u_j}{\partial x_j} = 0$.
- Assume μ = constant :

$$\frac{\partial \tau_{ik}}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

$$= -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_k \partial x_k} + \mu\frac{\partial}{\partial x_i}\underbrace{\left(\frac{\partial u_k}{\partial x_k}\right)}_{=0} \Rightarrow$$

Navier-Stoke's equation for incompressible fluid.

$$\rho\left(\frac{\partial u_i}{\partial t} + u_k\frac{\partial u_i}{\partial x_k}\right) = -\frac{\partial p}{\partial x_i} + \rho f_i + \mu\nabla^2 u_i$$

net pressure force

body force per unit volume

net viscous force.

Conservation of energies

1:s law of thermodynamics: Rate of change of total energy (thermal + mechanical) of material fluid volume equals rate of energy received by transport of heat and execution of work.

$$\frac{D}{Dt}\int_{V(t)} \rho\left(e+\tfrac{1}{2}u_k u_k\right)dV = \{RTT\} =$$

$$= \frac{d}{dt}\int_V \rho(\underset{\substack{\uparrow\\ \text{thermal}}}{e}+\tfrac{1}{2}\underset{\substack{\uparrow\\ \text{mechanical}}}{u_k u_k})\,dV + \underbrace{\oint_S \rho\left(e+\tfrac{1}{2}u_k u_k\right)u_j n_j\,dS}_{\text{net outflow of energy with velocity field}}$$

energy per unit mass

$$= \underbrace{\int_V u_k \rho f_k\,dV}_{\text{work rate by body force}} + \underbrace{\oint_S u_k \tau_{kj} n_j\,dS}_{\text{work rate by surface force}} - \underbrace{\oint_S q_j n_j\,dS}_{\text{net energy outflow by heat conduction}} \Rightarrow$$

$$\int_V \rho\frac{D}{Dt}\left(e+\tfrac{1}{2}u_k u_k\right)dV = \int_V \left(u_k\rho f_k + \frac{\partial}{\partial x_j}\left(u_k\tau_{kj}-q_j\right)\right)dV$$

Differential form of total energy equation.

$$\rho\frac{D}{Dt}\left(e+\tfrac{1}{2}u_k u_k\right) = u_k\rho f_k + \frac{\partial}{\partial x_j}\left(u_k\tau_{kj}\right) - \frac{\partial}{\partial x_j}q_j$$

Fouriers law for heat conduction (diffusion of heat).

$$q_j = -k\frac{\partial T}{\partial x_j} \quad ; \quad \underset{\sim}{q} = -k\nabla T \Rightarrow \text{heat flux density vector } \underset{\sim}{q}\ [J/m^2 s].$$

Thermal conductivity $k(T)$. $k_{H_2O} \sim 0{,}6\ J/msK$.

$k_{air} \sim 0{,}025\ J/msK$

Thermal diffusivity $\kappa = k/\rho c_p\ [m^2/s]$

$$-\frac{\partial q_j}{\partial x_j} = \frac{\partial}{\partial x_j}\left(k(T)\frac{\partial T}{\partial x_j}\right) = \{k = \text{const}\} = k\frac{\partial^2 T}{\partial x_j \partial x_j} = \underbrace{k\nabla^2 T}_{\text{diffusion of heat}}$$

Net work rate by surface forces

$$\frac{\partial}{\partial x_j}(u_k \tau_{kj}) = \underbrace{u_k \frac{\partial}{\partial x_j}\tau_{kj}}_{\text{Translational work rate}} + \underbrace{\tau_{kj}\frac{\partial u_k}{\partial x_j}}_{\text{Deformation work rate}}$$

We will show: Translational work rate changes mech. energy; Deformation work rate changes thermal energy

Mechanical energy equation: $u_k\left(\rho \frac{D}{Dt}u_k = \rho f_k + \frac{\partial}{\partial x_j}\tau_{kj}\right) \Rightarrow$

$$\rho\frac{D}{Dt}\left(\frac{u_k u_k}{2}\right) = u_k \rho f_k + u_k \frac{\partial}{\partial x_j}\tau_{kj} \Rightarrow \text{Subtract from total energy} \Rightarrow$$

$$\rho\frac{D}{Dt}e = \tau_{kj}\frac{\partial u_k}{\partial x_j} - \frac{\partial}{\partial x_j}q_j \Rightarrow \text{Insert } \tau_{kj} = -p\delta_{kj} + \tau_{kj} \Rightarrow$$

$$\tau_{kj}\frac{\partial u_k}{\partial x_j} = \underbrace{-p\frac{\partial u_k}{\partial x_k}}_{\substack{\text{work rate}\\ \text{by pressure from}\\ \text{isotropic exp.}\\ \text{(reversible)}}} + \underbrace{\tau_{kj} e_{kj}}_{\substack{\text{work rate}\\ \text{by visc. str.}\\ \text{from deformation}\\ \text{(irreversible)}}} + \underbrace{\underbrace{\tau_{kj}}_{S}\ \underbrace{\xi_{kj}}_{AS}}_{=0}$$

Dissipation function $\Phi = \tau_{kj}e_{kj} = \dots = 2\mu \tilde{e}_{kj}\tilde{e}_{kj} + \mu_B\left(\frac{\partial u_k}{\partial x_k}\right)^2 > 0$

Energy equations in differential form:

$$\rho\frac{D}{Dt}e + p\frac{\partial u_k}{\partial x_k} = -\frac{\partial q_j}{\partial x_j} + \Phi$$

$$\rho\frac{D}{Dt}\left(\frac{u_k u_k}{2}\right) + u_k\frac{\partial p}{\partial x_k} = u_k \rho f_k + \frac{\partial}{\partial x_j}(u_k \tau_{kj}) - \Phi$$

For perfect gas: $e = C_v T$, $p = \rho R T$.

Incompressible fluid $\rho = \rho_0 = \text{const}$; $e = cT$. $\Rightarrow$

$$\rho_0 C_p \frac{DT}{Dt} = k\nabla^2 T + \Phi \quad \text{or} \quad \frac{\partial T}{\partial t} + \underbrace{\underset{\sim}{u}\cdot\nabla T}_{\text{advection}} = \underbrace{\kappa\nabla^2 T}_{\text{diffusion}} + \underbrace{\Phi/\rho_0 C_p}_{\text{dissipation}}$$

Enthalpy $h = e + p/\rho$ inserted into thermal equation:

$$\Rightarrow \rho\frac{D}{Dt}h - \frac{D}{Dt}p = -\frac{\partial q_j}{\partial x_j} + \Phi$$

Perfect gas $h = C_p T$, $p = \rho R T$.

Details of dissipation function derivation:

$$\Phi = \tau_{ij} e_{ij} = (2\mu \overline{e_{ij}} + \mu_B \delta_{ij} \overline{\overline{e_{kk}}})(\overline{e_{ij}} + \tfrac{1}{3}\delta_{ij}\overline{\overline{e_{kk}}})$$

$$= 2\mu \overline{e_{ij}}\,\overline{e_{ij}} + 2\mu \tfrac{1}{3} \underbrace{\overline{e_{ii}}}_{=0} \overline{\overline{e_{kk}}} + \mu_B \overline{\overline{e_{kk}}}\, \underbrace{\overline{e_{ii}}}_{=0} + \mu_B \overline{\overline{e_{kk}}}^2 \frac{\overbrace{\delta_{ii}}^{=3}}{3}$$

$$= 2\mu \underbrace{\overline{e_{ij}}\,\overline{e_{ij}}}_{\geq 0} + \mu_B \underbrace{\overline{\overline{e_{kk}}}^2}_{\geq 0} \quad ; \quad \overline{\overline{e_{kk}}} = \frac{\partial u_k}{\partial x_k}.$$

Details of enthalpy derivation.

$$\frac{D}{Dt} h = \frac{D}{Dt} e + \frac{1}{\rho}\frac{Dp}{Dt} + p \underbrace{\frac{D}{Dt}\frac{1}{\rho}}_{-\frac{1}{\rho^2}\frac{D\rho}{Dt} = \frac{1}{\rho}\frac{\partial u_k}{\partial x_k}}$$

$$\frac{D}{Dt} h = \frac{De}{Dt} + \frac{1}{\rho}\frac{Dp}{Dt} + \frac{p}{\rho}\frac{\partial u_k}{\partial x_k}$$

Boussinesque approximation

$$\rho \frac{D\underset{\sim}{u}}{Dt} = -\nabla p + \rho \underset{\sim}{g} + \mu \nabla^2 \underset{\sim}{u} \quad ; \quad \rho \frac{Dh}{Dt} - \frac{Dp}{Dt} = k\nabla^2 T + \Phi$$

Assume $\rho = \rho_0 = \text{const}$, but. $\rho \underset{\sim}{g} = (\rho_0 - \rho_0 \alpha (T - T_0))\underset{\sim}{g}$. where

$$\alpha = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p \rightarrow \text{coefficient of thermal expansion.}$$

Mainly hydrostatic balance $0 = -\nabla p_0 + \rho_0 \underset{\sim}{g}$

Small departure : $p = p_0(\underset{\sim}{x}) + p'$, $p' \ll p_0$

$$\rho_0 \frac{D}{Dt}\underset{\sim}{u} = -\nabla p' - \rho_0 \alpha (T - T_0)\underset{\sim}{g} + \mu \nabla^2 \underset{\sim}{u}$$

$$\rho_0 C_p \frac{D}{Dt} T - \underset{\sim}{u} \nabla p_0 = k \nabla^2 T + \Phi$$

$$\underbrace{\rho_0 C_p \left(\frac{\partial T}{\partial t} + \underset{\sim}{u}\cdot\nabla T\right)}_{\sim \rho_0 C_p \frac{U\Delta T}{L}} + \underbrace{\underset{\sim}{u}\,\rho_0\,\underset{\sim}{g}}_{\sim U\rho_0 g} = k^* \nabla^2 T + \Phi$$

$$\frac{U\rho_0 g}{\rho_0 C_p U \Delta T / L} \sim \frac{gL}{C_p \Delta T} \sim \frac{10\,m/s^2 \cdot 1m}{10^3 J/kgK \cdot 1K} = 10^{-2} \ll 1.$$

$$\Phi \sim \mu \left(\frac{U}{L}\right)^2 \sim \frac{\mu U}{L^2}\cdot U \sim \rho_0 \alpha \Delta T g U$$

$$\frac{\Phi}{\rho C_p \underset{\sim}{u}\nabla T} \sim \frac{\rho_0 \alpha \Delta T g U}{\rho_0 C_p U \Delta T / L} \sim \underbrace{\alpha \Delta T}_{\ll 1} \underbrace{\frac{gL}{C_p \Delta T}}_{\ll 1} \ll 1.$$

$$\rho_0 C_p \frac{DT}{Dt} = k \nabla^2 T$$

for Boussinesque approx.

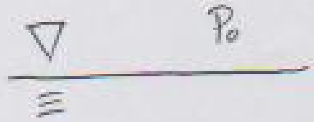
Lecture 5.

Navier-Stoke's equations.

Boundary conditions

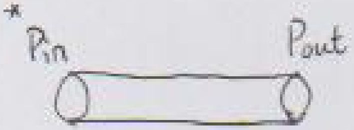
* $-u_i = 0$ on solid surfaces at rest

"no slip" condition of viscous fluid.



* Surface force given at boundary

-e.g. stagnant water surface at constant pressure.



* - Pressure given at the inlet/outlet of a pipe flow.

- Zero shear stress at free surface.

Thermal boundary condition.

* $T = T_{wall}$ temperature specified at solid surface

* $-k\nabla T \cdot \underset{\sim}{n} = 0$ thermal isolated surface.

Exact analytical solutions of the N-S equations: Rare example:

- Geometric idealizations.

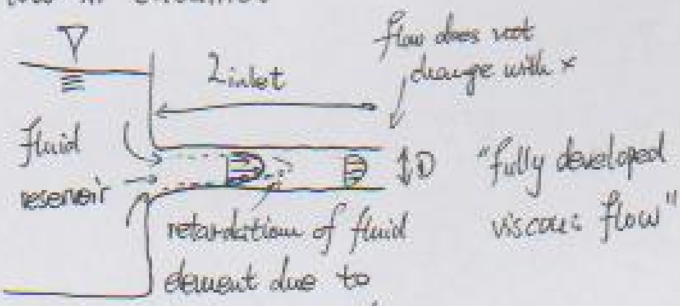
  * 2D flow * 1 infinite dimension * axial symmetry.

- Physical assumptions

  * Steady state * "fully developed flow" * incompressible fluid

  * constant viscosity / heat conductivity.

Flow in channel

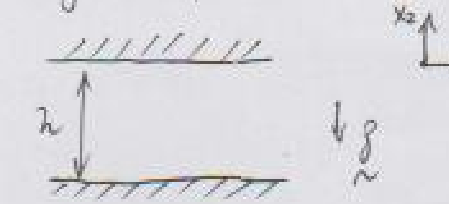


Inlet length for laminar flow

$L_{inlet} \sim D\,Re$; $Re = \frac{\bar{U}D}{\nu}$.

$Re \ll 1$, $L_{inlet} \sim D$.

Fully developed channel flow



* $\underset{\sim}{u} = (u_1(x_2), u_2(x_2), 0)$
* incompressible flow: $\underbrace{\frac{\partial u_1}{\partial x_1}}_{0} + \frac{\partial u_2}{\partial x_2} + \underbrace{\frac{\partial u_3}{\partial x_3}}_{0} = 0 \Rightarrow$

$u_2 = \text{const} = 0$ from b.c.

* Assume steady state flow. 2D momentum equation $\Rightarrow$

$$\rho\left(\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2}\right) = -\frac{\partial p}{\partial x_1} + \mu\frac{\partial^2 u_1}{\partial x_1^2} + \mu\frac{\partial^2 u_1}{\partial x_2^2}.$$

$$\rho\left(\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_2}{\partial x_2}\right) = -\frac{\partial p}{\partial x_2} - \rho g + \mu\frac{\partial^2 u_2}{\partial x_1^2} + \mu\frac{\partial^2 u_2}{\partial x_2^2} \Rightarrow$$

$$\begin{cases} 0 = -\frac{\partial p}{\partial x_1} + \mu\frac{\partial^2 u_1}{\partial x_2^2} \\ 0 = -\frac{\partial p}{\partial x_2} - \rho g \end{cases} \Rightarrow \quad \frac{dp}{dx_1} \text{ is constant.} \quad \frac{d^2u_1}{dx_2^2} = \frac{1}{\mu}\frac{dp}{dx_1} = \text{constant.}$$

We study two cases:

A) $\frac{dp}{dx_1} = 0$. Forcing by upper wall.

B) $\frac{dp}{dx_1} < 0$ Forcing by pressure difference. $P_{out} < P_{in}$.

A) Integral + b.c.: $u_1 = U_0 \frac{x_1}{h}$. Plane Couette flow.

Calculate shear stress $\tau_{12} = 2\mu\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \mu\frac{du_1}{dx_2} = \frac{\mu_0 U_0}{h}$.

B) Integrate + b.c.: $u_1(x_2) = -\frac{h^2}{2\mu}\frac{dp}{dx_1}\frac{x_2}{h}\left(1-\frac{x_2}{h}\right) = u_{max}\, 4\frac{x_2}{h}\left(1-\frac{x_2}{h}\right)$

Plane Poiseuille flow. $\quad u_{max} = -\frac{h^2}{8\mu}\frac{dp}{dx_1}$

Volumetric flow rate / unit width:

$$Q = \int_0^h u_1(x_2)dx_2 = u_{max}\int_0^h 4\frac{x_2}{h}\left(1-\frac{x_2}{h}\right) = U_{max}\frac{4h}{6} = -\frac{h^3}{12\mu}\frac{dp}{dx_1}$$

Average velocity $\bar{U} = \frac{Q}{h} = \frac{2}{3}U_{max}$

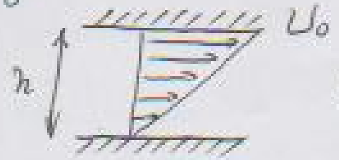
$$u_1(x_2) = \frac{3}{2}\bar{U}\,4\cdot\frac{x_2}{h}\left(1-\frac{x_2}{h}\right); \quad \tau_{12} = \mu\frac{du_1}{dx_2} = \mu\frac{3}{2}\frac{\bar{U}}{h}4\left(1-\frac{2x_2}{h}\right)$$

$$\Rightarrow \tau_{12}(h) = -\frac{6\mu\bar{U}}{h}, \quad \tau_{12}(h/2) = 0, \quad \tau_{12}(0) = \frac{6\mu\bar{U}}{h}$$

$$-\frac{dp}{dx_1}L\cdot h - 2\cdot 6\frac{\mu\bar{U}L}{h} = 0 \Rightarrow \bar{U} = -\frac{dp}{dx_1}\frac{h^2}{12\mu}.$$

Exact solution to energy equation

Plane Couette flow :



$u_1 = \frac{U_0 x_2}{h}$

$\tau_{12} = \mu \frac{U_0}{h}$ shear stress executes work rate / unit area of plate $U_0 \tau_{12} = \frac{\mu U_0^2}{h}$.

Energy equation: $0 = u_1 \frac{d}{dy}\tau_{12}$; $\frac{d}{dy}(u_1 \tau_{12}) - \tau_{12}\frac{du_1}{dy} = \frac{d}{dy}(u_1 \tau_{12}) - \underbrace{\mu\left(\frac{du_1}{dy}\right)^2}_{\Phi} = 0$

Dissipation function. $\Phi = 2\mu e_{ij} e_{ij}$ $[J/m^3 s]$.

$$e_{ij} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}\frac{du_1}{dx_2} \\ \frac{1}{2}\frac{du_1}{dx_2} & 0 \end{bmatrix} \Rightarrow \Phi = \mu\left(\frac{U_0}{h}\right)^2$$

$$\int_0^h \frac{d}{dy}(u_1 \tau_{12})\,dy - \int_0^h \Phi\,dy = 0 \quad \Rightarrow \quad U_0 \tau_{12} - \mu\left(\frac{U_0}{h}\right)^2 h = 0 \quad \Rightarrow$$

Work goes into heat by viscous dissipation

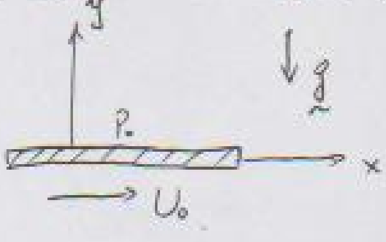
Lecture 6.

Navier-Stoke's equations in curvilinear coordinates.

Ex: Flow between coaxial rotating cylinder

Time-dependent flow with inertial effects.

Instantaneous start of infinite plate (Stoke's 1st problem).



Assume $\underset{\sim}{u} = (u(y,t), v)$

Continuity $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 + B.C. \Rightarrow v = 0$.

x-momentum : $\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$

y-momentum : $0 = -\frac{\partial p}{\partial y} - \rho g \Rightarrow p = p_0 - \rho g y$.

$\Rightarrow \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$; $\nu = \frac{\mu}{\rho}$ → diffusion equation.

Initial condition: $u(y, t=0) = 0$; $0 \leq y < \infty$

Boundary condition : $\begin{cases} u(y=0,t) = U_0 & t>0 \\ u(y\to\infty, t) = 0 & t\geq 0 \end{cases}$

Methods of solution:

* Seperation of variables. * Laplace trans * Similarity ansatz

Similarity solution of the form: $u(y,t) = U_0 f(y/\delta(t)) = U_0 f(\eta(y,t))$

$\delta(t)$ is the diffusion length (penetration length of b.c.)

$$\left(\frac{\partial u}{\partial t}\right)_y = U_0 \frac{df}{d\eta}\left(\frac{\partial \eta}{\partial t}\right)_y = U_0 \frac{df}{d\eta}\left(-\frac{y}{\delta^2}\right)\frac{d\delta}{dt} = -U_0 \frac{df}{d\eta}\eta\frac{1}{\delta}\frac{d\delta}{dt}$$

$$\left(\frac{\partial u}{\partial y}\right)_t = U_0 \frac{df}{d\eta}\left(\frac{\partial \eta}{\partial y}\right)_t = U_0 \frac{df}{d\eta}\frac{1}{\delta} \;;\; \left(\frac{\partial^2 u}{\partial y^2}\right)_t = U_0 \frac{d^2 f}{d\eta^2}\frac{1}{\delta^2} \quad \Rightarrow$$

$$-\frac{df}{d\eta}\eta\underbrace{\frac{1}{\delta}\frac{d\delta}{dt}} = \nu\frac{d^2f}{d\eta^2}\frac{1}{\delta^2}$$

both dependent of t, must be same function of t.

$$\Rightarrow \frac{1}{\delta(t)}\cdot\frac{d}{dt}\delta = C_1\frac{1}{\delta^2(t)} \Rightarrow \frac{d}{dt}\left(\frac{\delta^2}{2}\right) = C_1 = 2\nu \Rightarrow \delta^2 = 4\nu t.$$

O.D.E. $\quad \dfrac{\partial^2 f}{d\eta^2} + 2\eta\dfrac{df}{d\eta} = 0 \quad$ B.C. $\begin{cases} \dfrac{u(0,t)}{U_0} = f(0) = 1 \\ \dfrac{u(\infty,t)}{U_0} = f(\eta = \frac{y}{\delta} \to \infty) = 0 \end{cases} \Rightarrow$

I.C. $\dfrac{u(y,0)}{U_0} = f(y \to \infty) = 0$

$$\frac{f''}{f'} = -2\eta \Rightarrow \underbrace{\int_0^\eta \frac{f''}{f'}d\eta}_{\ln|f'(\eta)/f'(0)|} = -\eta^2 \Rightarrow f'(\eta) = f'(0)e^{-\eta^2} \Rightarrow f(\eta) = f'(0)\int_0^\eta e^{-z^2}dz + \underbrace{f(0)}_{=1}$$

$$\Rightarrow f(\eta) = 1 - \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-z^2}dz = 1 - \frac{2}{\sqrt{\pi}}\int_0^{y/\sqrt{4\nu t}} e^{-z^2}dz = 1 - \mathrm{erf}(y/\sqrt{4\nu t})$$

Shear stress $\tau_{xy} = \mu\dfrac{\partial u}{\partial y} = \mu U_0\dfrac{df}{d\eta}\dfrac{1}{\delta(t)} = -\dfrac{\mu U_0}{\sqrt{\pi\nu t}}e^{-(y/4\nu t)^2}$

$t \to 0 \Rightarrow |\tau_{xy}| \to \infty$ as infinite force is required to instantaneously accelerate the fluid.

Vorticity $\omega_z = \omega_3 = \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2} = -\dfrac{\partial u}{\partial y} = -\dfrac{U_0}{\delta}\dfrac{df}{d\eta} = -\dfrac{\tau_{xy}}{\mu} > 0.$

Analogy with diffusion of heat: $\dfrac{\partial u}{\partial t} = \nu\dfrac{\partial^2 u}{\partial y^2} \Rightarrow \dfrac{\partial T}{\partial t} = \kappa\dfrac{\partial^2 T}{\partial y^2}$.

$$\Rightarrow \frac{T - T_0}{T_0 - T_\infty} = 1 - \mathrm{erf}\{y/\sqrt{4\kappa t}\}.$$

Lecture 7.

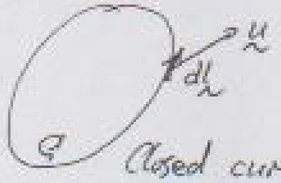
Vorticity dynamics

Basic vortex flows

$u_\theta = \frac{1}{2}\omega r$ Solid body rotation vorticity $\omega_z = \omega$ (Homogeneous distribution of vorticity)

$u_\theta = \frac{\Gamma}{2\pi r}$ Irrotational vortex / line vortex vorticity $\omega_z = 0$. (infinite vorticity at origin)

Definition: Circulation $\Gamma = \oint_C \underset{\sim}{u} \cdot d\underset{\sim}{l}$



Closed curve C.

Stoke's theorem:

$$\oint_C \underset{\sim}{u} \cdot d\underset{\sim}{l} = \iint_S \nabla \times \underset{\sim}{u} \cdot \underset{\sim}{n}\, dS = \iint_S \underset{\sim}{\omega} \cdot \underset{\sim}{n}\, dS = \iint_S |\underset{\sim}{\omega}| \cdot \underbrace{n_\omega dS}_{\text{projected } dS_\perp \text{ to } \underset{\sim}{\omega}}$$

$|\underset{\sim}{\omega}|$ is circulation/unit area.

Solid body rotation

$$u_\theta = \frac{1}{2}\omega r \qquad \Gamma = \oint u_\theta\, r d\theta = \frac{1}{2}\omega r^2 2\pi = \omega \underbrace{\pi r^2}_{\text{area}}.$$

$$\Gamma_\theta = \frac{1}{2}\omega r_2^2\theta - \frac{1}{2}\omega r_1^2\theta = \omega \underbrace{\frac{\theta}{2}(r_2^2 - r_1^2)}_{\text{area}}.$$

Line vortex

$$u_\theta = \frac{\Gamma}{2\pi r} \qquad \Gamma = \oint \frac{\Gamma}{2\pi r} r\, d\theta = \Gamma = \text{is the strength of line vortex.}$$

Although $\omega_z = 0$, $\Gamma \neq 0$ for all C enclosing the vortex.

Curve not enclosing (e.x.)

$$\Gamma_C = \frac{\Gamma}{2\pi r_2} r_2 \cdot 2\pi - \frac{\Gamma}{2\pi r_1} r_1 \cdot 2\pi = 0.$$

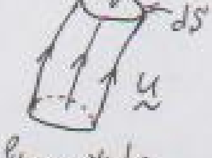
All vorticity concentrated to vortex centre.

Vortex lines

$d\Gamma = \underset{\sim}{\omega} \cdot \underset{\sim}{n}\, dS$

Vortex tube $\underset{\sim}{\omega} = \nabla \times \underset{\sim}{u} \Rightarrow \nabla \cdot \underset{\sim}{\omega} = 0$

Streamlines



$dQ = \underset{\sim}{u} \cdot \underset{\sim}{n}\, dS$.

Streamtube $\Rightarrow \nabla \cdot \underset{\sim}{u} = 0$ for incompressible

Viscous stress: $\tau_{ij} = 2\mu \overline{e_{ij}} + \mu_B \overline{\overline{e_{ij}}}$. Solid body rotation $\overline{\overline{e_{ij}}} = \overline{e_{ij}} = 0 \Rightarrow$ no deformation $\Rightarrow$ no viscous stress.

Irrotational flow deformation rate $\neq 0 \Rightarrow \tau_{ij} \neq 0$

Viscous force / unit volume: $\frac{\partial}{\partial x_j}(\tau_{ij}) = -\mu(\nabla \times \underset{\sim}{\omega})_i = 0$ for $\underset{\sim}{\omega} = \text{const}$ and for irrotational flow.

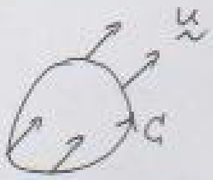
Kelvin's circulation theorem

If: a) Inviscous flow $\tau_{ij} = 0$ or $Re \to \infty$

b) Conservative body force $\underset{\sim}{f} = -\nabla \Pi$. e.g. $\Pi = -\underset{\sim}{g} \cdot \underset{\sim}{x}$.

c) Barotropic flow $\rho(p)$, e.g. ρ = constant or isentropic flow $P/\rho^{\gamma} = \text{const}$.

Then:



Material curve C

$$\Gamma = \oint_{C(t)} \underset{\sim}{u} \cdot d\underset{\sim}{l} \qquad \frac{D}{Dt}\Gamma = 0$$

Proof: $$\frac{D}{Dt}\Gamma = \frac{D}{Dt}\oint u_i dx_i = \underbrace{\oint \frac{Du_i}{Dt} dx_i}_{I} + \underbrace{\oint u_i \frac{D}{Dt} dx_i}_{II}$$

$$II = \oint_C u_i du_i = \oint_C d\left(\frac{u_i u_i}{2}\right) = 0 \qquad \Rightarrow$$

$$I = \oint_C \Big(-\frac{1}{\rho}\frac{\partial p}{\partial x_i} + f_i + \underbrace{\frac{1}{\rho}\frac{\partial}{\partial x_j}(\tau_{ij})}_{=0 \text{ for } \mu = 0}\Big) dx_i \qquad \Rightarrow$$

$$\oint_C \frac{1}{\rho(p)} dp = \oint \frac{dF}{dp}(\rho)\, dp = \oint dF = 0 \quad \Rightarrow \quad \oint -\frac{\partial}{\partial x_i}\Pi\, dx_i = -\oint d\Pi = 0 \quad \Rightarrow$$

$$\frac{D}{Dt}\Gamma = 0$$

Irrotational flow at $t=0$: $\underset{\sim}{\omega}(t=0) = 0$ (except at singular points of line vortex).

$\Gamma(t=0) = \iint \underset{\sim}{\omega} \cdot \underset{\sim}{n}\, dS = 0$ for any curve C (not enclosing the line vortex) and S.

then since $\frac{D}{Dt}\Gamma = 0$ for any curve $\Rightarrow \Gamma = \iint_S \underset{\sim}{\omega}(t) \cdot \underset{\sim}{n}\, dS = 0$ for any surface S.

Thus $\underset{\sim}{\omega}(t) = 0 \Rightarrow$ Irrotational flow remains irrotational under the condition of Kelvin's theorem.

Irrotational flow with line vortex Γ.

$\frac{D}{Dt}\Gamma_c = 0$, $\Gamma_c = \Gamma$ = constant. The material curve will enclose the vortex at all times.

Material curves not enclosing the vortex $\frac{D}{Dt}\Gamma_c = 0$, $\Gamma_c = 0$

Helmholz theorems (same conditions as Kelvin's).

- Vortex lines are material lines and move with the fluid.
- The strength of a line vortex, i.e. Γ, remains constant in time.
- The strength of a line vortex is constant along its length.
- A line vortex cannot end within the fluid. (Closed loop will end at solid wall)

Derivation of the vorticity equation.

$$\begin{cases} \frac{\partial}{\partial t} u + u\cdot\nabla u = -\frac{1}{\rho}\nabla p + f + \nu\nabla^2 u \\ \nabla\cdot u = 0 \end{cases}$$

$\uparrow$ allow for small variation of ρ.

$\omega = \nabla\times u$ vorticity vector

$\omega_i = \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}$

use: $u\cdot\nabla u = \frac{1}{2}\nabla(u\cdot u) - u\times(\nabla\times u) = \frac{1}{2}\nabla(u\cdot u) + \omega\times u$

[Since: $a\times(b\times c) = b\, a\cdot c - c\, a\cdot b$ then $u\times(\nabla\times u) = \nabla\left(\frac{u\cdot u}{2}\right) - u\cdot\nabla u$] $\Rightarrow$

$$\nabla\times\left[\frac{\partial u}{\partial t} + \frac{1}{2}\nabla(u\cdot u) + \omega\times u = -\frac{1}{\rho}\nabla p + f + \nu\nabla^2 u\right].$$ Note $\nabla\times\nabla(\cdot) = 0$

$$\nabla\times(\omega\times u) = \nabla\times(\omega\times u) + \nabla\times(\omega\times u) = u\cdot\nabla\omega - u\underbrace{\nabla\cdot\omega}_{=0} + \omega\underbrace{\nabla\cdot u}_{=0} - \omega\cdot\nabla u$$

$$\frac{\partial\omega_k}{\partial x_k} = \frac{\partial}{\partial x_k}\varepsilon_{kij}\frac{\partial u_j}{\partial x_i} = \varepsilon_{kij}\frac{\partial^2 u_j}{\partial x_k\partial x_i} = 0$$

$$\underbrace{\frac{\partial}{\partial t}\omega + \underbrace{u\cdot\nabla\omega}_{\text{advection of } \omega}}_{\frac{D\omega}{Dt}} = \underbrace{\omega\cdot\nabla u}_{\text{stretching \& turning of vortex lines}} + \underbrace{\frac{\nabla\rho\times\nabla p}{\rho^2}}_{\text{baroclinic generation of } \omega \text{ (=0 if } \rho(p) \text{ barotropic)}} + \underbrace{\nabla\times f}_{\text{body force generation (=0 if } f=-\nabla\psi)} + \underbrace{\nu\nabla^2\omega}_{\text{diffusion of } \omega}$$

Non-dimensional form (barotropic & constant body force) $\Rightarrow$

$$\frac{D}{Dt}\omega_i = \underbrace{\omega_j \frac{\partial u_i}{\partial x_j}}_{\text{Compare with equation for material line element.}} + \frac{1}{Re}\frac{\partial \omega_i}{\partial x_j \partial x_j} \qquad \frac{D}{Dt} dl_i = \frac{\partial u_i}{\partial x_j} dl_j$$

For $Re \to \infty$, $\frac{D}{Dt}\omega_i = \frac{\partial u_i}{\partial x_j}\omega_j$

Helmholtz theorem for $Re \to \infty$

- Vortex lines are material lines

i) Stretching of vortex line produce ω_i like stretching of dl_i produce length.

ii) Tilting of vortex line produce ω_i in one direction at presence of ω_i in another direction.

Study terms in the vorticity equation

2D: $\frac{\partial}{\partial t}\omega_3 + u_1\frac{\partial \omega_3}{\partial x_1} + u_2\frac{\partial \omega_3}{\partial x_2} = \nu\nabla^2\omega_3$. No tilting & stretching in 2D.

Ex: Diffusion of vorticity: Stoke's problem

$\omega_3 = \omega_z = \omega(y,t)$; $\frac{\partial \omega}{\partial t} = \nu\frac{\partial^2\omega}{\partial y^2}$; $\omega = \frac{U_0}{\sqrt{\pi\nu t}} e^{-y^2/4\nu t}$.

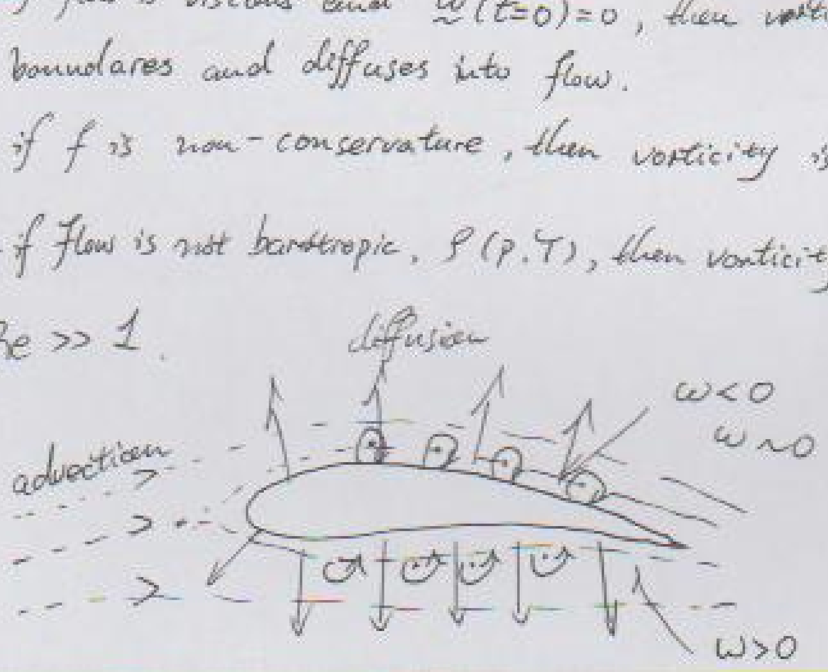
Diffusion length scale: $\delta(t) = \sqrt{4\nu t}$ $\Rightarrow$ Vorticity is generated at the wall and diffuses into fluid above (But $-\nu\frac{\partial \omega}{\partial y}\Big|_{y=0} = 0$; $t>0$)

Note: $\int_0^\infty \omega dy = \int_0^\infty -\frac{\partial u}{\partial y} dy = U_0 =$ constant. ω concentrated at the wall for $t>0$.

Generation of vorticity.

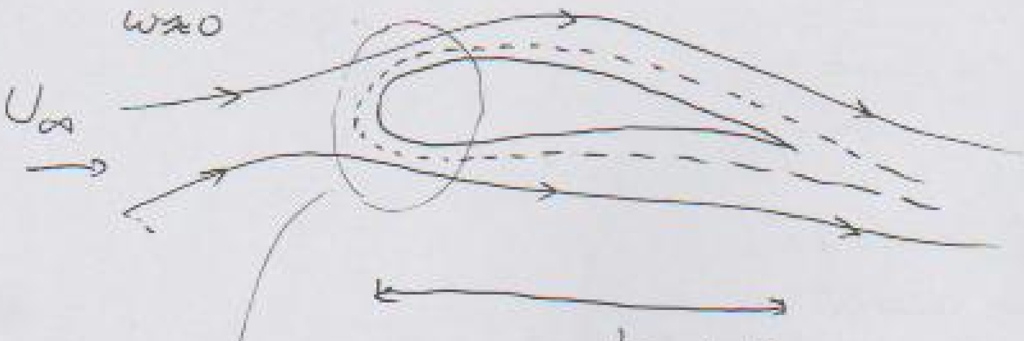
- if flow is viscous and $\underset{\sim}{\omega}(t=0)=0$, then vorticity is generated at solid boundares and diffuses into flow.
- if f is non-conservature, then vorticity is generated by $\nabla \times \underset{\sim}{f}$.
- if flow is not barotropic, $\rho(p,T)$, then vorticity is generated by $\nabla\rho \times \nabla p$.

$Re \gg 1$.

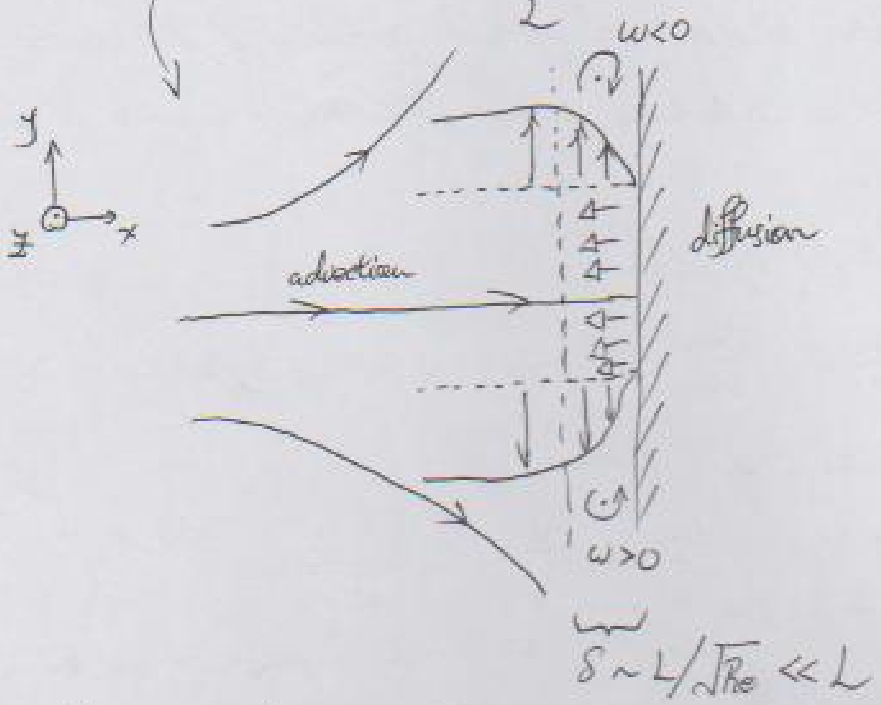


Lecture 8.

Flows at large Re

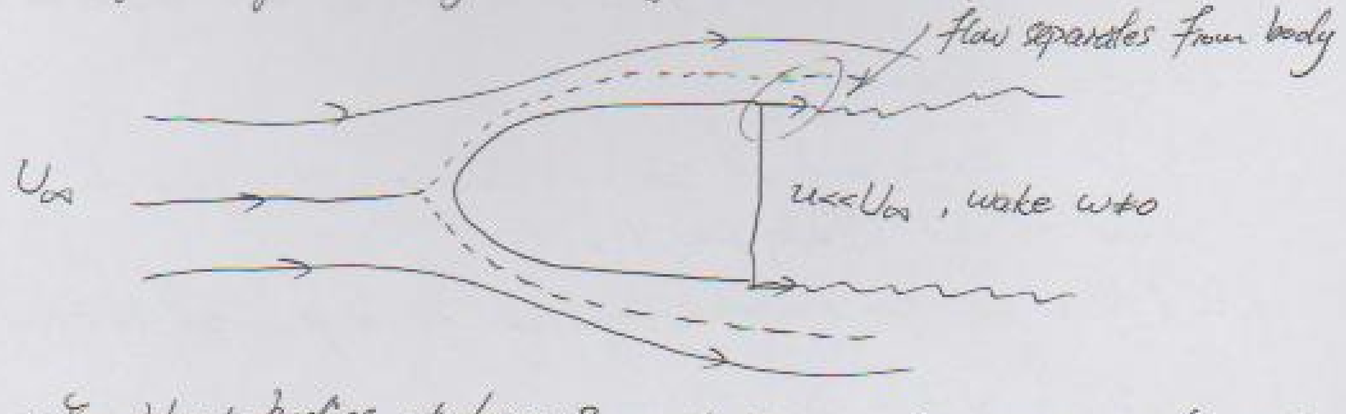


* Generation of vorticity at the wall boundary $\omega \neq 0$
* Advection with flow prevents diffusion far away from boundary



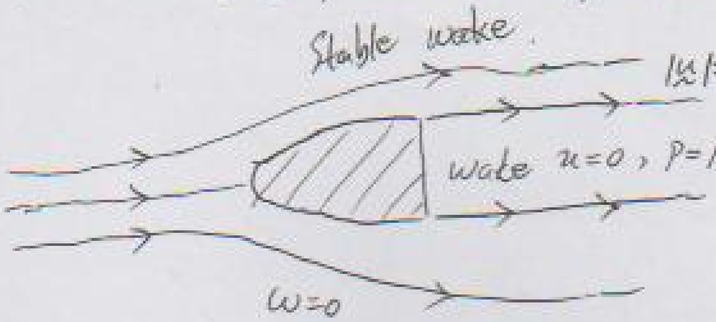
$$Re = \frac{UL}{\nu} \gg 1.$$

* For streamlined bodies, vorticity is concentrated close to the walls at large Re, generated by viscous effects and no slip-condition.



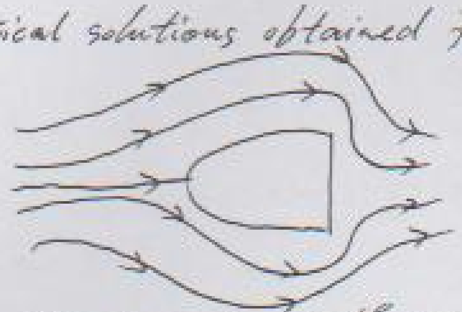
* For blunt bodies at large Re vorticity is also present in the wake.

The assumption of zero vorticity, irrotational flow, is only motivated for flows without "separation". (*Exception: known separation point stable wake)



Irrotational flow assumption, $\underset{\sim}{\omega} = 0$, for $Re \gg 1$.

* The viscous no slip condition cannot be fulfilled (generates vorticity)
* Non-physical solutions obtained for blunt body



Methods for solving irrotational flow problems.

Assume velocity field is obtained from potential $\Phi(\underset{\sim}{x}, t)$: $\underset{\sim}{u} = \nabla\Phi$ $\Rightarrow$

$\underset{\sim}{\omega} = \nabla \times \underset{\sim}{u} = \nabla \times (\nabla\Phi) = 0$

i) Potential flow is irrotational as required

ii) Potential flow has no net viscous force/unit volume.

$$\frac{\partial}{\partial x_j} \tau_{ij} = \mu \nabla^2 u_i = -\mu \varepsilon_{ijk} \frac{\partial \omega_k}{\partial x_j} = -\mu (\nabla \times \underset{\sim}{\omega}) = 0.$$

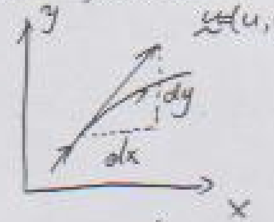
But cannot satisfy no slip-condition at solid wall.

iii) Potential flow is incompressible if $\nabla \cdot \underset{\sim}{u} = \nabla \cdot (\nabla\Phi) = \nabla^2 \Phi = 0$.

Solve Laplace equation for $\Phi(\underset{\sim}{x}, t)$ with condition of no normal velocity component at solid wall: $\underset{\sim}{u} \cdot \underset{\sim}{n} = \nabla\Phi \cdot \underset{\sim}{n} = 0$

Stream function

2D



Streamlines: tangent to velocity vector:

$$\frac{dy}{dx} = \frac{v}{u} \quad \text{or} \quad \frac{dx}{u} = \frac{dy}{v} \quad \text{or} \quad u\,dy - v\,dx = 0$$

$$\left(3D: \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}\right)$$

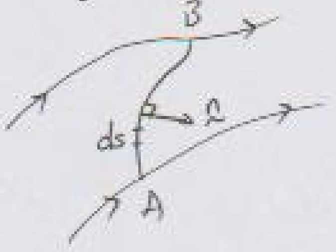
2D: $\exists$ stream function $\psi = \psi(x, y)$ such that $\psi = \text{const}$ on streamlines.

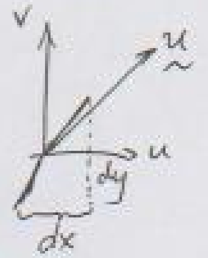
$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = \left\{ \frac{\partial\psi}{\partial x} = -v,\ \frac{\partial\psi}{\partial y} = u \right\} = -v\,dx + u\,dy = 0 \quad \text{on streamlines}$$

Consistent with continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial x\partial y} = 0$.

Def. $\Psi(x,y) = \int_0^{P(x,y)} (-v\,dx + u\,dy)$

Volume flux between two streamlines





$$dQ = u\,dy - v\,dx$$

$$Q_{AB} = \int_A^B dQ = \int_A^B u\,dy - v\,dx = \int_{(\cdot)}^{B} u\,dy - v\,dx - \int_{(\cdot)}^{A} u\,dy - v\,dx$$

$$= \psi_B - \psi_A.$$

Irrotational flow $\omega_z = 0$.

$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2} = -\nabla^2\psi = 0$. If flow is irrotational, the streamfunction must satisfy Laplace equation.

Bernoulli's equation (incompressible)

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\nabla^2 u_i - g\delta_{i3} \qquad \uparrow x_3 \quad \downarrow g$$

Rewrite using: $u_j\frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{1}{2}u_ju_j\right) - \varepsilon_{ijk}u_j\omega_k$ or $\underset{\sim}{u}\cdot\nabla\underset{\sim}{u} = \nabla\left(\frac{\underset{\sim}{u}\cdot\underset{\sim}{u}}{2}\right) - \underset{\sim}{u}\times\underset{\sim}{\omega}$

and $\nabla^2 u_i = \frac{\partial^2 u_i}{\partial x_j\partial x_j} = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\omega_k = -(\nabla\times\underset{\sim}{\omega})_i \quad \Rightarrow$

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_i}\left(\frac{1}{2}u_ju_j + \frac{p}{\rho} + gx_3\right) = \varepsilon_{ijk}u_j\omega_k - \nu\varepsilon_{ijk}\frac{\partial\omega_k}{\partial x_j} \qquad \text{or}$$

$$\frac{\partial\underset{\sim}{u}}{\partial t} + \nabla\left(\frac{1}{2}\underset{\sim}{u}\cdot\underset{\sim}{u} + \frac{p}{\rho} + gx_3\right) = \underset{\sim}{u}\times\underset{\sim}{\omega} - \nu\nabla\times\underset{\sim}{\omega}.$$

(1): Assume irrotational flow $\underset{\sim}{\omega} = 0$, thus we have a velocity potential Φ.

$u_i = \frac{\partial\Phi}{\partial x_i} \Rightarrow \frac{\partial}{\partial x_i}\left\{\frac{\partial\Phi}{\partial t} + \frac{1}{2}u_ju_j + \frac{p}{\rho} + gx_3\right\} = 0$ and

$\frac{\partial\Phi}{\partial t} + \frac{1}{2}u_ju_j + \frac{p}{\rho} + gx_3 = f(t)$. Require $Re \gg 1$, otherwise $\underset{\sim}{\omega}$ will diffuse from boundary.

(2): Assume $\underset{\sim}{\omega} \neq 0$, inviscid flow $Re \gg 1$ and stationary flow $\frac{\partial}{\partial t} = 0$

Integrate momentum equation along a streamline.

$\underset{\sim}{e_s} = \underset{\sim}{u}/|\underset{\sim}{u}|$

$$\int^S \underbrace{\underset{\sim}{e_s}\cdot\nabla}_{d/ds}\left(\frac{1}{2}\underset{\sim}{u}\cdot\underset{\sim}{u} + \frac{p}{\rho} + gx_3\right)ds = \int^S \underbrace{\underset{\sim}{e_s}}_{/\!/ \underset{\sim}{u}}\cdot\underbrace{(\underset{\sim}{u}\times\underset{\sim}{\omega})}_{\perp \underset{\sim}{u}}dS$$

$\Rightarrow \frac{1}{2}\underset{\sim}{u}\cdot\underset{\sim}{u} + \frac{p}{\rho} + gx_3 = $ constant along streamlines, differ between streamlines if $\underset{\sim}{\omega} \neq 0$

Bernolli's equation is often used to calculate p when $\underset{\sim}{u}$ is known.

i) Solve $\nabla^2\Phi = 0$ or $\nabla^2\Psi = 0$. ii) Calculate p from Bernoulli's equation.

Ex: Calculate the pressure, $P(r)$, of an irrotational vortex (vortex line) if the pressure at large distance is P_∞.

$$u_\theta = \frac{\Gamma}{2\pi r} \quad \omega_z = 0 \quad \frac{\partial}{\partial t} = 0 \Rightarrow \frac{1}{2}\rho u_\theta^2 + P(r) = \text{constant} = 0 + P_\infty \Rightarrow$$

$$P(r) = P_\infty - \frac{1}{2}\rho\left(\frac{\Gamma}{2\pi r}\right)^2 \text{; Singularity at } r=0.$$

Velocity potential: $\Phi = \frac{\Gamma}{2\pi}\theta + \text{const.}$

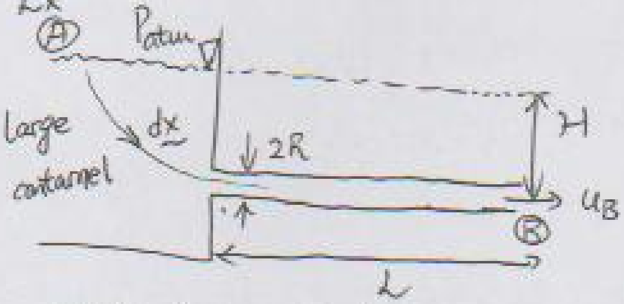
Stream function: $\Psi = -\frac{\Gamma}{2\pi}\ln r + \text{const.} \Rightarrow$

$$\begin{cases} u_r = \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = 0 \\ u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r} = \frac{\Gamma}{2\pi r} \end{cases}$$

One finds:

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \text{ ; } \nabla^2\Psi = 0$$

Ex:



Potential flow assumed

$$\frac{\partial}{\partial t}\Phi_A + \frac{1}{2}U_A^2 + \frac{P_{atm}}{\rho} + gH = \frac{\partial}{\partial t}\Phi_B + \frac{1}{2}U_B^2 + \frac{P_{atm}}{\rho} + gH$$

tube inlet

$u_B(t=0)=0$ is initial condition: $\Phi_B - \Phi_A = \int_A^B \nabla\Phi \cdot d\underset{\sim}{x} = \int_A^B \underset{\sim}{u}\cdot d\underset{\sim}{x} = \underbrace{\int_A \underset{\sim}{u}\cdot d\underset{\sim}{x}}_{\sim U_B R \ll U_B \cdot L} + u_B \cdot L$

$$\frac{\partial}{\partial t}\Phi_B - \frac{\partial}{\partial t}\Phi_A = \frac{d}{dt}U_B\cdot L \Rightarrow gH = \frac{d}{dt}U_B\cdot L + \frac{1}{2}U_B^2 - \underbrace{\frac{1}{2}U_A^2}_{\ll U_B^2/2}$$

Let $U_B = \sqrt{2gH}\,F(t) \Rightarrow gH = \sqrt{2gH}\frac{dF}{dt}L + \frac{1}{2}\cdot 2gH\cdot F^2(t) \Rightarrow \frac{1}{2} = \frac{1}{2}\frac{dF}{d\tilde{t}} + \frac{1}{2}F^2$.

Let $t = \frac{2L}{\sqrt{2gH}}\tilde{t}$ $[\tilde{t}] = 1$ $\Rightarrow \frac{dF}{d\tilde{t}} = 1 - F^2(\tilde{t}) = (1+F)(1-F) \Rightarrow \frac{1}{2}\frac{dF}{1+F} + \frac{1}{2}\frac{dF}{1-F} = d\tilde{t}$

Integral $\Rightarrow \frac{1}{2}\ln|1+F| - \frac{1}{2}\ln|1-F| = \tilde{t}$. $F(\tilde{t}=0)=1 \Rightarrow F = \tanh(\tilde{t})$

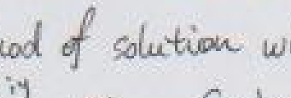
Lecture 9.

2D irrotational flow

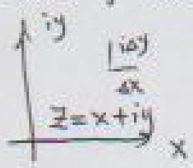
$\omega_z = 0$

Velocity potential: $\underset{\sim}{u} = \nabla\Phi$ Incompressible: $\nabla^2\Phi = 0$

Stream function: $u = \frac{\partial\psi}{\partial y}$; $v = -\frac{\partial\psi}{\partial x}$. $\omega_z = 0 \Rightarrow \nabla^2\Psi = 0$



Method of solution with analytic functions



Complex function $F(z)$ is analytic if $\frac{\partial F}{\partial z}(z)$ exists and is independent of direction. $F'(z) = \lim_{\Delta z \to 0}\left(\frac{F(z+\Delta z) - F(z)}{\Delta z}\right)$

Let $F(z) = \Phi(x,y) + i\Psi(x,y) \Rightarrow F' = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\{\Phi(x+\Delta x, y) + i\Psi(x+\Delta x, y) - \Phi(x,y) - i\Psi(x,y)\} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x}$

$F' = \lim_{i\Delta y \to 0} \frac{1}{i\Delta y}\{\phi(x, y+\Delta y) + i\psi(x, y+\Delta y) - \phi(x,y) - i\psi(x,y)\} = \frac{1}{i}\frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial y} - i\frac{\partial\phi}{\partial y}$

Real part: $\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$

Imaginary part: $\frac{\partial\psi}{\partial x} = -\frac{\partial\phi}{\partial y}$

Cauchy-Riemann equations.

One finds:

$\nabla^2\Phi = \nabla^2\Psi = 0$

$\nabla\Phi \cdot \nabla\Psi = 0 \Rightarrow$

Iso level surfaces of Φ & Ψ are orthogonal.

Any analytic function $F(z) = \Phi + i\Psi$ is a candidate to describe an irrotational, incompressible 2D flow field with: $u = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$, $v = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$.

We define the complex velocity $W(z) \to$ Complex conjugate of velocity vector.

$$W(z) = \frac{dF}{dz} = \frac{\partial\Phi}{\partial x} + i\frac{\partial\Psi}{\partial x} = u - iv = \frac{\partial\Psi}{\partial y} - i\frac{\partial\Phi}{\partial x}$$

Example:

$n=1$: Uniform flow over flat plate. $F = Az$, $W = A$. $u = A$.

$n=2$: stagnation point flow. $F = Az^2$. $W = 2A(x+iy)$. $\begin{cases} u = 2Ax \\ v = -2Ay \end{cases}$.

$n>2$: Narrow stagnation point flow

$1 \leq n \leq 2$: Flow towards a wedge (or in a corner), at surface $\theta = 0$: $u_r = nAr^{n-1}$

$n<1$: Flow around corner, at surface $\theta = 0$: $u_r = \frac{nA}{r^{1-n}}$, $\theta = \frac{\pi}{n}$: $u_r = -\frac{nA}{r^{1-n}}$

$n = \frac{1}{2}$: Flow around edge

Ex: Line source & line vortex: $F(z) = \left(\frac{m - i\Gamma}{2\pi}\right)\ln z$, $z = re^{i\theta}$ $\Rightarrow$

$$W = \frac{m - i\Gamma}{2\pi}\cdot\frac{1}{z} = (m - i\Gamma)\frac{1}{2\pi r}e^{-i\theta} \triangleq (u_r - iu_\theta)e^{-i\theta}$$

$\begin{cases} u_r = \frac{m}{2\pi r} & , m = 2\pi r u_r(r) \text{ is volume flux/width} \\ u_\theta = \frac{\Gamma}{2\pi r} & , \Gamma \text{ is circulation.} \end{cases}$ $\Rightarrow$ $\Phi = \frac{m}{2\pi}\ln r + \frac{\Gamma}{2\pi}\theta$, $\Psi = \frac{m}{2\pi}\theta - \frac{\Gamma}{2\pi}\ln r$

Ex: Mirror images in a plane. Mirror image superposed.

Ex: Line source $\oplus$ uniform stream: $F = Uz + \frac{m}{2\pi}\ln z = Ure^{i\theta} + \frac{m}{2\pi}(\ln r + i\theta)$

$\Rightarrow$ Flow past half-infinite body

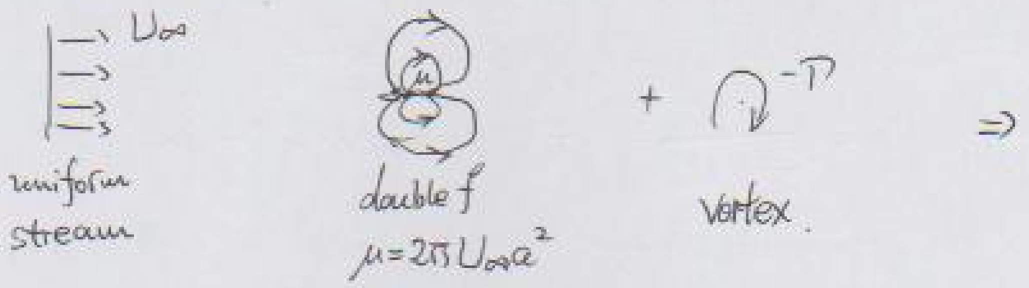
Ex: Dipole. $F = \frac{\mu}{2\pi z}$. μ is dipole strength. $\Rightarrow$

$$u_r = -\frac{\mu}{2\pi r^2}\cdot\cos\theta, \quad u_\theta = -\frac{\mu}{2\pi r^2}\cdot\sin\theta.$$

Lecture 10.

How to fly?

Flow past a circular cylinder with circulation.



$$F(z) = U_\infty\left(z + \frac{a^2}{z}\right) + \frac{i\Gamma}{2\pi}\ln z, \quad z = re^{i\theta} \quad \Rightarrow$$

$$\Psi = U_\infty \sin\theta\left(r - \frac{a^2}{r}\right) + \frac{\Gamma}{2\pi}\ln(r/a); \quad r = a \Rightarrow \Psi = 0 \Rightarrow \text{Streamline.}$$

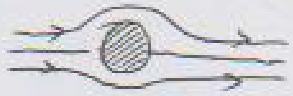
Complex velocity:

$$W = F'(z) = U_\infty\left(1 - \frac{a^2}{z^2}\right) + \frac{i\Gamma}{2\pi z} = \Big\{\underbrace{U_\infty\left(1 - \frac{a^2}{r^2}\right)\cos\theta}_{=u_r} + i\underbrace{\left[U_\infty\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}\right]}_{=-u_\theta}\Big\} e^{-i\theta}$$

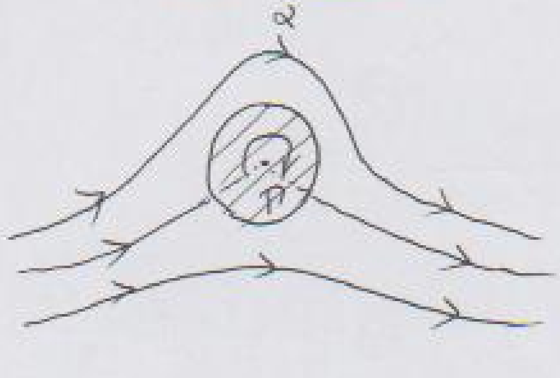
Stagnation points: $u_r = 0 \Rightarrow r = a$.

$$u_\theta = -2U_\infty \sin\theta_s - \frac{\Gamma}{2\pi a} = 0 \Rightarrow \sin\theta_s = -\frac{\Gamma}{4\pi a U_\infty} \quad \text{ok if } \Gamma < 4\pi a U_\infty.$$

$\Gamma = 0$: $\theta_s = 0, \pi$ flow around cylinder



$\Gamma < 4\pi a U_\infty$: $\theta_s = \underbrace{-\arcsin(\Gamma/4\pi a U_\infty)}_{\alpha}, \ \pi + \arcsin(\Gamma/4\pi a U_\infty)$



$\Gamma = 4\pi a U_\infty$

$\Gamma > 4\pi a U_\infty$

Pressure on cylinder surface

Bernoulli's equation $\Rightarrow P(\theta) + \frac{1}{2}\rho U_\theta^2(r=a,\theta) = P_\infty + \frac{1}{2}\rho U_\infty^2$.

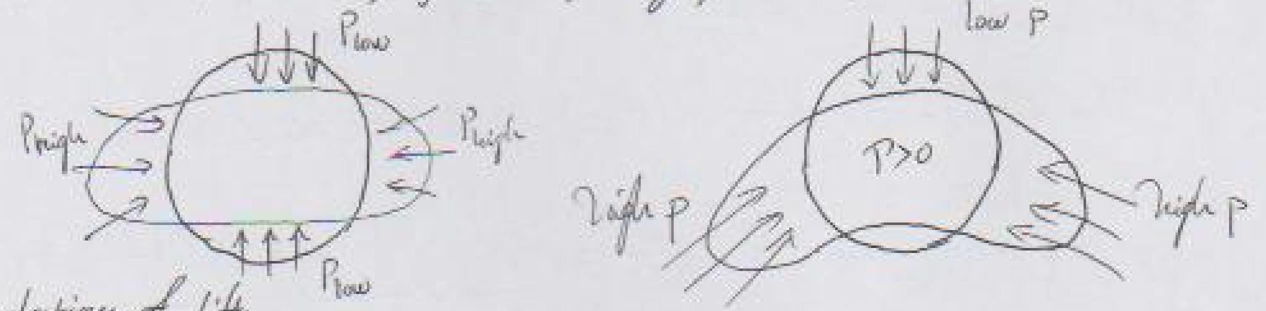
$$U_\theta^2(r=a,\theta) = \left(-2U_\infty \sin\theta - \frac{+\Gamma}{2\pi a}\right)^2 = 4U_\infty^2(\sin\theta + \sin\alpha)^2$$

$$= 4U_\infty^2 \sin^2\theta + 4U_\infty^2 \sin^2\alpha \quad ; \quad \sin\alpha = \frac{\Gamma}{4\pi U_\infty a}.$$

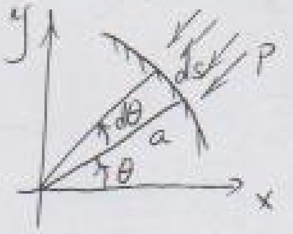
Pressure coefficient $C_P \triangleq \dfrac{P-P_\infty}{\frac{1}{2}\rho U_\infty} = 1 - 4\sin^2\theta - 4\sin^2\alpha - 8\sin\theta\sin\alpha$

$\Gamma = 0 \Rightarrow \sin\alpha = 0$ Symmetric pressure distribution. No lift force, no drag force.

$\Gamma > 0 \Rightarrow \sin\alpha > 0$ Lift force > 0, drag force $= 0$



Calculation of lift



$$dL_y' = -\underbrace{P(r=a)\,a\,d\theta}_{ds} \cdot \sin\theta$$

$$L_y' = \int_0^{2\pi} dL_y' = \frac{1}{2}\rho U_\infty^2 a \int_0^{2\pi} (-C_P)\sin\theta\, d\theta = \left\{ \int_0^{2\pi} \sin^2\theta\, d\theta = \pi, \text{ others} = 0 \right\}$$

$$= \frac{1}{2}\rho U_\infty^2 a \cdot 8\pi \cdot \sin\alpha. \quad \Rightarrow$$

$$C_L = \frac{L_y'}{\frac{1}{2}\rho U_\infty^2 \cdot 2a} = 4\pi \cdot \sin\alpha \leftarrow \text{"angle of attack"}. \quad \Rightarrow$$

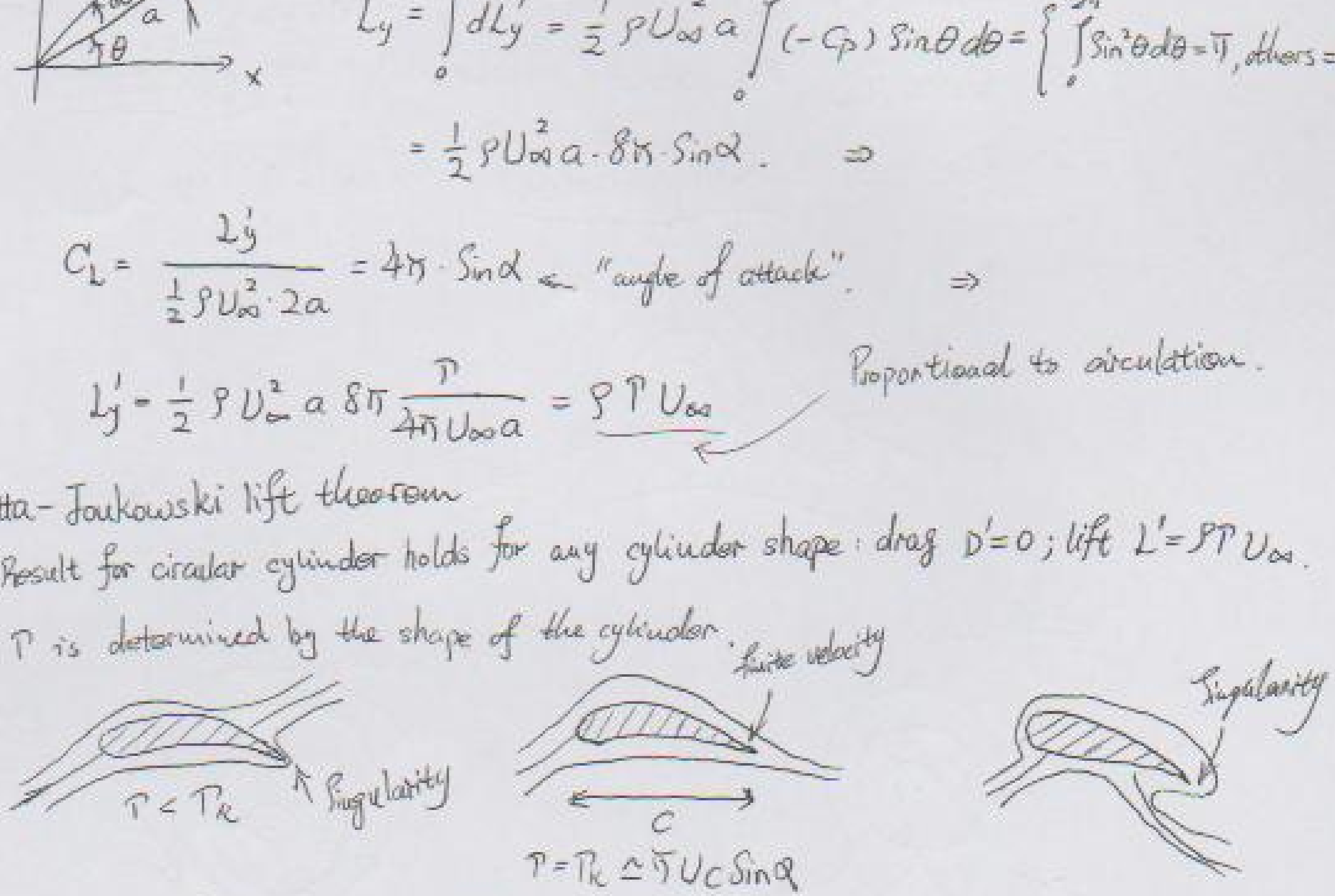
$$L_y' = \frac{1}{2}\rho U_\infty^2 a\, 8\pi \frac{\Gamma}{4\pi U_\infty a} = \underline{\rho \Gamma U_\infty}$$

Proportional to circulation.

Kutta-Joukowski lift theorem

Result for circular cylinder holds for any cylinder shape: drag $D' = 0$; lift $L' = \rho\Gamma U_\infty$.

Γ is determined by the shape of the cylinder.

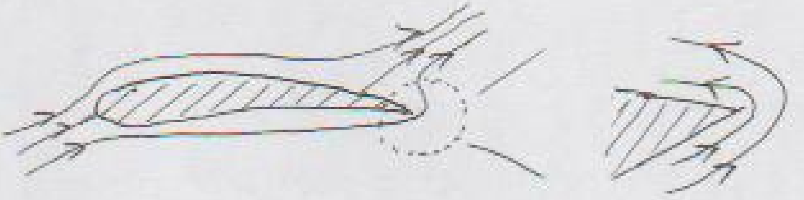


Kutta condition on circulation gives finite velocity at rear edge.

In reality, the circulation is generated by vortex shedding as a result of viscosity.

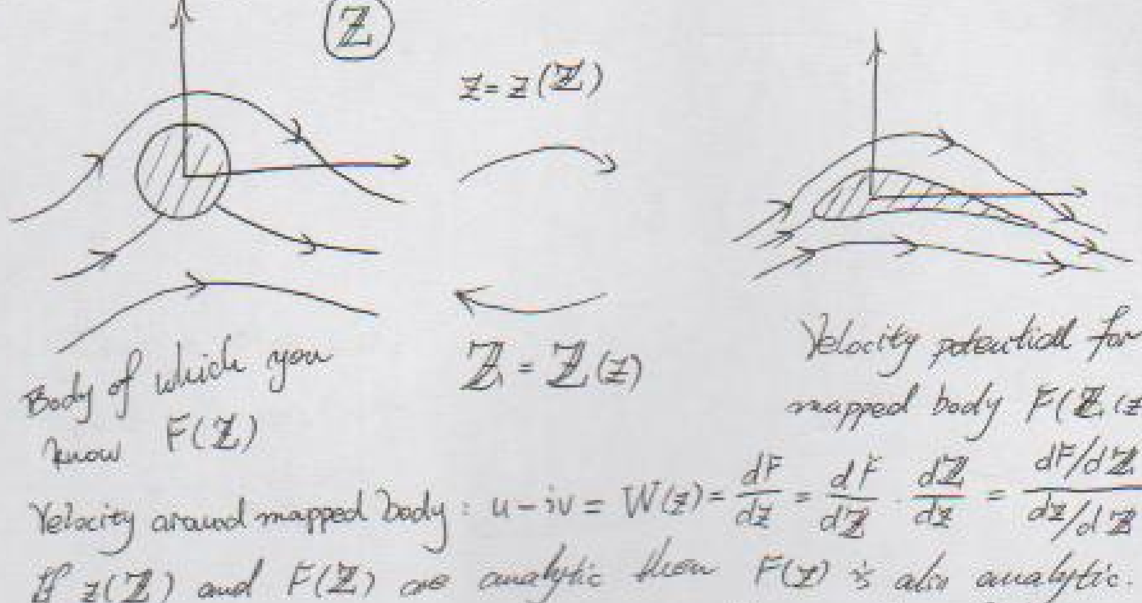
~~Generation~~ Generation of Circulation.

Recently started wing



Vorticity shed from boundary layer generates spiral vortex sheet of total circulation Γ.

Conformal mapping: Transforming one flow into another.



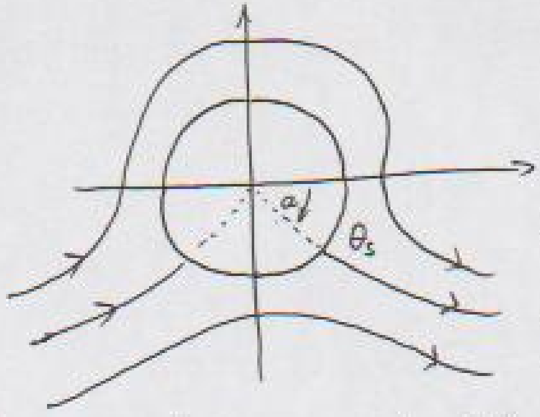
Body of which you know $F(Z)$

Velocity potential for mapped body $F(Z(z))$

Velocity around mapped body: $u - iv = W(z) = \frac{dF}{dz} = \frac{dF}{dZ} \cdot \frac{dZ}{dz} = \frac{dF/dZ}{dz/dZ}$.

If $z(Z)$ and $F(Z)$ are analytic then $F(z)$ is also analytic.

Study the transformation: $z_1 = z_2 + \frac{b^2}{z_2} = be^{i\theta_2} + \frac{b^2}{b \cdot e^{i\theta_2}} = b(e^{i\theta_2} + e^{-i\theta_2}) = 2b\cos\theta_2$



$$\mathcal{Z} = z_2 - z_{20}$$

$$z_1 = z_2 + \frac{b^2}{z_2}$$

$$\sin\theta_s = \frac{\Gamma}{4\pi a \cdot U_\infty}$$

$$F(\mathcal{Z}) = U_\infty\left(\mathcal{Z} + \frac{a^2}{\mathcal{Z}}\right) + \frac{i\Gamma}{2\pi}\ln\frac{\mathcal{Z}}{a}$$

Kutta condition: rear stagnation point must coincide with trailing edge, i.e. $\theta_s = \beta$.

$\Rightarrow \sin\beta = \frac{\Gamma_k}{4\pi a U_\infty}$. Lift force $L' = \rho\Gamma_k U_\infty = \rho U_\infty^2 4\pi \cdot a \cdot \sin\beta$.

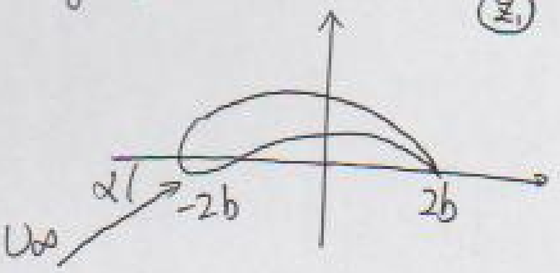
$$u - iv = \frac{dF}{dz_1} = \frac{dF}{d\mathcal{Z}} \Big/ \frac{dz_1}{d\mathcal{Z}} = \frac{dF}{d\mathcal{Z}} \Big/ \frac{dz_1}{dz_2} \cdot \underbrace{\frac{dz_2}{d\mathcal{Z}}}_{=1} = \frac{U_\infty\left(1 - \frac{a^2}{\mathcal{Z}^2}\right) + \frac{i\Gamma_k}{2\pi}\cdot\frac{1}{\mathcal{Z}}}{1 - b^2/z_2^2}$$

At trailing edge $z_2 = b \Rightarrow \frac{dF}{dz} \to \infty$ unless $\frac{dF}{dz} = 0$ at $\mathcal{Z} = a \cdot e^{-i\beta}$

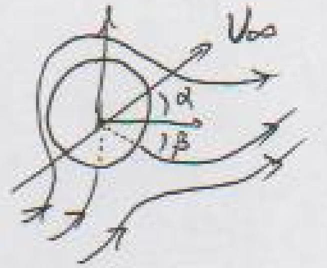
OK, because of Kutta condition

Angle of attack α:

z_1



$\mathcal{Z}$



$\Gamma_K = 4\pi a \cdot U_\infty \cdot \sin(\alpha + \beta)$ $\qquad$ $L' = \rho U_\infty^2 \cdot 4\pi a \cdot \sin(\alpha + \beta)$

Lift coefficient $C_L = \frac{L'}{\frac{1}{2}\rho U_\infty^2 C} = 2\pi\frac{a}{b}\sin(\alpha + \beta)$

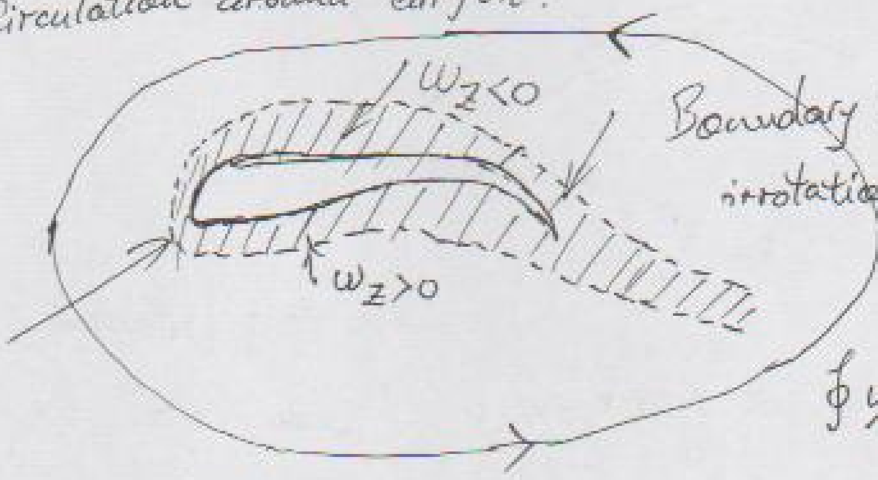
Lecture 11.

Generation of vorticity around airfoil.

Circulation $\oint_{C(t)} \underset{\sim}{u} \cdot d\underset{\sim}{l} = 0$ according to Kelvin.

There can be no net circulation generated for curve $C(t)$ along which $\underset{\sim}{u}(\underset{\sim}{x}, t)$ is irrotational.

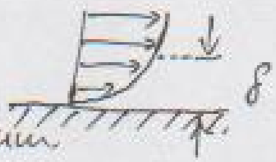
Circulation around airfoil.



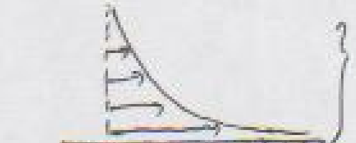
$$\oint \underset{\sim}{u} \cdot d\underset{\sim}{l} = -\Gamma = \iint \omega_z \, dS$$

Boundary layers.

In an essentrally inviscid flow, $Re \gg 1$, estimate thickness of viscous region, δ, along solid wall:



Stoke's problem with diffusion of momentum



$$\delta \sim \sqrt{\nu t}$$

Time for passage of the wing $t_L \sim L/U$. With $Re = \frac{UL}{\nu} \Rightarrow$

$$\delta \sim \sqrt{\frac{\nu L}{U}} = L\sqrt{\frac{\nu}{UL}} = \frac{L}{\sqrt{Re}}$$

. Note: $\frac{\delta}{L} \sim \frac{1}{\sqrt{Re}} \to 0$ as $Re \to \infty$.

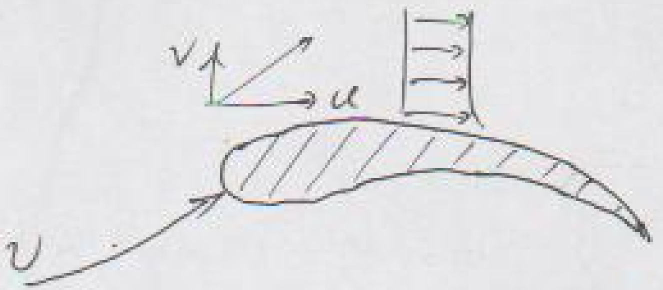
$\Rightarrow$ large velocity gradients near walls
(vorticity)

Derive approximate equation from Navier-Stokes valid in B.L.
δ follows from this analysis.

Division of flow: i) Outer irrotational part.

ii) Boundary layer part close to walls, where the no-slip condition is fullfilled

i) Irrotational, inviscid flow



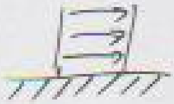
Typical scales $u \sim v \sim U$

Length scale $\sim L$.

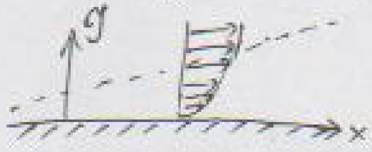
$$\begin{cases} \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} = -\nabla p + \cancel{\frac{1}{Re} \nabla^2 \underset{\sim}{u}} \quad Re \to \infty \\ \nabla \cdot u = 0 \end{cases}$$

Solved by e.g. potential flow theory

This solution does not satisfy the no-slip condition.



ii) Boundary layer



Typical scales:

$u \sim U$; $v \sim V \ll U$; $x \sim L$; $y \sim \delta \ll L$; $p \sim \rho U^2$.

Determine V from $\underbrace{\frac{\partial u}{\partial x}}_{\sim \frac{U}{L}} + \underbrace{\frac{\partial v}{\partial y}}_{\frac{V}{\delta}} = 0$ $\Rightarrow V \sim \frac{\delta}{L} U \ll U$.

y-momentum ↑:

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \frac{\partial^2 v}{\partial x^2} + \nu \frac{\partial^2 v}{\partial y^2}.$$

$$\sim U\frac{\delta U}{L^2} \quad \left(\frac{\delta U}{U}\right)^2 \frac{1}{\delta} \quad \frac{U^2}{\delta} \quad \nu \frac{\delta U}{L^3} \quad \nu \frac{\delta U}{L \delta^2}.$$

$$\left(\frac{\delta}{L}\right)^2 \quad \left(\frac{\delta}{L}\right)^2 \quad 1 \quad \frac{1}{Re}\left(\frac{\delta}{L}\right)^2 \quad \frac{1}{Re}$$

When $Re \to 0$ only pressure term ~ 1. $\Rightarrow \frac{\partial p}{\partial y} = 0$. $\Rightarrow$

$p = p_e(x)$
 ↳ edge of B.L. (external flow).

Pressure constant in B.L. given by inviscid outer flow.

x-momentum $\to x$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$\sim \frac{U^2}{L} \qquad \frac{\delta}{L}\frac{U^2}{\delta} \qquad \frac{U^2}{L} \qquad \nu\frac{U}{L^2} \qquad \nu\frac{U}{\delta^2}$$

$$\sim 1 \qquad 1 \qquad 1 \qquad 1/Re \qquad 1/Re\,(L/\delta)^2.$$

Balance as $Re \to \infty \Rightarrow \frac{1}{Re}\left(\frac{L}{\delta}\right)^2 \sim 1. \Rightarrow \frac{\delta}{L} \sim \frac{1}{\sqrt{Re}}$

Boundary layer equations:

$$\begin{cases} u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{dp_e}{dx} + \nu\dfrac{\partial^2 u}{\partial y^2} \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \end{cases}$$

Parabolic in x. Start at x_0: $u(x=x_0, y) = u_{in}(y)$

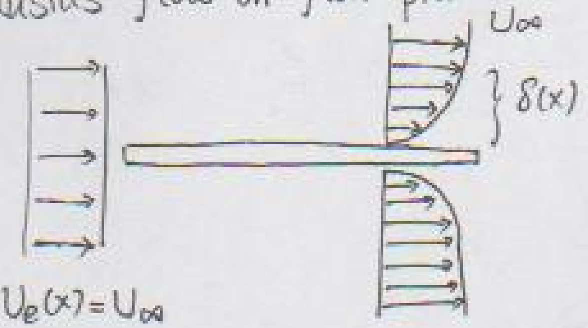
Boundary condition: $\begin{cases} u(x, y=0) = v(x, y=0) = 0 \\ u(x, y\to\infty) = U_e(x). \end{cases}$

$U_e(x)$ is the outer inviscid flow evaluated at the wall.
(Since $\delta/L \to 0$ as $Re \to \infty$).

Pressure, $P_e(x)$, given by Bernoulli's equation for outer flow.

$$P_e(x) + \frac{1}{2}\rho U_e^2(x) = \text{constant}.$$

Blausius flow on flat plate.



$$\begin{cases} u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = U_e\underbrace{\dfrac{dU_e}{dx}}_{=0} + \nu\dfrac{\partial^2 u}{\partial y^2} \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \end{cases}$$

$x=0$: $u = U_\infty$

$y=0$: $u=v=0$; $y\to\infty$: $u \to U_\infty$

Find similarity solution:

$$u = U_\infty f'(\eta) = \frac{\partial \psi}{\partial y} \qquad \eta = y/\delta(x)$$

Streamfunction $\psi = \int^y U_\infty f'(\eta)\,dy = \int^\eta U_\infty f'(\eta)\,d\eta\,\delta(x) = U_\infty \delta(x) f(\eta)$

$$v = -\left(\frac{\partial \psi}{\partial x}\right)_y = -\left(\frac{\partial \psi}{\partial x}\right)_\eta - \left(\frac{\partial \psi}{\partial \eta}\right)_x \left(\frac{\partial \psi}{\partial x}\right)_y = -U_\infty\left[\delta'(x) f(\eta) + \delta(x) f'\left(-\eta\frac{\delta'}{\delta}\right)\right)$$

$$= U_\infty \delta'(x)\left[-f + \eta f'\right]$$

u & v satisfies $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow$

$$\frac{\partial u}{\partial x} = U_\infty f'' \frac{\partial \eta}{\partial x} = -U_\infty f'' \eta \frac{\delta'}{\delta}$$

$$\frac{\partial u}{\partial y} = U_\infty f'' \frac{\partial \eta}{\partial y} = U_\infty f'' \frac{1}{\delta} \quad ; \quad \frac{\partial^2 u}{\partial y^2} = U_\infty f''' \frac{1}{\delta^2} \quad \Rightarrow$$

$$-U_\infty^2 f' f'' \eta \frac{\delta'}{\delta} + U_\infty^2 \frac{\delta'}{\delta}\left[-f + \eta f'\right] f'' = \nu U_\infty f''' \frac{1}{\delta^2} \quad \Rightarrow$$

$$f''' + \underbrace{\frac{U_\infty}{\nu}\delta\delta'}_{=\text{constant for similarity solution}} f f'' = 0$$

$$\frac{d}{dx}\frac{\delta^2}{2} = \frac{\nu}{U_\infty}\cdot \text{Constant} = \frac{\nu}{U_\infty}\cdot\frac{1}{2} \Rightarrow \delta^2 = \frac{\nu x}{U_\infty} + C \,.\; \delta(0) = 0 \Rightarrow$$

$$\delta = \sqrt{\nu x / U_\infty} \;.\Rightarrow\; f''' + \frac{1}{2} f f'' = 0 \quad \text{Blausius equation.}$$

$$f(0) = f'(0) = 0 \;;\; f'(\eta \to \infty) = 1.$$

$$\delta'(x) = \frac{1}{2}\sqrt{\frac{\nu}{U_\infty x}} \Rightarrow v = U_\infty \sqrt{\frac{\nu}{U_\infty x}}\,\frac{1}{2}\left[\eta f' - f\right].\quad v \ll U_\infty \text{ as } \frac{U_\infty x}{\nu} \gg 1$$

Numerical solution

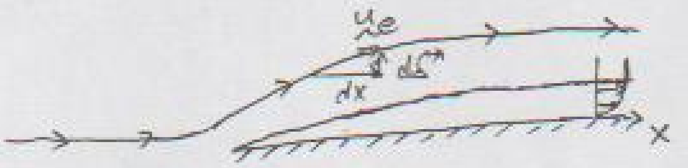
Boundary layer thickness $\delta_{99}(x) \sim \sqrt{x}$

$$f'(\eta_{99}) = 0.99 \Rightarrow \eta_{99} = 4.9 = \frac{\delta_{99}(x)}{\sqrt{\nu x / U_\infty}}$$

Boundary layer approximation invalid as $x \to 0$.

We require $Re_x = \frac{U_\infty x}{\nu} \gg 1$; $x \gg \nu/U_\infty$

Displacement of inviscid outer flow due to the boundary layer



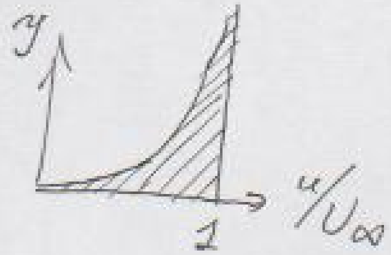
Streamline inclination

$$\frac{d\delta^*}{dx} = \frac{V_e(x)}{U_\infty} = 0{,}8604\sqrt{\frac{\nu}{U_\infty x}}$$

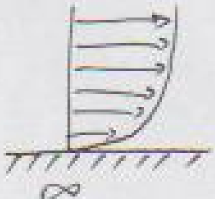
Total displacement: $\delta^* = \int_0^x \frac{V_e(x)}{U_\infty}dx = 0{,}8604 \cdot 2\sqrt{\frac{\nu x}{U_\infty}}$

Displacement thickness δ^*.

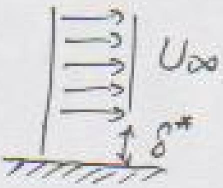
Blausius $\frac{\delta^*}{x} = \frac{1{,}7268}{\sqrt{Re_x}}$



General definition $\delta^* \triangleq \int_0^\infty (1-\frac{u}{U_\infty})dy$



$\Leftrightarrow$



$$\int_0^\infty (U_\infty - u)dy = U_\infty \delta^*$$

check:

$$\frac{d}{dx}\delta^* = \frac{d}{dx}\int_0^\infty (1-\frac{u}{U_\infty})dy = \int_0^\infty -\frac{1}{U_\infty}\frac{\partial u}{\partial x}dy$$

$$= \int_0^\infty \frac{1}{U_\infty}\frac{\partial v}{\partial y}dy = \frac{V_e(x)}{U_\infty}$$

Skin friction $\tau_w = \tau_{xy}(y=0) = \mu\frac{\partial u}{\partial y}(y=0) = \mu U_\infty \frac{\partial}{\partial y}(f'(\eta))_{\eta=0} =$

$$= \mu\frac{U_\infty}{\delta(x)}f''(0) = \frac{\mu U_\infty}{\sqrt{\nu x/U_\infty}}f''(0) = \rho U_\infty^2\sqrt{\frac{\nu}{U_\infty x}}f''(0) = \{\text{Blausius}\} =$$

$$= \frac{0{,}332}{\sqrt{Re_x}}$$