Fundamentals of Spaceflight.

Rocket Performance

Rocket Propulsion: we assuming:

- Force-free space
- Uniform steady flow over nozzle exit area
- Once the small mass leaves the nozzle, no force acts on it.

Properties: $t \to$ time, $V \to$ speed, $m \to$ mass,

$V_e \to$ exhaust speed w.r.t. rocket,

$\rho_e \to$ density of exhaust gas

$A_e \to$ nozzle exit area.

$\Rightarrow$ Mass of ejected hot gas during Δt is

$$\Delta m_e = \rho_e A_e V_e \Delta t > 0$$

$$(m + m_e)(V + \Delta V) + \Delta m_e (V - V_e) - mV = 0 \Rightarrow$$

$$m\frac{dV}{dt} = \frac{dm_e}{dt} \underbrace{V_e}_{\text{exhaust speed}}$$

Thrust equation: $T_m = \frac{dm_e}{dt} V_e = \underbrace{\dot{m}_e}_{\text{burn rate/mass flow}} V_e$

$$T_m = \dot{m}_e V_e = \rho_e A_e V_e^2 \sim q_e = \frac{1}{2}\rho_e V_e^2$$

Kinetic energy per unit volume $\approx$ dynamic pressure of exit flow

Potential energy of the compressible flow give

Pressure thrust $T_P = (P_e - P_a) A_e \Rightarrow$

$$T_{tot} = T_m + T_P = \dot{m}_e V_e + (P_e - P_a) A_e$$

Specifik impulse

fuel efficiency $\eta_{fuel} = T / \dot{m}_e = V_{eff}$

$$T_{tot} = \dot{m}_e V_e + (P_e - P_a) A_e = \dot{m}_e \left(V_e + (P_e - P_a)\frac{A_e}{\dot{m}_e}\right)$$

$= \dot{m}_e V_{eff}$, V_{eff} is "effective" exhaust speed.

Specifik impulse

$$I_{sp} = \int T_{tot} / W_e = I / m_e g_0 = \frac{V_{eff}\int \dot{m}_e dt}{m_e g_0} = \frac{V_{eff}\int dm_e}{m_e g_0}$$

$$= \frac{m_e V_{eff}}{m_e g_0} = V_{eff}/g_0$$

Number of seconds a unit weight of fuel would last if apply a unit force.

The Rocket equation: $m\frac{dV}{dt} = -\frac{dm}{dt}V_{eff}$

If $V_{eff} \approx$ constant, $\Delta V = -V_{eff} \ln \mu$, $\mu = \frac{m_f}{m_0}$ mass ratio

The speed increment ΔV only depends on the amount of fuel that has been burnt.

Single stage performance

m_* payload mass

m_s structural mass

m_p propellant mass

$m_0 = m_* + m_s + m_p$ initial mass

$m_f = m_* + m_s$ final mass

$\pi = \frac{m_*}{m_0}$ payload ratio

$\varepsilon = \frac{m_s}{m_s + m_p}$ structural ratio of booster

$\mu = \frac{m_f}{m_0}$ mass ratio $\Rightarrow$

$$\Delta V = -V_{eff} \ln[\varepsilon + (1-\varepsilon)\pi]$$

Multistage performance: $\varepsilon_k = \frac{m_{sk}}{m_{sk} + m_{pk}}$

$\Rightarrow \pi_k = \frac{m_{0,k+1}}{m_{0,k}}$. $\Rightarrow$ The overall payload ratio is:

$$\pi_* = \frac{m_*}{m_{0,1}} = \frac{m_*}{m_{0,N}} \frac{m_{0,N}}{m_{0,N-1}} \cdots \frac{m_{0,2}}{m_{0,1}} = \prod_{k=1}^{N} \pi_k = \pi_N \pi_{N-1} \cdots \pi_1$$

Overall burnout velocity:

$$V_* = \sum_{k=1}^{N} V_{eff,k} \ln\left(\frac{m_{0,k}}{m_{f,k}}\right) = -\sum_{k=1}^{N} V_{eff,k} \ln[\varepsilon_k + (1-\varepsilon_k)\pi_k]$$

$$= \sum_{k=1}^{N} \Delta V_k.$$

Optimization problem: maximize $\prod_{k=1}^{N} \pi_k$ subject to $V_* + \sum_{k=1}^{N} V_{eff,k} \ln[\varepsilon_k + (1-\varepsilon_k)\pi_k] = 0$

If $\varepsilon = \varepsilon_k$, $V_{eff} = V_{eff,k}$, then $\pi_k = \frac{e^{-\frac{V_*}{N V_{eff}}} - \varepsilon}{1 - \varepsilon}$

Rocket dynamics

The atmosphere: Hydrostatic equation: $\frac{dP}{dz} = \rho g$

$\Rightarrow \rho = \rho_0 e^{-H/H_0}$, $H_0 = RT/g$ is the called atmosphere scale height. ρ_0 density at $H=0$. "Exponential atmosphere"

Sounding rocket dynamics

$$\left.\begin{aligned} T &= -V_{eff}\frac{dm}{dt} = -g_0 I_{sp}\frac{dm}{dt} \\ D &= \frac{1}{2}\rho V^2 A C_D \\ W &= mg \end{aligned}\right\}$$

Equation of motion

$$m\frac{dV}{dt} = T - D - W \qquad \frac{dH}{dt} = V \quad \Rightarrow$$

$$V_* = \Delta V_{ideal} + \Delta V_{grav} + \Delta V_{drag}$$

$-V_{eff} \ln \frac{m_f}{m_0} > 0$

$-g\int_0^{t_b} dt = -g t_b < 0$

$-\int_0^{t_b} \frac{D}{m} dt = -\int_0^{t_b} \frac{\rho A C_D V^2}{2m} dt < 0$

Launcher dynamics (flat Earth)

γ flight path angle (between horisontal and $\bar{V}$)

α angle of attack (between $\bar{V}$ and body axis)

$\bar{L}$ aerodynamic lift ($\perp$ to $\bar{V}$)

$\bar{D}$ aerodynamic drag ($/\!/$ to $\bar{V}$)

$\bar{i}$ unit vectors.

$$\frac{d\bar{V}}{dt} = \ddot{X}\bar{i}_1 + \ddot{H}\bar{i}_2 = \frac{d\bar{V}_{/\!/}}{dt} + \frac{d\bar{V}_{\perp}}{dt}$$

$$= \frac{dV}{dt}\bar{s}_1 + V\frac{d\gamma}{dt}\bar{s}_2$$

$$\frac{d\gamma}{dt} = -\frac{g\cos\gamma}{V} \Leftarrow \begin{cases} \bar{s}_1: m\dfrac{dV}{dt} = T - D - mg\sin\gamma \\ \bar{s}_2: mV\dfrac{d\gamma}{dt} = -mg\cos\gamma \end{cases}$$

$$\bar{i}_1: \frac{dX}{dt} = V\cos\gamma \qquad \bar{s}_1 = \cos\gamma\,\bar{i}_1 + \sin\gamma\,\bar{i}_2$$

$$\bar{i}_2: \frac{dH}{dt} = V\cdot\sin\gamma \Rightarrow \bar{s}_2 = -\sin\gamma\,\bar{i}_1 + \cos\gamma\,\bar{i}_2$$

$$\Rightarrow \frac{{}^i d\bar{s}_1}{dt} = -\dot{\gamma}\sin\gamma\,\bar{i}_1 + \dot{\gamma}\cos\gamma\,\bar{i}_2 = \dot{\gamma}\bar{s}_2$$

$$\frac{{}^i d\bar{s}_2}{dt} = -\dot{\gamma}\cos\gamma\,\bar{i}_1 + \dot{\gamma}\sin\gamma\,\bar{i}_2 = -\dot{\gamma}\bar{s}_1$$

$$\Rightarrow \bar{V} = \dot{X}\bar{i}_1 + \dot{H}\bar{i}_2 = V\bar{s}_1 \Rightarrow$$

$$\bar{a} = \ddot{X}\bar{i}_1 + \ddot{H}\bar{i}_2 = \frac{dV}{dt}\bar{s}_1 + V\dot{\gamma}\bar{s}_2$$

We define the angular velocity vector that rotates the two frames $\bar{\omega}^{si} = \dot{\gamma}\bar{s}_3 \Rightarrow$

$$\frac{{}^i d\bar{V}}{dt} = \frac{dV}{dt}\bar{s}_1 + V\dot{\gamma}\bar{s}_2 = \frac{{}^s d\bar{V}}{dt} + \underbrace{\bar{\omega}^{si}\times\bar{V}}_{V\dot{\gamma}\bar{s}_2} = \frac{dV}{dt}\bar{s}_1$$

For all vectors $\bar{A}$: $\dfrac{{}^i d\bar{A}}{dt} = \dfrac{{}^s d\bar{A}}{dt} + \bar{\omega}^{si}\times\bar{A}$

The local horizon fram: $\bar{h}$ frame so that $\bar{h}_2$ always point at the laucher and position vector $\bar{r} = (R_\oplus + H)\bar{h}_2$. Angular velocity of the rotating $\bar{h}$ frame $\bar{\omega}^{hi} = -\dfrac{\dot{X}}{R_\oplus + H}\bar{h}_3 \Rightarrow$

$$\bar{V} = \frac{{}^i d\bar{r}}{dt} = \frac{{}^h d\bar{r}}{dt} + \bar{\omega}^{hi}\times\bar{r},\quad \bar{a} = \frac{{}^i d\bar{V}}{dt} = \frac{{}^h d\bar{V}}{dt} + \bar{\omega}^{hi}\times\bar{V}$$

$$\Rightarrow \bar{a} \approx \ddot{X}\bar{h}_1 + \left(\ddot{H} - \frac{\dot{X}^2}{R_\oplus + H}\right)\bar{h}_2 \Rightarrow$$

$$a_{tangent} = \frac{\dot{X}^2}{R_E + h}\sin\gamma,\quad a_{normal} = \frac{\dot{X}^2}{R_E + h}\cdot\cos\gamma,$$

$$a_{centripetal} = \frac{(V_c\cos\gamma)^2}{r} = \frac{\dot{X}^2}{R_E + h} \Rightarrow$$

A set of coupled nonlinear ordinary differential equations

$$\dot{V} = \frac{T}{m} - \frac{D}{m} - \left(g - \frac{\dot{X}^2}{R_\oplus + H}\right)\sin\gamma$$

$$\dot{\gamma} = -\frac{1}{V}\left(g - \frac{\dot{X}^2}{R_\oplus + H}\right)\cos\gamma$$

$$\dot{X} = V\cos\gamma$$

$$\dot{Y} = V\sin\gamma$$

$$\dot{m} = -\beta(t)$$

Steering laws: linear tangent laws

- Open loop linear steering laws

$$\tan\gamma(t) = \tan\gamma_0\left(1 - \frac{t}{t_{burnout}}\right) \Leftarrow \text{For flat earth}$$

$$\tan\gamma(t) = \tan\gamma_0\left[1 - \frac{t}{t_{burnout}} - \tan\beta\left(\frac{t}{t_{burnout}}\right)\right]$$

$\uparrow$ Taking into account the Earth curvature at burnout.

Orbital Mechanics

Newton's law of gravity: $\bar{F} = -\dfrac{GMm}{r^2}\dfrac{\bar{r}}{r}$

$$g = \frac{GM_\oplus}{r^2} = \frac{GM_\oplus}{(R_\oplus + H)^2} = \frac{\mu_\oplus}{(R_\oplus + H)^2}$$

$\mu_\oplus \approx 398600\ km^3/s^2$. If the Earth is not fixed, $\mu = G(M + m)$

$\Rightarrow \ddot{\bar{r}} + \dfrac{\mu}{r^3}\bar{r} = 0 \nearrow$ The motion of a small satellite does not depend on its mass.

The total energy per unit satellite mass: $E = \dfrac{V^2}{2} - \dfrac{\mu}{r}$ is constant in time for the two body problem. $\Rightarrow$

$$E_{circular} = \frac{V^2}{2} - \frac{\mu}{r} = \left\{\frac{V^2}{r} = g\right\} = \frac{gr}{2} - \frac{\mu}{r} = -\frac{\mu}{2r}$$

Escape velocity: $E = \dfrac{V^2}{2} - \dfrac{\mu}{r} = \{r \to \infty\} \to \dfrac{V_\infty^2}{2}$

$$\Rightarrow V_\infty \to 0 \Rightarrow E = \frac{V^2}{2} - \frac{\mu}{r} = 0 \Rightarrow V_{escape} = \sqrt{\frac{2\mu}{r}}$$

$E < 0$: bounded orbit ($v < V_{escape}$)

$E \geq 0$: escape orbit ($v \geq V_{escape}$)

Conservation of angular momentum: Kepler's 2:nd law the radius vector sweep out equal areas in equal time: $dA = \underbrace{\frac{1}{2}\cdot r\cdot r\dot{\theta}}_{constant}\cdot dt$

Orbital and trajectory solutions

The orbit equation: $r = \dfrac{H^2}{\mu(1+e\cos v)}$

$H \to$ angular momentum, $H^2 = \bar{r} \times \dot{\bar{r}} \cdot \bar{H}$

$e \to$ eccentricity, $v \to$ true anomaly

$e<1$ r is bounded and the orbit is an ellipse or circle.

$e \geq 1$ $r \to \infty$ possible and the trajectory is a parabola or an hyperbola.

$\Rightarrow E = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$. Very important: the total specific energy only depends on a.

$\Rightarrow$ The orbit period $P = 2\pi\sqrt{\frac{a^3}{\mu}} \Rightarrow$ Orbits with the same semimajor axes, but different eccentricities have the same orbit periods.

Orbital elements (Ω, i, ω)

Vernal equinox: the point on the sky where the sun crosses the equator from south to north the first day of spring.

Orientation of the orbital plane:

Ω: right ascension of the ascension (RAAN)

ω: argument of the perigee.

i : inclination

In the orbital plane:

a: semimajor axis, T_0: time of perigee passage

e: eccentricity,

Molniya orbit: $(i, \omega) = (63{,}4°, -90°)$

Earth Satellite Operations

The Hohmann transfer

First Δv is given by

$$\Delta v_1 = v(r_1) - v_{c1} = \sqrt{\frac{2\mu}{r_1} - \frac{2\mu}{r_1+r_2}} - \sqrt{\frac{\mu}{r_1}}$$

and the second Δv is given by

$$\Delta v_2 = v_{c1} - v(r_2) = \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu}{r_2} - \frac{2\mu}{r_1+r_2}}$$

Transfer time: $\Delta t = P_{ellipse}/2 = \pi\sqrt{\frac{a_t^3}{\mu}}$

Inclination change maneuvers

Law of cosines: $\Delta v = \sqrt{v_1^2 + v_2^2 - 2v_1 v_2 \cos\Delta i}$

For a pure inclination change. $(v_1 = v_2)$ we get

$\Delta v = 2v\sin\frac{\Delta i}{2}$, $\Delta v = v$ for $\Delta i = 60°$. Expensive!

Low-thrust orbit transfer

Acceleration $\bar{A} = \frac{\bar{T}}{m}$ assumed approximately constant. Spiral transfer between two circular orbits using low-thrust (but high I_{sp}) propulsion technology.

Equations of motion with thrust becomes

$$\ddot{\bar{r}} + \frac{\mu}{r^3}\bar{r} = \frac{\bar{T}}{m}$$

Consider time rate change of energy as in two-body problem:

$$\left(\ddot{\bar{r}} + \frac{\mu}{r^3}\bar{r}\right)\cdot\dot{\bar{r}} = \frac{d}{dt}\left(\frac{v^2}{2} - \frac{\mu}{r}\right) = \frac{dE}{dt} = \frac{\bar{T}\cdot\bar{v}}{m} =$$

$$= \left\{\begin{matrix}\text{Assume } \bar{T} \text{ and} \\ \bar{v} = \dot{\bar{r}} \text{ aligned}\end{matrix}\right\} = \frac{T}{m}v = Av \approx A\sqrt{\frac{\mu}{r}}$$

The total energy E increases with time due to the thrust, which is expected. The energy of the approximately circular orbit can also be expressed as a function of the radius r.

$E = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2r}$ and differentiation w.r.t. time leads to $\frac{dE}{dt} = \frac{d}{dt}\left(-\frac{\mu}{2r}\right) = \frac{\mu}{2r^2}\frac{dr}{dt} = A\sqrt{\frac{\mu}{r}}$

Equation of motion for radius of orbit: $\frac{dr}{dt} = 2A\sqrt{\frac{r^3}{\mu}}$

$$\frac{dr}{dt} = 2A\sqrt{\frac{r^3}{\mu}}$$

Low thrust inclination change

Earth oblateness effects.

Acceleration/gravity force due to the bulge will perturb the original two-body problem.

$$\ddot{\bar{r}} + \frac{\mu}{r^3}\bar{r} = \bar{A}_b \Rightarrow \frac{d\bar{H}}{dt} = \mathcal{M} = \bar{r} \times \bar{A}_b$$