Formelsamling

Vektoranalys

Gauss formel: $\int_\Omega \nabla\cdot\bar u\,dV = \oint_{\partial\Omega}\bar u\cdot d\bar S$

Stokes formel: $\int_S(\nabla\times\bar u)\cdot d\bar S = \oint_{\partial S}\bar u\cdot d\bar r$

Green formel I:

$$\int_\Omega \nabla u\cdot\nabla v\,dV = \oint_{\partial\Omega} u\frac{\partial v}{\partial n}dS - \int_\Omega u\Delta v\,dV$$

Greens formel II:

$$\int_\Omega(u\Delta v - v\Delta u)dV = \oint_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS$$

Laplaceoperatorn i cylinderkoordin

$$\Delta = \frac1r\partial_r r\partial_r + \frac1{r^2}\partial_\theta^2+\partial_z^2 = \partial_r^2+\frac1r\partial_r+\frac1{r^2}\partial_\theta^2+\partial_z^2$$

Laplaceoperatorn i sfäriska koord

$$\Delta = \frac1r\partial_r^2 r + \frac1{r^2}\Lambda = \frac1{r^2}\partial_r r^2\partial_r+\frac1{r^2}\Lambda = \partial_r^2+\frac2r\partial_r+\frac1{r^2}\Lambda$$

$$\Lambda = \frac{1}{\sin\theta}\partial_\theta\sin\theta\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2 = \partial_c(1-c^2)\partial_c+\frac{1}{1-c^2}\partial_\phi^2$$

där $c=\cos\theta$, $0<\theta<\pi$, $0<\phi<2\pi$

Ortogonalutvecklingar

$\langle u|v\rangle_w = \int\overline{u(x)}\,v(x)\,w(x)\,dx$

Om $\langle\varphi_j|\varphi_k\rangle=0,\ j\neq k$, så

$$\begin{cases} u=\sum c_k(u)\varphi_k \\ c_k(u)=\dfrac{\langle\varphi_k|u\rangle}{\rho_k},\ \rho_k=\langle\varphi_k|\varphi_k\rangle \end{cases}$$

Parseval:

$\langle u|v\rangle = \sum\frac{1}{\rho_k}\overline{\langle\varphi_k|u\rangle}\langle\varphi_k|v\rangle = \sum\rho_k\overline{c_k}c_k$

Sturm-Liouville: $\mathcal{A}u = \frac1w[-\nabla\cdot(p\nabla u)+qu]$

Speciella funktioner:

Gammafunktionen och Betafunktion

$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt$, $\Gamma(z+1)=z\Gamma(z)$

$\Gamma(n+1)=n!$, $\Gamma(1/2)=\sqrt\pi$

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Error funktion: $\mathrm{erf}(x)=\frac{2}{\sqrt\pi}\int_0^x e^{-y^2}dy$

Besselfunktioner:

$$e^{ir\sin\theta}=\sum_{-\infty}^{\infty}J_n(r)e^{in\theta}$$

$$J_n(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(z\sin\theta-n\theta)}d\theta$$

$$J_\nu(z)=\left(\frac z2\right)^\nu\sum_{k=0}^\infty\frac{1}{k!\,\Gamma(k+\nu+1)}\left(-\frac{z^2}{4}\right)^k$$

n hetal, $\nu\neq-1,-2$

Bessels differentialektion

$$u''+\frac1r u'+\left(\lambda-\frac{\nu^2}{r^2}\right)u=0$$

har den allmänna lösningen

$$\begin{cases} aJ_\nu(\sqrt\lambda r)+bY_\nu(\sqrt\lambda r) & \text{om } \lambda>0 \\ ar^\nu+br^{-\nu} & \text{om } \lambda=0,\nu\neq0 \\ a+b\ln r & \text{om } \lambda=0,\nu=0 \end{cases}$$

där Y_ν ej är begränsad i origo.

Normuttryck

$$\int_0^R\left|J_\nu\left(\frac rR\alpha_{\nu k}\right)\right|^2 r\,dr = \frac{R^2}{2}[J_{\nu+1}(\alpha_{\nu k})]^2 = \frac{R^2}{2}[J_\nu'(\alpha_{\nu k})]^2$$

Sfäriska besselfunktion

Differentialekvationen

$$u''+\frac2z u'+\left(\lambda-\frac{l(l+1)}{z^2}\right)u=0$$

har den allmänna lösningen

$$\begin{cases} aj_l(\sqrt\lambda z)+by_l(\sqrt\lambda z) & \text{om } \lambda>0 \\ az^l+bz^{-l-1} & \text{om } \lambda=0, l\neq-\frac12 \\ \dfrac{a+b\ln z}{\sqrt z} & \text{om } \lambda=0, l=-\frac12 \end{cases}$$

där $j_l(z)=\sqrt{\frac{\pi}{2z}}J_{l+1/2}(z)$,

$y_l(z)=\sqrt{\frac{\pi}{2\pi}}Y_{l+1/2}(z)$

Speciellt är

$j_0(z)=\frac{\sin z}{z}$, $j_1(z)=\frac{\sin z - z\cos z}{z^2}$

$y_0(z)=\frac{\cos z}{z}$, $y_1(z)=-\frac{\cos z - z\sin z}{z^2}$

Legendrefunktioner

Legendrepolynom $(P_l)_0^\infty$ ortogonala i $L_2(I)$, $I=(-1,1)$.

Legendres differentialekvation

$$\partial_x[(1-x^2)\partial_x u]+l(l+1)u=0,\quad l=0,1,\dots$$

har den allmänna lösningen

$u=aP_l(x)+bQ_l(x)$

där Q_l ej är begränsad och

$$P_l(x)=\frac{1}{2^l l!}D^l(x^2-1)^l$$

Rekursionsformel för Legendrepoly

$P_0(x)=1$, $P_1(x)=x$,

$$P_{l+1}=\frac{2l+1}{l+1}xP_l(x)-\frac{l}{l+1}P_{l-1}(x);$$

Associerade legendreekvationen

$$\partial_x[(1-x^2)\partial_x u]-\frac{m^2}{1-x^2}u+l(l+1)u=0$$

har den allmänna lösningen

$aP_l^m(x)+bQ_l^m(x)$

där Q_l^m ej är begränsad och

$$P_l^m(x)=(1-x^2)^{m/2}D^mP_l(x)$$

Greenfunktioner

Fundamentallösningar till

Laplaceoperatorn $(-\Delta K=\delta)$.

$K(\bar x)=-\frac{1}{2\pi}\ln|\bar x|$ i $\mathbb R^2$

$K(\bar x)=\frac{1}{4\pi|\bar x|}$ i $\mathbb R^3$

Poissonkärnor

$$P(r,\theta)=\frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos\theta}$$

(enhetscirkeln)

$$P(x,y)=\frac1\pi\frac{y}{x^2+y^2}$$

(halvplanet $y>0$)

Greenfunktion för Dirichlets problem

$$\begin{cases} -\Delta_{\bar x}G(\bar x;\bar\alpha)=\delta_{\bar\alpha}(\bar x), & \bar x\in\Omega \\ G(\bar x;\bar\alpha)=0, & \bar x\in\partial\Omega \end{cases}$$

Om $-\Delta u=f$ i Ω, $u=g$ på $\partial\Omega$ så gäller

$$u(\bar x)=\int_\Omega G(\bar x;\bar\alpha)f(\alpha)\,dV_\alpha - \oint_{\partial\Omega}\frac{\partial G}{\partial n_\alpha}g(\alpha)\,dS_\alpha$$

Konjugerade punkter med avseende på cirkel (sfär) $|\bar x|=\rho$

$|\bar\alpha||\tilde\alpha|=\rho^2$

$|\bar x-\bar\alpha|=\frac{|\bar\alpha|}{\rho}|\bar x-\tilde\alpha|$ då $|\bar x|=\rho$

Värmeledning

$$\begin{cases} G(x,t)=\dfrac{1}{\sqrt{4\pi at}}e^{-x^2/4at} \\ \partial_t G - a\partial_x^2 G=0, & x\in\mathbb R, t>0 \\ G(x,0)=\delta(x), & x\in\mathbb R \end{cases}$$

Vågutbredning

d'Alembert: $g(x)=u(x,0)$, $h=u_t(x,0)$

$$u(x,t)=\frac12[g(x-ct)+g(x+ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}h(y)\,dy$$

Karakteristikor

$$\begin{cases} a_{11}u_{xx}+2a_{12}u_{xy}+a_{22}u_{yy}+F(x,y,u,u_x,u_y)=0 \\ a_{11}dy^2-2a_{12}dx\,dy+a_{22}dx^2=0 \end{cases}$$

Nollställen till besselfunktionen $J_n(\alpha_{nk})$

k\n	0	1	2	3
1	2,405	3,832	5,136	6,380
2	5,520	7,016	8,417	9,761
3	8,654	10,173	11,620	13,015

Nollställen till $J_n'(x)$, $x<25$, $J_n'(\alpha_{nk}')$

k\n	0	1	2	3
1	0,000	1,841	3,054	4,201
2	3,832	5,331	6,706	8,051

Fouriertransformer

$$\mathcal Ff(\omega)=\hat f(\omega)=\int_{-\infty}^\infty e^{-i\omega t}f(t)\,dt$$

$$(\mathcal F^{-1}\hat f)(t)=f(t)=\frac{1}{2\pi}\int_{-\infty}^\infty e^{i\omega t}\hat f(\omega)\,d\omega$$

Parsevals formel:

$$\int_{-\infty}^\infty\overline{f(t)}g(t)\,dt=\frac1{2\pi}\int_{-\infty}^\infty\overline{\hat f(\omega)}\hat g(\omega)\,d\omega$$

Tabeln $\mathcal F$.

$\lambda f(t)+\mu g(t)$	$\lambda\hat f(\omega)+\mu\hat g(\omega)$
$f(at)$	$\frac{1}{\lvert a\rvert}\hat f\left(\frac{\omega}{a}\right)$
$f(t-t_0)$	$e^{-it_0\omega}\hat f(\omega)$
$e^{i\omega_0 t}f(t)$	$\hat f(\omega-\omega_0)$
$f'(t)$	$i\omega\hat f(\omega)$
$tf(t)$	$i\frac{d}{d\omega}\hat f(\omega)$
$f*g(t)$	$\hat f(\omega)\hat g(\omega)$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$e^{-t}\theta(t)$	$\frac{1}{1+i\omega}$
$e^{-\lvert t\rvert}$	$\frac{2}{1+\omega^2}$
$\frac{1}{1+t^2}$	$\pi e^{-\lvert\omega\rvert}$
e^{-t^2}	$\sqrt\pi e^{-\omega^2/4}$
$\theta(t+1)-\theta(t-1)$	$2\frac{\sin\omega}{\omega}$
$\theta(t)$	$\frac1i P\frac1\omega+\pi\delta$

Laplacetransformer

$$\mathcal Lf(s)=\int_{-\infty}^\infty e^{-st}f(t)\,dt$$

$$f(t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}e^{st}F(s)\,ds$$

$$\mathcal Ff(\omega)=\mathcal Lf(i\omega)$$

Tabeln. $s=\sigma+i\omega$

$\lambda f(t)+\mu g(t)$	$\lambda F(s)+\mu G(s)$
$f(at)$	$\frac{1}{\lvert a\rvert}F\left(\frac sa\right)$
$f(t-t_0)$	$e^{-t_0 s}F(s)$
$e^{at}f(t)$	$F(s-a)$
$f'(t)$	$sF(s)$
$tf(t)$	$-\frac{d}{ds}F(s)$
$f*g(t)$	$F(s)G(s)$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$\theta(t)$	$\frac1s$, $\sigma>0$
$\theta(t)-1$	$\frac1s$, $\sigma<0$
$t^k e^{at}\theta(t)$	$\frac{k!}{(s-a)^{k+1}}$, $\sigma>\mathrm{Re}\,a$
$\sin(bt)\theta(t)$	$\frac{b}{s^2+b^2}$, $\sigma>0$
$\cos(bt)\theta(t)$	$\frac{s}{s^2+b^2}$, $\sigma>0$
e^{-t^2}	$\sqrt\pi e^{s^2/4}$

Variationskalkyl

Euler-Lagranges ekvationer

Vid optimering av funktionalen

$$F[x]=\int L(x,\dot{x},t)\,dt$$

måste funktionen L uppfylla

Euler-Lagranges ekvation

$$\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}=0.$$

Modifierade Besselfunktioner

Bessels modifierade DE:

$$R''+\frac{1}{\rho}R'-\left(\lambda^2+\frac{\mu^2}{\rho^2}\right)R(\rho)=0 \Rightarrow$$

$$I_\mu(\rho\lambda)=i^{-\mu}J_\mu(i\rho\lambda)$$

$$K_\alpha(x)=\frac{\pi}{2}\,\frac{I_{-\alpha}(x)-I_\alpha(x)}{\sin(\alpha\pi)}$$

Kontinuitetsekvation

Ändring = Produktion − Utflöde

$$\int\frac{\partial \rho}{\partial t}dV=\int K\,dV-\int\nabla\cdot\bar{J}\,dV \Rightarrow$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot\bar{J}=K$$

Diffusion: $\bar{J}=-\lambda\nabla\rho \Rightarrow \partial_t\rho-\lambda\nabla^2\rho=K$

Värmeledning: $dq=c\rho\,dT \Rightarrow$

$$\partial_t T-\frac{\lambda}{c\rho}\nabla^2 T=\frac{K}{c\rho}$$

Överföring av inhomogeniteter:

$$u=\rho(x,t)-\rho_{stat}(x)$$

Membransvängning:

$[S]=\frac{\text{kraft}}{\text{längd}}$, $[\rho]=\frac{\text{massa}}{\text{area}}$, $c^2=\frac{S}{\rho}$

$\bar{F}=S\,d\bar{l}\times\hat{n} \Rightarrow u_{tt}-c^2\nabla^2 u=0$

Longitudinella svängning:

$u_{tt}-c^2u_{xx}=0,\quad c^2=E/\rho$

Separering av SL-problem:

I 2D: $\hat{L}=\hat{L}_x+f(x)\hat{L}_y$, $\hat{L}_yY_n=\mu_nY_n$

laplaceoperatorn i olika koord

Laplaceoperatorn som Sturm-Li

$$-\nabla^2=-\sum_{i=1}^{3}\partial_i^2$$

I kroklinigt koord-system

$$-\nabla^2=-\sum_i^{3}\frac{1}{h_1h_2h_3}\partial_i\frac{h_1h_2h_3}{h_i^2}\partial_i$$

$\Rightarrow w(x)=h_1h_2h_3$

För polära: $h_r=1$, $h_\varphi=r$

Distributioner

$f(x)$: $f[\varphi]=\int f(x)\varphi(x)\,dx$

$f'(x)$: $f'[\varphi]=-f[\varphi']$

Dimensionsanalys: $[\delta]=\frac{1}{L}$

Problem i oändliga rum:

I ändlig: $u(\bar{x},t)=\sum_n u_n(t)f_n(x)$

I oändliga rum:

$$u(x,t)=\int_{-\infty}^{\infty}\hat{u}(k,t)f_k(x)\,dk$$

$$=A\int_{-\infty}^{\infty}\hat{u}(k,t)e^{ikx}\,dk$$

Greenfunktioner & transformation

$\partial_t u-a\partial_x^2u=K(x,t) \Rightarrow$

$$u=\int_0^{\infty}dt'\int_{-\infty}^{\infty}dx'\,K(x',t')G(t-t',x-x')$$

$\hat{\hat{u}}(k,s)=\hat{\hat{K}}\hat{\hat{G}} \Rightarrow s\hat{\hat{u}}+ak^2\hat{\hat{u}}=\hat{\hat{K}}$

$\Rightarrow \hat{\hat{G}}=\frac{1}{s+k^2a}=\frac{\hat{\hat{u}}}{\hat{\hat{f}}}$

Greenfunktioner:

$$u(x,t)=\int_{-\infty}^{\infty}f(x')G_0(x-x',t)\,dx'$$

$$G_0(x-x_0,0)=\delta(x-x_0)$$

Fundamentallösningar i 2D

$\Delta G_0(\bar{r})=-\delta(\bar{r}) \Rightarrow -\frac{1}{2\pi}\ln\left(\frac{r}{r_0}\right)$

$\Delta G_0(\bar{r})+\mu^2G_0(\bar{r})=-\delta(\bar{r}) \Rightarrow -\frac{1}{4}Y_0(\mu r)$

$\Delta G_0(\bar{r})-\mu^2G_0(\bar{r})=-\delta(\bar{r}) \Rightarrow \frac{1}{2\pi}K_0(\mu r)$

$\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G_0(\bar{r},t)=-\delta(\bar{r})\delta(t) \Rightarrow$

$$\frac{1}{2\pi\sqrt{t^2-r^2/c^2}}\Theta(t-r/c)$$

$\left(D\Delta-\frac{\partial}{\partial t}\right)G_0(\bar{r},t)=-\delta(\bar{r})\delta(t) \Rightarrow$

$$\frac{1}{4\pi Dt}e^{-r^2/(4Dt)}\cdot\Theta(t)$$

Fundamentallösningar i 3D

$\Delta G_0(\bar{r})=-\delta(\bar{r}) \Rightarrow \frac{1}{4\pi r}$

$\Delta G_0(\bar{r})+\mu^2G_0(\bar{r})=-\delta(\bar{r}) \Rightarrow \frac{e^{\pm i\mu r}}{4\pi r}$

$\Delta G_0(\bar{r})-\mu^2G_0(\bar{r})=-\delta(r) \Rightarrow \frac{e^{-\mu r}}{4\pi r}$

$\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G_0(\bar{r},t)=-\delta(\bar{r})\delta(t)$

$\Rightarrow \frac{1}{4\pi r}\delta\left(\frac{r}{c}-t\right)\Theta(t)$

$\left(D\Delta-\frac{\partial}{\partial t}\right)G_0(\bar{r},t)=-\delta(\bar{r})\delta(t)$

$\Rightarrow \frac{1}{(4\pi Dt)^{3/2}}e^{-r^2/(4Dt)}\Theta(t)$

ODE: $y''+ay'+by=0 \Rightarrow$

(i) $r_1\neq r_2$ real: $y=C_1e^{r_1x}+C_2e^{r_2x}$

(ii) $r=r_1=r_2$: $y=(C_1x+C_2)e^{rx}$

(iii) $r_1=\alpha+i\beta$, $r_2=\alpha-i\beta$: $y=e^{\alpha x}(C_1\cos\beta x+C_2\sin\beta x)=e^{\alpha x}\cdot C\cdot\cos(\beta x+\theta)$

Differentiation formulations

$$\frac{d}{dx}[f\cdot g]=f\cdot g'+g\cdot f'$$

$$\frac{d}{dx}\left(\frac{f}{g}\right)=\frac{g\cdot f'-f g'}{[g]^2}$$

$$\frac{d}{dx}\tan x=\frac{1}{\cos^2 x},\quad \frac{d}{dx}\cot x=-\frac{1}{\sin^2 x}$$

$$\frac{d}{dx}\sin^{-1}x=\frac{1}{\sqrt{1-x^2}},\quad \frac{d}{dx}\cos^{-1}x=\frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x=\frac{1}{x^2+1},\quad \frac{d}{dx}\cot^{-1}x=\frac{-1}{x^2+1}$$

Integration formulations

$$\int a^x dx=\frac{1}{\ln a}a^x+C$$

$$\int \ln x\,dx=x\ln x-x+C$$

$$\int f(x)g(x)\,dx=F(x)g(x)-\int F(x)g'(x)\,dx$$

$$\int_a^b f(g(x))g'(x)\,dx=\int_{g(a)}^{g(b)}f(t)\,dt$$

Komplexa tal

$\ln(z)=\ln|z|+i\cdot\arg(z)$

$\sin(z)=(\exp(z)-\exp(-z))/2i$

$\cos(z)=(\exp(z)+\exp(z))/2$

Fourierserier

$$f(t)=\sum_{k=-\infty}^{\infty}c_ke^{ik\omega t}=c_0+\sum_{k=1}^{\infty}a_k\cos k\omega t+b_k\sin k\omega t$$

$\omega T=2\pi.\quad c_k=\frac{1}{T}\int_{period}e^{-ik\omega t}\cdot f(t)\,dt$

$$\begin{cases} a_k=\frac{2}{T}\int_{period}\cos(k\omega t)f(t)\,dt=c_k+c_{-k}\\ b_k=\frac{2}{T}\int_{period}\sin(k\omega t)f(t)\,dt=i(c_k-c_{-k})\end{cases}$$

Modellering:

$$\sin\theta=\frac{u_x}{\sqrt{1+u_x^2}}\simeq u_x$$

Besselfunktionerna

$$J_\alpha(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m+\alpha}$$

$$J_n(x)=\frac{1}{2\pi}\int_0^{2\pi}\cos(nt-x\sin t)\,dt$$

$$Y_\alpha(x)=\frac{J_\alpha(x)\cos(\alpha\pi)-J_{-\alpha}(x)}{\sin(\alpha\pi)}$$

Sfäriska Besselfunktioner

$$j_n(x)=\sqrt{\frac{\pi}{2x}}\,J_{n+1/2}(x)$$

$$y_n(x)=\sqrt{\frac{\pi}{2x}}\,Y_{n+1/2}(x)$$

$$A_nj_0(kr)+B_ny_0(kr)=C_n\frac{\sin(k(r-r_0))}{r}$$

Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)=\frac{\partial L}{\partial q_j}$$

$$L=T-V$$

Exempel på green funktion:

$$G(x,x',t)=G_0(x-x',t)-G_0(x+x',t)$$

$$=\frac{1}{\sqrt{4\pi at}}\left[e^{-\frac{(x-x')^2}{4at}}-e^{-\frac{(x+x')^2}{4at}}\right]$$

Beltrami identitet: då $\partial L/\partial x=0$

$$L-u'\frac{\partial L}{\partial u'}=0$$