SDM 366 Optimal Control and Estimation

LN1: Linear Algebra Review

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Why Linear Algebra:

- One of the most important tools for modern control theory

- Topics covered in this class, such as
  - State space model
  - Least squares
  - Stability analysis
  - Linear quadratic regulator (LQR)
  - Kalman filter

  can be viewed as applications of linear algebra

- Crucial for machine learning, robotics, computer vision, …

Facts about the students:

- Remember formulas without deep understanding of concepts
- Good at numerical calculations, but not analytical analysis

Goal:

- Learn to "Forget" the formulas or numerical techniques
- Review/rebuild fundamental concepts
- "Speak" the language of linear algebra: formulate/analyze/solve linear algebra problems without using formulas or numbers
  - Linear independence
  - Matrix rank
  - Vector space
  - Column space/null space
  - Solution $Ax = b$

- Just a short review. A good reference is the MIT course (Prof. Strang)

https://ocw.mit.edu/courses/mathematics/18-06sc-linear-algebra-fall-2011/index.htm

Review Outline

- **Part I:**
  - **Linear combination**
  - **Linear independence**
  - **Vector space**
- Part II:
  - Column space/Null space
  - Matrix rank
  - Solution space of Ax = b
- Part III
  - Inner product
  - Linear, conic, convex combinations
  - Some Geometric Sets

Key Concept: Linear Combination

- Linear combination of two vectors $v_1, v_2 \in R^2$

- Linear combination of $v_1, v_2, \ldots, v_k \in R^n$

Key Concept: Linear Combination

- Trivial and nontrivial linear combination:

- Span of a set of vectors:

  $span(v_1, v_2, \dots, v_k) = \{w \in R^n: w = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k, \text{for some scalars } \alpha_1, \alpha_2, \dots, \alpha_k\}$

Linear Independence

- Two vectors $\{v_1, v_2\}$ are **linearly dependent** if

- A set of vectors $\{v_1, \cdots, v_k\}$ is linearly **independent** if

No nontrivial linear combination = 0

- Equivalent definition: No vector v_i can be expressed as a linear combination of other vectors $v_1, \cdots, v_{i-1}, v_{i+1}, \dots, v_n$

Vector Space

- **Vector space $\boldsymbol{V}$**: set of elements for which "addition" and "multiplication by scalers" can be properly defined
  - element can be number, matrix, function, symbols …
  - "Addition" and "multiplication" can be defined as long as they satisfy certain Axioms.

- **Subspace** of a vector space V: subset of V that is "closed" under addition and multiplication
  - Span (v_1, v_2) :
  - $R_+^2 = \{x \in R^2 : x_1 \geq 0, x_2 \geq 0\}$:

Vector Space

- $\{v_1, v_2, \dots, v_k\}$ is a basis for a vector space V if
  1. $V = span(\{v_1, v_2, \dots, v_k\})$
  2. $\{v_1, v_2, \dots, v_k\}$ is linearly independent

- Dimension of a vector space:
  - Number of vectors in a basis
  - **Fact:** number of vectors in any basis of a finite-dim vector space is the same

Vector Space

- Coordinates of $w \in V$ with respect to a basis $\{v_1, v_2, \dots, v_k\}$

- Coordinates of a vector depend on the basis

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Key: Matrix vector multiplication as mixture of columns

- Let $y = Ax$, then y is a linear combination of the columns of A
  - Write matrix A in terms of its columns

$$A = [a_1 \quad a_2 \quad \dots \quad a_n], \text{ where } a_j \in R^m$$

  - Then $y = Ax$ can be written as

- Similarly, if $z = dB$, where d and z are row vectors, then z is a linear combination of the rows of B

Column Space (Range) of a Matrix $A \in R^{m\times n}$

$$\text{Col(A)} = \text{Range}(A) = \{ Ax \mid x \in R^n\} \subset R^m$$

- = **Span of columns of A**

- Example: $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

Null Space of a Matrix $A \in R^{m \times n}$

$$Null(A) = \{x \in R^n | Ax = 0\}$$

- Coefficients of linear combinations that result in a zero vector
- Zero null space implies: columns of A are independent
- Example of null space: $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

Rank of a matrix $A \in R^{m \times n}$

- Definition: $rank(A) = \dim(\text{Col}(A))$
  - i.e. number of independent columns of A

- Nontrivial facts
  - $rank(A) = rank(A^T)$

  - $rank(A) \leq \min(m, n)$: full rank means **rank(A) = min(m,n)**

  - $rank(A) + \dim(Null(A)) = n$
    - "**conservation of dimension**": Think about A as a linear mapping that maps $x \in R^n$ to a vector $y = Ax \in R^m$. Each dimension of input x is either crushed to zero or ends up in output

Example of "conservation of dimension":

- Find the $Null(A)$, where $A \in R^{10\times 4} = [a_1 \;\; a_2 \;\; a_3 \;\; a_4]$ satisfies

① a_1, a_3 independent

② $a_2 = 1a_1 + 2a_3$

③ $a_4 = 5a_1 - 36a_3$

Linear Equation $Ax = b,\ x \in R^n, b \in R^m$

- There exists a solution if

- There always exists a solution for any $b \in R^m$ if:

- There exists a unique solution if:

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Inner Product

- Inner product of vectors in R^n: $< v, w > =$

- Norm of $v \in R^n$

- Angle between $v, w \in R^n$

- Orthogonality:

Projection

- Projection of $v \in R^n$ along direction e

- $\{e_1, \ldots, e_k\}$ be orthonormal basis of vector space V, then any $v \in V$,

$$v = < v, e_1 > e_1 + \cdots + < v, e_k > e_k$$

General Inner Product

- General inner product $<\cdot,\cdot>: V \times V \to R$
  - maps each pair in a vector space to a scaler
  - satisfies several key properties: linearity, conjugate, positive definiteness …

- Inner product of matrices in $R^{m\times n}$: $< A, B > =$

- Inner product of two functions f, g on interval $[a, b]$:

Projection

- **Fourier series**: Consider a vector space of periodic functions:
$$V = \{\text{integrable functions over } [0,2\pi)\}$$

- Inner product: $< f, g > = \int_0^{2\pi} f(x)g(x)dx$

- Basis: $B = \{1,\ \cos x, \sin(x), \cos(2x),\ \sin(2x), \ldots\}$

- $f \in V$, then

Representation of Geometric Objects / Sets

- **Implicit method** via (sub)-level sets:

$$\{x \in R^n:\ f(x) = 0\} \quad \text{or} \quad \{x \in R^n: f(x) \leq 0\}$$

- **Explicit method**: $\{x(\alpha) \in R^n:\ \alpha \text{ satisfies certain conditions}\}$

Cone and Conic Combination

- **Cone**: A set S is called a cone if $x \in S \Rightarrow \lambda x \in S, \ \forall \lambda \geq 0$

- **Conic combination** of $v_1, \dots, v_k \in R^n$

$$cone(v_1, \dots, v_k) = \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k : \ \alpha_i \geq 0\}$$

Convex Set and Convex Combination

- Convex set: A set S is called convex if

$$v_1, v_2 \in S \Rightarrow \quad \alpha v_1 + (1-\alpha) v_2 \ \in S$$

- **Convex combination** of $v_1, \ldots, v_k \in R^n$

$$\{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k \colon \alpha_i \geq 0, \sum \alpha_i = 1\}$$

- Convex hull $\overline{co}(S)$: set of all convex combinations of points in S

Some Simple Geometric Sets - Line

- Line segment: Given $v_1 \neq v_2 \in R^n$: $\{v_2 + \alpha\,(v_1 - v_2): \ \alpha \in [0,1]\}$

- Line (explicit): $\{v_2 + \alpha\,(v_1 - v_2): \ \alpha \in R\}$

- Line: (implicit)
  - e.g. in R^2: $\{x \in R^2: \ a^T x = b\}$

Some Simple Geometric Sets - Hyperplanes

- Hyperplanes (Implicit): $\{x \in R^n : a^T x = b\}$

- Hyperplanes (Explicit): $\{x \in R^n : a^T x = b\} \Rightarrow \{x_0 + \sum_i \alpha_i v_i : \ \alpha_i \in R, i = 1, \dots, n-1\}$

- Halfspaces: $\{x \in R^n : a^T x \leq b\}$

Some Simple Geometric Sets - Polyhedron

- **(Convex) Polyhedron**: intersection of a finite number of half spaces

$$P = \{x \in R^n : Ax \leq b\}$$

- **Polyhedral cone**: intersection of finitely many halfspaces that contain the origin:

$$P = \{x \in R^n : Ax \leq 0\}$$

- **Polytope**: bounded polyhedron

Some Simple Geometric Sets - Polyhedron

- Polyhedron (vertex representation):

$$P = \overline{co}(v_1, \dots, v_m) \oplus cone(r_1, \dots, r_q)$$

Some Simple Geometric Sets - Quadratic Sets

- Euclidean balls: $B(x_c, r) = \{x \in R^n : \|x - x_c\|_2 \le r\}$

  or $B(x_c, r) = \{x_c + ru : u \in R^n, \|u\|_2 \le 1\}$

- Ellipsoids: $E = \{x \in R^n : (x - x_c)^T P^{-1} (x - x_c) \le 1\}$

  or $E = \{x_c + Au : u \in R^n, \|u\|_2 \le 1\}$