SDM366 Spring 2024 **Homework # 2**

Assigned: March 1, 2024 Due: Saturday, March 9, 2024

- Please turn in a hard copy of your solution to the TAs before 6pm on the due day.
- To receive credits, please write down all the necessary steps leading to final answer.

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1. *Cross Product.* Cross product is a linear operation that can be represented by a matrix. Let $w = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}^T$ and $r = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}^T$ be two vectors in $\mathbf{R}^3$. Let $v = w \times r$ be the cross product of w and r. Find the matrix A_w such that $v = A_w r$. (Note: the expression of A_w depends only on w and is called the skew-symmetric representation of w)

2. *Column and Null Space*: Define

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 2 & 0 \\ -1 & 0 & -2 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} -2 & -1 & 1 & 0 \\ 1 & 5 & 4 & 3 \\ 1 & -1 & -2 & -1 \\ 1 & 2 & 1 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$$

   (a) What are the dimensions of the null space and column space (i.e. range space) of A?
   (b) Find a set of basis vectors for null(A).
   (c) Find a set of basis vectors for col(A)
   (d) Is col(C) = col(A)? Justify your answer.
   (e) Find a matrix B of appropriate dimension such that $C = AB$. (You should be able to find B just by inspection).

   Hint: Let a_1, a_2, a_3 be the three columns of A and $c_1, \ldots, c_4$ be the four columns of C. By inspection (simple calculation), the following relations hold

$$a_3 = 2a_1 - a_2, \quad c_1 = -a_1 + a_2, \quad c_2 = a_1 + 2a_2$$
$$c_3 = 2a_1 + a_2, \quad c_4 = a_1 + a_2$$

3. *Speak the Matrix Language*: Express the following statements in matrix language. You can assume that all matrices mentioned have appropriate dimensions. Here is an example: "Every column of C is a linear combination of the columns of B" can be expressed as"$C = BF$ for some matrix F". There can be several answers; one is good enough. You are expect to justify all of your answers.

   (a) For each i, row i of Z is a linear combination of rows $i, \ldots, n$ of Y.
   (b) W is obtained from V by permuting adjacent odd and even columns (i.e., 1 and 2, 3 and 4,...).
   (c) Each column of P makes an acute angle with each column of Q.
   (d) Each column of P makes an acute angle with the corresponding column of Q.

(e) The first k columns of A are orthogonal to the remaining columns of A.

4. *Matrix Rank*:

   (a) Let $a \in \mathbf{R}^n$ be an n-dim vector. Show that the $n \times n$ matrix $A \triangleq aa^T$ is of rank 1.

   (b) Explain why the system $Ax = b$ has a solution if and only if $\text{rank}(A) = \text{rank}([A \;\; b])$.

5. *Ellipsoids:* Ellipsoid in $\mathbf{R}^n$ have two equivalent representations: (i) $E_1(P, x_c) = \{x \in \mathbf{R}^n : (x - x_c)P^{-1}(x - x_c) \leq 1\}$ and (ii) $E_2(A, x_c) = \{Au + x_c : \|u\|^2 \leq 1\}$. Given an eillipsoid $E_1(P, x_c)$ with P positive definite, its volume is $\nu_n\sqrt{det(P)}$ where ν_n is the volume of unit ball in $\mathbf{R}^n$, its semi-axes directions are given by the eigenvectors of P and the lengths of semi-axes are $\sqrt{\lambda_i}$, where λ_i are eigenvalues of P.

   (a) Given an Ellipsoid $E_1(P, x_c)$, find the corresponding (A, b) (in terms of P and x_c) such that $E_2(A, b)$ represents the same ellipsoid as $E_1(P, x_c)$

   (b) Draw the ellipse $E_1(P, x_c)$ with $P = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ and $x_c = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ by hand.