HW2

1. $v = w \times r = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \times \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} w_2 r_3 - w_3 r_2 \\ w_3 r_1 - w_1 r_3 \\ w_1 r_2 - w_2 r_1 \end{bmatrix} = \begin{bmatrix} & -w_3 r_2 & +w_2 r_3 \\ w_3 r_1 & & -w_1 r_3 \\ -w_2 r_1 & +w_1 r_2 & \end{bmatrix} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$

$\Rightarrow A_w = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}$

2.

(a) Define $A = [a_1 \ \ a_2 \ \ a_3]$, ($a_i$ are $\mathbb{R}^5$ vectors, $i = 1.2.3$)

It's obvious that $2a_1 - a_2 = a_3$ and $a_1 \neq \alpha a_2$ ($\alpha \in R$).

Hence, we may conclude that A has one linearly dependent column while the other two are linearly independent.

As a result:

dimension$\{null(A)\} = 1$, dimension$\{col(A)\} = 2$

(b) $2a_1 - a_2 - a_3 = 0 \quad \Rightarrow A \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = 0 \quad \Rightarrow$ a set of basis vectors for null(A) : $\left\{ \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right\}$

(c) according to "(a)" $\Rightarrow$ a set of basis vectors for col(A) : $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

(d) Define $C=[c_1\ c_2\ c_3\ c_4]$. ($c_i$ are $\mathbb{R}^5$ vectors, $i=1.2.3.4$)
We may find the interrelation of A and C that:

$$\left.\begin{array}{l} \begin{array}{l} c_1=-a_1+a_2 \\ c_2=a_1+2a_2 \\ c_3=2a_1+a_2 \\ c_4=a_1+a_2 \end{array} \Rightarrow c_1.c_2.c_3.c_4 \in \text{col}(A) \Rightarrow \text{col}(C)\subseteq \text{col}(A) \\ \\ \begin{array}{l} a_1=c_3-c_4 \\ a_2=-c_3+2c_4 \end{array} \Rightarrow a_1.a_2 \in \text{col}(C) \Rightarrow \text{col}(A)\subseteq \text{col}(C) \end{array}\right\} \Rightarrow \text{col}(C)=\text{col}(A)$$

(e) $B=\begin{bmatrix} -1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

3.
(a) $Z=FY$ for some matrix $F$ which is 'upper triangular' $F=\begin{bmatrix} \ddots & \vdots \\ 0 & \ddots \end{bmatrix}$

take transpose and we may find that $Z^T=Y^TF^T$,
which means column $i$ of $Z^T$ is a linear combination of columns $i,\cdots,n$ of $Y^T$

(b) $W = VF$, where $F = \begin{bmatrix} 0 & 1 & 0 & 0 & & & \cdots \\ 1 & 0 & 0 & 0 & & & \cdots \\ 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 1 & 0 & & & \\ & & & & 0 & 1 & \\ & \vdots & & & 1 & 0 & \\ & \vdots & & & & & \ddots \end{bmatrix}$

(c) $Q^T P = A$, and any element of $A$ is larger than zero

the statement $\Leftrightarrow$ any $p_i \in P$ and any $q_j \in Q$, $q_j^T p_i = a_{ji} > 0$, $i, j = 1, 2, 3, \cdots$

(d) $Q^T P = A$, and any diagonal element of $A$ is larger than zero

the statement $\Leftrightarrow$ any $p_i \in P$ and any $q_i \in Q$, $q_i^T p_i = a_{ii} > 0$, $i = 1, 2, 3, \cdots$

(e) Define $A_1 = [a_1 \ a_2 \cdots a_k]$, $A_2 = [a_{k+1} \ a_{k+2} \ \cdots]$
then $col(A_1) \subseteq null(A_2^T)$

the statement $\Leftrightarrow$ any $a_i \in A_1$ and $a_j \in A_2$, $a_j^T a_i = 0$, $i = 1, 2, \cdots, k$, $j = k+1, k+2 \cdots$

$\Leftrightarrow$ any $a_i \in A_1$, $A_2^T a_i = \begin{bmatrix} 0 \\ \vdots \end{bmatrix}$

$\Leftrightarrow$ $col(A_1) \subseteq null(A_2^T)$

4.

(a) $A \triangleq aa^T \Rightarrow$ each column of $A$ is a linear combination of vector '$a$', and '$a^T$' is the coefficient vector, hence, $A$ is of rank one

(b) $Ax = b$ has a solution $\iff b \in col(A) \iff rank(A) = rank([A\ b])$

5.

(a) According to properties of $P$, we could define that:

$P = Q\Lambda Q^T$, $P^{\frac{1}{2}} = Q\Lambda^{\frac{1}{2}}Q^T$ ($\Lambda$ is a diagonal matrix and $Q$ is an orthogonal matix)

then $(x-x_c)^T P^{-1}(x-x_c) \leq 1$

$\iff (x-x_c)^T(P^{-\frac{1}{2}}P^{-\frac{1}{2}})(x-x_c) \leq 1$

$\iff (x-x_c)^T(P^{-\frac{1}{2}})^T P^{-\frac{1}{2}}(x-x_c) \leq 1$

$\iff \left[P^{-\frac{1}{2}}(x-x_c)\right]^T\left[P^{-\frac{1}{2}}(x-x_c)\right] \leq 1$

$\iff u^T u \leq 1$

$\iff \|u\|^2 \leq 1$, $u = P^{-\frac{1}{2}}(x-x_c)$

$\iff \|u\|^2 \leq 1$, $x = P^{\frac{1}{2}}u + x_c$ $\Rightarrow A = P^{\frac{1}{2}}$, $b = x_c$

(b) let $|P-\lambda I|=0 \Rightarrow \lambda_1=3,\ \lambda_2=5$ . eigenvectors : $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$