SDM366 Optimal Control and Estimation

Lecture Note 3
Least Squares and Basic System Identification

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Outline

- **Least-Squares Problem Formulation**
- Solution to Linear Least-Squares Problems
- Linear Least-Squares Examples
- Applications to System ID
- Nonlinear Least Squares

- Last lecture: obtain discrete-time linear state space model from
  - physical process
  - given continuous time state space model
  - given nonlinear state space model

- The goal of this lecture note:
  - learn how to build model based on observed input-output data pairs
    - General case beyond the scope of this course
    - Focus on special case, where first obtain transfer function model from input-output data pairs, and then obtain the corresponding state space model

  - Main method: **Least Squares**

Least-Squares Problem Formulation:

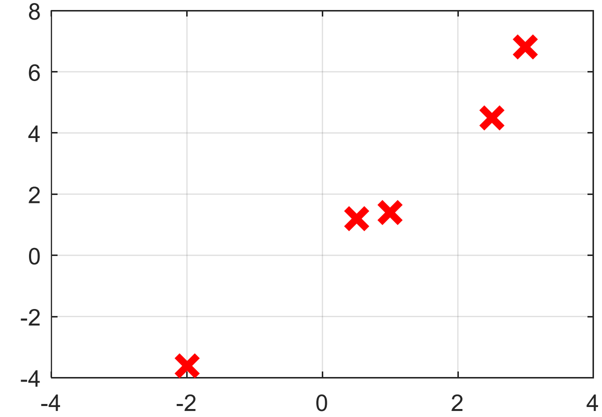
- Measurement Equation:

$$y = g(\theta) + v$$

  - $y \in R^m$: measurements data
  - $\theta \in \Theta \subseteq R^n$: parameter to be estimated, where Θ is the constraint set for feasible parameters
  - $v \in R^m$: unknown measurement noise
  - $g: R^n \to R^m$: known (possibly) nonlinear function relates θ with measurement y

- Least Squares Example:

i	1	2	3	4	5
x	1	0.5	-2	3	2.5
y	1.4	1.2	-3.6	6.8	4.5



Least-Squares Problem Formulation:

- **Problem Statement**: Find the best parameter in the constraint set Θ that minimizes the difference between the model and the measured data

$$\min_{\theta\in\Theta} J(\theta) = \min_{\theta\in\Theta} \left|\left|y - g(\theta)\right|\right|^2$$

- **Linear Least Squares**: $g(\theta) = H\theta$, where $H \in R^{m\times n}$ is a given deterministic matrix

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Optimization of multivariable function

- 1st –order necessary condition for optimality of $J(\theta)$

- Matrix calculus:
  - If $f: \mathrm{R}^n \to R^m$, then $\frac{\partial f}{\partial x}(\mathrm{x}) = Df(x) =$

Optimization of multivariable function

- **Gradient**: For scaler valued multivariate function $f: R^n \rightarrow R$, its gradient is defined as:

- For $f: R^n \rightarrow R$, notational convention $\nabla f(x) = \left(\frac{\partial f}{\partial x}(x)\right)^T$

- Some references use $\frac{\partial f}{\partial x}$ to denote gradient

- Directional derivative: $Df(x; d) = \lim_{\alpha \to 0} \frac{f(x+\alpha d) - f(x)}{h}$

- Some calculus examples:
  - $f(x) = Ax$

  - $f(x) = x^T A\, x$

  - **Exercise**: compute $\frac{\partial f}{\partial x}(x)$, where $x \in R^n$, and $f(x) = x^T x \cdot x$

- Derivation of linear least square solutions
  - Normal equation:

- Solution with full rank H

- Geometric interpretation of linear least squares

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- Linear Least Squares Example:

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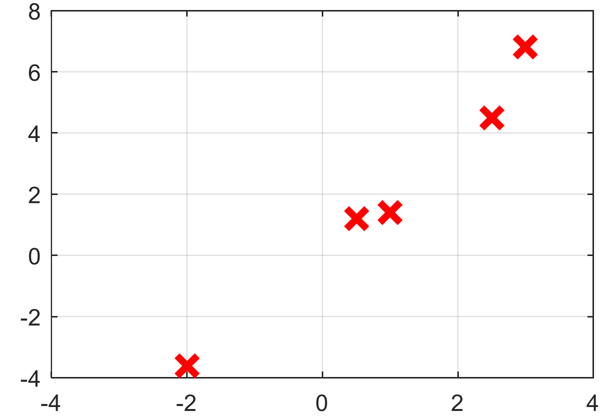
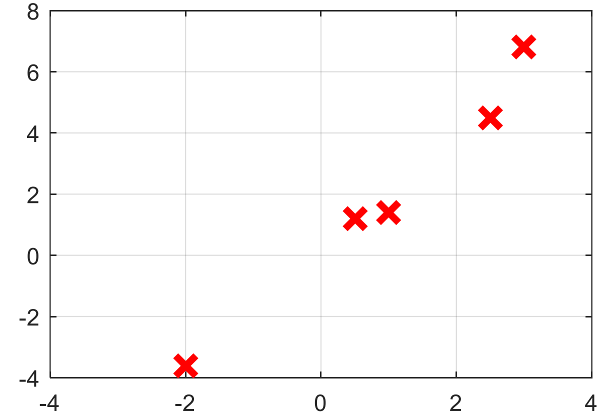
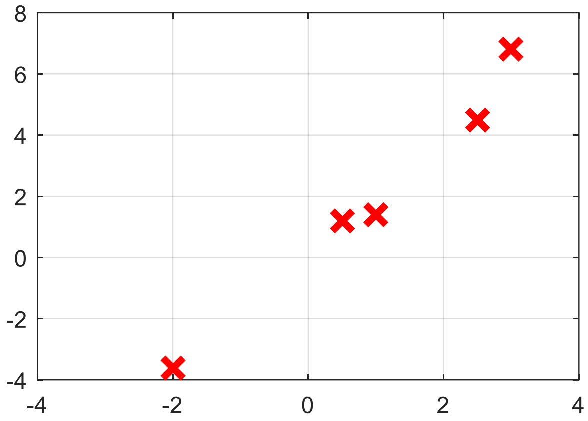


- Assume $y = \alpha x + \beta$

- Change hypothesis, assume $y = be^{ax}$



- Change hypothesis, assume that $y = \alpha_0 + \alpha_1 x + \alpha_2 x^2$



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Application to System Identification for Linear Systems

- ARX(p, q) model :(Autoregressive with exogenous input)

$$y(k) + \alpha_1 y(k-1) + \cdots + \alpha_p y(k-p) \\ = \beta_0 u(k) + \beta_1 u(k-1) + \cdots + \beta_q u(k-q) + v(k)$$

  - $v(k)$: noise signal
  - Model parameter: $\theta = \left[\alpha_1, \dots, \alpha_p, \beta_0, \beta_1, \dots, \beta_q\right]^T$
  - One-step predictor:

$$\hat{y}(k|\theta) = -\alpha_1 y(k-1) - \cdots - \alpha_p y(k-p) \\ +\beta_0 u(k) + \beta_1 u(k-1) + \cdots + \beta_q u(k-q)$$

- System ID problem for ARX model:

  Given data pairs $\{(u(k), y(k))\}_{k\leq N}$, find the parameter vector θ that minimizes cost:

  - $J(\hat{\theta}) = \sum_{k=1}^{N} ||\hat{y}(k|\hat{\theta}) - y(k)||^2$

- Formulate as least square problem:

  given data set, $(u_1, y_1), (u_2, y_2), \dots, (u_m, y_m)$

  - Regressor:

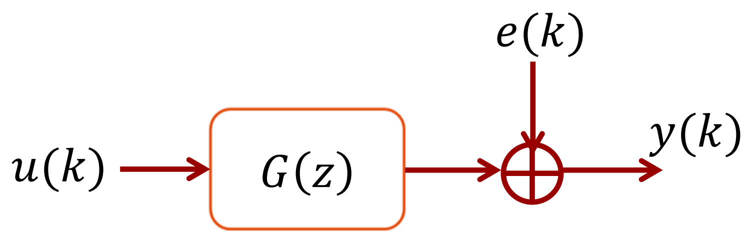
- Derivation continued

- **System ID Example I:**

$G(z) = \frac{(z^2+b)}{z^3+az}$, find best estimate for a, b,

given data set $(u_1, y_1), (u_2, y_2), \ldots, (u_{20}, y_{20})$

▪ System ID Example 2:



- $G(z) = \frac{z-1}{z-a}$, where a is an unknown scalar
- Data: $u(1) = 1, u(2) = \frac{1}{2}, u(3) = 1, y(1) = 2, y(2) = 1, y(3) = 2$

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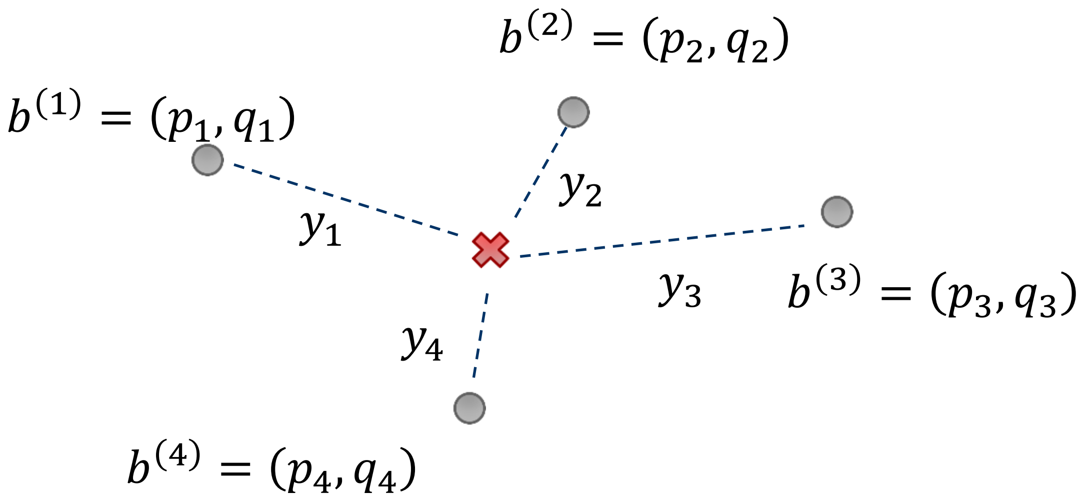
- Nonlinear Least Squares:

$$\min_{\theta\in\Theta} J(\theta) = ||y - g(\theta)||^2$$

  - For general nonlinear function $g(\theta)$, analytical solution to the above optimization is not available
  - Numerical optimization algorithms can be used to find the optimizer

$$\theta^* = argmin_{\theta\in\Theta} J(\theta)$$

- Nonlinear Least Square Example: Navigation by range measurement:



  - : beacons with known positions $b^{(i)} = (p_i, q_i)$
  - : target with unknown position $\theta = (\theta_1, \theta_2)$
  - y_i : known measured distance or range from beacon i:

    typical assumption: $y_i = ||b^{(i)} - \theta|| + v_i$

- Given measurements $y_1, y_2, \ldots, y_m$, find the best target location θ

- We can choose cost function: $J(\theta) = \sum_{i=1}^{m} \left(y_i - \left\| b^{(i)} - \theta \right\| \right)^2$

- Coding Example