SDM366 Optimal Control and Estimation

Lecture Note 2
State Space Model and Stability

~

Prof. Wei Zhang
Southern University of Science and Technology

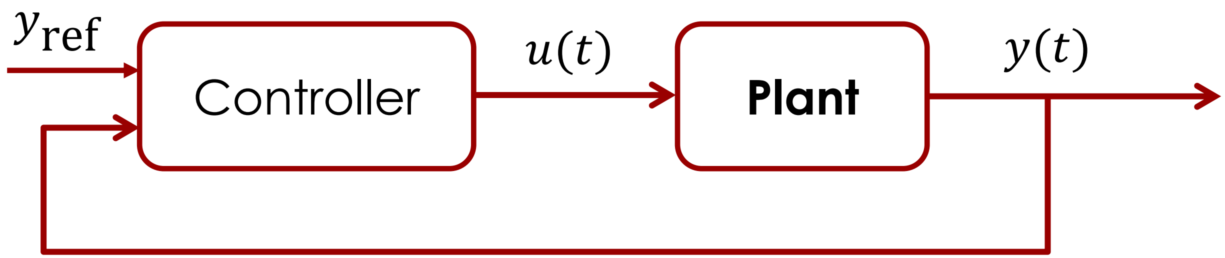
zhangw3@sustech.edu.cn
https://www.wzhanglab.site/

Outline

- **State space model: definition and examples**
- From continuous-time to discrete time model
- From nonlinear to linear model
- System solution and stability

State-space model based feedback control system:

- Goal: determine control input to achieve desired output

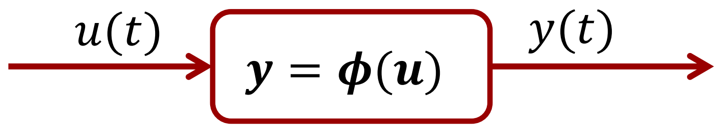


- Controller design is based on plant model
  - Model is different from the actual plant
  - "all models are wrong, but some are useful"

- Modeling approach:
  - First principle
  - Data driven (System ID)

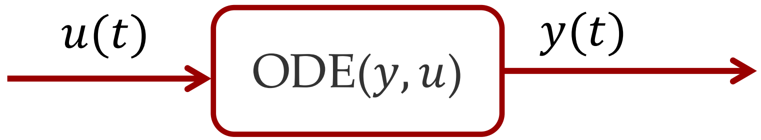
Static vs. Dynamic Systems

- Static system



  - $u(t)$ completely and immediately determines $y(t)$
  - Desired output y_{ref} can be perfectly tracked (in absence of disturbance) by open-loop plant inversion

- **Static vs. Dynamic Systems**
  - **Dynamic system:** differential or difference equation



- $u(t)$ does not fully determines $y(t)$
- At time t_0, the output $y(t_0)$ does not fully captures the system "behavior"

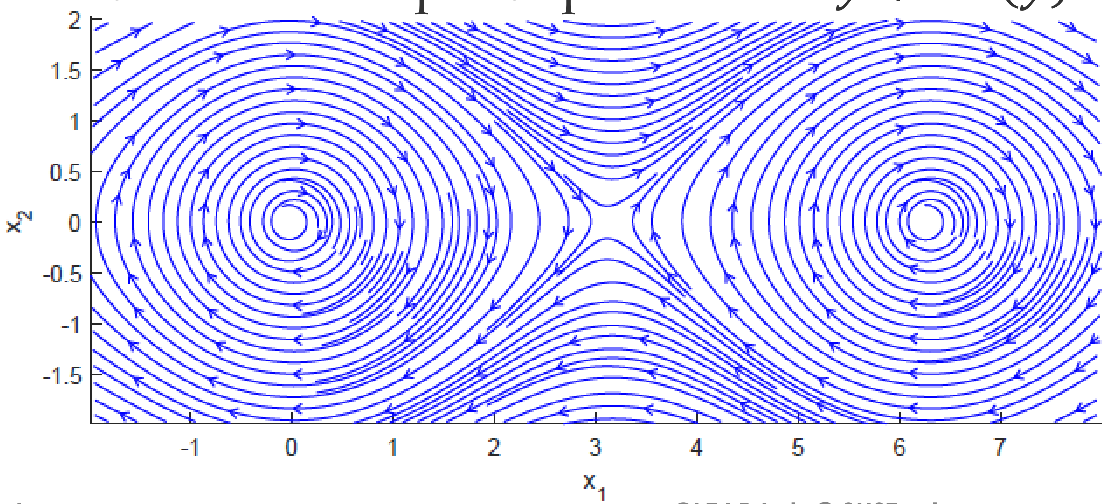
- **"State": info needed for future evolution,** it separates future from past
- **State** $x(t_0)$ at time t_0 and **input** $\boldsymbol{u}(\boldsymbol{t})$ over time $[t_o, t_f]$, **completely determines** the system behaviors

- **General continuous-time state space model**

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

  - $x \in R^n$ state vector, $u \in R^m$ control input, $y \in R^p$ output,
  - $f: R^n \times R^m \to R^n$: called **vector field**
  - $h: R^n \times R^m \to R^p$: output function
  - Called autonomous system if there is no control $f(x, u) = f(x)$
  - For autonomous sys, $\hat{x} \in \boldsymbol{R^n}$ is called **equilibrium** if $f(\hat{x}) = 0$

Vector field example of pendulum: $\ddot{y} + \sin(y) = 0$



- **General discrete-time state space model**

$$x(k+1) = f(x(k), u(k))$$
$$y(k) = h(x(k), u(k))$$

  - $x \in R^n$ state vector, $u \in R^m$ control input, $y \in R^p$ output
  - $f: R^n \times R^m \to R^n$: state update equation
  - $h: R^n \times R^m \to R^p$: output function
  - Called autonomous system if there is no control $f(x, u) = f(x)$
  - For autonomous sys, $\hat{x} \in \boldsymbol{R^n}$ is called **equilibrium** if $\hat{x} = f(\hat{x})$

- Discrete-time system:
  - Some discrete-time system is obtained from continuous time model by sampling
  - Some systems naturally evolve in discrete time.

- **Linear Systems:** system is called linear if:

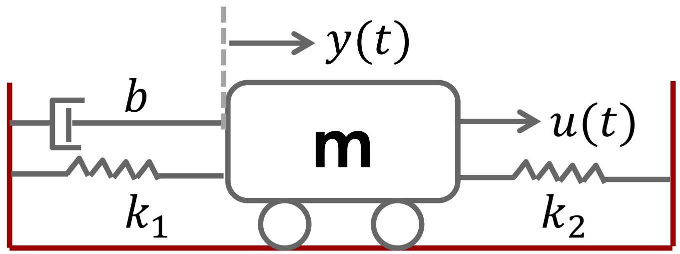
Continuous time
$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u},$$
$$\mathbf{y} = \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u},$$

Discrete time
$$\boldsymbol{x}(\boldsymbol{k}+\mathbf{1}) = \boldsymbol{f}(\boldsymbol{x}(\boldsymbol{k}), \boldsymbol{u}(\boldsymbol{k})) = \boldsymbol{A}\boldsymbol{x}(\boldsymbol{k}) + \boldsymbol{B}\boldsymbol{u}(\boldsymbol{k}),$$
$$\mathbf{y}(\boldsymbol{k}) = \boldsymbol{h}(\boldsymbol{x}(\boldsymbol{k}), \boldsymbol{u}(\boldsymbol{k})) = \boldsymbol{C}\boldsymbol{x}(\boldsymbol{k}) + \boldsymbol{D}\boldsymbol{u}(\boldsymbol{k}),$$

for some matrices A, B, C, D

- **State-space modeling:**
  - Find the functions $f(\cdot,\cdot), h(\cdot,\cdot)$
  - Or find A, B, C, D matrices if the system is linear

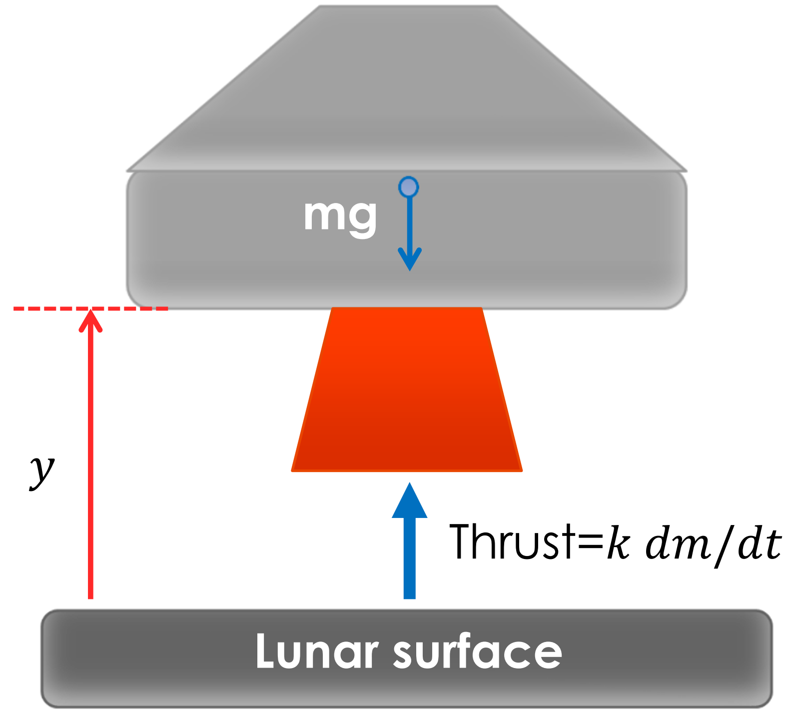
Example 1: Consider spring-damper cart system with zero initial conditions (initially at y = 0 and not moving). No friction



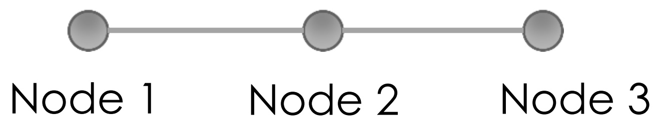
- Differential equation model

- State space model of Example 1 (infinitely many)

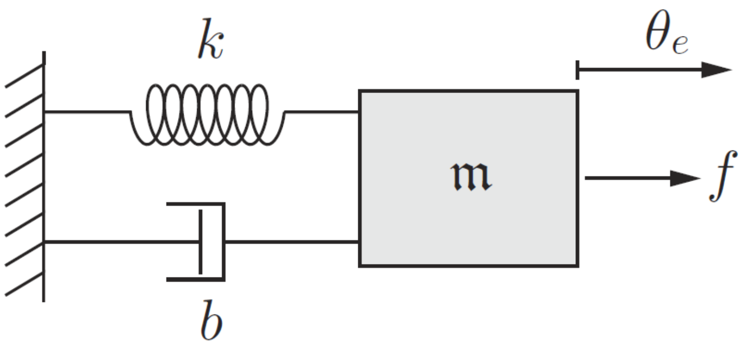
- **Example 2**: soft landing of a lunar module, $u = \frac{dm}{dt}$

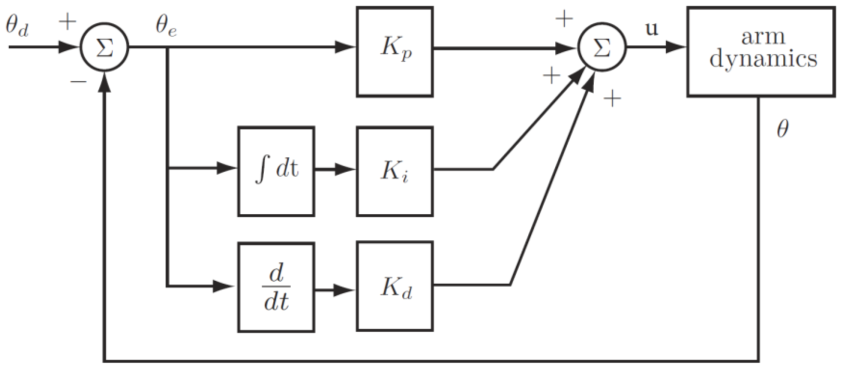


- **Example 3**: Sensor Network
  - Each iteration, exchange measurements with neighbors
  - The updated value is the average of its own value with the neighbors



▪ **Example 4**: PID for spring-damper system

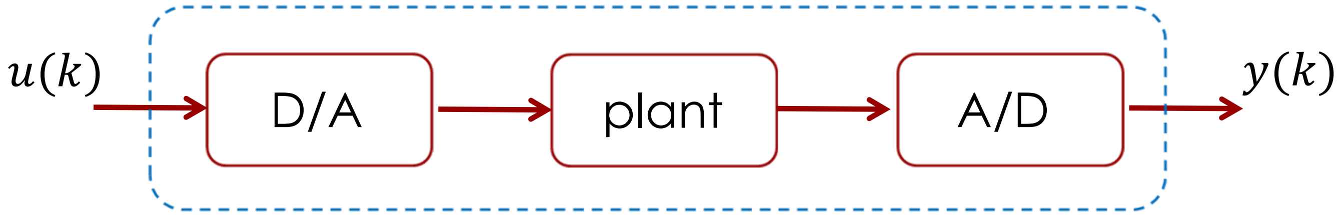




Outline

- State space model: definition and examples
- **From continuous-time to discrete time model**
- From nonlinear to linear model
- System solution and stability

From continuous time to discrete time model



- Approximate differential equation with difference equation
  - Euler forward rule:

From continuous-time to discrete-time model

- General nonlinear case:

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

From continuous-time to discrete-time model

- Linear case:

$$\dot{x} = A_c x + B_c u,$$
$$\text{y} = \; C_c x + D_c u,$$

Outline

- State space model: definition and examples
- From continuous-time to discrete time model
- **From nonlinear to linear model**
- System solution and stability

From nonlinear to linear

- Given model: $x(k+1) = f(x(k), u(k)),\ \ y(k) = h(x(k), u(k))$ and operating point: $(\hat{x}, \hat{u})$
- Goal: find a linearized model around $(\hat{x}, \hat{u})$

- Jacobian matrix of multivariable function $f: R^n \rightarrow R^m$

- Example of Jacobian matrix: $f(z) = \begin{bmatrix} 2z_1 + e^{z_2} \\ \log(z_3) + \frac{1}{z_2} \end{bmatrix}$, $\hat{z} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

- Taylor expansion of multivariate function
  - General expression: $f(z) = f(\hat{z}) + \left(\frac{\partial f}{\partial z}(z) \Big|_{z=\hat{z}} \right) \Delta z$+H.O.T

- Linearization around $(\hat{x}, \hat{u})$ using Taylor expansion:

$$f(x,u) \approx f(\hat{x},\hat{u}) + \overbrace{\left(\left. \frac{\partial f(x,u)}{\partial x} \right|_{x=\hat{x},u=\hat{u}} \right)}^{\hat{A}} \cdot \overbrace{(x-\hat{x})}^{\Delta x} + \overbrace{\left(\left. \frac{\partial f(x,u)}{\partial u} \right|_{x=\hat{x},u=\hat{u}} \right)}^{\hat{B}} \cdot \overbrace{(u-\hat{u})}^{\Delta u}$$

$$= \hat{A} \cdot \Delta x + \hat{B} \cdot \Delta u + f(\hat{x},\hat{u})$$

$$h(x,u) \approx h(\hat{x},\hat{u}) + \underbrace{\left(\left. \frac{\partial h(x,u)}{\partial x} \right|_{x=\hat{x},u=\hat{u}} \right)}_{\hat{C}} \cdot \underbrace{(x-\hat{x})}_{\Delta x} + \underbrace{\left(\left. \frac{\partial h(x,u)}{\partial u} \right|_{x=\hat{x},u=\hat{u}} \right)}_{\widehat{D}} \cdot \underbrace{(u-\hat{u})}_{\Delta u}$$

$$\Delta y \coloneqq y - h(\hat{x},\hat{u}) \approx \hat{C} \cdot \Delta x + \widehat{D} \cdot \Delta u$$

- **Example:** $$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \sin(x_2(k)) + \cos(u_2(k)) \\ x_1(k)x_2(k) + u_1u_2(k) \end{bmatrix}$$

$$y(k) = \cos(x_2(k)) + 2x_1(k) \qquad \hat{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \hat{u} = \begin{bmatrix} 0 \\ \frac{\pi}{2} \end{bmatrix}$$

Outline

- State space model: definition and examples
- From continuous-time to discrete time model
- From nonlinear to linear model
- **System solution and stability**

- General linear state space model:

$$x(k+1) = A(k)x(k) + B(k)u(k),$$
$$y(k) = C(k)x(k) + D(k)u(k)$$

- If system matrices $\big(A(k), B(k), C(k), D(k)\big)$ change over time k, then system is called **Linear Time Varying (LTV)** system

- If system matrices are constant w.r.t. to time, then the system is called a **Linear Time Invariant (LTI)** System

$$x(k+1) = Ax(k) + Bu(k),$$
$$y(k) = Cx(k) + Du(k)$$

- Derivation of Solution to LTI state space system:

$$x(k+1) = Ax(k) + Bu(k),$$
$$y(k) = Cx(k) + Du(k)$$

  - given initial state $x(0) = \hat{x}$, and control sequence $u(0), \dots, u(k)$, $k \geq 0$， we have $\boldsymbol{x(k) = A^k \hat{x} + \sum_{j=0}^{k-1} A^{k-j-1} B u(j)}$

- A large portion of control applications can be transformed into a ***regulation* problem**

  - Regulation problem: keep certain function of the state $x(k)$ or output $y(k)$ close to a known constant reference value under disturbances and model uncertainties

For example:

- Keep inverted pendulum at upright position ($\theta = 0$)
- Maintain a desired attitude of spacecraft or aircraft
- Air conditioner regulate temperate close to setpoint (e.g. 75F)
- Cruise control maintain a constant speed despite uncertain road conditions
- Converter maintains a desired voltage level for different loads

  - If reference $\mathrm{y}_{\mathrm{ref}}(t)$ is changing, this is no longer a regulation problem (becomes a **tracking problem**)

- **Internal Stability** (with $u(k) \equiv 0$, i.e. concerned with zero-input state response)

$$x(k+1) = Ax(k) + Bu(k),$$
$$y(k) = Cx(k) + Du(k)$$

  - Asymptotic stable: $||x(k)|| \to 0$, as $k \to \infty$, for all initial state $\hat{x}$
  - Marginal stable: $||x(k)|| \leq M$, for all $k = 1,2, \ldots$

- Recall state space solution for linear systems:

$$\boldsymbol{x(k) = A^k \hat{x} + \sum_{j=0}^{k-1} A^{k-j-1} B u(j)}$$

- Therefore, or linear system, the key for stability analysis is to understand how A^k behave as $k \to \infty$

- **Case 1:** diagonal matrix: e.g. $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

- **Case 2:** diagonalizable matrix, i.e. $\exists\ T$ such that $A = TDT^{-1}$

- **Case 3:** Unfortunately, ***not all*** square matrices are diagonalizable
  - e.g.: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is **not diagonalizable**

- **Theorem (Internal stability)**: LTI $(\boldsymbol{A}, \boldsymbol{B})$ is **asymptotically stable** if all eigs of A satisfies $|\boldsymbol{\lambda_i}| < \mathbf{1}$

- More discussions