

1. Q 5.12 textbook

Derive dual problem for: $\min_x - \sum_i \log(b_i - a_i^T x), x : a_i^T x \leq b_i, \forall i \in \{1, \dots, m\}$

$$y_i = b_i - a_i^T x$$

$$a_i^T x \leq b_i$$

$$L(x, y, \lambda, v) = - \sum_i \log y_i + \sum_i \lambda_i (a_i^T x - b_i) + \sum_i v_i (y_i + a_i^T x - b_i)$$

$$g(\lambda, v) = \inf_{x, y} L(x, y, \lambda, v)$$

$$g(\lambda, v) = \inf_{x, y} - \sum_i \log y_i + \sum_i \lambda_i (a_i^T x - b_i) + \sum_i v_i (y_i + a_i^T x - b_i)$$

$$(\exists i) \lambda_i \neq 0 \implies \lambda_i (a_i^T x - b_i) \text{ unbounded, so } \lambda = 0$$

$$g(\lambda, v) = \inf_{x, y} - \sum_i \log y_i + \sum_i v_i (y_i + a_i^T x - b_i)$$

$$g(\lambda, v) = \inf_{x, y} - \sum_i \log y_i + v^T y + v^T A x - v^T b$$

$$(\exists i) v_i < 0 \implies v^T y \text{ unbounded, so } v \succeq 0$$

$$\frac{\partial}{\partial y_i} - \sum_i \log y_i + v^T y + v^T A x - v^T b = -\frac{1}{y_i} + v_i = 0$$

$$y_i = \frac{1}{v_i}, v_i \neq 0$$

$$(\forall i) v_i \neq 0 \wedge v_i \geq 0 \implies v \succ 0$$

$$\frac{\partial}{\partial x} - \sum_i \log y_i + v^T y + v^T A x - v^T b = A^T v = 0$$

$$g(\lambda, v) = \begin{cases} - \sum_i \log \frac{1}{v_i} + \sum_i \frac{v_i}{v_i} - v^T b, & \text{if } A^T v = 0 \wedge v \succ 0 \\ -\infty, & \text{otherwise} \end{cases}$$

Dual problem:

$$\begin{aligned} \max_{\lambda, v} \quad & \sum_i \log v_i + m - v^T b = -(\min_{\lambda, v} - \sum_i \log v_i - m + v^T b) \\ \text{s.t.} \quad & A^T v = 0 \\ & -v_i \leq 0, \forall i \end{aligned}$$

2. Q 5.27 Equality constrained least squares

Give KKT conditions, derive expressions for primal and dual solutions.

$$\begin{aligned} \min_x & \|Ax - b\|_2^2 \\ \text{s.t. } & Gx = h \end{aligned}$$

$$f_0 = x^T A^T A x + 2b^T A x + b^T b$$

$$h_0 = Gx - h$$

$$L(x, \lambda, v) = f_0 + v^T h_0$$

$$L(x, \lambda, v) = x^T A^T A x - 2b^T A x + b^T b + v^T (Gx - h)$$

$$\frac{\partial L}{\partial x^*} = 0 = 2A^T A x^* - 2A^T b + G^T v$$

KKT conditions :

$$x^* = \frac{1}{2}(A^T A)^{-1}(2A^T b - G^T v)$$

$$Gx^* - h = 0$$

$$\begin{aligned} g(\lambda, v) &= \min_x L(x, \lambda, v) = \frac{1}{4}(2A^T b - G^T v)^T (A^T A)^{-1} (2A^T b - G^T v) \\ &\quad - 2b^T A \left(\frac{1}{2} (A^T A)^{-1} (2A^T b - G^T v) \right) + b^T b + v^T \left(G \frac{1}{2} (A^T A)^{-1} (2A^T b - G^T v) - h \right) \\ g(\lambda, v) &= \min_x L(x, \lambda, v) = \frac{1}{4}(2A^T b)^T (A^T A)^{-1} (2A^T b) - (A^T b)^T (A^T A)^{-1} (G^T v) \\ &\quad + \frac{1}{4}(G^T v)^T (A^T A)^{-1} (G^T v) + b^T A ((A^T A)^{-1} G^T v) - h^T v + v^T G \frac{1}{2} (A^T A)^{-1} (2A^T b - G^T v) \\ &\quad - b^T A (A^T A)^{-1} 2A^T b + b^T b \end{aligned}$$

rid of constants and simplify :

$$\begin{aligned} g(\lambda, v) &= \min_x L(x, \lambda, v) = -\frac{1}{4}(G^T v - 2A^T b)^T (A^T A)^{-1} (G^T v - 2A^T b) \\ &\quad - \frac{1}{2}(G^T v)^T (A^T A)^{-1} (G^T v) - h^T v \end{aligned}$$

Dual problem:

$$\begin{aligned} \max_{\lambda, v} g(\lambda, v) &= \max_v -\frac{1}{4}(G^T v - 2A^T b)^T (A^T A)^{-1} (G^T v - 2A^T b) \\ &\quad - \frac{1}{2}(G^T v)^T (A^T A)^{-1} (G^T v) - h^T v \\ \text{s.t. } & Gx^* - h = 0 \end{aligned}$$

Solve for v^* :

$$Gx^* - h = 0$$

$$x^* = \frac{1}{2}(A^T A)^{-1}(2A^T b - G^T v^*)$$

$$G\frac{1}{2}(A^T A)^{-1}(2A^T b - G^T v^*) - h = 0$$

$$v^* = 2G^{-T}(A^T b - A^T A G^{-1} h)$$

3. Q 5.35 Sensitivity analysis of GP

After log tranformation:

$$\begin{aligned} \min_y & f_0(y) \\ \text{s.t. } & \log f_i(y) \leq 0 \\ & \log h_i(y) = 0 \end{aligned}$$

Perturbed problem:

$$\begin{aligned} \min_y & f_0(y) \\ \text{s.t. } & \log f_i(y) \leq u_i \\ & \log h_i(y) = v_i \end{aligned}$$

where $y_i = \log(x_i)$

$L := \text{Lagrangian of original problem}$

$L' := \text{Lagrangian of perturbed problem}$

$$\begin{aligned} L'(y, \lambda, w) &= \log f_0(y) + \sum_i \lambda_i (\log f_i(y) - u_i) + \sum_i w_i (\log h_i(y) - v_i) \\ L'(y, \lambda, w) &= L(y, \lambda, w) - \sum_i \lambda_i u_i - \sum_i w_i v_i \\ p(u, v) &= \min_y L'(y, \lambda, w) = \min_y L(y, \lambda, w) - \lambda^T u - w^T v \end{aligned}$$

Dual:

$$\begin{aligned} \max_{\lambda, w} p(u, v) &= p^*(u, v) = \max_{\lambda, w} \min_y L(y, \lambda, w) - \lambda^T u - w^T v \\ \text{let } p(0, 0) &= \max_{\lambda, w} \min_y L(y, \lambda, w) \\ p^*(u, v) &= p(0, 0) + (\max_{\lambda, w} -\lambda^T u - w^T v) \end{aligned}$$

let λ^*, w^* be parameters for optimal $p(u, v)$

$$\begin{aligned}
 p^*(u, v) &= p(0, 0) - \lambda^{*T}u - w^{*T}v \\
 u \text{ not present in original problem} &\implies \\
 \frac{\partial}{\partial u_i} p(0, 0) &= 0, \forall i \\
 v \text{ not present in original problem} &\implies \\
 \frac{\partial}{\partial v_i} p(0, 0) &= 0, \forall i \\
 \frac{\partial}{\partial u_i} p^*(u, v) &= -\lambda_i^*, \forall i \\
 \frac{\partial}{\partial v_i} p^*(u, v) &= -w_i^*, \forall i
 \end{aligned}$$

Relaxation of i th constraint by α percent:

$$\partial p^*(u_i, 0) = -\lambda_i^* \partial u_i$$

$\partial u_i = \alpha \implies$ objective function of log transformed problem experiences a decrease in value by $\lambda_i^* \alpha$

Converting back to the original variable through identity $x = e^y$

z small $\implies e^z \approx 1 + z$ via Taylor expansion

$e^{-\lambda_i^* \alpha} \approx 1 - \lambda_i^* \alpha \implies$ objective function experiences an improvement of $\lambda_i^* \alpha$ percent since it is a minimization problem.

4. Q 5.42

5. Strong Duality for LP:

Primal:

$$\begin{aligned} \min_x \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Find the dual of the primal and argue that

- a) if the primal is unbounded then the dual is infeasible

$$L(x, \lambda, v) = c^T x + \lambda_1^T (b - Ax) + \lambda_2^T (-x)$$

$$L(x, \lambda, v) = (c^T - \lambda_1^T A - \lambda_2^T)x + \lambda_1^T b$$

$$g(\lambda, v) = \min_x L(x, \lambda, v) = \begin{cases} b^T \lambda_1, & c - A^T \lambda_1 - \lambda_2 = 0 \\ -\infty, & \text{o/w} \end{cases}$$

Dual :

$$\max_{\lambda_1, \lambda_2} b^T \lambda_1$$

$$\text{s.t. } c - A^T \lambda_1 - \lambda_2 = 0$$

$$g(\lambda, v) \leq c^T x^* \text{ (weak duality)}$$

$$\text{primal unbounded : } c^T x^* = -\infty \implies$$

$$g(\lambda, v) = -\infty \implies c - A^T \lambda_1 - \lambda_2 \neq 0 \implies \text{infeasible case of dual}$$

- b) if the primal is infeasible then the dual is either infeasible or unbounded

primal infeasible $\implies b - Ax \not\leq 0 \vee -x \not\leq 0$

$$\min_x c^T x + \lambda_1^T (b - Ax) + \lambda_2^T (-x) \leq c^T x$$

dual unbounded case :

suppose $c - A^T \lambda_1 - \lambda_2 = 0$, substitute for λ_1 :

$$\text{Dual : } \max_{\lambda_1, \lambda_2} b^T \lambda_1 = \max_{\lambda_1, \lambda_2} b^T A^{-T} c - b^T A^{-T} \lambda_2, \text{ such that :}$$

$$(\forall i)(b - Ax)_i > 0 \implies \lambda_{1_i} \text{ may be negative, and}$$

$$(\forall i)(-x)_i > 0 \implies \lambda_{2_i} \text{ may be negative, and}$$

$$c - A^T \lambda_1 - \lambda_2 = 0$$

$$\text{let } d = b^T A^{-T} c$$

$$\text{let } e^T = b^T A^{-T}$$

$$\text{Dual : } \max_{\lambda_1, \lambda_2} d - e^T \lambda_2$$

$(\exists i)e_i > 0$: select $\lambda_{2_i} \rightarrow -a$ and $(\exists i)e_i < 0$: select $\lambda_{2_i} \rightarrow a, a \rightarrow +\infty$ such that it satisfies :

$c - A^T \lambda_1 - \lambda_2 = 0$, then $e^T \lambda_2$ is unbounded

$$\text{eg : } c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, A = I, Ax \not\leq b, x \not\leq 0$$

$$\text{Dual : } \max_{\lambda_1, \lambda_2} d - I^T \begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \lambda_2, \text{ select } \lambda_2 = \begin{bmatrix} -a \\ a \end{bmatrix}$$

$$c - A^T \lambda_1 - \lambda_2 = 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \lambda_1 - \begin{bmatrix} -a \\ a \end{bmatrix} = 0$$

$$\text{select : } \lambda_1 = \begin{bmatrix} a \\ -a \end{bmatrix}$$

$$\text{let } a \rightarrow +\infty \implies \max_{\lambda_1, \lambda_2} d - I^T \begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \lambda_2 \Big|_{\lambda_2 = [-a, a]^T} = d + 2a = +\infty \implies \text{dual unbounded}$$

dual infeasible case :

suppose : $c > 0, x \not\leq 0, A = -1, Ax \geq b$

$$Ax \geq b \implies \lambda_1 \geq 0$$

$$x < 0 \implies \lambda_2 \leq 0$$

$$c - A^T \lambda_1 - \lambda_2 = c + \lambda_1 - \lambda_2$$

$$c + \lambda_1 > 0 \wedge -\lambda_2 \geq 0 \implies c - A^T \lambda_1 - \lambda_2 \neq 0 \implies \text{dual is not feasible}$$

6. Game Theory

$$\begin{aligned} & \min_x \max_y x^T P y \\ \text{s.t. } & Ax \leq b \\ & Cy \leq d \end{aligned}$$

$$\begin{aligned} & \max_y \min_x x^T P y \\ \text{s.t. } & Ax \leq b \\ & Cy \leq d \end{aligned}$$

- show that min-max problem has the same optimal value as the following minimization problem:

$$\begin{aligned} & \min_{\lambda, x} d^T \lambda \\ \text{s.t. } & C^T \lambda = P^T x \\ & Ax \leq b \\ & \lambda \geq 0 \end{aligned}$$

Inner optimization problem:

$$\begin{aligned} & \max_y (P^T x)^T y = -(\min_y -(P^T x)^T y) \\ \text{s.t. } & Ax \leq b \\ & Cy \leq d \end{aligned}$$

$$g(\lambda_1, \lambda_2) = \min_y -(P^T x)^T y = \begin{cases} \lambda_1^T (Ax - b) - \lambda_2^T d, & -P^T x + C^T \lambda_2 = 0 \\ -\infty, & \text{o/w} \end{cases}$$

$$\max_{\lambda_1, \lambda_2} g(\lambda_1, \lambda_2) = \max_{\lambda_1, \lambda_2} -\lambda_2^T d + \lambda_1^T (Ax - b)$$

$$\text{s.t. } -P^T x + C^T \lambda_2 = 0$$

$$\lambda_1, \lambda_2 \geq 0$$

$$\lambda_1^T (Ax - b) \leq 0 \implies$$

$$\max_{\lambda_1, \lambda_2} g(\lambda_1, \lambda_2) = \max_{\lambda_2} -\lambda_2^T d$$

$$-(\min_y -(P^T x)^T y) = -\max_{\lambda_1, \lambda_2} g(\lambda_1, \lambda_2) = -(-\min_{\lambda_2} \lambda_2^T d) = \min_{\lambda_2} \lambda_2^T d$$

rename λ_2 to λ and enclose with outer minimization :

$$\min_{x, \lambda} \lambda^T d$$

$$\text{s.t. } P^T x = C^T \lambda$$

$$Ax \leq b$$

$$\lambda \geq 0$$

- show that max-min problem has the same optimal value as the following maximization problem:

$$\begin{aligned}
 & \max_{y,v} -b^T v \\
 & s.t. \quad A^T v + P y = 0 \\
 & \quad \quad C y \leq d \\
 & \quad \quad v \geq 0
 \end{aligned}$$

Inner optimization problem:

$$\begin{aligned}
 & \min_x x^T P y \\
 & s.t. \quad A x \leq b \\
 & \quad \quad C y \leq d \\
 & g(\lambda_1, \lambda_2) = \min_x x^T P y + \lambda_1^T (A x - b) + \lambda_2^T (C y - d) \\
 & \quad = \begin{cases} -\lambda_1^T b + \lambda_2^T (C y - d), & P y + A^T \lambda_1 = 0 \\ -\infty, & \text{o/w} \end{cases} \\
 & \max_{\lambda_1, \lambda_2} g(\lambda_1, \lambda_2) = \max_{\lambda_1, \lambda_2} -\lambda_1^T b + \lambda_2^T (C y - d) \\
 & s.t. \quad P^T y + A^T \lambda_1 = 0 \\
 & \quad \quad \lambda_1 \geq 0 \\
 & \quad \quad \lambda_2 \geq 0 \\
 & \quad \quad A x - b \leq 0 \\
 & \quad \quad C y - d \leq 0 \\
 & \quad \quad \lambda_2^T (C y - d) \leq 0 \implies \\
 & \max_{\lambda_1, \lambda_2} g(\lambda_1, \lambda_2) = \max_{\lambda_1, \lambda_2} -\lambda_1^T b \\
 & s.t. \quad P^T y + A^T \lambda_1 = 0 \\
 & \quad \quad \lambda_1 \geq 0 \\
 & \quad \quad C y - d \leq 0 \\
 & \quad \text{rename variables and enclose with outer optimization :} \\
 & \max_{y,v} -b^T v \\
 & s.t. \quad P^T y + A^T v = 0 \\
 & \quad \quad C y \leq d \\
 & \quad \quad v \geq 0
 \end{aligned}$$

- show the above problems have the same optimal value (min-max equals max-min)

$$\begin{aligned}
 & \min_{x, \lambda} d^T \lambda \\
 & s.t. \quad C^T \lambda = P^T x \\
 & \quad Ax \leq b \\
 & \quad \lambda \geq 0
 \end{aligned}$$

$$\begin{aligned}
 g(a_1, a_2, w) &= \min_{x, \lambda} d^T \lambda + a_1^T (Ax - b) + a_2^T (-\lambda) + w^T (C^T \lambda - P^T x) \\
 & s.t. \quad a_1 \geq 0 \\
 & \quad a_2 \geq 0 \\
 & \quad Ax - b \leq 0 \\
 & \quad -\lambda \leq 0 \\
 & \quad C^T \lambda - P^T x = 0 \\
 g(a_1, a_2, w) &= \begin{cases} -a_1^T b, & d - a_2 + Cw = 0 \wedge A^T a_1 - Pw = 0 \\ -\infty, & \text{o/w} \end{cases}
 \end{aligned}$$

dual :

$$\begin{aligned}
 & \max_{a_1, a_2, w} g(a_1, a_2, w) = -b^T a_1 \\
 & s.t. \quad d - a_2 + Cw = 0 \\
 & \quad A^T a_1 - Pw = 0 \\
 & \quad a_2 \geq 0 \implies d + Cw \geq 0 \\
 & \quad a_1 \geq 0
 \end{aligned}$$

$$\text{let } v = a_1, y = -w$$

$$\begin{aligned}
 & \max_{v, y} g(v, y) = -b^T v \\
 & s.t. \quad A^T v + Py = 0 \\
 & \quad Cy \leq d \\
 & \quad v \geq 0
 \end{aligned}$$

We start with equivalent formulation of min-max and arrive at a equivalent formulation of max-min problem thus they obtain the same optimal value.