

1. Q 4.11(a,c,e) textbook
 (a) Minimize $\|Ax - b\|_\infty$

$$\begin{aligned} & \min \max_i |(Ax - b)_i| \\ \text{let } t = & \max_i |(Ax - b)_i| \\ & \min_{x,t} t, \text{ subject to :} \\ & |(Ax - b)_i| \leq t, \forall i \end{aligned}$$

LP Formulation:

$$\begin{aligned} & x \in R^n, t \in R \\ & \min_{x,t} t, \text{ subject to :} \\ & A_{i,:}x - t \leq b_i, \forall i \\ & -A_{i,:}x - t \leq -b_i, \forall i \end{aligned}$$

- (c) Minimize $\|Ax - b\|_1, \|x\|_\infty \leq 1$

$$\begin{aligned} & \min_x \sum_i |(Ax - b)_i|, \text{ subject to :} \\ & |x_i| \leq 1, \forall i \\ & \text{let } t_i = |(Ax - b)_i| \\ & \min_{x,t} \sum_i t_i, \text{ subject to :} \\ & |(Ax - b)_i| \leq t_i, \forall i \\ & |x_i| \leq 1, \forall i \end{aligned}$$

LP Formulation:

$$\begin{aligned} & x \in R^n, t \in R^n \\ & \min_{x,t} 1^T t, \text{ subject to :} \\ & A_{i,:}x - t_i \leq b_i \\ & -A_{i,:}x - t_i \leq -b_i \\ & x_i \leq 1, \forall i \\ & -x_i \leq 1, \forall i \end{aligned}$$

(e) Minimize $\|Ax - b\|_1 + \|x\|_\infty$

$$\begin{aligned} & \text{let } t_i = |Ax - b|_i \\ & \text{let } v = \max |x_i| \\ \min_{t,v,x} & \sum_i t_i + v, \text{ subject to :} \\ & |x_i| \leq v, \forall i \\ & |Ax - b|_i \leq t_i, \forall i \end{aligned}$$

LP formulation:

$$\begin{aligned} & x, t \in \mathbb{R}^n, v \in \mathbb{R} \\ \min_{t,v,x} & 1^T t + v, \text{ subject to :} \\ & x_i - v \leq 0, \forall i \\ & -x_i - v \leq 0, \forall i \\ & A_{i,:}x - t_i \leq +b_i, \forall i \\ & -A_{i,:}x - t_i \leq -b_i, \forall i \end{aligned}$$

2. Q 4.16 textbook

$$\begin{aligned} & x(t) \in \mathbb{R}^n, t \in \{0, \dots, N\} \\ & u(t) \in \mathbb{R}, t \in \{0, \dots, N\} \\ & x(t+1) = Ax(t) + bu(t), t \in \{0, \dots, N\} \\ & \text{given :} \\ & A \in \mathbb{R}^{n \times n} \\ & b \in \mathbb{R}^n \\ & x(0) = 0 \\ & \text{problem :} \end{aligned}$$

$$\min_u \sum_{t=0}^{N-1} f(u(t)), \text{ subject to :}$$

$$X(N) = x_{des}$$

$$f(a) = \begin{cases} |a|, & |a| \leq 1 \\ 2|a| - 1, & |a| > 1 \end{cases}$$

expanding $x(t)$:

$$x(0) = 0$$

$$x(1) = Ax(0) + bu(0) = bu(0)$$

$$x(2) = Ax(1) + bu(1) = A(bu(0)) + bu(1)$$

$$x(3) = Ax(2) + bu(2) = A(A(bu(0)) + bu(1)) + bu(2)$$

$$x(i) = A^{i-1}bu(0) + A^{i-2}bu(1) + \dots + A^0bu(i-1) = \sum_{j=0}^{i-1} A^jbu(i-1-j)$$

$$x(i) = \begin{bmatrix} A^{i-1}b & A^{i-2}b & \dots & b \end{bmatrix} \begin{bmatrix} u(0) \\ \vdots \\ u(i-1) \end{bmatrix}$$

$$x_{des} = x(N) = \begin{bmatrix} A^{N-1}b & A^{N-2}b & \dots & b \end{bmatrix} \begin{bmatrix} u(0) \\ \vdots \\ u(N-1) \end{bmatrix}$$

$$\text{let } v_t = |f(x(t))|$$

$$v_t = \max\{|u(t)|, 2|u(t)|-1\}$$

$$v_t \geq |u(t)|$$

$$v_t \geq 2|u(t)|-1$$

$$v_t \geq u_t$$

$$v_t \geq -u_t$$

$$v_t \geq 2u_t - 1$$

$$v_t \geq -2u_t - 1$$

LP formulation:

$$v_{0,\dots,N-1}, u_{0,\dots,N-1} \in \mathbb{R}^N$$

$$\min_{v,u} 1^T v, \text{ subject to :}$$

$$-v_t + u_t \leq 0, \forall t \in 0, \dots, N-1$$

$$-v_t - u_t \leq 0, \forall t \in 0, \dots, N-1$$

$$-v_t + 2u_t \leq 1, \forall t \in 0, \dots, N-1$$

$$-v_t - 2u_t \leq 1, \forall t \in 0, \dots, N-1$$

$$\begin{bmatrix} A^{N-1}b & A^{N-2}b & \dots & b \end{bmatrix} \begin{bmatrix} u_0 \\ \vdots \\ u_{N-1} \end{bmatrix} = x_{des}$$

3. Q 4.21(a) textbook

Find explicit solution for the QCQP:

minimize $c^T x$, subject to:

$$x^T A x \leq 1$$

$$A \in S_{++}^n, c \neq 0$$

$$x^T A x = x^T A^{1/2} A^{1/2} x = (A^{1/2} x)^T A^{1/2} x = \|A^{1/2} x\|_2^2$$

$$\|A^{1/2} x\|_2^2 \leq 1$$

$$\text{let } y = A^{1/2} x$$

$$x = A^{-1/2} y$$

$$c^T x = c^T A^{-1/2} y$$

$$\text{let } b^T = c^T A^{-1/2}$$

min $b^T y$ subject to :

$$\|y\|_2^2 \leq 1$$

$$\min b^T y = \frac{-b^T b \alpha}{\|b\|}, \alpha = \max \|y\|_2 = 1$$

$$y^* = \frac{-b}{\|b\|}$$

$$x^* = A^{-1/2} y^* = A^{-1/2} \frac{-b}{\|b\|}$$

$$= A^{-1/2} \frac{-(c^T A^{-1/2})^T}{\|(c^T A^{-1/2})^T\|}$$

$$= \frac{-A^{-1/2} A^{-1/2} c}{\|A^{-1/2} c\|} = \frac{-A^{-1} c}{(c^T A^{-1} c)^{1/2}}$$

$$c^T x^* = \frac{-c^T A^{-1} c}{(c^T A^{-1} c)^{1/2}}$$

$$c^T x^* = -(c^T A^{-1} c)^{1/2}$$

4. Q 4.25 textbook

$$\varepsilon_i = \{P_i u + q_i : \|u\|_2 \leq 1\}, i = 1, \dots, K + L, P_i \in S^n$$

Find a feasible hyperplane strictly separating $\varepsilon_1, \dots, \varepsilon_K$ from $\varepsilon_{K+1}, \dots, \varepsilon_{K+L}$.

$$a^T x + b > 0, x \in \bigcup_{i=1}^K \varepsilon_i$$

$$a^T x + b < 0, x \in \bigcup_{i=K+1}^{K+L} \varepsilon_i$$

let $\epsilon > 0$, a constant for strict separation

relax inequalities to:

$$a^T x + b \leq -\epsilon, x \in \bigcup_{i=1}^K \varepsilon_i$$

$$a^T x + b \geq \epsilon, x \in \bigcup_{i=K+1}^{K+L} \varepsilon_i$$

$$a^T(P_i u + q_i) + b \leq -\epsilon, \|u\|_2 \leq 1, i \in \{1, \dots, K\} \text{ becomes}$$

$$\sup_{\|u\|_2 \leq 1} a^T P_i u + a^T q_i + b \leq -\epsilon, i \in \{1, \dots, K\}$$

$$\sup_{\|u\|_2 \leq 1} a^T P_i u = \frac{a^T P_i (a^T P_i)^T}{\|a^T P_i\|_2} = \|a^T P_i\|_2$$

$$\|a^T P_i\|_2 + a^T q_i + b \leq -\epsilon, i \in \{1, \dots, K\}$$

$$\|a^T P_i\|_2 \leq -a^T q_i - b - \epsilon, i \in \{1, \dots, K\}$$

$$a^T(P_i u + q_i) + b \geq \epsilon, \|u\|_2 \leq 1, i = \{K + 1, \dots, K + L\} \text{ becomes}$$

$$\inf_{\|u\|_2 \leq 1} a^T P_i u + a^T q_i + b \geq \epsilon, i = \{K + 1, \dots, K + L\}$$

$$\inf_{\|u\|_2 \leq 1} a^T P_i u = \frac{a^T P_i (-a^T P_i)^T}{\|a^T P_i\|_2} = -\|a^T P_i\|_2$$

$$-\|a^T P_i\|_2 + a^T q_i + b \geq \epsilon, i = \{K + 1, \dots, K + L\}$$

$$\|a^T P_i\|_2 \leq a^T q_i + b - \epsilon, i = \{K + 1, \dots, K + L\}$$

Second Order Cone Programming formulation:

$$\begin{aligned}
 & \min_{a,b} 0 \\
 & \|a^T P_i\|_2 \leq -a^T q_i - b - \epsilon, i \in \{1, \dots, K\} \\
 & \|a^T P_i\|_2 \leq a^T q_i + b - \epsilon, i \in \{K+1, \dots, K+L\} \\
 & \text{where } \epsilon > 0
 \end{aligned}$$

5. Q 4.30 textbook

Express as Geometric Programming:

$$\begin{aligned}
 & T \in [T_{\min}, T_{\max}], T_{\min}, T_{\max} > 0 \\
 & r \in [r_{\min}, r_{\max}], r_{\min}, r_{\max} > 0 \\
 & w \in [w_{\min}, w_{\max}], w_{\min}, w_{\max} > 0 \\
 & w \leq 0.1r \\
 & T > 0 \\
 & r > 0 \\
 & w > 0 \\
 & \max \alpha_4 T r^2 \\
 & \alpha_1 \frac{Tr}{w} + \alpha_2 r + \alpha_3 r w \leq c_{\max} \\
 & \alpha_i > 0, \forall i
 \end{aligned}$$

$$\begin{aligned}
 \max \alpha_4 T r^2 &= \min \frac{1}{\alpha_4 T r^2} = \min \frac{1}{\alpha_4} T^{-1} r^{-2} \\
 & \frac{\alpha_1}{c_{\max}} T r w^{-1} + \frac{\alpha_2}{c_{\max}} r + \frac{\alpha_3}{c_{\max}} r w \leq 1 \\
 T \leq T_{\max} &\iff \frac{T}{T_{\max}} \leq 1 \\
 T \geq T_{\min} &\iff \frac{T_{\min}}{T} \leq 1 \\
 r \leq r_{\max} &\iff \frac{r}{r_{\max}} \leq 1 \\
 r \geq r_{\min} &\iff \frac{r_{\min}}{r} \leq 1 \\
 w \leq w_{\max} &\iff \frac{w}{w_{\max}} \leq 1 \\
 w \geq w_{\min} &\iff \frac{w_{\min}}{w} \leq 1 \\
 w \leq 0.1r &\iff \frac{10w}{r} \leq 1
 \end{aligned}$$

GP formulation:

$$\begin{aligned}
 \min_{T,r,w} \quad & \frac{1}{\alpha_4} T^{-1} r^{-2}, \text{ subject to :} \\
 & \frac{T}{T_{max}} \leq 1, \frac{T_{min}}{T} \leq 1 \\
 & \frac{r}{r_{max}} \leq 1, \frac{r_{min}}{r} \leq 1 \\
 & \frac{w}{w_{max}} \leq 1, \frac{w_{min}}{w} \leq 1 \\
 & \frac{10w}{r} \leq 1 \\
 & \frac{\alpha_1}{c_{max}} T r w^{-1} + \frac{\alpha_2}{c_{max}} r + \frac{\alpha_3}{c_{max}} r w \leq 1 \\
 & \text{given :} \\
 & \alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0 \\
 & T_{min}, T_{max}, r_{min}, r_{max}, w_{min}, w_{max} > 0
 \end{aligned}$$

6. Q 4.43(a-b) textbook

$$A : \mathbb{R}^n \rightarrow \mathbb{S}^m, A(x) = A_0 + x_1 A_1 + \dots + x_n A_n$$

Let $\lambda_1(x) \geq \dots \geq \lambda_m(x)$ denote the eigenvalues of $A(x)$.

Formulate problems as SDP.

- (a) Minimize the maximum eigenvalue
SDP formulation:

$$\begin{aligned}
 \min_{a,x} \quad & a, \text{ subject to :} \\
 & A(x) \leq_{\mathbb{S}^m_+} aI
 \end{aligned}$$

- (b) Minimize the spread of the eigenvalues

$$\begin{aligned}
 & \text{let } t = \lambda_1(x) - \lambda_m(x) \\
 \min_{t,x,a,b} \quad & t, \text{ subject to :} \\
 & A(x) \leq_{\mathbb{S}^m_+} bI \\
 & A(x) \geq_{\mathbb{S}^m_+} aI \\
 & t - b + a = 0
 \end{aligned}$$

7. reserved

8. reserved

9. Optimal Control of a Unit Mass

- a) given:

$$x(0) = 0$$

$$\dot{x}(0) = 0$$

$$f(t) = p_i, i - 1 < t \leq i, i = 1, \dots, 10$$

$$x(10) = 1$$

$$\dot{x}(10) = 0$$

$$\text{minimize } \sum_{i=1}^{10} p_i^2$$

$$\text{let } v = \dot{x}$$

$$v(i) = v(0) + \sum_{j=0}^i \frac{f(j)}{m} = v(0) + \sum_{j=1}^i p_j, m = 1$$

$$v(i) = \sum_{j=1}^i p_j = 1^T p, p \in \mathbb{R}^i$$

$$v(1) = p_1$$

$$v(2) = p_1 + p_2$$

...

$$v(10) = 1^T p = v_{des} = 0,$$

$$x(i) = x(0) + \sum_{j=0}^i v(j)$$

$$x(i) = \sum_{j=1}^i v(j) = 1^T v, v \in \mathbb{R}^i$$

$$x(i) = \sum_{j=1}^i \sum_{k=1}^j p_k$$

$$x(i) = \sum_{j=1}^i (i - j + 1) p_j$$

$$x(1) = p_1$$

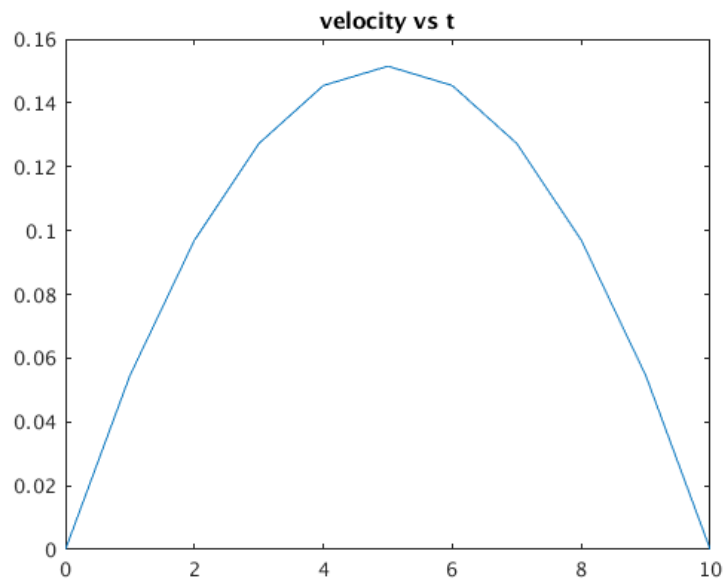
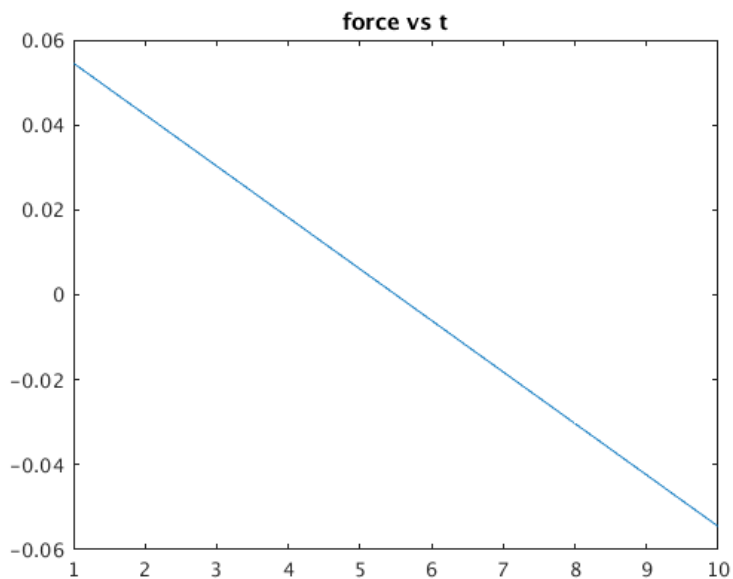
$$x(2) = 2p_1 + 1p_2$$

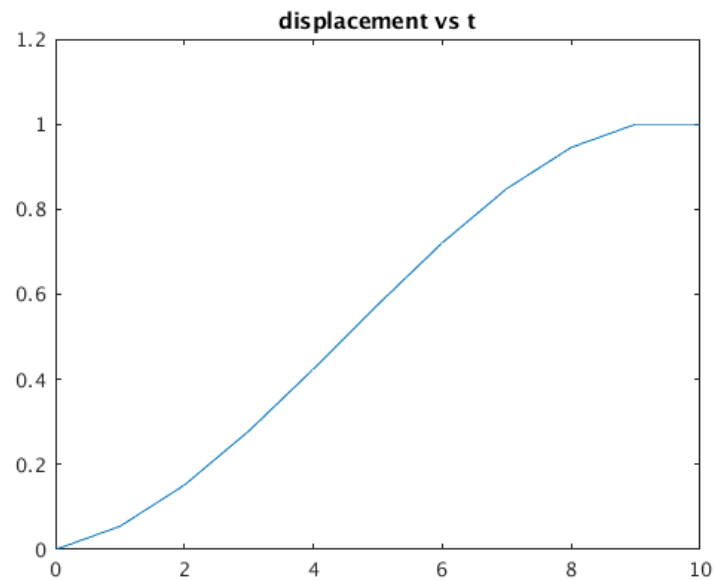
$$x(10) = 10p_1 + 9p_2 + \dots + 1p_{10}$$

QP formulation:

$$\min_f f^T I f, \text{ subject to :}$$

$$\begin{bmatrix} 10 & 9 & \dots & 2 & 1 \end{bmatrix} p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



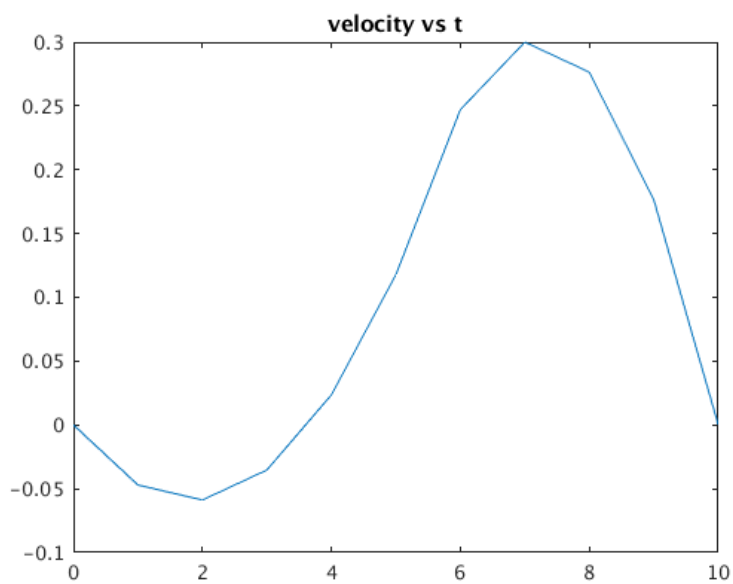
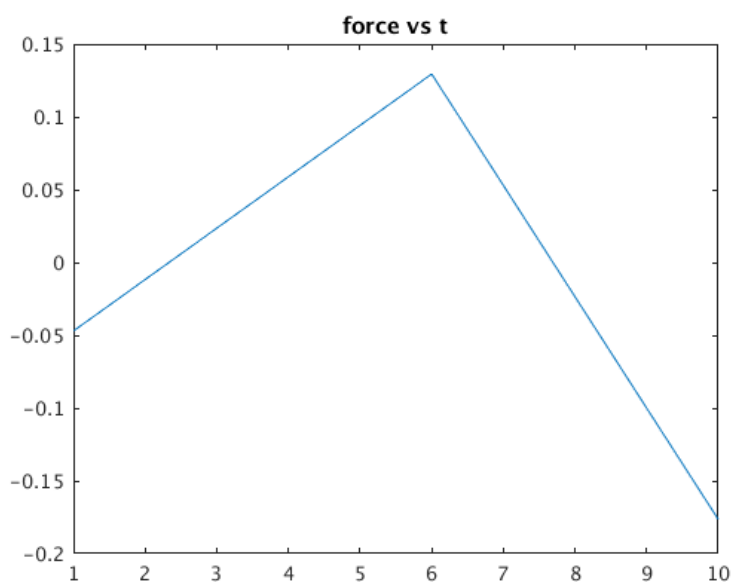


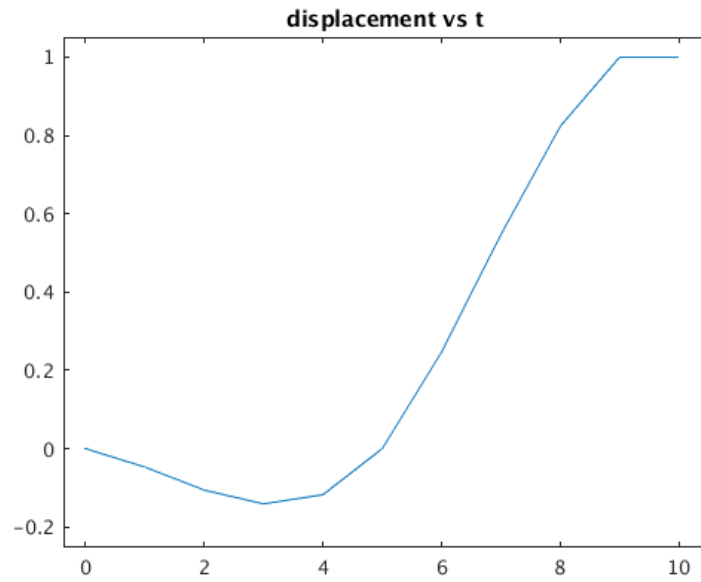
Optimal strategy in this case is to apply a smooth symmetrical force around $t = 5$, so that velocity is always non-negative and displacement is always towards the destination.

- b) additional constraint: $x(5) = 0$

$$\min_f f^T I f, \text{ subject to :}$$

$$\begin{bmatrix} & & & 1^T & & & & & \\ 10 & 9 & .. & & & & 2 & 1 & \\ 5 & 4 & 3 & 2 & 1 & 0 & .. & 0 & \end{bmatrix} p = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$





Optimal strategy in this case is to reverse direction and then go forward again to gain enough velocity timed so that the displacement at $t=5$ is 0.