1. Q 5.12 textbook
   Derive dual problem for: $min_x - \sum_i log(b_i - a_i^T x), x : a_i^T x < b_i, \forall i \in \{1, .., m\}$

$$
\begin{aligned}
& let\ y_i = b_i - a_i^T x > 0 \\
& a_i^T x < b_i\ relax\ to\ a_i^T x \le b_i \\
& L(x, y, \lambda, v) = -\sum_i log y_i + \sum_i \lambda_i (a_i^T x - b_i) + \sum_i v_i (y_i + a_i^T x - b_i) \\
& g(\lambda, v) = \inf_{x,y} L(x, y, \lambda, v) \\
& g(\lambda, v) = \inf_{x,y} -\sum_i log y_i + \sum_i \lambda_i (a_i^T x - b_i) + \sum_i v_i (y_i + a_i^T x - b_i) \\
& (\exists i)\lambda_i > 0 \wedge a_i^T x - b_i < 0 \implies \lambda_i (a_i^T x - b_i)\ unbounded,\ so\ \lambda \le 0 \\
& \lambda \ge 0 \wedge \lambda \le 0 \implies \lambda = 0 \\
& g(\lambda, v) = \inf_{x,y} -\sum_i log y_i + \sum_i v_i (y_i + a_i^T x - b_i) \\
& g(\lambda, v) = \inf_{x,y} -\sum_i log y_i + v^T y + v^T A x - v^T b \\
& y_i > 0 \implies -log y_i\ decreasing \\
& y_i > 0 \wedge (\exists v_i < 0) \implies \inf_{y_i} -log y_i + v_i y_i \ \to -\infty,\ so\ v \succeq 0 \\
& \frac{\partial}{\partial y_i}(-\sum_i log y_i + v^T y + v^T A x - v^T b) = -\frac{1}{y_i} + v_i = 0 \\
& (\forall i)\frac{\partial}{\partial y_i}(-\frac{1}{y_i} + v_i) > 0 \implies convexity \\
& y_i = \frac{1}{v_i}, v_i \ne 0 \\
& (\forall i) v_i \ne 0 \wedge v_i \ge 0 \implies v \succ 0 \\
& \frac{\partial}{\partial x}(-\sum_i log y_i + v^T y + v^T A x - v^T b) = A^T v = 0 \\
& g(\lambda, v) = \begin{cases} -\sum_i log \frac{1}{v_i} + \sum_i \frac{v_i}{v_i} - v^T b, & \text{if } A^T v = 0 \wedge v \succ 0 \\ -\infty, & \text{otherwise} \end{cases}
\end{aligned}
$$

Dual problem:

$$
\begin{aligned}
& \max_{\lambda,v} \sum_i log v_i + m - v^T b = -(\min_{\lambda,v} - \sum_i log v_i - m + v^T b) \\
& s.t.\ A^T v = 0 \\
& \quad - v_i \le 0, \forall i
\end{aligned}
$$

2. Q 5.27 Equality constrained least squares
Give KKT conditions, derive expressions for primal and dual solutions.

$$\min_x \|Ax - b\|_2^2$$
$$s.t.\ Gx = h$$

$$affine\ equality\ constraints\ only \implies Slater's\ condition\ satisfied$$
$$f_0 = x^T A^T Ax + 2b^T Ax + b^T b$$
$$h_0 = Gx - h$$
$$L(x, \lambda, v) = f_0 + v^T h_0$$
$$L(x, \lambda, v) = x^T A^T Ax - 2b^T Ax + b^T b + v^T(Gx - h)$$
$$\frac{\partial L}{\partial x^*} = 0 = 2A^T Ax^* - 2A^T b + G^T v$$

$KKT\ conditions:$
$$x^* = \frac{1}{2}(A^T A)^{-1}(2A^T b - G^T v)$$
$$Gx^* - h = 0$$

$$g(\lambda, v) = \inf_x L(x, \lambda, v) = \frac{1}{4}(2A^T b - G^T v)^T (A^T A)^{-1}(2A^T b - G^T v)$$
$$- 2b^T A(\frac{1}{2}(A^T A)^{-1}(2A^T b - G^T v)) + b^T b + v^T(G\frac{1}{2}(A^T A)^{-1}(2A^T b - G^T v) - h)$$
$$g(\lambda, v) = \inf_x L(x, \lambda, v) = \frac{1}{4}(2A^T b)^T (A^T A)^{-1}(2A^T b) - (A^T b)^T (A^T A)^{-1}(G^T v)$$
$$+ \frac{1}{4}(G^T v)^T (A^T A)^{-1}(G^T v) + b^T A((A^T A)^{-1} G^T v)) - h^T v + v^T G\frac{1}{2}(A^T A)^{-1}(2A^T b - G^T v)$$
$$- b^T A(A^T A)^{-1} 2A^T b) + b^T b$$
rid of constants and simplify :
$$g(\lambda, v) = \inf_x L(x, \lambda, v) = -\frac{1}{4}(G^T v - 2A^T b)^T (A^T A)^{-1}(G^T v - 2A^T b)$$
$$- \frac{1}{2}(Gv^T)^T (A^T A)^{-1}(G^T v) - h^T v$$

Dual problem:

$$\max_{\lambda, v} g(\lambda, v) = \max_v -\frac{1}{4}(G^T v - 2A^T b)^T (A^T A)^{-1}(G^T v - 2A^T b)$$
$$- \frac{1}{2}(Gv^T)^T (A^T A)^{-1}(G^T v) - h^T v$$
$$s, t.\ Gx^* - h = 0$$

Solve for v^*:

$$
\begin{aligned}
&Gx^* - h = 0 \\
&x^* = \frac{1}{2}(A^T A)^{-1}(2A^T b - G^T v^*) \\
&G\frac{1}{2}(A^T A)^{-1}(2A^T b - G^T v^*) - h = 0 \\
&v^* = 2G^{-T}(A^T b - A^T A G^{-1} h)
\end{aligned}
$$

3. Q 5.35 Sensitivity analysis of GP
After log tranformation:

$$\begin{aligned}\min_y \; & log f_0(y)\\ s.t. \; & log f_i(y) \leq 0\\ & log h_i(y) = 0\end{aligned}$$

Perturbed problem:

$$\begin{aligned}\min_y \; & log f_0(y)\\ s.t. \; & log f_i(y) \leq u_i\\ & log h_i(y) = v_i\end{aligned}$$

where $y_i = log(x_i), x_i > 0$

Let L:=Lagrangian of original problem after log transform
Let L':=Lagrangian of perturbed problem after log transform

$$\begin{aligned}L'(y, \lambda, w) &= log\ f_0(y) + \sum_i \lambda_i(log\ f_i(y) - u_i) + \sum_i w_i(log\ h_i(y) - v_i)\\ L'(y, \lambda, w) &= L(y, \lambda, w) - \sum_i \lambda_i u_i + - \sum_i w_i v_i\\ p(u, v) &= \inf_y L'(y, \lambda, w) = \inf_y L(y, \lambda, w) - \lambda^T u - w^T v\end{aligned}$$

Dual:

$$\begin{aligned}\max_{\lambda,w} p(u, v) &= p^*(u, v) = \max_{\lambda,w} \inf_y L(y, \lambda, w) - \lambda^T u - w^T v\\ let\ p(0, 0) &= \max_{\lambda,w} \inf_y L(y, \lambda, w)\\ p^*(u, v) &= p(0, 0) + (\max_{\lambda,w} -\lambda^T u - w^T v)\\ \lambda &\succeq 0\end{aligned}$$

let λ^*, w^* be parameters for optimal $p(u, v)$

$$\begin{aligned}&p^*(u, v) = p(0, 0) - \lambda^{*T} u - w^{*T} v\\ &u\ not\ present\ in\ original\ problem \implies \frac{\partial}{\partial u_i} p(0, 0) = 0, \forall i\\ &v\ not\ present\ in\ original\ problem \implies \frac{\partial}{\partial v_i} p(0, 0) = 0, \forall i\\ &\frac{\partial}{\partial u_i} p^*(u, v) = -\lambda_i^*, \forall i\\ &\frac{\partial}{\partial v_i} p^*(u, v) = -w_i^*, \forall i\end{aligned}$$

Inverse log transform $x \to e^x$ into original form of objective:

$$e^{\frac{\partial}{\partial u_i} p^*(u,v)} = e^{-\lambda_i^*}, \forall i$$
$$e^{\frac{\partial}{\partial v_i} p^*(u,v)} = e^{-w_i^*}, \forall i$$

Relaxation of ith constraint by α percent:
$\partial p^*(u_i, 0) = -\lambda_i^* \partial u_i$
$\partial u_i = \alpha \implies$ objective function of log transformed problem experiences a decrease in value by $\lambda_i \alpha$
Converting objective back via inverse of log: $x \to e^y$
z small $\implies e^z \approx 1 + z$ via Taylor expansion
$e^{-\lambda_i^* \alpha} \approx 1 - \lambda_i^* \alpha, \lambda_i^* \geq 0 \implies$ objective function experiences an improvement of $\lambda_i^* \alpha$ percent since it is a minimization problem.

4. Q 5.42. Find Lagrange dual problem in inequality form.

$$\begin{aligned} &\min_x \; c^T x \\ &s.t. \; Ax \preceq_K b \end{aligned}$$

$$\begin{aligned} L(x, \lambda, v) &= c^T x + \lambda^T (Ax - b) \\ \lambda &\succeq_{K^*} 0 \\ g(\lambda, v) &= \inf_x L(x, \lambda, v) \\ g(\lambda, v) &= \begin{cases} -\lambda^T b, & c + A^T \lambda = 0 \\ -\infty, & o/w \end{cases} \end{aligned}$$

Dual:

$$\begin{aligned} &\max_{\lambda, v} g(\lambda, v) = \max_{\lambda} \; -b^T \lambda \\ &s.t. \; c + A^T \lambda = 0 \\ &\quad \lambda \succeq_{K^*} 0 \\ \\ &let \; y = -\lambda \\ \\ &\max_y \; b^T y \\ &s.t. \; A^T y = c \\ &\quad y \preceq_{K^*} 0 \end{aligned}$$

5. Strong Duality for LP:
   Primal:

$$\begin{aligned}\min_x\ & c^T x\\ s.t.\ & Ax \geq b\\ & x \geq 0\end{aligned}$$

Find the dual of the primal and argue that

- a) if the primal is unbounded then the dual is infeasible

$$L(x,\lambda,v) = c^T x + \lambda_1^T(b - Ax) + \lambda_2^T(-x)$$
$$L(x,\lambda,v) = (c^T - \lambda_1^T A - \lambda_2^T)x + \lambda_1^T b$$
$$g(\lambda,v) = \inf_x L(x,\lambda,v) = \begin{cases} b^T\lambda_1, & c - A^T\lambda_1 - \lambda_2 = 0\\ -\infty, & o/w\end{cases}$$

$$Dual:$$
$$\begin{aligned}\max_{\lambda_1,\lambda_2}\ & b^T\lambda_1\\ s.t.\ & c - A^T\lambda_1 - \lambda_2 = 0\\ & \lambda_1 \geq 0\\ & \lambda_2 \geq 0\end{aligned}$$

$$g(\lambda,v) \leq c^T x^*$$
$$primal\ unbounded:\ c^T x^* = -\infty \implies g(\lambda,v) = -\infty \implies$$
$$c - A^T\lambda_1 - \lambda_2 \neq 0 \implies infeasible\ case\ of\ dual$$

- b) if the primal is infeasible then the dual is either infeasible or unbounded

primal infeasible:

$(\exists i)(b - Ax)_i > 0 \vee (\exists i)(-x)_i > 0$

$$
\begin{aligned}
&g(\lambda_1, \lambda_2) = \inf_x c^T x + \lambda_1^T (b - Ax) + \lambda_2^T (-x) \leq c^T x \\
&\lambda_1 \succeq 0 \\
&\lambda_2 \succeq 0 \\
&g(\lambda_1, \lambda_2) = \inf_x c^T x + \lambda_1^T (b - Ax) + \lambda_2^T (-x) \\
&dual : \max_{\lambda_1, \lambda_2} g(\lambda_1, \lambda_2) \\
&let\ x^*\ be\ optimal\ for\ g(\lambda_1, \lambda_2) = c^T x^* + \lambda_1^T (b - Ax^*) + \lambda_2^T (-x^*) \\
&if\ x^*\ is\ primal\ infeasible: \\
&\lambda_1 \succeq 0, \lambda_2 \succeq 0 \wedge (\exists i)(b - Ax^*)_i > 0 \wedge \lambda_{1_i} \to +\infty \implies \max_{\lambda_1, \lambda_2} g(\lambda_1, \lambda_2) \to +\infty \\
&\lambda_1 \succeq 0, \lambda_2 \succeq 0 \wedge (\exists i)(-x^*)_i > 0 \wedge \lambda_{2_i} \to +\infty \implies \max_{\lambda_1, \lambda_2} g(\lambda_1, \lambda_2) \to +\infty \\
&dual\ is\ unbounded\ in\ this\ case
\end{aligned}
$$

$$
\begin{aligned}
&suppose\ optimal\ value\ of\ dual\ exists\ with: \\
&\max_{\lambda_1, \lambda_2} c^T x^* + \lambda_1^T (b - Ax^*) + \lambda_2^T (-x^*) = c^T x^* \\
&c^T x^* + \lambda_1^{*T} (b - Ax^*) + \lambda_2^{*T} (-x^*) = c^T x^* \\
&\lambda_1^{*T} (b - Ax^*) + \lambda_2^{*T} (-x^*) = 0 \\
&primal\ infeasibility\ enforces: \\
&(\exists i)(b - Ax)_i > 0 \vee (\exists i)(-x)_i > 0 \\
&suppose: \\
&(\exists i)(b - Ax)_i > 0 \wedge (-x)_i < 0 \implies \\
&\lambda_1^{*T} (b - Ax^*)_i + \lambda_2^{*T} (-x^*)_i = 0,\ where\ \lambda_{1_i}^* = -\alpha \lambda_{2_i}^*, \alpha > 0 \\
&suppose: \\
&(\exists i, j, i \neq j)(b - Ax)_i > 0 \wedge (b - Ax)_j > 0 \wedge (b - Ax)_i = 0, (b - Ax)_j = 0 \implies \\
&\lambda_1^{*T} (b - Ax^*)_i + \lambda_1^{*T} (b - Ax^*)_j = 0,\ where\ \lambda_{1_i}^* = -\alpha \lambda_{1_j}^*, \alpha > 0 \\
&similiarily\ for:\ (\exists i, j)(b - Ax)_i > 0 \wedge (-x)_j > 0\ and\ (\exists i, j, i \neq j)(-x)_i > 0 \wedge (-x)_j > 0
\end{aligned}
$$

$\lambda_1, \lambda_2\ can\ be\ chosen\ such\ that\ \lambda_1 \not\succeq 0 \vee \lambda_2 \not\succeq 0,\ thus\ not\ satisfying\ dual\ feasibility.$

6. Game Theory

$$\begin{aligned} &\min_x \max_y x^T P y \\ s.t.\ & Ax \leq b \\ & Cy \leq d \end{aligned}$$

$$\begin{aligned} &\max_y \min_x x^T P y \\ s.t.\ & Ax \leq b \\ & Cy \leq d \end{aligned}$$

(a) show that min-max problem as the same optimal value as the following minimization problem:

$$\begin{aligned} &\min_{\lambda,x} d^T \lambda \\ s.t.\ & C^T \lambda = P^T x \\ & Ax \leq b \\ & \lambda \geq 0 \end{aligned}$$

Inner optimization problem:

$$\begin{aligned} &\max_y (P^T x)^T y = -(\min_y -(P^T x)^T y) \\ s.t.\ & Ax \leq b \\ & Cy \leq d \end{aligned}$$

$$g(\lambda_1, \lambda_2) = \inf_y -(P^T x)^T y = \begin{cases} \lambda_1^T (Ax - b) - \lambda_2^T d, & -P^T x + C^T \lambda_2 = 0 \\ -\infty, & o/w \end{cases}$$

$$\begin{aligned} &\max_{\lambda_1,\lambda_2} g(\lambda_1, \lambda_2) = \max_{\lambda_1,\lambda_2} -\lambda_2^T d + \lambda_1^T (Ax - b) \\ s.t.\ & -P^T x + C^T \lambda_2 = 0 \\ & \lambda_1, \lambda_2 \geq 0 \\ & \lambda_1^T (Ax - b) \leq 0 \implies \\ &\max_{\lambda_1,\lambda_2} g(\lambda_1, \lambda_2) = \max_{\lambda_2} -\lambda_2^T d \\ &-(\min_y -(P^T x)^T y) = -\max_{\lambda_1,\lambda_2} g(\lambda_1, \lambda_2) = -(-\min_{\lambda_2} \lambda_2^T d) = \min_{\lambda_2} \lambda_2^T d \\ & \textit{rename } \lambda_2 \textit{ to } \lambda \textit{ and enclose with outer minimization over } x: \\ &\min_{x,\lambda} \lambda^T d \\ s.t.\ & P^T x = C^T \lambda \\ & Ax \leq b \\ & \lambda \geq 0 \end{aligned}$$

(b) show that max-min problem as the same optimal value as the following maximization problem:

$$\begin{aligned}
&\max_{y,v} -b^T v \\
s.t.\ &A^T v + Py = 0 \\
&Cy \le d \\
&v \ge 0
\end{aligned}$$

Inner optimization problem:

$$\begin{aligned}
&\min_x \ x^T Py \\
&s.t.\ Ax \le b \\
&\quad Cy \le d \\
g(\lambda_1, \lambda_2) &= \inf_x x^T Py + \lambda_1^T(Ax - b) + \lambda_2^T(Cy - d) \\
&= \begin{cases} -\lambda_1^T b + \lambda_2^T(Cy - d), & Py + A^T\lambda_1 = 0 \\ -\infty, & o/w \end{cases} \\
max_{\lambda_1,\lambda_2} g(\lambda_1, \lambda_2) &= \max_{\lambda_1,\lambda_2} -\lambda_1^T b + \lambda_2^T(Cy - d) \\
&s.t.\ P^T y + A^T\lambda_1 = 0 \\
&\quad \lambda_1 \ge 0 \\
&\quad \lambda_2 \ge 0 \\
&\quad Cy - d \le 0 \implies \lambda_2^T(Cy - d) \le 0 \implies \\
&\max_{\lambda_1,\lambda_2} g(\lambda_1, \lambda_2) = \max_{\lambda_1,\lambda_2} -\lambda_1^T b \\
&s.t.\ P^T y + A^T\lambda_1 = 0 \\
&\quad \lambda_1 \ge 0 \\
&\quad Cy - d \le 0 \\
&\quad rename\ variables\ and\ enclose\ with\ outer\ optimization: \\
&\max_{y,v} - b^T v \\
&s.t.\ P^T y + A^T v = 0 \\
&\quad Cy \le d \\
&\quad v \ge 0
\end{aligned}$$

(c) show the above problems have the same optimal value (min-max equals max-min)

$$\begin{aligned}
&\min_{x,\lambda} d^T\lambda \\
&s.t.\ C^T\lambda = P^Tx \\
&\quad\ \ Ax \leq b \\
&\quad\ \ \lambda \geq 0
\end{aligned}$$

$$\begin{aligned}
g(a_1, a_2, w) &= \inf_{x,\lambda} d^T\lambda + a_1^T(Ax - b) + a_2^T(-\lambda) + w^T(C^T\lambda - P^Tx) \\
&s.t.\ a_1 \geq 0 \\
&\quad\ \ a_2 \geq 0 \\
g(a_1, a_2, w) &= \begin{cases} -a_1^Tb, & d - a_2 + Cw = 0 \wedge A^Ta_1 - Pw = 0 \\ -\infty, & o/w \end{cases}
\end{aligned}$$

$$\begin{aligned}
&Dual: \\
&\max_{a_1,a_2,w} g(a_1, a_2, w) = -b^Ta_1 \\
&s.t.\ d - a_2 + Cw = 0 \\
&\quad\ \ A^Ta_1 - Pw = 0 \\
&\quad\ \ a_2 \geq 0 \implies d + Cw \geq 0 \\
&\quad\ \ a_1 \geq 0
\end{aligned}$$

$$\begin{aligned}
&let\ v = a_1, y = -w \\
&\max_{v,y} g(v, y) = -b^Tv \\
&s.t.\ A^Tv + Py = 0 \\
&\quad\ \ Cy \leq d \\
&\quad\ \ v \geq 0
\end{aligned}$$

We start with equivalent formulation of min-max and arrive at a equivalent formulation of max-min problem thus they obtain the same optimal value.

7. Optimal control of a unit mass revisited

$$\|p\|_1 = \sum_{i=1}^{10} |p_i|$$
$$\|p\|_\infty = \max_{i=1,..,10} |p_i|$$

(a) Consider $\|p\|_1$ problem with same setup in Q9(1) of last assignment. Find optimal solution. Plot optimal force, position, and velocity. Write down the primal and dual solutions.

$$\min_p \sum_{i=1}^{10} |p_i|$$

Constraints: $x(0) = 0, x(10) = 1, \dot{x}(0) = 0, v(0) = 0, v(10) = 0$
let $v = \dot{x}$

$$v(i) = v(0) + \sum_{j=1}^{i} p_j/m, m = 1$$
$$v(10) = \sum_{j=1}^{i} p_j = 0$$
$$x(i) = x(0) + \sum_{j=0}^{i} v(j)$$
$$x(i) = \sum_{j=1}^{i} \sum_{k=1}^{j} p_k = \sum_{j=1}^{i} (i - j + 1) p_j$$
$$x(10) = [10, 9, .., 1]p = 1$$

Primal:

$$let\ t_i = |p_i|$$
$$let\ X = [p_1, .., p_{10}, t_1, .., t_{10}]^T$$
$$\min_X \begin{bmatrix} 1^T_{1:10} & 0^T_{1:10} \end{bmatrix} X$$
$$s.t.\ \begin{bmatrix} 1^T & 0^T \\ 10, 9, ..1 & 0^T \end{bmatrix} X = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} I_{10\times10} & -I_{10\times10} \\ -I_{10\times10} & -I_{10\times10} \end{bmatrix} X \leq \begin{bmatrix} 0_{20\times1} \end{bmatrix}$$

$$\begin{aligned}
L(p,t,\lambda,v) &= 1^T t + \lambda_1^T(p-t) + \lambda_2^T(-p-t) + v_1^T(1^T p) + v_2^T([10..1]p-1)\\
&= (1^T - \lambda_1^T - \lambda_2^T)t + (\lambda_1^T - \lambda_2^T + v_2^T[10..1] + v_1^T 1^T)p - v_2^T 1\\
g(\lambda,v) &= \inf_{p,t} L(p,t,\lambda,v)\\
&= \begin{cases} -v_2, & 1-\lambda_1-\lambda_2=0, \lambda_1-\lambda_2+[10..1]^T v_2 + 1v_1 = 0\\ -\infty, & o/w \end{cases}
\end{aligned}$$

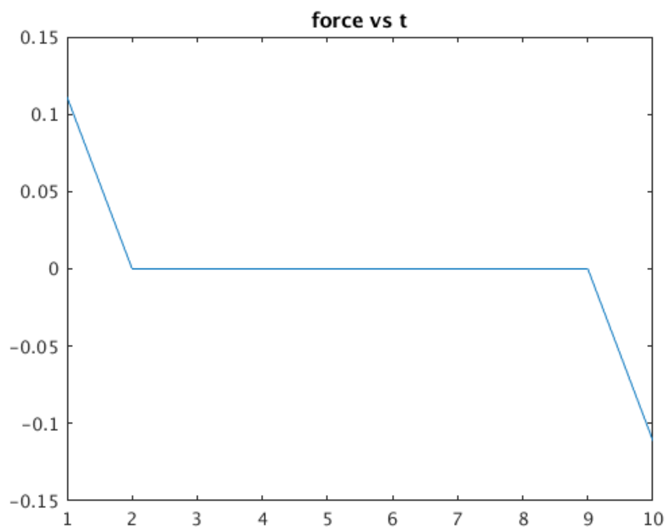
$$\begin{aligned}
&\max_{\lambda,v}\ -v_2\\
&s.t.\ 1-\lambda_1-\lambda_2=0\\
&\qquad \lambda_1-\lambda_2+\begin{bmatrix}10\\..\\1\end{bmatrix}v_2 + 1_{10\times1}v_1 = 0\\
&\quad \lambda_1 \geq 0\\
&\quad \lambda_2 \geq 0
\end{aligned}$$

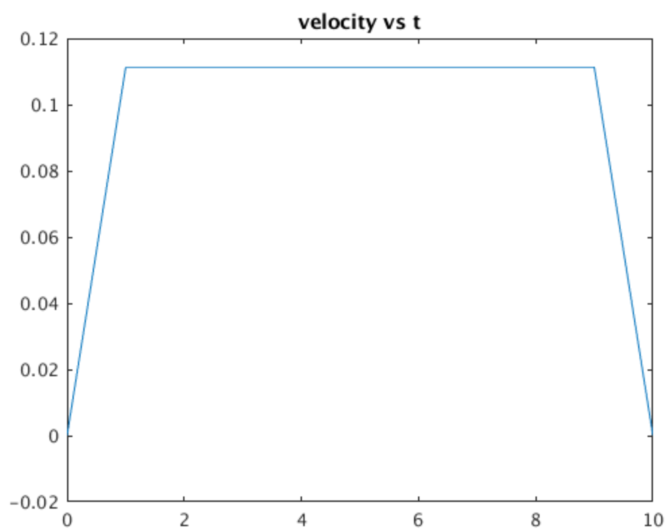
Dual:

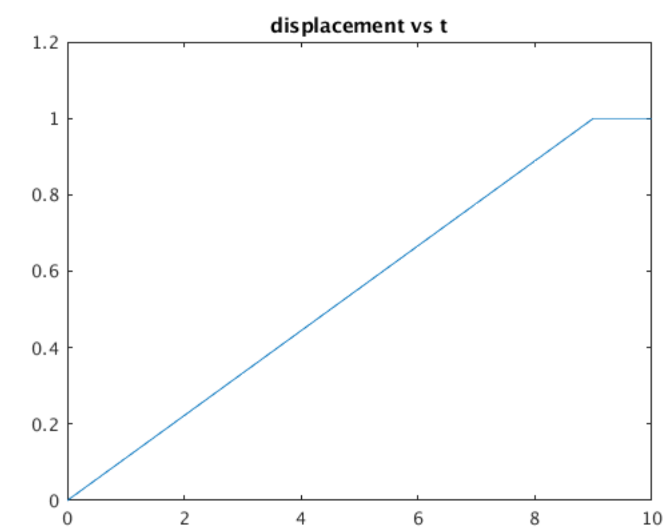
$$\begin{aligned}
&let\ X = [\lambda_1,\lambda_2,v_1,v_2]^T\\
&\lambda_1,\lambda_2 \in \mathbb{R}^{10}\\
&v_1,v_2 \in R\\
\max_X\ &\begin{bmatrix}0^T & 0^T & 0 & -1\end{bmatrix} X\\
s.t.\ &\begin{bmatrix} I_{10\times10} & -I_{10\times10} & 1_{10\times1} & [10..1]^T\\ I_{10\times10} & I_{10\times10} & 0 & 0 \end{bmatrix} X = \begin{bmatrix}0_{10\times1}\\1_{10\times1}\end{bmatrix}\\
&\begin{bmatrix}-I & -I & 0 & 0\end{bmatrix} X \leq 0
\end{aligned}$$

Solve using Matlab:
$cost_{optimal} = 0.2222$
Primal solution:
$p(1) = 0.1111, p(10) = -0.1111, (\forall i \notin \{1,10\})\ p(i) = 0$
$t(1) = t(10) = 0.1111, (\forall i \notin \{1,10\})\ t(i) = 0$

Dual solution:
$\lambda_1 = [1, 0.8889, 0.7778, 0.6667, 0.5556, 0.4444, 0.3333, 0.2222, 0.1111, 0]$
$\lambda_2 = [0, 0.1111, 0.2222, 0.3333, 0.4444, 0.5556, 0.6667, 0.7778, 0.8889, 1]$
$v_1 = 1.2222, v_2 = -0.2222$


force vs t
0.15
0.1
0.05
0
−0.05
−0.1
−0.15
1 2 3 4 5 6 7 8 9 10


velocity vs t
0.12
0.1
0.08
0.06
0.04
0.02
0
−0.02
0 2 4 6 8 10


displacement vs t
1.2
1
0.8
0.6
0.4
0.2
0
0 2 4 6 8 10

```
%% part a
f = [zeros(10,1); ones(10,1)]
A = [eye(10) -eye(10); -eye(10) -eye(10)]
b= [zeros(20,1)]

Aeq = [ones(1,10) zeros(1,10); 10:-1:1 zeros(1,10)]
beq = [0;1]
[xs,fval,exitflag,output,lambda] = linprog(f,A,b,Aeq,beq)

force = xs(1:10)
plot(1:10,force)
title('force vs t')

v = zeros(11,1)
for i=2:1:11
    v(i) = v(i-1) + force(i-1)
end

plot(0:1:10,v)
title('velocity vs t')

x = zeros(11,1)
for i=1:1:11
    if i > 1
        x(i) = x(i-1) + v(i)
    else
        x(i) = v(i)
    end
end

plot(0:1:10,x)
title('displacement vs t')
```

(b) i. Verify that for any vector v and w, we always have $|w^T v| \leq \|v\|_\infty \|w\|_1$

$$|w^T v| = |\sum_i w_i v_i| \leq \sum_i |w_i v_i| \leq \sum_i |w_i| v', v' = \max_i |v_i|$$
$$|w^T v| \leq v' \sum_i |w_i| = \|v\|_\infty \|w\|_1$$

ii. Let z be any solution of $Az = y$, explain why for any λ, we must have

$$\|z\|_1 \geq \frac{|\lambda^T y|}{\|A^T \lambda\|_\infty}$$

and thus z, λ for which the above inequality is satisfied with equality means z must be optimal.

$$\frac{|\lambda^T y|}{\|A^T\lambda\|_\infty} = \frac{|\lambda^T A z|}{\|A^T\lambda\|_\infty} = \frac{|z^T A^T \lambda|}{\|A^T\lambda\|_\infty}$$
$$|z^T A^T \lambda| \leq \|z\|_1 \|A^T \lambda\|_\infty$$
$$\frac{|\lambda^T y|}{\|A^T\lambda\|_\infty} \leq \frac{\|z\|_1 \|A^T\lambda\|_\infty}{\|A^T\lambda\|_\infty}$$
$$\frac{|\lambda^T y|}{\|A^T\lambda\|_\infty} \leq \|z\|_1$$

iii. Set λ to be the Lagrange multiplier associated with the equality constraint in part (a). Use the above inequality to directly verify that the bang-bang solution is optimal.

$$AX = y$$
$$\begin{bmatrix} 1^T & 0^T \\ 10, 9, ..1 & 0^T \end{bmatrix} X = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$let\ \lambda = [v_1, v_2]^T = [1.2222, -0.2222]^T$$
$$\|X\|_1 = \|[p_1, ..., p_{10}, t_1, .., t_{10}]\|_1 = 0.1111 - 0.1111 + 0.1111 + 0.1111 = 0.2222$$
$$|\lambda^T y| = \left| \lambda^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right| = 0.2222$$
$$\|A^T \lambda\|_\infty = \left\| \begin{bmatrix} 1^T & 0^T \\ 10, 9, ..1 & 0^T \end{bmatrix}^T \lambda \right\|_\infty = 1$$
$$\frac{|\lambda^T y|}{\|A^T\lambda\|_\infty} = 0.2222 = \|X\|_1$$
$$tight\ inequality \implies X\ is\ optimal$$

(c) Repeat part (a) for $\|*\|_\infty$ minimization problem.

$$\min_p \|p\|_\infty$$

$$\begin{aligned}
&let\ t = \max_i |p_i| \\
&\min_{p,t}\ t \\
&s.t.\ p_i \leq t, \forall i \\
&\quad\ \ p_i \geq -t, \forall i \\
&\quad\ \ 1^T p = 0 \\
&\quad\ \ [10, .., 1]p = 1
\end{aligned}$$

Primal:

$$\begin{aligned}
&let\ X = [p, t]^T \\
&p \in \mathbb{R}^{10}, t \in \mathbb{R} \\
\min_X\ &\begin{bmatrix} 0^T & 1 \end{bmatrix} X \\
s.t.\ &\begin{bmatrix} 1^T & 0 \\ 10..1 & 0 \end{bmatrix} X = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&\begin{bmatrix} I & -1_{10\times1} \\ -I & -1_{10\times1} \end{bmatrix} \leq 0_{20\times1}
\end{aligned}$$

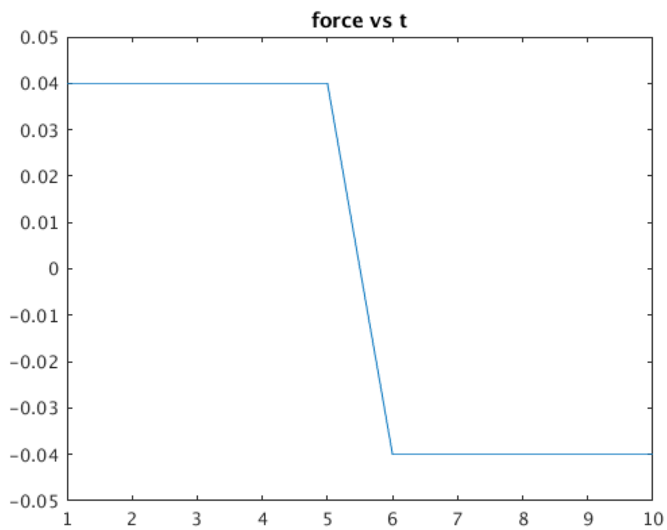
$$
\begin{aligned}
L(p,t,\lambda,v) &= t + \lambda_1^T(p-t) + \lambda_2^T(-p-t) + v_1^T 1^T p + v_2^T([10,..,1]p-1) \\
g(\lambda,v) &= \inf_{p,t} L(p,t,\lambda,v) \\
&= \begin{cases} -v_2, & \lambda_1 - \lambda_2 + 1v_1 + [10,..,1]^T v_2 = 0, 1 - 1^T\lambda_1 - 1^T\lambda_2 = 0 \\ -\infty, & o/w \end{cases}
\end{aligned}
$$

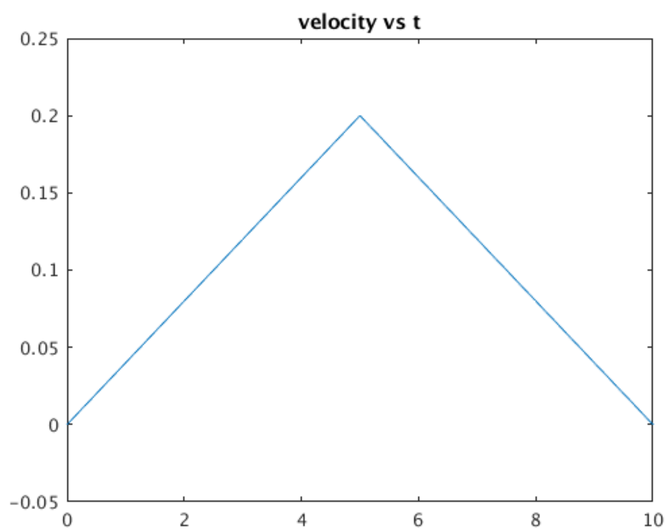
Dual:

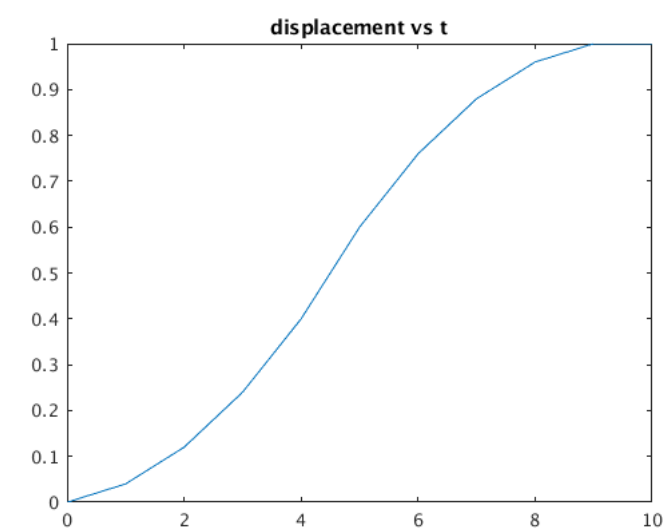
$$
\begin{aligned}
&let\ X = [\lambda_1, \lambda_2, v_1, v_2]^T \\
\max_X &\begin{bmatrix} 0^T & 0^T & 0 & -1 \end{bmatrix} X \\
s.t. &\begin{bmatrix} I_{10\times10} & -I_{10\times10} & 1_{10\times1} & [10..1]^T \\ -1^T & -1^T & 0 & 0 \end{bmatrix} X = \begin{bmatrix} 0_{10\times1} \\ -1 \end{bmatrix} \\
&\begin{bmatrix} -I_{10\times10} & 0_{10\times10} & 0_{10\times2} \\ 0_{10\times10} & -I_{10\times10} & 0_{10\times2} \end{bmatrix} X \leq 0_{20\times1}
\end{aligned}
$$

Solve using Matlab:
$cost_{optimal} = 0.04$
Primal solution:
$(\forall i \in \{1,..,5\})\ p(i) = 0.04$
$(\forall i \in \{6,..,10\})\ p(i) = -0.04$
$t = 0.04$

Dual solution:
$\lambda_1 = [0.2, 0.16, 0.12, 0.08, 0.04, 0, 0, 0, 0, 0]$
$\lambda_2 = [0, 0, 0, 0, 0, 0, 0.04, 0.08, 0.12, 0.16]$
$v_1 = 0.2, v_2 = -0.04$


force vs t
0.05
0.04
0.03
0.02
0.01
0
−0.01
−0.02
−0.03
−0.04
−0.05
1 2 3 4 5 6 7 8 9 10


velocity vs t
0.25
0.2
0.15
0.1
0.05
0
−0.05
0 2 4 6 8 10


displacement vs t
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 2 4 6 8 10

```
%% part c
f = [zeros(10,1); 1]
A = [eye(10) -ones(10,1); -eye(10) -ones(10,1)]
b= [zeros(20,1)]

Aeq = [ones(1,10) 0; 10:-1:1 0]
beq = [0;1]
[xs,fval,exitflag,output,lambda] = linprog(f,A,b,Aeq,beq)

force = xs(1:10)
plot(1:10,force)
title('force vs t')

v = zeros(11,1)
for i=2:1:11
    v(i) = v(i-1) + force(i-1)
end

plot(0:1:10,v)
title('velocity vs t')

x = zeros(11,1)
for i=1:1:11
    if i > 1
        x(i) = x(i-1) + v(i)
    else
        x(i) = v(i)
    end
end

plot(0:1:10,x)
title('displacement vs t')
```