

## 1. Q 5.12 textbook

Derive dual problem for:  $\min_x - \sum_i \log(b_i - a_i^T x), x : a_i^T x \leq b_i, \forall i \in \{1, \dots, m\}$

$$y_i = b_i - a_i^T x$$

$$a_i^T x \leq b_i$$

$$L(x, y, \lambda, v) = - \sum_i \log y_i + \sum_i \lambda_i (a_i^T x - b_i) + \sum_i v_i (y_i + a_i^T x - b_i)$$

$$g(\lambda, v) = \inf_{x, y} L(x, y, \lambda, v)$$

$$g(\lambda, v) = \inf_{x, y} - \sum_i \log y_i + \sum_i \lambda_i (a_i^T x - b_i) + \sum_i v_i (y_i + a_i^T x - b_i)$$

$$(\exists i) \lambda_i \neq 0 \implies \lambda_i (a_i^T x - b_i) \text{ unbounded, so } \lambda = 0$$

$$g(\lambda, v) = \inf_{x, y} - \sum_i \log y_i + \sum_i v_i (y_i + a_i^T x - b_i)$$

$$g(\lambda, v) = \inf_{x, y} - \sum_i \log y_i + v^T y + v^T A x - v^T b$$

$$(\exists i) v_i < 0 \implies v^T y \text{ unbounded, so } v \succeq 0$$

$$\frac{\partial}{\partial y_i} - \sum_i \log y_i + v^T y + v^T A x - v^T b = -\frac{1}{y_i} + v_i = 0$$

$$y_i = \frac{1}{v_i}, v_i \neq 0$$

$$(\forall i) v_i \neq 0 \wedge v_i \geq 0 \implies v \succ 0$$

$$\frac{\partial}{\partial x} - \sum_i \log y_i + v^T y + v^T A x - v^T b = A^T v = 0$$

$$g(\lambda, v) = \begin{cases} - \sum_i \log \frac{1}{v_i} + \sum_i \frac{v_i}{v_i} - v^T b, & \text{if } A^T v = 0 \wedge v \succ 0 \\ -\infty, & \text{otherwise} \end{cases}$$

Dual problem:

$$\begin{aligned} \max_{\lambda, v} \quad & \sum_i \log v_i + m - v^T b = -(\min_{\lambda, v} - \sum_i \log v_i - m + v^T b) \\ \text{s.t.} \quad & A^T v = 0 \\ & -v_i \leq 0, \forall i \end{aligned}$$

## 2. Q 5.27 Equality constrained least squares

Give KKT conditions, derive expressions for primal and dual solutions.

$$\begin{aligned} \min_x & \|Ax - b\|_2^2 \\ \text{s.t. } & Gx = h \end{aligned}$$

$$f_0 = x^T A^T A x + 2b^T A x + b^T b$$

$$h_0 = Gx - h$$

$$L(x, \lambda, v) = f_0 + v^T h_0$$

$$L(x, \lambda, v) = x^T A^T A x - 2b^T A x + b^T b + v^T (Gx - h)$$

$$\frac{\partial L}{\partial x^*} = 0 = 2A^T A x^* - 2A^T b + G^T v$$

KKT conditions :

$$x^* = \frac{1}{2}(A^T A)^{-1}(2A^T b - G^T v)$$

$$Gx^* - h = 0$$

$$\begin{aligned} g(\lambda, v) &= \inf_x L(x, \lambda, v) = \frac{1}{4}(2A^T b - G^T v)^T (A^T A)^{-1} (2A^T b - G^T v) \\ &\quad - 2b^T A \left( \frac{1}{2}(A^T A)^{-1} (2A^T b - G^T v) \right) + b^T b + v^T \left( G \frac{1}{2}(A^T A)^{-1} (2A^T b - G^T v) - h \right) \\ g(\lambda, v) &= \inf_x L(x, \lambda, v) = \frac{1}{4}(2A^T b)^T (A^T A)^{-1} (2A^T b) - (A^T b)^T (A^T A)^{-1} (G^T v) \\ &\quad + \frac{1}{4}(G^T v)^T (A^T A)^{-1} (G^T v) + b^T A ((A^T A)^{-1} G^T v) - h^T v + v^T G \frac{1}{2}(A^T A)^{-1} (2A^T b - G^T v) \\ &\quad - b^T A (A^T A)^{-1} 2A^T b + b^T b \end{aligned}$$

rid of constants and simplify :

$$\begin{aligned} g(\lambda, v) &= \inf_x L(x, \lambda, v) = -\frac{1}{4}(G^T v - 2A^T b)^T (A^T A)^{-1} (G^T v - 2A^T b) \\ &\quad - \frac{1}{2}(G^T v)^T (A^T A)^{-1} (G^T v) - h^T v \end{aligned}$$

Dual problem:

$$\begin{aligned} \max_{\lambda, v} g(\lambda, v) &= \max_v -\frac{1}{4}(G^T v - 2A^T b)^T (A^T A)^{-1} (G^T v - 2A^T b) \\ &\quad - \frac{1}{2}(G^T v)^T (A^T A)^{-1} (G^T v) - h^T v \\ \text{s.t. } & Gx^* - h = 0 \end{aligned}$$

Solve for  $v^*$ :

$$Gx^* - h = 0$$

$$x^* = \frac{1}{2}(A^T A)^{-1}(2A^T b - G^T v^*)$$

$$G\frac{1}{2}(A^T A)^{-1}(2A^T b - G^T v^*) - h = 0$$

$$v^* = 2G^{-T}(A^T b - A^T A G^{-1} h)$$

## 3. Q 5.35 Sensitivity analysis of GP

After log transformation:

$$\begin{aligned} \min_y & f_0(y) \\ \text{s.t. } & \log f_i(y) \leq 0 \\ & \log h_i(y) = 0 \end{aligned}$$

Perturbed problem:

$$\begin{aligned} \min_y & f_0(y) \\ \text{s.t. } & \log f_i(y) \leq u_i \\ & \log h_i(y) = v_i \end{aligned}$$

where  $y_i = \log(x_i)$

Let  $L$ :=Lagrangian of original problem after log transform

Let  $L'$ :=Lagrangian of perturbed problem after log transform

$$\begin{aligned} L'(y, \lambda, w) &= \log f_0(y) + \sum_i \lambda_i (\log f_i(y) - u_i) + \sum_i w_i (\log h_i(y) - v_i) \\ L'(y, \lambda, w) &= L(y, \lambda, w) - \sum_i \lambda_i u_i + - \sum_i w_i v_i \\ p(u, v) &= \inf_y L'(y, \lambda, w) = \inf_y L(y, \lambda, w) - \lambda^T u - w^T v \end{aligned}$$

Dual:

$$\begin{aligned} \max_{\lambda, w} p(u, v) &= p^*(u, v) = \max_{\lambda, w} \inf_y L(y, \lambda, w) - \lambda^T u - w^T v \\ \text{let } p(0, 0) &= \max_{\lambda, w} \inf_y L(y, \lambda, w) \\ p^*(u, v) &= p(0, 0) + (\max_{\lambda, w} -\lambda^T u - w^T v) \end{aligned}$$

let  $\lambda^*, w^*$  be parameters for optimal  $p(u, v)$

$$\begin{aligned}
 p^*(u, v) &= p(0, 0) - \lambda^{*T}u - w^{*T}v \\
 u \text{ not present in original problem} &\implies \\
 \frac{\partial}{\partial u_i} p(0, 0) &= 0, \forall i \\
 v \text{ not present in original problem} &\implies \\
 \frac{\partial}{\partial v_i} p(0, 0) &= 0, \forall i \\
 \frac{\partial}{\partial u_i} p^*(u, v) &= -\lambda_i^*, \forall i \\
 \frac{\partial}{\partial v_i} p^*(u, v) &= -w_i^*, \forall i
 \end{aligned}$$

Inverse log transform  $x \rightarrow e^x$  into original form of objective:

$$\begin{aligned}
 e^{\frac{\partial}{\partial u_i} p^*(u, v)} &= e^{-\lambda_i^*}, \forall i \\
 e^{\frac{\partial}{\partial v_i} p^*(u, v)} &= e^{-w_i^*}, \forall i
 \end{aligned}$$

Relaxation of  $i$ th constraint by  $\alpha$  percent:

$$\partial p^*(u_i, 0) = -\lambda_i^* \partial u_i$$

$\partial u_i = \alpha \implies$  objective function of log transformed problem experiences a decrease in value by  $\lambda_i^* \alpha$

Converting objective back via inverse of log:  $x \rightarrow e^x$

$z$  small  $\implies e^z \approx 1 + z$  via Taylor expansion

$e^{-\lambda_i^* \alpha} \approx 1 - \lambda_i^* \alpha \implies$  objective function experiences an improvement of  $\lambda_i^* \alpha$  percent since it is a minimization problem.

4. Q 5.42. Find Lagrange dual problem in inequality form.

$$\begin{aligned} \min_x \quad & c^T x \\ \text{s.t.} \quad & Ax \preceq_K b \end{aligned}$$

$$\begin{aligned} L(x, \lambda, v) &= c^T x + \lambda^T (Ax - b) \\ \lambda &\succeq_{K^*} 0 \\ g(\lambda, v) &= \inf_x L(x, \lambda, v) \\ g(\lambda, v) &= \begin{cases} -\lambda^T b, & c^T x + \lambda^T Ax = 0 \\ -\infty, & \text{o/w} \end{cases} \end{aligned}$$

Dual:

$$\begin{aligned} \max_{\lambda, v} \quad & g(\lambda, v) = \max_{\lambda} -b^T \lambda \\ \text{s.t.} \quad & c + A^T \lambda = 0 \\ & \lambda \succeq_{K^*} 0 \end{aligned}$$

$$\text{let } y = -\lambda$$

$$\begin{aligned} \max_y \quad & b^T y \\ \text{s.t.} \quad & A^T y = c \\ & y \preceq_{K^*} 0 \end{aligned}$$

## 5. Strong Duality for LP:

Primal:

$$\begin{aligned} \min_x \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Find the dual of the primal and argue that

- a) if the primal is unbounded then the dual is infeasible

$$L(x, \lambda, v) = c^T x + \lambda_1^T (b - Ax) + \lambda_2^T (-x)$$

$$L(x, \lambda, v) = (c^T - \lambda_1^T A - \lambda_2^T)x + \lambda_1^T b$$

$$g(\lambda, v) = \inf_x L(x, \lambda, v) = \begin{cases} b^T \lambda_1, & c - A^T \lambda_1 - \lambda_2 = 0 \\ -\infty, & \text{o/w} \end{cases}$$

Dual :

$$\max_{\lambda_1, \lambda_2} b^T \lambda_1$$

$$\text{s.t. } c - A^T \lambda_1 - \lambda_2 = 0$$

$$g(\lambda, v) \leq c^T x^* \text{ (weak duality)}$$

$$\text{primal unbounded : } c^T x^* = -\infty \implies$$

$$g(\lambda, v) = -\infty \implies c - A^T \lambda_1 - \lambda_2 \neq 0 \implies \text{infeasible case of dual}$$

- b) if the primal is infeasible then the dual is either infeasible or unbounded

*primal infeasible*  $\implies b - Ax \not\leq 0 \vee -x \not\leq 0$

$$\min_x c^T x + \lambda_1^T (b - Ax) + \lambda_2^T (-x) \leq c^T x$$

*dual unbounded case :*

suppose  $c - A^T \lambda_1 - \lambda_2 = 0$ , substitute for  $\lambda_1$  :

Dual :  $\max_{\lambda_1, \lambda_2} b^T \lambda_1 = \max_{\lambda_1, \lambda_2} b^T A^{-T} c - b^T A^{-T} \lambda_2$ , such that :

$(\forall i)(b - Ax)_i > 0 \implies \lambda_{1_i}$  may be negative, and

$(\forall i)(-x)_i > 0 \implies \lambda_{2_i}$  may be negative, and

$$c - A^T \lambda_1 - \lambda_2 = 0$$

let  $d = b^T A^{-T} c$

let  $e^T = b^T A^{-T}$

Dual :  $\max_{\lambda_1, \lambda_2} d - e^T \lambda_2$

$(\exists i)e_i > 0$  : select  $\lambda_{2_i} \rightarrow -a$  and  $(\exists i)e_i < 0$  : select  $\lambda_{2_i} \rightarrow a, a \rightarrow +\infty$  such that it satisfies :

$c - A^T \lambda_1 - \lambda_2 = 0$ , then  $e^T \lambda_2$  is unbounded

$$\text{eg : } c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, A = I, Ax \not\leq b, x \not\leq 0$$

Dual :  $\max_{\lambda_1, \lambda_2} d - I^T \begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \lambda_2$ , select  $\lambda_2 = \begin{bmatrix} -a \\ a \end{bmatrix}$

$$c - A^T \lambda_1 - \lambda_2 = 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \lambda_1 - \begin{bmatrix} -a \\ a \end{bmatrix} = 0$$

$$\text{select : } \lambda_1 = \begin{bmatrix} a \\ -a \end{bmatrix}$$

$$\text{let } a \rightarrow +\infty \implies \max_{\lambda_1, \lambda_2} d - I^T \begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \lambda_2 \Big|_{\lambda_2 = [-a, a]^T} = d + 2a = +\infty \implies \text{dual unbounded}$$

*dual infeasible case :*

suppose :  $c > 0, x \not\leq 0, A = -1, Ax \geq b$

$$Ax \geq b \implies \lambda_1 \geq 0$$

$$x < 0 \implies \lambda_2 \leq 0$$

$$c - A^T \lambda_1 - \lambda_2 = c + \lambda_1 - \lambda_2$$

$$c + \lambda_1 > 0 \wedge -\lambda_2 \geq 0 \implies c - A^T \lambda_1 - \lambda_2 \neq 0 \implies \text{dual is not feasible}$$



## 6. Game Theory

$$\begin{aligned} & \min_x \max_y x^T P y \\ \text{s.t. } & Ax \leq b \\ & Cy \leq d \end{aligned}$$

$$\begin{aligned} & \max_y \min_x x^T P y \\ \text{s.t. } & Ax \leq b \\ & Cy \leq d \end{aligned}$$

(a) show that min-max problem has the same optimal value as the following minimization problem:

$$\begin{aligned} & \min_{\lambda, x} d^T \lambda \\ \text{s.t. } & C^T \lambda = P^T x \\ & Ax \leq b \\ & \lambda \geq 0 \end{aligned}$$

Inner optimization problem:

$$\begin{aligned} & \max_y (P^T x)^T y = -(\min_y -(P^T x)^T y) \\ \text{s.t. } & Ax \leq b \\ & Cy \leq d \\ g(\lambda_1, \lambda_2) &= \inf_y -(P^T x)^T y = \begin{cases} \lambda_1^T (Ax - b) - \lambda_2^T d, & -P^T x + C^T \lambda_2 = 0 \\ -\infty, & \text{o/w} \end{cases} \\ \max_{\lambda_1, \lambda_2} g(\lambda_1, \lambda_2) &= \max_{\lambda_1, \lambda_2} -\lambda_2^T d + \lambda_1^T (Ax - b) \\ \text{s.t. } & -P^T x + C^T \lambda_2 = 0 \\ & \lambda_1, \lambda_2 \geq 0 \\ & \lambda_1^T (Ax - b) \leq 0 \implies \\ \max_{\lambda_1, \lambda_2} g(\lambda_1, \lambda_2) &= \max_{\lambda_2} -\lambda_2^T d \\ -(\min_y -(P^T x)^T y) &= -\max_{\lambda_1, \lambda_2} g(\lambda_1, \lambda_2) = -(-\min_{\lambda_2} \lambda_2^T d) = \min_{\lambda_2} \lambda_2^T d \end{aligned}$$

rename  $\lambda_2$  to  $\lambda$  and enclose with outer minimization over  $x$  :

$$\begin{aligned} & \min_{x, \lambda} \lambda^T d \\ \text{s.t. } & P^T x = C^T \lambda \\ & Ax \leq b \\ & \lambda \geq 0 \end{aligned}$$

- (b) show that max-min problem has the same optimal value as the following maximization problem:

$$\begin{aligned} & \max_{y,v} -b^T v \\ & s.t. \quad A^T v + Py = 0 \\ & \quad Cy \leq d \\ & \quad v \geq 0 \end{aligned}$$

Inner optimization problem:

$$\begin{aligned} & \min_x x^T Py \\ & s.t. \quad Ax \leq b \\ & \quad Cy \leq d \\ & g(\lambda_1, \lambda_2) = \inf_x x^T Py + \lambda_1^T (Ax - b) + \lambda_2^T (Cy - d) \\ & \quad = \begin{cases} -\lambda_1^T b + \lambda_2^T (Cy - d), & Py + A^T \lambda_1 = 0 \\ -\infty, & o/w \end{cases} \\ & \max_{\lambda_1, \lambda_2} g(\lambda_1, \lambda_2) = \max_{\lambda_1, \lambda_2} -\lambda_1^T b + \lambda_2^T (Cy - d) \\ & s.t. \quad P^T y + A^T \lambda_1 = 0 \\ & \quad \lambda_1 \geq 0 \\ & \quad \lambda_2 \geq 0 \\ & \quad Ax - b \leq 0 \\ & \quad Cy - d \leq 0 \\ & \quad \lambda_2^T (Cy - d) \leq 0 \implies \\ & \max_{\lambda_1, \lambda_2} g(\lambda_1, \lambda_2) = \max_{\lambda_1, \lambda_2} -\lambda_1^T b \\ & s.t. \quad P^T y + A^T \lambda_1 = 0 \\ & \quad \lambda_1 \geq 0 \\ & \quad Cy - d \leq 0 \\ & \quad \text{rename variables and enclose with outer optimization :} \\ & \max_{y,v} -b^T v \\ & s.t. \quad P^T y + A^T v = 0 \\ & \quad Cy \leq d \\ & \quad v \geq 0 \end{aligned}$$

(c) show the above problems have the same optimal value (min-max equals max-min)

$$\begin{aligned} \min_{x, \lambda} \quad & d^T \lambda \\ \text{s.t.} \quad & C^T \lambda = P^T x \\ & Ax \leq b \\ & \lambda \geq 0 \end{aligned}$$

$$g(a_1, a_2, w) = \inf_{x, \lambda} d^T \lambda + a_1^T (Ax - b) + a_2^T (-\lambda) + w^T (C^T \lambda - P^T x)$$

$$\begin{aligned} \text{s.t.} \quad & a_1 \geq 0 \\ & a_2 \geq 0 \\ & Ax - b \leq 0 \\ & -\lambda \leq 0 \\ & C^T \lambda - P^T x = 0 \end{aligned}$$

$$g(a_1, a_2, w) = \begin{cases} -a_1^T b, & d - a_2 + Cw = 0 \wedge A^T a_1 - Pw = 0 \\ -\infty, & \text{o/w} \end{cases}$$

*Dual :*

$$\begin{aligned} \max_{a_1, a_2, w} \quad & g(a_1, a_2, w) = -b^T a_1 \\ \text{s.t.} \quad & d - a_2 + Cw = 0 \\ & A^T a_1 - Pw = 0 \\ & a_2 \geq 0 \implies d + Cw \geq 0 \\ & a_1 \geq 0 \end{aligned}$$

$$\text{let } v = a_1, y = -w$$

$$\max_{v, y} g(v, y) = -b^T v$$

$$\text{s.t.} \quad A^T v + Py = 0$$

$$Cy \leq d$$

$$v \geq 0$$

We start with equivalent formulation of min-max and arrive at a equivalent formulation of max-min problem thus they obtain the same optimal value.

(a)

## 7. Optimal control of a unit mass revisited

$$\|p\|_1 = \sum_{i=1}^{10} |p_i|$$

$$\|p\|_\infty = \max_{i=1,\dots,10} |p_i|$$

- (a) Consider  $\|p\|_1$  problem with same setup in Q9(1) of last assignment. Find optimal solution. Plot optimal force, position, and velocity. Write down the primal and dual solutions.

$$\min_p \sum_{i=1}^{10} |p_i|$$

Constraints:  $x(0) = 0, x(10) = 1, \dot{x}(0) = 0, v(0) = 0, v(10) = 0$   
 let  $v = \dot{x}$

$$v(i) = v(0) + \sum_{j=1}^i p_j / m, m = 1$$

$$v(10) = \sum_{j=1}^{10} p_j = 0$$

$$x(i) = x(0) + \sum_{j=0}^{i-1} v(j)$$

$$x(i) = \sum_{j=1}^i \sum_{k=1}^j p_k = \sum_{j=1}^i (i - j + 1) p_j$$

$$x(10) = [10, 9, \dots, 1] p = 1$$

Primal:

$$\text{let } t_i = |p_i|$$

$$\text{let } X = [p_1, \dots, p_{10}, t_1, \dots, t_{10}]^T$$

$$\min_X [1_{1:10}^T \quad 0_{1:10}^T] X$$

$$\text{s.t.} \quad \begin{bmatrix} 1^T & 0^T \\ 10, 9, \dots, 1 & 0^T \end{bmatrix} X = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} I_{10 \times 10} & -I_{10 \times 10} \\ -I_{10 \times 10} & -I_{10 \times 10} \end{bmatrix} X \leq [0_{20 \times 1}]$$

$$\begin{aligned}
L(p, t, \lambda, v) &= 1^T t + \lambda_1^T (p - t) + \lambda_2^T (-p - t) + v_1^T (1^T p) + v_2^T ([10..1]p - 1) \\
&= (1^T - \lambda_1^T - \lambda_2^T)t + (\lambda_1^T - \lambda_2^T + v_2^T [10..1] + v_1^T 1^T)p - v_2^T 1 \\
g(\lambda, v) &= \inf_{p, t} L(p, t, \lambda, v) \\
&= \begin{cases} -v_2, & 1 - \lambda_1 - \lambda_2 = 0, \lambda_1 - \lambda_2 + [10..1]^T v_2 + 1v_1 = 0 \\ -\infty, & \text{o/w} \end{cases}
\end{aligned}$$

$$\begin{aligned}
&\max_{\lambda, v} -v_2 \\
&s.t. \ 1 - \lambda_1 - \lambda_2 = 0 \\
&\quad \lambda_1 - \lambda_2 + \begin{bmatrix} 10 \\ \cdot \\ 1 \end{bmatrix} v_2 + 1_{10 \times 1} v_1 = 0 \\
&\quad \lambda_1 \geq 0 \\
&\quad \lambda_2 \geq 0
\end{aligned}$$

Dual:

$$\begin{aligned}
&\text{let } X = [\lambda_1, \lambda_2, v_1, v_2]^T \\
&\lambda_1, \lambda_2 \in \mathbb{R}^{10} \\
&v_1, v_2 \in \mathbb{R} \\
&\max_X [0^T \ 0^T \ 0 \ -1] X \\
&s.t. \begin{bmatrix} I_{10 \times 10} & -I_{10 \times 10} & 1_{10 \times 1} & [10..1]^T \\ I_{10 \times 10} & I_{10 \times 10} & 0 & 0 \end{bmatrix} X = \begin{bmatrix} 0_{10 \times 1} \\ 1_{10 \times 1} \end{bmatrix} \\
&\quad [-I \ -I \ 0 \ 0] X \leq 0
\end{aligned}$$

Solve using Matlab:

$$cost_{optimal} = 0.2222$$

Primal solution:

$$p(1) = 0.1111, p(10) = -0.1111, (\forall i \notin \{1, 10\}) \ p(i) = 0$$

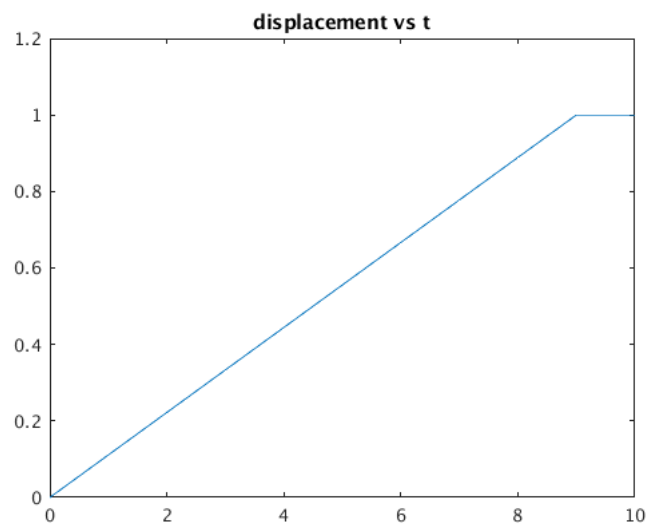
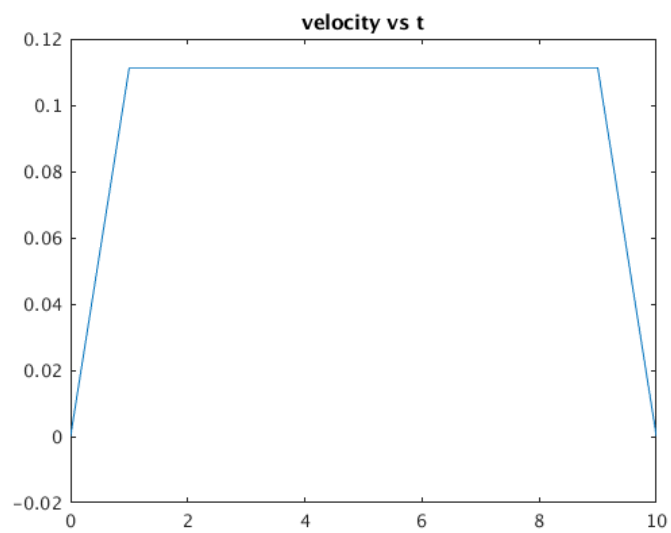
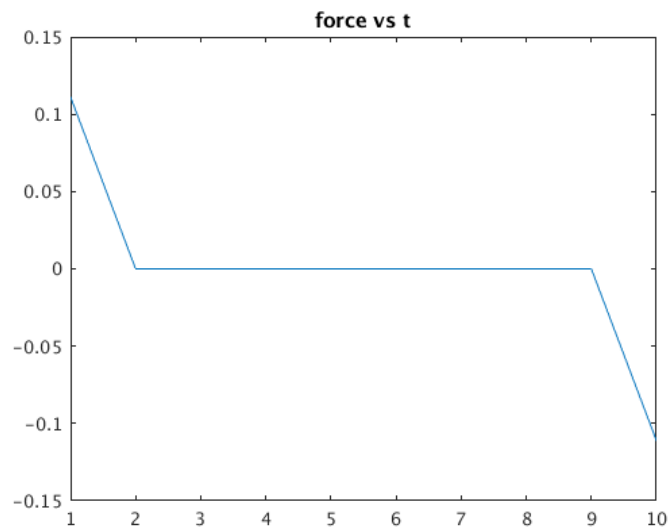
$$t(1) = t(10) = 0.1111, (\forall i \notin \{1, 10\}) \ t(i) = 0$$

Dual solution:

$$\lambda_1 = [1, 0.8889, 0.7778, 0.6667, 0.5556, 0.4444, 0.3333, 0.2222, 0.1111, 0]$$

$$\lambda_2 = [0, 0.1111, 0.2222, 0.3333, 0.4444, 0.5556, 0.6667, 0.7778, 0.8889, 1]$$

$$v_1 = 1.2222, v_2 = -0.2222$$



```
% part a
f = [zeros(10,1); ones(10,1)]
A = [eye(10) -eye(10); -eye(10) -eye(10)]
b = [zeros(20,1)]

Aeq = [ones(1,10) zeros(1,10); 10:-1:1 zeros(1,10)]
beq = [0;1]
[xs,fval,exitflag,output,lambda] = linprog(f,A,b,Aeq,beq)

force = xs(1:10)
plot(1:10,force)
title('force vs t')

v = zeros(11,1)
for i=2:1:11
    v(i) = v(i-1) + force(i-1)
end

plot(0:1:10,v)
title('velocity vs t')

x = zeros(11,1)
for i=1:1:11
    if i > 1
        x(i) = x(i-1) + v(i)
    else
        x(i) = v(i)
    end
end

plot(0:1:10,x)
title('displacement vs t')
```

- (b) i. Verify that for any vector  $v$  and  $w$ , we always have  $|w^T v| \leq \|v\|_\infty \|w\|_1$

$$\begin{aligned} |w^T v| &= \left| \sum_i w_i v_i \right| \leq \sum_i |w_i v_i| \leq \sum_i |w_i| v', \quad v' = \max_i |v_i| \\ |w^T v| &\leq v' \sum_i |w_i| = \|v\|_\infty \|w\|_1 \end{aligned}$$

- ii. Let  $z$  be any solution of  $Az = y$ , explain why for any  $\lambda$ , we must have

$$\|z\|_1 \geq \frac{|\lambda^T y|}{\|A^T \lambda\|_\infty}$$

and thus  $z, \lambda$  for which the above inequality is satisfied with equality means  $z$  must be optimal.

$$\begin{aligned} \frac{|\lambda^T y|}{\|A^T \lambda\|_\infty} &= \frac{|\lambda^T Az|}{\|A^T \lambda\|_\infty} = \frac{|z^T A^T \lambda|}{\|A^T \lambda\|_\infty} \\ |z^T A^T \lambda| &\leq \|z\|_1 \|A^T \lambda\|_\infty \\ \frac{|\lambda^T y|}{\|A^T \lambda\|_\infty} &\leq \frac{\|z\|_1 \|A^T \lambda\|_\infty}{\|A^T \lambda\|_\infty} \\ \frac{|\lambda^T y|}{\|A^T \lambda\|_\infty} &\leq \|z\|_1 \end{aligned}$$

- iii. Set  $\lambda$  to be the Lagrange multiplier associated with the equality constraint in part (a). Use the above inequality to directly verify that the bang-bang solution is optimal.

$$AX = y$$

$$\begin{bmatrix} 1^T & 0^T \\ 10, 9, \dots, 1 & 0^T \end{bmatrix} X = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{let } \lambda = [v_1, v_2]^T = [1.2222, -0.2222]^T$$

$$\|X\|_1 = \| [p_1, \dots, p_{10}, t_1, \dots, t_{10}] \|_1 = 0.1111 - 0.1111 + 0.1111 + 0.1111 = 0.2222$$

$$|\lambda^T y| = \left| \lambda^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right| = 0.2222$$

$$\left\| \begin{bmatrix} 1^T & 0^T \\ 10, 9, \dots, 1 & 0^T \end{bmatrix}^T \lambda \right\|_\infty = 1$$

$$\frac{|\lambda^T y|}{\|A^T \lambda\|_\infty} = 0.2222 = \|X\|_1$$

*tight inequality  $\implies X$  is optimal*



(c) Repeat part (a) for  $\|*\|_\infty$  minimization problem.

$$\min_p \|p\|_\infty$$

$$\text{let } t = \max_i |p_i|$$

$$\min_{p,t} t$$

$$\text{s.t. } p_i \leq t, \forall i$$

$$p_i \geq -t, \forall i$$

$$1^T p = 0$$

$$[10, \dots, 1]p = 1$$

Primal:

$$\text{let } X = [p, t]^T$$

$$p \in \mathbb{R}^{10}, t \in \mathbb{R}$$

$$\min_X [0^T \ 1] X$$

$$\text{s.t. } \begin{bmatrix} 1^T & 0 \\ 10..1 & 0 \end{bmatrix} X = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} I & -1_{10 \times 1} \\ -I & -1_{10 \times 1} \end{bmatrix} \leq 0_{20 \times 1}$$

$$\begin{aligned}
L(p, t, \lambda, v) &= t + \lambda_1^T(p - t) + \lambda_2^T(-p - t) + v_1^T 1^T p + v_2^T([10, \dots, 1]p - 1) \\
g(\lambda, v) &= \inf_{p, t} L(p, t, \lambda, v) \\
&= \begin{cases} -v_2, & \lambda_1 - \lambda_2 + 1v_1 + [10, \dots, 1]^T v_2 = 0, 1 - 1^T \lambda_1 - 1^T \lambda_2 = 0 \\ -\infty, & \text{o/w} \end{cases}
\end{aligned}$$

Dual:

$$\begin{aligned}
&\text{let } X = [\lambda_1, \lambda_2, v_1, v_2]^T \\
&\max_X [0^T \quad 0^T \quad 0 \quad -1] X \\
&s.t. \begin{bmatrix} I_{10 \times 10} & -I_{10 \times 10} & 1_{10 \times 1} & [10 \dots 1]^T \\ -1^T & -1^T & 0 & 0 \end{bmatrix} X = \begin{bmatrix} 0_{10 \times 1} \\ -1 \end{bmatrix} \\
&\quad \begin{bmatrix} -I_{10 \times 10} & 0_{10 \times 10} & 0_{10 \times 2} \\ 0_{10 \times 10} & -I_{10 \times 10} & 0_{10 \times 2} \end{bmatrix} X \leq 0_{20 \times 1}
\end{aligned}$$

Solve using Matlab:

$$cost_{optimal} = 0.04$$

Primal solution:

$$(\forall i \in \{1, \dots, 5\}) p(i) = 0.04$$

$$(\forall i \in \{6, \dots, 10\}) p(i) = -0.04$$

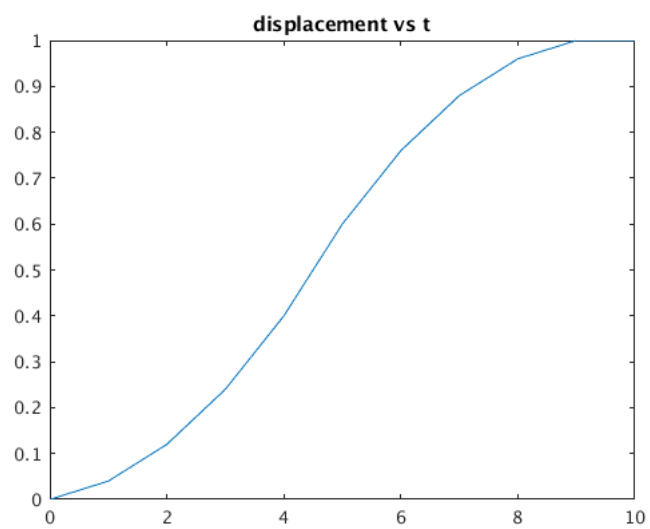
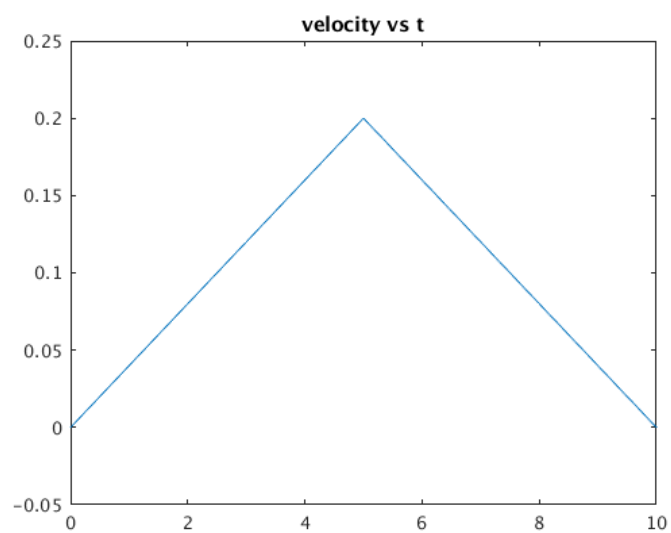
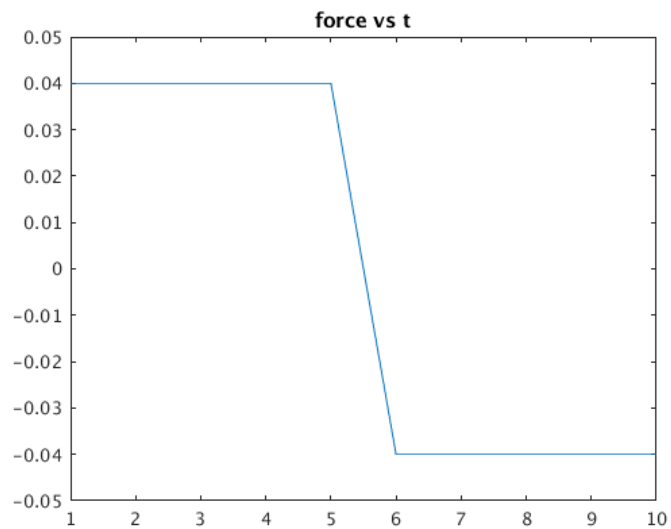
$$t = 0.04$$

Dual solution:

$$\lambda_1 = [0.2, 0.16, 0.12, 0.08, 0.04, 0, 0, 0, 0, 0]$$

$$\lambda_2 = [0, 0, 0, 0, 0, 0, 0.04, 0.08, 0.12, 0.16]$$

$$v_1 = 0.2, v_2 = -0.04$$



```
% part c
f = [zeros(10,1); 1]
A = [eye(10) -ones(10,1); -eye(10) -ones(10,1)]
b = [zeros(20,1)]

Aeq = [ones(1,10) 0; 10:-1:1 0]
beq = [0;1]
[xs,fval,exitflag,output,lambda] = linprog(f,A,b,Aeq,beq)

force = xs(1:10)
plot(1:10,force)
title('force vs t')

v = zeros(11,1)
for i=2:1:11
    v(i) = v(i-1) + force(i-1)
end

plot(0:1:10,v)
title('velocity vs t')

x = zeros(11,1)
for i=1:1:11
    if i > 1
        x(i) = x(i-1) + v(i)
    else
        x(i) = v(i)
    end
end

plot(0:1:10,x)
title('displacement vs t')
```