1. SDP Relaxation and Heuristics for Two-Way Partitioning Problem

    (a) Q 5.39 textbook

$$min\ x^T W x$$
$$s.t.\ x_i^2 = 1, \forall i \in \{1, .., n\}$$

    i. Show that the two-way partitioning problem can be cast as

$$min\ tr(WX)$$
$$s.t.\ X \succeq 0, rank(X) = 1$$
$$X_{ii} = 1, \forall i \in \{1, .., n\}$$

$x^T W x = tr(x^T W x) = tr(W x x^T)$

$let\ X = xx^T$

$(\forall i)x_i^2 = 1 \iff x_i = \{-1, 1\} \implies x^T I x = n$

$x^T I x = tr(xx^T) = n$

$(\forall i, j)X_{ij} = \{-1, 1\}$

$((\exists i)X_{ii} = -1 \implies tr(X) < n)$

$thus, for\ tr(X) = n : (\forall i)X_{ii} = 1$

$X = xx^T = x \begin{bmatrix} a_1 & a_2 & .. & a_n \end{bmatrix} = \begin{bmatrix} a_1 x & a_2 x & .. & a_n x \end{bmatrix}, a_i \in \mathbb{R}, x \in \mathbb{R}^n$

$(\forall i)(\exists j)\beta_{ij}a_i x = a_j x \implies \beta_{ij}a_i x - a_j x = 0$

$let\ \gamma_{ij} = \beta_{ij}a_i - a_j$

$\gamma_{ij}x = 0$

$x \neq 0 \implies ((\forall i)(\exists j)\gamma_{ij} = 0 \implies linear\ dependence\ between\ all\ column\ vectors\ of\ X)$

$thus,\ rank(X) = 1$

$(\forall w)w^T X w = w^T x x^T w = (x^T w)^T x^T w$

$(\forall i, w)(x^T w)_i(x^T w)_i \geq 0 \implies (\forall w)(x^T w)^T(x^T w) \geq 0 \iff X\ is\ SPD$

Combining all constraints and objective forms the desired result.

ii. SDP relaxation of two-way partitioning problem. Using the formulation in part (a), we can form the relaxation:

$$min \ tr(WX)$$
$$s.t. \ X \succeq 0$$
$$X_{ii} = 1, \forall i \in \{1, .., n\}$$

This problem is an SDP, and therefore can be solved efficiently. Explain why its optimal value gives a lower bound on the optimal value of the two-way partitioning problem (5.113). What can you say if an optimal point X^* for this SDP has rank one?

$$L(X, Z, v) = tr(WX) - tr(XZ) + tr(diag(v)diag(X) - I)$$
$$L(X, Z, v) = -tr(diag(v)I) + tr(WX) - tr(XZ) + tr(diag(v)diag(X))$$
$$g(Z, V) = \begin{cases} -1^T v, & W - Z + diag(v) \succeq 0 \\ -\infty, & o/w \end{cases}$$

dual problem :

$$\max_{Z,v} -1^T v$$
$$s.t. \ W - Z + diag(v) \succeq 0$$
$$Z \succeq 0$$

If an optimal point X^* for the relaxed problem has rank one:
X^* has minimal possible rank and $X^* \neq 0$, $X^* \succeq 0$.
$X^* \succeq 0$, so primal feasible.
Functions are all differentiable, KKT conditions apply at optimality where there exists a dual solution.
Dual of relaxed problem is feasible: $W + diag(v) \succeq Z, Z \succeq 0$
Using complementary slackness: $X^* \neq 0, -tr(X^* Z^*) = 0 \implies Z^* = 0$.

Dual problem of relaxed problem at optimality:

$$\max_{v,Z} -1^T v = [\max_{v} -1^T v]_{Z=Z^*}$$
$$s.t. \ W + diag(v) \succeq Z^*, Z^* = 0 \implies$$
$$s.t. \ W + diag(v) \succeq 0$$

This solution is equivalent to the solution of the dual of the original problem. Thus, if $rank(X^*) = 1$ of the relaxed problem, X^* obtains the same solution as the original problem where $x^* x^{*T} = X^*$ as required.

iii. We now have two SDPs that give a lower bound on the optimal value of the two-way partitioning problem (5.113): the SDP relaxation (5.115) found in part (b), and the Lagrange dual of the two-way partitioning problem, given in (5.114). What is the relation between the two SDPs? What can you say about the lower bounds found by them? Hint: Relate the two SDPs via duality.

$$(5.115) \ min \ tr(WX)$$
$$s.t. \ X \succeq 0$$
$$X_{ii} = 1, \forall i \in \{1, .., n\}$$

$$(5.114) \ maximize \ -1^T v$$
$$s.t. \ W + diag(v) \succeq 0$$

Taken from previous section, dual of relaxed problem:

$$L(X, Z, v) = tr(WX) - tr(XZ) + tr(diag(v)diag(X) - I)$$
$$L(X, Z, v) = -tr(diag(v)I) + tr(WX) - tr(XZ) + tr(diag(v)diag(X))$$
$$g(Z, V) = \begin{cases} -1^T v, & W - Z + diag(v) \succeq 0 \\ -\infty, & o/w \end{cases}$$

Dual problem:

$$\max_{Z,v} -1^T v$$
$$s.t. \ W - Z + diag(v) \succeq 0$$
$$Z \succeq 0$$

It is evident that solution to the dual of relaxed problem has a tightened generalized inequality constraint due to Z compared to (5.114):

$$W + diag(v) \succeq Z$$

This leads to potentially bigger v and hence potentially larger objective value of dual maximization problem (smaller objective value to the primal minimization problem). Thus, solution of relaxed problem provides a lower bound to the the original problem.

(b) Q 11.23(b-d) textbook

- b) heuristic partitioning

- c) randomized method

- d) greedy heuristic refinement

2. Interior Point Method