1. Q 4.11(a,c,e) textbook
   (a) Minimize $\|Ax - b\|_\infty$

$$\min \max_i |(Ax-b)_i|$$
$$\text{let } t = max_i |(Ax-b)_i|$$
$$\min_{x,t} t, subject\ to:$$
$$|(Ax-b)_i| \leq t, \forall i$$

LP Formulation:

$$x \in R^n, t \in R$$
$$\min_{x,t} t, subject\ to.:$$
$$A_{i,:}x - t \leq b_i, \forall i$$
$$-A_{i,:}x - t \leq -b_i, \forall i$$

(c) Minimize $\|Ax - b\|_1, \|x\|_\infty \leq 1$

$$\min_x \sum_i |(Ax-b)_i|, subject\ to:$$
$$|x_i| \leq 1, \forall i$$
$$let\ t_i = |(Ax-b)_i|$$
$$\min_{x,t} \sum_i t_i, subject\ to:$$
$$|(Ax-b)_i| \leq t_i, \forall i$$
$$|x_i| \leq 1, \forall i$$

LP Formulation:

$$x \in R^n, t \in R^n$$
$$\min_{x,t} 1^T t, subject\ to:$$
$$A_{i,:}x - t_i \leq b_i$$
$$-A_{i,:}x - t_i \leq -b_i$$
$$x_i \leq 1, \forall i$$
$$-x_i \leq 1, \forall i$$

(e) Minimize $\|Ax - b\|_1 + \|x\|_\infty$

$$\begin{aligned}
let\ t_i = |Ax - b|_i \\
let\ v = max|x_i| \\
\min_{t,v,x} \sum_i t_i + v, subject\ to: \\
|x_i| \leq v, \forall i \\
|Ax - b|_i \leq t_i, \forall i
\end{aligned}$$

LP formulation:

$$\begin{aligned}
x, t \in R^n, v \in R \\
\min_{t,v,x} 1^T t + v, subject\ to: \\
x_i - v \leq 0, \forall i \\
-x_i - v \leq 0, \forall i \\
A_{i,:}x - t_i \leq +b_i, \forall i \\
-A_{i,:}x - t_i \leq -b_i, \forall i
\end{aligned}$$

2. Q 4.16 textbook

$$
\begin{aligned}
x(t) \in \mathbb{R}^n, t \in \{0, .., N\} \\
u(t) \in \mathbb{R}, t \in \{0, .., N\} \\
x(t+1) = Ax(t) + bu(t), t \in \{0, .., N\} \\
given: \\
A \in \mathbb{R}^{n \times n} \\
b \in \mathbb{R}^n \\
x(0) = 0 \\
problem: \\
\min_u \sum_{t=0}^{N-1} f(u(t)), subject\ to: \\
X(N) = x_{des} \\
f(a) = \begin{cases} |a|, & |a| leq 1 \\ 2|a| - 1, & |a| > 1 \end{cases}
\end{aligned}
$$

expanding $x(t)$:

$$
\begin{aligned}
&x(0) = 0 \\
&x(1) = Ax(0) + bu(0) = bu(0) \\
&x(2) = Ax(1) + bu(1) = A(bu(0)) + bu(1) \\
&x(i) = A^{i-1}bu(0) + A^{i-2}bu(1) + ... + A^0 bu(i-1) = \sum_{j=0}^{i-1} A^j bu(i-1-j) \\
&x(i) = \begin{bmatrix} A^{i-1}b & A^{i-2}b & .. & ..b \end{bmatrix} \begin{bmatrix} u(0) \\ .. \\ u(i-1) \end{bmatrix} \\
&x_{des} = x(N) = \begin{bmatrix} A^{N-1}b & A^{N-2}b & .. & ..b \end{bmatrix} \begin{bmatrix} u(0) \\ .. \\ u(N-1) \end{bmatrix} \\
&let\ v_t = |f(x(t))| \\
&\quad v_t = max\{|u(t)|, 2|u(t)| - 1\} \\
&\quad v_t \geq |u(t)| \\
&\quad v_t \geq 2|u(t)| - 1 \\
&\quad v_t \geq u_t \\
&\quad v_t \geq -u_t \\
&\quad v_t \geq 2u_t - 1 \\
&\quad v_t \geq -2u_t - 1
\end{aligned}
$$

LP formulation:

$$
\begin{aligned}
& v_{0,..,N-1}, u_{0,..,N-1} \in \mathbb{R}^N \\
& \min_{v,u} 1^T v, \textit{subject to}: \\
& -v_t + u_t \le 0, \forall t \in 0, .., N-1 \\
& -v_t - u_t \le 0, \forall t \in 0, .., N-1 \\
& -v_t + 2u_t \le 1, \forall t \in 0, .., N-1 \\
& -v_t - 2u_t \le 1, \forall t \in 0, .., N-1 \\
& \begin{bmatrix} A^{N-1}b & A^{N-2}b & .. & ..b \end{bmatrix} \begin{bmatrix} u_0 \\ .. \\ u_{N-1} \end{bmatrix} = x_{des}
\end{aligned}
$$

3. Q 4.21(a) textbook
   Find explicit solution for the QCQP:
   minimize $c^T x$, subject to:
   $x^T A x \le 1$
   $A \in S^n_{++}, c \neq 0$

$$
\begin{aligned}
x^T A x &= x^T A^{1/2} A^{1/2} x = (A^{1/2}x)^T A^{1/2} x = \|A^{1/2}x\|_2^2 \\
\|A^{1/2}x\|_2^2 &\le 1 \\
\textit{let } y &= A^{1/2} x \\
x &= A^{-1/2} y \\
c^T x &= c^T A^{-1/2} y \\
\textit{let } b^T &= c^T A^{-1/2} \\
\min b^T y & \textit{ subject to}: \\
\|y\|_2^2 &\le 1 \\
\min b^T y &= \frac{-b^T b \alpha}{\|b\|}, \alpha = max\|y\|_2 = 1 \\
y^* &= \frac{-b}{\|b\|} \\
x^* &= A^{-1/2} y^* = A^{-1/2} \frac{-b}{\|b\|} \\
&= A^{-1/2} \frac{-(c^T A^{-1/2})^T}{\|(c^T A^{-1/2})^T\|} \\
&= \frac{-A^{-1/2} A^{-1/2} c}{\|A^{-1/2} c\|} = \frac{-A^{-1} c}{(c^T A^{-1} c)^{1/2}} \\
c^T x^* &= \frac{-c^T A^{-1} c}{(c^T A^{-1} c)^{1/2}} \\
c^T x^* &= -(c^T A^{-1} c)^{1/2}
\end{aligned}
$$

4. Q 4.25 textbook

$$\varepsilon_i = \{P_i u + q_i : \|u\|_2 \leq 1\}, i = 1, .., K + L, P_i \in S^n$$

Find a feasible hyperplane strictly separating $\varepsilon_1, .., \varepsilon_K$ from $\varepsilon_{K+1}, .., \varepsilon_{K+L}$.

$$a^T x + b > 0, x \in \bigcup_{i=1}^{K} \varepsilon_i$$

$$a^T x + b < 0, x \in \bigcup_{i=K+1}^{K+L} \varepsilon_i$$

let $\epsilon > 0$, a constant for strict separation

relax inequalities to:

$$a^T x + b \leq -\epsilon, x \in \bigcup_{i=1}^{K} \varepsilon_i$$

$$a^T x + b \geq \epsilon, x \in \bigcup_{i=K+1}^{K+L} \varepsilon_i$$

$$a^T (P_i u + q_i) + b \leq -\epsilon, \|u\|_2 \leq 1, i \in \{1, .., K\} \ \textit{becomes}$$

$$\sup_{\|u\|_2 \leq 1} a^T P_i u + a^T q_i + b \leq -\epsilon, i \in \{1, .., K\}$$

$$\sup_{\|u\|_2 \leq 1} a^T P_i u = \frac{a^T P_i (a^T P_i)^T}{\|a^T P_i\|_2} = \|a^T P_i\|_2$$

$$\|a^T P_i\|_2 + a^T q_i + b \leq -\epsilon, i \in \{1, .., K\}$$

$$\|a^T P_i\|_2 \leq -a^T q_i - b - \epsilon, i \in \{1, .., K\}$$

$$a^T (P_i u + q_i) + b \geq \epsilon, \|u\|_2 \leq 1, i = \{K + 1, .., K + L\} \ \textit{becomes}$$

$$\inf_{\|u\|_2 \leq 1} a^T P_i u + a^T q_i + b \geq \epsilon, i = \{K + 1, .., K + L\}$$

$$\inf_{\|u\|_2 \leq 1} a^T P_i u = \frac{a^T P_i (-a^T P_i)^T}{\|a^T P_i\|_2} = -\|a^T P_i\|_2$$

$$-\|a^T P_i\|_2 + a^T q_i + b \geq \epsilon, i = \{K + 1, .., K + L\}$$

$$\|a^T P_i\|_2 \leq a^T q_i + b - \epsilon, i = \{K + 1, .., K + L\}$$

Second Order Cone Programming formulation:

$$\begin{aligned}
&\min_{a,b} \ 0 \\
&\|a^T P_i\|_2 \leq -q_i^T a - b - \epsilon, i \in \{1, .., K\} \\
&\|a^T P_i\|_2 \leq q_i^T a + b - \epsilon, i \in \{K + 1, .., K + L\} \\
&\text{where } \epsilon > 0
\end{aligned}$$

5. Q 4.30 textbook
   Express as Geometric Programming:

$$
\begin{aligned}
&T \in [T_{min}, T_{max}], T_{min}, T_{max} > 0 \\
&r \in [r_{min}, r_{max}], r_{min}, r_{max} > 0 \\
&w \in [w_{min}, w_{max}], w_{min}, w_{max} > 0 \\
&w \leq 0.1r \\
&T > 0, r > 0, w > 0, c_{max} > 0 \\
&\max\ \alpha_4 T r^2,\ \textit{subject to}: \\
&\alpha_1 \frac{Tr}{w} + \alpha_2 r + \alpha_3 rw \leq c_{max} \\
&\alpha_i > 0, \forall i
\end{aligned}
$$

$$
\max\ \alpha_4 T r^2 = \min\ \frac{1}{\alpha_4 T r^2} = \min\ \frac{1}{\alpha_4} T^{-1} r^{-2}
$$

$$
\frac{\alpha_1}{c_{max}} T r w^{-1} + \frac{\alpha_2}{c_{max}} r + \frac{\alpha_3}{c_{max}} r w \leq 1
$$

$$
\begin{aligned}
T \leq T_{max} &\iff \frac{T}{T_{max}} \leq 1 \\
T \geq T_{min} &\iff \frac{T_{min}}{T} \leq 1 \\
r \leq r_{max} &\iff \frac{r}{r_{max}} \leq 1 \\
r \geq r_{min} &\iff \frac{r_{min}}{r} \leq 1 \\
w \leq w_{max} &\iff \frac{w}{w_{max}} \leq 1 \\
w \geq w_{min} &\iff \frac{w_{min}}{w} \leq 1 \\
w \leq 0.1r &\iff \frac{10w}{r} \leq 1
\end{aligned}
$$

GP formulation:

$$
\begin{aligned}
&\min_{T,r,w}\ \frac{1}{\alpha_4} T^{-1} r^{-2}, \textit{subject to}: \\
&\frac{T}{T_{max}} \leq 1, \frac{T_{min}}{T} \leq 1 \\
&\frac{r}{r_{max}} \leq 1, \frac{r_{min}}{r} \leq 1 \\
&\frac{w}{w_{max}} \leq 1, \frac{w_{min}}{w} \leq 1 \\
&\frac{10w}{r} \leq 1 \\
&\frac{\alpha_1}{c_{max}} T r w^{-1} + \frac{\alpha_2}{c_{max}} r + \frac{\alpha_3}{c_{max}} r w \leq 1
\end{aligned}
$$

6. Q 4.43(a-b) textbook
   $A : R^n \rightarrow S^m, A(x) = A_0 + x_1 A_1 + .. + x_n A_n$
   Let $\lambda_1(x) \geq ..\lambda_m(x)$ denote the eigenvalues of $A(x)$.
   Formulate problems as SDP.

   - (a) Minimize the maximum eigenvalue
     SDP formulation:

$$\min_{a,x} a, \ \textit{subject to} :$$
$$A(x) \leq_{S^m_+} aI$$

   - (b) Minimize the spread of the eigenvalues

$$\textit{let } t = \lambda_1(x) - \lambda_m(x)$$
$$\min_{t,x,a,b} t, \ \textit{subject to} :$$
$$A(x) \leq_{S^m_+} bI$$
$$A(x) \geq_{S^m_+} aI$$
$$t - b + a = 0$$

7. Formulate $\min_{x \in \mathbb{R}^n, z \in \mathbb{R}} \sum_{m=1}^{M} max(a_m^T x, z) + \tau \|x\|_2^2$ as LP/QP/SOCP/SDP.

$$\begin{aligned} let\ t_m = max(a_m^T x, z) \\ \min_{x \in \mathbb{R}^n, z \in \mathbb{R}, t \in \mathbb{R}^m} 1_{m\times 1}^T t_m + \tau \|x\|_2^2,\ subject\ to: \\ a_m^T x \leq t_m, \forall m \\ z \leq t_m, \forall m \end{aligned}$$

Quadratic Programming formulation:

$$X \in \mathbb{R}^{m+n+1} = \begin{bmatrix} t_{1:m}, & x_{1:n}, & z \end{bmatrix}$$

$$\min_X X^T \begin{bmatrix} & 0_{m\times(m+n+1)} & \\ 0_{n\times m} & \tau I_{n\times n} & 0_{n\times 1} \\ & 0_{m\times(m+n+1)} & \end{bmatrix} X + \begin{bmatrix} 1_{1\times m} & 0_{1\times(n+1)} \end{bmatrix} X,\ subject\ to:$$

$$\begin{bmatrix} -I_{m\times m} & A & 0_{m\times 1} \\ -I_{m\times m} & 0_{m\times 5} & 1_{m\times 1} \end{bmatrix} X \leq 0_{2m\times 1}, A = \begin{bmatrix} a_1^T \\ .. \\ a_m^T \end{bmatrix}$$

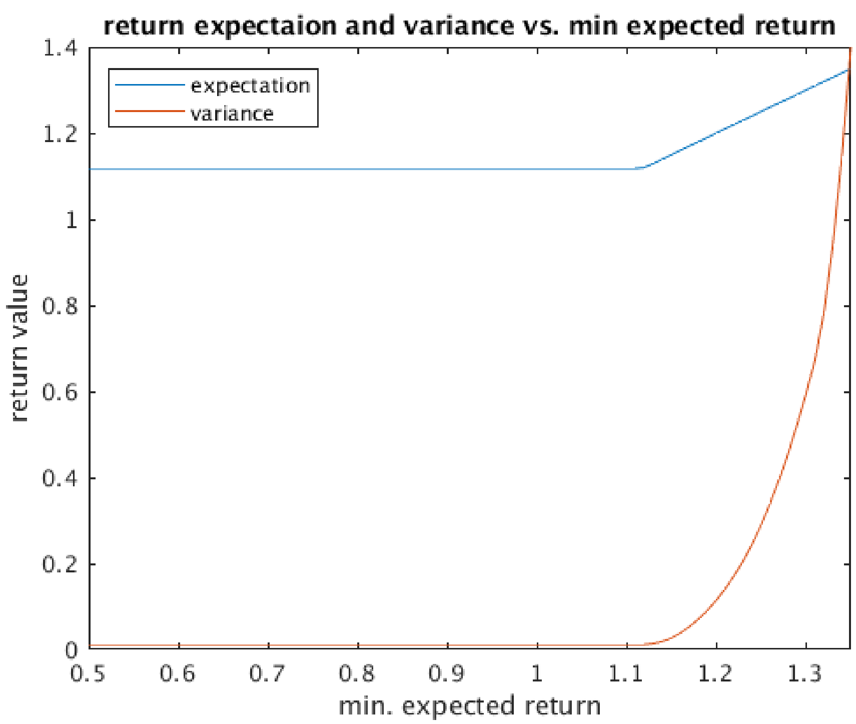
8. Portfolio Design

$$
\begin{aligned}
E[x^T p] &= \bar{x}p \\
E[(x^T p - \bar{x}^T p)(x^T p - \bar{x}^T p)^T] &= E[(p^T(x-\bar{x})(p^T(x-\bar{x})^T] \\
&= p^T E[(x-\bar{x})(x-\bar{x})^T]p \\
&= p^T \Sigma p
\end{aligned}
$$

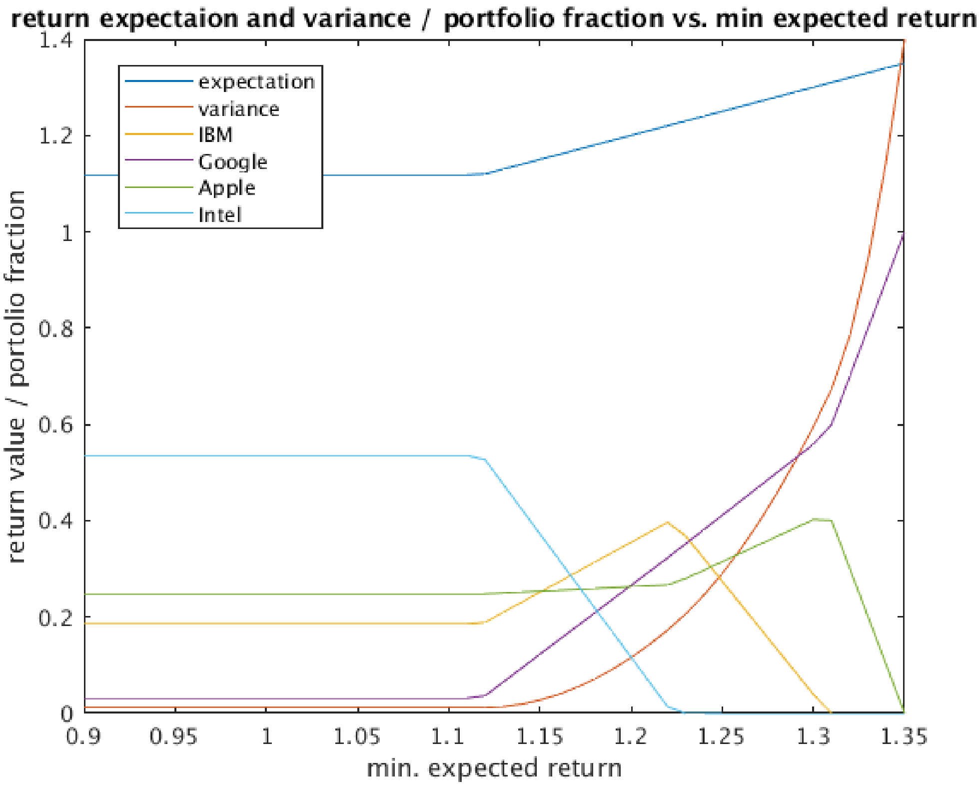
Quadratic Programming formulation:

$$
\begin{aligned}
\min_p p^T \Sigma p,\ \ subject\ to: \\
1^T p = 1 \\
-x^T p \leq -r \\
Diag(1_{4\times 1})p \leq 1 \\
-Diag(1_{4\times 1})p \leq 0
\end{aligned}
$$

Solving for a range of r yields:



Portfolio for a range of r:



Solver:

```
function [rs, s] = solve()
    x = [1.1, 1.35, 1.25, 1.05];
    H = [ 0.2 -0.2 -0.12 0.02;...
          -0.2 1.4 0.02 0.0;...
          -0.12 0.02 1 -0.4;...
           0.02 0 -0.4 0.2];

    Aeq = ones(1,4);
    beq = 1;
    A = -x;
    rs = 0.5:0.01:1.35
    lb = zeros(4,1);
    ub = ones(4,1);
    s = arrayfun(@(b) quadprog(H,[],A,-b,Aeq,beq,lb,ub),rs,...
        'UniformOutput',false);
end
```

```
[rs, ret] = solve();

ps = cell2mat(ret)';
x = [1.1, 1.35, 1.25, 1.05]';
H = [ 0.2 -0.2 -0.12 0.02;...
      -0.2 1.4 0.02 0.0;...
      -0.12 0.02 1 -0.4;...
       0.02 0 -0.4 0.2];
ms = ps * x;
vs = diag(ps * H * ps');

plot(rs,ms);
hold on;
plot(rs,vs);
title('return expectaion and variance vs. min expected return');
legend('expectation','variance');
xlabel('min. expected return');
ylabel('return value');
hold off;

%% additional plot for part b
plot(rs,ms);
hold on;
plot(rs,vs);
title('return expectaion and variance / portfolio fraction vs. min expected return'
xlabel('min. expected return');
ylabel('return value / portolio fraction');
plot(rs,ps(:,1));
plot(rs,ps(:,2));
plot(rs,ps(:,3));
plot(rs,ps(:,4));
legend('expectation','variance', 'IBM', 'Google', 'Apple', 'Intel');
hold off;
```

9. Optimal Control of a Unit Mass

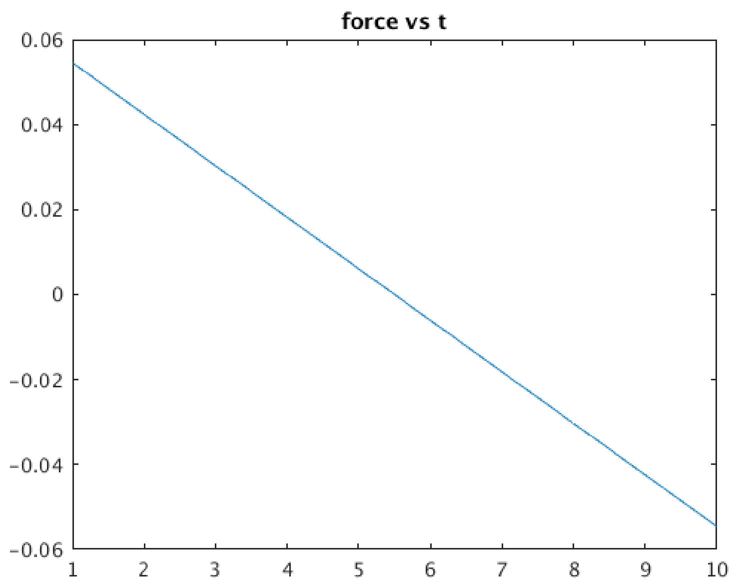
- a) given:

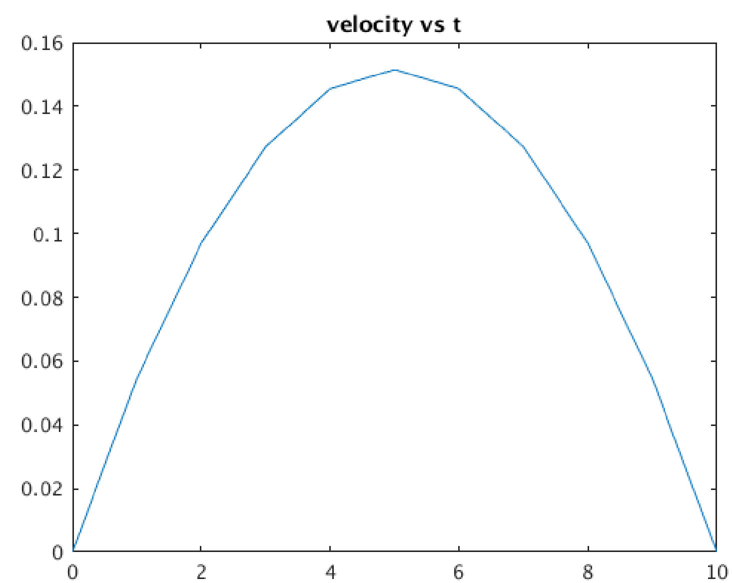
$$x(0) = 0$$
$$\dot{x}(0) = 0$$
$$f(t) = p_i, i-1 < t \le i, i = 1, .., 10$$
$$x(10) = 1$$
$$\dot{x}(10) = 0$$
$$minimize \sum_{i=1}^{10} p_i^2$$

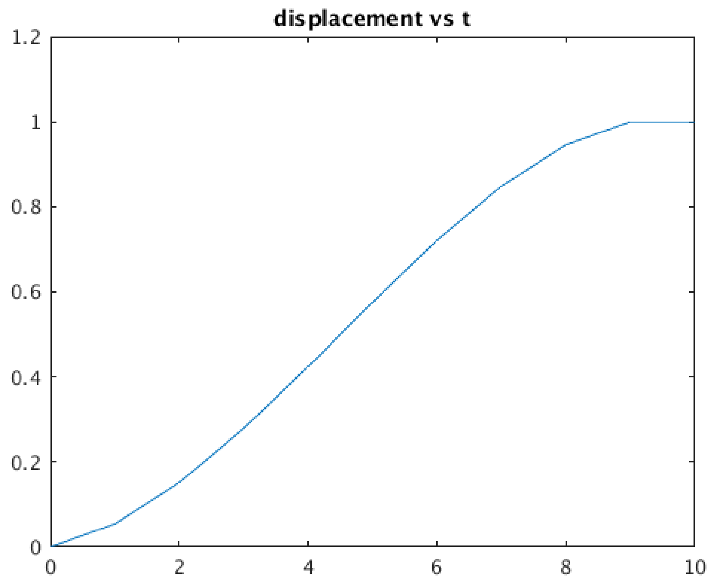
$$let\ v = \dot{x}$$
$$v(i) = v(0) + \sum_{j=0}^{i} \frac{f(j)}{m} = v(0) + \sum_{j=1}^{i} p_j, m = 1$$
$$v(i) = \sum_{j=1}^{i} p_j = 1^T p, p \in \mathbb{R}^i$$
$$v(1) = p_1$$
$$v(2) = p_1 + p_2$$
$$...$$
$$v(10) = 1^T p = v_{des} = 0,$$
$$x(i) = x(0) + \sum_{j=0}^{i} v(j)$$
$$x(i) = \sum_{j=1}^{i} v(j) = 1^T v, v \in \mathbb{R}^i$$
$$x(i) = \sum_{j=1}^{i} \sum_{k=1}^{j} p_k$$
$$x(i) = \sum_{j=1}^{i} (i-j+1) p_j$$
$$x(1) = p_1$$
$$x(2) = 2p_1 + 1p_2$$
$$x(10) = 10p_1 + 9p_2 + .. + 1p_{10}$$

QP formulation:

$$\min_f f^T I f, \; subject \; to:$$
$$\begin{bmatrix} & & 1^T & & \\ 10 & 9 & .. & 2 & 1 \end{bmatrix} f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$







Optimal strategy in this case is to apply a smooth symmetrical force around t = 5, so that velocity is always non-negative and displacement is always towards the destination.

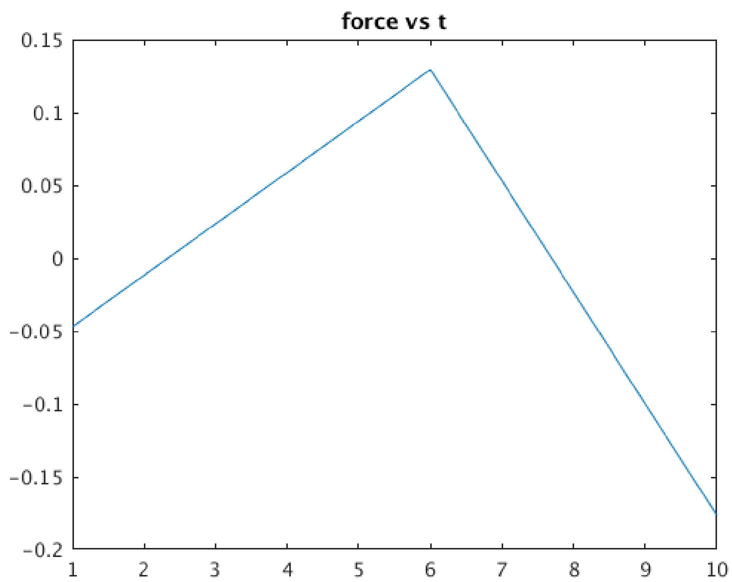
Solver:

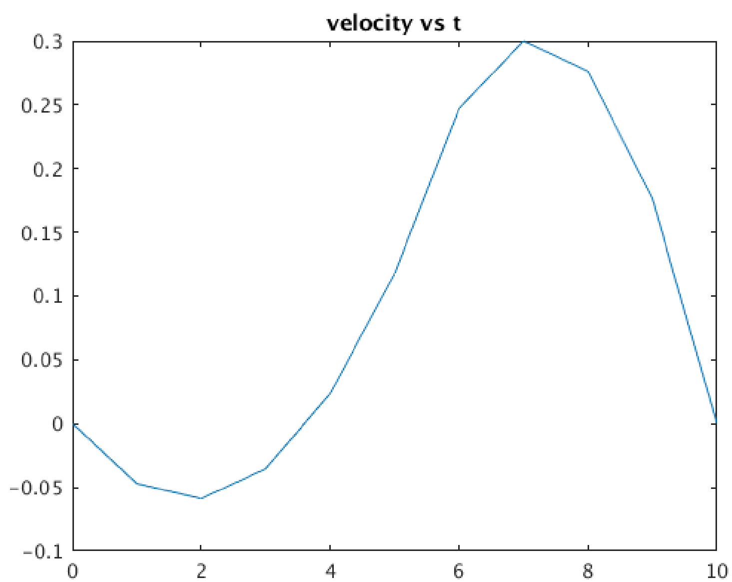
```
H = diag(ones(10,1))
f = []
A = []
b = []
Aeq = [ones(1,10); 10:-1:1]
beq = [0;1]
f = quadprog(H,f,A,b,Aeq,beq)

plot(1:10,f)
title('force vs t')

v = zeros(11,1)
for i=2:1:11
    v(i) = v(i-1) + f(i-1)
end

plot(0:1:10,v)
title('velocity vs t')

x = zeros(11,1)
for i=1:1:11
```

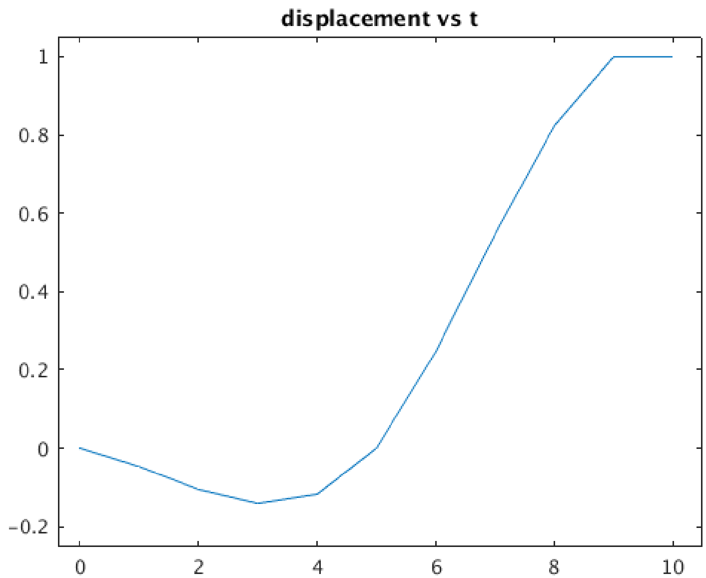
```
    if i > 1
        x(i) = x(i-1) + v(i)
    else
        x(i) = v(i)
    end
end

plot(0:1:10,x)
title('displacement vs t')
```

- b) additional constraint: $x(5) = 0$

$$\min_f f^T I f, \; subject\ to:$$
$$\begin{bmatrix} & & & & 1^T & & & \\ 10 & 9 & .. & & & & 2 & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 & .. & 0 \end{bmatrix} f = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$







Optimial strategy in this case is to reverse direction and then go forward again to gain enough velocity timed so that the displacement at t=5 is 0.
Solver:

```
H = diag(ones(10,1));
f = []; A = []; b = [];
Aeq = [ones(1,10); 10:-1:1; 5 4 3 2 1 0 0 0 0 0];
beq = [0;1;0];
f = quadprog(H,f,A,b,Aeq,beq);
plot(1:10,f)
title('force vs t')
v = zeros(11,1)
for i=2:1:11
    v(i) = v(i-1) + f(i-1)
end
plot(0:1:10,v)
title('velocity vs t')
x = zeros(11,1)
for i=1:1:11
    if i > 1
        x(i) = x(i-1) + v(i)
    else
        x(i) = v(i)
    end
end
plot(0:1:10,x)
title('displacement vs t')
```

10. Least-Square Deconvolution

- a) Find the deconvolution filter, g of size m, and the best D for the given impulse response, h, of the channel.

  A convolution matrix is constructed for use in QP. Constraints are also made for the convolution value $conv(g,h)[t=D]=1$ and filter to allow non-zero only in the appropriate indices in the system of equations. Then D is allowed to change and we collect feasible solutions for the filter and take the best one based on the lowest loss of the objective.

  Assume 1-indexing as in Matlab

$$
\begin{aligned}
&\text{given: } D \\
X &= \begin{bmatrix} 0_{n\times 1} \\ g_{m-1} \\ .. \\ g_0 \\ 0_{n\times 1} \end{bmatrix} \\
t_0 &= size(A,1)-1 \text{ //this row index corresponds to } conv(g,h)[0] \\
A &= \begin{bmatrix} h_0 & .. & h_{n-1} & 0_{1\times(m+n)} & \\ 0 & h_0 & .. & h_{n-1} & 0_{1\times(m+n-1)} \\ .. & & & & \\ & 0_{1\times(m+n)} & h_0 & .. & h_{n-1} \end{bmatrix} \\
A_2 &= A \\
A_2(t_0+D,:) &= 0 \\
H &= A_2^T A_2 \\
X_i &= 0, \forall i \in \{1,..,n\} \cup \{n+m+1,..,2n+m\} \\
A_{eq} &= A(t_0+D,:)
\end{aligned}
$$

  QP formulation for a given D:

$$
\begin{aligned}
&\min_X X^T H X,\ subject\ to: \\
&A_{eq}X = 1 \\
&X_i = 0, \forall i \in \{1,..,n\} \cup \{n+m+1,..,2n+m\}
\end{aligned}
$$

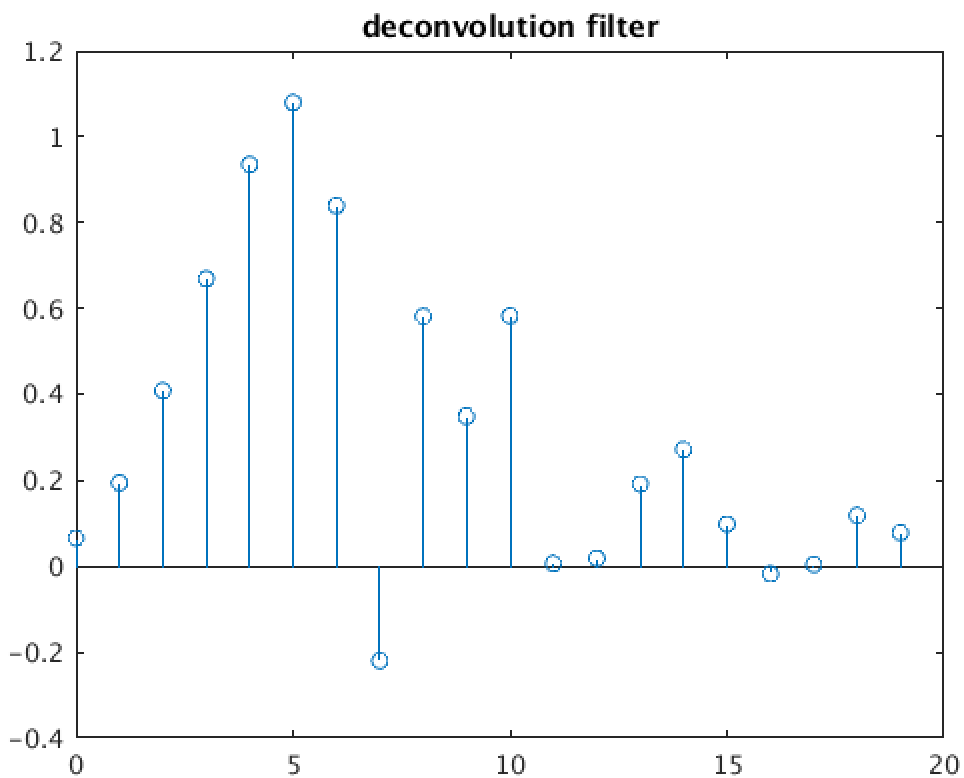
  Vary D in range $[-(m+n-1),1]$ and and solve QP to find best g using lowest $X^T H X$.

Solver:

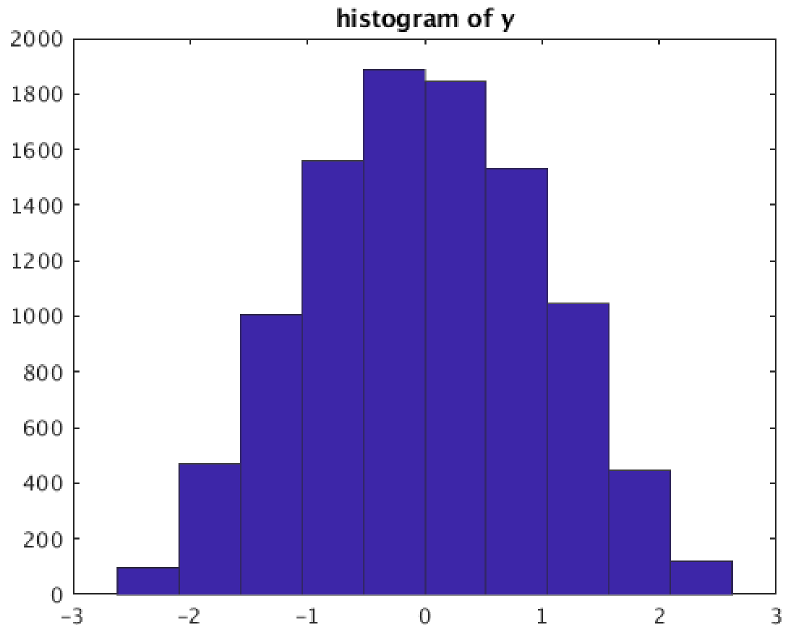
```
n = length(h);
m = 20;

% setup convolution matrix
e = zeros(m+2*n,n);
for i=1:10
    e(:,i) = ones(m+2*n,1) * h(i);
end
A = full(spdiags(e, 0:9, m+n+1, m+2*n));

% row index for conv(f,g)[t=0]
t_0 = size(A,1)-1;

gs = []; losses = []; ds = [];
% solve for different D offset from t_0 using QP
for D=-(m+n-1):1
    fprintf('d: %d\n',-D);
    A2 = [ A(1:t_0+D-1,:);
           zeros(1,m+2*n);
           A(t_0+D+1:end,:) ];
    H = A2'*A2;
    Aeq = A(t_0+D,:);
    beq = 1;
    lb = [zeros(n,1); ones(m,1) * -Inf; zeros(n,1)];
    ub = [zeros(n,1); ones(m,1) * Inf; zeros(n,1)];
    X = quadprog(H,[],[],[],Aeq,beq,lb,ub);
    if length(X) ~= 0 % guard for feasible answer
        g = flip(X(n+1:n+1+m-1)); % flip to get [g_0, ..]
        loss = X'*H*X;
        losses = [ losses; loss];
        gs = [gs; g'];
        ds = [ds; -D];
    end
end

% obtain the best answer for g
[~,idx] = sort(losses(:,1));
losses_sorted = losses(idx,:);
gs_sorted = gs(idx,:);
ds_sorted = ds(idx,:);
g_best = gs_sorted(1,:);
d_best = ds_sorted(1,:);
fprintf("D best: %d\n", d_best);
```

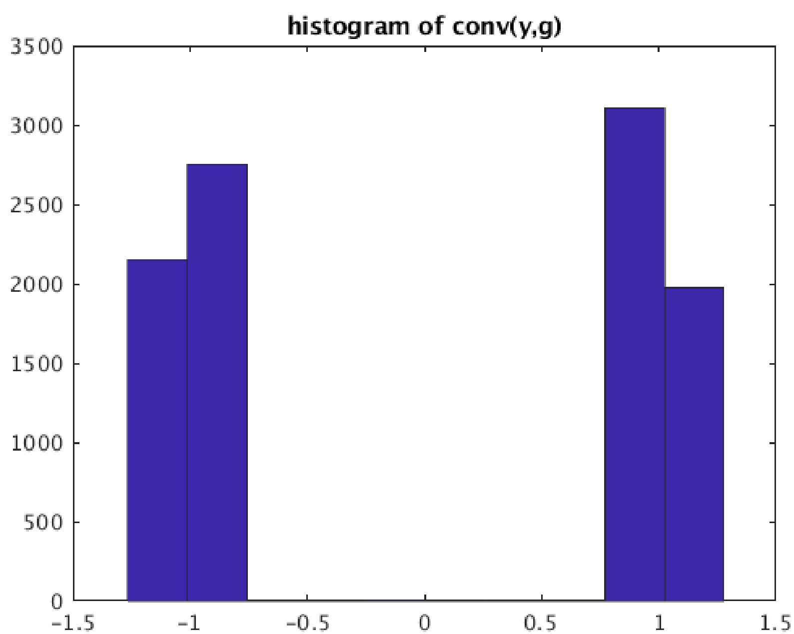
```
temp = conv(g_best,h);
assert(abs(temp(d_best+1)-1.0)<1e-15);
stem([0:length(temp)-1], temp);
title('conv(g,h)');

% verify on arbitrary data
samples = rand(10000,1);
output = conv(g_best, conv(h, samples));
[r,lags] = xcorr(samples,output);
[~,i]=max(r);
l = -lags(i);
assert(l==d_best);
```

Optimal parameters: D=8, g plotted below:



- b) Plot histogram of y (channel output) and z (filtered signal)





We have recovered the original signal. The output distribution is reasonable in that it has binary modes which is what we expect from given knowledge of the original signal input format.