1. Q 4.11(a,c,e) textbook
   (a) Minimize $\|Ax-b\|_\infty$

$$\min \max_i |(Ax-b)_i|$$
$$\text{let } t = max_i|(Ax-b)_i|$$
$$\min_{x,t} t, subject\ to:$$
$$|(Ax-b)_i| \leq t, \forall i$$

LP Formulation:

$$x \in R^n, t \in R$$
$$\min_{x,t} t, subject\ to.:$$
$$A_{i,:}x - t \leq b_i, \forall i$$
$$-A_{i,:}x - t \leq -b_i, \forall i$$

(c) Minimize $\|Ax-b\|_1, \|x\|_\infty \leq 1$

$$\min_x \sum_i |(Ax-b)_i|, subject\ to:$$
$$|x_i| \leq 1, \forall i$$
$$let\ t_i = |(Ax-b)_i|$$
$$\min_{x,t} \sum_i t_i, subject\ to:$$
$$|(Ax-b)_i| \leq t_i, \forall i$$
$$|x_i| \leq 1, \forall i$$

LP Formulation:

$$x \in R^n, t \in R^n$$
$$\min_{x,t} 1^T t, subject\ to:$$
$$A_{i,:}x - t_i \leq b_i$$
$$-A_{i,:}x - t_i \leq -b_i$$
$$x_i \leq 1, \forall i$$
$$-x_i \leq 1, \forall i$$

(e) Minimize $\|Ax-b\|_1 + \|x\|_\infty$

$$\begin{aligned} let\ t_i &= |Ax-b|_i \\ let\ v &= max|x_i| \\ \min_{t,v,x} \sum_i t_i + v, subject\ to: \\ |x_i| &\leq v, \forall i \\ |Ax-b|_i &\leq t_i, \forall i \end{aligned}$$

LP formulation:

$$\begin{aligned} x, t \in R^n, v \in R \\ \min_{t,v,x} 1^T t + v, subject\ to: \\ x_i - v \leq 0, \forall i \\ -x_i - v \leq 0, \forall i \\ A_{i,:}x - t_i \leq +b_i, \forall i \\ -A_{i,:}x - t_i \leq -b_i, \forall i \end{aligned}$$

2. Q 4.16 textbook

$$
\begin{aligned}
x(t) \in \mathbb{R}^n, t \in \{0, .., N\} \\
u(t) \in \mathbb{R}, t \in \{0, .., N\} \\
x(t+1) = Ax(t) + bu(t), t \in \{0, .., N\} \\
given: \\
A \in \mathbb{R}^{n \times n} \\
b \in \mathbb{R}^n \\
x(0) = 0 \\
problem: \\
\min_u \sum_{t=0}^{N-1} f(u(t)), subject\ to: \\
X(N) = x_{des} \\
f(a) = \begin{cases} |a|, & |a| leq 1 \\ 2|a| - 1, & |a| > 1 \end{cases}
\end{aligned}
$$

expanding $x(t)$:

$$
\begin{aligned}
&x(0) = 0 \\
&x(1) = Ax(0) + bu(0) = bu(0) \\
&x(2) = Ax(1) + bu(1) = A(bu(0)) + bu(1) \\
&x(i) = A^{i-1}bu(0) + A^{i-2}bu(1) + ... + A^0bu(i-1) = \sum_{j=0}^{i-1} A^j bu(i-1-j) \\
&x(i) = \begin{bmatrix} A^{i-1}b & A^{i-2}b & .. & ..b \end{bmatrix} \begin{bmatrix} u(0) \\ .. \\ u(i-1) \end{bmatrix} \\
&x_{des} = x(N) = \begin{bmatrix} A^{N-1}b & A^{N-2}b & .. & ..b \end{bmatrix} \begin{bmatrix} u(0) \\ .. \\ u(N-1) \end{bmatrix} \\
&let\ v_t = |f(x(t))| \\
&\quad v_t = max\{|u(t)|, 2|u(t)| - 1\} \\
&\quad v_t \geq |u(t)| \\
&\quad v_t \geq 2|u(t)| - 1 \\
&\quad v_t \geq u_t \\
&\quad v_t \geq -u_t \\
&\quad v_t \geq 2u_t - 1 \\
&\quad v_t \geq -2u_t - 1
\end{aligned}
$$

LP formulation:

$$
\begin{aligned}
& v_{0,..,N-1}, u_{0,..,N-1} \in \mathbb{R}^N \\
& \min_{v,u} 1^T v, subject\ to: \\
& -v_t + u_t \leq 0, \forall t \in 0,..,N-1 \\
& -v_t - u_t \leq 0, \forall t \in 0,..,N-1 \\
& -v_t + 2u_t \leq 1, \forall t \in 0,..,N-1 \\
& -v_t - 2u_t \leq 1, \forall t \in 0,..,N-1 \\
& \begin{bmatrix} A^{N-1}b & A^{N-2}b & .. & ..b \end{bmatrix} \begin{bmatrix} u_0 \\ .. \\ u_{N-1} \end{bmatrix} = x_{des}
\end{aligned}
$$

3. Q 4.21(a) textbook
   Find explicit solution for the QCQP:
   minimize $c^T x$, subject to:
   $x^T A x \leq 1$
   $A \in S^n_{++}, c \neq 0$

$$
\begin{aligned}
x^T A x &= x^T A^{1/2} A^{1/2} x = (A^{1/2}x)^T A^{1/2} x = \|A^{1/2}x\|_2^2 \\
\|A^{1/2}x\|_2^2 &\leq 1 \\
let\ y &= A^{1/2}x \\
x &= A^{-1/2}y \\
c^T x &= c^T A^{-1/2} y \\
let\ b^T &= c^T A^{-1/2} \\
\min b^T y\ subject\ to &: \\
\|y\|_2^2 &\leq 1 \\
\min b^T y &= \frac{-b^T b \alpha}{\|b\|}, \alpha = max\|y\|_2 = 1 \\
y^* &= \frac{-b}{\|b\|} \\
x^* &= A^{-1/2} y^* = A^{-1/2} \frac{-b}{\|b\|} \\
&= A^{-1/2} \frac{-(c^T A^{-1/2})^T}{\|(c^T A^{-1/2})^T\|} \\
&= \frac{-A^{-1/2} A^{-1/2} c}{\|A^{-1/2} c\|} = \frac{-A^{-1} c}{(c^T A^{-1} c)^{1/2}} \\
c^T x^* &= \frac{-c^T A^{-1} c}{(c^T A^{-1} c)^{1/2}} \\
c^T x^* &= -(c^T A^{-1} c)^{1/2}
\end{aligned}
$$

4. Q 4.25 textbook

$$\varepsilon_i = \{P_i u + q_i : \|u\|_2 \leq 1\}, i = 1, .., K+L, P_i \in S^n$$

Find a feasible hyperplane strictly separating $\varepsilon_1, .., \varepsilon_K$ from $\varepsilon_{K+1}, .., \varepsilon_{K+L}$.

$$a^T x + b > 0, x \in \bigcup_{i=1}^{K} \varepsilon_i$$
$$a^T x + b < 0, x \in \bigcup_{i=K+1}^{K+L} \varepsilon_i$$

let $\epsilon > 0$, a constant for strict separation

relax inequalities to:

$$a^T x + b \leq -\epsilon, x \in \bigcup_{i=1}^{K} \varepsilon_i$$
$$a^T x + b \geq \epsilon, x \in \bigcup_{i=K+1}^{K+L} \varepsilon_i$$
$$a^T(P_i u + q_i) + b \leq -\epsilon, \|u\|_2 \leq 1, i \in \{1, .., K\} \textit{ becomes}$$
$$\sup_{\|u\|_2 \leq 1} a^T P_i u + a^T q_i + b \leq -\epsilon, i \in \{1, .., K\}$$
$$\sup_{\|u\|_2 \leq 1} a^T P_i u = \frac{a^T P_i (a^T P_i)^T}{\|a^T P_i\|_2} = \|a^T P_i\|_2$$
$$\|a^T P_i\|_2 + a^T q_i + b \leq -\epsilon, i \in \{1, .., K\}$$
$$\|a^T P_i\|_2 \leq -a^T q_i - b - \epsilon, i \in \{1, .., K\}$$

$$a^T(P_i u + q_i) + b \geq \epsilon, \|u\|_2 \leq 1, i = \{K+1, .., K+L\} \textit{ becomes}$$
$$\inf_{\|u\|_2 \leq 1} a^T P_i u + a^T q_i + b \geq \epsilon, i = \{K+1, .., K+L\}$$
$$\inf_{\|u\|_2 \leq 1} a^T P_i u = \frac{a^T P_i (-a^T P_i)^T}{\|a^T P_i\|_2} = -\|a^T P_i\|_2$$
$$-\|a^T P_i\|_2 + a^T q_i + b \geq \epsilon, i = \{K+1, .., K+L\}$$
$$\|a^T P_i\|_2 \leq a^T q_i + b - \epsilon, i = \{K+1, .., K+L\}$$

Second Order Cone Programming formulation:

$$\min_{a,b} \; 0$$
$$\|a^T P_i\|_2 \leq -q_i^T a - b - \epsilon, i \in \{1, .., K\}$$
$$\|a^T P_i\|_2 \leq q_i^T a + b - \epsilon, i \in \{K+1, .., K+L\}$$
where $\epsilon > 0$

5. Q 4.30 textbook
   Express as Geometric Programming:

$$
\begin{aligned}
&T \in [T_{min}, T_{max}], T_{min}, T_{max} > 0 \\
&r \in [r_{min}, r_{max}], r_{min}, r_{max} > 0 \\
&w \in [w_{min}, w_{max}], w_{min}, w_{max} > 0 \\
&w \leq 0.1r \\
&T > 0, r > 0, w > 0, c_{max} > 0 \\
&\max \alpha_4 T r^2,\ subject\ to: \\
&\alpha_1 \frac{Tr}{w} + \alpha_2 r + \alpha_3 r w \leq c_{max} \\
&\alpha_i > 0, \forall i
\end{aligned}
$$

$$
\begin{aligned}
&\max\ \alpha_4 T r^2 = \min\ \frac{1}{\alpha_4 T r^2} = \min\ \frac{1}{\alpha_4} T^{-1} r^{-2} \\
&\frac{\alpha_1}{c_{max}} T r w^{-1} + \frac{\alpha_2}{c_{max}} r + \frac{\alpha_3}{c_{max}} r w \leq 1 \\
&T \leq T_{max} \iff \frac{T}{T_{max}} \leq 1 \\
&T \geq T_{min} \iff \frac{T_{min}}{T} \leq 1 \\
&r \leq r_{max} \iff \frac{r}{r_{max}} \leq 1 \\
&r \geq r_{min} \iff \frac{r_{min}}{r} \leq 1 \\
&w \leq w_{max} \iff \frac{w}{w_{max}} \leq 1 \\
&w \geq w_{min} \iff \frac{w_{min}}{w} \leq 1 \\
&w \leq 0.1r \iff \frac{10w}{r} \leq 1
\end{aligned}
$$

GP formulation:

$$
\begin{aligned}
&\min_{T,r,w} \frac{1}{\alpha_4} T^{-1} r^{-2}, subject\ to: \\
&\frac{T}{T_{max}} \leq 1, \frac{T_{min}}{T} \leq 1 \\
&\frac{r}{r_{max}} \leq 1, \frac{r_{min}}{r} \leq 1 \\
&\frac{w}{w_{max}} \leq 1, \frac{w_{min}}{w} \leq 1 \\
&\frac{10w}{r} \leq 1 \\
&\frac{\alpha_1}{c_{max}} T r w^{-1} + \frac{\alpha_2}{c_{max}} r + \frac{\alpha_3}{c_{max}} r w \leq 1
\end{aligned}
$$

6. Q 4.43(a-b) textbook
   $A : R^n \to S^m, A(x) = A_0 + x_1 A_1 + .. + x_n A_n$
   Let $\lambda_1(x) \geq ..\lambda_m(x)$ denote the eigenvalues of $A(x)$.
   Formulate problems as SDP.

   - (a) Minimize the maximum eigenvalue
     SDP formulation:

$$\min_{a,x} a, \ \ subject \ to :$$
$$A(x) \leq_{S^m_+} aI$$

   - (b) Minimize the spread of the eigenvalues

$$let \ t = \lambda_1(x) - \lambda_m(x)$$
$$\min_{t,x,a,b} t, \ subject \ to :$$
$$A(x) \leq_{S^m_+} bI$$
$$A(x) \geq_{S^m_+} aI$$
$$t - b + a = 0$$

7. Formulate $\min_{x\in\mathbb{R}^n, z\in\mathbb{R}} \sum_{m=1}^{M} max(a_m^T x, z) + \tau\|x\|_2^2$ as LP/QP/SOCP/SDP.

$$\begin{aligned} let\ t_m &= max(a_m^T x, z) \\ \min_{x\in\mathbb{R}^n, z\in\mathbb{R}, t\in\mathbb{R}^m} 1_{m\times 1}^T t_m + \tau\|x\|_2^2,\ subject\ to: \\ a_m^T x \le t_m, \forall m \\ z \le t_m, \forall m \end{aligned}$$

Quadratic Programming formulation:

$$X \in \mathbb{R}^{m+n+1} = \begin{bmatrix} t_{1:m}, & x_{1:n}, & z \end{bmatrix}$$

$$\min_X X^T \begin{bmatrix} & 0_{m\times(m+n+1)} & \\ 0_{n\times m} & \tau I_{n\times n} & 0_{n\times 1} \\ & 0_{m\times(m+n+1)} & \end{bmatrix} X + \begin{bmatrix} 1_{1\times m} & 0_{1\times(n+1)} \end{bmatrix} X,\ subject\ to:$$

$$\begin{bmatrix} -I_{m\times m} & A & 0_{m\times 1} \\ -I_{m\times m} & 0_{m\times 5} & 1_{m\times 1} \end{bmatrix} X \le 0_{2m\times 1}, A = \begin{bmatrix} a_1^T \\ .. \\ a_m^T \end{bmatrix}$$

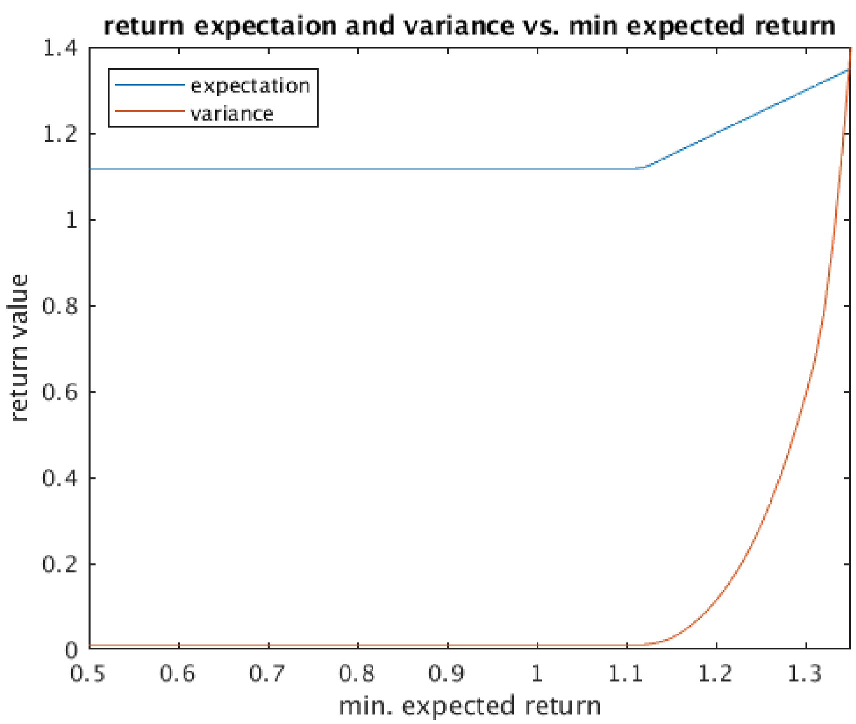
8. Portfolio Design

$$
\begin{aligned}
E[x^T p] &= \bar{x}p \\
E[(x^T p - \bar{x}^T p)(x^T p - \bar{x}^T p)^T] &= E[(p^T(x-\bar{x})(p^T(x-\bar{x})^T] \\
&= p^T E[(x-\bar{x})(x-\bar{x})^T]p \\
&= p^T \Sigma p
\end{aligned}
$$

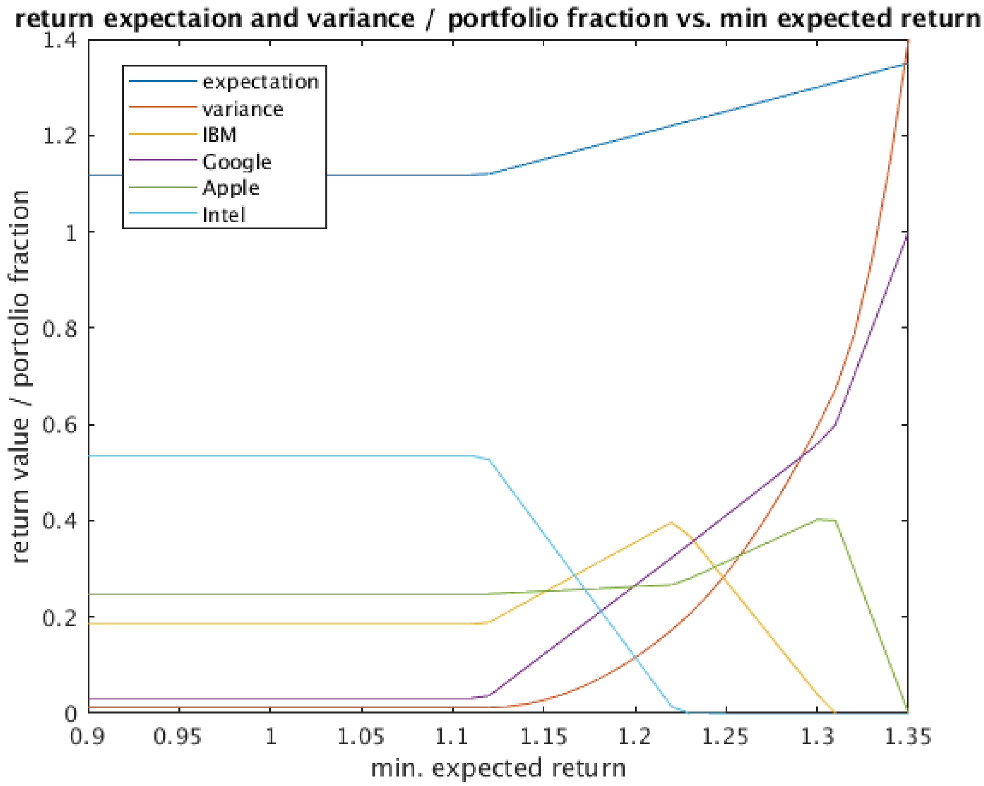
Quadratic Programming formulation:

$$
\begin{aligned}
\min_p p^T \Sigma p,\ subject\ to: \\
1^T p = 1 \\
-x^T p \leq -r \\
Diag(1_{4\times 1})p \leq 1 \\
-Diag(1_{4\times 1})p \leq 0
\end{aligned}
$$

Solving for a range of r yields:



Portfolio for a range of r:



Solver:

```
function [rs, s] = solve()
    x = [1.1, 1.35, 1.25, 1.05];
    H = [ 0.2 -0.2 -0.12 0.02;...
         -0.2 1.4 0.02 0.0;...
         -0.12 0.02 1 -0.4;...
          0.02 0 -0.4 0.2];

    Aeq = ones(1,4);
    beq = 1;
    A = -x;
    rs = 0.5:0.01:1.35
    lb = zeros(4,1);
    ub = ones(4,1);
    s = arrayfun(@(b) quadprog(H,[],A,-b,Aeq,beq,lb,ub),rs,...
        'UniformOutput',false);
end
```

```
[rs, ret] = solve();

ps = cell2mat(ret)';
x = [1.1, 1.35, 1.25, 1.05]';
H = [ 0.2 -0.2 -0.12 0.02;...
      -0.2 1.4 0.02 0.0;...
      -0.12 0.02 1 -0.4;...
       0.02 0 -0.4 0.2];
ms = ps * x;
vs = diag(ps * H * ps');

plot(rs,ms);
hold on;
plot(rs,vs);
title('return expectaion and variance vs. min expected return');
legend('expectation','variance');
xlabel('min. expected return');
ylabel('return value');
hold off;

%% additional plot for part b
plot(rs,ms);
hold on;
plot(rs,vs);
title('return expectaion and variance / portfolio fraction vs. min expected return'
xlabel('min. expected return');
ylabel('return value / portolio fraction');
plot(rs,ps(:,1));
plot(rs,ps(:,2));
plot(rs,ps(:,3));
plot(rs,ps(:,4));
legend('expectation','variance', 'IBM', 'Google', 'Apple', 'Intel');
hold off;
```

9. Optimal Control of a Unit Mass

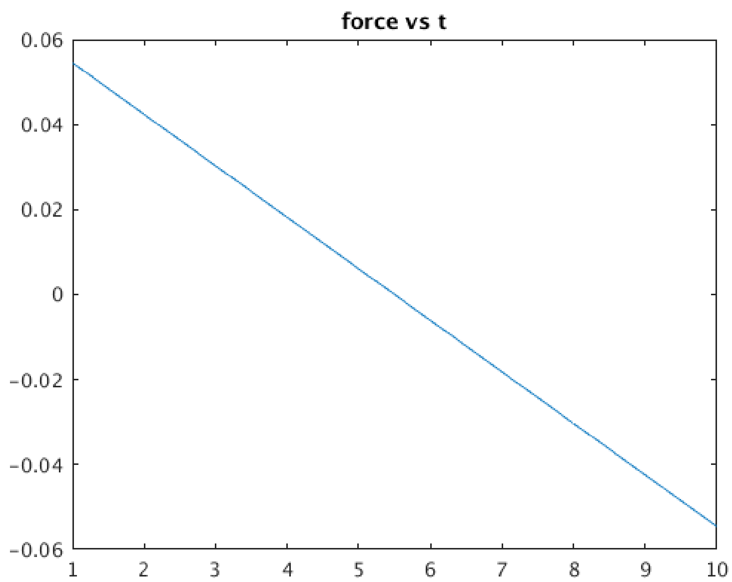
- a) given:

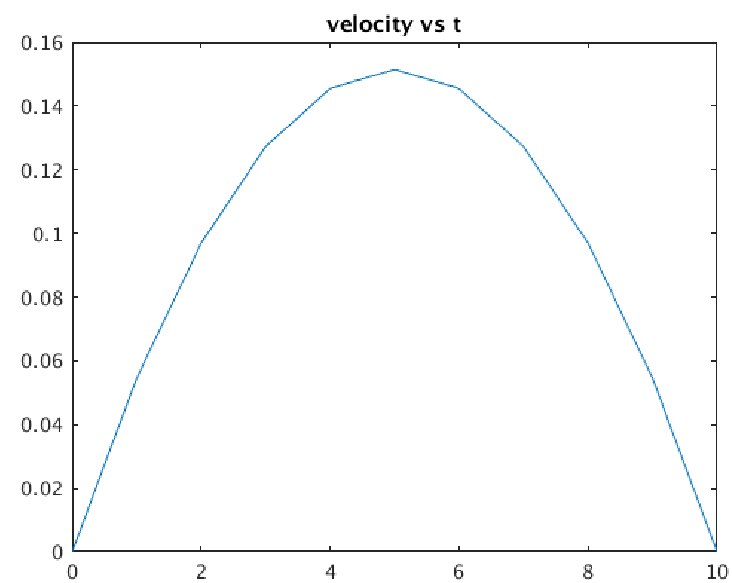
$$
\begin{aligned}
x(0) &= 0 \\
\dot{x}(0) &= 0 \\
f(t) &= p_i, i-1 < t \leq i, i = 1,..,10 \\
x(10) &= 1 \\
\dot{x}(10) &= 0 \\
minimize &\sum_{i=1}^{10} p_i^2
\end{aligned}
$$

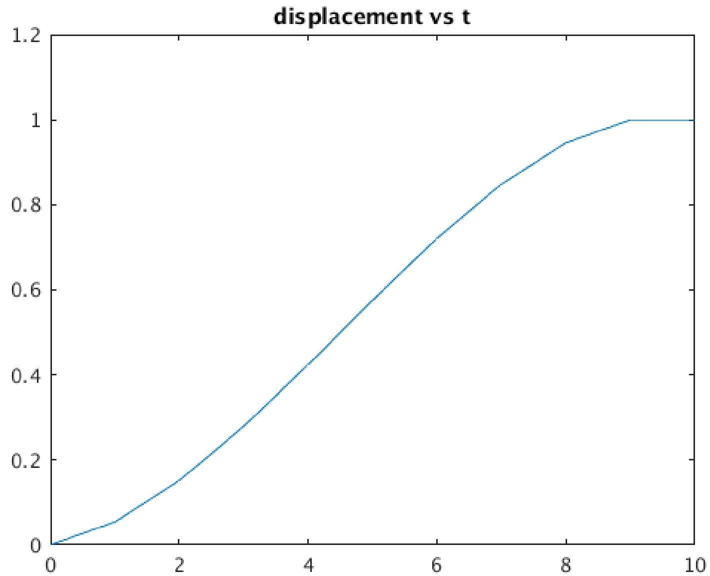
$$
\begin{aligned}
&let\ v = \dot{x} \\
v(i) &= v(0) + \sum_{j=0}^{i} \frac{f(j)}{m} = v(0) + \sum_{j=1}^{i} p_j, m = 1 \\
v(i) &= \sum_{j=1}^{i} p_j = 1^T p, p \in \mathbb{R}^i \\
v(1) &= p_1 \\
v(2) &= p_1 + p_2 \\
&... \\
v(10) &= 1^T p = v_{des} = 0, \\
x(i) &= x(0) + \sum_{j=0}^{i} v(j) \\
x(i) &= \sum_{j=1}^{i} v(j) = 1^T v, v \in \mathbb{R}^i \\
x(i) &= \sum_{j=1}^{i} \sum_{k=1}^{j} p_k \\
x(i) &= \sum_{j=1}^{i} (i-j+1) p_j \\
x(1) &= p_1 \\
x(2) &= 2p_1 + 1p_2 \\
x(10) &= 10p_1 + 9p_2 + .. + 1p_{10}
\end{aligned}
$$

QP formulation:

$$\min_f f^T I f, \ subject\ to:$$
$$\begin{bmatrix} & & 1^T & & \\ 10 & 9 & .. & 2 & 1 \end{bmatrix} f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$







Optimal strategy in this case is to apply a smooth symmetrical force around t = 5, so that velocity is always non-negative and displacement is always towards the destination.

Solver:

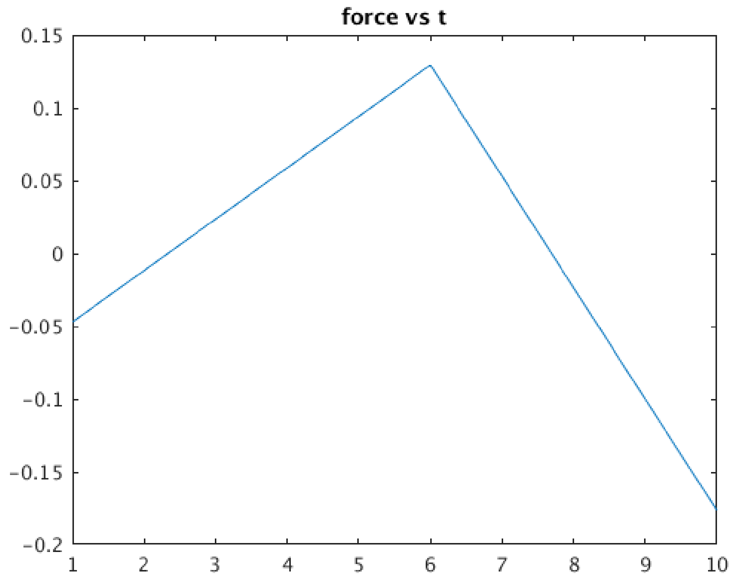
```
H = diag(ones(10,1))
f = []
A = []
b = []
Aeq = [ones(1,10); 10:-1:1]
beq = [0;1]
f = quadprog(H,f,A,b,Aeq,beq)

plot(1:10,f)
title('force vs t')

v = zeros(11,1)
for i=2:1:11
    v(i) = v(i-1) + f(i-1)
end

plot(0:1:10,v)
title('velocity vs t')

x = zeros(11,1)
for i=1:1:11
```

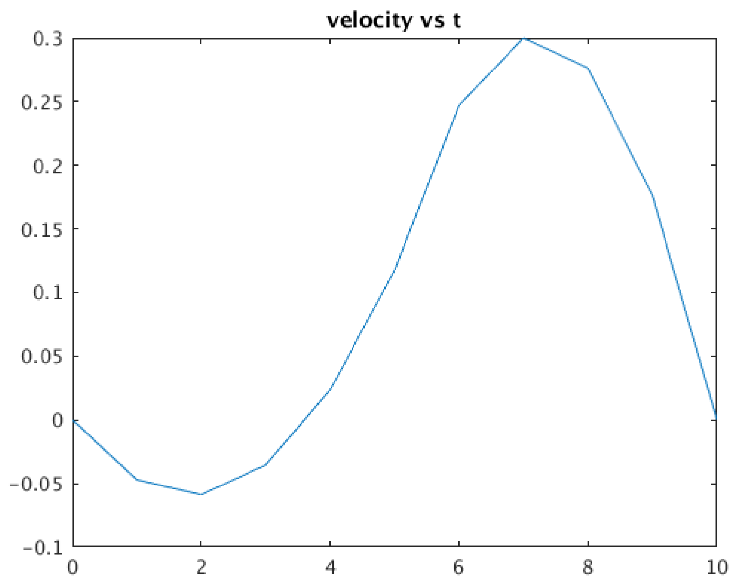
```
    if i > 1
        x(i) = x(i-1) + v(i)
    else
        x(i) = v(i)
    end
end

plot(0:1:10,x)
title('displacement vs t')
```

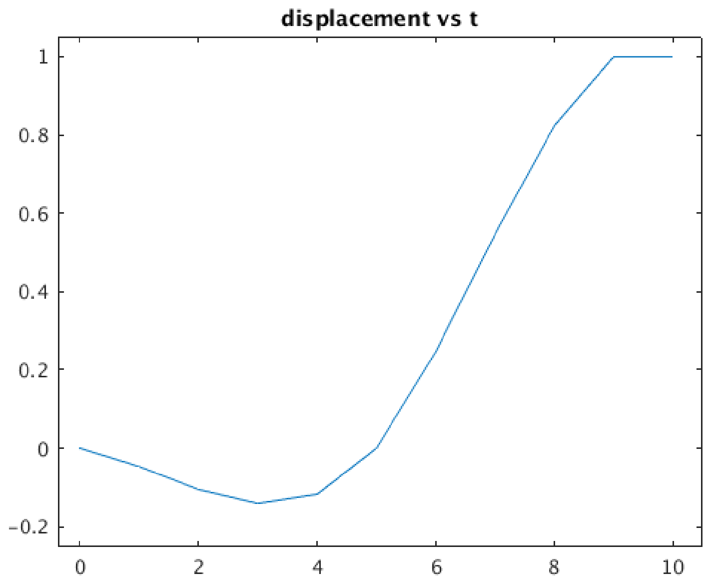
- b) additional constraint: $x(5) = 0$

$$\min_f f^T I f, \; subject\ to:$$

$$\begin{bmatrix} & & & 1^T & & & & \\ 10 & 9 & .. & & & & 2 & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 & .. & 0 \end{bmatrix} f = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$







Optimial strategy in this case is to reverse direction and then go forward again to gain enough velocity timed so that the displacement at t=5 is 0.

Solver:

```
H = diag(ones(10,1))
f = []
A = []
b = []
Aeq = [ones(1,10); 10:-1:1; 5 4 3 2 1 0 0 0 0 0]
beq = [0;1;0]
f = quadprog(H,f,A,b,Aeq,beq)

plot(1:10,f)
title('force vs t')
v = zeros(11,1)
for i=2:1:11
    v(i) = v(i-1) + f(i-1)
end
plot(0:1:10,v)
title('velocity vs t')

x = zeros(11,1)
for i=1:1:11
    if i > 1
        x(i) = x(i-1) + v(i)
    else
```

```
            x(i) = v(i)
        end
    end
    plot(0:1:10,x)
    title('displacement vs t')
```