1. SDP Relaxation and Heuristics for Two-Way Partitioning Problem
   (a) Q 5.39 textbook

$$\begin{aligned} min\ x^TWx \\ s.t.\ x_i^2 = 1, \forall i \in \{1,..,n\} \end{aligned}$$

i. Show that the two-way partitioning problem can be cast as

$$\begin{aligned} min\ tr(WX) \\ s.t.\ X \succeq 0, rank(X) = 1 \\ X_{ii} = 1, \forall i \in \{1,..,n\} \end{aligned}$$

$$\begin{aligned}
&x^TWx = tr(x^TWx) = tr(Wxx^T) \\
&let\ X = xx^T \\
&(\forall i)x_i^2 = 1 \iff x_i = \{-1,1\} \implies x^TIx = n \\
&x^TIx = tr(xx^T) = n \\
&((\exists i)X_{ii} = -1 \wedge (\forall i,j)X_{ij} = \{-1,1\} \implies tr(X) < n) \\
&thus, (\forall i)X_{ii} = 1\ for\ tr(X) = n
\end{aligned}$$

$$\begin{aligned}
&X = xx^T = x\begin{bmatrix} a_1 & a_2 & .. & a_n \end{bmatrix} = \begin{bmatrix} a_1x & a_2x & .. & a_nx \end{bmatrix}, a_i \in \mathbb{R}, x \in \mathbb{R}^n \\
&(\forall i)(\exists j)\beta_{ij}a_ix = a_jx \implies \beta_{ij}a_ix - a_jx = 0 \\
&let\ \gamma_{ij} = \beta_{ij}a_i - a_j \\
&\gamma_{ij}x = 0 \\
&x \neq 0 \implies (\forall i)(\exists j)\gamma_{ij} = 0 \implies linear\ dependence\ between\ column\ vectors\ of\ X \\
&thus,\ rank(X) = 1
\end{aligned}$$

$$\begin{aligned}
&(\forall w)w^TXw = w^Txx^Tw = (x^Tw)^Tx^Tw \\
&(\forall i)(\forall w)(x^Tw)_i(x^Tw)_i \geq 0 \implies (\forall w)(x^Tw)^T(x^Tw) \geq 0 \iff X\ is\ SPD
\end{aligned}$$

$Combining\ all\ constraints\ and\ objective\ forms\ the\ desired\ result$

ii. SDP relaxation of two-way partitioning problem. Using the formulation in part (a), we can form the relaxation:

$$\begin{aligned} min tr(WX) \\ s.t.\ X \succeq 0 \\ X_{ii} = 1, \forall i \in \{1, .., n\} \end{aligned}$$

This problem is an SDP, and therefore can be solved efficiently. Explain why its optimal value gives a lower bound on the optimal value of the two-way partitioning problem (5.113). What can you say if an optimal point X^* for this SDP has rank one?

$$X \succeq 0 \wedge X_{ii} = 1, \forall i \in \{1, .., n\} \implies (\forall i)\lambda_i(X) \geq 1$$

(b) Q 11.23(b-d) textbook

2. Interior Point Method