1. SDP Relaxation and Heuristics for Two-Way Partitioning Problem

   (a) Q 5.39 textbook

$$\begin{aligned} min\ x^TWx \\ s.t.\ x_i^2 = 1, \forall i \in \{1,..,n\} \end{aligned}$$

      i. Show that the two-way partitioning problem can be cast as

$$\begin{aligned} min\ tr(WX) \\ s.t.\ X \succeq 0, rank(X) = 1 \\ X_{ii} = 1, \forall i \in \{1,..,n\} \end{aligned}$$

$x^TWx = tr(x^TWx) = tr(Wxx^T)$
$let\ X = xx^T$
$(\forall i)x_i^2 = 1 \iff x_i = \{-1, 1\} \implies x^TIx = n$
$x^TIx = tr(xx^T) = n$
$(\forall i, j)X_{ij} = \{-1, 1\}$
$((\exists i)X_{ii} = -1 \implies tr(X) < n)$
$thus, for\ tr(X) = n : (\forall i)X_{ii} = 1$

$X = xx^T = x\begin{bmatrix} a_1 & a_2 & .. & a_n \end{bmatrix} = \begin{bmatrix} a_1x & a_2x & .. & a_nx \end{bmatrix}, a_i \in \mathbb{R}, x \in \mathbb{R}^n$
$(\forall i)(\exists j)\beta_{ij}a_ix = a_jx \implies \beta_{ij}a_ix - a_jx = 0$
$let\ \gamma_{ij} = \beta_{ij}a_i - a_j$
$\gamma_{ij}x = 0$
$x \neq 0 \implies ((\forall i)(\exists j)\gamma_{ij} = 0 \implies linear\ dependence\ between\ all\ column\ vectors\ of\ X)$
$thus,\ rank(X) = 1$

$(\forall w)w^TXw = w^Txx^Tw = (x^Tw)^Tx^Tw$
$(\forall i, w)(x^Tw)_i(x^Tw)_i \geq 0 \implies (\forall w)(x^Tw)^T(x^Tw) \geq 0 \iff X\ is\ SPD$

Combining all constraints and objective forms the desired result.

ii. SDP relaxation of two-way partitioning problem. Using the formulation in part (a), we can form the relaxation:

$$\begin{aligned} min\ tr(WX) \\ s.t.\ X \succeq 0 \\ X_{ii} = 1, \forall i \in \{1, .., n\} \end{aligned}$$

This problem is an SDP, and therefore can be solved efficiently. Explain why its optimal value gives a lower bound on the optimal value of the two-way partitioning problem (5.113). What can you say if an optimal point X^* for this SDP has rank one?

$$\begin{aligned} &L(X, Z, v) = tr(WX) - tr(XZ) + tr(diag(v)diag(X) - I) \\ &L(X, Z, v) = -tr(diag(v)I) + tr(WX) - tr(XZ) + tr(diag(v)diag(X)) \\ &g(Z, V) = \begin{cases} -1^T v, & W - Z + diag(v) \succeq 0 \\ -\infty, & o/w \end{cases} \\ &dual\ problem: \\ &\max_{Z,v} -1^T v \\ &s.t.\ W - Z + diag(v) \succeq 0 \\ &Z \succeq 0 \end{aligned}$$

If an optimal point X^* for the relaxed problem has rank one:
X^* has minimal possible rank and $X^* \neq 0$, $X^* \succeq 0$.
$X^* \succeq 0$, so primal feasible.
Functions are all differentiable, KKT conditions apply at optimality where there exists a dual solution.
Dual of relaxed problem is feasible: $W + diag(v) \succeq Z, Z \succeq 0$
Using complementary slackness: $X^* \neq 0, -tr(X^*Z^*) = 0 \implies Z^* = 0$.

Dual problem of relaxed problem at optimality:

$$\begin{aligned} &\max_{v,Z} -1^T v = [\max_{v}\ -1^T v]_{Z=Z^*} \\ &s.t.\ W + diag(v) \succeq Z^*, Z^* = 0 \implies \\ &s.t.\ W + diag(v) \succeq 0 \end{aligned}$$

This is equivalent to dual of the original problem, then X^* obtains same solution as the original problem where $x^* x^{*T} = X^*$.

iii. We now have two SDPs that give a lower bound on the optimal value of the two-way partitioning problem (5.113): the SDP relaxation (5.115) found in part (b), and the Lagrange dual of the two-way partitioning problem, given in (5.114). What is the relation between the two SDPs? What can you say about the lower bounds found by them? Hint: Relate the two SDPs via duality.

$$\begin{aligned}(5.115)\ & min\ tr(WX)\\ & s.t.\ X \succeq 0\\ & X_{ii} = 1, \forall i \in \{1, .., n\}\end{aligned}$$

$$\begin{aligned}(5.114)\ & maximize\ -1^T v\\ & s.t.\ W + diag(v) \succeq 0\end{aligned}$$

Taken from previous section, dual of relaxed problem:

$$L(X, Z, v) = tr(WX) - tr(XZ) + tr(diag(v)diag(X) - I)$$
$$L(X, Z, v) = -tr(diag(v)I) + tr(WX) - tr(XZ) + tr(diag(v)diag(X))$$
$$g(Z, V) = \begin{cases} -1^T v, & W - Z + diag(v) \succeq 0 \\ -\infty, & o/w \end{cases}$$

Dual problem:

$$\begin{aligned}&\max_{Z,v} -1^T v\\ &s.t.\ W - Z + diag(v) \succeq 0\\ &Z \succeq 0\end{aligned}$$

It is evident that solution to the dual of relaxed problem has a tightened generalized inequality constraint due to Z compared to (5.114):

$$W + diag(v) \succeq Z$$

This leads to potentially bigger v and hence potentially larger objective value of dual maximization problem (smaller objective value to the primal minimization problem). Thus, solution of relaxed problem provides a lower bound to the the original problem.

(b) Q 11.23(b-d) textbook

2. Interior Point Method