1. Q 5.12 textbook
   Derive dual problem for: $min_x - \sum_i log(b_i - a_i^T x), x : a_i^T x \leq b_i, \forall i \in \{1, .., m\}$

$$
\begin{aligned}
&y_i = b_i - a_i^T x \\
&a_i^T x \leq b_i \\
&L(x, y, \lambda, v) = -\sum_i logy_i + \sum_i \lambda_i(a_i^T x - b_i) + \sum_i v_i(y_i + a_i^T x - b_i) \\
&g(\lambda, v) = \inf_{x,y} L(x, y, \lambda, v) \\
&g(\lambda, v) = \inf_{x,y} -\sum_i logy_i + \sum_i \lambda_i(a_i^T x - b_i) + \sum_i v_i(y_i + a_i^T x - b_i) \\
&(\exists i)\lambda_i \neq 0 \implies \lambda_i(a_i^T x - b_i)\ unbounded,\ so\ \lambda = 0 \\
&g(\lambda, v) = \inf_{x,y} -\sum_i logy_i + \sum_i v_i(y_i + a_i^T x - b_i) \\
&g(\lambda, v) = \inf_{x,y} -\sum_i logy_i + v^T y + v^T Ax - v^T b \\
&(\exists i)v_i < 0 \implies v^T y\ unbounded,\ so\ v \succeq 0 \\
&\frac{\partial}{\partial y_i} -\sum_i logy_i + v^T y + v^T Ax - v^T b = -\frac{1}{y_i} + v_i = 0 \\
&y_i = \frac{1}{v_i}, v_i \neq 0 \\
&(\forall i)v_i \neq 0 \wedge v_i \geq 0 \implies v \succ 0 \\
&\frac{\partial}{\partial x} -\sum_i logy_i + v^T y + v^T Ax - v^T b = A^T v = 0 \\
&g(\lambda, v) = \begin{cases} -\sum_i log\frac{1}{v_i} + \sum_i \frac{v_i}{v_i} - v^T b, & \text{if } A^T v = 0 \wedge v \succ 0 \\ -\infty, & \text{otherwise} \end{cases}
\end{aligned}
$$

Dual problem:

$$
\begin{aligned}
&\max_{\lambda,v} \sum_i logv_i + m - v^T b = -(\min_{\lambda,v} -\sum_i logv_i - m + v^T b) \\
&s.t.\ A^T v = 0 \\
&\qquad -v_i \leq 0, \forall i
\end{aligned}
$$

2. Q 5.27 Equality constrained least squares
   Give KKT conditions, derive expressions for primal and dual solutions.

$$\min_x \|Ax - b\|_2^2$$
$$s.t.\ Gx = h$$

$f_0 = x^T A^T Ax + 2b^T Ax + b^T b$
$h_0 = Gx - h$
$L(x, \lambda, v) = f_0 + v^T h_0$
$L(x, \lambda, v) = x^T A^T Ax - 2b^T Ax + b^T b + v^T (Gx - h)$
$\dfrac{\partial L}{\partial x^*} = 0 = 2A^T Ax^* - 2A^T b + G^T v$

$KKT\ conditions:$
$x^* = \dfrac{1}{2}(A^T A)^{-1}(2A^T b - G^T v)$
$Gx^* - h = 0$

$g(\lambda, v) = \min_x L(x, \lambda, v) = \dfrac{1}{4}(2A^T b - G^T v)^T (A^T A)^{-1}(2A^T b - G^T v)$
$- 2b^T A(\dfrac{1}{2}(A^T A)^{-1}(2A^T b - G^T v)) + b^T b + v^T (G\dfrac{1}{2}(A^T A)^{-1}(2A^T b - G^T v) - h)$
$g(\lambda, v) = \min_x L(x, \lambda, v) = \dfrac{1}{4}(2A^T b)^T (A^T A)^{-1}(2A^T b) - (A^T b)^T (A^T A)^{-1}(G^T v)$
$+ \dfrac{1}{4}(G^T v)^T (A^T A)^{-1}(G^T v) + b^T A((A^T A)^{-1} G^T v)) - h^T v + v^T G\dfrac{1}{2}(A^T A)^{-1}(2A^T b - G^T v)$
$- b^T A(A^T A)^{-1} 2A^T b) + b^T b$
rid of constants and simplify :
$g(\lambda, v) = \min_x L(x, \lambda, v) = -\dfrac{1}{4}(G^T v - 2A^T b)^T (A^T A)^{-1}(G^T v - 2A^T b)$
$- \dfrac{1}{2}(Gv^T)^T (A^T A)^{-1}(G^T v) - h^T v$

Dual problem:

$$\max_{\lambda, v} g(\lambda, v) = \max_v -\frac{1}{4}(G^T v - 2A^T b)^T (A^T A)^{-1}(G^T v - 2A^T b)$$
$$-\frac{1}{2}(Gv^T)^T (A^T A)^{-1}(G^T v) - h^T v$$
$$s, t.\ Gx^* - h = 0$$

Solve for v^*:

$$\begin{aligned}
&Gx^* - h = 0 \\
&x^* = \frac{1}{2}(A^T A)^{-1}(2A^T b - G^T v^*) \\
&G\frac{1}{2}(A^T A)^{-1}(2A^T b - G^T v^*) - h = 0 \\
&v^* = 2G^{-T}(A^T b - A^T A G^{-1} h)
\end{aligned}$$

3. Q 5.35 Sensitivity analysis of GP

4. Q 5.42

5. Strong Duality for LP:
   Find the dual of the primal and argue that
   a) if the primal is unbounded then the dual is infeasible
   b) if the primal is infeasible then the dual is either infeasible or unbounded

   Primal:

$$\begin{aligned} &\min_x \; c^T x \\ &s.t. \; Ax \geq b \\ &\qquad x \geq 0 \end{aligned}$$

$$\begin{aligned} L(x, \lambda, v) &= c^T x + \lambda_1^T (b - Ax) + \lambda_2^T (-x) \\ L(x, \lambda, v) &= (c^T - \lambda_1^T A - \lambda_2^T) x + \lambda_1^T b \\ \min_x L(x, \lambda, v) &= \begin{cases} b^T \lambda_1, & c - A^T \lambda_1 - \lambda_2 = 0 \\ -\infty, & o/w \end{cases} \end{aligned}$$

   Dual:

$$\begin{aligned} &\max_{\lambda_1, \lambda_2} \; b^T \lambda_1 \\ &s.t. \; c - A^T \lambda_1 - \lambda_2 = 0 \end{aligned}$$