1. Q 5.12 textbook
   Derive dual problem for: $min_x - \sum_i log(b_i - a_i^T x), x : a_i^T x \leq b_i, \forall i \in \{1, .., m\}$

$$
\begin{aligned}
& y_i = b_i - a_i^T x \\
& a_i^T x \leq b_i \\
& L(x, y, \lambda, v) = -\sum_i log y_i + \sum_i \lambda_i (a_i^T x - b_i) + \sum_i v_i (y_i + a_i^T x - b_i) \\
& g(\lambda, v) = \inf_{x,y} L(x, y, \lambda, v) \\
& g(\lambda, v) = \inf_{x,y} -\sum_i log y_i + \sum_i \lambda_i (a_i^T x - b_i) + \sum_i v_i (y_i + a_i^T x - b_i) \\
& (\exists i) \lambda_i \neq 0 \implies \lambda_i (a_i^T x - b_i) \ unbounded, \ so \ \lambda = 0 \\
& g(\lambda, v) = \inf_{x,y} -\sum_i log y_i + \sum_i v_i (y_i + a_i^T x - b_i) \\
& g(\lambda, v) = \inf_{x,y} -\sum_i log y_i + v^T y + v^T A x - v^T b \\
& (\exists i) v_i < 0 \implies v^T y \ unbounded, \ so \ v \succeq 0 \\
& \frac{\partial}{\partial y_i} - \sum_i log y_i + v^T y + v^T A x - v^T b = -\frac{1}{y_i} + v_i = 0 \\
& y_i = \frac{1}{v_i}, v_i \neq 0 \\
& (\forall i) v_i \neq 0 \wedge v_i \geq 0 \implies v \succ 0 \\
& \frac{\partial}{\partial x} - \sum_i log y_i + v^T y + v^T A x - v^T b = A^T v = 0 \\
& g(\lambda, v) = \begin{cases} -\sum_i log \frac{1}{v_i} + \sum_i \frac{v_i}{v_i} - v^T b, & \text{if } A^T v = 0 \wedge v \succ 0 \\ -\infty, & \text{otherwise} \end{cases}
\end{aligned}
$$

Dual problem:

$$
\begin{aligned}
& \max_{\lambda, v} \sum_i log v_i + m - v^T b = -(\min_{\lambda, v} - \sum_i log v_i - m + v^T b) \\
& s.t. \ A^T v = 0 \\
& \qquad - v_i \leq 0, \forall i
\end{aligned}
$$

2. Q 5.27 Equality constrained least squares
   Give KKT conditions, derive expressions for primal and dual solutions.

$$\min_x \|Ax - b\|_2^2$$
$$s.t.\ Gx = h$$

$$f_0 = x^T A^T A x + 2b^T A x + b^T b$$
$$h_0 = Gx - h$$
$$L(x, \lambda, v) = f_0 + v^T h_0$$
$$L(x, \lambda, v) = x^T A^T A x - 2b^T A x + b^T b + v^T (Gx - h)$$
$$\frac{\partial L}{\partial x^*} = 0 = 2A^T A x^* - 2A^T b + G^T v$$

$$KKT\ conditions:$$
$$x^* = \frac{1}{2}(A^T A)^{-1}(2A^T b - G^T v)$$
$$Gx^* - h = 0$$

$$g(\lambda, v) = \inf_x L(x, \lambda, v) = \frac{1}{4}(2A^T b - G^T v)^T (A^T A)^{-1}(2A^T b - G^T v)$$
$$- 2b^T A(\frac{1}{2}(A^T A)^{-1}(2A^T b - G^T v)) + b^T b + v^T (G\frac{1}{2}(A^T A)^{-1}(2A^T b - G^T v) - h)$$
$$g(\lambda, v) = \inf_x L(x, \lambda, v) = \frac{1}{4}(2A^T b)^T (A^T A)^{-1}(2A^T b) - (A^T b)^T (A^T A)^{-1}(G^T v)$$
$$+ \frac{1}{4}(G^T v)^T (A^T A)^{-1}(G^T v) + b^T A((A^T A)^{-1} G^T v)) - h^T v + v^T G\frac{1}{2}(A^T A)^{-1}(2A^T b - G^T v)$$
$$- b^T A(A^T A)^{-1} 2A^T b) + b^T b$$

rid of constants and simplify :

$$g(\lambda, v) = \inf_x L(x, \lambda, v) = -\frac{1}{4}(G^T v - 2A^T b)^T (A^T A)^{-1}(G^T v - 2A^T b)$$
$$- \frac{1}{2}(Gv^T)^T (A^T A)^{-1}(G^T v) - h^T v$$

Dual problem:

$$\max_{\lambda, v} g(\lambda, v) = \max_v -\frac{1}{4}(G^T v - 2A^T b)^T (A^T A)^{-1}(G^T v - 2A^T b)$$
$$- \frac{1}{2}(Gv^T)^T (A^T A)^{-1}(G^T v) - h^T v$$
$$s, t.\ Gx^* - h = 0$$

Solve for v^*:

$$
\begin{aligned}
&Gx^* - h = 0 \\
&x^* = \frac{1}{2}(A^TA)^{-1}(2A^Tb - G^Tv^*) \\
&G\frac{1}{2}(A^TA)^{-1}(2A^Tb - G^Tv^*) - h = 0 \\
&v^* = 2G^{-T}(A^Tb - A^TAG^{-1}h)
\end{aligned}
$$

3. Q 5.35 Sensitivity analysis of GP
   After log tranformation:

$$\min_y f_0(y)$$
$$s.t.\ log f_i(y) \leq 0$$
$$log h_i(y) = 0$$

Perturbed problem:

$$\min_y f_0(y)$$
$$s.t.\ log f_i(y) \leq u_i$$
$$log h_i(y) = v_i$$

where $y_i = log(x_i)$

Let L:=Lagrangian of original problem after log transform
Let L':=Lagrangian of perturbed problem after log transform

$$\begin{aligned} L'(y,\lambda,w) &= log\ f_0(y) + \sum_i \lambda_i(log\ f_i(y) - u_i) + \sum_i w_i(log\ h_i(y) - v_i) \\ L'(y,\lambda,w) &= L(y,\lambda,w) - \sum_i \lambda_i u_i + -\sum_i w_i v_i \\ p(u,v) &= \inf_y L'(y,\lambda,w) = \inf_y L(y,\lambda,w) - \lambda^T u - w^T v \end{aligned}$$

Dual:

$$\begin{aligned} \max_{\lambda,w} p(u,v) = p^*(u,v) &= \max_{\lambda,w} \inf_y L(y,\lambda,w) - \lambda^T u - w^T v \\ let\ p(0,0) &= \max_{\lambda,w} \inf_y L(y,\lambda,w) \\ p^*(u,v) &= p(0,0) + (\max_{\lambda,w} -\lambda^T u - w^T v) \end{aligned}$$

let λ^*, w^* be parameters for optimal $p(u, v)$

$$\begin{aligned}
p^*(u,v) &= p(0,0) - {\lambda^*}^T u - {w^*}^T v \\
u \textit{ not present in original problem } &\implies \\
\frac{\partial}{\partial u_i} p(0,0) &= 0, \forall i \\
v \textit{ not present in original problem } &\implies \\
\frac{\partial}{\partial v_i} p(0,0) &= 0, \forall i \\
\frac{\partial}{\partial u_i} p^*(u,v) &= -\lambda_i^*, \forall i \\
\frac{\partial}{\partial v_i} p^*(u,v) &= -w_i^*, \forall i
\end{aligned}$$

Inverse log transform $x \to e^x$ into original form of objective:

$$\begin{aligned}
e^{\frac{\partial}{\partial u_i} p^*(u,v)} &= e^{-\lambda_i^*}, \forall i \\
e^{\frac{\partial}{\partial v_i} p^*(u,v)} &= e^{-w_i^*}, \forall i
\end{aligned}$$

Relaxation of ith constraint by α percent:
$\partial p^*(u_i, 0) = -\lambda_i^* \partial u_i$
$\partial u_i = \alpha \implies$ objective function of log transformed problem experiences a decrease in value by $\lambda_i \alpha$
Converting objective back via inverse of log: $x \to e^y$
z small $\implies e^z \approx 1 + z$ via Taylor expansion
$e^{-\lambda_i^* \alpha} \approx 1 - \lambda_i^* \alpha \implies$ objective function experiences an improvement of $\lambda_i^* \alpha$ percent since it is a minimization problem.

4. Q 5.42. Find Lagrange dual problem in inequality form.

$$\begin{aligned} &\min_x \; c^T x \\ &s.t. \; Ax \preceq_K b \end{aligned}$$

$$\begin{aligned} L(x, \lambda, v) &= c^T x + \lambda^T (Ax - b) \\ \lambda &\succeq_{K^*} 0 \\ g(\lambda, v) &= \inf_x L(x, \lambda, v) \\ g(\lambda, v) &= \begin{cases} -\lambda^T b, & c^T x + \lambda^T Ax = 0 \\ -\infty, & o/w \end{cases} \end{aligned}$$

Dual:

$$\begin{aligned} &\max_{\lambda, v} g(\lambda, v) = \max_{\lambda} \; -b^T \lambda \\ &s.t. \; c + A^T \lambda = 0 \\ &\quad \lambda \succeq_{K^*} 0 \\ \\ &let \; y = -\lambda \\ \\ &\max_y \; b^T y \\ &s.t. \; A^T y = c \\ &\quad y \preceq_{K^*} 0 \end{aligned}$$

5. Strong Duality for LP:
   Primal:

$$\begin{aligned} \min_x \; & c^T x \\ s.t. \; & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Find the dual of the primal and argue that

- a) if the primal is unbounded then the dual is infeasible

$$L(x, \lambda, v) = c^T x + \lambda_1^T (b - Ax) + \lambda_2^T (-x)$$
$$L(x, \lambda, v) = (c^T - \lambda_1^T A - \lambda_2^T) x + \lambda_1^T b$$
$$g(\lambda, v) = \inf_x L(x, \lambda, v) = \begin{cases} b^T \lambda_1, & c - A^T \lambda_1 - \lambda_2 = 0 \\ -\infty, & o/w \end{cases}$$

$$Dual:$$
$$\begin{aligned} \max_{\lambda_1, \lambda_2} \; & b^T \lambda_1 \\ s.t. \; & c - A^T \lambda_1 - \lambda_2 = 0 \end{aligned}$$

$$g(\lambda, v) \leq c^T x^* \; (weak\ duality)$$
$$primal\ unbounded: \; c^T x^* = -\infty \implies$$
$$g(\lambda, v) = -\infty \implies c - A^T \lambda_1 - \lambda_2 \neq 0 \implies infeasible\ case\ of\ dual$$

- b) if the primal is infeasible then the dual is either infeasible or unbounded

$$primal\ infeasible \implies b - Ax \not\leq 0 \vee -x \not\leq 0$$
$$\min_x c^T x + \lambda_1^T(b - Ax) + \lambda_2^T(-x) \leq c^T x$$

$$dual\ unbounded\ case:$$
$$suppose\ c - A^T\lambda_1 - \lambda_2 = 0,\ substitute\ for\ \lambda_1:$$
$$Dual: \max_{\lambda_1,\lambda_2}\ b^T\lambda_1 = \max_{\lambda_1,\lambda_2} b^T A^{-T} c - b^T A^{-T}\lambda_2,\ such\ that:$$
$$(\forall i)(b - Ax)_i > 0 \implies \lambda_{1_i}\ may\ be\ nagative,\ and$$
$$(\forall i)(-x)_i > 0 \implies \lambda_{2_i}\ may\ be\ nagative,\ and$$
$$c - A^T\lambda_1 - \lambda_2 = 0$$

$$let\ d = b^T A^{-T} c$$
$$let\ e^T = b^T A^{-T}$$
$$Dual: \max_{\lambda_1,\lambda_2}\ d - e^T\lambda_2$$
$$(\exists i)e_i > 0 : select\ \lambda_{2_i} \to -a\ and\ (\exists i)e_i < 0 : select\ \lambda_{2_i} \to a, a \to +\infty\ such\ that\ it\ satisfies:$$
$$c - A^T\lambda_1 - \lambda_2 = 0,\ then\ e^T\lambda_2\ is\ unbounded$$
$$eg: c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, A = I, Ax \not\geq b, x \not\geq 0$$
$$Dual:\ \max_{\lambda_1,\lambda_2}\ d - I^T \begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \lambda_2,\ select\ \lambda_2 = \begin{bmatrix} -a \\ a \end{bmatrix}$$
$$c - A^T\lambda_1 - \lambda_2 = 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \lambda_1 - \begin{bmatrix} -a \\ a \end{bmatrix} = 0$$
$$select:\ \lambda_1 = \begin{bmatrix} a \\ -a \end{bmatrix}$$
$$let\ a \to +\infty \implies \max_{\lambda_1,\lambda_2}\ d - I^T \begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \lambda_2 \Big|_{\lambda_2 = [-a,a]^T} = d + 2a = +\infty \implies dual\ unbounded$$

$$dual\ infeasible\ case:$$
$$suppose: c > 0, x \not\geq 0, A = -1, Ax \geq b$$
$$Ax \geq b \implies \lambda_1 \geq 0$$
$$x < 0 \implies \lambda_2 \leq 0$$
$$c - A^T\lambda_1 - \lambda_2 = c + \lambda_1 - \lambda_2$$
$$c + \lambda_1 > 0 \wedge -\lambda_2 \geq 0 \implies c - A^T\lambda_1 - \lambda_2 \neq 0 \implies dual\ is\ not\ feasible$$

6. Game Theory

$$\begin{aligned}
&\min_x \max_y x^T P y \\
s.t.\ &Ax \leq b \\
&Cy \leq d
\end{aligned}$$

$$\begin{aligned}
&\max_y \min_x x^T P y \\
s.t.\ &Ax \leq b \\
&Cy \leq d
\end{aligned}$$

(a) show that min-max problem as the same optimal value as the following minimization problem:

$$\begin{aligned}
&\min_{\lambda,x} d^T \lambda \\
s.t.\ &C^T \lambda = P^T x \\
&Ax \leq b \\
&\lambda \geq 0
\end{aligned}$$

Inner optimization problem:

$$\begin{aligned}
&\max_y (P^T x)^T y = -(\min_y -(P^T x)^T y) \\
s.t.\ &Ax \leq b \\
&Cy \leq d
\end{aligned}$$

$$g(\lambda_1, \lambda_2) = \inf_y -(P^T x)^T y = \begin{cases} \lambda_1^T (Ax - b) - \lambda_2^T d, & -P^T x + C^T \lambda_2 = 0 \\ -\infty, & o/w \end{cases}$$

$$\begin{aligned}
&\max_{\lambda_1,\lambda_2} g(\lambda_1, \lambda_2) = \max_{\lambda_1,\lambda_2} -\lambda_2^T d + \lambda_1^T (Ax - b) \\
s.t.\ &-P^T x + C^T \lambda_2 = 0 \\
&\lambda_1, \lambda_2 \geq 0 \\
&\lambda_1^T (Ax - b) \leq 0 \implies \\
&\max_{\lambda_1,\lambda_2} g(\lambda_1, \lambda_2) = \max_{\lambda_2} -\lambda_2^T d
\end{aligned}$$

$$-(\min_y -(P^T x)^T y) = -\max_{\lambda_1,\lambda_2} g(\lambda_1, \lambda_2) = -(-\min_{\lambda_2} \lambda_2^T d) = \min_{\lambda_2} \lambda_2^T d$$

rename λ_2 *to* λ *and enclose with outer minimization over* x :

$$\begin{aligned}
&\min_{x,\lambda} \lambda^T d \\
s.t.\ &P^T x = C^T \lambda \\
&Ax \leq b \\
&\lambda \geq 0
\end{aligned}$$

(b) show that max-min problem as the same optimal value as the following maximization problem:

$$\max_{y,v} -b^T v$$
$$s.t.\ A^T v + Py = 0$$
$$Cy \leq d$$
$$v \geq 0$$

Inner optimization problem:

$$\min_x x^T P y$$
$$s.t.\ Ax \leq b$$
$$Cy \leq d$$
$$g(\lambda_1, \lambda_2) = \inf_x x^T P y + \lambda_1^T (Ax - b) + \lambda_2^T (Cy - d)$$
$$= \begin{cases} -\lambda_1^T b + \lambda_2^T (Cy - d), & P^y + A^T \lambda_1 = 0 \\ -\infty, & o/w \end{cases}$$
$$max_{\lambda_1,\lambda_2} g(\lambda_1, \lambda_2) = \max_{\lambda_1,\lambda_2} -\lambda_1^T b + \lambda_2^T (Cy - d)$$
$$s.t.\ P^T y + A^T \lambda_1 = 0$$
$$\lambda_1 \geq 0$$
$$\lambda_2 \geq 0$$
$$Ax - b \leq 0$$
$$Cy - d \leq 0$$
$$\lambda_2^T (Cy - d) \leq 0 \implies$$
$$\max_{\lambda_1,\lambda_2} g(\lambda_1, \lambda_2) = \max_{\lambda_1,\lambda_2} -\lambda_1^T b$$
$$s.t.\ P^T y + A^T \lambda_1 = 0$$
$$\lambda_1 \geq 0$$
$$Cy - d \leq 0$$
$$rename\ variables\ and\ enclose\ with\ outer\ optimization:$$
$$\max_{y,v} - b^T v$$
$$s.t.\ P^T y + A^T v = 0$$
$$Cy \leq d$$
$$v \geq 0$$

(c) show the above problems have the same optimal value (min-max equals max-min)

$$\begin{aligned}
&\min_{x,\lambda} d^T\lambda \\
&s.t.\ C^T\lambda = P^x \\
&\quad Ax \le b \\
&\quad \lambda \ge 0
\end{aligned}$$

$$\begin{aligned}
g(a_1,a_2,w) &= \inf_{x,\lambda} d^T\lambda + a_1^T(Ax-b) + a_2^T(-\lambda) + w^T(C^T\lambda - P^T x) \\
s.t.\ & a_1 \ge 0 \\
& a_2 \ge 0 \\
& Ax - b \le 0 \\
& -\lambda \le 0 \\
& C^T\lambda - P^T x = 0 \\
g(a_1,a_2,w) &= \begin{cases} -a_1^T b, & d - a_2 + Cw = 0 \wedge A^T a_1 - Pw = 0 \\ -\infty, & o/w \end{cases} \\
& Dual: \\
\max_{a_1,a_2,w} & g(a_1,a_2,w) = -b^T a_1 \\
s.t.\ & d - a_2 + Cw = 0 \\
& A^T a_1 - Pw = 0 \\
& a_2 \ge 0 \implies d + Cw \ge 0 \\
& a_1 \ge 0
\end{aligned}$$

$$\begin{aligned}
&let\ v = a_1, y = -w \\
&\max_{v,y} g(v,y) = -b^T v \\
&s.t.\ A^T v + Py = 0 \\
&\quad Cy \le d \\
&\quad v \ge 0
\end{aligned}$$

We start with equivalent formulation of min-max and arrive at a equivalent formulation of max-min problem thus they obtain the same optimal value.

(a)

7. Optimal control of a unit mass revisited

$$\|p\|_1 = \sum_{i=1}^{10} |p_i|$$
$$\|p\|_\infty = \max_{i=1,..,10} |p_i|$$

(a) Consider $\|p\|_1$ problem with same setup in Q9(1) of last assignment. Find optimal solution. Plot optimal force, position, and velocity. Write down the primal and dual solutions.

(b) i. Verify that for any vector v and w, we always have $|w^T v| \leq \|v\|_\infty \|w\|_1$

ii. Let z be any solution of $Az = y$, explain why for any λ, we must have

$$\|z\|_1 \geq \frac{|\lambda^T y|}{\|A^T \lambda\|_\infty}$$

and thus z, λ for which the above inequality is satisfied with equality means z must be optimal.

iii. Set λ to be the Lagrange multiplier associated with the equality constraint in part (a). Use the above inequality to directly verify that the bang-bang solution is optimal.

(c) Repeat part (a) for $\|*\|_\infty$ minimization problem.