Please write legibly, explain clearly and present your solutions in an organised way. Some points will be given for the quality of presentation.

1. [*7 points*] Let A be irreducibly diagonally dominant with positive diagonal entries and nonpositive off-diagonal entries. Show that $A^{-1} > 0$. (That is, prove A is monotone.)
Notes: The fact that A is invertible has been shown in assignment 1, and is taken as granted here, together with all other facts shown in assignment 1. You are also welcome to use without proof the following fact, if it helps: If G is any (square) matrix with $\rho(G) < 1$, then $\mathbf{I} - G$ is nonsingular, and the series $\mathbf{I} + G + G^2 + \cdots$ converges to $(\mathbf{I} - G)^{-1}$.

2. *This is a continuation of question 5 of assignment 1.* Consider the two-point (one-dimensional) fourth-order Boundary Value Problem (BVP)

$$u_{xxxx} = g \quad \text{in} \quad (0, 1) \tag{1}$$

$$u = \gamma \quad \text{on} \quad x = 0,\ x = 1, \tag{2a}$$

$$u_{xx} = \zeta \quad \text{on} \quad x = 0,\ x = 1, \tag{2b}$$

where $u(x)$ is unknown, and $g(x)$, $\gamma(x)$ and $\zeta(x)$ are given functions. Consider also the discretization of the domain $(0, 1)$ by the gridpoints $x_i = i/n$, $i = 0, \cdots, n$, into n uniform subintervals with stepsize $h = 1/n$. Adopt the notation $u_i = u(x_i)$, $i = 0, \cdots, n$, whenever it is convenient. Let A be the matrix arising from the second-order finite difference approximation of the BVP (1)-(2) as developed in assignment 1 (including the $1/h^4$ factor). Let $B = trid\{1, -2, 1\}/h^2$.

(a) [*4 points*] Show that $-B$ and A are monotone.

(b) [*9 points*] Show that $\|B^{-1}\|_\infty$ and $\|A^{-1}\|_\infty$ are bounded from above independently of n. Give approximate bounds for $\|B^{-1}\|_\infty$ and $\|A^{-1}\|_\infty$. (The two bounds can be different.)
Hint: For the bound of $\|B^{-1}\|_\infty$, consider the vector $\bar{w}$ arising from evaluating $w(x) = x(x-1)/2$ at the interior grid points, in the order the rows of B are written. Then calculate $B\bar{w}$ and $\|\bar{w}\|_\infty$.

(c) [*8 points*] Using the bound for $\|A^{-1}\|_\infty$ developed in (b) and the finite difference approximations to u_{xxxx} developed in assignment 1, prove that $\max_{i=1}^{n-1} |u_i - \bar{u}_i| = O(h^2)$, where $\bar{u}_i$ are the approximations to u_i computed from the finite difference method of assignment 1.

(d) [*2 points*] Give the diagonalization of A. Describe an FFT-based (FST) algorithm for solving $A\bar{u} = \bar{g}$.

3. Consider the two-dimensional Boundary Value Problem (BVP)

$$\begin{aligned} u_{xxxx} + 2u_{xxyy} + u_{yyyy} &= g \quad \text{in} \quad (0, 1) \times (0, 1) \\ u &= \gamma \quad \text{on} \quad x = 0,\ x = 1,\ 0 \le y \le 1, \text{ and } y = 0,\ y = 1,\ 0 \le x \le 1 \\ u_{xx} &= \zeta \quad \text{on} \quad x = 0,\ x = 1,\ 0 \le y \le 1 \\ u_{yy} &= \eta \quad \text{on} \quad y = 0,\ y = 1,\ 0 \le x \le 1 \end{aligned}$$

where $u(x, y)$ is unknown and $g(x, y)$, $\gamma(x, y)$, $\zeta(x, y)$ and $\eta(x, y)$ are given functions. Consider also the discretization of the domain $(0, 1) \times (0, 1)$ by the gridpoints $(x_i, y_j) = (i/n, j/n)$, $i = 0, \cdots, n$, $j = 0, \cdots, n$, into $n \times n$ uniform subrectangles with stepsize $h = 1/n$, in both dimensions. Adopt the notation $u_{ij} = u(x_i, y_j)$, $i = 0, \cdots, n$, $j = 0, \cdots, n$, whenever it is convenient.

Extend the techniques of question 5, assignment 1, in order to derive a discrete form of the BVP, that is, a linear system $A\bar{u} = \bar{g}$ approximating the BVP, where $\bar{u}$ is a vector of approximate values of u on the nodes and $\bar{g}$ a vector made up of values of g, γ, ζ and η on the nodes. Number the nodes (as well as respective equations and unknowns) first bottom-up, then left-to-right. Thus $\bar{u}_k$ corresponds to $\bar{u}_{ij} \approx u_{ij}$, for $k = (m-1)(i-1) + j$.

(a) [*4 points*] Describe the properties (size, bandwidth, nonzero entries per row, sparsity pattern, block structure, etc) of the matrix A arising.

(b) [*4 points*] Write A in tensor product form, using only tridiagonal matrices and the identity matrix as components. (You can use regular matrix products, as well as tensor products.) Give explicit formulae for the eigenvalues and eigenvectors of A. Find the smallest and largest (algebraically) eigenvalues of A. For uniformity and for later purposes, you

must keep the $1/h^4$ factor with the matrix A.

(c) [*4 points*] Prove that A is symmetric positive definite. Consider now that instead of the operator $u_{xxxx} + 2u_{xxyy} + u_{yyyy}$ you have $\alpha u_{xxxx} + \beta u_{xxyy} + \gamma u_{yyyy}$, where α, β and γ are free parameters. Let $C = C(\alpha, \beta, \gamma)$ be the matrix arising. Adjust the formulae for the eigenvalues of A to obtain the respective formulae for the eigenvalues of C, in terms of α, β and γ. Under what conditions on α, β and γ is C spd?

(d) [*4 points*] Give an (approximate) bound for the Euclidean norm $\|A^{-1}\|_2$ of the inverse of A, and explain how you obtained it. Note that this bound should be independent of the size of A.

(e) [*4 points*] Give the (a) block-diagonalization of A. Describe an FFT-based (FST) algorithm for solving $A\bar{u} = \bar{g}$, that applies FST to one dimension and banded LU to the other.

4. [*30 points*] Write a programme (in MATLAB, FORTRAN, C, or any reasonable language) that implements the method to solve the above BVP (with $\alpha = 1$, $\beta = 2$ and $\gamma = 1$) using the method you derived in question 3. More specifically: write a programme (and subroutines/functions, if needed), which, given n, g, γ, ζ and η:
(α) Generates the matrix A and the respective right-side vector $\bar{g}$. The matrix should be generated and stored in sparse or banded format.
(β) Uses two linear solvers to solve $A\bar{u} = \bar{g}$ to obtain $\bar{u}$ on the (interior) gridpoints. The solvers are: (β1) Banded LU and (β2) FFT (FST) applied to one dimension and banded LU to the other. Computes and outputs the maximum norm of the difference of the two solution vectors.
(γ) Computes and outputs the maximum in absolute value error $e_n = \max_{i=1,j=1}^{n,n} |\bar{u}_{ij} - u_{ij}|$ of the approximation on the knots (assuming the exact solution u to the BVP is known).
(δ) For pairs of values of n of the form $(n, 2n)$, computes and outputs the order of convergence, $\log_2(e_n/e_{2n})$, corresponding to (e_n, e_{2n}).

For $n = 8, 16, 32, 64$, run your programme for the following choices of g, γ, ζ and η:
(i) Choose g, γ, ζ and η so that $u(x, y) = \sin x \sin y$
(ii) Choose g, γ, ζ and η so that $u(x, y) = x^{7/2} y^{7/2}$
For each value of n, and for each of the two problems i and ii, output the errors as stated in γ, and compute and output the orders of convergence corresponding to the errors output by the programme. Make a table showing the errors and orders of convergence in each case. (Or format your output to produce the results in the form of a table.) Do **not** output more than requested. Compact your output so that it is easily read (e.g. `format compact`, `fprintf`).

Comment on how the errors behave with n and whether the behaviour is expected from theory.
If you get any underflow message, use double precision. (MATLAB uses only double precision.)
Hand in a hard-copy of your code together with the output required, the table (if separate) and your comments.

Notes:
You are welcome to use the `kron` function in MATLAB to build the matrix for this problem. Make sure that at no point in the programme you store a non-sparse matrix.
MATLAB provides the function fft which applies the FFT to a vector or to each of the columns of a matrix, and the function dst which applies the FST to a vector or to each of the columns of a matrix.

5. [*8 points*] Consider the matrix $\mathbb{F}_n \in \mathbb{C}^{n \times n}$, defined by $(\mathbb{F}_n)_{kj} = e^{-\frac{2(j-1)(k-1)\pi i}{n}} = \cos \frac{2(j-1)(k-1)\pi}{n} - i \sin \frac{2(j-1)(k-1)\pi}{n}$, for $k = 1, \cdots, n$, $j = 1, \cdots, n$, and $i = \sqrt{-1}$. Show that $\frac{\mathbb{F}_n}{\sqrt{n}}$ is unitary.

6. [*12 points*] A $n \times n$ matrix with constant coefficients along the diagonals is called *Toeplitz*. Let a_j, $j = -n+1, \cdots, n-1$, be the constant coefficient along the jth diagonal, with $j = 0$ corresponding to the main diagonal, and the negative j's corresponding to subdiagonals (i.e. $j = -1$ corresponding to the first subdiagonal, etc). A Toeplitz matrix is called *circulant* if $a_{-j} = a_{n-j}$, $j = 1, \cdots, n-1$.

(a) Show that the circulant matrix C, with $a_1 = 1$, and $a_j = 0$, for $j = 0$, and $j = 2, \cdots, n-1$, is diagonalizable by the inverse of the matrix $\mathbb{F}_n$ (as defined in question 5). Through the diagonalization, give analytic formulae for the eigenvalues of C.

(b) Show that any circulant matrix is diagonalizable by the inverse of the matrix $\mathbb{F}_n$. Give analytic formulae for the eigenvalues of a circulant matrix in terms of its coefficients a_j, $j = 0, \cdots, n-1$, and explain how you got them.