Please write legibly, explain clearly and present your solutions in an organised way. Some points will be given for the quality of presentation.

1. Consider a tridiagonal matrix $A \in \mathbb{R}^{n\times n}$, with each off-diagonal element equal to 1, and each diagonal element equal to a, except $A_{n-1,n-1} = b$ and $A_{nn} = c$, where $a + 1 \le b$, and $b + 1 < c$.

(a) [*5 points*] Apply Gerschgorin's theorem to A to find a range where the eigenvalues of A lie. The range is to be given in terms of a and c.

(b) [*10 points*] Consider a similarity transformation $D_d A D_d^{-1}$, where D_d is a diagonal matrix, with all diagonal entries equal to 1, except $D_{nn} = d > 0$. Does there exist a d which gives the lowest (with the help of Gerschgorin's theorem again) upper bound for the eigenvalues of A? If yes, give a formula for this d (in terms of b and c), and explain how you got it. If no, explain why such a d does not exist.

2. [*10 points*] Let B be a $n \times n$ Hermitian matrix with eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. Show that, for any nonzero vector x,

$$\lambda_1 \le \frac{x^H Bx}{x^H x} \le \lambda_n.$$

3. [*10 points*] Show that, for any matrix A, $\|A\|_2 \le \sqrt{\|A\|_1 \|A\|_\infty}$.

4. If A and B are two $n \times m$ matrices, we say that $A \ge B$, if $a_{ij} \ge b_{ij}$ for all i, j. We say that a matrix M is *nonnegative*, $M \ge 0$, if all its entries are nonnegative, i.e. $M_{ij} \ge 0$ for all i, j. A real square matrix A is called *monotone*, if it is nonsingular and $A^{-1} \ge 0$.

A square matrix A of size n is *reducible*, if $n = 1$, or if there exists a permutation matrix P such that

$$P^{-1}AP = \begin{pmatrix} E & F \\ 0 & G \end{pmatrix}$$

where E and G are square matrices.

In other words, A is *reducible*, if $n = 1$, or if one can extract a subsystem of $Ax = b$ which preserves the correspondence between equations and unknowns, and which can be solved independently of the remaining subsystem.

An $n \times n$ matrix is *irreducible*, if it is not reducible.

A square matrix A of size n is *irreducibly diagonally dominant*, if it is diagonally dominant, with strict diagonal dominance on at least one row, and irreducible.

(a) [*4 points*] Let A, B and C be matrices of arbitrary but appropriate size, with C nonnegative and $A \le B$. Show that $AC \le BC$ and $CA \le CB$. If, in addition to $A \le B$, matrices A and B are monotone, show that $B^{-1} \le A^{-1}$.

(b) [*8 points*] Let A be irreducibly diagonally dominant. Show that A is nonsingular.

(c) [*8 points*] Let A be irreducibly diagonally dominant. Let D be the diagonal matrix with $D_{ii} = A_{ii}$, for all i. Show that $\rho(\mathbf{I} - D^{-1}A) < 1$.

5. Consider the two-point (one-dimensional) fourth-order Boundary Value Problem (BVP)

$$u_{xxxx} = g \quad \text{in } (0, 1) \tag{1}$$

$$u = \gamma \quad \text{on } x = 0, x = 1, \tag{2a}$$

$$u_{xx} = \zeta \quad \text{on } x = 0, x = 1, \tag{2b}$$

where $u(x)$ is unknown, and $g(x)$, $\gamma(x)$ and $\zeta(x)$ are given functions. Consider also the discretization of the domain $(0, 1)$ by the gridpoints $x_i = i/n$, $i = 0, \cdots, n$, into n uniform subintervals with stepsize $h = 1/n$. Adopt the notation $u_i = u(x_i)$, $i = 0, \cdots, n$, whenever it is convenient.

(a) [*6 points*] Using Taylor's series, derive a second order centered finite difference approximation to the **fourth** derivative u_{xxxx} of u at a point x. Note that the approximation will use five points, namely, $u(x - 2h)$, $u(x - h)$, $u(x)$, $u(x + h)$ and $u(x + 2h)$. State the smoothness assumptions you make on u, and explain why, under the assumptions, the approximation is second order.

(b) [*7 points*] The approximation derived in (a) can be applied to any point x_i, $i = 1, \cdots, n-1$. However, while for $i = 2, \cdots, n-2$, it gives rise to an equation relating some of the u_i's, $i = 1, \cdots, n-1$, to known quantities, for $i = 1$ and $i = n-1$, it gives rise to an equation involving some points outside the domain (0, 1) of the BVP. In order to derive a set of $n-1$ equations, relating u_i, $i = 1, \cdots, n-1$, to known quantities, we employ the boundary conditions (2b). It is known that, under certain assumptions, a second-order discrete form of (2b) is given by

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = \zeta_i, \; i = 0, \; n. \tag{3}$$

Consider the relation arising from applying the approximation in (a) to x_1 (which involves a point to the left of (0, 1)), and the relation (3) for $i = 0$, scaled appropriately, and subtract the latter from the former. In the resulting relation, there is no point outside (0, 1); only $u_0 = \gamma_0$, ζ_0, and u_i, $i = 1, 2, 3$ are involved. Derive the resulting relation, clearly showing the coefficients. With a similar treatment of the right boundary, a relation which involves only u_i, $i = n-3, n-2, n-1, z_n$, and $u_n = \gamma_n$ is derived. Indicate the relation.

(c) [*2 points*] Write the $n-3$ equations resulting from applying the approximation derived in (a) to x_i, $i = 2, \cdots, n-2$, and the two boundary equations derived in (b) in the form of a **banded** linear system of size $(n-1) \times (n-1)$. You now have a discrete form of the BVP (1)-(2), from which approximations $\bar{u}_i \approx u_i$, $i = 1, \cdots, n-1$, can be computed. Show the form of the system for $n = 6$ and the general form of the matrix for any n. Bring the system (if it is not already in that form) in a form so that a $1/h^4$ factor accompanies the matrix.

(d) [*3 points*] Let A be the matrix arising in (c) (including the $1/h^4$ factor). Find a relation between A and the matrix $B = 1/h^2 \; trid\{1, -2, 1\}$. Give an interpretation of this relation in terms of derivatives.

(e) [*7 points*] Give analytic formulae for the eigenvalues and eigenvectors of A. Give an approximate upper bound for the Euclidean norm $\|A^{-1}\|_2$ of the inverse of A, and explain how you obtained it. Note that this bound should be independent of the size $n-1$ of A.

(f) [*20 points*] Write a programme (in MATLAB, FORTRAN, C, or any reasonable language) to solve the above BVP using the equations in (c). More specifically: write a programme (and subroutines/functions, if needed), which, given n, g, γ and ζ:
(α) Generates the matrix and right-side of the equations in (c). The matrix should be generated and stored in pentadiagonal, or sparse or other equivalent format.
(β) Uses a **banded** (pentadiagonal) linear solver, that performs LU decomposition of the matrix (by Gauss Elimination), then forward and back substitutions, to solve the system and obtain $\bar{u}_i$ on the (interior) gridpoints. Note that, if the matrix is stored in sparse format (and is pentadiagonal), MATLAB automatically uses a pentadiagonal linear solver (including f/b/s), when the backslash "\" operator is used to solve the system.
(γ) Computes and outputs the maximum in absolute value error $e_n = \max_{i=1}^{n-1} |\bar{u}_i - u_i|$ of the approximation on the knots (assuming the exact solution u to the BVP is known).
(δ) For pairs of values of n of the form $(n, 2n)$, computes and outputs the order of convergence, $\log_2(e_n / e_{2n})$, corresponding to (e_n, e_{2n}), where the log should be base 2.

For $n = 8, 16, 32, 64, 128$, run your programme for the following choices of $g(x)$, $\gamma(x)$ and $\zeta(x)$:
(i) Choose $g(x)$, $\gamma(x)$ and $\zeta(x)$ so that $u(x) = x^3$
(ii) Choose $g(x)$, $\gamma(x)$ and $\zeta(x)$ so that $u(x) = \sin x$
(iii) Choose $g(x)$, $\gamma(x)$ and $\zeta(x)$ so that $u(x) = x^{9/2}$
For each value of n, and for each of the three problems i, ii and iii, output the errors as stated in γ. For each pair of values of n, (8, 16), (16, 32), (32, 64) and (64, 128), and for each of the two problems ii and iii, compute and output the orders of convergence corresponding to the errors output by the programme. Make a table showing the errors and orders of convergence in each case. (Or format your output to produce the results in the form of a table.) Do **not** output more than requested. Compact your output so that it is easily read. (In MATLAB, use format compact.)

Comment on the results (i.e. how the errors behave with n and whether the behaviour is expected from theory). If you get any underflow message, use double precision. (MATLAB uses only double precision.) Hand in a hard-copy of your code together with the output required, the table (if separate) and your comments.