(Bill) Yuan Liu, student #: 996954078

1. $u_{xxxx} + u = g$ where u is periodic on (a, b)

   (a) Let $a = 0, b = 2\pi, n = 6, u(x) = sin(x)$.
   Give A and $\bar{g}$.

$$\begin{aligned} A &\leftarrow \frac{1}{h^4} Toeplitz(\begin{bmatrix} 6 & -4 & 1 & 0 & 1 & -4 \end{bmatrix}) + I \\ A(:,1) &\leftarrow A(:,1) * 2 \end{aligned}$$

$$A = \frac{1}{h^4}\begin{bmatrix} 2 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} + \frac{1}{h^4}\begin{bmatrix} 12 & -4 & 1 & 0 & 1 & 4 \\ -8 & 6 & -4 & 1 & 0 & 1 \\ 2 & -4 & 6 & -4 & 1 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 \\ 2 & 0 & 1 & -4 & 6 & -4 \\ -8 & 2 & 0 & 1 & -4 & 6 \end{bmatrix}$$

$$\bar{g}_i = 2sin(\frac{2\pi i}{6}), i = 0, .., 5$$

(b) Adjust programme for assignment 1 to solve above with CG with tolerance 10^{-8} and zero vector as the initial guess

```
function [x,it] = mysolver(a, b, n, u, x)
% assumes u(x) is a symbolic function

    range = b-a;
    h = range/n;
    t = 0:1:n-1;
    xs = h.*t + a;

    g_symbol = diff(u,x,4) + u;
    g = eval(g_symbol(xs)');

    % construct A
    r = [6 -4 1 zeros(1,n-5) 1 -4];
    A = sparse(toeplitz(r));
    A=A./(h^4); % for u''''
    I=spdiags([ones(n)],0:0,n,n); %for u
    A=A+I;
    A(:,1) = A(:,1)*2; %for periodic boundary condition

    % solve with CG
    tol = 10^(-8);
    maxit = n;
    % x0 zero vector, preconditioner = I
    [x,flag,relres,iter] = pcg(A,g,tol,maxit,[],[],[]);
    it=iter;
end
```

```
clear
% run solver for each dim for each case and record max knot errors

dims = [ 8 16 32 64 128 ];

% note: reset errors before running
errors = [];
iterations = [];

b = 2*pi;
a = 0;
range = b-a;

% case: u(x)=sin(x)
for e = 1:length(dims)

    n=dims(e);

    h = range/n;
    t = 0:1:n-1;
    xs = range/n.*t + a;

    syms u(x);
    u(x)=sin(x);

    [approx,iter] = mysolver(a, b, n, u, x);
    exact = eval(u(xs)');

    max_error_knots = 0;
    max_error_knots = max(max_error_knots, max(abs(approx-exact)));
    errors = [errors max_error_knots];
    iterations = [iterations iter];

    fprintf("case 1, n: %d, max knot error: %f, iterations: %d\n", n, max_error
end
```

n	max absolute knot error	# of iter for convergence
8	0.051625	1
16	0.012867	1
32	0.003214	1
64	0.000803	1
128	0.000201	1

Table 1: Error and Convergence

Solution $\bar{u}$ is a multiple of $\bar{g}$ for matrix A, then u is a scaled eigenvector for A. Since preconditioner is I and initial guess is the zero vector, the starting residual is $\bar{g}$ which is a multiple of the eigenvector. With conjugate gradient method, if the residual coincides with the direction vector that is generated at each iteration, then the next iteration converges. This is because CG algorithm is designed to get rid of error components in terms of a direction vector at each iteration. This explains in all cases of n, the number of iterations needed for convergence is 1.

2. Let A be $n \times n$, spd matrix Let A be a n × n symmetric positive definite matrix, and assume that the Conjugate Gradient (CG) method is applied to Ax = b, for some n 1 vector b. Show that the direction vectors d (k) , and the residual vectors r (k) generated by the CG method satisfy:

$$r^{(k)} \in span(d^{(0)}, d^{(1)}, ..., d^{(k)}), for\ k = 0, ..., n1$$
$$Ad^{(k)} \in span(d^{(0)}, d^{(1)}, ..., d^{(k+1)}), for\ k = 0, ..., n2$$

and

$$\begin{aligned} span(d^{(0)}, d^{(1)}, ..., d^{(k)}) &= span(d^{(0)}, Ad^{(0)}, ..., A^k d^{(0)}) \\ &= span(r^{(0)}, Ar^{(0)}, ..., A^k r^{(0)}), for\ k = 0, ..., n1 \end{aligned}$$

By construction of CG algorithm and initial condition,

$$\begin{aligned} r^{(k)} &= r^{(k-1)} - \lambda_{k-1} Ad^{(k-1)} \\ d^{(k)} &= r^{(k)} + \alpha_k d^{(k-1)} \\ d^{(0)} &= r^{(0)} \end{aligned}$$

$d^{(0)} = r^{(0)} \in span(d^{(0)})$
Assume $r^{(k)} \in span(d^{(0)}, ..., d^{(k)})$ is true

From recursion of conjugate gradient:
$r^{(k+1)}$ is a linear combination of $r^{(k)}, Ad^{(k)} \implies$
$r^{(k+1)} \in span(d^{(0)}, ..., d^{(k)}, Ad^{(k)})$
$d^{(k+1)}$ is a linear combination of $r^{(k)}, d^{(k)}, Ad^{(k)} \implies$
$d^{(k+1)} \in span(r^{(k)}, d^{(k)}, Ad^{(k)}) = span(d^{(0)}, ..., d^{(k)}, Ad^{(k)})$
This means
$Ad^{(k)} \in span(d^{(0)}, ..., d^{(k)}, d^{(k+1)}), k = 0, .., n - 2$

Then
$r^{(k+1)} \in span(d^{(0)}, ..., d^{(k)}, Ad^{(k)}) = span(d^{(0)}, ..., d^{(k)}, d^{(k+1)})$
thus completing induction, $r^{(k)} \in span(d^{(0)}, ..., d^{(k)}, d^{(k)}), k = 0, .., n - 1$

Since CG is a CD method, by design d is A-orthogonal, $(d^{(i)}, Ad^{(j)}) = 0, \forall i < j,$ then
$d^{(k+1)} \in span(d^{(0)}, ..., d^{(k)}, Ad^{(k)}) = span(d^{(k)}, Ad^{(k)})$
recursive application results in
$d^{(1)} \in span(d^{(0)}, Ad^{(0)})$
$d^{(2)} \in span(d^{(1)}, Ad^{(1)}) = span(d^{(0)}, Ad^{(0)}, A^2 d^{(0)}))$
$d^{(k)} \in span(d^{(k-1)}, Ad^{(k-1)}) = span(d^{(0)}, ..., A^k d^{(0)})$

$d^{(i)} \in span(d^{(j)}), j > i$

Then
$span(d^{(0)}, ..., d^{(k)}) = span(d^{(0)}, Ad^{(0)}, ..., A^k d^{(0)})$

From earlier, since $d^{(k+1)}, r^{(k+1)} \in span(d^{(0)}, ..., d^{(k)}, d^{(k+1)}),$ then,
$d^{(k)}$ and $r^{(k)}$ lives in the same subspace at each iteration, thus:
$span(d^{(0)}, ..., d^{(k)}) = span(r^{(0)}, Ar^{(0)}, ..., A^k r^{(0)})$

3. Let A be a n $\times$ n symmetric positive definite matrix, and assume that the CG method is applied to $Ax = b$, for some n $\times$ 1 vector b. Show that the CG iterate $x^{(k+1)}$ minimized $\|x - \hat{x}\|_{A^{\frac{1}{2}}}$ over all $\hat{x} \in x^{(0)} + span(d^{(0)}, ..., d^{(k)})$.

Translation to linear space: $e^{(i)} = x^{(i)} - \hat{x}$, where $\hat{x}$ is the answer
Using CG algorithm to reduce e:
let $e^{(0)} = \sum_{i=0}^{n-1} c_i d^{(i)}$
$Ae^{(0)} = \sum_{i=0}^{n-1} c_i A d^{(i)}$
$(d^{(k)}, Ae^{(0)}) = \sum_{i=0}^{n-1} c_i (d^{(k)}, Ad^{(i)})$
Using A-orthogonality of d vectors: $(d^{(k)}, Ad^{(i)}) = 0, k < i$, c_k is chosen such that:
$(d^{(k)}, Ae^{(0)}) = c_k (d^{(k)}, Ad^{(k)})$
$c_k = \frac{(d^{(k)}, Ae^{(0)})}{(d^{(k)}, Ad^{(k)})}$
$c_k = \frac{(d^{(k)}, A(e^{(0)} + \sum_{i=0}^{k-1} - \frac{(d^{(i)}, Ae^{(i)})}{(d^{(i)}, Ad^{(i)})} d^{(i)}))}{(d^{(k)}, Ad^{(k)})}$ since d vectors are A-orthogonal
From CG algorithm: $x^{(i+1)} = x^{(i)} - \frac{(d^{(i)}, Ae^{(i)})}{(d^{(i)}, Ad^{(i)})} d^{(i)}$:
$c_k = \frac{(d^{(k)}, Ae^{(k)})}{(d^{(k)}, Ad^{(k)})}$
$x^{(i+1)} = x^{(i)} - c_k d^{(i)}$
$e^{(i)} = e^{(0)} - \sum_{j=0}^{i-1} c_j d^{(j)}$ which eliminiates a component of the error at each iteration
By the end of all iterations, $e^{(n-1)} = e^{(0)} - \sum_{j=0}^{n-1} c_j d^{(j)} = 0$
At each iteration, component $d^{(i)}$ of $e^{(0)}$ is reduced to 0.
With A-orthogonality, components $d^{(j)}, j \in 0, .., i$ of e are reduced to 0 and stays at 0 after iteration i, thus satisfying minimization of $\|e = x - \hat{x}\|$ in $span(d^{(0)}, ..., d^{(i)})$ after each iteration i.

4. Generalizing for assignment 2 question 3 and modifying boundary condition:

$$\begin{aligned} au_{xxxx} + bu_{xxyy} + cu_{yyyy} + du_{xx} + eu_{yy} + fu &= g \\ u &= \gamma \\ u_{xx} &= \zeta \\ u_y &= \eta \end{aligned}$$

(a) For $a = 1, b = 2, c = 1, d = 0, e = 0, f = 0$, modify algorithm from assignment 2 question to handle modified boundary condition.

Using method of undetermined coefficients for the new u_y boundary condition:

At $y = 0 (j = 0)$:

$$\begin{aligned} a_{-1}u_{i,j-1} &= a_{-1}(u_{i,j} + u^{(1)}_{i,j}(-h) + \frac{u^{(2)}_{i,j}(-h)^2}{2!} + \frac{u^{(3)}_{i,j}(-h)^3}{3!} + \frac{u^{(4)}_{i,j}(-h)^4}{4!}) + o(h^5) \\ a_0u_{i,j} &= a_0u_{i,j} \\ a_1u_{i,j+1} &= a_1(u_{i,j} + u^{(1)}_{i,j}(h) + \frac{u^{(2)}_{i,j}(h)^2}{2!} + \frac{u^{(3)}_{i,j}(h)^3}{3!} + \frac{u^{(4)}_{i,j}(h)^4}{4!}) + o(h^5) \\ a_2u_{i,j+2} &= a_2(u_{i,j} + u^{(1)}_{i,j}(2h) + \frac{u^{(2)}_{i,j}(2h)^2}{2!} + \frac{u^{(3)}_{i,j}(2h)^3}{3!} + \frac{u^{(4)}_{i,j}(2h)^4}{4!}) + o(h^5) \\ a_2u_{i,j+3} &= a_3(u_{i,j} + u^{(1)}_{i,j}(3h) + \frac{u^{(2)}_{i,j}(3h)^2}{2!} + \frac{u^{(3)}_{i,j}(3h)^3}{3!} + \frac{u^{(4)}_{i,j}(3h)^4}{4!}) + o(h^5) \end{aligned}$$

Solve:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 4 & 9 \\ -1 & 0 & 1 & 8 & 27 \\ 1 & 0 & 1 & 16 & 81 \end{bmatrix} \begin{bmatrix} a_{-1} \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{-1} \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -1/4 \\ -5/6 \\ 3/2 \\ -1/2 \\ 5/60 \end{bmatrix}$$

Stencil centered at $j = 0$:

$$\eta = \frac{1}{h}\begin{bmatrix} \frac{5}{60} \\ -\frac{1}{2} \\ \frac{3}{2} \\ -\frac{5}{6} \\ -\frac{1}{4} \\ 0 \\ 0 \end{bmatrix} + o(h^4)$$

At $y = 1 (j = n)$:

$$\begin{aligned}
a_{+1}u_{i,j+1} &= a_{+1}(u_{i,j} + u_{i,j}^{(1)}(h) + \frac{u_{i,j}^{(2)}(h)^2}{2!} + \frac{u_{i,j}^{(3)}(h)^3}{3!} + \frac{u_{i,j}^{(4)}(h)^4}{4!}) + o(h^5) \\
a_0u_{i,j} &= a_0u_{i,j} \\
a_{-1}u_{i,j-1} &= a_{-1}(u_{i,j} + u_{i,j}^{(1)}(-h) + \frac{u_{i,j}^{(2)}(-h)^2}{2!} + \frac{u_{i,j}^{(3)}(-h)^3}{3!} + \frac{u_{i,j}^{(4)}(-h)^4}{4!}) + o(h^5) \\
a_{-2}u_{i,j-2} &= a_{-2}(u_{i,j} + u_{i,j}^{(1)}(-2h) + \frac{u_{i,j}^{(2)}(-2h)^2}{2!} + \frac{u_{i,j}^{(3)}(-2h)^3}{3!} + \frac{u_{i,j}^{(4)}(-2h)^4}{4!}) + o(h^5) \\
a_{-3}u_{i,j-3} &= a_{-3}(u_{i,j} + u_{i,j}^{(1)}(-3h) + \frac{u_{i,j}^{(2)}(-3h)^2}{2!} + \frac{u_{i,j}^{(3)}(-3h)^3}{3!} + \frac{u_{i,j}^{(4)}(-3h)^4}{4!}) + o(h^5)
\end{aligned}$$

Solve:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & -2 & -3 \\ 1 & 0 & 1 & 4 & 9 \\ -1 & 0 & -1 & -8 & -27 \\ 1 & 0 & 1 & 16 & 81 \end{bmatrix} \begin{bmatrix} a_{+1} \\ a_0 \\ a_{-1} \\ a_{-2} \\ a_{-3} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{+1} \\ a_0 \\ a_{-1} \\ a_{-2} \\ a_{-3} \end{bmatrix} = \begin{bmatrix} 1/4 \\ 5/6 \\ -3/2 \\ 1/2 \\ -5/60 \end{bmatrix}$$

Stencil centered at $j = n$:

$$\eta = \frac{1}{h}\begin{bmatrix} 0 \\ 0 \\ \frac{1}{4} \\ \frac{5}{6} \\ -\frac{3}{2} \\ \frac{1}{2} \\ -\frac{5}{60} \end{bmatrix} + o(h^4)$$

Original boundary condition stencil applied from assignment 2 question 3 centered at $j = n$ and $j = 0$:

$$-\frac{1}{h^2}\eta_{orig} = \frac{1}{h^4}\begin{bmatrix} 0 \\ -1 \\ 2 \\ -1 \\ 0 \end{bmatrix} + o(1)$$

With original coefficient matrix, unapply original boundary stencil of u_{yy} and apply new boundary stencil of u_y onto the coefficient matrix:

At $j = 0$:

$$\frac{1}{h^2}\eta_{orig} + \frac{4}{h^3}\eta_{new} = \frac{1}{h^4}\begin{bmatrix} \frac{1}{3} \\ -2 \\ 1+6 \\ -2-\frac{10}{3} \\ 1-1 \\ 0 \\ 0 \end{bmatrix}$$

At $j = n$:

$$\frac{1}{h^2}\eta_{orig} - \frac{4}{h^3}\eta_{new} = \frac{1}{h^4}\begin{bmatrix} 0 \\ 0 \\ 1-1 \\ -2-\frac{10}{3} \\ 1+6 \\ -2 \\ \frac{1}{3} \end{bmatrix}$$

Changes applied to interior grid affecting each column of the grid:

$$\begin{bmatrix} 7 \\ -2 \\ \frac{1}{3} \\ 0 \\ ... \\ 0 \\ \frac{1}{3} \\ -2 \\ 7 \end{bmatrix}$$

This results in a delta (C) to the block diagonals of the coefficient matrix corresponding for each column as on top of the original values:

$$C = \begin{bmatrix} 7 & -2 & \frac{1}{3} & 0 & ... & & \\ 0 & ... & & & & & \\ & & & & & & \\ & & & & ... & & 0 \\ & & ... & 0 & \frac{1}{3} & -2 & 7 \end{bmatrix}$$

Overall delta to the coefficient matrix is $I_{n-1} \otimes C$

Boundary value constants moved to the RHS of the equation, Ax=g, for the changed boundary condition:

$j = 1$:
$-\frac{1}{h^4}(\frac{-16}{3}\gamma_{i,0} + 2\gamma_{i-1,0} + 2\gamma_{i+1,0}) + \frac{4}{h^3}\eta_{new}$

$j = n - 1$:
$-\frac{1}{h^4}(\frac{-16}{3}\gamma_{i,n} + 2\gamma_{i-1,n} + 2\gamma_{i+1,n}) - \frac{4}{h^3}\eta_{new}$

Reset of procedures is unchanged from the original problem.

Block diagonalization is modified by adding the change of the coefficient matrix represented by C.

$A = \frac{1}{h^4}(I_{n-1} \otimes tri\{1,-2,1\}^2 + tri\{1,-2,1\}^2 \otimes I_{n-1} + 2\, tri\{1,-2,1\} \otimes tri\{1,-2,1\} + I_{n-1} \otimes C)$

let $T = tri\{1,-2,1\}$

let V be a matrix of eigenvectors of $tri\{1,-2,1\}$
let $V^{-1}TV = D_T$ be diagonalization of $tri\{1,-2,1\}$
$D_T = diag(-4sin^2(\frac{l\pi}{2n})), l = 1,..,n-1$
let $(V^{-1}TV)^2 = V^{-1}T^2V = D_T^2 = D_B$ be diagonalization of matrix $(tridiag\{1,-2,1\})^2$

$$\begin{aligned}
I =& I_{n-1} \\
Block =& (V^{-1} \otimes I^{-1})A(V \otimes I) \\
Block =& (V^{-1} \otimes I^{-1})(\frac{1}{h^4}(I \otimes T^2 + T^2 \otimes I + 2\ T \otimes T + I \otimes C))(V \otimes I) \\
Block =& \frac{1}{h^4}((V^{-1} \otimes I^{-1})(I \otimes T^2)(V \otimes I) \\
&+ (V^{-1} \otimes I^{-1})(T^2 \otimes I)(V \otimes I) \\
&+ (V^{-1} \otimes I^{-1})(2\ T \otimes T)(V \otimes I)) \\
&+ (V^{-1} \otimes I^{-1})(I_{n-1} \otimes C)(V \otimes I)) \\
Block =& \frac{1}{h^4}((V^{-1}IV) \otimes (I^{-1}T^2I) \\
&+ (V^{-1}T^2V) \otimes (I^{-1}II) \\
&+ 2(V^{-1}TV) \otimes (I^{-1}TI)) \\
&+ ((V^{-1}IV) \otimes (I^{-1}CI) \\
Block =& \frac{1}{h^4}(I \otimes T^2 + D_B \otimes I + 2(D_T \otimes T) + I \otimes C)
\end{aligned}$$

FFT based procedure to solve u, from assignment 2, problem 3:

$A = (V \otimes I)Block(V^{-1} \otimes I^{-1})$
$A^{-1} = (V \otimes I)Block^{-1}(V^{-1} \otimes I^{-1})$
$\bar{u} = A^{-1}\bar{g} = (V \otimes I)Block^{-1}(V^{-1} \otimes I)\bar{g}$
$\bar{u} = (\mathbf{F}^{-1} \otimes I)Block^{-1}(\mathbf{F} \otimes I)\bar{g}$

DST of size $n-1$ to each column of $\bar{g}^T_{m-1 \times n-1}$:
$\bar{g}_{m-1 \times x-1} = reshape(\bar{g}, [m-1, n-1])$
$\bar{g}^{(1)}_{m-1 \times n-1} = (dst(\bar{g}^T_{m-1 \times x-1}))^T$
$stack(\bar{g}^{(1)}_{m-1,n-1}) = (\mathbf{F} \otimes I)\bar{g}$

solve $B\bar{g}^{(2)} = stack(\bar{g}^{(1)}_{m-1,n-1})$ for $\bar{g}^{(2)}$ via block solver:
$\bar{g}^{(2)} = B \backslash stack(\bar{g}^{(1)}_{m-1,n-1})$

inverse DST of size $n-1$ to each column of $(\bar{g}^{(2)}_{m-1,n-1})^T$:
$\bar{g}^{(2)}_{m-1.n-1} = reshape(\bar{g}^{(2)}, [m-1, n-1])$
$u_{m-1,n-1} = (idst((\bar{g}^{(2)}_{m-1,n-1})^T))^T$
$\bar{u} = stack(u_{m-1,n-1}) = (\mathbf{F}^{-1} \otimes I^{-1})\bar{g}^{(2)}$

(b) Write program for the generalized problem.

```
function [u_gmres,flag,relres,iter,resvec,A,g] = solver_pgmres(n,u,x,y,...
                                                    a,b,c,d,e,f)
%assume u(x,y),a,b,c,d,e,f are symbolic functions

dim=n-1;
h=1/n;

%a u_xxxx + b u_xxyy + c u_yyyy + d u_xx + e u_yy + f u = g

u_xxxx = diff(u,x,4);
u_yyyy = diff(u,y,4);
u_xxyy = diff(diff(u,x,2),y,2);
u_xx = diff(u,x,2);
u_yy = diff(u,y,2);

%boundary values
eta = sym('eta');
zeta = sym('zeta');

eta = diff(u,y,1);
zeta = diff(u,x,2);

gamma = sym('gamma');
gamma = u;

g = sym('g');
g(x,y) = a(x,y) * u_xxxx(x,y) + b(x,y) * u_xxyy(x,y) + c(x,y) * u_yyyy +...
         d(x,y) * u_xx(x,y) + e(x,y) * u_yy(x,y) + f(x,y) * u(x,y);

[A,g] = create_A_g_generic(n,x,y,a,b,c,d,e,f,g,...
    u_xxxx,u_xxyy,u_yyyy,u_xx,u_yy,eta,zeta,gamma);

%solver: gmres
maxit = 50;
tol = 10^(-9);
restart = 20;

[ret_u,ret_flag,ret_relres,ret_iter,resvec] = gmres(A,g,restart,tol,...
                                                   maxit,@precond);

u_gmres = ret_u;
flag = ret_flag;
```

```
    relres = ret_relres;
    iter = ret_iter;

        function u_solve = precond(r)
            h = 1/n;
            dim = n-1;
            [AA, B, T, C] = create_A_2(n);
            % use r from input instead of generating g vector
            % f_eta = diff(u,y,1);
            % f_zeta = diff(u,x,2);
            % %compute boundary values
            % g2 = compute_boundary_value_constants_2(n,f,f_eta,f_zeta);
            % %compute g(4th order mixed derivatives) on interior grid points
            % f_4th = diff(u,x,4) + 2*diff(diff(u,x,2), y, 2) + diff(u,y,4);
            % g1_grid = compute_grid(n,f_4th,x,y);
            % assert((size(g1_grid,1)==n-1) && (size(g1_grid,2)==n-1));
            % %remap grid points to 1D matrix
            % g1 = zeros(n-1*n-1,1);
            % for i=1:n-1
            %     for j=1:n-1
            %         g1(ij_to_k(dim, i, j),1)=g1_grid(i, j);
            %     end
            % end
            % g = g1+g2;
            %%%%%%%%%%%%%%%%%%%%%%%%%%%%
            % dst + block LU solver:
            % equivalent to:
            %     u = kron(F^-1_n, I_m) BlockDiag^-1 kron(F_n,I_m)g
            %     BlockDiag := block diagonalization of A

            I = eye(dim);
            j = 1:1:n-1;

            %compute eigenvalues of tridiag{1,-2,1}
            eigenvalues = -4*(sin(j*pi/(2*n))).^2;
            D_T = spdiags(eigenvalues', 0:0, dim, dim);
            % alternative: use eigs
            % need to reverse order to match up with ordering of dst, idst
            %D_T = spdiags([flip(eigs(T,dim))], 0:0, dim, dim); %alternative

            % kron(I,C) added below for the augmentation of
            % changed boundary condition from u_yy to u_y
            D_B = D_T^2;
            BlockDiag = (1/h^4)*(kron(I,T*T)+...
                                 kron(D_B,I)+...
```

```
                                     2*(kron(D_T,T))+...
                                     kron(I,C));

            %shape g to m by n matrix,
            %m being y-axis (fastest increasing axis)
            %of original grid
            g_m_by_n = reshape(r,[dim,dim]);
            %transform by eigenvectors
            g_1_n_by_m = dst(g_m_by_n');
            g_1_stacked = reshape(g_1_n_by_m',[dim*dim,1]);
            %solve BlockDiag g_2 = g_1
            g_2_stacked = BlockDiag\g_1_stacked;
            g_2_m_by_n = reshape(g_2_stacked, [dim, dim]);
            %inverse transform by eigenvectors
            u_n_by_m = idst(g_2_m_by_n');
            u_solve = reshape(u_n_by_m',[dim*dim,1]);
            %%%%%%%%%%%%%%%%%%%%%%%%%%%
        end
    end
```

```
function [A, B, T, C] = create_A_2(n)

    h=1/n;
    dim = n-1;
    I = eye(dim);

    %using composition of operators: u_xxxx+u_yyyy+2u_xxyy = (u_xx+u_yy)^2

    %construct tridiagonal matrix used for u_xx, u_yy
    %for each of dimensions x,y
    T = spdiags([ones(dim,1) -2*ones(dim,1) ones(dim,1)],...
                -1:1,...
                dim,dim);

    %let fastest increasing index to be along y dimension
    %kronecker product corresponding to difference operator u_yy
    B_yy = kron(I,T);
    %kronecker product corresponding to difference operator u_xx
    B_xx = kron(T,I);

    %matrix corresponding to (u_xx+u_yy)
    B = B_yy + B_xx;

    %alter values near boundary due to change of boundary condition
    %from u_yy = eta to u_y = eta
    C = spdiags([zeros(dim,1)], 0:0, dim, dim);

    %here we change matrix to revert origin boundary condition
    %corresponding to u_yy and apply the new boundary condition
    %for u_y
    %apply 4 * eta B.C. at bottom (j=1)
    %apply -4 * eta B.C. at top (j=n-1)
    D_bottom = 4*[-1/4 -5/6 3/2 -1/2 5/60];
    D_top = -4*flip([1/4 5/6 -3/2 1/2 -5/60]);

    C(1,1) = 1; %revert old boundary condition in y direction
    %apply new u_y boundary condition at j=1
    C(1,1:3) = C(1,1:3) + D_bottom(3:5);
    C(dim,dim) = 1; %revert old boundary condition in y direction
    %apply new u_y boundary condition at j=n-1
    C(dim,dim-2:dim) = C(dim,dim-2:dim) + D_top(1:3);

    CC = kron(I,C);
    A = 1/(h^4) * (CC + B^2);
end
```

```
function g2 = compute_boundary_value_constants_2(n,gamma,eta,zeta)
%compute boundary constants that touches or goes out of grid for
%the right hand side of Au=g
%returns negated value to be used for right side

%assumes gamma, eta, zeta are symbolic functions
%gamma: 0th order boundary condition
%eta: 1st order boundary condition along y dimension
%zeta: 2nd order boundary condition along x dimension

dim = n-1;
h = 1/n;

g2 = zeros(dim*dim,1);

for i = 1:n-1
    for j = 1:n-1

        if ( (3<=i) && (i<= n-3) && (3<=j) && (j<= n-3))
            continue;
        end

        if (i==1)
            g2(ij_to_k(dim,i,j)) = g2(ij_to_k(dim,i,j))...
                                   -1/(h^4)*(+2*gamma(0,(j-1)*h)...
                                             -6*gamma(0,j*h)...
                                             +2*gamma(0,(j+1)*h))...
                                   -1/(h^2)*(zeta(0,j*h));
        elseif (i==2)
            g2(ij_to_k(dim,i,j)) = g2(ij_to_k(dim,i,j))...
                                   -1/(h^4)*(gamma(0,j*h));
        end

        if (i==n-1)
            g2(ij_to_k(dim,i,j)) = g2(ij_to_k(dim,i,j))...
                                   -1/(h^4)*(+2*gamma(1,(j-1)*h)...
                                             -6*gamma(1,j*h)...
                                             +2*gamma(1,(j+1)*h))...
                                   -1/(h^2)*(zeta(1,j*h));
        elseif (i==n-2)
            g2(ij_to_k(dim,i,j))= g2(ij_to_k(dim,i,j))...
                                   -1/(h^4)*(gamma(1,j*h));
        end

        if (j==1)
```

```
                %apply o(h^4) accuracy u_y BC:
                % 4*eta/h^3 = +4*[-1/4 -5/6 3/2 -1/2 -5/60]*(1/h^4),
                % where 4*-5/6=-10/3 is the boundary entry
                g2(ij_to_k(dim,i,j)) = g2(ij_to_k(dim,i,j))...
                                        -1/(h^4)*(+2*gamma((i-1)*h,0)...
                                                 +(-8-10/3)*gamma(i*h,0)...
                                                 +2*gamma((i+1)*h,0))...
                                        +4/(h^3)*(eta(i*h,0));
            elseif (j==2)
                g2(ij_to_k(dim,i,j)) = g2(ij_to_k(dim,i,j))...
                                        -1/(h^4)*(gamma(i*h,0));
            end

            if (j==n-1)
                %apply o(h^4) accuracy u_y BC:
                % -4*eta/h^3 = -4*flip([1/4 5/6 -3/2 1/2 -5/60])*(1/h^4),
                % where -4*5/6=-10/3 is the boundary entry
                g2(ij_to_k(dim,i,j)) = g2(ij_to_k(dim,i,j))...
                                        -1/(h^4)*(+2*gamma((i-1)*h,1)...
                                                 +(-8-10/3)*gamma(i*h,1)...
                                                 +2*gamma((i+1)*h,1))...
                                        -4/(h^3)*(eta(i*h,1));
            elseif (j==n-2)
                g2(ij_to_k(dim,i,j)) = g2(ij_to_k(dim,i,j))...
                                        -1/(h^4)*(gamma(i*h,1));
            end
        end
    end

    %take care of overlap in corners in previous loops
    g2(ij_to_k(dim,1,1)) = g2(ij_to_k(dim,1,1)) + 1/(h^4)*(2*gamma(0,0));
    g2(ij_to_k(dim,1,n-1)) = g2(ij_to_k(dim,1,n-1)) + 1/(h^4)*(2*gamma(0,1));
    g2(ij_to_k(dim,n-1,1)) = g2(ij_to_k(dim,n-1,1)) + 1/(h^4)*(2*gamma(1,0));
    g2(ij_to_k(dim,n-1,n-1)) = g2(ij_to_k(dim,n-1,n-1)) + 1/(h^4)*(2*gamma(1,1));
```

```
function [A, rhs] = create_A_g_generic(n,x,y,a,b,c,d,e,f,g,...
                                       u_xxxx,u_xxyy,u_yyyy,...
                                       u_xx,u_yy,eta,zeta,gamma)
    %assumes a,b,c,d,e,f,u_xxxx,u_xxyy,u_yyyy,u_xx,u_yy are symbolic
    %a u_xxxx + b u_xxyy + c u_yyyy + d u_xx + e u_yy + f u = g

    h=1/n;
    dim = n-1;

    %padd 4 in each direction so that values at boundary and 1 grid outside
    %of boundary are captured
    pad = 4;
    pad_offset = 2;
    dim_mod = dim+4;

    rhs = sparse(zeros(dim_mod*dim_mod,1));

    gamma_aug = zeros(dim_mod,dim_mod);

    aa = zeros(dim,dim);
    bb = zeros(dim,dim);
    cc = zeros(dim,dim);
    dd = zeros(dim,dim);
    ee = zeros(dim,dim);
    ff = zeros(dim,dim);
    gg = zeros(dim,dim);

    for i=1:n-1
        for j=1:n-1
            aa(i,j) = double(a(i*h,j*h));
            bb(i,j) = double(b(i*h,j*h));
            cc(i,j) = double(c(i*h,j*h));
            dd(i,j) = double(d(i*h,j*h));
            ee(i,j) = double(e(i*h,j*h));
            ff(i,j) = double(f(i*h,j*h));
            gg(i,j) = double(g(i*h,j*h));
        end
    end

    aa_aug = zeros(dim_mod,dim_mod);
    bb_aug = zeros(dim_mod,dim_mod);
    cc_aug = zeros(dim_mod,dim_mod);
    dd_aug = zeros(dim_mod,dim_mod);
    ee_aug = zeros(dim_mod,dim_mod);
    ff_aug = zeros(dim_mod,dim_mod);
```

```
gg_aug = zeros(dim_mod,dim_mod);

for i=1:n-1
   aa_aug(:,2+i) = [ 0; 0; aa(:,i); 0; 0];
   bb_aug(:,2+i) = [ 0; 0; bb(:,i); 0; 0];
   cc_aug(:,2+i) = [ 0; 0; cc(:,i); 0; 0];
   dd_aug(:,2+i) = [ 0; 0; dd(:,i); 0; 0];
   ee_aug(:,2+i) = [ 0; 0; ee(:,i); 0; 0];
   ff_aug(:,2+i) = [ 0; 0; ff(:,i); 0; 0];
   gg_aug(:,2+i) = [ 0; 0; gg(:,i); 0; 0];
end

eta_eval = zeros(dim_mod,dim_mod);
zeta_eval = zeros(dim_mod,dim_mod);

for i=1:n-1+2
    point = (i-1)*h;
    eta_eval(i+1,2) = eta(point,0);
    eta_eval(i+1,dim_mod-1) = eta(point,1);

    zeta_eval(2,i+1) = zeta(0,point);
    zeta_eval(dim_mod-1,i+1) = zeta(1,point);

    gamma_aug(i+1,2) = gamma(point,0);
    gamma_aug(i+1,dim_mod-1) = gamma(point,1);
    gamma_aug(2,i+1) = gamma(0,point);
    gamma_aug(dim_mod-1,i+1) = gamma(1,point);
end

%stack by column (y-direction increasing fastest)
aa_stack = sparse(reshape(aa_aug',[dim_mod*dim_mod,1]));
bb_stack = sparse(reshape(bb_aug',[dim_mod*dim_mod,1]));
cc_stack = sparse(reshape(cc_aug',[dim_mod*dim_mod,1]));
dd_stack = sparse(reshape(dd_aug',[dim_mod*dim_mod,1]));
ee_stack = sparse(reshape(ee_aug',[dim_mod*dim_mod,1]));
ff_stack = sparse(reshape(ff_aug',[dim_mod*dim_mod,1]));
gg_stack = sparse(reshape(gg_aug',[dim_mod*dim_mod,1]));

eta_stack = sparse(reshape(eta_eval',[dim_mod*dim_mod,1]));
zeta_stack = sparse(reshape(zeta_eval',[dim_mod*dim_mod,1]));

gamma_stack = sparse(reshape(gamma_aug',[dim_mod*dim_mod,1]));

op = ones(dim_mod,1);
op_mod = ones(dim_mod,1);
```

```
op_mod(1:2,1)=0;
op_mod(dim_mod-1:dim_mod,1)=0;

%u_xxxx block pentadiag [. . . . .]
T = spdiags([op -4*op 6*op -4*op, op],...
              -2:2,...
              dim_mod,dim_mod);
T(1:2,:)=0;
T(dim_mod-1:dim_mod,:)=0;

Txxxx = kron(T, diag(op_mod));

%u_yyyy pentadiag [.....]
T = spdiags([op -4*op 6*op -4*op, op],...
              -2:2,...
              dim_mod,dim_mod);
T(1:2,:)=0;
T(dim_mod-1:dim_mod,:)=0;

Tyyyy = kron(diag(op_mod), T);

%u_xxyy:
T = spdiags([-2*op, 4*op, -2*op],...
              -1:1,...
              dim_mod,dim_mod);
T(1:2,:)=0;
T(dim_mod-1:dim_mod,:)=0;
Txxyy_a = kron(diag(op_mod), T); %same column

T = spdiags([op -2*op op],...
              -1:1,...
              dim_mod,dim_mod);
T(1:2,:)=0;
T(dim_mod-1:dim_mod,:)=0;

II = spdiags([op],...
              1,...
              dim_mod,dim_mod);
II(1:2,:)=0;
II(end-1:end,:)=0;
Txxyy_b = kron(II, T); %column to the right

II = spdiags([op],...
              -1,...
              dim_mod,dim_mod);
```

```
II(1:2,:)=0;
II(end-1:end,:)=0;
Txxyy_c = kron(II, T); %column to the left

Txxyy = Txxyy_a + Txxyy_b + Txxyy_c;

%u_xx tridiag [. . .]
T = spdiags([op -2*op op],...
             -1:1,...
             dim_mod,dim_mod);
T(1:2,:)=0;
T(dim_mod-1:dim_mod,:)=0;
Txx = kron(T, diag(op_mod));

%u_yy tridiag [...]
T = spdiags([op -2*op op],...
             -1:1,...
             dim_mod,dim_mod);
T(1:2,:)=0;
T(dim_mod-1:dim_mod,:)=0;
Tyy = kron(diag(op_mod), T);

%u
T = spdiags([op],...
            0,...
            dim_mod,dim_mod);
T(1:2,:)=0;
T(dim_mod-1:dim_mod,:)=0;
Tu = kron(diag(op_mod),T);

%multiply by provided coefficients a,b,c,d,e,f
%a u_xxxx + b u_xxyy + c u_yyyy + d u_xx + e u_yy + f u = g
Txxxx_scaled = bsxfun(@times,Txxxx,aa_stack);
Txxyy_scaled = bsxfun(@times,Txxyy,bb_stack);
Tyyyy_scaled = bsxfun(@times,Tyyyy,cc_stack);
Txx_scaled = bsxfun(@times,Txx,dd_stack);
Tyy_scaled = bsxfun(@times,Tyy,ee_stack);
Tu_scaled = bsxfun(@times,Tu,ff_stack);

%gamma_stack contains the values at boundary, dot product with each row
%to get boundary constants, move to right hand side of equation
rhs = rhs - 1/(h^4)*(Txxyy_scaled * gamma_stack);
rhs = rhs - 1/(h^2)*(Txx_scaled * gamma_stack);
rhs = rhs - 1/(h^2)*(Tyy_scaled * gamma_stack);
rhs = rhs - (Tu_scaled * gamma_stack);
```

```
%cancel value outside of grid for boundary condition in x-direction:
%Txxxx_scaled(:,1:dim_mod)
%Txxxx_scaled(:,dim_mod*dim_mod-dim_mod+1:dim_mod*dim_mod)
Tboundary_x = [1 -2 1];
for row=1:dim_mod*dim_mod
    for i=1:dim_mod
        val = Txxxx_scaled(row,i);
        if val ~= 0
            val_negate = -val;
            scale = val_negate/Tboundary_x(1);
            rhs(row) = rhs(row) + scale * 1/(h^2) * zeta_stack(i+dim_mod);
            Tboundary_x_scaled = scale.* Tboundary_x;
            for j=1:length(Tboundary_x_scaled)
                Txxxx_scaled(row,i+(j-1)*dim_mod) = ...
                   Txxxx_scaled(row,i+(j-1)*dim_mod) +...
                   Tboundary_x_scaled(j);
            end
            break;
        end
    end
    for i=dim_mod*dim_mod:-1:dim_mod*dim_mod-dim_mod+1
        val = Txxxx_scaled(row,i);
        if val ~= 0
            val_negate = -val;
            scale = val_negate/Tboundary_x(3);
            rhs(row) = rhs(row) + scale * 1/(h^2) * zeta_stack(i-dim_mod);
            Tboundary_x_scaled = scale.* Tboundary_x;
            for j=1:length(Tboundary_x_scaled)
                Txxxx_scaled(row,i-(j-1)*dim_mod) = ...
                   Txxxx_scaled(row,i-(j-1)*dim_mod) +...
                   Tboundary_x_scaled(3-j+1);
            end
            break;
        end
    end
end
%sanity check that all exterior grid points are zeroed out
assert(sum(sum(Txxxx_scaled(:,1:dim_mod)~=0))==0);
assert(sum(sum(Txxxx_scaled(:,dim_mod*dim_mod-dim_mod+1:dim_mod*dim_mod)~=0

%gamma_stack contains the values at boundary, dot product with each row
%to get boundary constants, move to right hand side of equation
rhs = rhs - 1/(h^4)*(Txxxx_scaled * gamma_stack);
```

```
%cancel value outside of grid for boundary condition in y-direction:
%col_indices = 1:dim_mod:dim_mod*dim_mod;
%Tyyyy_scaled(:,col_indices)
%Tyyyy_scaled(:,dim_mod:dim_mod:dim_mod*dim_mod)
%bottom: eta = 1/h*[-1/4 -5/6 3/2 -1/2 5/60]
%top: eta = 1/h*flip([1/4 5/6 -3/2 1/2 -5/60])
Tboundary_y_bottom = [-1/4 -5/6 3/2 -1/2 5/60];
Tboundary_y_top = flip([1/4 5/6 -3/2 1/2 -5/60]);
for row=1:dim_mod*dim_mod
    for i=1:dim_mod:dim_mod*dim_mod
        val = Tyyyy_scaled(row,i);
        if val ~= 0
        % apply once when first non-zero exterior grid point is detected
            val_negate = -val;
            scale = val_negate/Tboundary_y_bottom(1);
            rhs(row) = rhs(row) + scale * 1/(h^3) * eta_stack(i+1);
            Tboundary_y_scaled = scale.* Tboundary_y_bottom;
            %apply BC stencil values
            for j=1:length(Tboundary_y_scaled)
                Tyyyy_scaled(row,i+j-1) = Tyyyy_scaled(row,i+j-1) +...
                    Tboundary_y_scaled(j);
            end
            break;
        end
    end
    for i=flip(dim_mod:dim_mod:dim_mod*dim_mod)
        val = Tyyyy_scaled(row,i);
        if val ~= 0
        % apply once when first non-zero exterior grid point is detected
            val_negate = -val;
            scale = val_negate/Tboundary_y_top(end);
            rhs(row) = rhs(row) + scale * 1/(h^3) * eta_stack(i-1);
            Tboundary_y_scaled = scale.* Tboundary_y_top;
            %apply BC stencil values
            for j=1:length(Tboundary_y_scaled)
                Tyyyy_scaled(row,i-(j-1)) = Tyyyy_scaled(row,i-(j-1)) +...
                    Tboundary_y_scaled(length(Tboundary_y_scaled)-j+1);
            end
            break;
        end
    end
end
%sanity check that all exterior grid points are zeroed out
assert(sum(sum(Tyyyy_scaled(:,1:dim_mod:dim_mod*dim_mod)~=0))==0);
assert(sum(sum(Tyyyy_scaled(:,dim_mod:dim_mod:dim_mod*dim_mod)~=0))==0);
```

```
    %gamma_stack contains the values at boundary, dot product with each row
    %to get boundary constants, move to right hand side of equation
    rhs = rhs - 1/(h^4)*(Tyyyy_scaled * gamma_stack);

    rhs = rhs + gg_stack;

    Atemp = 1/(h^4)*(Txxxx_scaled + Txxyy_scaled + Tyyyy_scaled) +...
        1/(h^2)*(Txx_scaled + Tyy_scaled) + Tu_scaled;

    %get only interior points
    Atemp = Atemp(2*dim_mod+1:dim_mod*dim_mod-dim_mod*2,...
                  2*dim_mod+1:dim_mod*dim_mod-dim_mod*2);
    A = sparse(zeros(dim*dim,dim*dim));

    for i =1:dim
         offset_i = (i-1) * dim_mod + 1;
         for j =1:dim
             offset_j = (j-1) * dim_mod + 1;
             A((i-1)*dim+1:(i-1)*dim+1+dim-1,...
               (j-1)*dim+1:(j-1)*dim+1+dim-1) = ...
                       Atemp(offset_i+2:offset_i+2+dim-1,...
                             offset_j+2:offset_j+2+dim-1);

         end
    end

    rhs_m_by_n = reshape(rhs,[dim_mod,dim_mod]);
    rhs_temp = rhs_m_by_n(3:3+dim-1, 3:3+dim-1);
    rhs = reshape(rhs_temp,[dim*dim,1]);

end
```

```
function grid = compute_grid(n,f,x,y)
    % assumes f is a symbolic function taking in x and y as inputs
    % return evaluations at interior grid points as f(grid(x,y))

    h = 1/n;

    points = 1:1:n-1;
    assert(size(points,2)==n-1);

    domain = points .* h;

    grid = zeros(n-1,n-1);

    for i=1:1:n-1
        grid(:,i) = f(domain, ones(1,n-1).* domain(i));
    end
end

function k = ij_to_k(dim,i,j)
    assert(i>=1);
    assert(i<=dim);
    assert(j>=1);
    assert(j<=dim);
    k=dim*(i-1)+j;
end

function [i,j] = k_to_ij(dim,k)
    assert(k>=1);
    assert(k<=dim*dim);
    i=fix((k-1)/dim)+1;
    j=mod(k-1,dim)+1;
end

function out = convert_to_cartesian(dim, input)
    %convert to cartesian coordinate
    out = zeros(dim,dim);
    for c = 1:dim*dim
        [i,j] = k_to_ij(dim,c);
        out(i,j) = input(c);
    end
end
```

```
syms x y;
syms u(x,y);
u(x,y) = x^(9/2)*y^(9/2);

syms a b c d e f;
a(x,y)=1+exp(x+y);
b(x,y)=1+1/(x+y);
c(x,y)=3+x*y;
d(x,y)=-sin(x)*sin(y);
e(x,y)=-1-exp(x+y);
f(x,y)=x+y;

ns = [8,16,32,64,128];
errors = zeros(length(ns),1);

for i=1:length(ns)
    n = ns(i);
    dim = n-1;
    fprintf("n: %d\n",n);

    [u_gmres,flag,relres,iter,resvec,A,g] = solver_pgmres(n,u,x,y, ...
                                                a,b,c,d,e,f);

    semilogy(0:length(resvec)-1,resvec./norm(g),'-o');
    xlabel('Iteration number');
    ylabel('Relative residual');

    u_exact = full(compute_grid(n,u,x,y));
    u_exact_stacked = reshape(u_exact', [dim*dim,1]);

    %block LU
    u_block = A\g;

    fprintf("norm of difference of banded LU and PGMRES: ...
        %f\n", norm(u_gmres-u_block));
    fprintf("banded LU: max abs knot error:...
        %f\n", max(abs(u_block-u_exact_stacked)));
    fprintf("PGMRES: max abs knot error: ...
        %f\n", max(abs(u_gmres-u_exact_stacked)));
    fprintf("PGMRES: converge flag: %d\n", flag);
    fprintf("PGMRES: iter:(%d, %d)\n", iter(1), iter(2));
    errors(i) = max(abs(u_gmres-u_exact_stacked));
end

order_of_convergence = zeros(length(ns)-1,1);
```

```
for i=1:length(ns)-1
    order_of_convergence(i)=log2(errors(i)/errors(i+1));
    fprintf("order of convergence: (n: %d, 2n: %d): %f\n", ...
        ns(i), ns(i+1), order_of_convergence(i));
end
```

Results for:
$u = x^{\frac{9}{2}} y^{\frac{9}{2}}$,
$a = 1 + exp(x + y)$,
$b = 1 + 1/(x + y)$,
$c = 3 + xy$,
$d = -sin(x)sin(y)$,
$e = -1 - exp(x + y)$,
$f = x + y$
$gmres\ tol = 10^{-9}, restart = 20$

n	banded LU max abs knot error
8	0.000276791847
16	0.000064385517
32	0.000015620426
64	0.000003859295
128	0.000000958319

Table 2: Statistics using Banded LU

n	PGMRES max abs knot error	iter outer	iter inner	norm of sol'n diff w/ banded LU
8	0.000276791853	1	11	$1.29 * 10^{-10}$
16	0.000064385534	1	13	$6.95 * 10^{-10}$
32	0.000015620429	1	15	$1.738 * 10^{-9}$
64	0.000003859311	1	15	$14.746 * 10^{-9}$
128	0.000000958382	1	17	$14.536 * 10^{-9}$

Table 3: Statistics using PGMRES

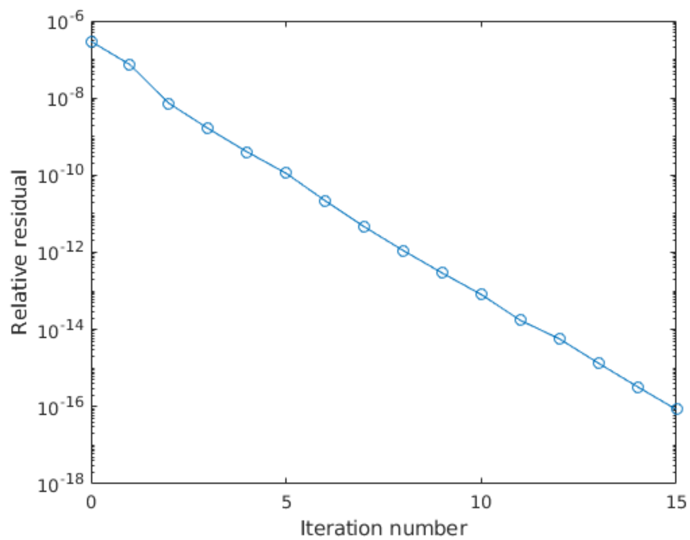
n	2n	order of convergence
8	16	2.103993
16	32	2.043303
32	64	2.017019
64	128	2.009671

Table 4: Order of Convergence using PGMRES

The order of convergence using PMGRES is around 2.

It seems there is an entry threshold of work in terms of number of iterations for PGMRES for a given tolerance. As n is increased, the inner iteration gradually increases and stabilizes before slightly increasing again from n=64 to n=128. As n increases, it takes longer for PGMRES to achieve a given tolerance especially from 64 to 128. In addition, as n increases, the solution of PMGRES seems to degrade when compared to Banded LU in terms of max absolute knot error.

Residual history shows a monotonic decrease as expected since GMRES's direction vectors generation is based on the Gram-Schmidt method. Result for n=32 is shown below.



(c) Consider BVP II with $g = 1, \gamma = \eta = \zeta = 0$. Exact solution of this BVP is not known. Solve with banded LU and solver from part (a). Output approximate solutions at point (0.5,0.5) of the domain for n=8,16,32,64,128. What is the order of convergence of the computed solutions? How big an n should we take in order to obtain the solution at (0.5,0.5) with 10^{-9} precision?

n	part (a) solver	Banded LU
8	0.0018806907	0.0018806907
16	0.0019054534	0.0019054534
32	0.0019140108	0.0019140108
64	0.0019163418	0.0019163418
128	0.0019169380	0.0019169380

Table 5: evaluation at (0.5,0.5)

From the previous part, we see PGMRES has an order of convergence of around 2 and the Banded LU solver gives a slightly better answer to the exact solution, meaning it has an order of convergence of 2. Part (a) solver gives an answer that is nearly identical to the Banded LU solver, which is known to have an order of convergence of 2. This means the part (a) solver also has an order of convergence of 2.

Find n for 10^{-9} precision at $(0.5, 0.5)$:

let $v = |u_{16} - u_8| = 2.47627 * 10^{-5}$
$\sum_{i=j}^{\infty} v2^{-2i} \leq 10^{-9}$
$\sum_{i=0}^{\infty}(\frac{1}{4})^i - \sum_{i=0}^{j-1}(\frac{1}{4})^i \leq \frac{10^{-9}}{v}$
$j \geq -\frac{1}{2}log_2(\frac{3*10^{-9}}{4v}) \approx 7.505$
let $j = 8$
$n = 8 * 2^j = 2^{11}$

```
ns = [8,16,32,64,128];
for i=1:length(ns)
    n = ns(i);
    dim = n-1;
    g = ones(dim*dim,1);
    fprintf("n: %d\n",n);
    [A, u] = my_precond(n,g);
    mid = n/2;
    k = ij_to_k(dim,mid,mid);
    u_LU = A\g;
    u_mid = u(k);
    u_mid_lu = u_LU(k);
    fprintf("u(midpoint): %.12f\n", u_mid);
    fprintf("LU u(midpoint): %.12f\n", u_mid_lu);
end
```