- A.13

  Show $\mathbb{R}^n$ is open:
  Since $(\forall p \in \mathbb{R}^n)(\forall \delta > 0)(B_\delta^n(p)) \subset \mathbb{R}^n$, where $B_\delta^n(p) := \{x \in \mathbb{R}^n : \|x - p\| < \delta\}$ is true. This trivially satisfies the definition of an open set: $(\forall p \in \mathbb{R}^n)(\exists \delta > 0)(B_\delta^n(p)) \subset \mathbb{R}^n$. Thus $\mathbb{R}^n$ is open.

  Show $\mathbb{R}^n$ is closed:
  Since all limit points $\{p\}$, where $(\forall p \in \mathbb{R}^n)(\forall \delta > 0)(\forall q \in B_\delta^n(p), q \neq p)$ trivially satisfies $q \in \mathbb{R}^n)$. This satisfies the definition of a closed set. Thus $\mathbb{R}^n$ is closed.

  Show $\emptyset$ is open:
  $(\forall p \in \emptyset)(\exists \delta > 0)(B_\delta^n(p) \subset \emptyset)$ is trivially upheld since p is non-existent and thus need not be evaluated, thus $\emptyset$ is open.

  Show $\emptyset$ is closed:
  Since $\emptyset$ is empty, the limit points set satisfying $\{p : (\forall \delta > 0)(\exists q \in B_\delta^n(p), q \neq p)\}$ is empty and thus trivially satisfies $\{p : (\forall \delta > 0)(\exists q \in B_\delta^n(p), q \neq p)(q \in \emptyset)\}$. Thus $\emptyset$ contains all its limit points and is closed.

- A.14

  Show X is closed iff $X^c = \{x \in \mathbb{R}^n : x \notin X\}$ is open

  Show X closed $\rightarrow X^c$ open:
  X closed $\rightarrow \partial(X) \subset X$
  $X = interior(X) \cup \partial(X)$
  $X^c = (interior(X) \cup \partial(X))^c$
  $X^c = (interior(X))^c \cap (\partial(X))^c$
  $X^c = closure(X^c) \cap (\partial(X))^c$
  $(\partial(X))^c = interior(X) \cup interior(X^c)$
  $X^c = closure(X^c) \cap (interior(X) \cup interior(X^c))$
  $X^c = interior(X^c)$
  Then X^c is open.

  Show X^c open $\rightarrow X$ closed:
  $X^c = closure(X^c) \cap (\partial(X))^c = interior(X^c)$
  $(X^c)^c = X = (interior(X^c))^c = closure((X^c)^c) = closure(X)$
  Then X is closed

- A.15

  Show X is closed iff X contains all its boundary points

  Show X closed $\rightarrow$ X contains all its boundary points:
  From the definition of a boundary point, it has for all open neighbourhood with at least one point in the neighbourhood that is also in X. Then, boundary point is also a limit point assuming the boundary point is not the only singular point within X around the neighbourhood. In the degenerate case, the boundary point is a member of X anyways. Thus, X is closed implies it contains all its limits points which implies it contains all boundary points plus the degenerate case.

  Show X contains all its boundary points $\rightarrow$ X closed:
  $X = interior(X) \cup \partial(X)$
  $(\forall p : \text{p is a limit point}) p \in (\text{interior}(X) \cup \partial(X))$ where $\text{interior}(X) \cap \partial(X) = \emptyset$
  If X includes all the boundary points in addition to its interior points, then it is clear that all limit points of X are contained within X, hence X is closed.

- A.16

  1. $\mathbb{Z} \times \{0\}$:
     not open
     closed
     limit points $= \emptyset$
     boundary points $= \mathbb{Z} \times \{0\}$

  2. $\{x \in \mathbb{R}^2 : x_1 = 1/n, n \in \mathbb{N}, x_2 = 0\}$:
     not open
     not closed
     limit points $= \{[0, 0]^T\}$
     boundary points $= \{[1/n, 0]^T\},\ n \in \mathbb{N}$

  3. $\{x \in \mathbb{R}^2 : 1 \leq x_1 \leq 2\}$:
     not open
     closed
     limit points $= \{[x_1, x_2]^T\}, 1 \leq x_1 \leq 2, x_2 \in \mathbb{R}$
     boundary points $= \{[1, x_2]^T, [2, x_2]^T\}, x_2 \in \mathbb{R}$

  4. $\{x \in \mathbb{R}^2 : 1 < x_1 < 2\}$:
     open
     not closed
     limit points $= \{[x_1, x_2]^T\}, 1 \leq x_1 \leq 2, x_2 \in \mathbb{R}$
     boundary points $= \{[1, x_2]^T, [2, x_2]^T\}, x_2 \in \mathbb{R}$

5. $\{x \in \mathbb{R}^2 : 1 \leq \|x\|_2 \leq 2\}$:
   not open
   closed
   limit points $\{x \in \mathbb{R}^2 : 1 \leq \|x\|_2 \leq 2\}$
   boundary points: 2 circles of radius 1,2 centered at origin$=\{x : x \in \mathbb{R}^2, \|x\|_2 = 1 \vee \|x\|_2 = 2\}$

- A.17

1. Is $X = \{\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathbb{R}^2 : |x_2| < |x_1|\}^T$ connected?

   let $X = X_1 \cup X_2$, where:
   $X_1 = \{\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T : |x_2| < |x_1|, x_1 < 0\}$
   $X_2 = \{\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T : |x_2| < |x_1|, x_1 \geq 0\}$

   $closure(X_1) = \bar{X}_1 = \{\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T : |x_2| \leq |x_1|, x_1 \leq 0\}$
   $closure(X_2) = \bar{X}_2 = \{\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T : |x_2| \leq |x_1|, x_1 \geq 0\}$

   $X_1 \cap \bar{X}_2 = \emptyset$
   $\bar{X}_1 \cap X_2 = \emptyset$

   Then, X_1 and X_2 are separated. X can be partitioned as a union of 2 non-empty separated sets. Thus X is not connected.

2. Is $X \cup \{\begin{bmatrix} 0 & 0 \end{bmatrix}^T\}$ connected?

   let $X = X_1 \cup X_2$, where:
   $X_1 = \{\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T : |x_2| < |x_1|, x_1 < 0\}$
   $X_2 = \{\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T : |x_2| < |x_1|, x_1 \geq 0\} \cup \begin{bmatrix} 0 & 0 \end{bmatrix}$

   $closure(X_1) = \bar{X}_1 = \{\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T : |x_2| \leq |x_1|, x_1 \leq 0\}$
   $closure(X_2) = \bar{X}_2 = \{\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T : |x_2| \leq |x_1|, x_1 \geq 0\}$

   $X_1 \cap \bar{X}_2 = \emptyset$
   $\bar{X}_1 \cap X_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$

   Then, X_1 and X_2 are not separated. $X \cup \{\begin{bmatrix} 0 & 0 \end{bmatrix}^T\}$ is a union of 2 non-empty non-separated sets. Thus, $X \cup \{\begin{bmatrix} 0 & 0 \end{bmatrix}^T\}$ is connected.

- A.18

  $X = \{x \in \mathbb{R}^2 : 0 < \|x\|_2 < 1\} \cup \{\begin{bmatrix}0 & 0\end{bmatrix}^T\}$.

  1. Find all limit points.
     $\{x \in \mathbb{R}^2 : 0 < \|x\|_2 < 1\} \cap closure(\{\begin{bmatrix}0 & 0\end{bmatrix}\}) \neq \emptyset$
     $\{x \in \mathbb{R}^2 : 0 < \|x\|_2 < 1\}$ and $\{\begin{bmatrix}0 & 0\end{bmatrix}\}$ are connected
     $\{x \in \mathbb{R}^2 : 0 \leq \|x\|_2 < 1\} = \{x \in \mathbb{R}^2 : 0 < \|x\|_2 < 1\} \cup \{\begin{bmatrix}0 & 0\end{bmatrix}\}$
     all limit points of $X = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$
  2. Find all boundary points.
     all boundary points($\{x \in \mathbb{R}^2 : 0 \leq \|x\|_2 < 1\}$)=$\{x \in \mathbb{R}^2 : \|x\|_2 = 1\}$

- A.22

  1. Using Definition A.23, write definition that a function $f : X \to Y$ is not Lipschitz continuous at $x_0 \in X$.
     f is not Lipschitz is the true statement that:
     $\neg(\exists \delta > 0)(\exists L > 0)(\forall x \in B_\delta^n(x_0))(\forall y \in B_\delta^n(x_0))(\|f(x) - f(y)\| \leq L\|x - y\|)$
     $= (\forall \delta > 0)\neg(\exists L > 0)(\forall x \in B_\delta^n(x_0))(\forall y \in B_\delta^n(x_0))(\|f(x) - f(y)\| \leq L\|x - y\|)$
     $= (\forall \delta > 0)(\forall L > 0)\neg(\forall x \in B_\delta^n(x_0))(\forall y \in B_\delta^n(x_0))(\|f(x) - f(y)\| \leq L\|x - y\|)$
     $= (\forall \delta > 0)(\forall L > 0)(\exists x \in B_\delta^n(x_0))\neg(\forall y \in B_\delta^n(x_0))(\|f(x) - f(y)\| \leq L\|x - y\|)$
     $= (\forall \delta > 0)(\forall L > 0)(\exists x \in B_\delta^n(x_0))(\exists y \in B_\delta^n(x_0))\neg(\|f(x) - f(y)\| \leq L\|x - y\|)$
     $= (\forall \delta > 0)(\forall L > 0)(\exists x \in B_\delta^n(x_0))(\exists y \in B_\delta^n(x_0))(\|f(x) - f(y)\| > L\|x - y\|)$

  2. Prove that $f : \mathbb{R} \to \mathbb{R}, f(x) = x^{1/3}$ is not Lipschitz continuous at $x_0 = 0$.
     $f(x_0) = 0$
     need to show:
     $(\forall \delta > 0)(\forall L > 0)(\exists x \in B_\delta^n(0))(\exists y \in B_\delta^n(0))(\|f(x) - f(y)\| > L\|x - y\|)$
     let $y = x_0 = 0$
     $(\forall \delta > 0)(\forall L > 0)(\exists x \in B_\delta^n(0))(\|f(x) - 0\| > L\|x - 0\|)$
     using 2-norm, $\|f(x)\| = |x^{1/3}|$
     let $x = c$
     $f(c) = |c^{1/3}|$
     $\|c - 0\| = |c|$
     $\|f(c) - 0\| = |c^{1/3}| = \frac{|c|}{|c^{2/3}|} = |c^{2/3}|^{-1}\|c - 0\|$
     $\|f(x) - f(y)\| = |c^{2/3}|^{-1}\|x - y\|$
     let c go towards 0:
     $L_c = \lim_{c \to 0}|c^{2/3}|^{-1} = \infty$
     $\|f(x) - f(y)\| = L_c\|x - y\|$, at $y = x_0$, x near neighbourhood of x_0
     Therefore, it is true that:
     $(\forall \delta > 0)(\forall L > 0)(L_c\|x - y\| > L\|x - y\|)$
     which satisfy:
     $(\forall \delta > 0)(\forall L > 0)(\exists x \in B_\delta^n(0))(\exists y \in B_\delta^n(0))(\|f(x) - f(y)\| > L\|x - y\|)$
     Thus, $f(x) = x^{1/3}$ is not Lipschitz continuous at $x_0 = 0$.

- A.24 let $A : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto Ax$.

1. Show this function is continuously differentiable

   Show $x \mapsto Ax$ is differentiable:
   $\lim_{h\to 0} \frac{\|A(x+h)-A(x)-Lh\|}{\|h\|} = 0$
   $\lim_{h\to 0} \frac{\|Ax+Ah-Ax-Lh\|}{\|h\|} = 0$
   $\lim_{h\to 0} \frac{\|Ah-Lh\|}{\|h\|} = 0$
   $\lim_{h\to 0}(A-L)h = 0$ by positivity of norm
   $A = L$
   $L = f'(x_0) = \frac{\partial(Ax)}{\partial x}(x_0) = A$
   $(\forall x_0 \in X) f'(x_0) = L = A$
   Then, $x \mapsto Ax$ is differentiable.

   Show $x \mapsto df_x$ is continuous:
   $df_x = A$
   use of Lipschitz continuity $\to$ continuous:
   show $(\forall x_0 \in X)(\exists \delta > 0)(\exists L > 0)(\forall x, y \in B^n_\delta(x_0))\|Ax - Ay\| \leq L\|x - y\|$
   since $\|A\| = sup\{\frac{\|A(x-y)\|}{\|x-y\|} : x - y \neq 0\}$
   $\|A\| \geq \frac{\|A(x-y)\|}{\|x-y\|}, x - y \neq 0$
   let $L = \|A\|$
   $L \geq \frac{\|A(x-y)\|}{\|x-y\|}, x - y \neq 0$
   $L\|x - y\| \geq \|A(x - y)\|$
   Then it is true that:
   $(\forall x_0 \in X)(\exists \delta > 0)(\exists L > 0)(\forall x, y \in B^n_\delta(x_0))\|Ax - Ay\| \leq L\|x - y\|$
   Then, $x \mapsto df_x$ is Lipschitz continous and is continuous.
   $A : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto Ax$ is continuously differentiable.

2. Show this function's differential is $\mathbf{dA}_{x_0}(h) = Ah$
   $\lim_{h\to 0} \frac{\|A(x_0+h)-A(x_0)-df_{x_0}(h)\|}{\|h\|} = 0$
   $\lim_{h\to 0} \frac{\|Ah-df_{x_0}(h)\|}{\|h\|} = 0$
   $Ah - df_{x_0}(h) = 0$
   $df_{x_0}(h) = Ah$

- A.25 Consider a quadratic form $Q(x) : \mathbb{R}^n \to \mathbb{R}$

1. Show it is differentiable with differential of $\mathbf{dQ_{x_0}}(h) = 2x_0^T Qh$ so that matrix representation of its differential is $dQ_{x_0} = 2x_0^T Q$
$\lim_{h\to 0} \frac{\|f(x_0+h)-f(x_0)-df_{x_0}(h)\|}{\|h\|} = 0$
$\lim_{h\to 0} \frac{\|(x_0+h)^T Q(x_0+h)-x_0^T Qx_0-df_{x_0}(h)\|}{\|h\|} = 0$
$\lim_{h\to 0} \frac{\|(x_0^T Qx_0+x_0^T Qh+h^T Qx_0+h^T Qh-x_0^T Qx_0-df_{x_0}(h)\|}{\|h\|} = 0$
$\lim_{h\to 0} \frac{\|(2x_0^T Qh+h^T Qh-df_{x_0}(h)\|}{\|h\|} = 0$
$\frac{h^T Qh}{\|h\|}$ vanishes as $h \to 0$
$\lim_{h\to 0} \frac{\|(2x_0^T Qh-df_{x_0}(h)\|}{\|h\|} = 0$
$2x_0^T Qh - df_{x_0}(h) = 0$ by positivity of norm
Thus, the differential of a quadratic form at x_0 is $df_{x_0}(h) = 2x_0^T Qh$
$(\forall x_0 \in \mathbb{R}^n) df_{x_0}(h) = 2x_0^T Qh$ exists and a quadratic form is differentiable.
Thus, matrix representation of its differential is $dQ_{x_0} = 2x_0^T Q$

2. Show it is C^1
From previous part, Q is shown to be continuous. Then we need to show its differential function is also continuous.
Using Lipschitz continuity $\to$ continuous:
Show:
$(\forall x_0 > 0)(\exists \delta > 0)(\exists L > 0)(\forall x, y \in B^n_\delta(x_0))\|f(x) - f(y)\| \leq L\|x - y\|$
let $f = dQ_{x_0} = 2x_0^T Q$
$(\forall x_0 > 0)(\exists \delta > 0)(\exists L > 0)(\forall x, y \in B^n_\delta(x_0))\|(2x^T Q)^T - (2y^T Q)^T\| \leq L\|x - y\|$
$\|(2x^T Q)^T - (2y^T Q)^T\| = 2\|Q^T x - Q^T y\|$
Q is symmetric: $Q^T = Q$
$\|(2x^T Q)^T - (2y^T Q)^T\| = 2\|Q(x - y)\|$
use upper bound of $\|Q(x - y)\|$ with matrix norm:
$\|Q\| = sup\{\frac{\|Q(x-y)\|}{\|x-y\|}, x - y \neq 0\}$
$\|Q\|\|x - y\| \geq \|Q(x - y)\|$
$\|(2x^T Q)^T - (2y^T Q)^T\| = 2\|Q(x - y)\| \leq 2\|Q\|\|x - y\|$
let $L = 2\|Q\|$
$\|(2x^T Q)^T - (2y^T Q)^T\| \leq L\|x - y\|$
It is true that:
$(\forall x_0 > 0)(\exists \delta > 0)(\exists L > 0)(\forall x, y \in B^n_\delta(x_0))\|(2x^T Q)^T - (2y^T Q)^T\| \leq L\|x - y\|$
Then, the differential of $Q(x)$, $2x_0^T Q$, is Lipschitz continuous and thus continuous. Since $Q(x)$ is continuous and its differential is continuous, then Q is continuously differentiable (C^1).