- 3.8

$$
\begin{aligned}
\text{let } \tilde{x} &= x - \hat{x} \\
\dot{\tilde{x}} &= Ax + f(x) - (A\hat{x} + f(\hat{x}) + L(y - C\hat{x}) \\
&= (A - LC)\tilde{x} + f(x) - f(\hat{x}) \\
\text{Desired property of observer: } & lim_{t\to\infty}\tilde{x}(t) = 0 \\
\text{let } V(\tilde{x}) &= \tilde{x}^T P \tilde{x} \\
V(\tilde{x}^* = 0) &= 0 \\
\text{using theorem 3.24:} & \\
\forall\sigma(A - LC) < 0 &\implies (\exists P, Q) P(A - LC) + (A - LC)^T P = -Q, \\
& \text{where } Q, P \text{ are spd} \\
&\implies V(\tilde{x}) \text{ is a pd function} \\
\text{using Lyapunov direct method:} & \\
L_{\dot{\tilde{x}}}V(\tilde{x}) &< 0 \text{ at } \tilde{x}^* \\
L_{\dot{\tilde{x}}}V(\tilde{x}) &= 2\tilde{x}^T P((A - LC)\tilde{x} + f(x) - f(\tilde{x})) \\
L_{\dot{\tilde{x}}}V(\tilde{x}) &= -\tilde{x}^T Q \tilde{x} + 2\tilde{x}^T P(f(x) - f(\hat{x})) < 0 \\
2\tilde{x}^T P(f(x) - f(\hat{x})) &< \tilde{x}^T Q \tilde{x} \\
\|2\tilde{x}^T P(f(x) - f(\hat{x}))\| &< \|\tilde{x}^T Q \tilde{x}\| \\
\|2\tilde{x}^T P(f(x) - f(\hat{x}))\| &\leq 2\|\tilde{x}^T\|\|P\|K\|\tilde{x}\| < \tilde{x}^T Q \tilde{x} \\
K &< \frac{\tilde{x}^T Q \tilde{x}}{2\|\tilde{x}^T\|\|P\|\|\tilde{x}\|}
\end{aligned}
$$

if $Q = I, Q$ is Hermitian, use Rayleigh Quotient :

$$
\begin{aligned}
1 = \lambda_{min}(I) \leq & \frac{x^T Q x}{x^T x} \leq \lambda_{max}(I) = 1 \\
K &< \frac{\|\tilde{x}\|_2^2}{2\|\tilde{x}^T\|\|P\|\|\tilde{x}\|} \\
\|*\|_2 &= \text{spectral radius}(*) \\
K &< \frac{1}{2\|P\|_2} = \frac{1}{2\ \lambda_{max}(P)}
\end{aligned}
$$

This gives the bound on f's Lipschitz constant with chosen Q, P of the Lyapunov function.

With V positive definite and $L_{\dot{\tilde{x}}}V(\tilde{x})$ negative definite at $\tilde{x}^*$, $\tilde{x}^*$ is an asymptotically stable equilibrium so an observer $\dot{\hat{x}} = A\hat{x} + f(\hat{x}) + L(y - C\hat{x})$ exists.

- 3.9
  $f = \dot{x} = Ax + g(x)$, $\|g(x)\| \leq \gamma \|x\|_2^2$, g is locally Lipschitz

  1. prove exponentially stable equilibrium exists at the origin

     use Lyapunov indirect method:
     $lim_{\|x\| \to x^*=0} \frac{\|f(x)-f(0)-dfx\|}{\|x\|} = 0$
     $f(0) = A0 + g(0) = 0$
     linearization near $x^* = 0$ with Taylor expansion:
     $f(x \text{ near } 0) \approx A0 + g(0) + A + \dot{g}|_{x=0} = A + (O(\gamma\|x\|_2^2)I)|_{x=0} = A$
     $df|_{x=0} = A$
     $\forall \sigma(A) < 0 \implies$ linearization $\dot{x} = df|_{x=0}x$ is asymptotically stable, then $x^* = 0$ is an exponentially stable equilibrium of f.

  2. estimate domain of attraction

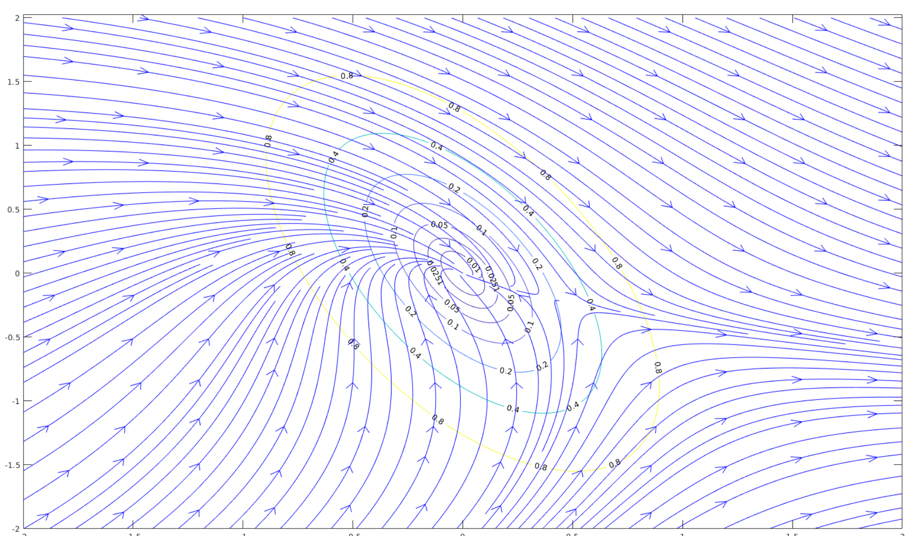
     from part 1, df_{x^*} is Hurwitz
     pick a spd Q=I, solve spd P for Lyapunov equation $PA + A^T P = -Q$
     let $V(x) = x^T P x$
     $\frac{\partial V}{\partial t} = (\frac{\partial V}{\partial (x-x^*)})(\frac{\partial}{t}(x - x^*))$
     near x^*:
     let $\tilde{x} = x - x^*$
     $\frac{\partial V}{\partial t} = ((P\tilde{x})^T + \tilde{x}^T P)(df_{x^*} + f(x) - df_{x^*}\tilde{x})$
     $\frac{\partial V}{\partial t} = (\tilde{x}^T P df_{x^*} + \tilde{x}^T P df_{x^*}) + 2\tilde{x}^T P(f(x) - df_{x^*}\tilde{x})$
     $\frac{\partial V}{\partial t} = (\tilde{x}^T PA\tilde{x} + (\tilde{x}^T PA\tilde{x})^T) + 2\tilde{x}^T P(f(x) - df_{x^*}\tilde{x})$
     $\frac{\partial V}{\partial t} = -\tilde{x}^T Q\tilde{x} + 2\tilde{x}^T P(f(x) - df_{x^*}\tilde{x})$
     $\frac{\partial V}{\partial t} = -x^T Qx + 2x^T P(Ax + g(x) - (A(x - 0)))$
     $\frac{\partial V}{\partial t} = -x^T Qx + 2x^T Pg(x)$
     let $D = \{x : -x^T Qx + 2x^T Pg(x) \leq 0\}$ be connected and $0 \in D$
     $Q = I$ is Hermitian $\implies \lambda_{min}(Q) \leq \frac{x^T Qx}{x^T x} \leq \lambda_{max}(Q)$
     $\frac{x^T Qx}{x^T x} = 1$, $x^T Qx = x^T x$
     $-x^T x + 2x^T Pg(x) \leq 0$
     $-x^T x + 2x^T Pg(x) \leq -x^T x + 2\|x^T P\|\gamma\|x\|_2^2 \leq 0$
     $-1 + 2\|x\|_2\|P\|_2\gamma \leq 0$
     $\|x\|_2 \leq \frac{1}{2\gamma\|P\|_2}$
     $D = \{x : \|x\|_2 \leq \frac{1}{2\gamma\|P\|_2}\}$
     let $c^* = inf_{x \in \partial D}\{(x-0)^T P(x-0)\}$
     $\lambda_{min}(P)x^T x \leq x^T Px \leq \lambda_{max}(P)x^T x, \|x\|_2 \leq \frac{1}{2\gamma\|P\|_2}$
     $\lambda_{min}(P)x^T x, \|x\|_2 = \frac{1}{2\gamma\|P\|_2}, x^T x = \|x\|_2^2 \implies \lambda_{min}(P)x^T x = \frac{\lambda_{min}(P)}{4\gamma^2\lambda_{max}(P)^2}$
     $c^* = \frac{\lambda_{min}(P)}{4\gamma^2\lambda_{max}(P)^2}$
     domain of attraction estimate: $D_{x^*} = \{x : x^T Px < c^*\}$

3. consider:
let $n = 2$
$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$
$g(x) = \begin{bmatrix} x_1^2 + x_2^2 \\ 0 \end{bmatrix}$

let $Q = I$
$PA + A^T = -Q$
$P = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

calculate γ:
$\|g(x)\|_2 = ((x_1^2 + x_2^2)^2 + 0^2)^{0.5} = x_1^2 + x_2^2$
$\|x\|_2 = (x_1^2 + x_2^2)^{0.5}$
$\|g(x)\|_2 \leq \gamma \|x\|_2^2$
$x_1^2 + x_2^2 \leq \gamma((x_1^2 + x_2^2)^{0.5})^2$
$x_1^2 + x_2^2 \leq \gamma(x_1^2 + x_2^2)$
$\gamma = 1$

$c^* = \frac{\lambda_{min}(P)}{4\gamma^2 \lambda_{max}(P)^2}$
$c^* = \frac{\lambda_{min}(P)}{4\lambda_{max}(P)^2} = \frac{0.2929}{4(1.7071)^2} = 0.0251$
domain of attraction estimate: $D_{x^*} = \{x : x^T P x < 0.0251\}$



From phase portrait, the estimate is conservative as it does not include a large region to the bottom left that are attractived to the origin.

- 3.10

$$x = \begin{bmatrix} x_1 - \pi \\ x_2 \\ \hat{\theta} - \theta \end{bmatrix}$$

$$V = \begin{bmatrix} x_1 - \pi & x_2 & \hat{\theta} - \theta \end{bmatrix} P \begin{bmatrix} x_1 - \pi \\ x_2 \\ \hat{\theta} - \theta \end{bmatrix}$$

$$P = \begin{bmatrix} p_1 & p_2 & 0 \\ p_2 & p_3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$\frac{\partial V}{\partial x} = \begin{bmatrix} 2p_1(x_1 - \pi) + 2x_2 p_2 \\ 2p_2(x_1 - \pi) + 2x_2 p_3 \\ \hat{\theta} - \theta \end{bmatrix}$$

$$\frac{\partial x}{\partial t} = \begin{bmatrix} x_2 \\ -c_1(x_1 - \pi) - c_2 x_2 + (\hat{\theta} - \theta) sin(x_1) \\ \varphi(x_1, x_2) \end{bmatrix}$$

$c_1, c_2 > 0$

$L_f V = 2p_1(x_1 - \pi)x_2 + 2p_2 x_2^2 + (2p_2(x_1 - \pi) + 2p_3 x_2)((\hat{\theta} - \theta) sin(x_1) - c_1(x_1 - \pi) - c_2 x_2) + (\hat{\theta} - \theta)\varphi(x_1, x_2)$

let $\varphi(x_1, x_2) = -(2p_2(x_1 - \pi) + 2p_3 x_2) sin(x_1)$

make $L_f V$ to be negative semidefinite at $x = \begin{bmatrix} \pi \\ 0 \\ * \end{bmatrix}$:

$L_f V = -2p_2 c_1 (x_1 - \pi)^2 + x^2(2p_2 - 2p_3 c_2) + x_2(x_1 - \pi)(2p_1 - 2p_2 c_2 - 2p_3 c_1) \leq 0$
$p_2 \geq 0$
$p_2 - p_3 c_2 \leq 0$
$p_1 - p_2 c_2 - p_3 c_1 = 0$

let $p_2 = 1$
$p_3 \geq 1/c_2$
let $p_3 = 2/c_2$
$p_1 = c_2 + \frac{2c_1}{c_2}$

$$V = \begin{bmatrix} x_1 - \pi & x_2 & \hat{\theta} - \theta \end{bmatrix} \begin{bmatrix} c_2 + \frac{2c_1}{c_2} & 1 & 0 \\ 1 & \frac{2}{c_2} & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 - \pi \\ x_2 \\ \hat{\theta} - \theta \end{bmatrix}$$

let diagonal entries be big enough to dominate other entries in each row so that all eigenvalues are positive by Gersgorin Circle theorem: $c_1 = \frac{1}{2}, c_2 = 1$

$$P = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

P is symmetric and all eigenvalues are positive so P is spd.

P is spd, so V is positive definite at $\begin{bmatrix} \pi \\ 0 \\ \theta \end{bmatrix}$

$sin(x_1)$ is C^1 and rest of terms in $\frac{\partial x}{\partial t}$ are linear and so f is locally Lipschitz

V is a quadratic and differentiates to a linear function and is C^1

With chosen constants, $L_f V$ is negative semidefinite at $\begin{bmatrix} \pi \\ 0 \\ \theta \end{bmatrix}$

By Lyapunov direct method, $\begin{bmatrix} \pi \\ 0 \\ \theta \end{bmatrix}$ is a stable equilibrium, so all trajectories are bounded.

From earlier, $L_f V = 0$ for $\begin{bmatrix} \pi \\ 0 \\ * \end{bmatrix}$ so the straight line at $x_1 = \pi, x_2 = 0$ is an invariant set.

Substituting any of these values into f gives equilibrium for the original system, so the line at $x_1 = \pi, x_2 = 0$ is a set of equilibria for the closed loop system.

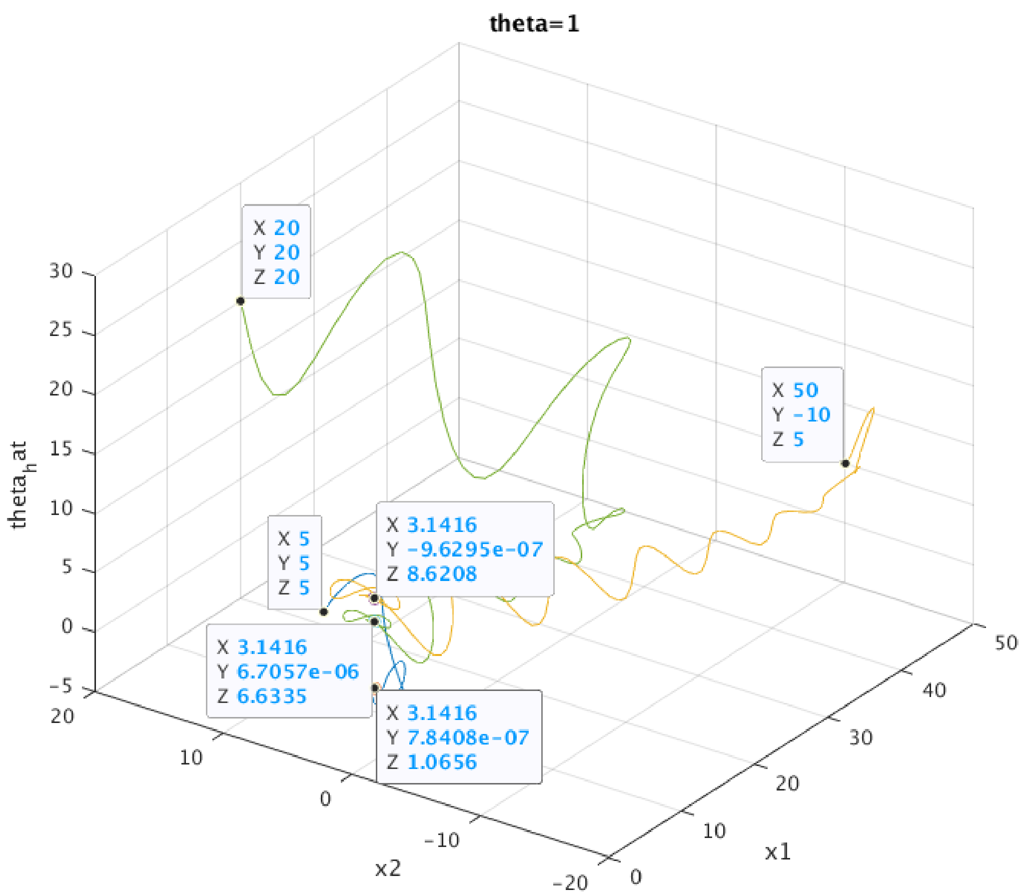
$\Omega = \{x : x_1 = \pi, x_2 = 0\}$ is the largest invariant subset of the level set for $L_f V = 0$, since $L_f V \leq 0$, then all solutions approaches Ω as $t \to \infty$.

$u = ml^2(-c_1(x_1 - \pi) - c_2 x_2 + \hat{\theta} sin(x_1)) = ml^2(-1/2(x_1 - \pi) - x_2 + \hat{\theta} sin(x_1))$

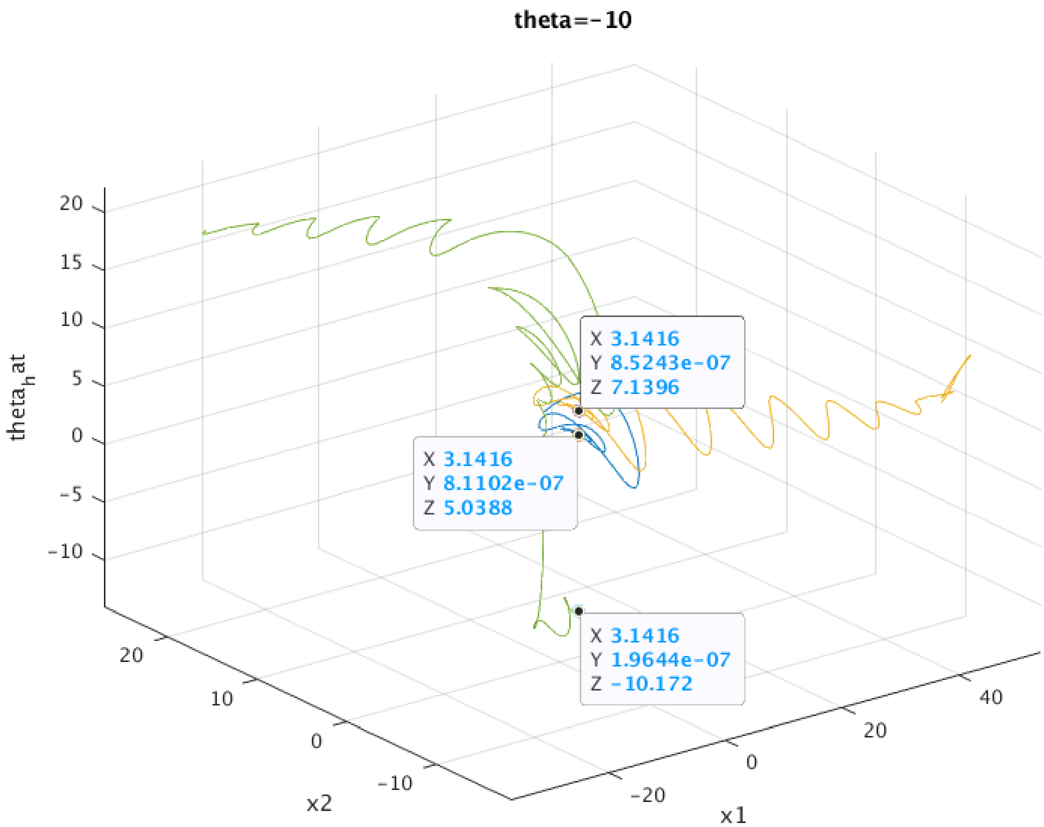
$$\frac{\partial x}{\partial t} = \begin{bmatrix} x_2 \\ -\theta sin(x_1) + \frac{1}{ml^2} u \\ -(2p_2(x_1 - \pi) + 2p_3 x_2) sin(x_1) \end{bmatrix} = \begin{bmatrix} x_2 \\ -\theta sin(x_1) - 1/2(x_1 - \pi) - x_2 + \hat{\theta} sin(x_1) \\ -(2(x_1 - \pi) + 4x_2) sin(x_1) \end{bmatrix}$$

```
function dx = ss_prob3(t,x)
    theta = 1;
    dx1 = x(2);
    dx2 = -theta*sin(x(1))-1/2*(x(1)-pi)-x(2)+x(3)*sin(x(1));
    dx3 = -(2*(x(1)-pi)+2*2*x(2))*sin(x(1));
    dx = [dx1; dx2; dx3];
end
t=[0 500];
ini_cond= ... ;
[~,X]=ode45(@ss_prob3,t,ini_cond);
u = X(:,1);
v = X(:,2);
w = X(:,3);
plot3(u,v,w);
```

$\theta = 1, x_0 = [5, 5, 5], [20, 20, 20], [50, -10, 5]$:



$\theta = -10, x_0 = [5, 5, 5], [50, -10, 5], [-30, 20, 20]$:



From the plot, $x_1 \to \pi, x_2 \to 0, \hat{\theta} - \theta \not\to 0$

- 3.11

$$V = x^T P x$$

$$x = \begin{bmatrix} x_1 - \pi \\ x_2 \\ \hat{a_1} - a_1 \\ \hat{a_2} - a_2 \end{bmatrix}$$

$$\frac{\partial x}{\partial t} = \begin{bmatrix} x_2 \\ sin(x_1)(\theta_2 \hat{a_1} - \theta_1) - \theta_2 \hat{a_2} \\ c_1(x_1 - \pi) + c_2 x_2 \\ \varphi_1 \\ \varphi_2 \end{bmatrix}$$

$$P = \begin{bmatrix} p_1 & p_2 & 0 & 0 \\ p_2 & p_3 & 0 & 0 \\ 0 & 0 & \frac{\theta}{2} & 0 \\ 0 & 0 & 0 & \frac{\theta}{2} \end{bmatrix}$$

$a_1 = \frac{\theta_1}{\theta_2}$
$a_2 = \frac{1}{\theta_2}$
$\theta_2 > 0$

$$\frac{\partial V}{\partial t} = 2 \begin{bmatrix} (x_1 - \pi)p_1 + p_2 x_2 \\ (x_1 - \pi)p_2 + p_3 x_2 \\ \theta_2(\hat{a_1} - a_1) \\ \theta_2(\hat{a_2} - a_2) \end{bmatrix}^T \begin{bmatrix} x_2 \\ sin(x_1)(\theta_2 \hat{a_1} - \theta_1) - \theta_2 \hat{a_2} \\ c_1(x_1 - \pi) + c_2 x_2 \\ \varphi_1 \\ \varphi_2 \end{bmatrix} \leq 0$$

$$\begin{aligned} \frac{\partial V}{\partial t} =& 2(x_2^2 p_2 + \\ & x_2(x_1 - \pi)p_1 + \\ & sin(x_1)\theta_2 \hat{a_1}(x_1 - \pi)p_2 + \\ & sin(x_1)(-\theta_1)(x_1 - \pi)p_2 + \\ & sin(x_1)(\theta_2 \hat{a_1})p_3 x_2 + \\ & sin(x_1)(-\theta_1)p_3 x_2 + \\ & \theta_2(\hat{a_1} - a_1)\varphi_1 + \theta_2(\hat{a}_2 - a_2)\varphi_2 \\ & - \theta_2 \hat{a_2}((x_1 - \pi)^2 c_1 p_2 + c_2 p_3 x_2^2 + x_2(x_1 - \pi)(c_2 p_2 + c_1 p_3))) \leq 0 \end{aligned}$$

let $\varphi_2 = (x_1-\pi)^2 c_1 p_2 + c_2 p_3 x_2^2 + x_2(x_1-\pi)(c_2p_2+c_1p_3)$

$\theta_2 a_2 = 1$

$\theta_2(\hat{a}_2 - a_2)\varphi_2 - \theta_2\hat{a}_2((x_1-\pi)^2 c_1 p_2 + c_2 p_3 x_2^2 + x_2(x_1-\pi)(c_2p_2+c_1p_3))$
$= \theta_2\hat{a}_2((x_1-\pi)^2 c_1 p_2 + c_2 p_3 x_2^2 + x_2(x_1-\pi)(c_2p_2+c_1p_3))$
$- \theta_2\hat{a}_2((x_1-\pi)^2 c_1 p_2 + c_2 p_3 x_2^2 + x_2(x_1-\pi)(c_2p_2+c_1p_3))$
$- \theta_2 a_2((x_1-\pi)^2 c_1 p_2 + c_2 p_3 x_2^2 + x_2(x_1-\pi)(c_2p_2+c_1p_3))$
$= -((x_1-\pi)^2 c_1 p_2 + c_2 p_3 x_2^2 + x_2(x_1-\pi)(c_2p_2+c_1p_3))$

let $\varphi_1 = -sin(x_1)((x_1-\pi)p_2 + p_3 x_2)$
$\theta_2 a_1 = \theta_2 \frac{\theta_1}{\theta_2} = \theta_1$

$\theta_2(\hat{a}_1 - a_1)\varphi_1$
$= (\theta_2\hat{a}_1 - \theta_1)\varphi_1$
$= -sin(x_1)(\theta_2\hat{a}_1)((x_1-\pi)p_2 + p_3x_2) + sin(x_1)(\theta_1)((x_1-\pi)p_2 + p_3x_2)$

$sin(x_1)\theta_2\hat{a}_1(x_1-\pi)p_2 +$
$sin(x_1)(-\theta_1)(x_1-\pi)p_2 +$
$sin(x_1)(\theta_2\hat{a}_1)p_3x_2 +$
$sin(x_1)(-\theta_1)p_3x_2 +$
$+ \theta_2(\hat{a}_1 - a_1)\varphi_1$
$= 0$

Then,

$x_2^2 p_2 +$
$x_2(x_1-\pi)p_1 +$
$sin(x_1)\theta_2\hat{a}_1(x_1-\pi)p_2 +$
$sin(x_1)(-\theta_1)(x_1-\pi)p_2 +$
$sin(x_1)(\theta_2\hat{a}_1)p_3x_2 +$
$sin(x_1)(-\theta_1)p_3x_2 +$
$\theta_2(\hat{a}_1 - a_1)\varphi_1 + \theta_2(\hat{a}_2 - a_2)\varphi_2$
$- \theta_2\hat{a}_2((x_1-\pi)^2 c_1 p_2 + c_2 p_3 x_2^2 + x_2(x_1-\pi)(c_2p_2+c_1p_3)) \le 0$
simplifies to:

$x_2^2 p_2 + x_2(x_1-\pi)p_1 - ((x_1-\pi)^2 c_1 p_2 + c_2 p_3 x_2^2 + x_2(x_1-\pi)(c_2p_2+c_1p_3)) \le 0$

solve constraints:

$x_2^2(p_2 - c_2 p_3) \leq 0$
$-(x_1 - \pi)^2 c_1 p_2 \leq 0$
get rid of saddle points:
$x_2(x_1 - \pi)(p_1 - c_2 p_2 - c_1 p_3) = 0$

$c_1 p_2 \geq 0$
$p_2 - c_2 p_3 \leq 0$
$p_1 - c_2 p_2 - c_1 p_3 = 0$

let $p_2 = 1$
$p_3 \geq \frac{1}{c_2}$
let $p_3 = \frac{2}{c_2}$
$p_1 = c_2 + \frac{2c_1}{c_2}$

let $c_2 = c_1 = 1$ so diagonals dominate and all eigenvalues are positive

$$P = \begin{bmatrix} c_2 + \frac{2c_1}{c_2} & 1 & 0 & 0 \\ 1 & \frac{2}{c_2} & 0 & 0 \\ 0 & 0 & \frac{\theta}{2} & 0 \\ 0 & 0 & 0 & \frac{\theta}{2} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & \frac{\theta}{2} & 0 \\ 0 & 0 & 0 & \frac{\theta}{2} \end{bmatrix}$$

P is symmetric and all eigenvalues are positive then P is spd

V is postive definite at $x^* = \begin{bmatrix} \pi & 0 & a_1 & a_2 \end{bmatrix}^T$

$L_f V$ is negative semidefinite at $\begin{bmatrix} \pi & 0 & * & * \end{bmatrix}^T$

By Lyapunov direct method, $\begin{bmatrix} \pi & 0 & a_1 & a_2 \end{bmatrix}^T$ is a stable equilibrium, so all trajectories are bounded.

$L_f V = 0$ for $\begin{bmatrix} \pi & 0 & * & * \end{bmatrix}^T$ and evaluation with f yields equilibrium for the original system at $x_1 = \pi, x_2 = 0$, so the straight line at $x_1 = \pi, x_2 = 0$ is a set of equilibriua for the closed loop system.

$\Omega = \{x : x_1 = \pi, x_2 = 0\}$ is the largest invariant subset of the level set for $L_f V = 0$, since $L_f V \leq 0$, then all solutions approaches Ω as $t \to \infty$.

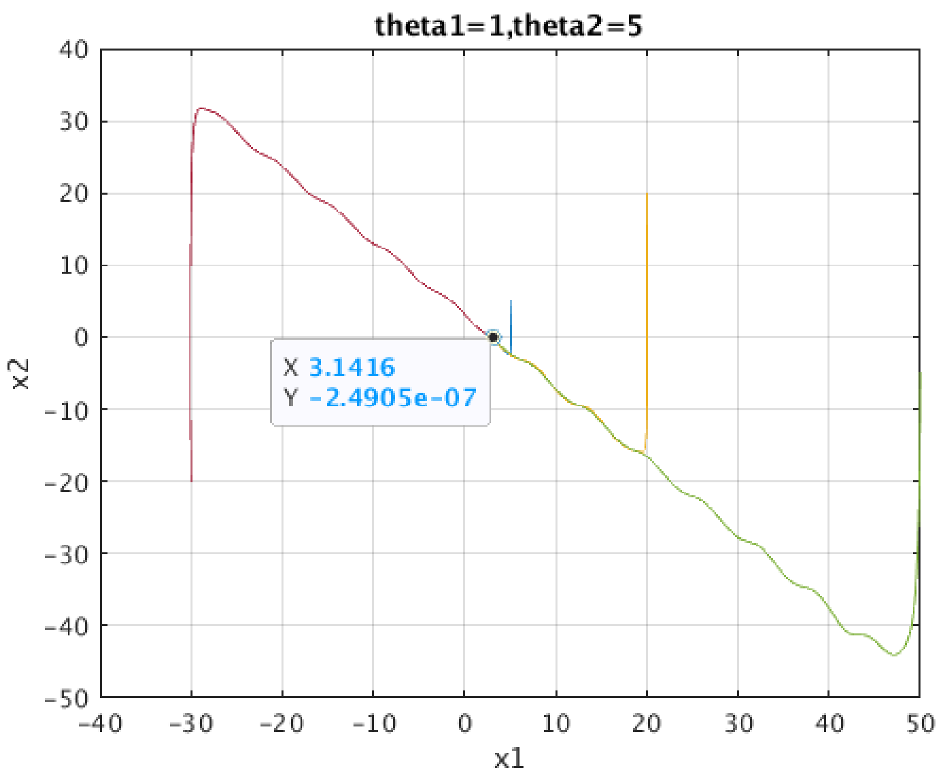
$u = \hat{a}_1 sin(x_1) - \hat{a}_2(c_1(x_1 - \pi) + c_2 x_2)$
$u = \hat{a}_1 sin(x_1) - \hat{a}_2((x_1 - \pi) + x_2)$

$$\frac{\partial x}{\partial t} = \begin{bmatrix} x_2 \\ -\theta_1 sin(x_1) + \theta_2 u \\ -sin(x_1)((x_1-\pi)p_2 + p_3 x_2) \\ (x_1-\pi)^2 c_1 p_2 + c_2 p_3 x_2^2 + x_2(x_1-\pi)(c_2 p_2 + c_1 p_3) \end{bmatrix}$$

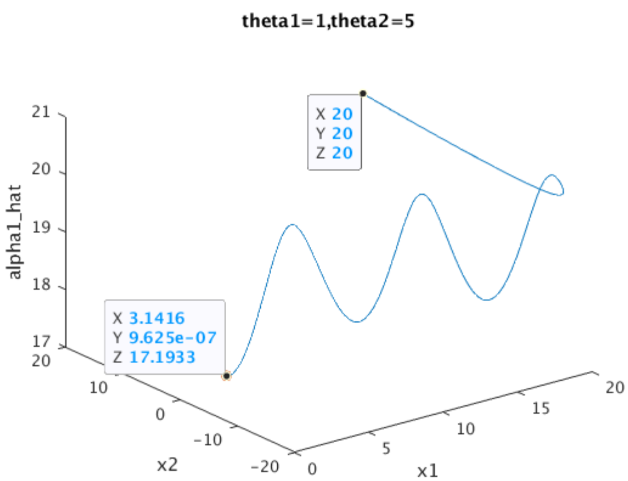
$$\frac{\partial x}{\partial t} = \begin{bmatrix} x_2 \\ -\theta_1 sin(x_1) + \theta_2(\hat{a}_1 sin(x_1) - \hat{a}_2((x_1-\pi)+x_2)) \\ -sin(x_1)((x_1-\pi)+2x_2) \\ (x_1-\pi)^2 + 2x_2^2 + 3x_2(x_1-\pi) \end{bmatrix}$$

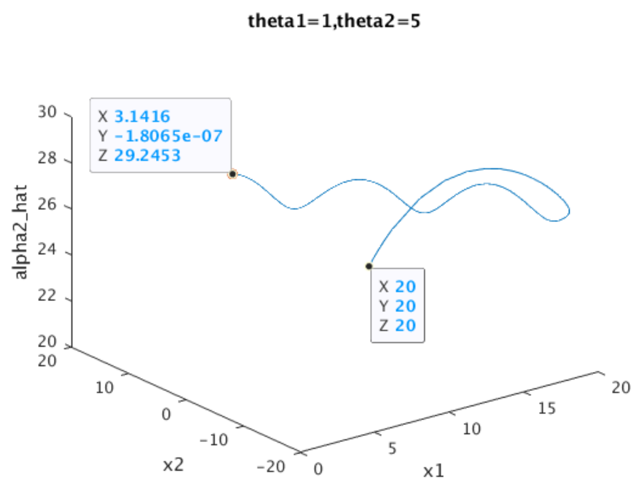
simulation:

$\theta_1 = 1, \theta_2 = 5, x_0 = [5,5,5,5], [20,20,20,20], [50,-5,10,5], [-30,-20,-10,5]:$



$x_0 = [20, 20, 20, 20]$:





From plots, $x_1 \to \pi, x_2 \to 0, \hat{\alpha_1} - \alpha_1 \not\to 0, \hat{\alpha_2} - \alpha_2 \not\to 0$