- 1.5
  From the graph of x vs. t, it is true that,
  $(\exists t_A)(\exists t_B)(t_A \neq t_B) \wedge (x(t_A, x_0) = x(t_B, x_0)) \wedge (\frac{\partial}{\partial t}x(t_A, x_0) \neq \frac{\partial}{\partial t}x(t_B, x_0))$,
  which occurs in the inflection region of the graph. Thus, the vector field is time varying ($f(t,x)$). x(t) cannot be a solution of a scalar differential equation with locally Lipschitz autonomous vector field.

- 1.10

  let $\chi = \begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R} \times \mathcal{X}$

  let $\bar{f}(\chi) = \begin{bmatrix} 1 \\ f(\chi) \end{bmatrix} = \dot{\chi}$

  $f(\chi) = f(\begin{bmatrix} t \\ x \end{bmatrix})$

  if map $(t,x) \mapsto f(t,x)$ is locally Lipschitz,
  show $\phi(t_0, t_0, x_0) = x_0$

  $(t,x) \mapsto f(t,x)$ is locally Lipschitz:
  $(\forall \chi)(\exists L) \| f(\chi) - f(\chi_0) \| \leq L \| \chi - \chi_0 \|$ for each $\chi_0$

  $\| \bar{f}(\chi) - \bar{f}(\chi_0) \| = \left\| \begin{bmatrix} 1 \\ f(\chi) \end{bmatrix} - \begin{bmatrix} 1 \\ f(\chi_0) \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 \\ f(\chi) - f(\chi_0) \end{bmatrix} \right\|$
  $\| \bar{f}(\chi) - \bar{f}(\chi_0) \| = \| f(\chi) - f(\chi_0) \|$
  $(\forall \chi)(\exists L) \| \bar{f}(\chi) - \bar{f}(\chi_0) \| \leq L \| \chi - \chi_0 \|$ for each $\chi_0$

  $\bar{f}$ is locally Lipschitz

  let $\bar{\phi} : \mathbb{R} \times (\mathbb{R} \times \mathcal{X}) \to \mathbb{R} \times \mathcal{X}$
  $\phi = \mathbb{R} \times \mathbb{R} \times \mathcal{X} \to \mathcal{X}$
  $\phi = \text{project}_x \bar{\phi}$, where $\text{project}_x \bar{\phi} : \mathbb{R} \times \mathcal{X} \to \mathcal{X}$, $\text{project}_x \bar{\phi} = \text{map } \begin{bmatrix} t \\ x \end{bmatrix} \mapsto x$

  use theorem 1.20: if $\bar{f}$ is locally Lipschitz on $\mathbb{R} \times \mathcal{X}$ then, $(\forall \chi_0)(\exists \chi(t))$ $\chi(t)$ is unique and maximal with $\chi(t_0) = \chi_0 = \begin{bmatrix} t_0 \\ x_0 \end{bmatrix}$

  let $\bar{\phi}(t, \chi_0)$ be that solution satisfying initial condition and $\bar{f}$

  $\phi(\tau, t_0, x_0) = project_x(\bar{\phi}(\tau, \chi_0))$

$\phi(t_0, t_0, x_0) = project_x(\bar{\phi}(t_0, \chi_0)) = project_x(\chi_0) = project_x(\begin{bmatrix} t_0 \\ x_0 \end{bmatrix}) = x_0$

show $\partial_{t'}\phi(t', t_0, x_0) = f(t', \phi(t', t_0, x_0))$

$\partial_{t'}\phi(t', t_0, x_0) = \frac{\partial}{\partial t'}\left(project_x \int_0^{t'} \bar{f}(\chi(\tau))d\tau + \chi_0\right)$

$\partial_{t'}\phi(t', t_0, x_0) = project_x\bar{f}(\chi(t')) = project_x \begin{bmatrix} 1 \\ f(\chi(t')) \end{bmatrix} \Big|_{\chi(0)=\chi_0}$

$\partial_{t'}\phi(t', t_0, x_0) = f(\chi(t')) = f\left(\begin{bmatrix} t(t') \\ x(t') \end{bmatrix}\right) \Big|_{\chi(0)=\chi_0}$

$t(t') = \int_0^{t'} 1 d\tau = t' + t_0 = t'$

$\partial_{t'}\phi(t', t_0, x_0) = f\left(\begin{bmatrix} t' \\ \phi(t', \chi_0) \end{bmatrix}\right) = f\left(\begin{bmatrix} t' \\ \phi(t', [t_0, x_0]) \end{bmatrix}\right) = f(t', \phi(t', t_0, x_0))$

show $(t, t_0, x_0) \mapsto \phi(t, t_0, x_0)$ is continuous

Using theorem 1.26: $\bar{f}$ is locally Lipschitz on $X = \mathbb{R} \times \mathcal{X}$, $W = \{(t, \chi_0)\} \in \mathbb{R} \times (\mathbb{R} \times \mathcal{X}), t \in T_{\chi_0}$:=maximal interval of existence, $\bar{\phi}(t, \chi_0)$ being maximal solution with initial condition $\chi_0$, then W is open and $(t, \chi_0) \mapsto \bar{\phi}(t, \chi_0)$ is continuous. Then, $project_x\bar{\phi} = \phi$, and $(t, t_0, x_0) \mapsto \phi(t, t_0, x_0)$ is continuous.

- 1.11

extending previous question to include constants $\lambda \in \mathbb{R}^m$

let $\chi = \begin{bmatrix} t \\ x \\ \lambda \end{bmatrix} \in \mathbb{R} \times \mathcal{X} \times \mathbb{R}^m$

let $\bar{f}(\chi) = \begin{bmatrix} 1 \\ f(\chi) \\ 0^{m\times 1} \end{bmatrix} = \dot{\chi}$

$f(\chi) = f\left(\begin{bmatrix} t \\ x \\ 0^{m\times 1} \end{bmatrix}\right)$

if map $(t, x, \lambda) \mapsto f(t, x, \lambda)$ is locally Lipschitz,
show $\phi(t, t_0, x_0, \lambda_0)$ is continuous satisfying $\phi(t_0, t_0, x_0, \lambda_0) = x_0$

since map $(t, x, \lambda) \mapsto f(t, x, \lambda)$ is locally Lipschitz:
$(\forall\chi)(\exists L)\|f(\chi) - f(\chi_0)\| \leq L\|\chi - \chi_0\|$ for each $\chi_0$

$\|\bar{f}(\chi) - \bar{f}(\chi_0)\| = \left\| \begin{bmatrix} 1 \\ f(\chi) \\ 0^{m\times 1} \end{bmatrix} - \begin{bmatrix} 1 \\ f(\chi_0) \\ 0^{m\times 1} \end{bmatrix} \right\|$

$\|\bar{f}(\chi) - \bar{f}(\chi_0)\| = \left\| \begin{bmatrix} 0 \\ f(\chi) - f(\chi_0) \\ 0^{m\times 1} \end{bmatrix} \right\| = \left\| \begin{bmatrix} f(\chi) - f(\chi_0) \end{bmatrix} \right\|$

$(\forall\chi)(\exists L)\|\bar{f}(\chi) - \bar{f}(\chi_0)\| = \|f(\chi) - f(\chi_0)\| \leq L\|\chi - \chi_0\|$ for each $\chi_0$

$\bar{f}$ is locally Lipschitz

We can proceed down the same general method as in the previous question and get the resulting $\phi = project_x(\bar{\phi})$ and in the end $\bar{\phi}(t, \chi_0)$ is continuous satisfying initial condition and $\phi(t, \chi_0)$ is continuous satisying the initial condition.

let $\bar{\phi} : \mathbb{R} \times \mathcal{X} \times \mathbb{R}^m \to \mathcal{X} \times \mathbb{R}^m$, where $\bar{\phi}(t, \chi_0)$ is a maximal solution satisfying initial condition $\chi_0$ via theorem 1.20

use theorem 1.26 with $\bar{f}$ being locally Lipschitz and $\bar{\phi}(t, \chi_0)$ being a maximal solution so that W is open and $(t, \chi_0) \mapsto \bar{\phi}(t, \chi_0)$ is continuous.

$\phi = project_x(\bar{\phi})$, where $project_x(\bar{\phi}) : \mathcal{X} \times \mathbb{R}^m \to \mathcal{X}$
Then, $project_x((t, \chi_0) \mapsto \bar{\phi}(t, \chi_0)) = \phi$ is continuous.

- 1.13

1)
$t \in \mathbb{R}$ small enough that domain of definition of $\phi_t$ is non-empty.
Determine domain and codomain of the map $\phi_t$.
domain: $x_0 : (\forall x_0)(t', x_0) \in W \neq \emptyset,\ \forall t' \in [0, t]\}|_{t\ fixed}$
codomain: $\{\phi(t') : x(0) = x_0,\ \forall t' \in [0, t],\ x_0 \in d\}$

injective:
if x goes through $x_a$, $x_b$ such that $x_a \neq x_b$, then $\phi(0, x_a) = \phi(t_c, x_b), t_c \neq 0$.
$\phi(t, x_a) = \phi(t, \phi(t_c, x_b))$
$\phi(t, x_a) = \phi(t + t_c, x_b)) \neq \phi(t, x_b)$.
Then it is injective: $(\forall x_a, x_b)x_a \neq x_b \wedge \phi(t, x_a) \neq \phi(t, x_b)$

surjective:
let $\phi_t(x) = x_2$
$\phi(0, x_2) = \phi(t, x)$
$\phi(-t, x_2) = \phi(0, x) = x$
Since the domain of definition of $\phi_t$ is non-empty, it guarantees that elements in codomain as atleast a corresponding element in the domain.
$(\forall x_2 = \{\phi(t') : x(0) = x_0,\ \forall t' \in [0, t],\ x_0 \in d\})(\exists x_0)(x_0 \mapsto \phi_t(x_0) = x_2)$

Then $\phi_t$ is injective and surjective, then $\phi_t$ is bijective.
An inverse exists so that $y \mapsto \phi_t^{-1}(y) = x$:
$\phi_t(x) = \phi(t, x) = y = \phi(0, y)$
$\phi(0, x) = \phi(-t, y)$
$\phi_t^{-1}(y) = \phi(-t, y) = x$

2)
let $(\exists T) T > 0 \land x_1 \in \mathcal{X} \land x_1 = \phi(T, x_1)$
Show that:
$(\forall t \in \mathbb{R})\phi(t+T, x_1) = \phi(t, x_1)$, system keeps looping around without possibly moving on to $x_2$

let $\phi(T_A, x_1) = x_2$
let $T_A = T_1 + T_2 + ... + T_n$, where $T_1, ..., T_{n-1} = T$, $T_n = T_A \ mod \ T$

$\phi(T_A, x_1) = \phi(T_1 + T_2 + ... + T_n, x_1)$
$\phi(T_A, x_1) = \phi(T_2 + ... + T_n, \phi(T_1, x_1)) = \phi(T_2 + ... + T_n, x_1)$
...
$\phi(T_A, x_1) = \phi(T_n, x_1)$
Then, $\phi(T_n, x_1)$ is within cyclic path of $\phi(0, x_1)$ and $\phi(T, x_1)$. $x_2$ may not be reached for any T.

3)
let $(\forall x_1, x_2 \in \mathcal{X})(\exists t_1, t_2 \in \mathbb{R})(x_1 \neq x_2 \land \phi(t_1, x_1) = \phi(t_2, x_2))$
show $t_1 \neq t_2$

if a curve ever connects $x_1$ with $x_2$, then: $\phi(0, x_1) = \phi(\epsilon, x_2), \epsilon \neq 0$
$\phi(t_1, x_1) = \phi(t_1, \phi(\epsilon, x_2)) = \phi(t_2, x_2))$

if $t_1 = t_2$:
$\phi(t, x_1) = \phi(t, \phi(\epsilon, x_2)) = \phi(t, x_2))$
$\phi(t, x_1) = \phi(t + \epsilon, x_2) = \phi(t, x_2))$
it is not true that $\phi(t + \epsilon, x_2) = \phi(t, x_2)), \epsilon \neq 0$

then, $(\forall x_1, x_2 \in \mathcal{X})(\exists t_1, t_2 \in \mathbb{R})(x_1 \neq x_2 \land \phi(t_1, x_1) = \phi(t_2, x_2)) \rightarrow t_1 \neq t_2$

consider $t_1 \neq t_2$, show flow carries $x_1$ to $x_2$ or vice versa, in positive time.

let $t_1 + \delta = t_2, \delta \neq 0$

$\phi(t_1, x_1) = \phi(t_2, x_2)$
$\phi(t_1, x_1) = \phi(t_1 + \delta, x_2)$
$\phi(0, x_1) = \phi(\delta, x_2)$
$x_1 = \phi(\delta, x_2)$
$\delta > 0 : x_2 \rightarrow x_1$ via flow
$\phi(-\delta, x_1) = \phi(x_2)$
$\phi(-\delta, x_1) = x_2$
$\delta < 0 : x_2 = \phi(-\delta, x_1) = \phi(|\delta|, x_1), x_1 \rightarrow x_2$ via flow

- 1.15

  part 2:

  $$\phi(t,x) = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} x$$

  $W = (t, x_0) \in \mathbb{R} \times \mathcal{X}, t \in T_{x_0}$ is open.

  $$\phi(0,x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x = x$$

  Consistency is satisfied.

  $$\phi(s+t,x) = \begin{bmatrix} e^{s+t} & 0 \\ 0 & e^{2s+2t} \end{bmatrix} x$$

  $$\phi(s,\phi(t,x)) = \begin{bmatrix} e^s & 0 \\ 0 & e^{2s} \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} x = \begin{bmatrix} e^{s+t} & 0 \\ 0 & e^{2s+2t} \end{bmatrix} x = \phi(s+t,x)$$

  Semigroup property is satisfied.

  Interval of maximal existence $T_{x_0} = (-\infty, \infty)$. Then $W = \mathbb{R} \times \mathcal{X}$ and $\phi(t,x)$ is a phase flow.

  vector field:

  $$\frac{\partial}{\partial t}\phi(t,x) = f(x)|_t = \begin{bmatrix} e^t & 0 \\ 0 & 2e^{2t} \end{bmatrix} x$$

  part 6:

  $$\phi(t,x) = e^{-t} \begin{bmatrix} cos(2t) & sin(2t) \\ -sin(2t) & cos(2t) \end{bmatrix} x$$

  $W = (t, x_0) \in \mathbb{R} \times \mathcal{X}, t \in T_{x_0}$ is open.

  $$\phi(0,x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x = x$$

  Consistency is satisfied.

$$\phi(s+t,x) = e^{-(s+t)} \begin{bmatrix} cos(2s+2t) & sin(2s+2t) \\ -sin(2s+2t) & cos(2s+2t) \end{bmatrix} x$$

$$\phi(s,\phi(t,x)) = e^{-s} \begin{bmatrix} cos(2t) & sin(2t) \\ -sin(2t) & cos(2t) \end{bmatrix} e^{-t} \begin{bmatrix} cos(2s) & sin(2s) \\ -sin(2s) & cos(2s) \end{bmatrix} x$$

$$\phi(s,\phi(t,x)) = e^{-(s+t)} \begin{bmatrix} cos(2s)cost(2t) - sin(2s)sin(2t) & cos(2s)sin(2t) + sin(2s)cos(2t) \\ -sin(2s)cos(2t) - cos(2s)sin(2t) & -sin(2s)sin(2t) + cos(2s)cos(2t) \end{bmatrix} x$$

$cos(a \pm b) = cos(a)cos(b) \mp sin(a)sin(b)$
$sin(a \pm b) = sin(a)cos(b) \pm cos(a)sin(b)$

$$\phi(s,\phi(t,x)) = e^{-(s+t)} \begin{bmatrix} cos(2s+2t) & sin(2s+2t) \\ -sin(2s+2t) & cos(2s+2t) \end{bmatrix} x = \phi(s+t,x)$$

Semigroup property is satisfied.

Interval of maximal existence $T_{x_0} = (-\infty, \infty)$. Then $W = \mathbb{R} \times \mathcal{X}$ and $\phi(t,x)$ is a phase flow.

vector field:

$$\tfrac{\partial}{\partial t}\phi(t,x) = f(x)|_t = \begin{bmatrix} -e^{-t}cos(2t) - 2e^{-t}sin(2t) & -e^{-t}sin(2t) + 2e^{-t}cos(2t) \\ e^{-t}sin(2t) - 2e^{-t}cos(2t) & -e^{-t}cos(2t) - 2e^{-t}sin(2t) \end{bmatrix} x$$

- 1.16

1)
$\phi(t, x) = x$
$\phi(0, x) = x$
Consistency is satisfied.

$\phi(s + t, x) = x$
$\phi(s, \phi(t, x)) = x$
Semigroup property is satisfied.

$W = (t, x_0) \in \mathbb{R} \times \mathcal{X}, t \in T_{x_0} = (-\infty, +\infty)$
$W = \mathbb{R} \times \mathcal{X}$
$\phi(t, x) = x$ is a phase flow.

2)
$\phi(t, x) = tx$
$(\forall x)\phi(0, x) = 0 \neq x$
Consistency is not satisfied.

$\phi(s + t, x) = (s + t)x$
$(\forall s, t)\phi(s, \phi(t, x)) = t(sx) \neq \phi(s + t, x)$
Semigroup property is not satisfied.

$\phi(t, x) = tx$ is not a phase flow.

3)
$\phi(t, x) = (t + 1)x$
$\phi(0, x) = x$
Consistency is satisfied.

$\phi(s + t, x) = (s + t + 1)x$
$(\forall s, t)\phi(s, \phi(t, x)) = (s + 1)(t + 1)x = (st + s + t + 1)x \neq \phi(s + t, x)$
Semigroup property is not satisfied.

$\phi(t, x) = tx$ is not a phase flow.

4)
$\phi(t,x) = e^{t+1}x$
$(\forall x)\phi(0,x) = e^{1}x \neq x$
Consistency is not satisfied.

$\phi(s+t,x) = e^{s+t+1}x$
$\phi(s,\phi(t,x)) = e^{s+1}e^{t+1}x = e^{s+t+2}x \neq \phi(s+t,x)$
Semigroup property is not satisfied.

$\phi(t,x) = e^{t+1}x$ is not a phase flow.

5)
$\phi(t,x) = e^{t}x$
$(\forall x)\phi(0,x) = x$
Consistency is satisfied.

$\phi(s+t,x) = e^{s+t}x$
$\phi(s,\phi(t,x)) = e^{s}e^{t}x = e^{s+t}x = \phi(s+t,x)$
Semigroup property is satisfied.

$W = (t,x_0) \in \mathbb{R} \times \mathcal{X}, t \in T_{x_0} = (-\infty,+\infty)$
$W = \mathbb{R} \times \mathcal{X}$
$\phi(t,x) = e^{t}x$ is a phase flow.