- 2.2
  show $\dot{x} = Ax$ cannot exhibit limit cycles (orbit that is closed, isolated, not an equilibrium)

  - find closed orbit of $\dot{x} = Ax$ that is not an equilibrium:
    closed orbit := not an equilibrium and is a closed unparameterized curve
    equilibrium, x^*, of $\dot{x} = Ax$:
    $f(x^*) = 0$
    $Ax^* = 0$
    $x^* \in 0 \cup Ker(A)$

    closed orbit,of $\dot{x} = Ax, x(0) = x_0$:
    $O(x_0) : (\exists T > 0)\phi(T, x_0) = \phi(x_0) \wedge x_0 \notin \{0 \cup Ker(A)\}$
    the general solution of LTI for some x_0 is: $x(t) = e^{At}x_0$
    assuming non-degeneracy of A, and using similarity transform, $P\Lambda = AP$:
    $x'(t) = e^{\Lambda t}x'_0$, Λ := some block diagonal matrix
    consider simplification to $\mathbb{R}^2$ domain:
    using periodicity of $e^{it} = cos(t) + isin(t), e^{-it} = cos(t) - isin(t)$
    we observe eigenvalues $\lambda = i, -i$
    from similarity transform, the eigenvalues of A are same as that of Λ
    then closed orbit of $\dot{x} = Ax$ is the image of $\phi(t, x_0) = x_0e^{At}, \lambda_A = \pm i, \forall t \in [0, T]$

  - show closed orbits of $\dot{x} = Ax$ cannot be isolated:
    isolated orbit, $O(x_0)$, satisfies:
    $(\forall p \in O(x_0))(\exists \delta > 0)(\forall q \in B^n_\delta(p))q \notin O(x_0) \implies O(q)$ not closed

    a non-isolated orbit, $O(x_0)$, satisfies:
    $\neg(\forall p \in O(x_0))(\exists \delta > 0)(\forall q \in B^n_\delta(p))q \notin O(x_0) \implies O(q)$ not closed
    $(\exists p \in O(x_0))\neg(\exists \delta > 0)(\forall q \in B^n_\delta(p))q \notin O(x_0) \implies O(q)$ not closed
    $(\exists p \in O(x_0))(\forall \delta > 0)\neg(\forall q \in B^n_\delta(p))q \notin O(x_0) \implies O(q)$ not closed
    $(\exists p \in O(x_0))(\forall \delta > 0)(\exists q \in B^n_\delta(p))\neg(q \notin O(x_0) \implies O(q)$ not closed)
    $(\exists p \in O(x_0))(\forall \delta > 0)(\exists q \in B^n_\delta(p))(q \notin O(x_0)) \wedge O(q)$ closed)

    let $x'_0 = x_0 + \epsilon, \epsilon \neq 0$ such that x'_0 is not part of $O(x_0)$ and $x'_0 \in B^n_\delta(p)$

    f is a linear transform with eigenvalues $\lambda_A = \pm i$, the eigenvalues are invariant for all x

    using simplied domain of $\mathbb{R}^2$:
    in the coordinate system after similarity transform

    in polar form, $e^{\pm it}$ maintains a magnitude of $(cos^2(t) + sin^2(t))^{1/2} = 1$, for all t

since the phase of $e^{\pm it}x_0'$ and $e^{\pm it}x_0$ are in sync with a same period, $\phi(t, x_0')$ and $\phi(t, x_0)$ never meets and maintain a distance of $\|e^{it}\|\|x_0' - x_0\| = \|x_0' - x_0\|$

then $\phi(t, x_0')$ is periodic and orbit $O(x_0')$ is closed for all suitable ϵ

then, this satisfies $(\exists p \in O(x_0))(\forall \delta > 0)(\exists q \in B_\delta^n(p))(q \notin O(x_0)) \wedge O(q)$ closed), where q corresponds to suitable x_0' earlier

so, closed orbits of $\dot{x} = Ax$ is not isolated, then the system cannot have limit cycles

- 2.3

  - Give necessary and sufficient conditions for a closed set Ω to be negatively invariant for (2.1).

    f is locally Lipschitz on the domain
    let $f_2(x) = -f(x)$
    $(\forall x \in \Omega) f_2(x) \in T_\Omega(x) \iff \Omega$ negatively invariant

    Prove that your conditions are correct using Theorem 2.8:

    since $f(x)$ is locally Lipschitz at x_0 in domain, then:
    $(\exists L, \delta > 0)(\forall x, y \in B^n_\delta(x_0)) \|f(x) - f(y)\| \leq L\|x - y\|$

    $\|f_2(x) - f_2(y)\| = \|-f(x) + f(y)\| = \|f(x) - f(y)\|$
    so, $f_2(x)$ is also locally Lipschitz at x_0

    using theorem 2.8, since f_2 is locally Lipschitz,
    $(\forall x \in \Omega) f_2(x) \in T_\Omega(x) \iff \Omega$ positively invariant

    since $\phi(t, x_0), t \in \mathbb{R}_{\leq 0}$ with $f(x)$ is equivalent to $\phi(t, x_0), t \in \mathbb{R}_{\geq 0}$ with $-f(x)$, then the above is equivalent to:
    $(\forall x \in \Omega) f(x) \in T_\Omega(x) \iff \Omega$ negatively invariant

  - Give necessary and sufficient conditions for a closed set Ω to be invariant for (2.1).

    f is locally Lipschitz on the domain
    $(\forall x \in \Omega) - f(x) \in T_\Omega(x) \wedge f(x) \in T_\Omega(x) \iff \Omega$ invariant

    Prove that your conditions are correct using Theorem 2.8

    the first half corresponds to Ω being negatively invariant and the 2nd half corresponds to Ω being positively invariant which is given by theorm 2.8 itself

    the 2 local intervals are joined together since x_0 is part of the local solution by both intervals and by uniqueness of maximal solution, we extend to include both intervals

    by joining 2 time parameter intervals $T^+_{x_0} \cup T^-_{x_0} = T_{x_0}$ for any $x_0 \in \mathcal{X}$ where each half interval satisfies the respective definitions of Ω being positively / negatively invarant makes the entire interval satisfied for Ω being invariant.

- 2.4
  use theorem 2.14 to show Lie derivative is 0 for level set Ω satisfying $\varphi(x) = 0$ and that $\partial\varphi_x$ does not lose rank

  let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

  let $\varphi = E' = E - k = \int_0^y f(\tau)d\tau + \frac{1}{2}M\dot{y}^2 - k = \int_0^{x_1} f(\tau)d\tau + \frac{1}{2}Mx_2^2 - k$

  check φ is C^1:

  $\lim_{h\to 0} \frac{\varphi(x_1+h,x_2)-\varphi(x_1,x_2)}{h} = \frac{\partial\varphi}{\partial x_1}$
  $\varphi(x_1+h, x_2) - \varphi(x_1, x_2) = F(x_1+h) - F(0) + \frac{1}{2}Mx_2^2 - k - (F(x_1) - F(0) + \frac{1}{2}Mx_2^2 - k) = F(x_1 + h) - F(x_1)$
  $\lim_{h\to 0} \frac{F(x_1+h)-F(x_1)}{h} = \frac{\partial\varphi}{\partial x_1}$
  $f(x_1) = \frac{\partial\varphi}{\partial x_1}$
  f is locally Lipschitz $\implies$ $\frac{\partial\varphi}{\partial x_1}$ is locally continuous

  $\lim_{h\to 0} \frac{\varphi(x_1,x_2+h)-\varphi(x_1,x_2+h)}{h} = \frac{\partial\varphi}{\partial x_2}$
  $\varphi(x_1, x_2+h) - \varphi(x_1, x_2) = F(x_1) - F(0) + \frac{1}{2}M(x_2+h)^2 - k - (F(x_1) - F(0) + \frac{1}{2}Mx_2^2 - k) = \frac{1}{2}M(x_2^2 + 2x_2h + h^2 - x_2^2) = \frac{1}{2}M(2x_2h + h^2)$
  $\lim_{h\to 0} \frac{\frac{1}{2}M(2x_2h+h^2)}{h} = \lim_{h\to 0} \frac{1}{2}M(2x_2 + h) = Mx_2 = \frac{\partial\varphi}{\partial x_2}$
  $\frac{\partial\varphi}{\partial x_2}$ is linear and is continuous

  then, φ is C^1

  let $\Omega_k = \text{level\_set}_k(\varphi) = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \int_0^{x_1} f(\tau)d\tau + \frac{1}{2}Mx_2^2 - k = 0 \right\}$
  $\partial\varphi_x = \begin{bmatrix} f(x_1) & Mx_2 \end{bmatrix}$

  since $f(x_1) = 0$ for $x_1 = 0$, need to check $Mx_2 \neq 0$ at $x_1 = 0$
  $\lim_{x_1\to 0} \int_0^{x_1} f(\tau)d\tau + \frac{1}{2}Mx_2^2 = lim_{x_1\to 0}k$
  $\lim_{x_1\to 0} x_1 f(x_1) + \frac{1}{2}Mx_2^2 = k$
  using $(\forall y \neq 0) yf(y) > 0$ and since f is Lipschitz:
  as $x_1 \to 0$, $\epsilon + \frac{1}{2}Mx_2^2 = k, \epsilon \to 0^+$
  $x_2 = (\frac{2}{M}(k - \epsilon))^{\frac{1}{2}}, \epsilon \to 0^+$

  then for $x_1 \to 0$:
  $(\forall k \neq 0 \implies x_2 \neq 0) \implies \partial\varphi_x$ is full row rank
  $(k = 0 \implies x_2 = 0) \implies$ equilibrium point and thus $\partial\varphi_x$ need not be concerned

for the case of $k \neq 0$:
$M\ddot{y} = -f(y)$
$\dot{x} = \begin{bmatrix} x_2 \\ -\frac{1}{M} f(x_1) \end{bmatrix}$
$\partial \varphi_x \dot{x} = f(x_1)x_2 - x_2 f(x_1) = 0$

then, all energy level sets are invariant

- 2.5
  use theorem 2.17 to show level set Ω satisfying $\varphi(x) \leq 0$ and Lie derivative $= 0$ for those satisfying $\varphi(x) = 0$ and that $\partial\varphi_x$ does not lose rank

  let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

  let $\varphi = E' = E - k = \int_0^y f(\tau)d\tau + \frac{1}{2}M\dot{y}^2 - k = \int_0^{x_1} f(\tau)d\tau + \frac{1}{2}Mx_2^2 - k$

  φ is C^1 as in 2.4

  let $\Omega_k = \text{level\_set}_k(\varphi) = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \int_0^{x_1} f(\tau)d\tau + \frac{1}{2}Mx_2^2 - k \leq 0 \right\}$
  $\partial\varphi_x = \begin{bmatrix} f(x_1) & Mx_2 \end{bmatrix}$

  since $f(x_1) = 0$ for $x_1 = 0$, need to check $Mx_2 \neq 0$ at $x_1 = 0$
  $\lim_{x_1 \to 0} \int_0^{x_1} f(\tau)d\tau + \frac{1}{2}Mx_2^2 = lim_{x_1 \to 0} k$
  $\lim_{x_1 \to 0} x_1 f(x_1) + \frac{1}{2}Mx_2^2 = k$
  using $(\forall y \neq 0) y f(y) > 0$ and since f is Lipschitz:
  as $x_1 \to 0$, $\epsilon + \frac{1}{2}Mx_2^2 = k, \epsilon \to 0^+$
  $x_2 = (\frac{2}{M}(k - \epsilon))^{\frac{1}{2}}, \epsilon \to 0^+$

  then for $x_1 \to 0$:
  $(\forall k \neq 0 \implies x_2 \neq 0) \implies \partial\varphi_x$ is full row rank
  $(k = 0 \implies x_2 = 0) \implies$ equilibrium point and thus $\partial\varphi_x$ need not be concerned

  for the case of $k \neq 0$:
  $M\ddot{y} = -f(y) - g(\dot{y})$
  $\dot{x} = \begin{bmatrix} x_2 \\ -\frac{1}{M}f(x_1) - \frac{1}{M}g(x_2) \end{bmatrix}$
  $\partial\varphi_x \dot{x} = f(x_1)x_2 - x_2 f(x_1) - x_2 g(x_2) = -x_2 g(x_2)$
  given $(\forall y \neq 0) \dot{y} g(\dot{y}) > 0$, then:
  $\partial\varphi_x \dot{x} \leq 0$

  then, all energy sublevel sets are positively invariant

- 2.8

1. show E is nonnegative continuously differentiable function

   $\dot{x_1} = x_2$
   $\dot{x_2} = -f(x_1)$

   $\varphi = E = \int_0^{x_1} f(\tau)d\tau + \frac{1}{2}x_2^2$

   check φ is C^1:

   $\lim_{h\to 0} \frac{\varphi(x_1+h,x_2)-\varphi(x_1,x_2)}{h} = \frac{\partial \varphi}{\partial x_1}$
   $\varphi(x_1+h, x_2) - \varphi(x_1, x_2) = F(x_1+h) - F(0) + \frac{1}{2}x_2^2 - k - (F(x_1) - F(0) + \frac{1}{2}x_2^2 - k) = F(x_1+h) - F(x_1)$
   $\lim_{h\to 0} \frac{F(x_1+h)-F(x_1)}{h} = \frac{\partial \varphi}{\partial x_1}$
   $f(x_1) = \frac{\partial \varphi}{\partial x_1}$
   given f is $C^1 \implies \frac{\partial \varphi}{\partial x_1}$ is locally continuous

   $\lim_{h\to 0} \frac{\varphi(x_1,x_2+h)-\varphi(x_1,x_2+h)}{h} = \frac{\partial \varphi}{\partial x_2}$
   $\varphi(x_1, x_2+h) - \varphi(x_1, x_2) = F(x_1) - F(0) + \frac{1}{2}(x_2+h)^2 - k - (F(x_1) - F(0) + \frac{1}{2}x_2^2 - k) = \frac{1}{2}(x_2^2 + 2x_2h + h^2 - x_2^2) = \frac{1}{2}(2x_2h + h^2)$
   $\lim_{h\to 0} \frac{\frac{1}{2}(2x_2h+h^2)}{h} = x_2 = \frac{\partial \varphi}{\partial x_2}$
   $\frac{\partial \varphi}{\partial x_2}$ is linear and is continuous
   then, φ is C^1

   check nonnegativity of φ:

   given $f(0) = 0$, $(\forall x_1 \neq 0) x_1 f(x_1) > 0 \implies \int_0^{x_1} f(\tau)d\tau \geq 0$
   $\frac{1}{2}x_2^2 \geq 0$
   then, $\varphi \geq 0$

2. let $\lim_{x_1\to\pm\infty} \int_0^{x_1} f(\tau)d\tau = \infty$, show $(\forall c \geq 0) E_c = \{(x_1, x_2) : E(x_1, x_2) = c\}$ is compact and invariant

   let $\varphi' = E - c = \int_0^{x_1} f(\tau)d\tau + \frac{1}{2}x_2^2 - c$

   $$E_c = \text{level\_set}_c(\varphi') = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \varphi' = 0 \iff \int_0^{x_1} f(\tau)d\tau + \frac{1}{2}x_2^2 - c = 0 \right\}$$

   use result from question 2.4 for φ' is C^1 and partial derivatives since φ' differ from φ of 2.4 by some constants/coefficients

$\partial\varphi'_x = \begin{bmatrix} f(x_1) & x_2 \end{bmatrix}$

check rank of φ':

since $f(x_1) = 0$ for $x_1 = 0$, check $x_2 \neq 0$ at $x_1 = 0$
$\lim_{x_1 \to 0} \int_0^{x_1} f(\tau)d\tau + \frac{1}{2}x_2^2 = lim_{x_1 \to 0} k$
$\lim_{x_1 \to 0} x_1 f(x_1) + \frac{1}{2}x_2^2 = k$
using $(\forall y \neq 0) y f(y) > 0$ and f is $C^1 \implies f$ locally Lipschitz:
as $x_1 \to 0$, $\epsilon + \frac{1}{2}x_2^2 = k, \epsilon \to 0^+$
$x_2 = (2(k-\epsilon))^{\frac{1}{2}}, \epsilon \to 0^+$

then for $x_1 \to 0$:
$(\forall k \neq 0 \implies x_2 \neq 0) \implies \partial\varphi_x$ is full row rank
$(k = 0 \implies x_2 = 0) \implies$ equilibrium point

for the case of $k \neq 0$:
$\dot{x} = \begin{bmatrix} x_2 \\ -f(x_1) \end{bmatrix}$
$\partial\varphi'_x \dot{x} = f(x_1)x_2 - x_2 f(x_1) = 0$
then, $(\forall c \geq 0) E_c$ are invariant

show all sets E_c are bounded:
$V(x_1) = \int_0^{x_1} f(\tau)d\tau$
$\varphi' = 0$: $V(x_1) + \frac{1}{2}x_2^2 - c = 0$
$\frac{1}{2}x_2^2 \geq 0$
$V(x_1) \geq 0$
$0 \leq \frac{1}{2}x_2^2 \leq c$
$x_2 \leq (2c)^{\frac{1}{2}}$
$0 \leq V(x_1) \leq c$
V convex with minimum at $V(0)$ satisfies:
$(\exists x : V(x) = c)(\forall V(x_1) \leq c)|x_1| \leq |x|$
$\|x\| \leq \left\| \begin{bmatrix} 0 \\ (2c)^{\frac{1}{2}} \end{bmatrix} \right\| + \left\| \begin{bmatrix} c \\ 0 \end{bmatrix} \right\|$
then, all sets E_c are bounded

show all sets E_c are closed:
codomain of energy level set constraint: $\varphi'(x) = \{0\}$
$(\forall y \in \{(\varphi'(x))^c : x \in \mathbb{R}^2\} = 0^c)(\forall \delta > 0)(\forall y' \in B^1_\delta(y)) y' \in 0^c$
$\{\varphi'(x))^c : x \in \mathbb{R}^2\}$ is open $\implies \{\varphi'(x)) : x \in \mathbb{R}^2\}$ is closed
$\{\varphi'(x)) : x \in \mathbb{R}^2\}$ is closed $\wedge\ \varphi'$ is continuous $\implies \varphi'^{-1}(\varphi'(x))) \in \mathbb{R}^2$ is closed
the domain of φ' is E_c, so all sets E_c are closed

all sets E_c are closed and bounded $\implies$ all sets E_c are compact

3. use part 2, show $\forall c > 0$, all orbits through points in E_C are closed curves

   E'_c where c=0 corresponds to an equilibrium

   let $E'_c = \{E_c : \forall c \neq 0\}$ be a set of all sets of dynamic system where each contains no equilibrium

   using Poincare-Bendixson theorem:

   $(\forall E \in E'_c)$ E is non-empty, invariant, compact, C^1 planar dynamical systems with no equilibrium $\implies (\forall x_0 \in E) L(x_0)$ is a closed orbit

4. Prove that all orbits of the ODE are closed curves

   E_0 corresponding to $c = 0$ is compact, invariant and an equilibrium
   $(\forall x \in E_0)\phi(T_{\mathbb{R}}, x) = x \iff (\forall x \in E_0)O(x) = \{x\}$
   E_0 is compact and $(\forall x \in E_0)O(x) = \{x\} \implies (\forall x \in E_0)x$ is a curve of a singular point and it is closed

   using $((\forall E \in E'_c)(\forall x_0 \in E)L(x_0)$ is a closed orbit) and $((\forall x \in E_0)O(x)$ is closed), all orbits of ODE are closed curves

- 2.9 Consider the system of Exercise 2.8 and suppose the system has dissipation

  $\dot{x_1} = x_2$
  $\dot{x_2} = -f(x_1) - g(x_2)$
  g is locally Lipschitz
  $g(0) = 0$
  $g'(x_2) > 0$ for all $x_2 \neq 0$

  Show that the system cannot have any closed orbits

  $\dot{x} = \begin{bmatrix} x_2 \\ -f(x_1) - g(x_2) \end{bmatrix}$

  $\frac{\partial \dot{x}_1}{\partial x_1} = 0$
  $\frac{\partial \dot{x}_2}{\partial x_2} = -\frac{\partial g}{\partial x_2}$
  $\frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2} = -\dot{g}$

  $-\dot{g}(x_2) \leq 0$
  $-\dot{g}(x_2) \neq 0, \forall x_2$:

  using Bendixson's Criterion, the C^1 planar system on $\mathbb{R}^2$ domain has $\frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2}$ that is not identically zero and does not change sign $\implies$ the system has no closed orbit in the domain