

0.1 Symbols

$S^n \equiv$ set of all symmetric matrices

$M_+^n \equiv$ set of all positive definite matrices (PD)

$S_+^n \equiv$ set of all symmetric positive semidefinite matrices (SPSD)

$S_{++}^n \equiv$ set of all symmetric positive definite matrices (SPD)

$cl(X) \equiv$ closure of X

$relint(X) \equiv$ relative interior of X

$int(X) \equiv$ interior of X

$[x]^+ \equiv \max(x, 0)$

0.2 Preliminary

Consider $f: \mathbb{R}^n \rightarrow \mathbb{R}$

Gradient of f : $\nabla f(x) = \begin{bmatrix} \partial f / \partial x_1 \\ \vdots \end{bmatrix}$

$$f(x) = a^T x \implies \nabla f(x) = a$$

$$f(x) = x^T P x, P = P^T \implies \nabla f(x) = 2Px$$

$$f(x) = x^T P x \implies \nabla f(x) = 2\left(\frac{P^T + P}{2}\right)x = (P^T + P)x$$

Taylor expansion approximation:

$$f(x) \approx f(x_0) + \nabla^T f(x_0)(x - x_0) + o((x - x_0)^2)$$

$$f(x + \delta x) \approx f(x_0) + \nabla^T f(x_0)\delta x + o((\delta x)^2)$$

Chain rule:

$$f: \mathbb{R} \rightarrow \mathbb{R}, g: \mathbb{R} \rightarrow \mathbb{R}, h(x) = f(g(x))$$

$$\nabla h(x) = g'(f(x))\nabla f(x)$$

$$g: \mathbb{R}^m \rightarrow \mathbb{R}, g(x) = f(Ax + b)$$

$$\nabla g(x) = A^T \nabla f(Ax + b)$$

2nd derivative:

$$\nabla^2 f(x) = \begin{bmatrix} \partial^2 f / \partial x_1 \partial x_1 & \cdots \\ \vdots & \partial^2 f / \partial x_n \partial x_n \end{bmatrix}$$

$$\nabla f(x) = Px + g$$

$$\nabla^2 f(x) = P$$

Hessian gives the 2nd order approximation:

$$f(x) \approx f(x_0) + \nabla^T f(x_0)(x - x_0) +$$

$$\frac{1}{2}(x - x_0)^T \nabla^2 f(x_0)(x - x_0)$$

Matrices:

$A \in \mathbb{R}^{m \times n}$: set of all real matrices

$$\text{inner product: } \sum_i \sum_j x_{ij} y_{ij} = \text{trace}(XY^T) =$$

$$\text{trace}(Y^T X) = \sum_i (XY)_{ii}$$

note trace has cyclic property

$$\text{frobenius norm: } \|X\|_F = (\sum_i \sum_j X_{ij}^2)^{\frac{1}{2}}$$

range: $R(A) = \{Ax : x \in \mathbb{R}^n\} = \sum_i a_i x_i$, where a_i is

ith column (column space of A)

null space: $N(A) = \{x : Ax = 0\}$

SVD:

$$A_{m \times n} = U_{m \times m} \Sigma_{m \times n} V_{n \times n}^T$$

U and V are left and right eigenvector matrixes

U and V are orthogonal matrixes ($BB^T = B^T B = I$)

Σ is rectangular diagonal matrix of eigenvalues

$$A_{m \times n} x_n$$

linear transformation: $U \Sigma V^T x$

rotation - scaling - rotation

PSD matrix:

$$A \text{ PSD} \iff (\forall x) x^T A x \geq 0 \iff (\forall i) \lambda_i(A) \geq 0$$

$$A \text{ PSD} \implies A^{1/2} \text{ exists}$$

Real symmetric matrices have real eigenvalues:

$$Av = \lambda v$$

$$v^* A v = v^* \lambda v = \lambda \|v\|_2^2$$

$$(v^* A v)^* = v^* A^* v = \lambda^* \|v\|_2^2 \implies \lambda = \lambda^*$$

Affine sets

$$\text{A set } C \subseteq \mathbb{R}^n \text{ is affine if } (\forall x_1, x_2 \in C)(\forall \theta \in \mathbb{R}) \implies \theta x_1 + (1 - \theta)x_2 \in C$$

Convex sets

$$\text{A set } C \subseteq \mathbb{R}^n \text{ is convex if } (\forall x_1, x_2 \in C)(\forall \theta \in \mathbb{R}) 0 \leq \theta \leq 1 \implies \theta x_1 + (1 - \theta)x_2 \in C$$

Operations preserving convex sets:

- partial sum
- sum
- coordinate projection
- scaling
- translation
- intersection between any convex sets

Separating Hyperplanes: if $S, T \subset \mathbb{R}^n$ are convex and disjoint, then $\exists a \neq 0, b$ such that:

$$a^T x \geq b, \forall x \in S$$

$$a^T x \leq b, \forall x \in T$$

Supporting Hyperplane:

if S is convex, $\forall x_0 \in \partial S$ (boundary of S), then $\exists a \neq 0$ such that $a^T x \leq a^T x_0, \forall x \in S$

Convex combination:

$$\sum_i \theta_i x_i, \forall \theta_i \in \mathbb{R}, \sum_i \theta_i = 1, \theta_i \geq 0$$

Convex hull:

The set of all convex combinations of points in C , the hull is convex

Hyperplane

$$C = \{x : a^T x = b\}, a \in \mathbb{R}^n, a \neq 0, b \in \mathbb{R}$$

Halfspaces

$$C = \{x : a^T x \leq b\}, a \in \mathbb{R}^n, a \neq 0, b \in \mathbb{R}$$

$$\text{let } a^T x_c = b$$

$$C = \{x : a^T (x - x_c) \leq 0\}, a \in \mathbb{R}^n, a \neq 0$$

Ellipse

$$E(x_c, P) = \{x : (x - x_c)^T P^{-1} (x - x_c) \leq 1\}, P > 0$$

$$P = r^2 I \implies \text{Euclidean Ball}$$

$$P = Q \begin{bmatrix} \lambda_1 & \dots \\ \dots & \lambda_n \end{bmatrix} Q^T$$

$$(x - x_c)^T (Q \begin{bmatrix} \lambda_1 & \dots \\ \dots & \lambda_n \end{bmatrix} Q^T)^{-1} (x - x_c) \leq 1$$

$$\tilde{x}^T \begin{bmatrix} \frac{1}{\lambda_1} & \dots \\ \dots & \frac{1}{\lambda_n} \end{bmatrix} \tilde{x} \leq 1$$

$$\tilde{x}^T \begin{bmatrix} \frac{1}{\lambda_1} & \dots \\ \dots & \frac{1}{\lambda_n} \end{bmatrix} \tilde{x} = \frac{\tilde{x}_1^2}{\lambda_1} + \dots + \frac{\tilde{x}_n^2}{\lambda_n} \leq 1$$

volum of ellipsoid proportional to $\sqrt{\det(P)} = \sqrt{\prod_i \lambda_i}$

0.3 Problem Types

LP

standard, inequality, general forms

$$\begin{aligned} \min_x c^T x \text{ s.t. :} \\ Ax = b \\ x \succeq 0 \end{aligned}$$

$$\begin{aligned} \min_x c^T x \text{ s.t. :} \\ Ax \preceq b \end{aligned}$$

$$\begin{aligned} \min_x c^T x + d \text{ s.t. :} \\ Gx \preceq h \\ Ax = b \end{aligned}$$

QP

$$\begin{aligned} \min_x \frac{1}{2} x^T P x + q^T x + r \text{ s.t. :} \\ Gx \leq h \\ Ax = b \end{aligned}$$

QCQP

$$\begin{aligned} \min_x \frac{1}{2} x^T P_0 x + q_0^T x + r_0, \text{ s.t. :} \\ \frac{1}{2} x^T P_i x + q_i^T x + r_i \leq 0, \forall i \\ Ax = b \end{aligned}$$

SOCP

$$\begin{aligned} \min_x f^T x \text{ s.t. :} \\ \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \forall i \\ Fx = g \end{aligned}$$

$$\begin{aligned} (\forall i) b_i = 0 &\implies LP \\ (\forall i) c_i = 0 &\implies QCQP \end{aligned}$$

GP

$$\min_x f_0(x) \text{ s.t. :}$$

$$f_i(x) \leq 1, \forall i$$

$$h_i(x) = 1, \forall i$$

$$f_i \text{ is a posynomial} := \sum_i h_i$$

$$h_i \text{ is a monomial} := cx_1^{a_1} x_2^{a_2} \dots, c > 0, a_i \in \mathbb{R}$$

Use transform of objective and constraint functions:

$$y_i = \log x_i, x_i = e^{y_i}$$

h_i becomes exponential of affine function

$\tilde{f}_i = \log(f_i)$ becomes log sum exp (convex)

If all constraints and objective are monomials, reduces to LP after transform.

SDP

general, standard, inequality forms

$$\min_x c^T x \text{ s.t. :}$$

$$LMI : \sum_i^n x_i F_i + G \preceq_K 0$$

$$Ax = b$$

$$x \in \mathbb{R}^n$$

$$F_i, G \in S^m, K \in S_+^m$$

$$\min_X \text{tr}(CX) \text{ s.t. :}$$

$$\text{tr}(A_i X) = b_i, \forall i$$

$$X \succeq 0$$

$$\min_x c^T x \text{ s.t. :}$$

$$\sum_i^n x_i A_i \preceq_K B$$

$$Ax = b$$

$$B, A_i \in S^m, K \in S_+^m$$

concatenating constraints:

$$F^{(i)}(x) = \sum_j x_j F_i^{(j)} + G^{(i)} \preceq 0$$

$$Gx \preceq h$$

$$\implies$$

$$\text{diag}(Gx - h, F^{(1)}(x), \dots, F^{(m)}(x)) \preceq 0$$

if all matrices are diagonal, reduces to LP

0.4 Convex/Concave Functions

- Affine
- Pointwise supremum of convex function
 - distance to farthest point in a set
 - support function of set
- Norm
- Non-negative weighted sum of convex functions

0.4.1 log det X, concave

$$\begin{aligned}
 \text{let } X &= Z + tV \succ 0 \\
 f &= \log \det(Z + tV) \\
 f &= \log \det(Z^{-0.5}(I + tZ^{-0.5}VZ^{0.5})Z^{0.5}) \\
 f &= \log(\det(Z^{-0.5})\det(I + tZ^{-0.5}VZ^{0.5})\det(Z^{0.5})) \\
 f &= \log(\det(Z^0)\det(I + tZ^{-0.5}VZ^{0.5})) \\
 f &= \log \det(I + tZ^{-0.5}VZ^{0.5}) \\
 f &= \log \Pi_i(1 + \lambda_i t) \\
 f &= \sum_i \log(1 + \lambda_i t) \\
 \frac{\partial f}{\partial t} &= \sum_i \frac{\lambda_i}{1 + \lambda_i t} \\
 \frac{\partial^2 f}{\partial t^2} &= \sum_i \frac{-\lambda_i^2}{(1 + \lambda_i t)^2} = -\sum_i \frac{\lambda_i^2}{(1 + \lambda_i t)^2} \leq 0 \\
 \nabla^2 f &\leq 0 \iff f \text{ concave}
 \end{aligned}$$

0.4.2 log $\sum_i \exp(x_i)$, convex

$$\begin{aligned}
 \nabla^2 f &= \frac{1}{(1^T z)^2} (1^T z \text{diag}(z) - zz^T) \\
 v^T zz^T v &= \det(v^T zz^T v) = \det(vv^T zz^T) \\
 v^T zz^T v &= \sum_j \sum_i z_j z_i v_j v_i \\
 v^T zz^T v &= \left(\sum_j z_j z_j\right) \left(\sum_i z_i v_i\right) \\
 v^T zz^T v &= \left(\sum_i z_i v_i\right)^2
 \end{aligned}$$

use Holder's Inequality :

$$\begin{aligned}
 \|a\|_2^2 \|b\|_2^2 &\geq |a^T b|^2 \\
 \text{let } a &= z_i^{0.5}, b = v_i z_i^{0.5} \\
 1^T z \left(\sum_i v_i^2 z_i\right) - \left(\sum_i z_i v_i\right)^2 &\geq 0 \\
 v^T \nabla^2 f v &= \frac{1}{(1^T z)^2} \left(1^T z \left(\sum_i v_i^2 z_i\right) - \left(\sum_i z_i v_i\right)^2\right) \geq 0 \\
 \nabla^2 f &\geq 0 \iff f \text{ convex}
 \end{aligned}$$

0.4.3 geometric mean on R_{++}^n , concave

$$\begin{aligned}
 f &= (\Pi_i x_i)^{\frac{1}{n}} \\
 \frac{\partial}{\partial x_i} f &= \frac{1}{n} (\Pi_i x_i)^{\frac{1}{n}-1} \Pi_{j \neq i} x_j \\
 \frac{\partial^2}{\partial x_i^2} f &= \frac{1}{n} \left(\frac{1}{n} - 1\right) (\Pi_i x_i)^{\frac{1}{n}-2} (\Pi_{j \neq i} x_j)^2 \\
 \frac{\partial^2}{\partial x_i^2} f &= \frac{1}{n} \left(\frac{1}{n} - 1\right) \frac{(\Pi_i x_i)^{\frac{1}{n}}}{x_i^2} \\
 \frac{\partial^2}{\partial x_i x_k} f &= \frac{1}{n^2} \frac{(\Pi_i x_i)^{\frac{1}{n}}}{x_i x_k}, i \neq k \\
 \frac{\partial^2}{\partial x_i x_k} f &= \frac{1}{n^2} \frac{(\Pi_i x_i)^{\frac{1}{n}}}{x_i x_k} - \delta_{ik} \frac{1}{n} \frac{(\Pi_i x_i)^{\frac{1}{n}}}{x_i^2} \\
 v^T \nabla^2 f v &= \frac{-(\Pi_i x_i)^{\frac{1}{n}}}{n^2} \left(n \sum_i \frac{v_i^2}{x_i^2} - \left(\sum_i \frac{v_i}{x_i}\right)^2\right) \\
 \text{apply Cauchy Schwartz Inequality :} \\
 \text{let } a &= \mathbf{1}, b_i = \frac{v_i}{x_i} \\
 \|\mathbf{1}\|_2^2 \left(\sum_i \frac{v_i^2}{x_i}\right) &\geq \left(\sum_i \frac{v_i}{x_i}\right)^2 \\
 n \sum_i \frac{v_i^2}{x_i^2} - \left(\sum_i \frac{v_i}{x_i}\right)^2 &\geq 0 \\
 v^T \nabla^2 f v &\leq 0 \iff f \text{ concave}
 \end{aligned}$$

0.4.4 quadratic over linear, convex

$$f(x, y) = \frac{h(x)}{g(y)}, g(y) \text{ linear}, g(y) \in R_+$$

$$\nabla^2 f = vv^T \text{ is PSD} \iff f \text{ convex}$$

0.5 Composition of functions

Mnemonic derivation from scalar composite function

$$f = h(g(x))$$

$$f' = g'(x)h'(g(x))$$

$$f'' = g''(x)h'(g(x)) + (g'(x))^2 h''(g(x))$$

h convex & non-decreasing, g convex \implies f convex

$$h'' \geq 0, g''(x) \geq 0, h'(g(x)) \geq 0 \implies f'' \geq 0$$

h convex & non-increasing, g concave \implies f convex

$$h'' \geq 0, g''(x) \leq 0, h'(g(x)) \leq 0 \implies f'' \geq 0$$

h concave & non-decreasing, g concave \implies f concave

$$h'' \leq 0, g''(x) \leq 0, h'(g(x)) \geq 0 \implies f'' \leq 0$$

h concave & non-increasing, g convex \implies f concave

$$h'' \leq 0, g''(x) \geq 0, h'(g(x)) \leq 0 \implies f'' \leq 0$$

0.6 Convexity Preservation of Sets

0.6.1 Intersection

$$(\forall \alpha \in A) S_\alpha \text{ is convex cone} \implies \\ \cap_{\alpha \in A} S_\alpha \text{ is convex cone}$$

Any closed convex set can be represented by possibly infinitely many half spaces.

0.6.2 Affine functions

let $f(x) = Ax + b, f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

then if S is a convex set we have:

- project forward: $f(S) = \{f(X) : X \in S\}$ is convex
- project back: $f^{-1}(S) = \{X : f(X) \in S\}$ is convex

Example:

$$C = \{y : y = Ax + b, \|x\| \leq 1\}$$

$\|x\| \leq 1$ is convex, $Ax + b$ is affine $\implies C$ is convex

Example:

$$C = \{x : \|Ax + b\| \leq 1\}$$

$\{y : \|y\| \leq 1\}$ is convex $\wedge y$ is an affine function of $x \implies C$ is convex

0.7 Constraint Qualifications

0.7.1 Slater's Constraint Qual.

Optimal solution is in relative interior: $x^* \in \text{relint}(S)$

Inequalities $(\forall i) f_i(x)$ convex $\wedge f_i(x) < 0 \implies$
Slater's constraint satisfied.

Inequalities $(\forall i) f_i(x)$ affine $\implies (\forall i) f_i(x) \leq 0 \wedge (\exists i) f_i(x) < 0 \implies$ Slater's constraint satisfied.

Achieving Slater's constraint implies 0 duality gap.

0.7.2 KKT

Assumes optimality achieved with 0 duality gap:
 $\nabla L(x^*, \lambda^*, v^*) = 0$

$$L(x^*, \lambda^*, v^*) = f_0(x^*) + \sum_i \lambda_i^* f_i(x^*) + \sum_i v_i h_i(x^*)$$

$$\nabla L(x^*, \lambda^*, v^*) = \nabla f_0(x^*) + \sum_i \lambda_i^* \nabla f_i(x^*) + \sum_i v_i \nabla h_i(x^*)$$

We have the constraints:

$$f_i(x^*) \leq 0$$

$$\lambda_i^* \geq 0$$

$$h_i(x^*) = 0$$

$$\lambda_i^* f_i(x^*) = 0$$

$$\nabla L(x^*, \lambda^*, v^*) = \nabla f_0(x^*) + \sum_i \lambda_i^* \nabla f_i(x^*) + \sum_i v_i \nabla h_i(x^*)$$

Primal inequality constraints convex and equality constraints affine and KKT satisfied \implies 0 duality gap with specified points for primal and dual. (sufficient).

If Slater's constraint satisfied then the above is sufficient and necessary:

Primal inequality constraints convex and equality constraints affine and KKT satisfied \iff 0 duality gap with specified points for primal and dual.

0.8 Definitions

0.8.1 Convex Function

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y), \forall \theta = [0, 1]$$

For convenience we sometimes define an extended value function:

$$\tilde{f}(x) = \begin{cases} f(x), & x \in \text{dom}(f) \\ \infty, & \text{other wise} \end{cases}$$

if $f(x)$ convex, then \tilde{f} is also convex

Sublevel set of a function

$$C(\alpha) = \{x \in \text{dom}(f) : f(x) \leq \alpha\}$$

For convex function, all sublevel sets are convex ($\forall \alpha$).
Converse is not true.

Quasi-convex function: if its sublevel sets are all convex.

Epigraph of functions:

$$\text{epi}(f) = \{(x, t) : x \in \text{dom}(f), f(x) \leq t\} \in \mathbb{R}^{n+1}, \\ f \in \mathbb{R}^n \rightarrow \mathbb{R}.$$

f is convex function $\iff \text{epi}(f)$ is convex set

0.8.2 First order condition

Suppose f is differentiable and domain of f is convex.
Then:

f convex \iff

$$(\forall x, x_0 \in \text{dom}(f)) f(x) \geq f(x_0) + \nabla f(x_0)^T(x - x_0)$$

rough proof:

suppose $f(x)$ is convex but $(\exists x, x_0) f(x) < f(x_0) + \nabla f(x_0)^T(x - x_0)$

then this means the function should bend across the tangent line which violates the convexity

proof for converse direction:

suppose that $(\exists x, x_0) f(x) \geq f(x_0) + \nabla f(x_0)^T(x - x_0)$

to show that $f(x)$ is convex lets take $x, y \in \text{dom}(f), z = \theta x + (1 - \theta)y$
 $\theta f(x) + (1 - \theta)f(y) \geq f(z) + \nabla f(z)^T(\theta x - \theta z + (1 - \theta)y - (1 - \theta)z)$
 $\theta f(x) + (1 - \theta)f(y) \geq f(\theta x + (1 - \theta)y)$
 $f(x)$ is convex

0.8.3 Second order condition

Suppose f is twice differentiable and $\text{dom}(f)$ is convex,
then $f(x)$ is convex $\iff \nabla^2 f(x) \geq 0$ (PSD, eg: wrt. S_+^n)

proof for scalar case:

suppose that $f(x)$ is convex, then the first-order condition holds

$$\text{for } x, y \in \text{dom}(f) : f(x) \geq f(y) + f'(y)(x - y)$$

$$\text{for } y, x \in \text{dom}(f) : f(y) \geq f(x) + f'(x)(y - x)$$

$$f'(x)(y - x) \leq f(y) - f(x) \leq f'(y)(y - x)$$

$$f'(x)(y - x) \leq f'(y)(y - x) \implies 0 \leq (y - x)(f'(y) - f'(x))$$

$$\text{take } y \rightarrow x : 0 \leq f''(x)$$

$$f''(x) \geq \frac{f'(x+\delta x) - f'(x)}{\delta x}$$

conversely, suppose that $f'(z) \geq 0, \forall z \in \text{dom}(f)$,
take $x, y \in \text{dom}(f)$ WLOG $x < y$

$$\int_x^y f''(z)(y - z)dz \geq 0$$

$$f''(z) \geq 0, (y - z) \geq 0$$

$$I_1 = \int_x^y f''(z)ydz = y(f'(y) - f'(x))$$

$$I_2 = -\int_x^y zf''(z)dz$$

$$dv = f''(z)dz \implies v = f'(z)$$

$$u = z \implies du = dz$$

$$I_2 = -[zf'(z)]_x^y + \int_x^y f'(z)dz = -yf'(y) + xf'(x) + f(y) - f(x)$$

$$I_1 + I_2 = yf'(y) - yf'(x) - yf'(y) + xf'(x) + f(y) - f(x) \geq 0$$

$$\implies f(y) \geq f(x) + f'(x)(y - x) \text{ first order condition:}$$

$$x < y$$

$$\text{first order condition holds} \implies f(x) \text{ convex}$$

0.8.4 Inequalities

$$x \preceq_K y \iff y - x \in K$$

x is a minimum in S wrt. cone K :

$$x \in S : (\forall y \in S) f(y) \succeq_K f(x) \iff f(y) - f(x) \in S \\ S \subseteq x + K$$

x is a minimal in S wrt. cone K :

$$x \in S : (\forall y \in S) f(y) \preceq_K f(x) \implies x = y \\ x \in S : (\forall y \in S) f(x) - f(y) \in K \implies x = y \\ (x - K) \cap S = \{x\}$$

0.8.5 Cone

$$(\forall x \in C, \forall \theta \geq 0) \theta x \in C$$

0.8.6 Convex Cone

Eg: S^n, S_+^n are convex cones
convexity check for S_+^N :

$$x_1 \in S_+^n \implies v^T x_1 v \geq 0 \\ x_2 \in S_+^n \implies v^T x_2 v \geq 0 \\ x = \theta x_1 + (1 - \theta) x_2, \theta \in [0, 1] \\ v^T (\theta x_1 + (1 - \theta) x_2) v \geq 0 \\ v^T \theta x_1 v + (1 - \theta) v^T x_2 v \implies x \in S_+^n$$

convexity check for cone:

$$x_1 \in S_+^n \implies \theta x_1 \in S_+^n, \theta \geq 0 \\ (\forall v) v^T x v \geq 0 \implies v^T (\theta x) v \geq 0 \implies \text{cone}$$

0.8.7 Proper Cone

Definition:

- convex
- closed (contains all limit points)
- solid (non-empty interior)
- pointed (contains no line): $x \in K \implies -x \notin K$

Then the proper cone K defines a generalized inequality (\preceq_K) in \mathbb{R}^n

$$x \preceq_K y \implies y - x \in K \\ x \prec_K y \implies y - x \in \text{int}(K)$$

Example: $K = R_+^n$ (non-negative orthant):

$$n = 2 \\ x \preceq_{R_+^2} y \implies y - x \in R_+^2$$

Cone provides partial ordering using difference of 2 objects.

$$X \preceq_{S_+^n} Y \iff Y - X \in S_+^n \iff Y - X \text{ is PSD}$$

0.8.8 Norm Cone

$$K = \{(x, t) \in \mathbb{R}^{n+1} : \|x\| \leq t\}, x \in \mathbb{R}^n$$

0.8.9 Dual norm

$$\|z\|_* := \sup_x \{z^T x : \|x\|_p \leq 1\}$$

Dual of L1-norm:

$$\|z\|_* := \sup_x \{z^T x : \|x\|_1 \leq 1\}$$

$$\max \sum_i z_i x_i,$$

$$\text{subject to : } \sum_i \|x_i\| \leq 1$$

select x_i corresponding to z_i with maximum absolute value

$$\text{equivalent to } \|z\|_* = \|z\|_\infty$$

Dual of L- ∞ -norm:

$$\|z\|_* := \sup_x \{z^T x : \|x\|_\infty \leq 1\}$$

$$\max \sum_i z_i x_i,$$

$$\text{subject to : } \|x_i\| \leq 1, \forall i$$

choose $x_i = 1$ if $z_i \geq 0$ and $x_i = -1$ if $z_i < 0$

$$\text{equivalent to } \|z\|_* = \|z\|_1$$

Dual norm of Lp-norm: Lq-norm where $1/p + 1/q = 1$

Properties of Dual Cone:

- K^* closed and convex
- $K_1 \subseteq K_2 \implies K_2^* \subseteq K_1^*$
- K has non-empty interior $\implies K^*$ pointed
- $cl(K)$ pointed $\implies K^*$ has non-empty interior
- $K^{**} = cl(\text{convhull}(K))$, useful for relaxed optimization
- K convex and closed $\implies K = K^{**}$

0.8.10 Operator norm

$$\|X\|_{a,b} = \sup\{\|Xu\|_a : \|u\|_b \leq 1\}, X \in \mathbb{R}^{m \times n}$$

0.8.11 Dual cone

K is a cone

$$K^* = \{y : x^T y \geq 0, \forall x \in K\}$$

0.8.12 Dual norm cone

$$K^* = \{(u, v) : \|u\|_* \leq v\}$$

$$\text{where } K = \{(x, t) : \|x\| \leq t\}$$

0.8.13 support function of a set

$$S_C(x) = \sup\{x^T y : y \in C\}$$

$$\text{dom}(S_C) = \{x : \sup_{y \in C} x^T y < \infty\}$$

It is pointwise supremum of convex function, so it is convex.

0.9 Regularized Approximation

Noise sensitivity of different objectives:

- robust least squares / Huber penalty
- log barrier
- deadzone linear
- quadratic

Least norm problems

- L2 norm objective with equality constraint
- sparsity inducing norms (eg: L1)
- norm ball constraint
- probability distribution
 - convex comb. of columns of A to fit b
- variable constraints
 - box
 - one sided bound

Multicriterion Formulation

Tikhonov Regularization
todo..

0.10 Descent Methods

Algorithm 1: Descent Overview

```

1 init  $x_0 \in \text{dom} f$ ;
2 while stopping criterion is not satisfied
3    $\Delta x \leftarrow$  Compute Descent Direction;
4    $t \leftarrow$  Compute Descent Step Size;
5    $x_{k+1} \leftarrow x_k + t\Delta x$ 

```

0.10.1 Search Step Size

Given Δx , step direction

Search step size:

Exact Line Search: $t = \operatorname{argmin}_{s \geq 0} f(x + s\Delta x)$

Algorithm 2: Backtracking Line Search

```

 $\Delta x$ : Search direction
 $t$  : Step size
1  $\alpha \in (0, 0.5)$ ;
2  $\beta \in (0, 1)$ ;
3  $t \leftarrow 1$ ;
4 while  $f(x + t\Delta x) > f(x) + \alpha t \nabla f(x)^T \Delta x$ 
5    $t \leftarrow \beta t$ ;

```

0.10.2 Search Direction - 1st Order Methods

Steepest Descent

$$\operatorname{argmin}_{\Delta x} f(x) + \nabla f(x)^T \Delta x_{sd}$$

$$\operatorname{argmin}_{\Delta x} \nabla f(x)^T \Delta x_{sd}$$

Normalized Steepest Descent

$$\Delta x_{nsd} = \operatorname{argmin}_v \{ \nabla f(x)^T v : \|v\| \leq 1 \}$$

$$\Delta x_{nsd} = \frac{\Delta x_{sd}}{\|\nabla f(x)\|_*}$$

$$\|\nabla f(x)\|_* = \sup_y \{ \nabla f(x)^T y : \|y\| \leq 1 \} \quad (\text{dual norm})$$

$$\nabla f(x)^T \Delta x_{sd} = \nabla f(x)^T \Delta x_{nsd} \|\nabla f(x)\|_* = \|\nabla f(x)\|_*^2$$

Steepest Descent for L2-Norm

$$\Delta x = -\nabla f(x)$$

$$\Delta x_{nsd} = \frac{-\nabla f(x)}{\|\nabla f(x)\|_*}$$

Steepest Descent for Quadratic Norm

$$\|z\|_P = (z^T P z)^{\frac{1}{2}} = \|P^{\frac{1}{2}} z\|_2, P \in S_{++}^n$$

$$\Delta x_{nsd} = -(\nabla f(x)^T P^{-1} \nabla f(x))^{-\frac{1}{2}} P^{-1} \nabla f(x)$$

$$\Delta x_{sd} = -P^{-1} \nabla f(x)$$

Normalized Steepest Descent for L1-norm

$$\Delta x_{nsd} = \operatorname{argmin}_v \{ \nabla f(x)^T v : \|v\|_1 \leq 1 \}$$

$$\Delta x_{nsd} = -\operatorname{sign}\left(\frac{\partial f(x)}{\partial x_i}\right) e_i, i : \|\nabla f(x)\|_\infty = |(\nabla f(x))_i|$$

Search Direction - 2nd Order Methods, Unconstrained

Newton's Method

$$\Delta x_{nt} = -\nabla^2 f(x)^{-1} \nabla f(x)$$

$$\nabla^2 f(x)^{-1} \succ 0 \implies$$

$$\nabla f(x)^T \Delta x_{nt} = -\nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x) < 0$$

Affine invariance of Newton's method: auto-scaling of level curves to enable better descent direction

2nd order Taylor approximation of function is minimized with Δx_{nt} , so quadratic model gives good approximation near minimizer.

Newton Decrement: used for convergence/stopping criterion

Norm of Newton Step:

$$\lambda(x) = (\Delta x_{nt}^T \nabla^2 f(x) \Delta x_{nt})^{\frac{1}{2}} = \|(\nabla^2 f(x) \Delta x_{nt})^{\frac{1}{2}}\|_2$$

Bounding difference of lower bound of 2nd order model and $f(x)$:

$$f(x) - \inf\{f(x) + \nabla f(x)^T v + \frac{1}{2} v^T \nabla^2 f(x) v : A(x+v) = b\} = \frac{1}{2} \lambda(x)^2$$

$$\frac{1}{2} \lambda(x)^2 \approx f(x) - p^*$$

$$\nabla f(x)^T \Delta x_{nt} = -\nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x)$$

$$\nabla f(x)^T \Delta x_{nt} = -\lambda(x)^2 = \frac{d}{dt} f(x + \Delta x_{nt} t)|_{t=0}$$

Algorithm 3: Newton Method Descent

```

1 init  $x_0 \in \text{dom} f$ ;
2 do
3    $\Delta x_{nt} \leftarrow -\nabla^2 f(x)^{-1} \nabla f(x)$ ;
4    $\lambda(x)^2 \leftarrow \nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x)$ ;
5    $t \leftarrow \text{Compute Step Size (line search)}$ ;
6    $x_{k+1} \leftarrow x_k + t \Delta x_{nt}$ 
7 while  $\frac{1}{2} \lambda(x)^2 > \epsilon$ ;

```

Assumptions of above algorithm: KKT matrix invertible. Descent method is decreasing wrt. f .

Convergence: split into damped phase and quadratically convergent phase. Fast convergence once step size is 1.

BFGS: a Quasi-newton method
Direction:

$$p_k = -B_k^{-1} \nabla f_k$$

Properties for B :

$$B \succ 0$$

$$B = B^T$$

Satisfaction of Secant equation:

$$B_{k+1}(x_{k+1} - x_k) = \nabla f_{k+1} - \nabla f_k$$

$$B_{k+1} s_k = y_k$$

Curvature condition:

$$B_{k+1} s_k = y_k$$

$$s_k^T B_{k+1} s_k = s_k^T y_k$$

$$B_{k+1} \succ 0 \implies s_k^T y_k > 0$$

This need to be enforced if doing optimization on a non-convex function

General problem of find B

$$\min_B \|B - B_k\|$$

$$s.t. \ B = B^T, B s_k = y_k$$

$$\text{BFGS Update: } B_{k+1} = B_k + \frac{y_k y_k^T}{y_k^T s_k} - \frac{B_k s_k s_k^T B_k^T}{s_k^T B_k s_k}$$

Algorithm 4: BFGS Descent

```

1 init  $x_0 \in \text{dom} f$ 
2 init  $B_0$  (eg:  $I$ )
3 while  $\|y_k\| > \epsilon$ 
4   solve for  $p_k$  in  $B_k p_k = -\nabla f(x_k)$ 
5    $t \leftarrow \text{argmin}_s f(x_k + s p_k)$  (or backtrack search)
6    $s_k \leftarrow t p_k$ 
7    $x_{k+1} \leftarrow x_k + s_k$ 
8    $y_k = \nabla f(x_{k+1}) - \nabla f(x_k)$ 
9    $B_{k+1} = B_k + \frac{y_k y_k^T}{y_k^T s_k} - \frac{B_k s_k s_k^T B_k^T}{s_k^T B_k s_k}$ 

```

Inversion of B via Sherman-Morrison:

$$B_{k+1}^{-1} = \left(I - \frac{s_k y_k^T}{y_k^T s_k} \right) B_k^{-1} \left(I - \frac{y_k s_k^T}{y_k^T s_k} \right) + \frac{s_k s_k^T}{y_k^T s_k}.$$

Sherman-Morrison:

$$A^{-1} \text{ exists: } (A + uv^T)^{-1} \text{ exists} \iff 1 + v^T A^{-1} u \neq 0$$

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1} u v^T A^{-1}}{1 + v^T A^{-1} u}$$

0.11 Equality Constrained Optimization

Approaches:

- elimination of equality constraint, variable substitution into objective, unconstrained opt.
- Newton's Method with equality constraint, and assumed feasible start, unconstrained opt.
- Newton's Method with equality constraint, infeasible start, equivalence using primal-dual residual method

KKT optimality, with x assumed to be feasible:

$$\Delta x = \underset{v}{\operatorname{argmin}} f(x) + \nabla f(x)^T v + \frac{1}{2} v^T \nabla^2 f(x) v$$

$$\text{s.t. } A(x + v) = b$$

optimality condition :

$$L(v, w) = f(x) + \nabla f(x)^T v + \frac{1}{2} v^T \nabla^2 f(x) v + w^T (A(x + v) - b)$$

$$\frac{\partial L(v, w)}{\partial v} = \nabla f(x) + \nabla^2 f(x) v + A^T w = 0$$

$$A(x + v) = b$$

$$Ax = b \implies Av = 0$$

KKT system :

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = K \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} v \\ w \end{bmatrix} = K^{-1} \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix}, K^{-1} \text{ exists}$$

$$\Delta x = v$$

Alternatively Solve via elimination

$$\nabla f(x) + \nabla^2 f(x) v + A^T w = 0$$

$$v + \nabla^2 f(x)^{-1} A^T w + \nabla^2 f(x)^{-1} \nabla f(x) = 0$$

$$Av + A \nabla^2 f(x)^{-1} A^T w + A \nabla^2 f(x)^{-1} \nabla f(x) = 0$$

$$A \nabla^2 f(x)^{-1} A^T w + A \nabla^2 f(x)^{-1} \nabla f(x) = 0$$

$$w = -(A \nabla^2 f(x)^{-1} A^T)^{-1} A \nabla^2 f(x)^{-1} \nabla f(x)$$

$$v = \nabla^2 f(x)^{-1} (-\nabla f(x) - A^T w)$$

$$\Delta x = v$$

$\nabla^2 f(x)$ not invertible, augment KKT system s.t.

$$(\exists Q) \nabla^2 f(x) + AQA > 0$$

$$\begin{bmatrix} \nabla^2 f(x) + AQA & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix}$$

$$Av = 0$$

$$AQA v = 0 \implies \text{solution same to original problem}$$

Algorithm 5: Newton Method w/ Equality Constraint

```

1 init  $x_0 \in \operatorname{dom} f, Ax_0 = b$ ;
2 do
3    $\Delta x_{nt} \leftarrow$  Solve KKT System / Elimination;
4    $\lambda(x)^2 \leftarrow \Delta x_{nt}^T \nabla^2 f(x)^{-1} \Delta x_{nt}$ ;
5    $t \leftarrow$  Compute Step Size (eg: backtrack);
6    $x_{k+1} \leftarrow x_k + t \Delta x_{nt}$ 
7 while  $\frac{1}{2} \lambda(x)^2 > \epsilon$ ;
```

0.11.1 Infeasible Start Newton Method

Modify KKT system for residual: $Av = b - Ax$

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = K \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ b - Ax \end{bmatrix}$$

Once step length becomes 1, all following iterates are feasible.

Equivalence with Primal-Dual residual update:

$$\min_x f(x), \text{ s.t. } Ax = b$$

Dual:

$$L(x, v) = f(x) + v^T (Ax - b)$$

$$\text{s.t. } Ax - b = 0$$

optimality :

$$\frac{\partial L}{\partial x} = \nabla f(x) + A^T v = 0$$

residual of primal and dual:

$$r = \begin{bmatrix} r_{dual} \\ r_{pri} \end{bmatrix} = \begin{bmatrix} \nabla f(x) + A^T v \\ Ax - b \end{bmatrix}$$

$$y = \begin{bmatrix} x \\ v \end{bmatrix}$$

$$\Delta y = \begin{bmatrix} \Delta x \\ \Delta v \end{bmatrix}$$

$$r(y + \Delta y) \approx r(y) + Dr(y) \Delta y$$

$$Dr(y) := \text{Gradient of } r(y)$$

$$Dr(y) = \begin{bmatrix} \nabla_x r_{dual}^T & \nabla_v r_{dual}^T \\ \nabla_x r_{pri}^T & \nabla_v r_{pri}^T \end{bmatrix}$$

$$Dr(y) = \begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix}$$

goal: $r(y + \Delta y) \rightarrow 0$

$$r(y) + Dr(y)\Delta y = 0$$

$$\begin{bmatrix} \nabla f(x) + A^T v \\ Ax - b \end{bmatrix} + \begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta v \end{bmatrix} = 0$$

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta v \end{bmatrix} = \begin{bmatrix} -\nabla f(x) - A^T v \\ b - Ax \end{bmatrix}$$

comparison with definition of infeasible start Newton Method:

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ b - Ax \end{bmatrix}$$

Equivalence if for infeasible start, we select:

$$\begin{aligned} v &= \Delta x \\ A^T(\Delta v + v) &= A^T w = 0 \\ w &= v + \Delta v \end{aligned}$$

Solving using primal-dual formulation, obtain $\Delta v, \Delta x$
Backtrack line search using $\|r\|$ instead of f

Algorithm 6: Newton Method w/ Infeasible Start

```

1 init  $x_0 \in \text{dom} f$ 
2  $\beta \in (0, 1)$ 
3  $\alpha \in (0, 0.5)$ 
4 do
5    $\Delta v, \Delta x \leftarrow$  Solve primal-dual KKT system
6   Backtrack Step Size Search with  $\|r\|$ :
7    $t \leftarrow 1$ 
8   while
9      $\|r(x + t\Delta x, v + t\Delta v)\| > (1 - \alpha t)\|r(x, v)\|$ 
9      $t \leftarrow \beta t$ 
10   $x_{k+1} \leftarrow x_k + t\Delta x$ 
11   $v_{k+1} \leftarrow v_k + t\Delta v$ 
12 while  $\|r(x, v)\| > \epsilon \vee Ax \neq b$ ;
```

0.12 Inequality Constrained Optimization

$I(u)$ convex, non-decreasing, but non-differentiable.

Ideas:

- gradient projection: update x with descent direction, then project back onto feasibility set. Assume feasibility set is convex.

$x \leftarrow \text{descent update}$

$\tilde{s} = \underset{s}{\operatorname{argmin}} \|x - s\|, \text{ s.t. } s \in X = \text{feasibility set}$

let $[x]^+ = \underset{s}{\operatorname{argmin}} \|x - s\|, \text{ s.t. } s \in X$

$x_{k+1} \leftarrow [x_k + t\Delta x]^+, \text{ eg: } \Delta x = \nabla f(x)$

Issues: projection hard in general, line search becomes harder. Ok for simple projection (eg: box constraints).

- Adapt Newton's Method with projection.

$$\min_v \nabla f(x)^T v + \frac{1}{2} v^T \nabla^2 f(x) v$$

$$\text{s.t. } x_{k+1} = x_k + v \in X$$

Issues: Hard to solve.

- elimination inequality constraint and augment objective (eg: Interior Point)

Interior Point with Inequality & Equality Constraints

$$\min_x f_0(x)$$

$$\text{s.t. } f_i(x) \leq 0, \forall i \in \{1, \dots, m\}$$

$$Ax = b$$

Assumptions:

- solution exists
- strict feasibility (Slater's cond. hold), so strong duality
- objective and inequality functions differentiable and convex

Barrier Method:

$$\min_x f_0(x) + \sum_{i=1}^m I(f_i(x))$$

$$\text{s.t. } Ax = b$$

$$I(u) = \begin{cases} 0 & , u \leq 0 \\ +\infty & , o/w \end{cases}$$

Use log barrier function to approximate $I(u)$

$$\hat{I}(u) = -\frac{1}{t} \log(-u)$$

$$\hat{I}(u)|_{t \rightarrow +\infty} \rightarrow I(u)$$

Log barrier:

$$\Phi(x) = -\sum_i \log(-f_i(x))$$

$$\min_x f_0(x) - \frac{1}{t} \sum_{i=1}^m \log(-f_i(x))$$

$$\min_x f_0(x) + \frac{1}{t} \Phi(x)$$

$$\min_x t f_0(x) + \Phi(x)$$

$$\text{s.t. } Ax = b$$

Approach: fix t , optimize problem to obtain $x^*(t)$. Repeat for t increasing in value with previously solved intermediate solution so that $x^*(t)|_{t \rightarrow +\infty} = x^*$.

Find a starting strictly feasible point x . Set $t_0 > 0$. Inner optimization problem, solve $x^*(t)$ (via iterative algo such as Newton):

Algorithm 7: Log Barrier Method

```

1 do
2    $x^*(t) = \min_x t f_0(x) + \Phi(x), \text{ s.t. } Ax = b$ 
3   update:
4    $x = x^*(t)$ 
5    $t \leftarrow \mu t, \mu > 1$  (eg: 10)
6 while  $\frac{m}{t} < \epsilon$  not met;
```

Central Path:

Successive solving of inner optimization problem with varying t traces out a path, leading closer to the optimal final solution.

Central path lies in the interior of the feasibility region, thus potential optimal solution on the boundary of the feasibility set is never exactly reached.

Inner optimization usually uses 2nd order methods for auto-scaling of level set of optimization objective when t is large.

Search for Initial Feasible Point (Phase 1):

Formulate problem as:

$$\begin{aligned} \min_{x,s} \quad & s \\ \text{s.t.} \quad & f_i(x) \leq s, \forall i \\ & Ax = b \end{aligned}$$

If $s^* < 0$, x^* is strictly feasible. Else infeasibility.

Initialization of Phase 1 problem: set $s > \max_i f_i(x)$ so that starting point is interior of feasible set. Then proceed to solve using Interior Point method.

If $s^* < 0$, then original problem is solved using x^* as the strictly feasible starting point.

Interpretation of Interior Point Method with KKT:

Primal of Original:

$$\begin{aligned} \min_x \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0, \forall i \in \{1, \dots, m\} \\ & Ax = b \end{aligned}$$

Lagrangian of Primal:

$$f_0(x) + \sum_i \lambda_i f_i(x) + w^T(Ax - b)$$

KKT optimality conditions:

$$\begin{aligned} \nabla f_0(x) + \sum_i \lambda_i \nabla f_i(x) + A^T w &= 0 \\ \lambda_i &\geq 0 \\ Ax - b &= 0 \\ \lambda_i f_i(x) &= 0 \\ f_i(x) &\leq 0 \end{aligned}$$

Primal of Log-Barrier:

$$\begin{aligned} \min_x \quad & f_0(x) - \frac{1}{t} \sum_i \log(-f_i(x)) \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

KKT optimality conditions for Log-Barrier problem:

$$\begin{aligned} \nabla f_0(x) - \frac{1}{t} \sum_i \frac{\nabla f_i(x)}{f_i(x)} + A^T w &= 0 \\ Ax - b &= 0 \\ f_i(x) &\leq 0 \text{ (domain of log)} \\ \text{let } \lambda_i &= -\frac{1}{tf_i(x)} \\ \nabla f_0(x) + \sum_i \lambda_i \nabla f_i(x) + A^T w &= 0 \\ Ax - b &= 0 \end{aligned}$$

$$t > 0, f_i(x) \leq 0 \implies \lambda_i = -\frac{1}{tf_i(x)} \geq 0$$

Overall (Modified) KKT conditions for Log-Barrier:

$$\begin{aligned} \nabla f_0(x) + \sum_i \lambda_i \nabla f_i(x) + A^T w &= 0 \\ f_i(x) &\leq 0 \\ Ax - b &= 0 \\ \lambda_i &\geq 0 \\ \lambda_i f_i(x) &= -\frac{1}{t} \end{aligned}$$

This approaches true KKT conditions as $t \rightarrow +\infty$.

Obtaining a bound on duality gap of the modified problem from original:

Dual of original:

$$\begin{aligned} g(\lambda) &= \min_{Ax=b} f_0(x) + \sum_i \lambda_i f_i(x) \\ g(\lambda) &\leq f_0(x) + \sum_i \lambda_i f_i(x), Ax = b \\ g(\lambda) &\leq f_0(x^*) + \sum_i \lambda_i f_i(x^*), Ax^* = b \\ (\forall i) f_i(x^*) &\leq 0, \lambda_i \geq 0 \implies \\ g(\lambda) &\leq f_0(x^*) + \sum_i \lambda_i f_i(x^*) \leq f_0(x^*) \\ g(\lambda) &\leq f_0(x^*), \lambda \geq 0 \end{aligned}$$

Log-Barrier Problem:

$$\begin{aligned} \min_x f_0(x) - \frac{1}{t} \sum_i \log(-f_i(x)) \\ \text{s.t. } Ax = b \end{aligned}$$

KKT optimality conditions:

$$\begin{aligned} \nabla f_0(x) + \sum_i -\frac{1}{tf_i(x)} \nabla f_i(x) + A^T w &= 0 \\ f_i(x) &\leq 0 \\ Ax - b &= 0 \\ \text{from dual :} \\ g(\lambda) &\leq f_0(x) + \sum_i \lambda_i f_i(x), Ax = b \\ \lambda_i &= -\frac{1}{tf_i(x)}, f_i(x) \leq 0 \implies \text{same KKT conditions} \\ x^*(t) \text{ minimizes for particular } \lambda_i^*(t) &= -\frac{1}{tf_i(x)} \\ g(\lambda^*(t)) &= f_0(x^*(t)) + \sum_i \lambda_i^*(t) f_i(x^*(t)), Ax^*(t) = b \\ g(\lambda^*(t)) &= f_0(x^*(t)) + \sum_i -\frac{1}{tf_i(x^*(t))} f_i(x^*(t)) \\ g(\lambda^*(t)) &= f_0(x^*(t)) + \sum_{i=1}^m -\frac{1}{t}, Ax^*(t) = b \\ g(\lambda^*(t)) &= f_0(x^*(t)) - \frac{m}{t} \leq f_0(x^*) \\ f_0(x^*(t)) &\leq f_0(x^*) + \frac{m}{t} \\ f_0(x^*) &\leq f_0(x^*(t)) \implies \text{gives bound of gap : } \frac{m}{t} \end{aligned}$$

$x^*(t)$ gives solution that is bounded by $\frac{m}{t}$ away from optimal wrt. objective.

$\frac{m}{t}$ usable as stopping criterion.

Primal-Dual Interior Point Method (Alternative to Barrier Method)

Overall (Modified) KKT conditions for Log-Barrier:

$$\begin{aligned} \nabla f_0(x) + \sum_i \lambda_i \nabla f_i(x) + A^T v &= 0 \\ f_i(x) &\leq 0 \\ Ax - b &= 0 \\ \lambda_i &\geq 0 \\ -\lambda_i f_i(x) - \frac{1}{t} &= 0 \\ y &= \begin{bmatrix} x \\ \lambda \\ v \end{bmatrix} \\ r(y) &= \begin{bmatrix} \nabla f_0(x) + \sum_i \lambda_i \nabla f_i(x) + A^T v \\ -\lambda_1 f_1(x) - \frac{1}{t} = 0 \\ \vdots \\ -\lambda_m f_m(x) - \frac{1}{t} = 0 \\ Ax - b \end{bmatrix} \end{aligned}$$

Goal: drive $r(y)$ to 0

1st order approx of $r(y + \Delta y)$:

$$r(y + \Delta y) \approx r(y) + Dr(y)\Delta y = 0$$

$$Dr(y) = \begin{bmatrix} \nabla^2 f_0(x) + \sum_i \lambda_i \nabla^2 f_i(x) & f'_1(x) & \dots & f'_m(x) & A^T \\ -\lambda_1 f'_1(x) & -f_1(x) & 0 & \dots & \\ \vdots & \vdots & \vdots & \vdots & \\ -\lambda_m f'_m(x) & 0 & -f_m(x) & 0 & \\ A & 0 & \dots & \end{bmatrix}$$

$$Dr(y)\Delta y = -r(y)$$

Overall algorithm:

Algorithm 8: IP Primal-Dual Method

- 1 init strictly feasible
 $x, \lambda > 0, \eta = -\sum_i \lambda_i f_i(x), \mu = 10$
 - 2 $\eta = -\sum_i \lambda_i f_i(x)$
 - 3 **do**
 - 4 $t \leftarrow \frac{\mu m}{\eta}$
 - 5 Solve for Δy in $Dr(y)\Delta y = -r(y)$
 - 6 Line search for step size, s , using norm of residual $\|r\|$, s.t. $\lambda > 0, f(x) < 0$
 - 7 $y \leftarrow y + s\Delta y$
 - 8 $\eta = -\sum_i \lambda_i f_i(x)$ (proxy for duality gap, the smaller the better)
 - 9 **while** $\|r_{\text{prim}}\| > \epsilon \vee \|r_{\text{dual}}\| > \epsilon \vee \eta > \epsilon_2$;
-

Note: one loop only, but stopping in middle of algorithm does not guarantee a feasible solution.

0.13 Ellipsoid Method

f is convex, C^1 only

$$(\forall x, x_0) f(x) \geq f(x_0) + \nabla f(x_0)^T (x - x_0)$$

$$\nabla f(x_0)^T (x - x_0) \geq 0 \implies f(x) \geq f(x_0)$$

Idea: use of halfspace to perform elimination of search space

Selection of potential search points: use centroid of feasible polyhedron to maximize elimination space

Volumne reduction: $(X_{max} - X_{min})/2^k$

Algorithm 9: Halfspace Elimination

```

1 do
2   update select  $x^{(k)}$  using center of  $C^{(k)}$ 
   where  $x^* \in C^{(k-1)}$ 
3    $(\forall i) C^{(k-1)} = \{x : \nabla f(x^{(i)})^T (x - x^{(i)}) \leq 0\}$ 
4   bisection:
5    $f'(x^{(k)})(x - x^{(k)})$  eliminates 1 side of
   halfspace
6    $C^{(k)} = C^{(k-1)} \cap \{x : \nabla f(x^{(k)})^T (x - x^{(k)}) \leq 0\}$ 
7 while Not Satisfy stopping criterion;
```

Pros:

- centroid selection easy
- bisection evaluation easy
- exponential rate

Implementation issues: centroid search, intersection in high dimension

Simplify: ellipsoid instead of polyhedral for localiza-

Algorithm 10: Ellipsoid Method

```

1 do
2   set  $x^{(k+1)}$  as center of  $\varepsilon^{(k)}$ 
3   eval  $\nabla f(x^{(k+1)})$ 
4   find min. volume ellipsoid covering:
5    $S = \{x : \nabla f(x^{(k+1)})^T (x - x^{(k+1)}) \leq 0\}$ 
6    $\varepsilon^{(k+1)} = \varepsilon \cap S$ 
7 while Not Satisfy stopping criterion;
```

Intersection search:

$$g = \nabla f(x^{(k+1)})$$

$$\tilde{g} = \frac{g}{\sqrt{g^T P g}}$$

$$\varepsilon^{(k)} = \{x : (x - x^{(k+1)})^T P^{-1} (x - x^{(k+1)}) \leq 1\}$$

$$\varepsilon^{(k+1)} = \{x : (x - x^+)^T (P^+)^{-1} (x - x^+) \leq 1\}$$

$$x^+ = x^{(k-1)} - \frac{1}{n+1} P \tilde{g}, n = \dim \text{ of } X$$

$$P^+ = \frac{n^2}{n^2-1} (P - \frac{2}{n+1} P \tilde{g} \tilde{g}^T P)$$

$$\text{Vol}(\varepsilon^{(k+1)}) \leq e^{-\frac{1}{2n}} \text{Vol}(\varepsilon^{(k)})$$

0.14 Subgradient Method

f convex, but not C^1, C^2

Idea: need a g such that $(\forall x) f(x) \geq f(x_0) + g^T (x - x_0)$

$g :=$ a subgradient

Can use g in ellipsoid method

0.14.1 Application

Decomposition of LP into decoupled systems:

Original LP:

$$\begin{aligned}
 & \min_{u,v} c^T u + d^T v \\
 & s.t. Au \leq b \\
 & Pv \leq q \\
 & Fu + Gv \leq h
 \end{aligned}$$

Augment objective w/ constraint, partial Lagrangian

$$L(u, v, \lambda) = c^T u + d^T v + \lambda^T (Fu + Gv - h)$$

$$g(\lambda) = \min_{u,v} c^T u + d^T v + \lambda^T (Fu + Gv - h)$$

$$s.t. Au \leq b$$

$$Pv \leq q$$

For fixed λ , we can solve 2 separate LPs involving only u and v

$$\min_u c^T u + \lambda^T Fu$$

$$s.t. Au \leq b$$

$$\min_v d^T v + \lambda^T Gv$$

$$s.t. Pv \leq q$$

$\max_{\lambda} g(\lambda)$ s.t. $\lambda \geq 0$ gives optimal λ^*

$g(\lambda)$ concave, but not differentiable

Use subgradient, s , s.t. $g(\tilde{\lambda}) \leq g(\lambda) + s^T (\tilde{\lambda} - \lambda), \forall \tilde{\lambda}$

(u^*, v^*) for a fixed λ :

$S = Fu^* + Gv^* - h$ is a subgradient of g at λ

$$g(\tilde{\lambda}) = \min_{u,v} c^T u + d^T v + \tilde{\lambda}^T (Fu + Gv - h) \leq c^T u^* + d^T v^* + \tilde{\lambda}^T (Fu^* + Gv^* - h)$$

For $\forall u, v$ feasible:

$$\begin{aligned} g(\tilde{\lambda}) &= \min_{u,v} c^T u + d^T v + \tilde{\lambda}^T (Fu + Gv - h) \\ \text{s.t. } Au &\leq b \\ Pv &\leq q \end{aligned}$$

$$\begin{aligned} g(\tilde{\lambda}) &\leq c^T u^* + d^T v^* + \tilde{\lambda}^T (Fu^* + Gv^* - h) \\ &\quad + \lambda^T (Fu^* + Gv^* - h) \\ &\quad - \lambda^T (Fu^* + Gv^* - h) \\ g(\lambda) &= c^T u^* + d^T v^* + \lambda^T (Fu^* + Gv^* - h) \\ g(\tilde{\lambda}) &= g(\lambda) + (\tilde{\lambda} - \lambda)^T (Fu^* + Gv^* - h) \\ &= g(\lambda) + s^T (\tilde{\lambda} - \lambda) \end{aligned}$$

$s = Fu^* + Gv^* - h$ is subgradient of g at point λ
 $\max_{\lambda \geq 0} g(\lambda)$ for expression of subgradient

Update rule for λ :
 $\lambda^{(k+1)} = [\lambda^{(k)} + \alpha_k s^{(k)}]^+$

Interpretation of subgradient $s = Fu^* + Gv^* - h$

Original problem $Fu + Gv - h \leq 0$

λ , pricing variable associated w/ constraint

Penalize objective w/ $\lambda^T (Fu + Gv - h)$

$(\exists u^s, v^*) Fu^* + Gv^* \leq h \implies \lambda$ decreases
 $(\exists u^s, v^*) Fu^* + Gv^* \geq h \implies \lambda$ increases

Summary for decoupling LP:

$$\begin{aligned} \min_{u,v} c^T u + d^T v \\ \text{s.t. } Au &\leq b \\ Pv &\leq q \\ Fu + Gv &\leq h \end{aligned}$$

Subproblems:

$$\begin{aligned} \min_u c^T u + \lambda^T Fu \\ \text{s.t. } Au &\leq b \end{aligned}$$

$$\begin{aligned} \min_v d^T v + \lambda^T Gv \\ \text{s.t. } Pv &\leq q \end{aligned}$$

Algorithms:

Algorithm 11: Decoupling LP with Subgradient

```

1 set  $\lambda^{(0)} > 0$ 
2 do
3   solve 2 LP problems for fixed
      $\lambda^{(k)} \rightarrow (u^*, v^*)$ 
4   calculate  $s^{(k)} = Fu^* + Gv^* - h$ 
5   update  $\lambda^{(k+1)} = [\lambda^{(k)} + \alpha_k s^{(k)}]^+$ 
6 while not meeting  $\|f(u, v) - g(\lambda)\| < \epsilon$ ;
```

Choosing α_k , derivation ultimately comes to:

$\sum_{k=1}^{\infty} \alpha_k^2 < \infty, \sum_{k=1}^{\infty} \alpha_k \rightarrow \infty \implies$
 $\lambda^{(k)} \rightarrow \lambda^*, k \rightarrow \infty$
 eg: $\alpha_k = \frac{1}{k}$

Issues with subgradient method:

- can be slow
- step size selection difficult

0.15 Practical Optimization Strategies

Ellipsoid Method:

- slower
- easier to code

Interior Point Method:

- faster
- harder to code

Tradeoff:

program vs. run time leverage existing software: use problem conversion (eg: LP to SDP)

LP \iff SDP:

$$\begin{aligned} \min_x c^T x \\ s.t. Ax \leq b \end{aligned}$$

$$\begin{aligned} \min_x c^T x \\ s.t. \text{diag}(Ax - b) \preceq 0 \iff \\ \sum_i x_i F_i \preceq 0 \end{aligned}$$

SOC \iff SDP:

$$\begin{aligned} \min_x f^T x \\ s.t. \|Ax + b\|_2 \leq c^T x + d \end{aligned}$$

$$\begin{aligned} \min_x f^T x \\ s.t. \begin{bmatrix} (c^T x + d)I & Ax + b \\ (Ax + b)^T & c^T x + d \end{bmatrix} \succeq 0 \end{aligned}$$

use Schur Complement:

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

$$X \succeq 0 \iff A \succeq 0 \wedge C - B^T A^{-1} B \succeq 0$$

$$X \succeq 0 \iff C \succeq 0 \wedge A - B C^{-1} B^T \succeq 0$$

$$\begin{aligned} \text{let } X = \begin{bmatrix} (c^T x + d)I & Ax + b \\ (Ax + b)^T & c^T x + d \end{bmatrix} \succeq 0 \iff \\ \begin{cases} c^T x + d \geq 0 \\ c^T x + d - \frac{(Ax + b)^T I (Ax + b)}{c^T x + d} \geq 0 \end{cases} \\ (c^T x + d)^2 \geq \|Ax + b\|_2^2 \\ c^T x + d \geq \|Ax + b\|_2 \end{aligned}$$

0.15.1 Coordinate Descent

Unconstrained minimization problem

$$\begin{aligned} \min_{x_1, x_2} f(x_1, x_2) \\ \text{assume } x_1^*, x_2^* \text{ exist} \end{aligned}$$

Algorithm 12: Coordinate Descent

```

1 do
2   | fix  $x_1$ , min. over  $x_2$ 
3   | fix  $x_2$ , min. over  $x_1$ 
4 while not until convergence;
```

Convergence: yes

Sequence of objective value is non-increasing

Optimal value exists and bounded below

CD algo. guaranteed to converge to a (local) minima whether or not objective function is convex

global optimal: $\begin{cases} \text{No,} & \text{nonconvex obj} \\ \text{Yes,} & \text{convex, differentiable obj} \end{cases}$

Assume CD converges to (x_1^*, x_2^*) , ∇f exists.

Piecing together:

$$\begin{aligned} \nabla_{x_1} f|_{x_1^*, x_2^*} &= 0 \\ \nabla_{x_2} f|_{x_1^*, x_2^*} &= 0 \end{aligned}$$

$$\nabla f = \begin{bmatrix} \nabla_{x_1} f \\ \nabla_{x_2} f \end{bmatrix} \Big|_{x_1^*, x_2^*} = 0 \implies \text{optimal}$$

Non-differentiable objective function, iterate may get stuck.

Constrained problem

CD works if constraints are separable

eg:

$$\begin{aligned} \min_{x_1, x_2} f(x_1, x_2) \\ s.t. \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} b \\ g \end{bmatrix} \end{aligned}$$

x_1, x_2 decoupled, separable

Solve separate problems, concatenate results

At optimality:

$$\begin{aligned}\nabla_{x_1} f(x_1^*, x_2^*) + A^T \lambda_1 &= 0 \\ \nabla_{x_2} f(x_1^*, x_2^*) + F^T \lambda_2 &= 0 \\ \implies \\ \nabla_x f(x_1^*, x_2^*) + \begin{bmatrix} A^T & 0 \\ 0 & F^T \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \\ &= \nabla_x f(x_1^*, x_2^*) + \tilde{A}^T \lambda = 0\end{aligned}$$

CD works for separable constraints due to:

- convergence guarantee
- at convergence, KKT condition of orig. problem satisfied

0.15.2 Sequential Quadratic Programming (SQP)

$\min_x f_0(x)$, f_0 nonconvex, differentiable

Idea: per iteration, approximate $f_0(x)$ by a convex function (eg: quadratic)

Solves for a local optimum

At point x :

$$f_0(x+v) \approx f_0(x) + \nabla f_0(x)^T v + \frac{1}{2} v^T \nabla^2 f_0(x) v$$

$$\min_v \nabla f_0(x)^T v + \frac{1}{2} v^T \nabla^2 f_0(x) v$$

$$\Delta x = v^* \implies x = x + \alpha \Delta x, \alpha \text{ via line search}$$

Constrained problem:

Idea: form Lagrangian $L(x, \lambda, v) = f_0(x) + \sum_i \lambda_i f_i(x) + \sum_i v_i h_i(x)$

$$\min_x L(x, \lambda, v)$$

$$s.t. f_i(x) \leq 0$$

$$h_i(x) = 0$$

At λ, v optimal dual variable

$$\min_v (\nabla_x L)^T v + \frac{1}{2} v^T (\nabla_x^2 L) v$$

$$s.t. \nabla f_i(x)^T v + f_i(x) \leq 0$$

$$\nabla h_i(x)^T v + h_i(x) \leq 0$$

Updating λ, v ?

Select $\Delta \lambda, \Delta v$ in direction of subgradient

For fixed λ, v : $x^{(k+1)} \leftarrow x + \alpha \Delta x, \Delta x = v$

Algo:

Algorithm 13: Sequential Quadratic Programming

```

1 do
2   solve QP to obtain  $(\Delta x, \Delta \lambda, \Delta v)$ 
3   line search
    $(x, \lambda, v) = (x, \lambda, v) + \alpha(\Delta x, \Delta \lambda, \Delta v)$ 
4   reapproximate by QP
5 while not until convergence;
```

0.16 Appendix

0.16.1 Gradient of Log Det

$$f(x) = \log(\det X)$$

$$\begin{aligned} f(X + \delta X) &= \log \det(X + \delta X) \\ &= \log \det((X^{\frac{1}{2}}(I + X^{-\frac{1}{2}}\delta X X^{-\frac{1}{2}})X^{\frac{1}{2}}) \\ &= \log \det(X) + \log \det(I + X^{-\frac{1}{2}}\delta X X^{-\frac{1}{2}}) \\ \text{let } M &= X^{-\frac{1}{2}}\delta X X^{-\frac{1}{2}} \\ &= \log \det(X) + \log \det(I + M) \end{aligned}$$

claim eigenvalues of $I + M$: $1 + \lambda_i$

$$\begin{aligned} Mv_i &= \lambda_i v_i \\ (I + M)v_i &= (1 + \lambda_i)v_i \\ \det(M) &= \prod_i (1 + \lambda_i) \end{aligned}$$

$$\begin{aligned} f(X + \delta X) &= \log \det(X) + \log \prod_i (1 + \lambda_i) \\ &= \log \det(X) + \sum_i \log(1 + \lambda_i) \\ &\approx \log \det(X) + \sum_i \lambda_i \text{ since } \delta X \text{ is small} \\ &\approx \log \det(X) + \text{trace}(X^{-\frac{1}{2}}\delta X X^{\frac{1}{2}}) \\ &\approx \log \det(X) + \text{trace}(X^{-1}\delta X) \end{aligned}$$

$$\text{trace}(X^{-1}\delta X) = (X^{-T})^T \delta X$$

$$f(X + \delta X) = f(X) + (\nabla f(X))^T \delta X \implies \nabla f(X) = X^{-1}$$

$$\log \det(X) = \log(X) \implies f'(X) = \frac{1}{X}$$

0.16.2 2nd order approximation of Log Det

$$f(X + \delta X) = f(X) + \langle \nabla f(X), \delta X \rangle + \frac{1}{2} \langle \delta X, \nabla^2 f(X) \delta X \rangle$$

first look at first order approximation: $g(X) = X^{-1}$

$$\begin{aligned} g(X + \delta X) &= (X + \delta X)^{-1} = (X^{\frac{1}{2}}(I + X^{-\frac{1}{2}}\delta X X^{-\frac{1}{2}})X^{\frac{1}{2}})^{-1} \\ &= X^{-\frac{1}{2}}(I + X^{-\frac{1}{2}}\delta X X^{-\frac{1}{2}})^{-1}X^{-\frac{1}{2}} \\ \text{for small } A (\text{small eigenvalues}): (I + A)^{-1} &\approx I - A \\ &= X^{-\frac{1}{2}}(I - X^{-\frac{1}{2}}\delta X X^{-\frac{1}{2}})X^{-\frac{1}{2}} \\ &= X^{-1} - X^{-1}\delta X X^{-1} \end{aligned}$$

$$\log \det(X + \delta X) = \log \det(X) + \text{tr}(X^{-1}\delta X) - \frac{1}{2} \text{tr}(\delta X X^{-1}\delta X X^{-1})$$

0.17 Miscellaneous Properties

$$(a+x)^{-1} \approx 1-x$$

$$\lim_{t \rightarrow 0} \frac{f(x+\epsilon t) - f(x)}{t} = \frac{\partial f(x)}{\partial x} \epsilon$$

0.17.1 Pseudo-inverse

Overconstrained case:

Cast as L2 norm approximation problem

$$\min_x \|Ax - b\|_2^2$$

$$(Ax - b)^T (Ax - b) = x^T A^T Ax - 2x^T A^T b + b^T b$$

$$\frac{\partial}{\partial x} (x^T A^T Ax - 2x^T A^T b + b^T b) = 2A^T Ax - 2A^T b$$

$$x = (A^T A)^{-1} A^T b$$

Underconstrained case:

Cast as a least-norm problem w/ equality constraint

$$\min_x \|x\|_2^2$$

$$s.t. : Ax = b$$

$$\min_x L(x, \lambda, v) = x^T x - v^T (Ax - b)$$

$$\frac{\partial (x^T x - v^T (Ax - b))}{\partial x} = 2x - A^T v = 0$$

$$x = \frac{1}{2} A^T v$$

$$g(\lambda, v) = [x^T x - v^T (Ax - b)]_{x = \frac{1}{2} A^T v}$$

$$g(\lambda, v) = -\frac{1}{4} v^T A A^T v + v^T b$$

$$dual : \max_{\lambda, v} g(\lambda, v) = -\min_{\lambda, v} g(\lambda, v)$$

$$\frac{\partial (\frac{1}{4} v^T A A^T v + v^T b)}{\partial v} = 0$$

$$v = \frac{1}{2} A A^T v + b = 0$$

$$v = -2(AA)^{-1}b$$

$$x = -\frac{1}{2} A^T v|_{v = -2(AA)^{-1}b} = A^T (AA^T)^{-1} b$$

0.18 Problems

0.18.1 min. vol. covering ball

Find minimal volume norm ball B covering $B_j, \forall j$, where B_j is a norm ball. Consider 2-norm:

Let c_j, r_j be center, radius of norm ball B_j
Let c, r be center, radius of B

$$\min_{c, r} r$$

$$s.t. \|c - c_j + r_j\|_2 \leq r, \forall j$$

$$c^c + (-c_j + r_j)^T(-c_j + r_j) + 2c^T(-c_j + r_j) \leq r, \forall j$$

$$c^c + (c_j + r_j)^T(c_j + r_j) + 2c^T(c_j + r_j) \leq r, \forall j$$

QCQP problem

Find minimal volume norm ball B covering $B_j, \forall j$, where B_j is a norm ball. Consider ∞ -norm:

$$\min_{c, r} r$$

$$s.t. \|c - c_j + r_j\|_\infty \leq r, \forall j$$

$$c - c_j + r_j \leq r, \forall j$$

$$-c + c_j + r_j \leq r, \forall j$$

LP problem

0.18.2 polyhedron intersection

$$D = \{x \in \mathbb{R}^n : Cx \leq d\}$$

$$G = \{x \in \mathbb{R}^n : Cx \leq d\}$$

Solve $D \cap G = \emptyset$? using LP:

$$\min_{a, b, x} 0$$

$$s.t. a^T x = b$$

$$Cx - d \not\leq 0$$

$$Hx - g \not\leq 0$$

$$\min_{a, b, x, e} \sum_i e_i$$

$$s.t. a^T x = b$$

$$Cx - d + e \leq 0$$

$$Hx - g + e \leq 0$$

$$e \leq 0$$

f^* feasible and $f^* < 0 \implies D \cap G = \emptyset$

otherwise, $\implies D \cap G \neq \emptyset$

Solve $D \subseteq G$? using LP:

Algorithm 14: Descent Overview

1 *verts* of $D \leftarrow$ solve intersection of halfspace equations of D

2 **for** $\forall \text{vert} \in D$ **do**

3 $\min 0$

4 s.t. $Hv \leq g$

5 $f^* = \begin{cases} NaN \implies \text{return } D \not\subseteq G \\ o/w \implies \text{continue} \end{cases}$

6 **return** $D \subseteq G$
