1. Convex Functions
   Definition: convex function:

   $$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y), \forall\theta = [0,1]$$

   strict version:

   $$f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta)f(y), \forall\theta = (0,1)$$

   chord lies above the function
   Definition concave function:

   $$f(\theta x + (1-\theta)y) \geq \theta f(x) + (1-\theta)f(y)$$

   chord lies below the function
   For convenience we sometimes define an extended value function:

   $$\tilde{f}(x) = \begin{cases} f(x), & x \in dom(f) \\ \infty, & other\ wise \end{cases}$$

   if $f(x)$ convex, then $\tilde{f}$ is also convex

2. examples:
   linear and affine functions are both convex and concave
   $\frac{1}{x}$ neither convex nor concave on domain of $\mathbb{R}$
   $\|*\|$ on $\mathbb{R}^n$ is convex

3. sublevel set of a function

   $$C(\alpha) = \{x \in dom(f) : f(x) \leq \alpha\}$$

   For convex function, all sublevel sets are convex ($\forall\alpha$). Converse is not true.

4. quasi-convex function: if its sublevel sets are all convex

5. epigraph of functions: $epi(f) = \{(x,t) : x \in dom(f), f(x) \leq t\} \in \mathbb{R}^{n+1}, f \in \mathbb{R}^n \to \mathbb{R}$
   the set of space above the function
   f is convex function $\iff$ $epi(f)$ is convex set

6. differentiable convex function
   First order condition: suppose f is differentiable and domain of f is convex. Then:
   f is convex $\iff (\forall x, x_0 \in dom(f)) f(x) \geq f(x_0) + \nabla f(x_0)^T(x - x_0)$
   $f(x) + \nabla f(x_0)^T(x - x_0)$ is the 1st order approximation
   the local information $(f(x), \nabla f(x_0))$ gives global information about the function (global underestimator)

   rough proof:
   suppose $f(x)$ is convex but $(\exists x, x_0) f(x) < f(x) + \nabla f(x_0)^T(x - x_0)$
   then this means the function should bend across the tangent line which violates the convexity

   proof for converse direction:
   suppose that $(\exists x, x_0) f(x) \geq f(x) + \nabla f(x_0)^T(x - x_0)$
   to show that $f(x)$ is convex lets take $x, y \in dom(f), z = \theta x + (1 - \theta)y$
   for x, z: $f(x) + \nabla f(z)^T(x - z)$
   for y, z: $f(y) + \nabla f(z)^T(y - z)$
   $\theta f(x) + (1 - \theta)f(y) \geq f(z) + \nabla f(z)^T(\theta x - \theta z + (1 - \theta)y - (1 - \theta)z)$
   $\theta f(x) + (1 - \theta)f(y) \geq f(\theta x + (1 - \theta)y)$
   $f(x)$ is convex

7. second order condition: suppose f is twice differentiable and $dom(f)$ is convex, then $f(x)$ is convex $\iff \nabla^2 f(x) \geq 0$ (PSD, eg: wrt. S_+^n)

   proof for scalar case:
   suppose that $f(x)$ is convex, then the first-order condition holds
   for $x, y \in dom(f) : f(x) \geq f(y) + f'(y)(x - y)$
   for $y, x \in dom(f) : f(y) \geq f(x) + f'(x)(y - x)$
   $f'(x)(y - x) \leq f(y) - f(x) \leq f'(y)(y - x)$
   $f'(x)(y - x) \leq f'(y)(y - x) \implies 0 \leq (y - x)(f'(x) - f'(y))$
   take $y \to x : 0 \leq f''(x)$
   $f''(x) \geq \frac{f'(x+\delta x) - f'(x)}{\delta x}$

   conversely, suppose that $f'(z) \geq 0, \forall z \in dom(f)$, take $x, y \in dom(f)$ WLOG $x < y$
   $\int_x^y f''(z)(y - z)dz \geq 0$
   $f''(z) \geq 0, (y - z) \geq 0 = I_1 + I_2$
   $I_1 = \int_x^y f''(z)ydz - yf'(z)|_x^y = y(f'(y) - f'(x))$
   $I_2 = -\int_x^y f''(z)dz$
   $dv = f''(z)dz \implies v = f''(z)$
   $u = z \implies du = dz$
   $I_2 = -zf'(z)|_x^y + \int_x^y f(z)dz = -yf'(y) + xf'(x) + f(y) - f(x)$
   $I_1 + I_2 = yf'(y) - yf'(x) - yf'(y) + xf'(x) + f(y) - f(x) \geq 0$
   $\implies f(y) \geq f(x) + f'(x)(y - x)$ first order condition: $x < y$

first order condition holds $\implies f(x)$ convex

8. basic properties of convex functions:
   (a) $f(x)$ is convex $\implies (\exists\alpha)\alpha f(x)$ is convex
   (b) $f_1, ..f_n$ convex $\implies f_1 + ...f_n$ is convex
   (c) if $f(x)$ is convex $\implies \forall g(x)$ affine, $f(g(x))$ is convex
   eg: $f(x) = \|Ax + b\|$ is convex. $f(y) = \|y\|$ is convex
   (d) $f_1, ..f_n$ convex $\implies max\{f_1 + ...f_n\}$ is convex