0.1 Problem Types

LP

standard, inequality, general forms

$$\min_x c^T x \ s.t.: \quad Ax = b \quad x \succeq 0$$

$$\min_x c^T x \ s.t.: \quad Ax \preceq b$$

$$\min_x c^T x + d \ s.t.: \quad Gx \preceq h \quad Ax = b$$

QP

$$\min_x \frac{1}{2}x^T P x + q^T x + r \ s.t.: \quad Gx \leq h \quad Ax = b$$

QCQP

$$\min_x \frac{1}{2}x^T P_0 x + q_0^T x + r_0, s.t.: \quad \frac{1}{2}x^T P_i x + q_i^T x + r_i \leq 0, \forall i \quad Ax = b$$

SOCP

$$\min_x f^T x \ s.t.: \quad A_i x + b_{i2} \leq c_i^T x + d_i, \forall i \quad Fx = g$$

$$(\forall i) b_i = 0 \implies LP$$
$$(\forall i) c_i = 0 \implies QCQP$$

GP

$$\min_x f_0(x) \ s.t.: \quad f_i(x) \leq 1, \forall i \quad h_i(x) = 1, \forall i$$
$$f_i \ is \ a \ posynomial := \sum_i h_i$$
$$h_i \ is \ a \ monomial := c x_1^{a_1} x_2^{a_2} .., c > 0, a_i \in \mathbb{R}$$

Use transform of objective and constraint functions:
$y_i = log x_i, x_i = e^{y_i}$
$\tilde{h}_i$ becomes exponential of affine function
$\tilde{f}_i = log(f_i)$ becomes log sum exp (convex)
If all constraints and objective are monomials, reduces to LP after transform.

SDP

general, standard, inequality forms

$$\min_x \ c^T x \ s.t.: \quad LMI: \sum_i^n x_i F_i + G \preceq_K 0 \quad Ax = b \quad x \in \mathbb{R}^n \quad F_i, G \in S^m, K \in S_+^m$$

$$\min_X \ tr(CX) s.t.: \quad tr(A_i X) = b_i, \forall i \quad X \succeq 0$$

$$\min_x \ c^T x \ s.t.: \quad \sum_i^n x_i A_i \preceq_K B \quad Ax = b \quad B, A_i \in S^m, K \in S_+^m$$

concatenating constraints:

$$F^(i)(x) = \sum_j x_j F_i^{(i)} + G^{(i)} \preceq 0$$
$$Gx \preceq h$$
$$\implies$$
$$diag(Gx - h, F^{(1)}(x), .., F^{(m)}(x)) \preceq 0$$

if all matrices are diagonal, reduces to LP

0.2 Convexity/Concavity

0.2.1 log det X, concave

$$let\ X = Z + tV \succ 0$$
$$f = logdet(Z + tV)$$
$$f = logdet(Z^{-0.5}(I + tZ^{-0.5}VZ^{0.5})Z^{0.5})$$
$$f = log(det(Z^{-0.5})det(I + tZ^{-0.5}VZ^{0.5})det(Z^{0.5}))$$
$$f = log(det(Z^{0})det(I + tZ^{-0.5}VZ^{0.5}))$$
$$f = logdet(I + tZ^{-0.5}VZ^{0.5})$$
$$f = log\Pi_i(1 + \lambda_i t)$$
$$f = \sum_i log(1 + \lambda_i t)$$
$$\frac{\partial f}{\partial t} = \sum_i \frac{\lambda_i}{1 + \lambda_i t}$$
$$\frac{\partial^2 f}{\partial t^2} = \sum_i \frac{-\lambda_i^2}{(1 + \lambda_i t)^2} = -\sum_i \frac{\lambda_i^2}{(1 + \lambda_i t)^2} \leq 0$$
$$\nabla^2 f \leq 0 \iff f\ concave$$

0.2.2 log $\sum_i exp(x_i)$, convex

$$\nabla^2 f = \frac{1}{(1^T z)^2}(1^T z diag(z) - zz^T)$$
$$v^T zz^T v = det(v^T zz^T v) = det(vv^T zz^T)$$
$$v^T zz^T v = \sum_j \sum_i z_j z_i v_j v_i$$
$$v^T zz^T v = (\sum_j z_j z_j)(\sum_i z_i v_i)$$
$$v^T zz^T v = (\sum_i z_i v_i)^2$$
$$use\ Holder's\ Inequality:$$
$$\|a\|_2^2 \|b\|_2^2 \geq |a^T b|^2$$
$$let\ a = z_i^{0.5}, b = v_i z_i^{0.5}$$
$$1^T z(\sum_i v_i^2 z_i) - (\sum_i z_i v_i)^2 \geq 0$$
$$v^T \nabla^2 f v = \frac{1}{(1^T z)^2}\left(1^T z(\sum_i v_i^2 z_i) - (\sum_i z_i v_i)^2\right) \geq 0$$
$$\nabla^2 f \geq 0 \iff f\ convex$$

0.2.3 geometric mean on R^n_{++}, concave

$$f = (\Pi_i x_i)^{\frac{1}{n}}$$
$$\frac{\partial}{\partial x_i} f = \frac{1}{n}(\Pi_i x_i)^{\frac{1}{n}-1}\Pi_{j\neq i} x_j$$
$$\frac{\partial^2}{\partial x_i^2} f = \frac{1}{n}(\frac{1}{n} - 1)(\Pi_i x_i)^{\frac{1}{n}-2}(\Pi_{j\neq i} x_j)^2$$
$$\frac{\partial^2}{\partial x_i^2} f = \frac{1}{n}(\frac{1}{n} - 1)\frac{(\Pi_i x_i)^{\frac{1}{n}}}{x_i^2}$$
$$\frac{\partial^2}{\partial x_i x_k} f = \frac{1}{n^2}\frac{(\Pi_i x_i)^{\frac{1}{n}}}{x_i x_k}, i \neq k$$
$$\frac{\partial^2}{\partial x_i x_k} f = \frac{1}{n^2}\frac{(\Pi_i x_i)^{\frac{1}{n}}}{x_i x_k} - \delta_{ik}\frac{1}{n}\frac{(\Pi_i x_i)^{\frac{1}{n}}}{x_i^2}$$
$$v^T \nabla^2 f v = \frac{-(\Pi_i x_i)^{\frac{1}{n}}}{n^2}(n\sum_i \frac{v_i^2}{x_i^2} - (\sum_i \frac{v_i}{x_i})^2)$$
$$apply\ Cauchy\ Schwartz\ Inequality:$$
$$let\ a = \mathbf{1}, b_i = \frac{v_i}{x_i}$$
$$\|\mathbf{1}\|_2^2(\sum_i \frac{v_i^2}{x_i}) \geq (\sum_i \frac{v_i}{x_i})^2$$
$$n\sum_i \frac{v_i^2}{x_i^2} - (\sum_i \frac{v_i}{x_i})^2 \geq 0$$
$$v^T \nabla^2 f v \leq 0 \iff f\ concave$$

0.2.4 quadratic over linear, convex

$$f(x, y) = \frac{h(x)}{g(y)}, g(y)\ linear, g(y) \in R_+$$
$$\nabla^2 f = vv^T\ is\ PSD \iff f\ convex$$

todo..

0.2.5 affine mapping

0.2.6 pointwise supremum over convex functions

0.2.7 norm

0.2.8 non-negative weighted sum and expectation of convex/concave functions

0.3 Definitions

0.3.1 dual norm

0.3.2 operator norm

0.3.3 dual cone

$$\begin{aligned} &K \textit{ is a cone} \\ K^* &= \{y : x^T y \geq 0, \forall x \in K\} \end{aligned}$$

0.3.4 support function of a set

0.4 Composition of functions

$$\begin{aligned} f &= h(g(x)) \\ f' &= g'(x)h'(g(x)) \\ f'' &= g''(x)h'(g(x)) + (g'(x))^2 h''(g(x)) \end{aligned}$$

$$\begin{aligned} \text{h convex \& non-decreasing, g convex} &\implies \text{f convex} \\ h'' \geq 0, g''(x) \geq 0, h'(g(x)) \geq 0 &\implies f'' \geq 0 \\ \text{h convex \& non-increasing, g concave} &\implies \text{f convex} \\ h'' \geq 0, g''(x) \leq 0, h'(g(x)) \leq 0 &\implies f'' \geq 0 \\ \text{h concave \& non-decreasing, g concave} &\implies \text{f concave} \\ h'' \leq 0, g''(x) \leq 0, h'(g(x)) \geq 0 &\implies f'' \leq 0 \\ \text{h concave \& non-increasing, g convex} &\implies \text{f concave} \\ h'' \leq 0, g''(x) \geq 0, h'(g(x)) \leq 0 &\implies f'' \leq 0 \end{aligned}$$

0.5 Inequalities

0.5.1 geometric mean inequality

0.6 Identities