1. Polyhedral

$$P = \{x : Ax \leq b, Cx = D\}$$

2. Euclidean Ball

$$B(x_c, x) = \{x : \|x - x_c\|_2 \leq r\}$$

   Can use affine combination and triangular inequality of norm to prove convexity.

3. Elipse

$$\begin{aligned}
E(x_c, P) = \{x : (x - x_c)^T P^{-1}(x - x_c) \leq 1\}, P > 0 \\
P = r^2 I \implies \text{ Euclidean Ball} \\
P = Q \begin{bmatrix} \lambda_1 & .. \\ .. & \lambda_n \end{bmatrix} Q^T \\
(x - x_c)^T Q \begin{bmatrix} \lambda_1 & .. \\ .. & \lambda_n \end{bmatrix} Q^T (x - x_c) \leq 1 \\
\tilde{x}^T \begin{bmatrix} \lambda_1 & .. \\ .. & \lambda_n \end{bmatrix} \tilde{x}^T \leq 1 \\
\tilde{x}^T \begin{bmatrix} \frac{1}{\lambda_1} & .. \\ .. & \frac{1}{\lambda_n} \end{bmatrix} \tilde{x} = \frac{\tilde{x_1}^2}{\lambda_1} + .. + \frac{\tilde{x_n}^2}{\lambda_n} \leq 1 \\
\text{volum of elipsoid proportional to } \sqrt{det(P)} = \sqrt{\Pi_i \lambda_i}
\end{aligned}$$

4. Norm Ball

$$C = \{x : \|x\| \leq 1\} \text{ for any } \|*\|$$

   Example: lp-norm

5. Cone

$$(\forall x \in C, \forall \theta \geq 0) \theta x \in C$$

6. Convex Cone Eg: S^n, S_+^N are convex cones
   convexity check for S_+^N:

$$\begin{aligned}
x_1 \in S_+^n \implies v^T x_1 v \geq 0 \\
x_2 \in S_+^n \implies v^T x_2 v \geq 0 \\
v^T(\theta x_1 + (1 - \theta)x_2)v \geq 0 \\
v^T \theta x_1 v + (1 - \theta)v^T x_2 v \implies x \in S_+^n
\end{aligned}$$

convexity check for cone:

$$x_1 \in S_+^n \implies \theta x_1 \in S_+^n, \theta \geq 0$$
$$(\forall v) v^T x v \geq 0 \implies v^T(\theta x)v \geq 0 \implies \text{ cone}$$

7. Proper Cone
   Definition:
   - convex
   - closed(contains all boundary points)
   - solid(non-empty interior)
   - pointed(contains no line): $x \in K \implies -x \in K$

   Then the proper cone K fefines a generalized inequality ($\leq_K$) in $\mathbb{R}^n$

   $$x \leq_K y \implies y - x \in K$$
   $$x <_K y \implies y - x \in int(K)$$

   Example: $K = R_+^n$ (non-negative orthant):

   $$n = 2$$
   $$x \leq_{R_+^2} y \implies y - x \in R_+^2$$

   Cone provides partial ordering using difference of 2 objects Example: $X\ leq_{S_+^n} Y \implies Y - X \in S_+^n$ (Y-X is PSD)

8. Operations Preserving Convexity

   - intersection

$$S\alpha \text{ is affine, convex, convex cone} \forall \alpha \in A$$
$$\cap_{\alpha \in A} S_\alpha \text{ is affine, convex, convex cone}$$

     Example:Polyhedral is the intersection of some halfspaces and hyperplanes, so it is convex.
     Any cloed convex set can be represented by possibly infinitely many half spaces.

   - affine functions
     let $f(x) = Ax + b, f : \mathbb{R}^n \to \mathbb{R}^m$
     then if S is a convex set we have:

     – project forward: $f(S) = \{f(X) : X \in S\}$ is convex

     – project back: $f^{-1}(S) = \{X : f(X) \in S\}$ is convex

     Example:

$$C = \{y : y = Ax + b, \|x\| \leq 1\}$$

     $\|x\| \leq 1$ is convex, $Ax + b$ is affine $\implies$ C is convex
     Example:

$$C = \{x : \|Ax + b\| \leq 1\}$$

     $\{y : \|y\| \leq 1\}$
     y is an affine function of $x \implies$ C is convex

9. Affine Functions $A(x) = \sum_i x_i A_i + B, A_i \in S^n, B \in S^n, x \in \mathbb{R}^n$
   is $\{X : \sum_i x_i A_i = \tilde{A}(x) \leq B\}$ convex?
   let $y = B - \tilde{A}(x)$
   $\{y : 0 \leq y\}$ is convex
   We know $\{y : 0 \leq y\}$ is convex. Further y is an affine function of x $\implies$
   $\{X : \sum_i x_i A_i \leq B\}$ is also convex
   $\{X : \sum_i x_i A_i \leq B\}$ is a Linear Matrix Inequality

10. Properties of convex sets

    - Separating Hyperplanes: if $S, T \subset \mathbb{R}^n$ are convex and disjoint, then $\exists a \neq 0, b$ such that:

$$a^T x \geq b, forall x \in S$$
$$a^T x \leq b, forall x \in T$$

11. Supporting Hyperplane:
    if S is convex, $\forall x_0 \in \partial S$ (boundary of S), then $\exists a \neq 0$ such that $a^T x \leq a^T x_0, \in S$