1. Polyhedral

$$P = \{x : Ax \le b, Cx = D\}$$

2. Euclidean Ball

$$B(x_c, x) = \{x : \|x - x_c\|_2 \le r\}$$

Can use affine combination and triangular inequality of norm to prove convexity.

3. Elipse

$$E(x_c, P) = \{x : (x - x_c)^T P^{-1}(x - x_c) \le 1\}, P > 0$$
$$P = r^2 I \implies \text{Euclidean Ball}$$
$$P = Q \begin{bmatrix} \lambda_1 & .. \\ .. & \lambda_n \end{bmatrix} Q^T$$
$$(x - x_c)^T (Q \begin{bmatrix} \lambda_1 & .. \\ .. & \lambda_n \end{bmatrix} Q^T)^{-1}(x - x_c) \le 1$$
$$\tilde{x}^T \begin{bmatrix} \frac{1}{\lambda_1} & .. \\ .. & \frac{1}{\lambda_n} \end{bmatrix} \tilde{x} \le 1$$
$$\tilde{x}^T \begin{bmatrix} \frac{1}{\lambda_1} & .. \\ .. & \frac{1}{\lambda_n} \end{bmatrix} \tilde{x} = \frac{\tilde{x_1}^2}{\lambda_1} + .. + \frac{\tilde{x_n}^2}{\lambda_n} \le 1$$

volum of elipsoid proportional to $\sqrt{det(P)} = \sqrt{\Pi_i \lambda_i}$

4. Norm Ball

$$C = \{x : \|x\| \le 1\} \text{ for any } \|*\|$$

Example: lp-norm

5. Cone

$$(\forall x \in C, \forall \theta \ge 0)\theta x \in C$$

6. Convex Cone
   Eg: S^n, S_+^n are convex cones
   convexity check for S_+^N:

$$x_1 \in S_+^n \implies v^T x_1 v \ge 0$$
$$x_2 \in S_+^n \implies v^T x_2 v \ge 0$$
$$v^T(\theta x_1 + (1 - \theta)x_2)v \ge 0$$
$$v^T \theta x_1 v + (1 - \theta)v^T x_2 v \implies x \in S_+^n$$

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convexity check for cone:

$$x_1 \in S_+^n \implies \theta x_1 \in S_+^n, \theta \geq 0$$
$$(\forall v)v^T x v \geq 0 \implies v^T(\theta x)v \geq 0 \implies \text{cone}$$

7. Proper Cone
   Definition:

   - convex
   - closed(contains all limit points)
   - solid(non-empty interior)
   - pointed(contains no line): $x \in K \implies -x \notin K$

   Then the proper cone K defines a generalized inequality ($\leq_K$) in $\mathbb{R}^n$

   $$x \leq_K y \implies y - x \in K$$
   $$x <_K y \implies y - x \in int(K)$$

   Example: $K = R_+^n$ (non-negative orthant):

   $$n = 2$$
   $$x \leq_{R_+^2} y \implies y - x \in R_+^2$$

   Cone provides partial ordering using difference of 2 objects.
   Example: $X \leq_{S_+^n} Y \iff Y - X \in S_+^n \iff$ Y-X is PSD

8. Operations Preserving Convexity

- intersection

$$(\forall \alpha \in A)S_\alpha \text{ is affine, convex, convex cone} \implies$$
$$\cap_{\alpha \in A}S_\alpha \text{ is affine, convex, convex cone}$$

   Example: Polyhedral is the intersection of some halfspaces and hyperplanes, so it is convex.
   Any cloed convex set can be represented by possibly infinitely many half spaces.

- affine functions
   let $f(x) = Ax + b, f : \mathbb{R}^n \to \mathbb{R}^m$
   then if S is a convex set we have:

   - project forward: $f(S) = \{f(X) : X \in S\}$ is convex

   - project back: $f^{-1}(S) = \{X : f(X) \in S\}$ is convex

   Example:

$$C = \{y : y = Ax + b, \|x\| \le 1\}$$

   $\|x\| \le 1$ is convex, $Ax + b$ is affine $\implies$ C is convex
   Example:

$$C = \{x : \|Ax + b\| \le 1\}$$

   $\{y : \|y\| \le 1\}$ is convex $\wedge$ y is an affine function of x $\implies$ C is convex

9. Affine Functions $A(x) = \sum_i x_i A_i + B, A_i \in S^n, B \in S^n, x \in \mathbb{R}^n$
   is $\{X : \sum_i x_i A_i = \tilde{A}(x) \le B\}$ convex?
   let $y = B - \tilde{A}(x)$
   $\{y : 0 \le y\}$ is convex
   We know $\{y : 0 \le y\}$ is convex. Further y is an affine function of x $\implies$
   $\{X : \sum_i x_i A_i \le B\}$ is also convex
   $\{X : \sum_i x_i A_i \le B\}$ is a Linear Matrix Inequality

10. Properties of convex sets

- Separating Hyperplanes: if $S, T \subset \mathbb{R}^n$ are convex and disjoint, then $\exists a \neq 0, b$ such that:

$$a^T x \geq b, \forall x \in S$$
$$a^T x \leq b, \forall x \in T$$

11. Supporting Hyperplane:
    if S is convex, $\forall x_0 \in \partial S$ (boundary of S), then $\exists a \neq 0$ such that $a^T x \leq a^T x_0, \forall x \in S$