1 2nd order conditions

Theorem 1.1. *If x^* is a local minimizer of $f : \mathbb{R}^n \to \mathbb{R}$ and $\nabla^2 f(x)$ exists and is continuous in neighbourhood of x^*, then $\nabla f(x^*) =$ and $\nabla^2 f(x^*)$ is symmetric positive semi-definite.*

Proof. By contrdiction.
Note from previous theorem $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is symmetric. So contrdiction must be $\nabla^2 f(x^*)$ is not PSD: $(\exists p) p^T \nabla^2 f(x^*) p < 0$.

By continuity of $\nabla^2 f(x)$ must have that there is an $\alpha > 0$ such that:

$$\phi(t) = p^T \nabla^2 f(x^* + t\alpha p)p < 0$$

for all $t \in (0, 1)$.
$\phi(t) = \nabla f(x^* + t\alpha p)^T p < 0$
$f(x^* + \alpha p) = f(x^*) + \alpha \nabla f(x^*)^T p + \frac{\alpha^2}{2} p^T \nabla^2 f(x^* + t)p$ for some $t \in (0, 1)$
$\nabla f(x^*)^T = 0$, $\frac{\alpha^2}{2} p^T \nabla^2 f(x^* + t)p < 0$
We can always choose α small enough such that $x^* + \alpha p \in B(x^*, r)$ □

Theorem 1.2. *2nd order Sufficient condition for a minimizer.*
if $\nabla^2 f(x)$ exists and is continuous in a neighbourhood of x^* and $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is symmetric positive definite, then x^* is a local minimizer of f.

Proof. Since $\nabla^2 f(x^*)$ is symmetric postiive definite and $\nabla^2 f(x)$ is continuous in a neighbourhood of x^* there exists a $\delta > 0$ s.t. $\nabla^2 f(x * + p)$ is SPD for all $\|p\| < \delta$.
$u^T \nabla^2 f(x^* + p)u > 0, \forall u \in \mathbb{R}^n, u \neq 0$
Choose any point $x \in B(x^*, \delta), x \neq x^*$
Let $p = x - x^*$, $\|p\| < \delta$
So $f(x) = f(x^* + p) = f(x^*) + \nabla f(x^*)^T p + \frac{1}{2} p^T \nabla^2 f(x^* + tp)p, t \in (0, 1)$
$\nabla f(x^*)^T = 0$, $\frac{1}{2} p^T \nabla f(x^*)^T p > 0$, so:
$f(x) > f(x^* + p)$ □

Theorem 1.3. *Then if f is convex, any loval minimer is a global minizer (does not imply a local minimizer is unique).*

Proof. By contradiction.
Suppose x^* is a local minimizer and there is some y^* s.t. $f(y^*) < f(x^*)$

$$\begin{aligned} f(\alpha x^*) + (1 - \alpha)y^*) &\leq \alpha f(x^*) + (1 - \alpha) f(y^*) \\ &< \alpha f(x^*) + (1 - \alpha) f(x^*) = f(x^*), \alpha < 1 \\ z &= \alpha x^* + (1 - \alpha)y^* \\ f(z) &< f(x^*) \end{aligned}$$

For any $B(x^*, r), r > 0$, we can choose $\alpha < 1$ s.t. $z \in B(x^*, r)$ □

Theorem 1.4. *If f is convex, then any local minimizer is a global minimizer. In addition, if f is continuously differentiable in a neighbourhood of x^* and $\nabla f(x^*) = 0$ then x^* is a global minimizer of $f(x)$.*