1. convex function (strict)

$$\forall x, y \in D, D \text{ convex}, \alpha \in (0,1) f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha) f(y)$$

eg:

$$f(x) = c^T x + b$$
$$f(x) = x^T A x + b^T x + c$$

definition 1: A is SPD $\iff$ $x^T A x > 0$, for all $x \neq 0$, A is symmetric
A is SPSD $\iff$ $x^T A x \geq 0$, for all $x \neq 0$, A is symmetric
A is SPSD $\implies$ f is convex
A is SPD $\implies$ f is strictly convex
definition 2: A is SPD $\iff$ A is symmetric and $(\exists a > 0)(\forall x \in \mathbb{R}) x^T A x \geq a x^T x$

show def 2 $\implies$ def 1:
so $\exists a > 0$ s.t. $x^T A x \geq a x^T x > 0$
$x \neq 0 \implies x^T x > 0$

show def 1 $\implies$ 2:
from def 1: $x^T A x > 0, \forall x \neq 0$
let $B = \{x : \|x\| = 1\}$
consider $\min_{\|x\|_2 = 1} x^T A x = \min_{x \in B} x^T A x = \bar{x}^T A \bar{x}$, $\|\bar{x}\| = 1$
$x^T A x$ is continuous for any x
definition for continuous: $(\forall \epsilon > 0)(\exists \delta > 0) \|x - y\| < \delta \implies |\delta(x) - \delta(y)| < \epsilon$
use compactness to bound sequence limit
show B is compact:

(a) B is closed:
B is closed $\iff$ $\forall \{x_n\} \subset B \land x_n \to x^* \implies x^* \in B$
suppose $\{x_n\}, x_n \in B(\|x_n\|_x = 1) \land x_n \to x^*$
want to show $x^* \in B(ie : \|x_n\|_x = 1)$
since $x_n \to x^*$, for any $\epsilon > 0, \exists N$ s.t. for $n \in N$, $\|x_n - x^*\| < \epsilon$
now consider $|\|x^*\|_2 - 1| = |\|x^*\|_2 - \|x_n\|_2| \leq \|x^* - x_n\|$
$|\|x^*\|_2 - 1| = 0 \implies \|x^*\|_2 = 1 \implies x^* \in B$

(b) B is bounded:
B is bounded $\iff$ $(\exists M)(\forall x \in B)\|x\| \leq M$
choose any $x \in \mathbb{R}^n, x \neq 0$
let $y = \frac{x}{\|x\|_2}$
$\|y\|_2 = \|\frac{x}{\|x\|_2}\| = \frac{1}{\|x\|_2} x$
$y^T A y \geq \alpha$
$y^T A y = \frac{x^T}{\|x\|_2} A \frac{x}{\|x\|_2} = \frac{1}{\|x\|_2^2} = \alpha x^T x$

N skew symmetric $\iff N^T = -N$
$x^T N x$, N skew symmetric $\implies x^T N x = 0$
$(x^T N x)^T = x^T N^T x = -x^T N x$
$x^T N x = -x^T N x \implies N = 0$
$A = N + S$, A SPD
$x^T (N + S) x^T = x^T S x$