Non-Malleability of the Fiat–Shamir Transform Revisited for Multi-round SRS-Based Protocols

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Abstract. Faust, Kohlweiss, Marson, and Venturi (INDOCRYPT 2012) showed that non-interactive zero knowledge (NIZK) proof systems obtained by applying the Fiat–Shamir transformation to a public-coin sigma protocol are simulation sound and simulation extractable under lenient conditions. In this paper, we extend this work and formally define simulation extractability for protocols in the random oracle model (ROM) which also use a structured reference string (SRS). Furthermore, we show that NIZK proof systems obtained by applying the Fiat–Shamir transformation to a public-coin multi-round interactive protocol with SRS are simulation-extractable under lenient conditions. A consequence of our result is that, in the ROM, we obtain non-malleable NIZKs essentially for free from a much wider class of protocols than Faust et al. Importantly, we show that two popular zero knowledge SNARKs — Plonk [29] and Sonic [45] — are simulation extractable out-of-the-box.

1 Introduction

The Fiat–Shamir (FS) transform takes a public-coin interactive protocol and makes it interactive by hashing the current protocol transcript to compute the verifier's public coins. While in principle justifiable in the random oracle model (ROM) [9], it is theoretically unsound [32] and so only a heuristic that should be used with care. Nevertheless, the FS transform is a popular design tool when it comes to constructing *zero knowledge succinct arguments of knowledge* (zkSNARKs). In recent years, zkSNARKs have seen indisputable progress [16, 22, 30, 34, 35, 42, 43, 46] Their succinctness make them especially useful for deployment in real systems [4, 20, 21, 48, 53]. However, many works, including Sonic [45] Plonk [29], and Marlin [19] are designed and proven secure as multi-round interactive protocols. Security is then only *conjectured* for their non-interactive variants by employing the FS transform.

Security conjectures via the Fiat–Shamir transform are not the only Achilles heel of zkSNARKs. While the security of the FS transform is of interest primarily to cryptographers, other concerns are more practical: The most efficient zkSNARKs require a trusted third party generated SRS and provide no security if the SRS, which needs to be generated afresh for every new relation, is subverted. This issue has been tackled, e.g., by Plonk and Sonic which feature an updatable and universal SRS [36]. Instead of using a single SRS-generating entity, this framework allows a universal SRS to be generated by a set of parties with the assurance that it remains secure if at least one of these parties is honest.

A further problem with existing security models is proof malleability. Arguably, in the real world the adversary has access to many proofs provided by other parties with the same zero-knowledge scheme. In fact, in the most popular applications of zkSNARKs, like privacy-preserving blockchains, proofs made by all blockchain-participants are (usually) public. Thus, it is only reasonable to require a zero-knowledge proof system to be resilient to attacks that adapt proofs generated by different parties. Indeed, the malleability of ECDSA signatures led to transaction malleability attack against Bitcoin payments and such issues can also affect private crypto-currencies such as Zcash. Nevertheless, most zkSNARKs are only shown to satisfy a (standard) knowledge soundness definition.

Unfortunately, to the best of our knowledge, there are zkSNARKs that are proven to be simulation-extractable [3,14,35,37] and zkSNARKs with a universal and updateable SRS [19, 29, 36, 45]. However, no zkSNARKs are known to enjoy both of these properties out-of-the-box (even for a weaker notion of simulation extractability). Obviously, given a universal and updateable zkSNARK one could lift it to be simulation-extractable using techniques described, e.g., in [2,41], but such a lift comes with inevitable efficiency loss.

Given the proliferation of zkSNARK systems and their frequently used to secure crypto-currency assets of immense value [4, 20, 21, 48, 53] analyzing their security is of great importance. We investigate the advanced security properties of a class of multi-round SRS-based zkSNARK protocols. As in the case of sigma protocols we show that it is sufficient for protocol designers to follow certain design principles and satisfy basic security properties. Using the Fiat-Shamir transform, these properties are then lifting to practice relevant security notions.

1.1 Our contribution

We show that a class of forking sound[1] interactive proofs of knowledge that are (honest-verifier) zero-knowledge in the standard model and have a unique response property *are simulation-extractable out-of-the box* in the random oracle model when the Fiat–Shamir transformation is applied to them. In contrast to Faust et al. [25] who focused on special-sound 3-message sigma protocols, our result can be applied to a wider class of protocols and is applicable to zkSNARKs.

We follow Faust et al. in their definition of simulation extractability. They called their simulation-extractability *weak* which relates to the fact that the extractor's probability of returning a witness depends on the adversary's probability acc of producing an acceptable proof. More precisely, the extractor is not guaranteed to succeed if the adversary outputs an acceptable proof with probability acc smaller than knowledge error ν. On the other hand, we show that this ν can be arbitrarily small, even negligible. However, for acc close to ν the extractor becomes fairly inefficient. Since our extractor is based on a *forking lemma*, we call notion *forking* simulation extractability. This is also to avoid confusion with

[1] Forking soundness is a variant of special soundness, where the transcripts provided to the extractor were obtained through interaction with an honest verifier. We define it later.

weakly simulation extractable protocols, which can be re-randomizable. Since our notion prevents proof malleability, in this work we often simply refer to it as simulation extractability.

Our extractor is non-blackbox as it needs to rewind the adversary (to run it with different random oracle responses). On the other hand, the extractor is non-whitebox either as it only requires oracle access to the adversary and does not depend on the adversary's code. Moreover, it does not rely on knowledge assumptions. Our proof, however, requires the protocols to satisfy a number of properties, which can be proven in the algebraic group model (AGM).

We note that when the extractor rewinds the adversary, it is expected that its success probability dependent on the adversary's probability of returning a valid proof. Unfortunately, the extraction technique we use is quite inefficient as the security loss is exponential in the number of transcripts required to extract the witness. Our analysis thus also serves as a warning—conjecturing security based on the Fiat–Shamir transformation can result in a huge security loss.

In addition to simulation extractability we also give a proof of simulation soundness for the same generic class of protocols. On the up side, we show that despite the fairly inefficient extraction we can prove a much tighter result for simulation soundness.

To show that our result are useful and practical we prove that two of the most efficient updatable and universal zkSNARKs—Plonk and Sonic—are simulation sound and forking simulation-extractable. To obtain these results, despite not changing anything at all in these protocols, we define new intermediary properties satisfied by these multi-round computationally sound protocols and their building blocks: forking soundness, generalized unique response property, and a generalized forking lemma. These conceptual insights into interactive SNARK systems help us overcome a number of challenges, as we explain next.

1.2 Our techniques

We note that Plonk and Sonic—as originally presented in [29] and [45]—are interactive proofs of knowledge that are made non-interactive by the Fiat–Shamir transform. In the following, we denote the underlying interactive protocols by $\mathbf{P}$ (for Plonk) and $\mathbf{S}$ (for Sonic) and the resulting non-interactive proof systems by $\mathbf{P}_{\mathsf{FS}}$ and $\mathbf{S}_{\mathsf{FS}}$, respectively.

Forking soundness. First, following [25], we need to show for the protocols we consider that one can extract a witness from sufficiently many valid proof transcripts with a common prefix. However many protocols do not meet the standard definition of special soundness for sigma protocols. First of all, the definition requires extraction of a witness from any two transcripts, each containing three messages and sharing the first message. For $\mathbf{P}$ and $\mathbf{S}$ that is not enough. The definition needs to be adapted to cover protocols with more than three messages. Furthermore, the number of transcripts required is much greater. Concretely, $(3\mathsf{n} + 1)$—where n is the number of constraints in the proven circuit—for $\mathbf{P}$ and $(\mathsf{n} + \mathsf{Q} + 1)$—where n and Q are the numbers of multiplicative and linear constraints—for $\mathbf{S}$. Hence, we do not have a *pair of transcripts*, but a *tree of transcripts*.

Second, both protocols rely on structured reference strings which come with trapdoors. If in possession of the trapdoor, an adversary can produce multiple valid proof transcripts without knowing the witness and potentially for false statements. Recall that the standard special soundness definition requires witness extraction from *any* tree of acceptable transcripts that share a common root. This means that there are no such trees for incorrect statements. In this paper we define a different, forking lemma-related, version of soundness which we call *forking soundness*. That is, we show that it is possible to extract a witness from all but negligibly many trees of acceptable transcripts produced by probabilistic polynomial time (PPT) adversaries, given that the trees are generated as interactions between a (possibly malicious) prover and an honest verifier. We show how to use an adversary from which we cannot extract the witness but that produces such a tree to break the underlying computational assumption.

Unique response property. Another property we require is the unique response property which states, as expressed in [26] for 3-messages sigma protocols, that all but the first message sent by the prover are deterministic (intuitively, the prover can only employ fresh randomness in the first message of the protocol). Again, we can not use this definition right out of the box. P_{FS} does not satisfy it—both the first and the second prover's messages are randomised. Although S prover is deterministic after it picks its first message, the protocol has more than 3 rounds, hence it also does not fulfill the definition. We thus propose a generalisation of the definition which states that a protocol is i-ur if the prover is deterministic starting from its $(i+1)$-th message. For our proof it is sufficient that this property is met by P for $i = 2$. Since Sonic's prover is deterministic from the second message on, it is 1-ur.

To be able to show the unique response property (for both of the protocols) we also had to show that the modified KZG polynomial commitment schemes [40] proposed in [29] and [45] have a *unique opening property* which requires that for a commitment to a polynomial $p(X)$ it is infeasible for any PPT adversary to provide two different acceptable openings witnesses at point z even for the consistent evaluation $p(z)$.

HVZK. In order to show our result we also show that (interactive) P and S are honest verifier zero-knowledge in the standard model, i.e. the simulator is able to produce a transcript indistinguishable from a transcript produced by an honest prover and verifier without any additional knowledge, esp. without knowing the SRS trapdoor. Although both Sonic and Plonk are zero-knowledge, their simulators utilise trapdoors. For our reduction to work, we need simulators able to provide indistinguishable proofs relying only on reordering the messages and picking suitable verifier challenges. That is, any PPT party should be able to produce a simulated proof by its own. (Note that this property does not necessary break soundness of the protocol as the simulator is required only to produce a transcript and is not involved in a real conversation with a real verifier). This property allows us to build simulators for P_{FS} and S_{FS} that rely only on the programmability of the random oracle. Note that this programmability is also only needed from some round i onward.

Generalising Boneh-Boyen-Goh [12] uber assumption. To show that Plonk is zero-knowledge we rely on a variant of BBG's *uber assumption*. In its original version, the assumption assures that some polynomial evaluation (on a random, unknown point) represented in a bilinear pairing target group $\mathbb{G}_T$ is indistinguishable from a random element. In our variant, we modify two things. (1) Firstly, the polynomial evaluation can be represented in other groups than $\mathbb{G}_T$; here we use $\mathbb{G}_1$; (2) Secondly, the distinguisher is given not only a single polynomial evaluation, but a number of polynomials evaluations. That is, it either gets the result of k polynomial evaluations or k random numbers. We show security of the generalized uber assumption directly in the generic group model.

Generalisation of the general forking lemma. Consider an interactive 3-message special-sound protocol $\mathbf{\Psi}$ and its non-interactive version $\mathbf{\Psi}_{\mathsf{FS}}$ obtained by the Fiat–Shamir transform. The general forking lemma provides an instrumental lower bound for the probability of extracting a witness from an adversary who provides two proofs for the same statement that share the first message. Since $\mathbf{P}$ and $\mathbf{S}$ have more than 3 messages and are not special-sound, the forking lemma of Bellare and Neven [8], cannot be used directly. We thus propose a modification that covers multi-message protocols where witness extraction requires more transcripts than merely two. Unfortunately, we also observe that the security gap grows with the number of transcripts and the probability that the extractor succeeds diminishes significantly. (That said, we have to note that the security loss is polynomial, albeit big.)

We note that some modern zkSNARKs, like [16,45], heavily rely on the Fiat–Shamir transform and thus potentially the forking lemma. First, an interactive protocol is proposed and its security and forking soundness analysed. Second, one uses an argument that the Fiat–Shamir transform can be used to get a protocol that is non-interactive and shares the same security properties.

We see our generalized forking lemma as contributing to a critical assessment of this approach. The analysis of the interactive protocol is not enough and one has to consider the security loss implied by the Fiat-Shamir transform for the target security notion. Thus one has to either rely on our generalisation of the forking lemma or disclose a transformation that does not suffer this loss. We note that the security loss may also apply when knowledge soundness is proven. That is the case for the original Sonic paper, whose security proof relies on so-called witness-extended emulation. The authors of Plonk and recent work on Sonic [31] worked around this problem by proving knowledge soundness directly in the algebraic group model.

Simulation soundness and forking simulation-extractability. Given our modified, less restrictive, definition for forking soundness and the unique response property, and our generalised forking lemma we obtain our main results—proofs for the simulation soundness and forking simulation extractability of $\mathbf{P}_{\mathsf{FS}}$ and $\mathbf{S}_{\mathsf{FS}}$. The proofs are inspired by simulation-extractability and simulation-soundness proofs from [25], with major modifications, which were required as [25] considers only sigma protocols that are undoubtedly much simpler protocols than the considered proof systems.

1.3 Structure of the paper

In the next section we present necessary preliminaries. Section 3 gives definitions and lemmas for multi-round SRS-based protocol needed to instantiate our framework. In Section 4, we present our main result, a proof of simulation-soundness and simulation-extraction for multi-round SRS-based zero-knowledge proofs systems. In Section 5 and Supp. Mat. D we show that Plonk and Sonic fulfill the requirements of our framework and are in fact simulation sound and forking simulation extractable.

1.4 Related Work

There are many results on simulation extractability for non-interactive zero-knowledge proofs (NIZKs). First, Groth [33] noticed that a (black-box) simulation extractable NIZK is universally-composable (UC) [18]. Then Dodis et al. [24] introduced a notion of (black-box) *true simulation extractability* and showed that no NIZK can be UC-secure if it does not satisfy this property. In the context of zkSNARKs it is important to mention such works as the first simulation-extractable zkSNARK by Groth and Maller [37] and SE zkSNARK for QAP by Lipmaa [44]. Kosba's et al. [41] give a general transformation from a NIZK to a black-box SE NIZK. Although their transformation works for zkSNARKs as well, succinctness of the proof system is not preserved as the statement's witness is encrypted. Recently, Abdolmaleki et al. [2] showed another transformation that obtains non-black-box simulation extractability but also preserves succinctness of the argument.

Independently, some authors focused on obtaining simulation extractability of known zkSNARKs, like Groth16 [35], by introducing minor modifications and using stronger assumptions [3,14]. Interestingly, although such modifications hurt performance of the proof system, the resulting zkSNARKs are still more efficient than the first SE zkSNARK [37], see [3]. Recently, [5] showed that the original Groth's proof system from [35] is weakly SE and randomisable.

Forking lemma generalizations. There are several task specific variants, e.g., [6, 7, 38], of the general forking lemma [8, 47] for analyzing the forking behavior of random-oracle based executions. In [13] Bootle et al. proposed a novel inner-product argument which security relies on, so-called, witness-extended emulation. To show that property, the authors proposed a new version of forking lemma, which gives a lower bound on probability that a tree finding algorithm is able to produce a tree of acceptable transcripts by rewinding a conversation between a (potentially malicious) prover and verifier.

Although the result in that paper is dubbed a "forking lemma" it differs from forking lemmas known from e.g. [8, 47]. First of all, the forking lemmas in these papers analyse the probability of building a tree of acceptable transcripts for Fiat–Shamir based non-interactive proof systems, while the protocol presented by Bootle et al. is intended to work for interactive proof systems.

Importantly, it is not obvious how the result of Bootle et al. can be used to show security of non-interactive protocols as it relies on interactive provers whose proving strategies are more limited than proving strategies of non-interactive

provers. For example, if a challenge given by the verifier does not suit an interactive prover, it can only try to finish a proof with it or abort. On the other hand, a non-interactive prover has far wider scope of possible actions—when the protocol is non-interactive the prover may adapt its strategy based on the random oracle outputs.

State restoration attack security [11, 39] defines a stronger soundness notion for interactive protocols, that allows for a tighter reduction from interactive to non-interactive protocols. [31] combine this approach with the algebraic group model to obtain tight security guarantees for the non-interactive versions of Bulletproof [16] and Sonic. A similar approach is also taken by [17]. These works however are focused on showing security of concrete proof systems, while we are show security of an arguably wide class of protocols. They also do not consider simulation soundness and simulation extractability. Last, but not least, our high-level security proof relies only on random oracle, while [17, 31] make use of random oracle *and* the AGM. The AGM is only used to prove low-level properties about specific protocols.

2 Preliminaries

Let PPT denote probabilistic polynomial-time and $\lambda \in \mathbb{N}$ be the security parameter. All adversaries are stateful. For an algorithm $\mathcal{A}$, let $\mathrm{im}(\mathcal{A})$ be the image of $\mathcal{A}$ (the set of valid outputs of $\mathcal{A}$), let $\mathsf{R}(\mathcal{A})$ denote the set of random tapes of correct length for $\mathcal{A}$ (assuming the given value of λ), and let $r \leftarrow\!\!\$\, \mathsf{R}(\mathcal{A})$ denote the random choice of the randomiser r from $\mathsf{R}(\mathcal{A})$. We denote by $\mathsf{negl}(\lambda)$ ($\mathsf{poly}(\lambda)$) an arbitrary negligible (resp. polynomial) function.

Probability ensembles $X = \{X_\lambda\}_\lambda$ and $Y = \{Y_\lambda\}_\lambda$, for distributions X_λ, Y_λ, have *statistical distance* Δ equal $\epsilon(\lambda)$ if $\sum_{a \in \mathrm{Supp}(X_\lambda \cup Y_\lambda)} |\Pr[X_\lambda = a] - \Pr[Y_\lambda = a]| = \epsilon(\lambda)$. We write $X \approx_\lambda Y$ if $\Delta(X_\lambda, Y_\lambda) \leq \mathsf{negl}(\lambda)$. For values $a(\lambda)$ and $b(\lambda)$ we write $a(\lambda) \approx_\lambda b(\lambda)$ if $|a(\lambda) - b(\lambda)| \leq \mathsf{negl}(\lambda)$.

For a probability space $(\Omega, \mathcal{F}, \mu)$ and event $\mathsf{E} \in \mathcal{F}$ we denote by $\overline{\mathsf{E}}$ an event that is complementary to E, i.e. $\overline{\mathsf{E}} = \Omega \setminus \mathsf{E}$.

Denote by $\mathcal{R} = \{\mathbf{R}\}$ a family of relations. We assume that if $\mathbf{R}$ comes with any auxiliary input, it is benign. Directly from the description of $\mathbf{R}$ one learns security parameter λ and other necessary information like public parameters p containing description of a group $\mathbb{G}$, if the relation is a relation of group elements (as it usually is in case of zkSNARKs).

Bilinear groups. A bilinear group generator $\mathsf{Pgen}(1^\lambda)$ returns public parameters $\mathsf{p} = (p, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T, \hat{e}, [1]_1, [1]_2)$, where $\mathbb{G}_1$, $\mathbb{G}_2$, and $\mathbb{G}_T$ are additive cyclic groups of prime order $p = 2^{\Omega(\lambda)}$, $[1]_1, [1]_2$ are generators of $\mathbb{G}_1$, $\mathbb{G}_2$, resp., and $\hat{e} : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$ is a non-degenerate PPT-computable bilinear pairing. We assume the bilinear pairing to be Type-3, i.e., that there is no efficient isomorphism from $\mathbb{G}_1$ to $\mathbb{G}_2$ or from $\mathbb{G}_2$ to $\mathbb{G}_1$. We use the by now standard bracket notation, i.e., we write $[a]_\iota$ to denote ag_ι where g_ι is a fixed generator of $\mathbb{G}_\iota$. We denote $\hat{e}([a]_1, [b]_2)$ as $[a]_1 \bullet [b]_2$. Thus, $[a]_1 \bullet [b]_2 = [ab]_T$. We freely use the bracket notation with matrices, e.g., if $\boldsymbol{AB} = \boldsymbol{C}$ then $\boldsymbol{A}[\boldsymbol{B}]_\iota = [\boldsymbol{C}]_\iota$ and $[\boldsymbol{A}]_1 \bullet [\boldsymbol{B}]_2 = [\boldsymbol{C}]_T$. Since every

algorithm $\mathcal{A}$ takes as input the public parameters we skip them when describing $\mathcal{A}$'s input. Similarly, we do not explicitly state that each protocol starts with generating these parameters by Pgen.

2.1 Computational assumptions.

Discrete-log assumptions. Security of Plonk and Sonic relies on two discrete-log based security assumptions—(q_1, q_2)-dlog assumption and its variant that allows for negative exponents (q_1, q_2)-ldlog assumption[2]. We omit here description of the assumptions and refer to Supp. Mat. C.1.

Proofs by Game-Hoping. Proofs by *game hoping* is a method of writing proofs popularised by e.g. Shoup [51] and Dent [23]. The method relies on the following lemma.

Lemma 1 (Difference lemma, [51, Lemma 1]). *Let* $\mathsf{A}, \mathsf{B}, \mathsf{F}$ *be events defined in some probability space, and suppose that* $\mathsf{A} \wedge \overline{\mathsf{F}} \iff \mathsf{B} \wedge \overline{\mathsf{F}}$. *Then* $|\Pr[\mathsf{A}] - \Pr[\mathsf{B}]| \leq \Pr[\mathsf{F}]$.

2.2 Algebraic Group Model

The algebraic group model (AGM) introduced in [27] lies between the standard model and generic bilinear group model. In the AGM it is assumed that an adversary $\mathcal{A}$ can output a group element $[y] \in \mathbb{G}$ if $[y]$ has been computed by applying group operations to group elements given to $\mathcal{A}$ as input. It is further assumed, that $\mathcal{A}$ knows how to "build" $[y]$ from that elements. More precisely, the AGM requires that whenever $\mathcal{A}([\boldsymbol{x}])$ outputs a group element $[y]$ then it also outputs $\boldsymbol{c}$ such that $[y] = \boldsymbol{c}^\top \cdot [\boldsymbol{x}]$. Both Plonk and Sonic have been shown secure using the AGM. An adversary that works in the AGM is called *algebraic*.

2.3 Polynomial commitment

In the polynomial commitment scheme $\mathbf{PC} = (\mathsf{KGen}, \mathsf{Com}, \mathsf{Op}, \mathsf{Vf})$ the committer C can convince the receiver R that some polynomial f which C committed to evaluates to s at some point z chosen by R. Plonk and Sonic use variants of the KZG polynomial commitment scheme [40]. We denote the first by $\mathbf{PC_P}$ and the latter by $\mathbf{PC_S}$ Due to page limit, we omit their presentation here and refer to Fig. 2 and Fig. 3 in the Supp. Mat. A.1. In this paper we use evaluation binding, commitment of knowledge, and, newly introduced, unique opening properties. Formal definitions of these could be find in Supp. Mat. A.1, here we briefly introduce them.

Evaluation binding intuitively, this property assures that no adversary could provide two valid openings for two different evaluations of the same commitment in the same point.

Commitment of knowledge when a commitment scheme is "of knowledge" then if an adversary produces a (valid) commitment c, which it can open, then it also knows the underlying polynomial f which commits to that value.

[2] Note that [45] dubs their assumption *a dlog assumption*. We changed that name to distinguish it from the more standard dlog assumption used in [29]. "l" in *ldlog* relates to use of Laurent polynomials in the assumption.

[45] shows, using AGM, that $\mathbf{PC_S}$ is a commitment of knowledge. The same reasoning could be used to show that property for $\mathbf{PC_P}$.

Unique opening that property assures that there is only one valid opening for the committed polynomial and given evaluation point. This property is crucial in showing forking simulation-extractability of Plonk and Sonic. We show that the Plonk's and Sonic's polynomial commitment schemes satisfy this requirement in Lemma 3 and Lemma 11 respectively.

2.4 Zero knowledge

In a zero-knowledge proof system, a prover convinces the verifier of veracity of a statement without leaking any other information. The zero-knowledge property is proven by constructing a simulator that can simulate the view of a cheating verifier without knowing the secret information—witness—of the prover. A proof system has to be sound as well, i.e. for a malicious prover it should be infeasible to convince a verifier on a false statement. Here, we focus on proof systems that guarantee soundness against PPT malicious provers.

More precisely, let $\mathcal{R}(1^\lambda) = \{\mathbf{R}\}$ be a family of NP relations. Denote by $\mathcal{L}_{\mathbf{R}}$ the language determined by $\mathbf{R}$. Let P and V be PPT algorithms, the former called *prover* and the latter *verifier*. We allow our proof system to have a setup, i.e. there is a KGen algorithm that takes as input the relation description $\mathbf{R}$ and outputs a common reference string srs. We denote by $\langle \mathsf{P}(\mathbf{R}, \mathsf{srs}, \mathsf{x}, \mathsf{w}), \mathsf{V}(\mathbf{R}, \mathsf{srs}, \mathsf{x})\rangle$ a *transcript* (also called *proof*) π of a conversation between P with input $(\mathbf{R}, \mathsf{srs}, \mathsf{x}, \mathsf{w})$ and V with input $(\mathbf{R}, \mathsf{srs}, \mathsf{x})$. We write $\langle \mathsf{P}(\mathbf{R}, \mathsf{srs}, \mathsf{x}, \mathsf{w}), \mathsf{V}(\mathbf{R}, \mathsf{srs}, \mathsf{x})\rangle = 1$ if in the end of the transcript the verifier V returns 1 and say that V accepts it. We sometimes abuse notation and write $\mathsf{V}(\mathbf{R}, \mathsf{srs}, \mathsf{x}, \pi) = 1$ to denote a fact that π is accepted by the verifier. (This is especially handy when the proof system is non-interactive, i.e. the whole conversation between the prover and verifier consists of a single message π sent by P).

A proof system $\boldsymbol{\Psi} = (\mathsf{KGen}, \mathsf{P}, \mathsf{V}, \mathsf{Sim})$ for $\mathcal{R}$ is required to have three properties: completeness, soundness and zero knowledge, which are defined as follows:

Completeness. An interactive proof system $\boldsymbol{\Psi}$ is *complete* if an honest prover always convinces an honest verifier, that is for all $\mathbf{R} \in \mathcal{R}(1^\lambda)$ and $(\mathsf{x}, \mathsf{w}) \in \mathbf{R}$

$$\Pr[\langle \mathsf{P}(\mathbf{R}, \mathsf{srs}, \mathsf{x}, \mathsf{w}), \mathsf{V}(\mathbf{R}, \mathsf{srs}, \mathsf{x})\rangle = 1 \mid \mathsf{srs} \leftarrow \mathsf{KGen}(\mathbf{R})] = 1\,.$$

Soundness. We say that $\boldsymbol{\Psi}$ for $\mathcal{R}$ is *sound* if no PPT prover $\mathcal{A}$ can convince an honest verifier V to accept a proof for a false statement $\mathsf{x} \notin \mathcal{L}$. More precisely, for all $\mathbf{R} \in \mathcal{R}(1^\lambda)$

$$\Pr[\langle \mathcal{A}(\mathbf{R}, \mathsf{srs}, \mathsf{x}), \mathsf{V}(\mathbf{R}, \mathsf{srs}, \mathsf{x})\rangle = 1 \mid \mathsf{srs} \leftarrow \mathsf{KGen}(\mathbf{R}), \mathsf{x} \leftarrow \mathcal{A}(\mathbf{R}, \mathsf{srs}); \mathsf{x} \notin \mathcal{L}_{\mathbf{R}}] \leq \mathsf{negl}(\lambda)\,;$$

Sometimes a stronger notion of soundness is required—except requiring that the verifier rejects proofs of statements outside the language, we request from the prover to know a witness corresponding to the proven statement. This property is called *knowledge soundness*.

Zero knowledge. We call a proof system $\mathbf{\Psi}$ *zero-knowledge* if for any $\mathbf{R} \in \mathcal{R}(1^\lambda)$, $(\mathsf{x}, \mathsf{w}) \in \mathbf{R}$, and adversary $\mathcal{A}$ there exists a PPT simulator Sim such that

$$\left\{ \langle \mathsf{P}(\mathbf{R}, \mathsf{srs}, \mathsf{x}, \mathsf{w}), \mathcal{A}(\mathbf{R}, \mathsf{srs}, \mathsf{x}, \mathsf{w}) \rangle \;\middle|\; \mathsf{srs} \leftarrow \mathsf{KGen}(\mathbf{R}) \right\} \approx_\lambda \left\{ \mathsf{Sim}^{\mathcal{A}}(\mathbf{R}, \mathsf{srs}, \mathsf{x}) \;\middle|\; \mathsf{srs} \leftarrow \mathsf{KGen}(\mathbf{R}) \right\}.$$

We call zero knowledge *perfect* if the distributions are equal and *computational* if they are indistinguishable for any NUPPT distinguisher.

Alternatively, zero-knowledge can be defined by allowing the simulator to use the trapdoor td that is generated along the srs. In this paper we distinguish simulators that requires a trapdoor to simulate and those that do not. We call the former *SRS-simulators*. We say that a protocol is zero knowledge in the standard model if its simulator does not require the trapdoor.

Occasionally, a weaker version of zero knowledge is sufficient. So called *honest verifier zero knowledge* (HVZK) assumes that the verifier's challenges are picked at random from some predefined set. Although weaker, this definition suffices in many applications. Especially, an interactive zero-knowledge proof that is HVZK and *public-coin* (i.e. the verifier outputs as challenges its random coins) can be made non-interactive and zero-knowledge in the random oracle model by using the Fiat–Shamir transformation.

In our simulation soundness proof (but not simulation extractability) we need an additional property of the zero-knowledge proof system which we call k-programmable ZK.

Definition 1 (k-programmable ZK). *Let $\mathbf{\Psi}$ be a $(2\mu + 1)$-message ZK proof system and let $\mathbf{\Psi}_{\mathsf{FS}}$ be its Fiat–Shamir variant. We say that $\mathbf{\Psi}_{\mathsf{FS}}$ is k-programmable ZK if there exists a simulator $\mathsf{Sim}_{\mathsf{FS}}$ that*

1. *produces proofs indistinguishable from proofs output by an honest prover;*
2. $\mathsf{Sim}_{\mathsf{FS}}$ *program the random oracle* only *for challenges from round k to $\mu + 1$.*

We note that $\mathbf{P}$ is 2-programmable ZK and $\mathbf{S}$ is 1-programmable. This follows directly from the proofs of their standard model zero-knowledge property in Lemmas 6 and 14.

Idealised verifier and verification equations Let $(\mathsf{KGen}, \mathsf{P}, \mathsf{V})$ be a protocol—in this paper, either a proof system or a polynomial commitment scheme. Observe that the KGen algorithm provides an SRS which can be interpreted as a set of group representation of polynomials evaluated at trapdoor elements. E.g. for a trapdoor χ the SRS contains $[\mathsf{p}_1(\chi), \ldots, \mathsf{p}_k(\chi)]_1$, for some polynomials $\mathsf{p}_1(X), \ldots, \mathsf{p}_k(X) \in \mathbb{F}_p[X]$. On the other hand, the verifier V accepts if a (possibly set of) verification equation $\mathsf{ve}_{\mathsf{x},\pi}$ (note that the verification equation changes relate to the instance x and proof π), which can also be interpreted as a polynomial in $\mathbb{F}_p[X]$ and which coefficients depend on messages sent by the prover, zeroes at χ. Following [29] we call verifiers who checks that $\mathsf{ve}_{\mathsf{x},\pi}(\chi) = 0$ *real verifiers* as opposed to *ideal verifiers* who accepts only when $\mathsf{ve}_{\mathsf{x},\pi}(X) = 0$. That is, while a real verifier accepts when a polynomial *evaluates* to zero, an ideal verifier accepts only when the polynomial *is* zero.

Although ideal verifiers are impractical, the are very useful in our proofs. More precisely, we show that

1. the idealised verifier accepts an incorrect proof (what "incorrect" means depends on the situation) with at most negligible probability (and many cases—never);
2. when the real verifier accepts, but not the idealised one, then we show how to use a malicious P to break the underlying security assumption (in our case—a variant of dlog.)

Sigma protocols A sigma protocol $\Sigma = (\mathsf{P}, \mathsf{V}, \mathsf{Sim})$ for a relation $\mathbf{R} \in \mathcal{R}(1^\lambda)$ is a special case of an interactive proof which transcript compounds of three messages (a, b, z), the middle being a challenge provided by the verifier. Sigma protocols are honest verifier zero-knowledge in the standard model and specially-sound. That is, there exists an extractor Ext which given two accepting transcripts (a, b, z), (a, b', z') for a statement x can recreate the corresponding witness if $b \neq b'$. Formally, *Special soundness.* A sigma protocol Σ is *specially-sound* if for any adversary $\mathcal{A}$ the probability

$$\Pr\left[\begin{array}{l|l} \mathsf{w} \leftarrow \mathsf{Ext}(\mathbf{R}, \mathsf{x}, (a, b, z), (a, b', z')), & (\mathsf{x}, (a, b, z), (a, b', z')) \leftarrow \mathcal{A}(\mathbf{R}), b \neq b', \\ \mathbf{R}(\mathsf{x}, \mathsf{w}) = 0 & \mathsf{V}(\mathbf{R}, \mathsf{x}, (a, b, z)) = \mathsf{V}(\mathbf{R}, \mathsf{x}, (a, b', z')) = 1, \end{array}\right]$$

is upper-bounded by some negligible function $\mathsf{negl}(\lambda)$.

Another property that sigma protocols may have is a unique response property [26] which states that no PPT adversary can produce two accepting transcripts that differ only on the last element. More precisely, *Unique response property.* Let $\Sigma = (\mathsf{P}, \mathsf{V}, \mathsf{Sim})$ be a sigma-protocol for $\mathbf{R} \in \mathcal{R}(1^\lambda)$ which proofs compound of three messages (a, b, z). We say that Σ is has a unique response property if for all PPT algorithms $\mathcal{A}$ holds

$$\Pr[\mathsf{V}(\mathbf{R}, \mathsf{x}, (a, b, z)) = \mathsf{V}(\mathbf{R}, \mathsf{x}, (a, b, z')) = 1 \mid (\mathsf{x}, a, b, z, z') \leftarrow \mathcal{A}(\mathbf{R}), z \neq z'] \leq \mathsf{negl}(\lambda)\,.$$

If this property holds even against unbounded adversaries, it is called *strict*, cf. [25]. Later on we call protocols that follows this notion *ur-protocols*. For the sake of completeness we note that many sigma protocols, like e.g. Schnorr's protocol [49], fulfil this property.

2.5 From interactive to non-interactive—the Fiat–Shamir transform

Consider a $(2\mu + 1)$-message, public-coin, honest verifier zero-knowledge interactive proof system $\boldsymbol{\Psi} = (\mathsf{KGen}, \mathsf{P}, \mathsf{V}, \mathsf{Sim})$ for $\mathbf{R} \in \mathcal{R}(1^\lambda)$. Let π be a proof performed by the prover P and verifier V compound of messages $(a_1, b_1, \ldots, a_{\mu-1}, b_\mu, a_{\mu+1})$, where a_i comes from P and b_i comes from V. Denote by $\mathcal{H}$ a random oracle. Let $\boldsymbol{\Psi}_{\mathsf{FS}} = (\mathsf{KGen}_{\mathsf{FS}}, \mathsf{P}_{\mathsf{FS}}, \mathsf{V}_{\mathsf{FS}}, \mathsf{Sim}_{\mathsf{FS}})$ be a proof system such that

- $\mathsf{KGen}_{\mathsf{FS}}$ behaves as KGen.
- P_{FS} behaves as P except after sending message a_i, $i \in [1\,..\,\mu]$, the prover does not wait for the message from the verifier but computes it locally setting $b_i = \mathcal{H}(\pi[0..i])$, where $\pi[0..j] = (\mathsf{x}, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j)$. (Importantly, $\pi[0..\mu + 1] = (\mathsf{x}, \pi)$).

- V_{FS} behaves as V but does not provide challenges to the prover's proof. Instead it computes the challenges locally as P_{FS} does. Then it verifies the resulting transcript π as the verifier V would.
- $\mathsf{Sim}_{\mathsf{FS}}$ behaves as Sim, except when Sim picks challenge b_i before computing message $\pi[0,i]$, $\mathsf{Sim}_{\mathsf{FS}}$ programs the random oracle to output b_i on $\pi[0,i]$.

The Fiat–Shamir heuristic states that $\boldsymbol{\Psi}_{\mathsf{FS}}$ is a zero-knowledge non-interactive proof system for $\mathbf{R} \in \mathcal{R}(1^\lambda)$.

2.6 Non-malleability definitions for NIZKs

Real life applications often require a NIZK proof system to be non-malleable. That is, no adversary seeing a proof π for a statement x should be able to provide a new proof π' related to π. *Simulation extractability* formalizes a strong version of non-malleability by requiring that no adversary can produce a valid proof without knowing the corresponding witness. This must hold even if the adversary is allowed to see polynomially many simulated proofs for any statements it wishes.

Definition 2 (Forking simulation-extractable NIZK, [25]). *Let* $\boldsymbol{\Psi} = (\mathsf{KGen}, \mathsf{P}, \mathsf{V}, \mathsf{Sim})$ *be a forking-sound HVZK proof and* $\boldsymbol{\Psi}_{\mathsf{FS}} = (\mathsf{KGen}_{\mathsf{FS}}, \mathsf{P}_{\mathsf{FS}}, \mathsf{V}_{\mathsf{FS}}, \mathsf{Sim}_{\mathsf{FS}})$ *be* $\boldsymbol{\Psi}$ *transformed by the Fiat–Shamir transform. We say that* $\boldsymbol{\Psi}_{\mathsf{FS}}$ *is* forking simulation-extractable *with* extraction error ν *if for any* PPT *adversary* $\mathcal{A}$ *that is given oracle access to a random oracle* $\mathcal{H}$ *and simulator* $\mathsf{Sim}_{\mathsf{FS}}$*, and produces an accepting transcript of* $\boldsymbol{\Psi}$ *with probability* acc*, that is*

$$\mathsf{acc} = \Pr\left[\begin{array}{l|l} \mathsf{V}_{\mathsf{FS}}(\mathbf{R}, \mathsf{srs}, \mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}) = 1, & \mathsf{srs} \leftarrow \mathsf{KGen}_{\mathsf{FS}}(\mathbf{R}), r \leftarrow\!\!\$\, \mathsf{R}(\mathcal{A}), \\ (\mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}) \notin Q & (\mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}) \leftarrow \mathcal{A}^{\mathsf{Sim}_{\mathsf{FS}}, \mathcal{H}}(\mathbf{R}, \mathsf{srs}; r) \end{array}\right],$$

probability

$$\mathsf{ext} = \Pr\left[\begin{array}{l|l} \mathsf{V}_{\mathsf{FS}}(\mathbf{R}, \mathsf{srs}, \mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}) = 1, & \mathsf{srs} \leftarrow \mathsf{KGen}_{\mathsf{FS}}(\mathbf{R}), r \leftarrow\!\!\$\, \mathsf{R}(\mathcal{A}), \\ (\mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}) \notin Q, & (\mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}) \leftarrow \mathcal{A}^{\mathsf{Sim}_{\mathsf{FS}}, \mathcal{H}}(\mathbf{R}, \mathsf{srs}; r) \\ \mathbf{R}(\mathsf{x}_{\mathcal{A}}, \mathsf{w}_{\mathcal{A}}) = 1 & \mathsf{w}_{\mathcal{A}} \leftarrow \mathsf{Ext}_{\mathsf{se}}(\mathbf{R}, \mathsf{srs}, \mathcal{A}, r, \mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}, Q, Q_{\mathcal{H}}) \end{array}\right]$$

is at at least

$$\mathsf{ext} \geq \frac{1}{\mathsf{poly}(\lambda)}(\mathsf{acc} - \nu)^d - \varepsilon(\lambda)\,,$$

for some polynomial $\mathsf{poly}(\lambda)$*, constant* d *and negligible* $\varepsilon(\lambda)$ *whenever* $\mathsf{acc} \geq \nu$*. List* Q *contains all* (x, π) *pairs where* x *is an instance provided to the simulator by the adversary and* π *is the simulator's answer. List* $Q_{\mathcal{H}}$ *contains all* $\mathcal{A}$*'s queries to* $\mathcal{H}$ *and* $\mathcal{H}$*'s answers.*

Simulation sound NIZKs. Another notion for non-malleable NIZKs is *simulation soundness*. It allows the adversary to see simulated proof, however, in contrast to simulation extractability it does not require an extractor to provide a witness for the proven statement. Instead, it is only necessary, that an adversary who sees simulated proofs cannot make the verifier accept a proof of an incorrect statement. More precisely,

Definition 3 (Simulation soundness). *Let* $\mathbf{\Psi} = (\mathsf{KGen}, \mathsf{P}, \mathsf{V}, \mathsf{Sim})$ *be a NIZK proof and* $\mathbf{\Psi}_{\mathsf{FS}} = (\mathsf{KGen}_{\mathsf{FS}}, \mathsf{P}_{\mathsf{FS}}, \mathsf{V}_{\mathsf{FS}}, \mathsf{Sim}_{\mathsf{FS}})$ *be* $\mathbf{\Psi}$ *transformed by the Fiat–Shamir transform. We say that* $\mathbf{\Psi}_{\mathsf{FS}}$ *is* simulation-sound *for any* PPT *adversary* $\mathcal{A}$ *that is given oracle access to a random oracle* $\mathcal{H}$ *and simulator* $\mathsf{Sim}_{\mathsf{FS}}$*, probability*

$$\mathsf{ssnd} = \Pr\left[\begin{array}{l|l} \mathsf{V}_{\mathsf{FS}}(\mathbf{R}, \mathsf{srs}, \mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}) = 1, & \mathsf{srs} \leftarrow \mathsf{KGen}(\mathbf{R}), r \leftarrow\!\!\$\, \mathsf{R}(\mathcal{A}), \\ (\mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}) \notin Q, & (\mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}) \leftarrow \mathcal{A}^{\mathsf{Sim}_{\mathsf{FS}}, \mathcal{H}}(\mathbf{R}, \mathsf{srs}; r) \\ \neg\exists \mathsf{w}_{\mathcal{A}} : \mathbf{R}(\mathsf{x}_{\mathcal{A}}, \mathsf{w}_{\mathcal{A}}) = 1 & \end{array}\right]$$

is at most negligible. List Q *contains all* (x, π) *pairs where* x *is an instance provided to the simulator by the adversary and* π *is the simulator's answer.*

We note that the probability ssnd Definition 3 can be expressed in terms of simulation-extractability. More precisely, the condition $\neg\exists \mathsf{w} : \mathbf{R}(\mathsf{x}_{\mathcal{A}}, \mathsf{w}_{\mathcal{A}}) = 1$ can be substituted with $\mathbf{R}(\mathsf{x}_{\mathcal{A}}, \mathsf{w}_{\mathcal{A}}) = 0$, where $\mathsf{w}_{\mathcal{A}}$, returned by a possibly unbounded extractor, is either a witness to $\mathsf{x}_{\mathcal{A}}$ (if there exists any) or $\perp$ (if there is none). More precisely,

$$\mathsf{ssnd} = \Pr\left[\begin{array}{l|l} \mathsf{V}_{\mathsf{FS}}(\mathbf{R}, \mathsf{srs}, \mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}) = 1, & \mathsf{srs} \leftarrow \mathsf{KGen}(\mathbf{R}), r \leftarrow\!\!\$\, \mathsf{R}(\mathcal{A}), \\ (\mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}) \notin Q, & (\mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}) \leftarrow \mathcal{A}^{\mathsf{Sim}_{\mathsf{FS}}, \mathcal{H}}(\mathbf{R}, \mathsf{srs}; r) \\ \mathbf{R}(\mathsf{x}_{\mathcal{A}}, \mathsf{w}_{\mathcal{A}}) = 0 & \mathsf{w}_{\mathcal{A}} \leftarrow \mathsf{Ext}(\mathbf{R}, \mathsf{srs}, \mathcal{A}, r, \mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}, Q, Q_{\mathcal{H}},) \end{array}\right].$$

The only necessary input to the unbounded extractor Ext is the instance $\mathsf{x}_{\mathcal{A}}$ (the rest is given for the consistency with the simulation extractability definition). With the probabilities in Definition 2 holding regardless of whether the extractor is unbounded or not, we obtain the following equality $\mathsf{ssnd} = \mathsf{acc} - \mathsf{ext}$.

3 Definitions and lemmas for multi-round SRS-based protocols

The result of Faust et al. [25] do not apply to our setting since the protocols we consider have and SRS and more than three messages, require more than just two transcripts for standard model extraction and are not special sound. We thus adapt special soundness to forking soundness, and generalize the forking lemma and the unique response property to make them compatible with multi-round SRS-based protocols.

3.1 Generalised forking lemma.

First of all, although dubbed "general", Lemma 7 is not general enough for our purpose as it is useful only for protocols where witness can be extracted from just two transcripts. To be able to extract a witness from, say, an execution of $\mathbf{P}$ we need to obtain at least $(3\mathsf{n}+1)$ valid proofs, and $(\mathsf{n}+\mathsf{Q}+1)$ for $\mathbf{S}$. Here we propose a generalisation of the general forking lemma that given probability of producing an accepting transcript acc lower-bounds the probability of generating a *tree of accepting transcripts* T, which allows to extract a witness.

Definition 4 (Tree of accepting transcripts, cf. [13]). *Consider a* $(2\mu+1)$*-message interactive proof system* $\mathbf{\Psi}$*. An* $(n_1, \ldots, n_\mu)$*-tree of accepting transcript*

$\mathsf{GF}^m_{\mathcal{Z}}(y, h^1_1, \ldots, h^1_q)$

$\rho \leftarrow\!\$\, \mathsf{R}(\mathcal{Z})$
$(i, s_1) \leftarrow \mathcal{Z}(y, h^1_1, \ldots, h^1_q; \rho)$
$i_1 \leftarrow i$
if $i = 0$ **return** $(0, \bot)$
for $j \in [2\,..\,m]$
  $h^j_1, \ldots, h^j_{i-1} \leftarrow h^{j-1}_1, \ldots, h^{j-1}_{i-1}$
  $h^j_i, \ldots, h^j_q \leftarrow\!\$\, H$
  $(i_j, s_j) \leftarrow \mathcal{Z}(y, h^j_1, \ldots, h^j_{i-1}, h^j_i, \ldots, h^j_q; \rho)$
  if $i_j = 0 \vee i_j \neq i$ **return** $(0, \bot)$
if $\exists (j, j') \in [1\,..\,m]^2, j \neq j' : (h^j_i = h^{j'}_i)$ **return** $(0, \bot)$
else return $(1, \boldsymbol{s})$

Fig. 1: Generalised forking algorithm $\mathsf{GF}^m_{\mathcal{Z}}$

is a tree where each node on depth i, for $i \in [1\,..\,\mu + 1]$, is an i-th prover's message in an acceptable transcript; edges between the nodes are labeled with verifier's challenges, such that no two edges on the same depth have the same label; and each node on depth i has $n_i - 1$ siblings and n_{i+1} children. Altogether, the tree consists of $N = \prod_{i=1}^{\mu} n_i$ branches, which makes N acceptable transcripts. We require $N = \mathsf{poly}(\lambda)$.

Lemma 2 (General forking lemma II). *Fix $q \in \mathbb{Z}$ and set H of size $h \geq m$. Let $\mathcal{Z}$ be a PPT algorithm that on input $y, h_1, \ldots, h_q$ returns (i, s) where $i \in [0\,..\,q]$ and s is called a side output. Denote by IG a randomised instance generator. We denote by acc the probability*

$$\Pr[i \neq 0 \mid y \leftarrow \mathsf{IG};\ h_1, \ldots, h_q \leftarrow\!\$\, H;\ (i, s) \leftarrow \mathcal{Z}(y, h_1, \ldots, h_q)]\,.$$

Let $\mathsf{GF}^m_{\mathcal{Z}}$ denote the algorithm described in Fig. 1 then the probability $\mathsf{frk} := \Pr[b = 1 \mid y \leftarrow \mathsf{IG};\ h_1, \ldots, h_q \leftarrow\!\$\, H;\ (b, \boldsymbol{s}) \leftarrow \mathsf{GF}^m_{\mathcal{Z}}(y, h_1, \ldots, h_q)]$ is at least

$$\frac{\mathsf{acc}^m}{q^{m-1}} - \mathsf{acc} \cdot \left(1 - \frac{h!}{(h-m)! \cdot h^m}\right).$$

The proof goes similarly to [8, Lemma 1] with some modifications required by the fact that the protocol has more than 3 rounds and the number of transcripts required is larger. Due to page limit the proof is presented in Supp. Mat. C.4.

To highlight importance of the generalised forking lemma we describe how we use it in our forking simulation-extractability proof. Let $\boldsymbol{\Psi}$ be a forking sound proof system where for an instance x the corresponding witness can be extracted from an $(1, \ldots, 1, n_k, 1, \ldots, 1)$-tree of accepting transcripts. Let $\mathcal{A}$ be

the simulation-extractability adversary that outputs an acceptable proof with probability at least acc. (Although we use the same acc to denote probability of $\mathcal{Z}$ outputing a non-zero i and probability of $\mathcal{A}$ outputing an acceptable proof we claim that these probabilities are exactly the same what comes from how we define $\mathcal{Z}$.) Let $\mathcal{A}$ produce an acceptable proof $\pi_{\mathcal{A}}$ for instance $\mathsf{x}_{\mathcal{A}}$; r be $\mathcal{A}$'s randomness; Q the list of queries submitted by $\mathcal{A}$ along with simulator's Sim answers; and $Q_{\mathcal{H}}$ be the list of all random oracle queries made by $\mathcal{A}$. All of these are given to the extractor Ext that internally runs the forking algorithm $\mathsf{GF}^{n_k}_{\mathcal{Z}}$. Algorithm $\mathcal{Z}$ takes $(\mathbf{R}, \mathsf{srs}, \mathcal{A}, Q, r)$ as input y and $Q_{\mathcal{H}}$ as input $h^1_1, \ldots, h^1_q$. (For the sake of completeness, we allow $\mathsf{GF}^{n_k}_{\mathcal{Z}}$ to pick $h^1_{l+1}, \ldots, h^1_q$ responses if $Q_{\mathcal{H}}$ has only $l < q$ elements.)

Next, $\mathcal{Z}$ runs internally $\mathcal{A}(\mathbf{R}, \mathsf{srs}; r)$ and responds to its random oracle and simulator queries by using $Q_{\mathcal{H}}$ and Q. Note that $\mathcal{A}$ makes the same queries as it did before it output $(\mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}})$ as it is run on the same random tape and with the same answers from the simulator and random oracle. After $\mathcal{A}$ finishes its acceptable proof $\pi_{\mathcal{A}}$, algorithm $\mathcal{Z}$ outputs $(i, \pi_{\mathcal{A}})$, where i is the index of a random oracle query submitted by $\mathcal{A}$ to get the challenge after k-th message from the prover—a message where the tree of transcripts branches. Then, after the first run of $\mathcal{A}$ is done, the extractor runs $\mathcal{Z}$ again, but this time it provides fresh random oracle responses $h^2_i, \ldots, h^2_q$. Note that this is equivalent to rewinding $\mathcal{A}$ to a point just before $\mathcal{A}$ is about to ask its i-th random oracle query. The probability that the adversary produces an acceptable transcript with the fresh random oracle responses is at least acc. This continues until the required number of transcripts is obtained.

We note that in the original forking lemma the forking algorithm F, cf. Fig. 4, gets only as input y and elements $h^1_1, \ldots, h^1_q$ are randomly picked from H internally by F. However, assuming that $h^1_1, \ldots, h^1_q$ are random oracle responses, and thus random, makes the change only notational.

We also note that the general forking lemma proposed in Lemma 2 works for protocols with an extractor that can obtain the witness from a $(1, \ldots, 1, n_k, 1, \ldots, 1)$-tree of acceptable transcripts. This limitation however does not affect the main result of this paper, i.e. showing that both Plonk and Sonic are forking simulation extractable.

3.2 Unique-response protocols

Another problem comes with another assumption required by Faust et al.—the unique response property of the transformed sigma protocol. The original Fischlin's formulation, although suitable for applications presented in [25, 26], is not enough in our case. First of all, the property assumes that the protocol has three messages, with the middle being the challenge from the verifier. That is not the case we consider here. Second, it is not entirely clear how to generalize the property. Should one require that after the first challenge from the verifier the prover's responses are fixed? That could not work since the prover needs to answer differently on different verifier's challenges, as otherwise the protocol could have fewer rounds. Another problem arises when the protocol contains some message, but not the first one, where the prover randomises its message.

In that case unique-responsiveness can not hold as well. Last but not least, the protocols we consider here are not designed to be in the standard model, but utilises SRS what also complicates things considerably.

We walk around these obstacles by providing a generalised notion of the unique response property. More precisely, we say that a $(2\mu+1)$-message protocol has *unique responses from* i, and call it an i-ur-protocol, if it follows the definition below:

Definition 5 (i-ur-protocol). *Let $\boldsymbol{\Psi}$ be a $(2\mu+1)$-message public coin proof system $\boldsymbol{\Psi} = (\mathsf{KGen}, \mathsf{P}, \mathsf{V}, \mathsf{Sim})$. Let $\boldsymbol{\Psi}_{\mathsf{FS}}$ be $\boldsymbol{\Psi}$ after the Fiat–Shamir transform and $\mathcal{H}$ the random oracle. Denote by $a_1, \ldots, a_\mu, a_{\mu+1}$ protocol messages output by the prover, We say that $\boldsymbol{\Psi}$ has* unique responses from i on *if for any* PPT *adversary $\mathcal{A}$:*

$$\Pr\left[\begin{array}{l|l} \mathsf{x}, \boldsymbol{a} = (a_1, \ldots, a_{\mu+1}), \boldsymbol{a'} = (a'_1, \ldots, a'_{\mu+1}) \leftarrow \mathcal{A}^{\mathcal{H}}(\mathbf{R}, \mathsf{srs}), & \\ \boldsymbol{a} \neq \boldsymbol{a'}, a_1, \ldots, a_i = a'_1, \ldots, a'_i, & \mathsf{srs} \leftarrow \mathsf{KGen}_{\mathsf{FS}}(\mathbf{R}) \\ \mathsf{V}^{\mathcal{H}}_{\mathsf{FS}}(\mathbf{R}, \mathsf{srs}, \mathsf{x}, \boldsymbol{a}) = \mathsf{V}^{\mathcal{H}}_{\mathsf{FS}}(\mathbf{R}, \mathsf{srs}, \mathsf{x}, \boldsymbol{a'}) = 1 & \end{array}\right]$$

is upper-bounded by some negligible function $\mathsf{negl}(\lambda)$.

Intuitively, a protocol is i-ur if it is infeasible for a PPT adversary to produce a pair of acceptable and different proofs π, π' that are the same on first i messages. We note that the definition above is also meaningful for protocols without an SRS. Intuitively in that case srs is the empty string.

3.3 Forking soundness

Note that the special soundness property (as usually defined) holds for all—even computationally unbounded—adversaries. Unfortunately, since a simulation trapdoors for $\mathbf{P}$ and $\mathbf{S}$ exist, the protocols cannot be special sound in that regard. This comes since an unbounded adversary could reveal the trapdoor and build a number of simulated proofs for a fake statement. Hence, we provide a weaker, yet sufficient, definition of *forking soundness*. More precisely, we state that an adversary that is able to answer correctly multiple challenges either knows the witness or can be used to break some computational assumption.

However, differently from the standard definition of special soundness, we do not require from the extractor to be able to extract the witness from *any* tree of acceptable transcripts. We require that the tree is produced honestly, e.g. all challenges are picked randomly—exactly as an honest verifier would pick them. Intuitively, the tree is as it would be generated by a GF algorithm from the generalized forking lemma.

Definition 6 ($(\varepsilon(\lambda), k, n)$-forking soundness). *Let $\boldsymbol{\Psi} = (\mathsf{KGen}, \mathsf{P}, \mathsf{V}, \mathsf{Sim})$ be an $(2\mu+1)$-message proof system for a relation $\mathbf{R}$.*

Let $\mathcal{T}$, called tree creator, be the algorithm below that rewinds the PPT *adversary $\mathcal{A}^{\mathsf{Sim}_{\mathsf{FS}}, \mathcal{H}}(\mathbf{R}, \mathsf{srs}; r)$ to produce a $(1, \ldots, n, \ldots, 1)$-tree of transcripts such that none of the n challenges in round k were used in simulated proofs.*

$\mathcal{T}$ has oracle access to $\mathcal{A}$ and provides it with (oracle) access to random oracle $\mathcal{H}$ and simulator $\mathsf{Sim}_{\mathsf{FS}}$ – more precisely $\mathcal{T}$ has an internal procedure $\mathcal{B}$ that provided srs and random oracle queries' responses $h_1, \ldots, h_1$ gives $\mathcal{A}$ access to the random oracle and simulates proof for it. In the end, $\mathcal{B}$ returns the index i of the random oracle query made for challenge k, the set Q of simulator random oracle indexes, the instance x, and the proof $\mathcal{A}$ returns. Eventually, $\mathcal{T}$ returns a $(1, \ldots, n, \ldots, 1)$ tree of acceptable transcripts T.

$\mathcal{T}(\mathcal{A}, \mathsf{srs} \leftarrow\!\$\, \mathsf{KGen}(\mathbf{R}))$

$h_1^1, \ldots, h_q^1 \leftarrow\!\$\, H$
$(i, Q, \mathsf{x}, \pi_1) \leftarrow \mathcal{B}(\mathcal{A}, \mathsf{srs}, h_1^1, \ldots, h_q^1)$
if $i \in Q \vee \mathsf{V}(\mathsf{srs}, \mathsf{x}, \pi_1) = 0$ **return** $(0, \bot)$
for $j \in [2\,..\,m]$
  $h_1^j, \ldots, h_{i-1}^j \leftarrow h_1^{j-1}, \ldots, h_{i-1}^{j-1}$
  $h_i^j, \ldots, h_q^j \leftarrow\!\$\, H$
  $(i_j, Q_j, \mathsf{x}_j, \pi_j) \leftarrow \mathcal{B}(\mathcal{A}, \mathsf{srs}, h_1^j, \ldots, h_{i-1}^j, h_i^j, \ldots, h_q^j)$
  if $i \neq i_j \vee i_j \in Q_j \vee \mathsf{x} \neq \mathsf{x}_j \vee \mathsf{V}(\mathsf{srs}, \mathsf{x}_j, \pi_i) = 0$ **return** $(0, \bot)$
else return $(1, \mathsf{T} = (\mathsf{x}, \boldsymbol{\pi}))$

We say that $\boldsymbol{\Psi}$ is $(\varepsilon(\lambda), k, n)$-forking sound if for any PPT adversary the probability that

$$\Pr\left[\mathsf{w} \leftarrow \mathsf{Ext}_{\mathsf{tree}}(\mathsf{T}), \mathbf{R}(\mathsf{T.x}, \mathsf{w}) = 0 \mid \mathsf{srs} \leftarrow\!\$\, \mathsf{KGen}(\mathbf{R}), (1, \mathsf{T}) \leftarrow \mathcal{T}(\mathcal{A}, \mathsf{srs})\right] \leq \varepsilon(\lambda).$$

4 Simulation soundness and forking simulation-extractability—the general result

Equipped with definitional framework of Section 3 we are ready to present the main result of this paper—a proof of simulation soundness and forking simulation extractability of Fiat-Shamir NIZK based on multi-round protocols.

The proofs go by game hoping. The games are controlled by an environment $\mathcal{E}$ that internally runs a simulation extractability adversary $\mathcal{A}$, provides it with access to a random oracle and simulator, and when necessary rewinds it. The games differ by various breaking points, i.e. points where the environment decides to abort the game.

Denote by $\pi_{\mathcal{A}}, \pi_{\mathsf{Sim}}$ proofs returned by the adversary and the simulator respectively. We use $\pi[i]$ to denote prover's message in the i-th round of the proof (counting from 1), i.e. $(2i-1)$-th message exchanged in the protocol. $\pi[i].\mathsf{ch}$ denotes the challenge that is given to the prover after $\pi[i]$, and $\pi[i..j]$ to denote all messages of the proof including challenges between rounds i and j, but not challenge $\pi[j].\mathsf{ch}$. When it is not explicitly stated we denote the proven instance x by $\pi[0]$ (however, there is no following challenge $\pi[0].\mathsf{ch}$).

Without loss of generality, we assume that whenever the accepting proof contains a response to a challenge from a random oracle, then the adversary queried the oracle to get it. It is straightforward to transform any adversary that violates this condition into an adversary that makes these additional queries to the random oracle and wins with the same probability.

Theorem 1 (Simulation soundness). *Assume that* $\mathbf{\Psi}$ *is* k*-programmable HVZK in the standard model, that is* $\varepsilon_{\mathsf{s}}(\lambda)$*-sound and* k-ur *with security* $\varepsilon_{\mathsf{ur}}(\lambda)$*. Then the probability that a* PPT *adversary* $\mathcal{A}$ *breaks simulation soundness of* $\mathbf{\Psi}_{\mathsf{FS}}$ *is upper-bounded by* $\varepsilon_{\mathsf{ur}}(\lambda) + q_{\mathcal{H}}^{\mu}\varepsilon_{\mathsf{s}}(\lambda)$ *, where* q *is the total number of queries made by the adversary* $\mathcal{A}$ *to a random oracle* $\mathcal{H}\colon \{0,1\}^* \to \{0,1\}^{\lambda}$*.*

Proof. **Game** G_0**:** This is a simulation soundness game played between an adversary $\mathcal{A}$ who has given access to a random oracle $\mathcal{H}$ and simulator $\mathbf{\Psi}_{\mathsf{FS}}.\mathsf{Sim}$. $\mathcal{A}$ wins if it manages to produce an acceptable proof for a false statement. In the following game hops we upper-bound the probability that this happens.

Game G_1**:** This is identical to G_0 except now the game is aborted if there is a simulated proof π_{Sim} for $\mathsf{x}_{\mathcal{A}}$ such that $(\mathsf{x}_{\mathcal{A}}, \pi_{\mathsf{Sim}}[1..k]) = (\mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}[1..k])$. That is, the adversary in its final proof reuses at least k messages from a simulated proof it saw before and the proof is acceptable. Denote that event by $\mathsf{Err}_{\mathsf{ur}}$.

$\mathsf{G}_0 \mapsto \mathsf{G}_1$**:** We have, $\Pr\left[\mathsf{G}_0 \wedge \overline{\mathsf{Err}_{\mathsf{ur}}}\right] = \Pr\left[\mathsf{G}_1 \wedge \overline{\mathsf{Err}_{\mathsf{ur}}}\right]$ and, from the difference lemma, cf. Lemma 1, $|\Pr[\mathsf{G}_0] - \Pr[\mathsf{G}_1]| \leq \Pr[\mathsf{Err}_{\mathsf{ur}}]$. Thus, to show that the transition from one game to another introduces only minor change in probability of $\mathcal{A}$ winning it should be shown that $\Pr[\mathsf{Err}_{\mathsf{ur}}]$ is small.

We can assume that $\mathcal{A}$ queried the simulator on the instance it wishes to output, i.e. $\mathsf{x}_{\mathcal{A}}$. We show a reduction $\mathcal{R}_{\mathsf{ur}}$ that utilises $\mathcal{A}$ to break the k-ur property of $\mathbf{\Psi}$. Let $\mathcal{R}_{\mathsf{ur}}$ run $\mathcal{A}$ internally as a black-box:

- The reduction answers both queries to the simulator $\mathbf{\Psi}_{\mathsf{FS}}.\mathsf{Sim}$ and to the random oracle. It also keeps lists Q, for the simulated proofs, and $Q_{\mathcal{H}}$ for the random oracle queries.
- When $\mathcal{A}$ makes a fake proof $\pi_{\mathcal{A}}$ for $\mathsf{x}_{\mathcal{A}}$, $\mathcal{R}_{\mathsf{ur}}$ looks through lists Q and $Q_{\mathcal{H}}$ until it finds $\pi_{\mathsf{Sim}}[0..k]$ such that $\pi_{\mathcal{A}}[0..k] = \pi_{\mathsf{Sim}}[0..k]$ and a random oracle query $\pi_{\mathsf{Sim}}[k].\mathsf{ch}$ on $\pi_{\mathsf{Sim}}[0..k]$.
- $\mathcal{R}_{\mathsf{ur}}$ returns two proofs for $\mathsf{x}_{\mathcal{A}}$:

$$\pi_1 = (\pi_{\mathsf{Sim}}[1..k], \pi_{\mathsf{Sim}}[k].\mathsf{ch}, \pi_{\mathsf{Sim}}[k+1..\mu+1])$$
$$\pi_2 = (\pi_{\mathsf{Sim}}[1..k], \pi_{\mathsf{Sim}}[k].\mathsf{ch}, \pi_{\mathcal{A}}[k+1..\mu+1])$$

If $\pi_1 = \pi_2$, then $\mathcal{A}$ fails to break simulation soundness, as $\pi_2 \in Q$. On the other hand, if the proofs are not equal, then $\mathcal{R}_{\mathsf{ur}}$ breaks k-ur-ness of $\mathbf{\Psi}$. This happens only with negligible probability $\varepsilon_{\mathsf{ur}}(\lambda)$, hence $\Pr[\mathsf{Err}_{\mathsf{ur}}] \leq \varepsilon_{\mathsf{ur}}(\lambda)$.

Game G_2**:** This is identical to G_1 except that now the environment aborts if the instance the adversary proves is not in the language.

$\mathsf{G}_1 \mapsto \mathsf{G}_2$**:** We show that $|\Pr[\mathsf{G}_1] - \Pr[\mathsf{G}_2]| \leq q^{\mu} \cdot \varepsilon_{\mathsf{s}}(\lambda)$, where $\varepsilon_{\mathsf{s}}(\lambda)$ is the probability of breaking soundness of the underlying *interactive* protocol $\mathbf{\Psi}$. Note that

$|\Pr[\mathsf{G}_1] - \Pr[\mathsf{G}_2]|$ is the probability that $\mathcal{A}$ outputs an acceptable proof for a false statement which does not break the unique response property (such proofs have been excluded by G_1). Consider a soundness adversary $\mathcal{A}'$ who initiates a proof with $\boldsymbol{\Psi}$'s verifier $\boldsymbol{\Psi}.\mathsf{V}$, internally runs $\mathcal{A}$ and proceeds as follows:

- It guesses indices $i_1, \ldots, i_\mu$ such that random oracle queries $h_{i_1}, \ldots, h_{i_\mu}$ are the queries used in the $\pi_{\mathcal{A}}$ proof eventually output by $\mathcal{A}$. This is done with probability at least $1/q^\mu$ (since there are μ challenges from the verifier in $\boldsymbol{\Psi}$).
- On input h for the i-th, $i \notin \{i_1, \ldots, i_\mu\}$, random oracle query, $\mathcal{A}'$ returns randomly picked y, sets $\mathcal{H}(h) = y$ and stores (h, y) in $Q_{\mathcal{H}}$ if h is sent to $\mathcal{H}$ the first time. If that is not the case, $\mathcal{A}$ finds h in $Q_{\mathcal{H}}$ and returns the corresponding y.
- On input h_{i_j} for the i_j-th, $i_j \in \{i_1, \ldots, i_\mu\}$, random oracle query, $\mathcal{A}'$ parses h_{i_j} as a partial proof transcript $\pi_{\mathcal{A}}[1..j]$ and runs $\boldsymbol{\Psi}$ using $\pi_{\mathcal{A}}[j]$ as a $\boldsymbol{\Psi}.\mathsf{P}$'s j-th message to $\boldsymbol{\Psi}.\mathsf{V}$. The verifier responds with a challenge $\pi_{\mathcal{A}}[j].\mathsf{ch}$. $\mathcal{A}'$ sets $\mathcal{H}(h_{i_j}) = \pi_{\mathcal{A}}[j].\mathsf{ch}$. If we guessed the indices correctly we have that $h_{i_{j'}}$, for $j' \leq j$, parsed as $\pi_{\mathcal{A}}[1..j']$ is a prefix of $\pi_{\mathcal{A}}[1..j]$.
- On query $\mathsf{x}_{\mathsf{Sim}}$ to Sim, $\mathcal{A}'$ runs the simulator $\boldsymbol{\Psi}.\mathsf{Sim}$ internally. Note that we require a simulator that only programs the random oracle for $j \geq k$. If the simulator makes a previously unanswered random oracle query with input $\pi_{\mathsf{Sim}}[1..j]$, $1 \leq j < k$, and this is the i_j-th query, it generates $\pi_{\mathsf{Sim}}[j].\mathsf{ch}$ by invoking $\boldsymbol{\Psi}.\mathsf{V}$ on $\pi_{\mathsf{Sim}}[j]$ and programs $\mathcal{H}(h_{i_j}) = \pi_{\mathsf{Sim}}[j].\mathsf{ch}$. It returns π_{Sim}.
- Answers $\boldsymbol{\Psi}.\mathsf{V}$'s final challenge $\pi_{\mathcal{A}}[\mu].\mathsf{ch}$ using the answer given by $\mathcal{A}$, i.e. $\pi_{\mathcal{A}}[\mu]$.

That is, $\mathcal{A}'$ manages to break soundness of $\boldsymbol{\Psi}$ if $\mathcal{A}$ manages to break simulation soundness without breaking the unique response property and $\mathcal{A}'$ correctly guesses the indices of $\mathcal{A}$ random oracle queries. This happens with probability upper-bounded by $|\Pr[\mathsf{G}_1] - \Pr[\mathsf{G}_2]| \cdot 1/q^\mu$. Hence $|\Pr[\mathsf{G}_1] - \Pr[\mathsf{G}_2]| \leq q^\mu \cdot \varepsilon_{\mathsf{s}}(\lambda)$.

Note that in G_2 the adversary cannot win. Thus the probability that $\mathcal{A}_{\mathsf{ss}}$ is successful is upper-bounded by $\varepsilon_{\mathsf{ur}}(\lambda) + q^\mu \cdot \varepsilon_{\mathsf{s}}(\lambda)$. □

We conjecture that based on the recent results on state restoration attack secure soundness [31], which effectively allows to query the verifier multiple times on different overlapping transcripts the q^μ loss could be avoided. However, this would reduce the class of protocols covered by our results.

Theorem 2 (Forking simulation-extractable multi-message protocols). *Let $\boldsymbol{\Psi} = (\mathsf{KGen}, \mathsf{P}, \mathsf{V}, \mathsf{Sim})$ be an interactive $(2\mu + 1)$-message proof system for $\mathcal{R}(1^\lambda)$ that is honest verifier zero-knowledge in the standard model[3], has k-ur property with security $\varepsilon_{\mathsf{ur}}(\lambda)$, and is $(\varepsilon_{\mathsf{s}}(\lambda), k, n)$-forking sound. Let $\mathcal{H}\colon \{0,1\}^* \to \{0,1\}^\lambda$ be a random oracle. Then $\boldsymbol{\Psi}_{\mathsf{FS}}$ is forking simulation-extractable with extraction error $\varepsilon_{\mathsf{ur}}(\lambda)$ against PPT algebraic adversaries that makes up to q random oracle queries and returns an acceptable proof with probability at least acc.*

[3] Crucially, we require that one can provide an indistinguishable simulated proof without any additional knowledge, as e.g knowledge of a SRS trapdoor.

The extraction probability ext *is at least* $\mathsf{ext} \geq \frac{1}{q^{n-1}}(\mathsf{acc} - \varepsilon_{\mathsf{ur}}(\lambda))^n - \varepsilon(\lambda)$, *for some negligible* $\varepsilon(\lambda)$.

Proof. **Game G_0:** This is a simulation extraction game played between an adversary $\mathcal{A}$ who has given access to a random oracle $\mathcal{H}$ and simulator $\mathbf{\Psi}_{\mathsf{FS}}.\mathsf{Sim}$. There is also an extractor Ext that, from a proof $\pi_{\mathcal{A}}$ for instance $\mathsf{x}_{\mathcal{A}}$ output by the adversary and from transcripts of $\mathcal{A}$'s operations is tasked to extract a witness $\mathsf{w}_{\mathcal{A}}$ such that $\mathbf{R}(\mathsf{x}_{\mathcal{A}}, \mathsf{w}_{\mathcal{A}})$ holds. $\mathcal{A}$ wins if it manages to produce an acceptable proof and the extractor fails to reveal the corresponding witness. In the following game hops we upper-bound the probability that this happens.

Game G_1: This is identical to G_0 except that now the game is aborted if there is a simulated proof π_{Sim} for $\mathsf{x}_{\mathcal{A}}$ such that $(\mathsf{x}_{\mathcal{A}}, \pi_{\mathsf{Sim}}[1..k]) = (\mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}[1..k])$. That is, the adversary in its final proof reuses at least k messages from a simulated proof it saw before and the proof is acceptable. Denote that event by $\mathsf{Err}_{\mathsf{ur}}$.

$\mathsf{G}_0 \mapsto \mathsf{G}_1$**:** $\Pr[\mathsf{Err}_{\mathsf{ur}}] \leq \varepsilon_{\mathsf{ur}}(\lambda)$. The proof goes exactly as in Corollary 2.

Game G_2: This is identical to G_1 except that now the environment aborts also when it fails to build a $(1, \ldots, 1, n, 1, \ldots, 1)$-tree of accepting transcripts T by rewinding $\mathcal{A}$. Denote that event by $\mathsf{Err}_{\mathsf{frk}}$.

$\mathsf{G}_1 \mapsto \mathsf{G}_2$**:** Note that for every acceptable proof $\pi_{\mathcal{A}}$, we may assume that whenever $\mathcal{A}$ outputs in Round k message $\pi_{\mathcal{A}}[k]$, then the $(\mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}[1..k])$ random oracle query was made by the adversary, not the simulator[4], i.e. there is no simulated proof π_{Sim} on $\mathsf{x}_{\mathsf{Sim}}$ such that $(\mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}[1..k]) = (\mathsf{x}_{\mathsf{Sim}}, \pi_{\mathsf{Sim}}[1..k])$. Otherwise, the game would be already interrupted by the error event in Game G_1. As previously, $|\Pr[\mathsf{G}_1] - \Pr[\mathsf{G}_2]| \leq \Pr[\mathsf{Err}_{\mathsf{frk}}]$.

We describe our extractor Ext here. The extractor takes as input relation $\mathbf{R}$, SRS srs, $\mathcal{A}$'s code, its randomness r, the output instance $\mathsf{x}_{\mathcal{A}}$ and proof $\pi_{\mathcal{A}}$, as well as the list Q of simulated proofs (and their instances) and the list of random oracle queries and responses $Q_{\mathcal{H}}$. Then, Ext starts a forking algorithm $\mathsf{GF}^n_{\mathcal{Z}}(y, h_1, \ldots, h_q)$ for $y = (\mathbf{R}, \mathsf{srs}, \mathcal{A}, r, \mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}, Q)$ where we set $h_1, \ldots, h_q$ to be the consecutive queries from list $Q_{\mathcal{H}}$. We run $\mathcal{A}$ internally in $\mathcal{Z}$.

To assure that in the first execution of $\mathcal{Z}$ the adversary $\mathcal{A}$ produce the same $(\mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}})$ as in the extraction game, $\mathcal{Z}$ provides $\mathcal{A}$ with the same randomness r and answers queries to the random oracle and simulator with pre-recorded responses in $Q_{\mathcal{H}}$ and Q. Note, that since the view of the adversary when run inside $\mathcal{Z}$ is the same as its view with access to the real random oracle and simulator, it produces exactly the same output. After the first run, $\mathcal{Z}$ outputs the index i of a random oracle query that was used by $\mathcal{A}$ to compute the challenge $\pi[k].\mathsf{ch} = \mathcal{H}(\pi_{\mathcal{A}}[0..k])$ it had to answer in the $(k+1)$-th round and adversary's transcript, denoted by s_1 in GF's description. If no such query took place $\mathcal{Z}$ outputs $i = 0$.

Then new random oracle responses are picked for queries indexed by $i, \ldots, q$ and the adversary is rewound to the point just prior to when it gets the response

[4] [25] calls these queries *fresh*.

to RO query $\pi_{\mathcal{A}}[0..k]$. The adversary gets a random oracle response from a new set of responses $h_i^2, \ldots, h_q^2$. If the adversary requests a simulated proof after seeing h_i^2 then $\mathcal{Z}$ computes the simulated proof on its own. Eventually, $\mathcal{Z}$ outputs index i' of a query that was used by the adversary to compute $\mathcal{H}(\pi_{\mathcal{A}}[0..k])$, and a new transcript s_2. $\mathcal{Z}$ is run n times with different random oracle responses. If a tree T of n transcripts is built then Ext runs internally the tree extractor $\mathsf{Ext}_{\mathsf{tree}}(\mathsf{T})$ and outputs what it returns.

We emphasize here the importance of the unique response property. If it does not hold then in some j-th execution of $\mathcal{Z}$ the adversary could reuse a challenge that it learned from observing proofs in Q. In that case, $\mathcal{Z}$ would output $i = 0$, making the extractor fail. Fortunately, the case that the adversary breaks the unique response property has already been covered by the abort condition in G_1.

Denote by $\widetilde{\mathsf{acc}}$ the probability that $\mathcal{A}$ outputs a proof that is accepted and does not break k-ur-ness of $\boldsymbol{\Psi}$. Denote by $\widetilde{\mathsf{acc}}'$ the probability that algorithm $\mathcal{Z}$, defined in the lemma, produces an accepting proof with a fresh challenge after Round k. Given the discussion above, we can state that $\widetilde{\mathsf{acc}} = \widetilde{\mathsf{acc}}'$.

Next, from the generalised forking lemma, cf. Lemma 2, we get that

$$\Pr[\mathsf{Err}_{\mathsf{frk}}] \leq 1 - \widetilde{\mathsf{acc}} \cdot \left(\widetilde{\mathsf{acc}}^{n-1}/q^{n-1} + (2^\lambda)!/((2^\lambda - n)! \cdot (2^\lambda)^n) - 1 \right). \quad (1)$$

Game G_3: This game is identical to G_2 except that it aborts if $\mathsf{Ext}_{\mathsf{tree}}(\mathsf{T})$ run by Ext fails to extract the witness.

$\mathsf{G}_2 \mapsto \mathsf{G}_3$: Since $\boldsymbol{\Psi}$ is forking-sound the probability that $\mathsf{Ext}_{\mathsf{tree}}(\mathsf{T})$ fails is upper-bounded by $\varepsilon_{\mathsf{f}}(\lambda)$.

Since Game G_3 is aborted when it is impossible to extract the correct witness from T, hence the adversary $\mathcal{A}$ cannot win. Thus, by the game-hoping argument,

$$|\Pr[\mathsf{G}_0] - \Pr[\mathsf{G}_4]| \leq 1 - \left(\frac{\widetilde{\mathsf{acc}}^n}{q^{n-1}} + \widetilde{\mathsf{acc}} \cdot \frac{(2^\lambda)!}{(2^\lambda - n)! \cdot (2^\lambda)^n} - \widetilde{\mathsf{acc}} \right) + \varepsilon_{\mathsf{ur}}(\lambda) + \varepsilon_{\mathsf{f}}(\lambda).$$

Thus the probability that extractor $\mathsf{Ext}_{\mathsf{ss}}$ succeeds is at least

$$\frac{\widetilde{\mathsf{acc}}^n}{q^{n-1}} + \widetilde{\mathsf{acc}} \cdot \frac{(2^\lambda)!}{(2^\lambda - n)! \cdot (2^\lambda)^n} - \widetilde{\mathsf{acc}} - \varepsilon_{\mathsf{ur}}(\lambda) - \varepsilon_{\mathsf{f}}(\lambda).$$

Since $\widetilde{\mathsf{acc}}$ is probability of $\mathcal{A}$ outputting acceptable transcript that does not break k-ur-ness of $\boldsymbol{\Psi}$, then $\widetilde{\mathsf{acc}} \geq \mathsf{acc} - \varepsilon_{\mathsf{ur}}(\lambda)$, where acc is the probability of $\mathcal{A}$ outputing an acceptable proof as defined in Definition 2. It thus holds

$$\mathsf{ext} \geq \frac{(\mathsf{acc} - \varepsilon_{\mathsf{ur}}(\lambda))^n}{q^{n-1}} - \underbrace{(\mathsf{acc} - \varepsilon_{\mathsf{ur}}(\lambda)) \cdot \left(1 - \frac{(2^\lambda)!}{(2^\lambda - n)! \cdot (2^\lambda)^n} \right) - \varepsilon_{\mathsf{ur}}(\lambda) - \varepsilon_{\mathsf{f}}(\lambda)}_{\varepsilon(\lambda)}. \quad (2)$$

Note that the part of Eq. (2) denoted by $\varepsilon(\lambda)$ is negligible as $\varepsilon_{\mathsf{ur}}(\lambda), \varepsilon_{\mathsf{f}}(\lambda)$ are negligible, and $\frac{(2^\lambda)!}{(2^\lambda - n)! \cdot (2^\lambda)^n} \geq \left((2^\lambda - n)/2^\lambda\right)^n$ is overwhelming. Thus,

$$\mathsf{ext} \geq q^{-(n-1)} (\mathsf{acc} - \varepsilon_{\mathsf{ur}}(\lambda))^n - \varepsilon(\lambda).$$

and $\boldsymbol{\Psi}_{\mathsf{FS}}$ is forking simulation extractable with extraction error $\varepsilon_{\mathsf{ur}}(\lambda)$. □

5 Non-Malleability of $\mathbf{P}_{\mathsf{FS}}$

In this section we show that $\mathbf{P}_{\mathsf{FS}}$ is simulation-sound and forking simulation-extractable. To that end, we proceed as follows. First we show that the version of the KZG polynomial commitment scheme that is proposed in the Plonk paper has the unique opening property, cf. Section 2.3 and Lemma 3. This is then used to show that $\mathbf{P}$ has the 2-ur property, cf. Lemma 4.

Next, we show that $\mathbf{P}$ is forking-sound. That is, given a number of acceptable transcripts which match on the first 3 rounds of the protocol we can either reveal a correct witness for the proven statement or use one of the transcripts to break the dlog assumption. This result is shown in the AGM, cf. Lemma 5.

Given forking-soundness of $\mathbf{P}$, we use the fact that it is also 2-ur and show, in a similar fashion to [25], that it is simulation-extractable. That is, we build reductions that given a simulation extractability adversary $\mathcal{A}$ either break the protocol's unique response property or based on forking soundness break the dlog assumption, if extracting a valid witness from a tree of transcripts is impossible. See Corollary 1.

Due to page limit, we omit description of Plonk here and refer to Supp. Mat. B.1. Unfortunately, we also have to move some of the proofs to the Supplementary Materials as well, cf. Supp. Mat. B

5.1 Unique opening property of $\mathbf{PC}_{\mathsf{P}}$

Lemma 3. *Let $\mathbf{PC}_{\mathsf{P}}$ be a batched version of a batched KZG polynomial commitment, cf. Fig. 2, then $\mathbf{PC}_{\mathsf{P}}$ has the unique opening property in the AGM with security $\varepsilon_{\mathsf{op}}(\lambda) \leq 2\varepsilon_{\mathsf{dlog}}(\lambda)+1/|\mathbb{F}_p|$, where $\varepsilon_{\mathsf{dlog}}(\lambda)$ is security of the $(\mathsf{n}+2,1)$-dlog assumption and $\mathbb{F}_p$ is the field used in $\mathbf{PC}_{\mathsf{P}}$.*

The proof has been referred to Supp. Mat. B.2.

5.2 Unique response property

Lemma 4. *Let $\mathbf{PC}_{\mathsf{P}}$ be commitment of knowledge with security $\varepsilon_{\mathsf{k}}(\lambda)$, $\varepsilon_{\mathsf{bind}}(\lambda)$-binding and has unique opening property with security $\varepsilon_{\mathsf{op}}(\lambda)$, then probability that a PPT adversary $\mathcal{A}$ breaks $\mathbf{P}$'s 2-ur property is at most $\varepsilon_{\mathsf{k}}(\lambda) + 2 \cdot \varepsilon_{\mathsf{bind}}(\lambda) + \varepsilon_{\mathsf{op}}(\lambda)$.*

The proof has been referred to Supp. Mat. B.3.

5.3 Forking soundness

Lemma 5. *Assume an idealised $\mathbf{P}$ verifier fails at most with probability $\varepsilon_{\mathsf{id}}(\lambda)$ and the discrete logarithm advantage is bounded by $\varepsilon_{\mathsf{dlog}}(\lambda)$, $\mathbf{P}$ is $(\varepsilon_{\mathsf{f}}(\lambda), 3, 3\mathsf{n}+1)$-forking sound for $\varepsilon_{\mathsf{f}}(\lambda) \leq \varepsilon_{\mathsf{id}}(\lambda) + \varepsilon_{\mathsf{dlog}}(\lambda)$ against algebraic adversaries.*

Proof. Let srs be $\mathbf{P}$'s SRS and denote by srs_1 all SRS's $\mathbb{G}_1$-elements; that is, $\mathsf{srs}_1 = \left[1, \chi, \ldots, \chi^{\mathsf{n}+2}\right]_1$. Let $\mathcal{T}$ be an algebraic adversary that produces a statement x and a $(1, 1, 3\mathsf{n}+1, 1)$-tree of acceptable transcripts T. Note that in all transcripts the instance x, proof elements $[\mathsf{a}(\chi), \mathsf{b}(\chi), \mathsf{c}(\chi), \mathsf{z}(\chi), \mathsf{t}(\chi)]_1$ and challenges α, β, γ are common as the transcripts share the first three rounds. The tree

branches after the third round of the protocol where the challenge $\mathfrak{z}$ is presented, thus tree T is build using different values of $\mathfrak{z}$.

We consider two mutually disjunctive events. The first, E holds when all of the transcripts are acceptable by the idealised verification equation, i.e. $\mathsf{ve}_{\mathsf{x},\pi}(X) = 0$, cf. Eq. (5). The second, $\overline{\mathsf{E}}$ holds when there is a transcript that is acceptable by the real verifier, but not by the ideal verifier. That is, for that particular transcript it holds that $\mathsf{ve}_{\mathsf{x},\pi}(\chi) = 0$, but $\mathsf{ve}_{\mathsf{x},\pi}(X) \neq 0$. When event E holds we show that the extractor $\mathsf{Ext}_{\mathsf{tree}}$ is successful. If it does not hold, we show a reduction $\mathcal{R}_{\mathsf{dlog}}$ that breaks the dlog assumption.

When E holds: Let $(1, \mathsf{T}) \leftarrow \mathcal{T}(\mathbf{R}, \mathsf{srs})$ and x be a relation proven in T. Since the protocol $\mathbf{P}$, instantiated with the idealised verification equation, is perfectly sound, except with negligible probability of the idealised verifier failure $\varepsilon_{\mathsf{id}}(\lambda)$, for a valid proof π of a statement x there exists a witness w, such that $\mathbf{R}(\mathsf{x}, \mathsf{w})$ holds. Note that since the $\mathcal{T}$ produces $(3\mathsf{n}+1)$ acceptable transcripts for different challenges $\mathfrak{z}$, it obtains the same number of different evaluations of polynomials $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{z}, \mathsf{t}$. Since the transcripts are acceptable by an idealised verifier, the equality between polynomial t and combination of polynomials $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{z}$ described in Round 3 of the protocol holds. Hence, $\mathsf{a}, \mathsf{b}, \mathsf{c}$ encodes the valid witness for the proven statement. Since $\mathsf{a}, \mathsf{b}, \mathsf{c}$ are of degree at most $(\mathsf{n} + 2)$ and there is more than $(\mathsf{n} + 2)$ their evaluations known, $\mathsf{Ext}_{\mathsf{tree}}$ can recreate their coefficients by interpolation and reveal the witness with probability 1. Hence, the probability that extraction fails in that case is upper-bounded by probability of an idealised verifier failing $\varepsilon_{\mathsf{id}}(\lambda)$, which is negligible.

When $\overline{\mathsf{E}}$ holds: Let $\mathcal{T}$ be an adversary that for relation $\mathbf{R}$ and randomly picked $\mathsf{srs} \leftarrow\!\!\$\, \mathsf{KGen}(\mathbf{R})$ produces a tree of acceptable transcripts such that $\overline{\mathsf{E}}$ holds. Let $\mathcal{R}_{\mathsf{dlog}}$ be a reduction that gets as input an $(\mathsf{n}+2, 1)$-dlog instance $[1, \ldots, \chi^{\mathsf{n}}]_1, [\chi]_2$ and is tasked to output χ. The reduction proceeds as follows—it gives the input instance to the adversary as the SRS. Let $(1, \mathsf{T})$ be the output returned by $\mathcal{A}$. Let x be a relation proven in T. Consider a transcript $\pi \in \mathsf{T}$ such that $\mathsf{ve}_{\mathsf{x},\pi}(X) \neq 0$, but $\mathsf{ve}_{\mathsf{x},\pi}(\chi) = 0$. Since the adversary is algebraic, all group elements included in T are extended by their representation as a combination of the input $\mathbb{G}_1$-elements. Hence all coefficients of the verification equation polynomial $\mathsf{ve}_{\mathsf{x},\pi}(X)$ are known and $\mathcal{R}_{\mathsf{dlog}}$ can find its zero points. Since $\mathsf{ve}_{\mathsf{x},\pi}(\chi) = 0$, the targeted discrete log value χ is among them. Hence, the probability that this event happens is upper-bounded by $\varepsilon_{\mathsf{dlog}}(\lambda)$. □

5.4 Honest verifier zero-knowledge

Lemma 6. *Let $\mathbf{P}$ be zero knowledge with security $\varepsilon_{\mathsf{zk}}(\lambda)$. Let $(\mathsf{R}, \mathsf{S}, \mathsf{T}, \mathsf{f}, 1)$-uber assumption for $\mathsf{R}, \mathsf{S}, \mathsf{T}, \mathsf{f}$ as defined in Eq. (3) hold with security $\varepsilon_{\mathsf{uber}}(\lambda)$. Then $\mathbf{P}$ is computationally honest verifier zero-knowledge with simulator Sim that does not require a SRS trapdoor with security $\varepsilon_{\mathsf{zk}}(\lambda) + \varepsilon_{\mathsf{uber}}(\lambda)$.*[5]

[5] The simulator works as a simulator for proofs that are zero-knowledge in the standard model. However, we do not say that Plonk is HVZK in the standard model as proof of that *requires* the SRS simulator.

Proof. The proof goes by game-hoping. The environment that controls the games provides the adversary with a SRS srs, then the adversary outputs an instance–witness pair (x, w) and, depending on the game, is provided with either real or simulated proof for it. In the end of the game the adversary outputs either 0 if it believes that the proof it saw was provided by the simulator and 1 in the other case.

Game G_0: In this game $\mathcal{A}(\mathbf{R}, \mathsf{srs})$ picks an instance–witness pair (x, w) and gets a real proof π for it.

Game G_1: In this game for $\mathcal{A}(\mathbf{R}, \mathsf{srs})$ picks an instance–witness pair (x, w) and gets a proof π that is simulated by a simulator Sim_χ which utilises for the simulation the SRS trapdoor and proceeds as described in Supp. Mat. B.1. $\mathsf{G}_0 \mapsto$

G_1**:** Since Plonk is zero-knowledge, probability that $\mathcal{A}$ outputs a different bit in both games is negligible. Hence $|\Pr[\mathsf{G}_0] - \Pr[\mathsf{G}_1]| \leq \varepsilon_{\mathsf{zk}}(\lambda)$.

Game G_2: In this game $\mathcal{A}(\mathbf{R}, \mathsf{srs})$ picks an instance–witness pair (x, w) and gets a proof π simulated by the simulator Sim which proceeds as follows.

In Round 1 the simulator picks randomly both the randomisers $b_1, \ldots, b_6$ and sets $\mathsf{w}_i = 0$ for $i \in [1\,..\,3\mathsf{n}]$. Then Sim outputs $[\mathsf{a}(\chi), \mathsf{b}(\chi), \mathsf{c}(\chi)]_1$. For the first round challenge, the simulator picks permutation argument challenges β, γ randomly.

In Round 2, the simulator computes $\mathsf{z}(X)$ from the newly picked randomisers b_7, b_8, b_9 and coefficients of polynomials $\mathsf{a}(X), \mathsf{b}(X), \mathsf{c}(X)$. Then it evaluates $\mathsf{z}(X)$ honestly and outputs $[\mathsf{z}(\chi)]_1$. Challenge α that should be sent by the verifier after Round 2 is picked by the simulator at random.

In Round 3 the simulator starts by picking at random a challenge $\mathfrak{z}$, which in the real proof comes as a challenge from the verifier sent *after* Round 3. Then Sim computes evaluations $\mathsf{a}(\mathfrak{z}), \mathsf{b}(\mathfrak{z}), \mathsf{c}(\mathfrak{z}), \mathsf{S}_{\sigma 1}(\mathfrak{z}), \mathsf{S}_{\sigma 2}(\mathfrak{z}), \mathsf{PI}(\mathfrak{z}), \mathsf{L}_1(\mathfrak{z}), \mathsf{Z}_\mathsf{H}(\mathfrak{z})$, $\mathsf{z}(\mathfrak{z}\omega)$ and computes $\mathsf{t}(X)$ honestly. Since for a random $\mathsf{a}(X), \mathsf{b}(X), \mathsf{c}(X), \mathsf{z}(X)$ the constraint system is (with overwhelming probability) not satisfied and the constraints-related polynomials are not divisible by $\mathsf{Z}_\mathsf{H}(X)$, hence $\mathsf{t}(X)$ is a rational function rather than a polynomial. Then, the simulator evaluates $\mathsf{t}(X)$ at $\mathfrak{z}$ and picks randomly a degree-$(3\mathsf{n}-1)$ polynomial $\tilde{\mathsf{t}}(X)$ such that $\mathsf{t}(\mathfrak{z}) = \tilde{\mathsf{t}}(\mathfrak{z})$ and publishes a commitment $[\tilde{\mathsf{t}}_{\mathsf{lo}}(\chi), \tilde{\mathsf{t}}_{\mathsf{mid}}(\chi), \tilde{\mathsf{t}}_{\mathsf{hi}}(\chi)]_1$. After this round the simulator outputs $\mathfrak{z}$ as a challenge.

In the next round, the simulator computes polynomial $\mathsf{r}(X)$ as an honest prover would, cf. Supp. Mat. B.1 and evaluates $\mathsf{r}(X)$ at $\mathfrak{z}$.

The rest of the evaluations are already computed, thus Sim simply outputs $\mathsf{a}(\mathfrak{z}), \mathsf{b}(\mathfrak{z}), \mathsf{c}(\mathfrak{z}), \mathsf{S}_{\sigma 1}(\mathfrak{z}), \mathsf{S}_{\sigma 2}(\mathfrak{z}), \mathsf{t}(\mathfrak{z}), \mathsf{z}(\mathfrak{z}\omega)$. After that it picks randomly the challenge v, proceeds in the last round as an honest prover would proceed and outputs the final challenge, u, by picking it at random as well.

$\mathsf{G}_1 \mapsto \mathsf{G}_2$**:** We now describe the reduction $\mathcal{R}$ which relies on the $(\mathsf{R}, \mathsf{S}, \mathsf{T}, \mathsf{F}, 1)$-uber assumption, cf. Supp. Mat. C.2 where $\mathsf{R}, \mathsf{S}, \mathsf{T}, \mathsf{F}$ are polynomials over variables $\boldsymbol{B} = B_1, \ldots, B_9$ and are defined as follows. Let $E = \{\{2\}, \{3, 4\}, \{5, 6\}, \{7, 8, 9\}\}$

and $E' = E \setminus \{2\}$. Let

$$\begin{aligned}
\mathsf{F}(\boldsymbol{B}) &= \{B_1\} \cup \{B_1 B_i \mid i \in A,\ A \in E'\} \cup \{B_1 B_i B_j \mid i \in A, j \in B,\ A, B \in E', B \neq A\} \cup \\
&\quad \{B_1 B_i B_j B_k \mid i \in A, j \in B, k \in C,\ A, B, C \in E', A \neq B \neq C \neq A\}\,, \\
\mathsf{R}(\boldsymbol{B}) &= \{B_i \mid i \in A,\ A \in E\} \cup \{B_i B_j \mid i \in A, j \in B,\ A \neq B, A, B \in E\} \cup \\
&\quad \{B_i B_j B_k \mid i \in A,\ j \in B,\ k \in C,\ A, B, C \text{ all different and in } E\} \cup \\
&\quad \{B_i B_j B_k B_l \mid i \in A,\ j \in B,\ k \in C,\ l \in D,\ A, B, C, D \text{ all different and in } E\} \\
&\quad \setminus \mathsf{F}(\boldsymbol{B})\,, \\
\mathsf{S}(\boldsymbol{B}) &= \emptyset, \qquad \mathsf{T}(\boldsymbol{B}) = \emptyset.
\end{aligned} \tag{3} \tag{4}$$

That is, the elements of R are all singletons, pairs, triplets and quadruplets of B_i variables that occur in polynomial $\mathsf{t}(\boldsymbol{B})$ except the challenge element $\mathsf{f}(\boldsymbol{B})$ which are all elements that depends on a variable B_1. Variables $\boldsymbol{B}$ are evaluated to randomly picked $\boldsymbol{b} = b_1, \ldots, b_9$.

The reduction $\mathcal{R}$ learns $[\mathsf{R}]_1$ and challenge $[\boldsymbol{w}]_1 = [w_1, \ldots, w_{12}]_1$ where $\boldsymbol{w}$ is either a vector of evaluations $\mathsf{F}(\boldsymbol{b})$ or a sequence of random values $y_1, \ldots, y_{12}$, for the sake of concreteness we state $w_1 = b_1$ or $w_1 = y_1$ (depending on the chosen random bit). Then it picks $\chi, \mathfrak{z}$ and computes the SRS srs from χ. Elements b_i are interpreted as polynomials in X that are evaluated at χ, i.e. $b_i = b_i(\chi)$. Next, $\mathcal{R}$ sets for $\xi_i, \zeta_i \leftarrow\!\!\$\, \mathbb{F}_p$ $\left[\tilde{\mathsf{b}}_1(X)\right]_1 = (X-\mathfrak{z})(X-\omega\mathfrak{z})\,[w_1]_1\,(X) + \xi_i(X-\mathfrak{z})\,[1]_1 + \zeta_i(X-\omega\mathfrak{z})\,[1]_1$,, and $\left[\tilde{\mathsf{b}}_i(X)\right]_1 = (X-\mathfrak{z})(X-\omega\mathfrak{z})\,[b_i]_1\,(X) + \xi_i(X-\mathfrak{z})\,[1]_1 + \zeta_i(X-\omega\mathfrak{z})\,[1]_1$, for $i \in [2\,..\,9]$.

Denote by $\tilde{b}_i$ evaluations of $\tilde{\mathsf{b}}_i$ at χ. The reduction computes all $\left[\tilde{b}_i\tilde{b}_j\right]_1$, $\left[\tilde{b}_i\tilde{b}_j\tilde{b}_k\right]_1$, $\left[\tilde{b}_i\tilde{b}_j\tilde{b}_k\tilde{b}_l\right]_1$ such that $[B_iB_j, B_iB_jB_k, B_iB_jB_kB_l]_1 \in \mathsf{R}$. This is possible since $\mathcal{R}$ knows all singletons $[w_1, b_2, \ldots, b_9]_1$ and pairs $[b_ib_j]_1 \in \mathsf{R}$ which can be used to compute all required pairs $\left[\tilde{b}_i\tilde{b}_j\right]_1$:

$$\begin{aligned}
\left[\tilde{b}_i\tilde{b}_j\right]_1 &= ((\chi-\mathfrak{z})(\chi-\omega\mathfrak{z})\,[b_i]_1 + \xi_i(\chi-\mathfrak{z})\,[1]_1 + \zeta_i(\chi-\omega\mathfrak{z})\,[1]_1) \cdot \\
&\quad ((\chi-\mathfrak{z})(\chi-\omega\mathfrak{z})\,[b_j]_1 + \xi_j(\chi-\mathfrak{z})\,[1]_1 + \zeta_j(\chi-\omega\mathfrak{z})\,[1]_1) = \\
&\quad ((\chi-\mathfrak{z})(\chi-\omega\mathfrak{z}))^2\,[b_ib_j]_1 + ((\chi-\mathfrak{z})(\chi-\omega\mathfrak{z})\,[b_i]_1\,(\xi_j(\chi-\mathfrak{z})\,[1]_1 + \zeta_j(\chi-\omega\mathfrak{z})\,[1]_1) + \\
&\quad ((\chi-\mathfrak{z})(\chi-\omega\mathfrak{z})\,[b_j]_1\,(\xi_i(\chi-\mathfrak{z})\,[1]_1 + \zeta_i(\chi-\omega\mathfrak{z})\,[1]_1) + \psi,
\end{aligned}$$

where ψ compounds of $\xi_i, \xi_j, \zeta_i, \zeta_j, \mathfrak{z}, \omega\mathfrak{z}, \chi$ which are all known by $\mathcal{R}$ and no b_i nor b_j. Analogously for the triplets and quadruplets and elements dependent on $\boldsymbol{w}$.

Next the reduction runs the adversary $\mathcal{A}(\mathbf{R}, \mathsf{srs})$ and obtains from $\mathcal{A}$ an instance–witness pair (x, w). $\mathcal{R}$ now prepares a simulated proof as follows:

Round 1 $\mathcal{R}$ computes $[\mathsf{a}(\chi)]_1$ using as randomisers $\left[\tilde{b}_1\right]_1$, $\left[\tilde{b}_2\right]_1$ and setting $\mathsf{w}_i = 0$, for $i \in [1\,..\,3\mathsf{n}]$. Similarly it computes $[\mathsf{b}(\chi)]_1, [\mathsf{c}(\chi)]_1$. $\mathcal{R}$ publishes the obtained values and picks a Round 1 challenge β, γ at random. Note that regardless $w_1 = b_1$ or a random element, $[a(\chi)]_1$ is random. Thus $\mathcal{R}$'s output has the same distribution as output of a real prover.

Round 2 $\mathcal{R}$ computes $[\mathsf{z}(\chi)]_1$ using $\tilde{b}_7, \tilde{b}_8, \tilde{b}_9$ and publishes it. Then it picks randomly the challenge α. This round output is independent on b_1 thus $\mathcal{R}$'s output is indistinguishable from the prover's.

Round 3 The reduction computes $\mathsf{t}_{\mathsf{lo}}(\chi), \mathsf{t}_{\mathsf{mid}}(\chi), \mathsf{t}_{\mathsf{hi}}(\chi)$, which all depend on b_1. To that end $\left[\tilde{b}_1\right]_1$ is used. Note that if $\boldsymbol{w}$ is a vector of $\mathsf{F}(b_1, \ldots, b_9)$ evaluations then $[\mathsf{t}_{\mathsf{lo}}(\chi), \mathsf{t}_{\mathsf{mid}}(\chi), \mathsf{t}_{\mathsf{hi}}(\chi)]_1$ is the same as the real prover's. Alternatively, if $\boldsymbol{w}$ is a vector of random values, then $\mathsf{t}_{\mathsf{lo}}(\chi), \mathsf{t}_{\mathsf{mid}}(\chi), \mathsf{t}_{\mathsf{hi}}(\chi)$ are all random polynomials which evaluates at $\mathfrak{z}$ to the same value as the polynomials computed by the real prover. That is, in that case $\mathsf{t}_{\mathsf{lo}}(\chi), \mathsf{t}_{\mathsf{mid}}(\chi), \mathsf{t}_{\mathsf{hi}}(\chi)$ are as the simulator Sim would compute. Eventually, $\mathcal{R}$ outputs $\mathfrak{z}$.

Round 4 The reduction outputs $\mathsf{a}(\mathfrak{z}), \mathsf{b}(\mathfrak{z}), \mathsf{c}(\mathfrak{z}), \mathsf{S}_{\sigma 1}(\mathfrak{z}), \mathsf{S}_{\sigma 2}(\mathfrak{z}), \mathsf{t}(\mathfrak{z}), \mathsf{z}(\omega\mathfrak{z})$. For the sake of concreteness, denote by $S = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{t}, \mathsf{z}\}$. Although for a polynomial $\mathsf{p} \in S$, reduction $\mathcal{R}$ does not know $\mathsf{p}(\chi)$ or even do not know all the coefficients of p, the polynomials in S was computed such that the reduction always knows their evaluation at $\mathfrak{z}$ and $\omega\mathfrak{z}$.

Round 5 $\mathcal{R}$ computes the openings of the polynomial commitments assuring that evaluations at $\mathfrak{z}$ it provided were computed honestly.

If the adversary $\mathcal{A}$'s output distribution differ in Game G_1 and G_2 then the reduction uses it to distinguish between $\boldsymbol{w} = \mathsf{F}(b_1, \ldots, b_9)$ and $\boldsymbol{w}$ being random, thus $|\Pr[\mathsf{G}_1] - \Pr[\mathsf{G}_2]| \leq \varepsilon_{\mathsf{uber}}(\lambda)$. Eventually, $|\Pr[\mathsf{G}_0] - \Pr[\mathsf{G}_2]| \leq \varepsilon_{\mathsf{zk}}(\lambda) + \varepsilon_{\mathsf{uber}}(\lambda)$. □

5.5 Simulation soundness and simulation extractability of P_{FS}

Since Lemmas 4 and 5 hold, P is 2-ur and forking sound. We now make use of Corollary 2 and Theorem 2 and show that P_{FS} is simulation sound and forking simulation-extractable as defined in Section 2.6.

Corollary 1 (Forking simulation extractability of P_{FS}). *Assume an idealised* P *verifier fails at most with probability* $\varepsilon_{\mathsf{id}}(\lambda)$*, the discrete logarithm advantage is bounded by* $\varepsilon_{\mathsf{dlog}}(\lambda)$ *and the* PC_{P} *is a commitment of knowledge with security* $\varepsilon_{\mathsf{k}}(\lambda)$*, binding security* $\varepsilon_{\mathsf{bind}}(\lambda)$ *and has unique opening property with security* $\varepsilon_{\mathsf{op}}(\lambda)$*. Let* $\mathcal{H}\colon \{0,1\}^* \to \{0,1\}^\lambda$ *be a random oracle. Let* $\mathcal{A}$ *be a* PPT *adversary that can make up to* q *random oracle queries and outputs an acceptable proof for* P_{FS} *with probability at least* acc*. Then* P_{FS} *is forking simulation-extractable with extraction error* $\eta = \varepsilon_{\mathsf{ur}}(\lambda)$*. The extraction probability* ext *is at least*

$$\mathsf{ext} \geq \frac{1}{q^{3(\varepsilon_{\mathsf{id}}(\lambda)+\varepsilon_{\mathsf{dlog}}(\lambda))}}(\mathsf{acc} - \varepsilon_{\mathsf{k}}(\lambda) - 2 \cdot \varepsilon_{\mathsf{bind}}(\lambda) - \varepsilon_{\mathsf{op}}(\lambda))^{3\mathsf{n}+1} - \varepsilon(\lambda)\,,$$

for some negligible $\varepsilon(\lambda)$ *and* n *being the number of constrains in the proven circuit.*

Corollary 2 (Simulation soundness of P_{FS}). *Assume that* P *is* 2*-programmable HVZK in the standard model, that is* $\varepsilon_{\mathsf{s}}(\lambda)$*-sound and the* PC_{P} *is a commitment of knowledge with security* $\varepsilon_{\mathsf{k}}(\lambda)$*, binding security* $\varepsilon_{\mathsf{bind}}(\lambda)$ *and has unique opening property with security* $\varepsilon_{\mathsf{op}}(\lambda)$*. Then the probability that*

a PPT *adversary* $\mathcal{A}$ *breaks simulation soundness of* $\mathbf{\Psi}_{\mathsf{FS}}$ *is upper-bounded by* $\varepsilon_{\mathsf{k}}(\lambda) + 2 \cdot \varepsilon_{\mathsf{bind}}(\lambda) + \varepsilon_{\mathsf{op}}(\lambda) + q_{\mathcal{H}}^4 \varepsilon_{\mathsf{s}}(\lambda)$, *where* q *is the total number of queries made by the adversary* $\mathcal{A}$ *to a random oracle* $\mathcal{H}\colon \{0,1\}^* \to \{0,1\}^\lambda$.

6 Further work

We identify a number of problems which we left as further work. First of all, the generalised version of the forking lemma presented in this paper can be generalised even further to include protocols where forking soundness holds for protocols where $\mathsf{Ext}_{\mathsf{tree}}$ extracts a witness from a $(n_1, \ldots, n_\mu)$-tree of acceptable transcripts, where more than one $n_j > 1$. I.e. to include protocols that for witness extraction require transcripts that branch at more than one point.

Although we picked Plonk and Sonic as examples for our framework, it is not limited to SRS-based NIZKs. Thus, it would be interesting to apply it to known so-called transparent zkSNARKs like Bulletproofs [16], Aurora [10] or AuroraLight [28].

Since the rewinding technique and the forking lemma used to show simulation extractability of $\mathbf{P}_{\mathsf{FS}}$ and $\mathbf{S}_{\mathsf{FS}}$ come with security loss, it would be interesting to show SE of these protocols directly in the algebraic group model.

Although we focused here only on zkSNARKs, it is worth to investigating other protocols that may benefit from our framework, like e.g. identification schemes.

Last, but not least, this paper would benefit greatly if a more tight version of the generalised forking lemma was provided. However, we have to note here that some of the inequalities used in the proof are already tight, i.e. for specific adversaries, some of the inequalities are already equalities.

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Supplementary Material

A Omitted protocols descriptions

A.1 Polynomial commitment schemes

Figs. 2 and 3 present variants of KZG polynomial commitment schemes used in Plonk and Sonic. The key generation algorithm KGen takes as input a security parameter 1^λ and a parameter max which determines the maximal degree of the committed polynomial. We assume that max can be read from the output SRS.

We emphasize the following properties of a secure polynomial commitment **PC**:

Evaluation binding: A PPT adversary $\mathcal{A}$ which outputs a commitment $\boldsymbol{c}$ and evaluation points $\boldsymbol{z}$ has at most negligible chances to open the commitment to two different evaluations $\boldsymbol{s}, \boldsymbol{s'}$. That is, let $k \in \mathbb{N}$ be the number of committed polynomials, $l \in \mathbb{N}$ number of evaluation points, $\boldsymbol{c} \in \mathbb{G}^k$ be the commitments, $\boldsymbol{z} \in \mathbb{F}_p^l$ be the arguments the polynomials are evaluated at, $\boldsymbol{s}, \boldsymbol{s'} \in \mathbb{F}_p^k$ the evaluations, and $\boldsymbol{o}, \boldsymbol{o'} \in \mathbb{F}_p^l$ be the commitment openings. Then for every PPT adversary $\mathcal{A}$

$$\Pr\left[\begin{array}{l|l} \mathsf{Vf}(\mathsf{srs}, \boldsymbol{c}, \boldsymbol{z}, \boldsymbol{s}, \boldsymbol{o}) = 1, & \mathsf{srs} \leftarrow \mathsf{KGen}(1^\lambda, \mathsf{max}), \\ \mathsf{Vf}(\mathsf{srs}, \boldsymbol{c}, \boldsymbol{z}, \boldsymbol{s'}, \boldsymbol{o'}) = 1, & (\boldsymbol{c}, \boldsymbol{z}, \boldsymbol{s}, \boldsymbol{s'}, \boldsymbol{o}, \boldsymbol{o'}) \leftarrow \mathcal{A}(\mathsf{srs}) \\ \boldsymbol{s} \neq \boldsymbol{s'} & \end{array}\right] \leq \mathsf{negl}(\lambda)\,.$$

We say that **PC** has the unique opening property if the following holds:

Opening uniqueness: Let $k \in \mathbb{N}$ be the number of committed polynomials, $l \in \mathbb{N}$ number of evaluation points, $\boldsymbol{c} \in \mathbb{G}^k$ be the commitments, $\boldsymbol{z} \in \mathbb{F}_p^l$ be the arguments the polynomials are evaluated at, $\boldsymbol{s} \in \mathbb{F}_p^k$ the evaluations, and $\boldsymbol{o} \in \mathbb{F}_p^l$ be the commitment openings. Then for every PPT adversary $\mathcal{A}$

$$\Pr\left[\begin{array}{l|l} \mathsf{Vf}(\mathsf{srs}, \boldsymbol{c}, \boldsymbol{z}, \boldsymbol{s}, \boldsymbol{o}) = 1, & \mathsf{srs} \leftarrow \mathsf{KGen}(1^\lambda, \mathsf{max}), \\ \mathsf{Vf}(\mathsf{srs}, \boldsymbol{c}, \boldsymbol{z}, \boldsymbol{s}, \boldsymbol{o'}) = 1, & (\boldsymbol{c}, \boldsymbol{z}, \boldsymbol{s}, \boldsymbol{o}, \boldsymbol{o'}) \leftarrow \mathcal{A}(\mathsf{srs}) \\ \boldsymbol{o} \neq \boldsymbol{o'} & \end{array}\right] \leq \mathsf{negl}(\lambda)\,.$$

Intuitively, opening uniqueness assures that there is only one valid opening for the committed polynomial and given evaluation point. This property is crucial in showing forking simulation-extractability of Plonk and Sonic. We show that the Plonk's and Sonic's polynomial commitment schemes satisfy this requirement in Lemma 3 and Lemma 11 respectively.

Commitment of knowledge For every PPT adversary $\mathcal{A}$ who produces commitment c, evaluation s and opening o there exists a PPT extractor Ext such that

$$\Pr\left[\begin{array}{l|l} \deg \mathsf{f} \leq \mathsf{max} & \mathsf{srs} \leftarrow \mathsf{KGen}(1^\lambda, \mathsf{max}), \\ c = \mathsf{Com}(\mathsf{srs}, \mathsf{f}), & c \leftarrow \mathcal{A}(\mathsf{srs}), z \leftarrow\!\!\$\, \mathbb{F}_p \\ \mathsf{Vf}(\mathsf{srs}, c, z, s, o) = 1 & (s, o) \leftarrow \mathcal{A}(\mathsf{srs}, c, z), \\ & \mathsf{f} = \mathsf{Ext}_{\mathcal{A}}(\mathsf{srs}, c) \end{array}\right] \geq 1 - \varepsilon_{\mathsf{k}}(\lambda).$$

In that case we say that $\mathbf{PC}$ is $\varepsilon_{\mathsf{k}}(\lambda)$-knowledge.

Intuitively when a commitment scheme is "of knowledge" then if an adversary produces a (valid) commitment c, which it can open, then it also knows the underlying polynomial f which commits to that value. [45] shows, using AGM, that $\mathbf{PC_S}$ is a commitment of knowledge. The same reasoning could be used to show that property for $\mathbf{PC_P}$.

$\mathsf{KGen}(1^\lambda, \mathsf{max})$

$\chi \leftarrow\$ \mathbb{F}_p^2$
return $\left[1, \ldots, \chi^{\mathsf{n}+2}\right]_1, [\chi]_2$

$\mathsf{Com}(\mathsf{srs}, \mathbf{f}(X))$

return $[\boldsymbol{c}]_1 = [\mathbf{f}(\chi)]_1$

$\mathsf{Op}(\mathsf{srs}, \boldsymbol{\gamma}, \boldsymbol{z}, \boldsymbol{s}, \mathbf{f}(X))$

for $i \in [1 \,..\, |\boldsymbol{z}|]$ **do**
$\quad \mathsf{o}_i(X) \leftarrow \sum_{j=1}^{t_i} \gamma_i^{j-1} \frac{\mathsf{f}_{i,j}(X) - \mathsf{f}_{i,j}(z_i)}{X - z_i}$
return $\boldsymbol{o} = [\mathbf{o}(\chi)]_1$

$\mathsf{Vf}(\mathsf{srs}, [c]_1, \boldsymbol{z}, \boldsymbol{s}, [\mathsf{o}(\chi)]_1)$

$\boldsymbol{r} \leftarrow \mathbb{F}_p^{|\boldsymbol{z}|}$
for $i \in [1 \,..\, |\boldsymbol{z}|]$ **do**
$\quad$ **if** $\sum_{i=1}^{|\boldsymbol{z}|} r_i \cdot \left[\sum_{j=1}^{t_j} \gamma_i^{j-1} c_{i,j} - \sum j = 1^{t_j} s_{i,j}\right]_1 \bullet [1]_2 +$
$\quad \sum_{i=1}^{|\boldsymbol{z}|} r_i z_i o_i \bullet [1]_2 \neq \left[-\sum_{i=1}^{|\boldsymbol{z}|} r_i o_i\right]_1 \bullet [\chi]_2$ **then**
$\quad\quad$ **return** 0
return 1.

Fig. 2: $\mathbf{PC_P}$ polynomial commitment scheme.

B Non-malleability of Plonk, omitted proofs and descriptions

B.1 Plonk protocol rolled out

The constrain system Assume C is a fan-in two arithmetic circuit, which fan-out is unlimited and has n gates and m wires ($\mathsf{n} \leq \mathsf{m} \leq 2\mathsf{n}$). Plonk's constraint system is defined as follows:

- Let $\boldsymbol{V} = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$, where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in [1 \,..\, \mathsf{m}]^{\mathsf{n}}$. Entries $\boldsymbol{a}_i, \boldsymbol{b}_i, \boldsymbol{c}_i$ represent indices of left, right and output wires of circuits i-th gate.
- Vectors $\boldsymbol{Q} = (\boldsymbol{q_L}, \boldsymbol{q_R}, \boldsymbol{q_O}, \boldsymbol{q_M}, \boldsymbol{q_C}) \in (\mathbb{F}^{\mathsf{n}})^5$ are called *selector vectors*:
  - If the i-th gate is a multiplicative gate then $\boldsymbol{q_L}_i = \boldsymbol{q_R}_i = 0$, $\boldsymbol{q_M}_i = 1$, and $\boldsymbol{q_O}_i = -1$.
  - If the i-th gate is an addition gate then $\boldsymbol{q_L}_i = \boldsymbol{q_R}_i = 1$, $\boldsymbol{q_M}_i = 0$, and $\boldsymbol{q_O}_i = -1$.
  - $\boldsymbol{q_C}_i = 0$ always.

$\mathsf{KGen}(1^\lambda, \mathsf{max})$

$\alpha, \chi \leftarrow\!\$\, \mathbb{F}_p^2$

return $\left[\{\chi^i\}_{i=-\mathsf{n}}^{\mathsf{n}}, \{\alpha\chi^i\}_{i=-\mathsf{n}, i\neq 0}^{\mathsf{n}}\right]_1$,
$\left[\{\chi^i, \alpha\chi^i\}_{i=-\mathsf{n}}^{\mathsf{n}}\right]_2, [\alpha]_T$

$\mathsf{Com}(\mathsf{srs}, \mathsf{max}, \mathsf{f}(X))$

$\mathsf{c}(X) \leftarrow \alpha \cdot X^{\mathsf{d}-\mathsf{max}}\mathsf{f}(X)$

return $[c]_1 = [\mathsf{c}(\chi)]_1$

$\mathsf{Op}(\mathsf{srs}, z, s, f(X))$

$\mathsf{o}(X) \leftarrow \dfrac{\mathsf{f}(X) - \mathsf{f}(z)}{X - z}$

return $[\mathsf{o}(\chi)]_1$

$\mathsf{Vf}(\mathsf{srs}, \mathsf{max}, [c]_1, z, s, [\mathsf{o}(\chi)]_1)$

if $[\mathsf{o}(\chi)]_1 \bullet [\alpha\chi]_2 + [s - z\mathsf{o}(\chi)]_1 \bullet [\alpha]_2 =$
$[c]_1 \bullet \left[\chi^{-\mathsf{d}+\mathsf{max}}\right]_2$ **then return** 1

else return 0.

Fig. 3: $\mathbf{PC_S}$ polynomial commitment scheme.

We say that vector $\boldsymbol{x} \in \mathbb{F}^{\mathsf{m}}$ satisfies constraint system if for all $i \in [1 \,..\, \mathsf{n}]$

$$\boldsymbol{q_L}_i \cdot \boldsymbol{x}_{\boldsymbol{a}_i} + \boldsymbol{q_R}_i \cdot \boldsymbol{x}_{\boldsymbol{b}_i} + \boldsymbol{q_O} \cdot \boldsymbol{x}_{\boldsymbol{c}_i} + \boldsymbol{q_M}_i \cdot (\boldsymbol{x}_{\boldsymbol{a}_i} \boldsymbol{x}_{\boldsymbol{b}_i}) + \boldsymbol{q_C}_i = 0.$$

Algorithms rolled out Plonk argument system is universal. That is, it allows to verify computation of any arithmetic circuit which has no more than n gates using a single SRS. However, to make computation efficient, for each circuit there is allowed a preprocessing phase which extend the SRS with circuit-related polynomial evaluations.

For the sake of simplicity of the security reductions presented in this paper, we include in the SRS only these elements that cannot be computed without knowing the secret trapdoor χ. The rest of the SRS—the preprocessed input—can be computed using these SRS elements thus we leave them to be computed by the prover, verifier, and simulator.

Plonk *SRS generating algorithm* $\mathsf{KGen}(\mathbf{R})$*:* The SRS generating algorithm picks at random $\chi \leftarrow\!\$\, \mathbb{F}_p$, computes and outputs

$$\mathsf{srs} = \left(\left[\{\chi^i\}_{i=0}^{\mathsf{n}+2}\right]_1, [\chi]_2\right).$$

Preprocessing: Let $H = \{\omega^i\}_{i=1}^{\mathsf{n}}$ be a (multiplicative) n-element subgroup of a field $\mathbb{F}$ compound of n-th roots of unity in $\mathbb{F}$. Let $\mathsf{L}_i(X)$ be the i-th element of an n-elements Lagrange basis. During the preprocessing phase polynomials $\mathsf{S}_{\mathsf{id}\mathsf{j}}, \mathsf{S}_{\sigma\mathsf{j}}$,

for $\mathsf{j} \in [1\,..\,3]$, are computed:

$$\mathsf{S}_{\mathsf{id1}}(X) = X, \qquad \mathsf{S}_{\sigma 1}(X) = \sum_{i=1}^{\mathsf{n}} \sigma(i)\mathsf{L}_i(X),$$
$$\mathsf{S}_{\mathsf{id2}}(X) = k_1 \cdot X, \qquad \mathsf{S}_{\sigma 2}(X) = \sum_{i=1}^{\mathsf{n}} \sigma(\mathsf{n}+i)\mathsf{L}_i(X),$$
$$\mathsf{S}_{\mathsf{id3}}(X) = k_2 \cdot X, \qquad \mathsf{S}_{\sigma 3}(X) = \sum_{i=1}^{\mathsf{n}} \sigma(2\mathsf{n}+i)\mathsf{L}_i(X).$$

Coefficients k_1, k_2 are such that $H, k_1 \cdot H, k_2 \cdot H$ are different cosets of $\mathbb{F}^*$, thus they define $3 \cdot \mathsf{n}$ different elements. [29] notes that it is enough to set k_1 to a quadratic residue and k_2 to a quadratic non-residue.

Furthermore, we define polynomials $\mathsf{q}_\mathsf{L}, \mathsf{q}_\mathsf{R}, \mathsf{q}_\mathsf{O}, \mathsf{q}_\mathsf{M}, \mathsf{q}_\mathsf{C}$ such that

$$\mathsf{q}_\mathsf{L}(X) = \sum_{i=1}^{\mathsf{n}} \boldsymbol{q_L}_i \mathsf{L}_i(X), \qquad \mathsf{q}_\mathsf{O}(X) = \sum_{i=1}^{\mathsf{n}} \boldsymbol{q_O}_i \mathsf{L}_i(X),$$
$$\mathsf{q}_\mathsf{R}(X) = \sum_{i=1}^{\mathsf{n}} \boldsymbol{q_R}_i \mathsf{L}_i(X), \qquad \mathsf{q}_\mathsf{C}(X) = \sum_{i=1}^{\mathsf{n}} \boldsymbol{q_C}_i \mathsf{L}_i(X).$$
$$\mathsf{q}_\mathsf{M}(X) = \sum_{i=1}^{\mathsf{n}} \boldsymbol{q_M}_i \mathsf{L}_i(X),$$

Plonk *prover* $\mathsf{P}(\mathbf{R}, \mathsf{srs}, \mathsf{x}, \mathsf{w} = (\mathsf{w}_i)_{i \in [1\,..\,3\cdot\mathsf{n}]})$.

Round 1 Sample $b_1, \ldots, b_9 \leftarrow\!\$\, \mathbb{F}_p$; compute $\mathsf{a}(X), \mathsf{b}(X), \mathsf{c}(X)$ as

$$\mathsf{a}(X) = (b_1 X + b_2)\mathsf{Z}_\mathsf{H}(X) + \sum_{i=1}^{\mathsf{n}} \mathsf{w}_i \mathsf{L}_i(X)$$
$$\mathsf{b}(X) = (b_3 X + b_4)\mathsf{Z}_\mathsf{H}(X) + \sum_{i=1}^{\mathsf{n}} \mathsf{w}_{\mathsf{n}+i} \mathsf{L}_i(X)$$
$$\mathsf{c}(X) = (b_5 X + b_6)\mathsf{Z}_\mathsf{H}(X) + \sum_{i=1}^{\mathsf{n}} \mathsf{w}_{2\cdot\mathsf{n}+i} \mathsf{L}_i(X)$$

Output polynomial commitments $[\mathsf{a}(\chi), \mathsf{b}(\chi), \mathsf{c}(\chi)]_1$.

Round 2 Get challenges $\beta, \gamma \in \mathbb{F}_p$

$$\beta = \mathcal{H}(\pi[0..1], 0)\,, \qquad \gamma = \mathcal{H}(\pi[0..1], 1)\,.$$

Compute permutation polynomial $\mathsf{z}(X)$

$$\mathsf{z}(X) = (b_7 X^2 + b_8 X + b_9)\mathsf{Z}_\mathsf{H}(X) + \mathsf{L}_1(X) +$$
$$+ \sum_{i=1}^{\mathsf{n}-1} \left(\mathsf{L}_{i+1}(X) \prod_{j=1}^{i} \frac{(\mathsf{w}_j + \beta\omega^{j-1} + \gamma)(\mathsf{w}_{\mathsf{n}+j} + \beta k_1 \omega^{j-1} + \gamma)(\mathsf{w}_{2\mathsf{n}+j} + \beta k_2 \omega^{j-1} + \gamma)}{(\mathsf{w}_j + \sigma(j)\beta + \gamma)(\mathsf{w}_{\mathsf{n}+j} + \sigma(\mathsf{n}+j)\beta + \gamma)(\mathsf{w}_{2\mathsf{n}+j} + \sigma(2\mathsf{n}+j)\beta + \gamma)} \right)$$

Output polynomial commitment $[\mathsf{z}(\chi)]_1$

Round 3 Get the challenge $\alpha = \mathcal{H}(\pi[0..2])$, compute the quotient polynomial

$$\begin{aligned}
&\mathsf{t}(X) = \\
&(\mathsf{a}(X)\mathsf{b}(X)\mathsf{q}_\mathsf{M}(X) + \mathsf{a}(X)\mathsf{q}_\mathsf{L}(X) + \mathsf{b}(X)\mathsf{q}_\mathsf{R}(X) + \mathsf{c}(X)\mathsf{q}_\mathsf{O}(X) + \mathsf{PI}(X) + \mathsf{q}_\mathsf{C}(X))\frac{1}{\mathsf{Z}_\mathsf{H}(X)} + \\
&+ ((\mathsf{a}(X) + \beta X + \gamma)(\mathsf{b}(X) + \beta k_1 X + \gamma)(\mathsf{c}(X) + \beta k_2 X + \gamma)\mathsf{z}(X))\frac{\alpha}{\mathsf{Z}_\mathsf{H}(X)} \\
&- (\mathsf{a}(X) + \beta\mathsf{S}_{\sigma 1}(X) + \gamma)(\mathsf{b}(X) + \beta\mathsf{S}_{\sigma 2}(X) + \gamma)(\mathsf{c}(X) + \beta\mathsf{S}_{\sigma 3}(X) + \gamma)\mathsf{z}(X\omega))\frac{\alpha}{\mathsf{Z}_\mathsf{H}(X)} \\
&+ (\mathsf{z}(X) - 1)\mathsf{L}_1(X)\frac{\alpha^2}{\mathsf{Z}_\mathsf{H}(X)}
\end{aligned}$$

Split $\mathsf{t}(X)$ into degree less then n polynomials $\mathsf{t}_\mathsf{lo}(X), \mathsf{t}_\mathsf{mid}(X), \mathsf{t}_\mathsf{hi}(X)$, such that

$$\mathsf{t}(X) = \mathsf{t}_\mathsf{lo}(X) + X^\mathsf{n}\mathsf{t}_\mathsf{mid}(X) + X^{2\mathsf{n}}\mathsf{t}_\mathsf{hi}(X)\,.$$

Output $[\mathsf{t}_\mathsf{lo}(\chi), \mathsf{t}_\mathsf{mid}(\chi), \mathsf{t}_\mathsf{hi}(\chi)]_1$.

Round 4 Get the challenge $\mathfrak{z} \in \mathbb{F}_p$, $\mathfrak{z} = \mathcal{H}(\pi[0..3])$. Compute opening evaluations

$$\mathsf{a}(\mathfrak{z}), \mathsf{b}(\mathfrak{z}), \mathsf{c}(\mathfrak{z}), \mathsf{S}_{\sigma 1}(\mathfrak{z}), \mathsf{S}_{\sigma 2}(\mathfrak{z}), \mathsf{t}(\mathfrak{z}), \mathsf{z}(\mathfrak{z}\omega),$$

Compute the linearisation polynomial

$$\begin{aligned}
\mathsf{r}(X) = \; &\mathsf{a}(\mathfrak{z})\mathsf{b}(\mathfrak{z})\mathsf{q}_\mathsf{M}(X) + \mathsf{a}(\mathfrak{z})\mathsf{q}_\mathsf{L}(X) + \mathsf{b}(\mathfrak{z})\mathsf{q}_\mathsf{R}(X) + \mathsf{c}(\mathfrak{z})\mathsf{q}_\mathsf{O}(X) + \mathsf{q}_\mathsf{C}(X) \\
&+ \alpha \cdot ((\mathsf{a}(\mathfrak{z}) + \beta\mathfrak{z} + \gamma)(\mathsf{b}(\mathfrak{z}) + \beta k_1\mathfrak{z} + \gamma)(\mathsf{c}(\mathfrak{z}) + \beta k_2\mathfrak{z} + \gamma) \cdot \mathsf{z}(X)) \\
&- \alpha \cdot ((\mathsf{a}(\mathfrak{z}) + \beta\mathsf{S}_{\sigma 1}(\mathfrak{z}) + \gamma)(\mathsf{b}(\mathfrak{z}) + \beta\mathsf{S}_{\sigma 2}(\mathfrak{z}) + \gamma)\beta\mathsf{z}(\mathfrak{z}\omega) \cdot \mathsf{S}_{\sigma 3}(X)) \\
&+ \alpha^2 \cdot \mathsf{L}_1(\mathfrak{z}) \cdot \mathsf{z}(X)
\end{aligned}$$

Output $\mathsf{a}(\mathfrak{z}), \mathsf{b}(\mathfrak{z}), \mathsf{c}(\mathfrak{z}), \mathsf{S}_{\sigma 1}(\mathfrak{z}), \mathsf{S}_{\sigma 2}(\mathfrak{z}), \mathsf{t}(\mathfrak{z}), \mathsf{z}(\mathfrak{z}\omega), \mathsf{r}(\mathfrak{z})$.

Round 5 Compute the opening challenge $v \in \mathbb{F}_p$, $v = \mathcal{H}(\pi[0..4])$. Compute the openings for the polynomial commitment scheme

$$\begin{aligned}
\mathsf{W}_\mathfrak{z}(X) &= \frac{1}{X - \mathfrak{z}}\begin{pmatrix}
\mathsf{t}_\mathsf{lo}(X) + \mathfrak{z}^\mathsf{n}\mathsf{t}_\mathsf{mid}(X) + \mathfrak{z}^{2\mathsf{n}}\mathsf{t}_\mathsf{hi}(X) - \mathsf{t}(\mathfrak{z}) \\
+ v(\mathsf{r}(X) - \mathsf{r}(\mathfrak{z})) \\
+ v^2(\mathsf{a}(X) - \mathsf{a}(\mathfrak{z})) \\
+ v^3(\mathsf{b}(X) - \mathsf{b}(\mathfrak{z})) \\
+ v^4(\mathsf{c}(X) - \mathsf{c}(\mathfrak{z})) \\
+ v^5(\mathsf{S}_{\sigma 1}(X) - \mathsf{S}_{\sigma 1}(\mathfrak{z})) \\
+ v^6(\mathsf{S}_{\sigma 2}(X) - \mathsf{S}_{\sigma 2}(\mathfrak{z}))
\end{pmatrix} \\
\mathsf{W}_{\mathfrak{z}\omega}(X) &= \frac{\mathsf{z}(X) - \mathsf{z}(\mathfrak{z}\omega)}{X - \mathfrak{z}\omega}
\end{aligned}$$

Output $[\mathsf{W}_\mathfrak{z}(\chi), \mathsf{W}_{\mathfrak{z}\omega}(\chi)]_1$.

Plonk **verifier** $\mathsf{V}(\mathbf{R}, \mathsf{srs}, \mathsf{x}, \pi)$**:**
The Plonk verifier works as follows

Step 1 Validate all obtained group elements.
Step 2 Validate all obtained field elements.
Step 3 Validate the instance $\mathsf{x} = \{\mathsf{w}_i\}_{i=1}^{\mathsf{n}}$.
Step 4 Compute challenges $\beta, \gamma, \alpha, \mathfrak{z}, v, u$ from the transcript.
Step 5 Compute zero polynomial evaluation $\mathsf{Z}_\mathsf{H}(\mathfrak{z}) = \mathfrak{z}^\mathsf{n} - 1$.
Step 6 Compute Lagrange polynomial evaluation $\mathsf{L}_1(\mathfrak{z}) = \frac{\mathfrak{z}^\mathsf{n}-1}{\mathsf{n}(\mathfrak{z}-1)}$.
Step 7 Compute public input polynomial evaluation $\mathsf{PI}(\mathfrak{z}) = \sum_{i\in[1\,..\,\mathsf{n}]} \mathsf{w}_i \mathsf{L}_i(\mathfrak{z})$.
Step 8 Compute quotient polynomials evaluations

$$\mathsf{t}(\mathfrak{z}) = \frac{1}{\mathsf{Z}_\mathsf{H}(\mathfrak{z})}\Big(\mathsf{r}(\mathfrak{z}) + \mathsf{PI}(\mathfrak{z}) - (\mathsf{a}(\mathfrak{z}) + \beta\mathsf{S}_{\sigma 1}(\mathfrak{z}) + \gamma)(\mathsf{b}(\mathfrak{z}) + \beta\mathsf{S}_{\sigma 2}(\mathfrak{z}) + \gamma) \\ (\mathsf{c}(\mathfrak{z}) + \gamma)\mathsf{z}(\mathfrak{z}\omega)\alpha - \mathsf{L}_1(\mathfrak{z})\alpha^2\Big)\,.$$

Step 9 Compute batched polynomial commitment $[D]_1 = v\,[r]_1 + u\,[z]_1$ that is

$$[D]_1 = v\begin{pmatrix}\mathsf{a}(\mathfrak{z})\mathsf{b}(\mathfrak{z})\cdot[\mathsf{q}_\mathsf{M}]_1 + \mathsf{a}(\mathfrak{z})\,[\mathsf{q}_\mathsf{L}]_1 + \mathsf{b}\,[\mathsf{q}_\mathsf{R}]_1 + \mathsf{c}\,[\mathsf{q}_\mathsf{O}]_1 + \\ +((\mathsf{a}(\mathfrak{z}) + \beta\mathfrak{z} + \gamma)(\mathsf{b}(\mathfrak{z}) + \beta k_1\mathfrak{z} + \gamma)(\mathsf{c} + \beta k_2\mathfrak{z} + \gamma)\alpha + \mathsf{L}_1(\mathfrak{z})\alpha^2) + \\ -(\mathsf{a}(\mathfrak{z}) + \beta\mathsf{S}_{\sigma 1}(\mathfrak{z}) + \gamma)(\mathsf{b}(\mathfrak{z}) + \beta\mathsf{S}_{\sigma 2}(\mathfrak{z}) + \gamma)\alpha\beta\mathsf{z}(\mathfrak{z}\omega)\,[\mathsf{S}_{\sigma 3}(\chi)]_1)\end{pmatrix} + \\ + u\,[\mathsf{z}(\chi)]_1\,.$$

Step 10 Computes full batched polynomial commitment $[F]_1$:

$$[F]_1 = \big([\mathsf{t}_{\mathsf{lo}}(\chi)]_1 + \mathfrak{z}^\mathsf{n}\,[\mathsf{t}_{\mathsf{mid}}(\chi)]_1 + \mathfrak{z}^{2\mathsf{n}}\,[\mathsf{t}_{\mathsf{hi}}(\chi)]_1\big) + u\,[\mathsf{z}(\chi)]_1 + \\ + v\begin{pmatrix}\mathsf{a}(\mathfrak{z})\mathsf{b}(\mathfrak{z})\cdot[\mathsf{q}_\mathsf{M}]_1 + \mathsf{a}(\mathfrak{z})\,[\mathsf{q}_\mathsf{L}]_1 + \mathsf{b}(\mathfrak{z})\,[\mathsf{q}_\mathsf{R}]_1 + \mathsf{c}(\mathfrak{z})\,[\mathsf{q}_\mathsf{O}]_1 + \\ +((\mathsf{a}(\mathfrak{z}) + \beta\mathfrak{z} + \gamma)(\mathsf{b}(\mathfrak{z}) + \beta k_1\mathfrak{z} + \gamma)(\mathsf{c}(\mathfrak{z}) + \beta k_2\mathfrak{z} + \gamma)\alpha + \mathsf{L}_1(\mathfrak{z})\alpha^2) + \\ -(\mathsf{a}(\mathfrak{z}) + \beta\mathsf{S}_{\sigma 1}(\mathfrak{z}) + \gamma)(\mathsf{b}(\mathfrak{z}) + \beta\mathsf{S}_{\sigma 2}(\mathfrak{z}) + \gamma)\alpha\beta\mathsf{z}(\mathfrak{z}\omega)\,[\mathsf{S}_{\sigma 3}(\chi)]_1)\end{pmatrix} \\ + v^2\,[\mathsf{a}(\chi)]_1 + v^3\,[\mathsf{b}(\chi)]_1 + v^4\,[\mathsf{c}(\chi)]_1 + v^5\,[\mathsf{S}_{\sigma 1}(\chi)]_1 + v^6\,[\mathsf{S}_{\sigma 2}(\chi)]_1\,.$$

Step 11 Compute group-encoded batch evaluation $[E]_1$

$$[E]_1 = \frac{1}{\mathsf{Z}_\mathsf{H}(\mathfrak{z})}\begin{bmatrix}\mathsf{r}(\mathfrak{z}) + \mathsf{PI}(\mathfrak{z}) + \alpha^2\mathsf{L}_1(\mathfrak{z}) + \\ -\alpha\,((\mathsf{a}(\mathfrak{z}) + \beta\mathsf{S}_{\sigma 1}(\mathfrak{z}) + \gamma)(\mathsf{b}(\mathfrak{z}) + \beta\mathsf{S}_{\sigma 2}(\mathfrak{z}) + \gamma)(\mathsf{c}(\mathfrak{z}) + \gamma)\mathsf{z}(\mathfrak{z}\omega))\end{bmatrix}_1 \\ + \big[v\mathsf{r}(\mathfrak{z}) + v^2\mathsf{a}(\mathfrak{z}) + v^3\mathsf{b}(\mathfrak{z}) + v^4\mathsf{c}(\mathfrak{z}) + v^5\mathsf{S}_{\sigma 1}(\mathfrak{z}) + v^6\mathsf{S}_{\sigma 2}(\mathfrak{z}) + u\mathsf{z}(\mathfrak{z}\omega)\big]_1\,.$$

Step 12 Check whether the verification equation holds

$$\big([\mathsf{W}_\mathfrak{z}(\chi)]_1 + u\cdot[\mathsf{W}_{\mathfrak{z}\omega}(\chi)]_1\big)\bullet[\chi]_2 - \big(\mathfrak{z}\cdot[\mathsf{W}_\mathfrak{z}(\chi)]_1 + u\mathfrak{z}\omega\cdot[\mathsf{W}_{\mathfrak{z}\omega}(\chi)]_1 + [F]_1 - [E]_1\big)\bullet[1]_2 = 0\,. \tag{5}$$

The verification equation is a batched version of the verification equation from [40] which allows the verifier to check openings of multiple polynomials in two points (instead of checking an opening of a single polynomial at one point).

Plonk simulator $\mathsf{Sim}_\chi(\mathbf{R}, \mathsf{srs}, \mathsf{td} = \chi, \mathsf{x})$:
The Plonk simulator proceeds as an honest prover would, except:

1. In the first round, it sets $\mathsf{w} = (\mathsf{w}_i)_{i\in[1\,..\,3\mathsf{n}]} = \mathbf{0}$, and at random picks $b_1, \ldots, b_9$. Then it proceeds with that all-zero witness.
2. In Round 3, it computes polynomial $\mathsf{t}(X)$ honestly, however uses trapdoor χ to compute commitments $\mathsf{t}_{\mathsf{lo}}(\chi), \mathsf{t}_{\mathsf{mid}}(\chi), \mathsf{t}_{\mathsf{hi}}(\chi)$.

B.2 Proof of PC_P unique opening property (Lemma 3)

Proof. Let $\boldsymbol{z} = (z, z') \in \mathbb{F}_p^2$ be the two points the polynomials are evaluated at, $k \in \mathbb{N}$ be the number of the committed polynomials to be evaluated at z, and $k' \in \mathbb{N}$ be the number of the committed polynomials to be evaluated at z', $\boldsymbol{c} \in \mathbb{G}^k, \boldsymbol{c'} \in \mathbb{G}^{k'}$ be the commitments, $\boldsymbol{s} \in \mathbb{F}_p^k, \boldsymbol{s'} \in \mathbb{F}_p^{k'}$ the evaluations, and $\boldsymbol{o} = (o, o') \in \mathbb{F}_p^2$ be the commitment openings. We need to show that the probability a PPT $\mathcal{A}$ opens the same commitment in two different ways is negligible. is at most $\varepsilon_{\mathsf{op}}(\lambda)$.

Step 1: First, consider a case where the commitment is limited to commit to multiple polynomials which are evaluated at the same point z. As noted in [29, Lemma 2.2] it is enough to upper bound the probability of the adversary succeeding using the idealised verification equation. This holds since an adversary that manages to provide a commitment opening that holds for the real verifier, but does not hold for the idealised verifier can be used to break the dlog assumption and reveal the secret trapdoor used to produce the commitment's SRS, the probability of such event is bounded by the $(\mathsf{n}+2, 1)$-dlog security $\varepsilon_{\mathsf{dlog}}(\lambda)$, cf. [29, Lemma 2.2] for more details.

For polynomials $\mathbf{f} = \mathsf{f}_1, \ldots, \mathsf{f}_k$, evaluation point z, evaluation result $\boldsymbol{s} = s_1, \ldots, s_k$, random γ, and opening $\mathsf{o}(X)$ the idealised check verifies that $\sum_{i=1}^k \gamma^{i-1}\mathsf{f}_i(X) - \sum_{i=1}^k \gamma^{i-1}s_i \equiv \mathsf{o}(X)(X-z)$. Since $\mathsf{o}(X)(X-z) \in \mathbb{F}_p[X]$ then from the uniqueness of polynomial composition, there is only one $\mathsf{o}(X)$ that fulfils the equation above.

Step 2: Second, consider a case when the polynomials are evaluated on two points $\boldsymbol{z} = (z, z')$ and the adversary is asked to provide two openings $\boldsymbol{o} = (o, o')$. Similarly, we analyse the case of the ideal verification. In that scenario, the verifier checks whether the following equality, for γ, r' picked at random, holds:

$$\sum_{i=1}^{k} \gamma^{i-1} \cdot \mathsf{f}_i(X) - \sum_{i=1}^{k} \gamma^{i-1} \cdot s_i + r' \left(\sum_{i=1}^{k'} \gamma'^{i-1} \cdot \mathsf{f}'_i(X) - \sum_{i=1}^{k'} \gamma'^{i-1} \cdot s'_i \right) \equiv \mathsf{o}(X)(X-z) + r'\mathsf{o}'(X)(X-z'). \quad (6)$$

Since r' has been picked at random from $\mathbb{F}$, probability that Eq. (6) holds while either $\sum_{i=1}^k \gamma^{i-1} \cdot \mathsf{f}_i(X) - \sum_{i=1}^k \gamma^{i-1} \cdot s_i \not\equiv \mathsf{o}(X)(X-z)$, or $\sum_{i=1}^{k'} \gamma'^{i-1} \cdot \mathsf{f}'_i(X) - \sum_{i=1}^{k'} \gamma'^{i-1} \cdot s'_i \not\equiv \mathsf{o}'(X)(X-z')$ is $1/|\mathbb{F}_p|$ cf. [29]. This brings the proof back to Step 1 above. Altogether, Step 2 gives the following bound for the unique opening property $\varepsilon_{\mathsf{op}}(\lambda) \leq 2\varepsilon_{\mathsf{dlog}}(\lambda) + 1/\mathbb{F}_p$. □

B.3 Proof of Plonk unique response property (Lemma 4)

Proof (sketch). Let $\mathcal{A}(\mathbf{R}, \mathsf{srs} = (\left[1, \chi, \ldots, \chi^{\mathsf{n}+2}\right]_1, [\chi]_2))$ be an adversary tasked to break the 2-ur-ness of $\mathbf{P}$. It is sufficient to observe that the first 2 rounds of the protocol determines, along with the verifiers challenges, the rest of it.

In Round 3 the adversary outputs a commitment to polynomial $\mathsf{t}(X)$ which assures that all constraints of the system are fulfilled. Since the commitment scheme is deterministic, there is only one value c that is a commitment to $\mathsf{t}(X)$. Assume that $\mathcal{A}$ outputs $c' \neq c$ and is later able to open c' to $y = \mathsf{t}(\mathfrak{z})$, where $\mathfrak{z}$ is a random point determined later. Using the AGM and arguments similar to [45], we argue that $\mathbf{PC}_{\mathbf{P}}$ is a commitment of knowledge. That is, an AGM adversary $\mathcal{A}$ that outputs a commitment c' which it can later open, knows a polynomial f of degree-($\leq \mathsf{n}+2$) such that $[\mathsf{f}(\chi)]_1 = c'$. Denote by $\varepsilon_{\mathsf{k}}(\lambda)$ probability that $\mathcal{A}$ breaks knowledge soundness of the commitment. Thus if $c' \neq c$ and the commitment scheme is evaluation binding with security $\varepsilon_{\mathsf{bind}}(\lambda)$ then $\mathcal{A}$ when picking c' picks it as a commitment to f which evaluates at $\mathfrak{z}$ to $\mathsf{t}(\mathfrak{z})$. The probability of that is negligible as there can only be no more than $\mathsf{n}+2$ overlapping points between $\mathsf{f}(X)$ and $\mathsf{t}(X)$, and $\mathfrak{z}$ remains random for $\mathcal{A}$ when it computes c'. Hence, the probability that $\mathcal{A}$ is able to produce two different outputs of Round 3 is upper-bounded by $\varepsilon_{\mathsf{k}}(\lambda) + \varepsilon_{\mathsf{bind}}(\lambda)$.

In Round 4 the prover is asked to give evaluations of predefined polynomials at some point $\mathfrak{z}$. Naturally, for the given polynomials only one value at $\mathfrak{z}$ is correct. Assume $\mathcal{A}$ is able to produce two different outputs in that round: $\boldsymbol{r_4} = (\widetilde{\mathsf{a}}, \widetilde{\mathsf{b}}, \widetilde{\mathsf{c}}, \widetilde{\mathsf{S}_{\sigma 1}}, \widetilde{\mathsf{S}_{\sigma 2}}, \widetilde{\mathsf{r}}, \widetilde{\mathsf{z}})$ and $\boldsymbol{r_4} = (\widetilde{\mathsf{a}}', \widetilde{\mathsf{b}}', \widetilde{\mathsf{c}}', \widetilde{\mathsf{S}_{\sigma 1}}', \widetilde{\mathsf{S}_{\sigma 2}}', \widetilde{\mathsf{r}}', \widetilde{\mathsf{z}}')$ which suppose to be evaluations at $\mathfrak{z}$ of polynomials $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{S}_{\sigma 1}, \mathsf{S}_{\sigma 2}, \mathsf{r}$ and an evaluation at $\mathfrak{z}\omega$ of z. Clearly, at least one of $\boldsymbol{r_4}$, $\boldsymbol{r'_4}$ has to be incorrect, thus if both evaluations are acceptable by the $\mathbf{PC}_{\mathbf{P}}.\mathsf{Vf}$ then the evaluation binding property of $\mathbf{PC}_{\mathbf{P}}$ is broken. This happens with probability upper-bounded by $\varepsilon_{\mathsf{bind}}(\lambda)$.

In the last round of the protocol the prover provides openings for the polynomial commitment evaluations done before. Assume $\mathcal{A}$ is able to produce two different polynomial commitment openings pairs: $\boldsymbol{r_5} = (\widetilde{\mathsf{W}_{\mathfrak{z}}}, \widetilde{\mathsf{W}_{\mathfrak{z}\omega}})$ and $\boldsymbol{r'_5} = (\widetilde{\mathsf{W}_{\mathfrak{z}}}', \widetilde{\mathsf{W}_{\mathfrak{z}\omega}}')$. Since $\mathbf{PC}_{\mathbf{P}}$ has unique opening property, one of the openings has to be incorrect and should be rejected by the polynomial commitment verifier. This happens except with probability $\varepsilon_{\mathsf{op}}(\lambda)$.

Hence, the probability that after fixing the two first rounds, the adversary is able to produce two different outputs in one of the following rounds is upper-bounded by $\varepsilon_{\mathsf{k}}(\lambda) + 2 \cdot \varepsilon_{\mathsf{bind}}(\lambda) + \varepsilon_{\mathsf{op}}(\lambda)$. □

C Additional preliminaries, lemmas and proofs

C.1 Dlog assumptions

Definition 7 ((q_1, q_2)-dlog assumption). *Let $\mathcal{A}$ be a* PPT *adversary that gets as input $[1, \chi, \ldots, \chi^{q_1}]_1, [1, \chi, \ldots, \chi^{q_2}]_2$, for some randomly picked $\chi \in \mathbb{F}_p$, then*

$$\Pr[\chi \leftarrow \mathcal{A}([1, \chi, \ldots, \chi^{q_1}]_1, [1, \chi, \ldots, \chi^{q_2}]_2) \mid \chi \leftarrow\!\$\, \mathbb{F}_p] \leq \mathsf{negl}(\lambda).$$

Definition 8 (**(q_1, q_2)-ldlog assumption).** *Let $\mathcal{A}$ be a* PPT *adversary that gets as input $[\chi^{-q_1}, \ldots, 1, \chi, \ldots, \chi^{q_1}]_1, [\chi^{-q_2}, \ldots, 1, \chi, \ldots, \chi^{q_2}]_2$, for some randomly picked $\chi \in \mathbb{F}_p$, then*

$$\Pr\left[\chi \leftarrow \mathcal{A}(\left[\chi^{-q_1}, \ldots, 1, \chi, \ldots, \chi^{q_1}\right]_1, \left[\chi^{-q_2}, \ldots, 1, \chi, \ldots, \chi^{q_2}\right]_2) \;\middle|\; \chi \leftarrow\!\$\, \mathbb{F}_p\right] \leq \mathsf{negl}(\lambda).$$

C.2 Uber assumption

BBG uber assumption. Also, to be able to show computational honest verifier zero knowledge of Plonk in the standard model, what is required by our reduction, we rely on the *uber assumption* introduced by Boneh et al. [12] as presented by Boyen in [15].

Let $r, s, t, c \in \mathbb{N} \setminus \{0\}$, Consider vectors of polynomials $\mathsf{R} \in \mathbb{F}_p[X_1, \ldots, X_c]^r$, $\mathsf{S} \in \mathbb{F}_p[X_1, \ldots, X_c]^s$ and $\mathsf{T} \in \mathbb{F}_p[X_1, \ldots, X_c]^t$. Write $\mathsf{R} = (\mathsf{r}_1, \ldots, \mathsf{r}_r)$, $\mathsf{S} = (\mathsf{s}_1, \ldots, \mathsf{s}_s)$ and $\mathsf{T} = (\mathsf{t}_1, \ldots, \mathsf{t}_t)$ for polynomials $\mathsf{r}_i, \mathsf{s}_j, \mathsf{t}_k$.

For a function f and vector $(x_1, \ldots, x_c)$ we write $f(\mathsf{R})$ to denote application of f to each element of R, i.e. $f(\mathsf{R}) = (f(\mathsf{r}_1(x_1, \ldots, x_c), \ldots, f(\mathsf{r}_r(x_1, \ldots, x_c))$. Similarly for applying f to S and T.

Definition 9 (Independence of $\mathsf{R}, \mathsf{S}, \mathsf{T}$). *Let $\mathsf{R}, \mathsf{S}, \mathsf{T}$ be defined as above. We say that polynomial $\mathsf{f} \in \mathbb{F}_p[X_1, \ldots, X_c]$ is* dependent *on $\mathsf{R}, \mathsf{S}, \mathsf{T}$ if there exists $rs + t$ constants $a_{i,j}, b_k$ such that $\mathsf{f} = \sum_{i=1}^{r} \sum_{j=1}^{s} a_{i,j} \mathsf{r}_i \mathsf{s}_j + \sum_{k=1}^{t} b_k \mathsf{t}_k$. We say that f is* independent *if it is not dependent.*

To show (standard-model) zero knowledge of Plonk we utilize a generalization of Boneh-Boyen-Goh's *uber assumption* [12] stated as follows (the changed element has been put into a dashbox)

Definition 10 (**$(\mathsf{R}, \mathsf{S}, \mathsf{T}, \mathsf{F}, 1)$-uber assumption).** *Let $\mathsf{R}, \mathsf{S}, \mathsf{T}$ be defined as above, $(x_1, \ldots, x_c, y_1, \ldots, y_d) \leftarrow\!\$\, \mathbb{F}_p^{c+d}$ and let F be a cardinality-d set of pair-wise independent polynomials which are also independent of $(\mathsf{R}, \mathsf{S}, \mathsf{T})$, cf. Definition 9. Then, for any* PPT *adversary $\mathcal{A}$*

$$\Pr\left[\mathcal{A}([\mathsf{R}(x_1, \ldots x_c)]_1, [\mathsf{S}(x_1, \ldots, x_c)]_2, [\mathsf{T}(x_1, \ldots, x_c)]_T, \boxed{[\mathsf{F}(x_1, \ldots, x_c)]_1}) = 1\right] \approx_\lambda$$
$$\Pr\left[\mathcal{A}([\mathsf{R}(x_1, \ldots x_c)]_1, [\mathsf{S}(x_1, \ldots, x_c)]_2, [\mathsf{T}(x_1, \ldots, x_c)]_T, \boxed{[y_1, \ldots, y_d]_1}) = 1\right].$$

Compared to the original uber assumptions, there are two major changes. First, we require not target group $\mathbb{G}_T$ elements to be indistinguishable, but elements of $\mathbb{G}_1$. Second, Boneh et al.'s assumption works for distinguishers who are given only one challenge polynomial f, i.e. $|\mathsf{F}| = 1$.

We show security of our version of the uber assumption using the generic group model as introduced by Shoup [50] where all group elements are represented by random binary strings of length λ. That is, there are random encodings ξ_1, ξ_2, ξ_T which are injective functions from $\mathbb{Z}_p^+$ to $\{0,1\}^\lambda$. We write $\mathbb{G}_i = \{\xi_i(x) \mid x \in \mathbb{Z}_p^+\}$, for $i \in \{1, 2, T\}$. For the sake of clarity we denote by $\xi_{i,j}$ the j-th encoding in group $\mathbb{G}_i$.

Let $\mathsf{P}_i = \{p_1, \ldots, p_{\tau_i}\} \subset \mathbb{F}_p[X_1, \ldots, X_n]$, for $i \in \{1, 2, T\}, \tau_i, n \in \mathbb{N}$, be sets of multivariate polynomials. Denote by $\mathsf{P}_i(x_1, \ldots, x_n)$ a set of evaluations of polynomials in $\mathsf{P_i}$ at $(x_1, \ldots, x_n)$. Denote by $L_i = \{(p_j, \xi_{i,j}) \mid j \leq \tau_i\}$.

Let $\mathcal{A}$ be an algorithm that is given encodings ξ_{i,j_i} of polynomials in P_i for $i \in \{1, 2, T\}, j_i = \tau_i$. There is an oracle O that allows to perform $\mathcal{A}$ the following queries:

Group operations in $\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T$: On input $(\xi_{i,j}, \xi_{i,j'}, i, op)$, $j, j' \leq \tau_i$, $op \in \{\mathsf{add}, \mathsf{sub}\}$, O sets $\tau_i' \leftarrow \tau_i + 1$, computes $p_{i,\tau_i'} = p_{i,j}(x_1, \ldots, x_n) \pm p_{i,j'}(x_1, \ldots, x_n)$ respectively to *op*. If there is an element $p_{i,k} \in L_i$ such that $p_{i,k} = p_{\tau_i'}$, then the oracle returns encoding of $p_{i,k}$. Otherwise it sets the encoding $\xi_{i,\tau_i'}$ to a new unused random string, adds $(p_{i,\tau_i'}, \xi_{i,\tau_i'})$ to L_i, and returns $\xi_{i,\tau_i'}$.

Bilinear pairing: On input $(\xi_{1,j}, \xi_{2,j'})$ the oracle sets $\tau' \leftarrow \tau_T + 1$ and computes $r_{\tau'} \leftarrow p_{i,j}(x_1, \ldots, x_n) \cdot p_{i,j'}(x_1, \ldots, x_n)$. If $r_{\tau'} \in L_T$ then return encoding found in the list L_T, else pick a new unused random string and set $\xi_{T,\tau'}$ to it. Return the encoding to the algorithm.

Given that, we are ready to show security of our variant of the Boneh et al. uber assumption. The proof goes similarly to the original proof given in [12] with minor differences.

Theorem 3 (Security of the uber assumption). *Let $\mathsf{P}_i \in \mathbb{F}_p[X_1, \ldots, X_n]^{m_i}$, for $i \in \{1, 2, T\}$ be τ_i tuples of n-variate polynomials over $\mathbb{F}_p$ and let $\mathsf{F} \in \mathbb{F}_p[X_1, \ldots, X_n]^m$. Let ξ_0, ξ_1, ξ_T, $\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T$ be as defined above. If polynomials $f \in \mathsf{F}$ are pair-wise independent and are independent of $\mathsf{P}_1, \mathsf{P}_2, \mathsf{P}_T$, then for any $\mathcal{A}$ that makes up to q queries to the GGM oracle holds:*

$$\left| \Pr\left[\mathcal{A}\begin{pmatrix} \xi_1(\mathsf{P}_1(x_1, \ldots, x_n)), \\ \xi_2(\mathsf{P}_2(x_1, \ldots, x_n)), \\ \xi_T(\mathsf{P}_T(x_1, \ldots, x_n)), \\ \xi_1(\mathsf{F}_0), \xi_1(\mathsf{F}_1) \end{pmatrix} = b \middle| \begin{array}{r} x_1, \ldots, x_n, y_1, \ldots, y_m \leftarrow\$ \mathbb{F}_p, \\ b \leftarrow\$ \{0, 1\}, \\ \mathsf{F}_b \leftarrow \mathsf{F}(x_1, \ldots, x_n), \\ \mathsf{F}_{1-b} \leftarrow (y_1, \ldots, y_m) \end{array} \right] - \frac{1}{2} \right| \leq \frac{d(q + m_1 + m_2 + m_T + m)^2}{2p}$$

Proof. Let $\mathcal{C}$ be a challenger that plays with $\mathcal{A}$ in the following game. $\mathcal{C}$ maintains three lists

$$L_i = \{(p_j, \xi_{i,j}) \mid j \in [1 \,..\, \tau_i]\},$$

for $i \in \{1, 2, T\}$. Invariant τ states that $\tau_1 + \tau_2 + \tau_T = \tau + m_1 + m_2 + m$.

Challenger $\mathcal{C}$ answers $\mathcal{A}$'s oracle queries. However, it does it a bit differently that the oracle O would:

Group operations in $\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T$: On input $(\xi_{i,j}, \xi_{i,j'}, i, op)$, $j, j' \leq \tau_i$, $op \in \{\mathsf{add}, \mathsf{sub}\}$, $\mathcal{C}$ sets $\tau' \leftarrow \tau_i + 1$, computes $p_{i,\tau'}(X_1, \ldots, X_n) = p_{i,j}(X_1, \ldots, X_n) \pm p_{i,j'}(X_1, \ldots, X_n)$ respectively to *op*. If there is a polynomial $p_{i,k}(X_1, \ldots, X_n) \in L_i$ such that $p_{i,k}(X_1, \ldots, X_n) = p_{\tau'}(X_1, \ldots, X_n)$, then the challenger returns encoding of $p_{i,k}$. Otherwise it sets the encoding $\xi_{i,\tau'}$ to a new unused random string, adds $(p_{i,\tau'}, \xi_{i,\tau'})$ to L_i, and returns $\xi_{i,\tau'}$.

Bilinear pairing: On input $(\xi_{1,j}, \xi_{2,j'})$ the challenger sets $\tau' \leftarrow \tau_T + 1$ and computes $r_{\tau'}(X_1, \ldots, X_n) \leftarrow p_{i,j}(X_1, \ldots, X_n) \cdot p_{i,j'}(X_1, \ldots, X_n)$. If $r_{\tau'}(X_1, \ldots, X_n) \in L_T$, $\mathcal{C}$ returns encoding found in the list L_T. Else it picks a new unused random string and set $\xi_{T,\tau'}$ to it. Finally it returns the encoding to the algorithm.

After at most q queries to the oracle, the adversary returns a bit b'. At that point the challenger $\mathcal{C}$ chooses randomly $x_1, \ldots, x_n, y_1 \ldots, y_m$, random bit b, and sets $X_i = x_i$, for $i \in [1\,..\,n]$, and $Y_i = y_i$, for $i \in [1\,..\,m]$; furthermore, $\mathsf{F}_b \leftarrow \mathsf{F}(x_1, \ldots, x_n)$ and $\mathsf{F}_{1-b} \leftarrow (y_1, \ldots, y_m)$. Note that $\mathcal{C}$ simulates perfectly unless the chosen values $x_1, \ldots, x_n, y_1, \ldots, y_m$ result in equalities between polynomial evaluations that are not equalities between the polynomials. That is, the simulation is perfect unless for some i, j, j' holds

$$p_{i,j}(x_1, \ldots, x_n) - p_{i,j'}(x_1, \ldots, x_n) = 0,$$

for $p_{i,j}(X_1, \ldots, X_n) \neq p_{i,j'}(X_1, \ldots, X_n)$. Denote by bad an event that at least one of the three conditions holds. When bad happens, the answer $\mathcal{C}$ gives to $\mathcal{A}$ differs from an answer that a real oracle would give. We bound the probability that bad occurs in two steps.

First we set $\mathsf{F}_b = \mathsf{F}(X_1, \ldots, X_n)$. Note that symbolic substitutions do not introduce any new equalities in $\mathbb{G}_1$. That is, if for all j, j' holds $p_{1,j} \neq p_{1,j'}$, then $p_{1,j} \neq p_{1,j'}$ even after setting $\mathsf{F}_b = \mathsf{F}(X_1, \ldots, X_n)$. This follows since all polynomials in F are pairwise independent and F independent on $\mathsf{P}_1, \mathsf{P}_2, \mathsf{P}_T$. Indeed, $p_{1,j} - p_{1,j'}$ is a polynomial of the form

$$\sum_{j=1}^{m_1} a_j p_{1,j} + \sum_{j=1}^{m} b_j f_j(X_1, \ldots, X_n),$$

for some constants a_j, b_j. If the polynomial is non-zero, but setting $\mathsf{F}_b = \mathsf{F}(X_1, \ldots, X_n)$ makes this polynomial vanish, then some f_k must be dependent on some $\mathsf{P}_1, \mathsf{F} \setminus \{f_k\}$.

Now we set $X_1 \ldots, X_n, \mathsf{F}_{1-b}$ and bound probability that for some i and j, j' holds $(p_{i,j}(x_1, \ldots, x_n) - p_{i,j'}(x_1, \ldots, x_n) = 0$ for $p_{i,j} \neq p_{i,j'}$. By the construction, the maximum total degree of these polynomials is $d = \max(d_{\mathsf{P}_1} + d_{\mathsf{P}_2}, d_{\mathsf{P}_T}, d_{\mathsf{F}})$, where d_f is the total degree of some polynomial f and for a set of polynomials $F = \{f_1, \ldots, f_k\}$, we write $d_F = \{d_{f_1}, \ldots, d_{f_k}\}$. Thus, for a given j, j' probability that a random assignment to $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ is a root of $p_{i,j} - p_{i,j'}$ is, by the Schwartz-Zippel lemma, bounded by d/p, which is negligible. There is at most $2 \cdot \binom{q+m_0+m_1+m}{2}$ such pairs $p_{i,j}, p_{i,j'}$ we have that

$$\Pr[\mathsf{bad}] \leq \binom{q + m_0 + m_1 + m}{2} \cdot \frac{2d}{p} \leq (q + m_0 + m_1 + m)^2 \frac{d}{p}.$$

As noted, if bad does not occur then the simulation is perfect. Also the bit b has been chosen independently on the $\mathcal{A}$'s view, thus $\Pr[b = b' \mid \neg\mathsf{bad}] = 1/2$.

Hence,

$$\Pr[b = b'] \leq \Pr[b = b' \mid \neg\mathsf{bad}](1 - \Pr[\mathsf{bad}]) + \Pr[\mathsf{bad}] = \frac{1}{2} + \frac{\Pr[\mathsf{bad}]}{2}$$
$$\Pr[b = b'] \geq \Pr[b = b' \mid \neq \mathsf{bad}](1 - \Pr[\mathsf{bad}]) = \frac{1}{2} - \frac{\Pr[\mathsf{bad}]}{2}.$$

Finally,

$$\left|\Pr[b = b'] - \frac{1}{2}\right| \leq \Pr[\mathsf{bad}]/2 \leq (q + m_0 + m_1 + m)^2 \frac{d}{2p}$$

as required.

C.3 Special simulation-extractability of sigma protocols and forking lemma

Theorem 4 (Special simulation extractability of the Fiat–Shamir transform [25]). *Let $\Sigma = (\mathsf{P}, \mathsf{V}, \mathsf{Sim})$ be a non-trivial sigma protocol with unique responses for a language $\mathcal{L} \in \mathsf{NP}$. In the random oracle model, the NIZK proof system $\Sigma_{\mathsf{FS}} = (\mathsf{P}_{\mathsf{FS}}, \mathsf{V}_{\mathsf{FS}}, \mathsf{Sim}_{\mathsf{FS}})$ resulting by applying the Fiat–Shamir transform to Σ is special simulation extractable with extraction error $\eta = q/h$ for the simulator Sim. Here, q is the number of random oracle queries and h is the number of elements in the range of $\mathcal{H}$.*

The theorem relies on the following *general forking lemma* [47].

Lemma 7 (General forking lemma, cf. [8, 25]). *Fix $q \in \mathbb{Z}$ and a set H of size $h > 2$. Let $\mathcal{Z}$ be a PPT algorithm that on input $y, h_1, \ldots, h_q$ returns (i, s), where $i \in [0\,..\,q]$ and s is called a* side output*. Denote by IG a randomised instance generator. We denote by acc the probability*

$$\Pr[i > 0 \mid y \leftarrow \mathsf{IG}; h_1, \ldots, h_q \leftarrow\!\$\, H; (i, s) \leftarrow \mathcal{Z}(y, h_1, \ldots, h_q)]\,.$$

Let $\mathsf{F}_{\mathcal{Z}}(y)$ denote the algorithm described in Fig. 4, then the probability frk defined as $\mathsf{frk} := \Pr[b = 1 \mid y \leftarrow \mathsf{IG}; (b, s, s') \leftarrow \mathsf{F}_{\mathcal{Z}}(y)]$ holds

$$\mathsf{frk} \geq \mathsf{acc}\left(\frac{\mathsf{acc}}{q} - \frac{1}{h}\right).$$

C.4 Proof of the generalized forking lemma (Lemma 2)

Proof. First denote by $\mathsf{acc}(y)$ and $\mathsf{frk}(y)$ the following probabilities

$$\mathsf{acc}(y) = \Pr[i \neq 0 \mid h_1, \ldots, h_q \leftarrow\!\$\, H;\ (i, s) \leftarrow \mathcal{Z}(y, h_1, \ldots, h_q)]\,.$$
$$\mathsf{frk}(y) = \Pr[b = 1 \mid (b, \boldsymbol{s}) \leftarrow \mathsf{GF}^m_{\mathcal{Z}}(y, h_1, \ldots, h_q)]\,.$$

We start by claiming that for all y

$$\mathsf{frk}(y) \geq \frac{\mathsf{acc}(y)^m}{q^{m-1}} - \mathsf{acc}(y) \cdot \left(1 - \frac{h!}{(h-m)! \cdot h^m}\right) \tag{7}$$

$\mathsf{F}_{\mathcal{Z}}(y)$

$\rho \leftarrow\!\$\, \mathsf{R}(\mathcal{Z})$
$h_1, \ldots, h_q \leftarrow\!\$\, H$
$(i, s) \leftarrow \mathcal{Z}(y, h_1, \ldots, h_q; \rho)$
if $i = 0$ **return** $(0, \bot, \bot)$
$h'_i, \ldots, h'_q \leftarrow\!\$\, H$
$(i', s') \leftarrow \mathcal{Z}(y, h_1, \ldots, h_{i-1}, h'_i, \ldots, h'_q; \rho)$
if $(i = i') \wedge (h_i \neq h'_i)$ **return** $(1, s, s')$
else return $(0, \bot, \bot)$

Fig. 4: Forking algorithm $\mathsf{F}_{\mathcal{Z}}$

Then with the expectation taken over $y \leftarrow\!\$\, \mathsf{IG}$, we have

$$\mathsf{frk} = \mathbb{E}\left[\mathsf{frk}(y)\right] \geq \mathbb{E}\left[\frac{\mathsf{acc}(y)^m}{q^{m-1}} - \mathsf{acc}(y) \cdot \left(1 - \frac{h!}{(h-m)! \cdot h^m}\right)\right] \tag{8}$$

$$\geq \frac{\mathbb{E}\left[\mathsf{acc}(y)\right]^m}{q^{m-1}} - \mathbb{E}\left[\mathsf{acc}(y)\right] \cdot \left(1 - \frac{h!}{(h-m)! \cdot h^m}\right) \tag{9}$$

$$= \frac{\mathsf{acc}^m}{q^{m-1}} - \mathsf{acc} \cdot \left(1 - \frac{h!}{(h-m)! \cdot h^m}\right). \tag{10}$$

Where Eq. (8) comes from Eq. (7); Eq. (9) comes from linearity of expected value and Lemma 8; and Eq. (10) holds by the fact that $\mathbb{E}\left[\mathsf{acc}(y)\right] = \mathsf{acc}$.

We now show Eq. (7). Denote by $J = [1\,..\,m]^2 \setminus \{(j, j)\}_{j \in [1\,..\,m]}$. For any input y, with probabilities taken over the coin tosses of $\mathsf{GF}^m_{\mathcal{Z}}$ we have

$$\begin{aligned}
\mathsf{frk}(y) &= \Pr\left[i_j = i_{j'} \wedge i_j \geq 1 \wedge h^j_{i_j} \neq h^{j'}_{i_{j'}} \text{ for } (j, j') \in J\right] \\
&\geq \Pr[i_j = i_{j'} \wedge i_j \geq 1 \text{ for } (j, j') \in J] - \Pr\left[i_j \geq 1 \wedge h^j_{i_j} = h^{j'}_{i_{j'}} \text{ for some } (j, j') \in J\right] \\
&= \Pr[i_j = i_{j'} \wedge i_j \geq 1 \text{ for } (j, j') \in J] - \Pr[i_j \geq 1] \cdot \left(1 - \frac{h!}{(h-m)! \cdot h^m}\right) \\
&= \Pr[i_j = i_{j'} \wedge i_j \geq 1 \text{ for } (j, j') \in J] - \mathsf{acc}(y) \cdot \left(1 - \frac{h!}{(h-m)! \cdot h^m}\right).
\end{aligned}$$

Probability that for some $(j, j') \in J$ and $i_j = i_{j'}$ holds $h^j_{i_j} \neq h^{j'}_{i_{j'}}$ equals

$$\frac{h \cdot (h-1) \cdot \ldots \cdot (h-m-1)}{h^m} = \frac{h!}{(h-m)! \cdot h^m}.$$

That is, it equals the number of all m-element strings where each element is different divided by the number of all m-element strings, where elements are taken from a set of size h.

It remains to show that $\Pr[i_j = i_{j'} \wedge i_j \geq 1 \text{ for } (j,j') \in J] \geq \mathsf{acc}(y)^m/q^{m-1}$. Let $\mathsf{R}(\mathcal{Z})$ denote the set from which $\mathcal{Z}$ picks its coins at random. For each $\iota \in [1\,..\,q]$ let $X_\iota \colon \mathsf{R}(\mathcal{Z}) \times H^{\iota-1} \to [0,1]$ be defined by setting $X_\iota(\rho, h_1, \ldots, h_{\iota-1})$ to

$$\Pr[i = \iota \mid h_\iota, \ldots, h_q \leftarrow\!\!\$\, H; (i,s) \leftarrow \mathcal{Z}(y, h_1, \ldots, h_q; \rho)]$$

for all $\rho \in \mathsf{R}(\mathcal{Z})$ and $h_1, \ldots, h_{\iota-1} \in H$. Consider X_ι be a random variable over the uniform distribution on its domain. Then

$$\begin{aligned}
&\Pr[i_j = i_{j'} \wedge i_j \geq 1 \text{ for } (j,j') \in J] = \sum_{\iota=1}^{q} \Pr[i_1 = \iota \wedge \ldots \wedge i_m = \iota] \\
&= \sum_{\iota=1}^{q} \Pr[i_1 = \iota] \cdot \Pr[i_2 = \iota \mid i_1 = \iota] \cdot \ldots \cdot \Pr[i_m = \iota \mid i_1 = \ldots = i_{m-1} = \iota] \\
&= \sum_{\iota=1}^{q} \sum_{\rho, h_1, \ldots, h_{\iota-1}} X_\iota(\rho, h_1, \ldots, h_{\iota-1})^m \cdot \frac{1}{|\mathsf{R}(\mathcal{Z})| \cdot |H|^{\iota-1}} = \sum_{\iota=1}^{q} \mathbb{E}\left[X_\iota^m\right] .
\end{aligned}$$

Importantly, $\sum_{\iota=1}^{q} \mathbb{E}\left[X_\iota\right] = \mathsf{acc}(y)$.

By Lemma 8 we get

$$\sum_{\iota=1}^{q} \mathbb{E}\left[X_\iota^m\right] \geq \sum_{\iota=1}^{q} \mathbb{E}\left[X_\iota\right]^m .$$

Note that for e.g. $X_i = 1$, $i \in [1\,..\,q]$ the inequality becomes equality, that is, it is tight.

We now use the Hölder inequality, cf. Lemma 9, for $x_i = \mathbb{E}\left[X_i\right]$, $y_i = 1$, $p = m$, and $q = m/(m-1)$ obtaining

$$\left(\sum_{i=1}^{q} \mathbb{E}\left[X_i\right]\right)^m \leq \left(\sum_{i=1}^{q} \mathbb{E}\left[X_i\right]^m\right) \cdot q^{m-1} \tag{11}$$

$$\frac{1}{q^{m-1}} \cdot \mathsf{acc}(y)^m \leq \sum_{i=1}^{q} \mathbb{E}\left[X_i\right]^m . \tag{12}$$

Finally, we get

$$\mathsf{frk}(y) \geq \frac{\mathsf{acc}(y)^m}{q^{m-1}} - \mathsf{acc}(y) \cdot \left(1 - \frac{h!}{(h-m)! \cdot h^m}\right) .$$

□

Lemma 8. *Let $\mathsf{R}(\mathcal{Z})$ denote the set from which $\mathcal{Z}$ picks its coins at random. For each $\iota \in [1\,..\,q]$ let $X_\iota \colon \mathsf{R}(\mathcal{Z}) \times H^{\iota-1} \to [0,1]$ be defined by setting $X_\iota(\rho, h_1, \ldots, h_{\iota-1})$ to*

$$\Pr[i = \iota \mid h_\iota, \ldots, h_q \leftarrow\!\!\$\, H; (i,s) \leftarrow \mathcal{Z}(y, h_1, \ldots, h_q; \rho)]$$

for all $\rho \in \mathsf{R}(\mathcal{Z})$ and $h_1, \ldots, h_{\iota-1} \in H$. Consider X_ι as a random variable over the uniform distribution on its domain. Then $\mathbb{E}\left[X_\iota^m\right] \geq \mathbb{E}\left[X_\iota\right]^m$.

Proof. First we recall the Jensen inequality [52], if for some random variable X holds $|\mathbb{E}[X]| \leq \infty$ and f is a Borel convex function then

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)] .$$

Finally, we note that $|\mathbb{E}[X]| \leq \infty$ and taking to the m-th power is a Borel convex function on $[0, 1]$ interval. □

Lemma 9 (Hölder's inequality. Simplified.). *Let x_i, y_i, for $i \in [1 .. q]$, and p, q be real numbers such that $1/p + 1/q = 1$. Then*

$$\sum_{i=1}^{q} x_i y_i \leq \left(\sum_{i=1}^{q} x_i^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{q} y_i^p\right)^{\frac{1}{q}} . \tag{13}$$

Remark 1 (Tightness of the Hölder inequality). In is important to note that Inequality (13) is tight. More precisely, for $\mathbb{E}[X_i] = x$, $i \in [1 .. q]$ we have

$$\begin{gathered}\sum_{i=1}^{q} x = \left(\sum_{i=1}^{q} x^m\right)^{\frac{1}{m}} \cdot \left(\sum_{i=1}^{q} 1^{\frac{m}{m-1}}\right)^{\frac{m-1}{m}} \\ qx = (qx^m)^{\frac{1}{m}} \cdot q^{\frac{m-1}{m}} \\ (qx)^m = qx^m \cdot q^{m-1} \\ (qx)^m = (qx)^m .\end{gathered}$$

Lemma 10. *Let $\mathsf{f}(X)$ be a random degree-d polynomial over $\mathbb{F}_p[X]$. Then the probability that $\mathsf{f}(X)$ has roots in $\mathbb{F}_p$ is at least $1/d!$.*

Proof. First observe that there is p^d canonical polynomials in $\mathbb{F}_p[X]$. Each of the polynomials may have up to d roots. Consider polynomials which are reducible to polynomials of degree 1, i.e. polynomials that have all d roots. The roots can be picked in $\bar{C}_d^p$ ways, where $\bar{C}_k^n$ is the number of k-elements combinations with repetitions from n-element set. That is,

$$\bar{C}_k^n = \binom{n+k-1}{k} .$$

Thus, the probability that a randomly picked polynomial has all d roots is

$$\begin{aligned}p^{-d} \cdot \bar{C}_d^p = p^{-d} \cdot \binom{p+d-1}{d} = p^{-d} \cdot \frac{(p+d-1)!}{(p+d-1-d)! \cdot d!} = \\ p^{-d} \cdot \frac{(p+d-1) \cdot \ldots \cdot p \cdot (p-1)!}{(p-1)! \cdot d!} = p^{-d} \cdot \frac{(p+d-1) \cdot \ldots \cdot p}{d!} \\ \geq p^{-d} \cdot \frac{p^d}{d!} = \frac{1}{d!}\end{aligned}$$

□

D Non-malleability of $\mathsf{S_{FS}}$

D.1 Sonic protocol rolled out

In this section we present Sonic's constraint system and algorithms. Reader familiar with them may jump directly to the next section.

The constraint system Sonic's system of constraints composes of three n-long vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ which corresponds to left and right inputs to multiplication gates and their outputs. It hence holds $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{c}$.

There is also Q linear constrains of the form

$$\boldsymbol{a}\boldsymbol{u_q} + \boldsymbol{b}\boldsymbol{v_q} + \boldsymbol{c}\boldsymbol{w_q} = k_q,$$

where $\boldsymbol{u_q}, \boldsymbol{v_q}, \boldsymbol{w_q}$ are vectors for the q-th linear constraint with instance value $k_q \in \mathbb{F}_p$. Furthermore define polynomials

$$\begin{aligned} \mathsf{u_i}(Y) &= \sum_{q=1}^{\mathsf{Q}} Y^{q+\mathsf{n}} u_{q,i}\,, \qquad \mathsf{w_i}(Y) = -Y^i - Y^{-i} + \sum_{q=1}^{\mathsf{Q}} Y^{q+\mathsf{n}} w_{q,i}\,, \\ \mathsf{v_i}(Y) &= \sum_{q=1}^{\mathsf{Q}} Y^{q+\mathsf{n}} v_{q,i}\,, \qquad \mathsf{k}(Y) = \sum_{q=1}^{\mathsf{Q}} Y^{q+\mathsf{n}} k_q. \end{aligned} \tag{14}$$

In Sonic we will use commitments to the following polynomials.

$$\begin{aligned} \mathsf{r}(X,Y) &= \sum_{i=1}^{\mathsf{n}} \left(a_i X^i Y^i + b_i X^{-i} Y^{-i} + c_i X^{-i-\mathsf{n}} Y^{-i-\mathsf{n}}\right) \\ \mathsf{s}(X,Y) &= \sum_{i=1}^{\mathsf{n}} \left(u_i(Y) X^{-i} + v_i(Y) X^i + w_i(Y) X^{i+\mathsf{n}}\right) \\ \mathsf{t}(X,Y) &= \mathsf{r}(X,1)(\mathsf{r}(X,Y) + \mathsf{s}(X,Y)) - \mathsf{k}(Y)\,. \end{aligned}$$

Algorithms rolled out Sonic *SRS generation* $\mathsf{KGen}(\mathbf{R})$. The SRS generating algorithm picks randomly $\alpha, \chi \leftarrow\!\!\$\, \mathbb{F}_p$ and outputs

$$\mathsf{srs} = \left(\left[\{\chi^i\}_{i=-\mathsf{d}}^{\mathsf{d}}, \{\alpha\chi^i\}_{i=-\mathsf{d}, i\neq 0}^{\mathsf{d}}\right]_1, \left[\{\chi^i, \alpha\chi^i\}_{i=-\mathsf{d}}^{\mathsf{d}}\right]_2, [\alpha]_T\right)$$

Sonic *prover* $\mathsf{P}(\mathbf{R}, \mathsf{srs}, \mathsf{x}, \mathsf{w} = \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$.

Round 1 The prover picks randomly randomisers $c_{\mathsf{n}+1}, c_{\mathsf{n}+2}, c_{\mathsf{n}+3}, c_{\mathsf{n}+4} \leftarrow\!\!\$\, \mathbb{F}_p$. Set $\mathsf{r}(X,Y) \leftarrow \mathsf{r}(X,Y) + \sum_{i=1}^{4} c_{\mathsf{n}+i} X^{-2\mathsf{n}-i}$. Commits to $\mathsf{r}(X,1)$ and outputs $[r]_1 \leftarrow \mathsf{Com}(\mathsf{srs}, \mathsf{n}, \mathsf{r}(X,1))$. Then it gets challenge y from the verifier.

Round 2 P commits to $\mathsf{t}(X,y)$ and outputs $[t]_1 \leftarrow \mathsf{Com}(\mathsf{srs}, \mathsf{d}, \mathsf{t}(X,y))$. Then it gets a challenge z from the verifier.

Round 3 The prover computes commitment openings. That is, it outputs

$$[o_a]_1 = \mathsf{Op}(\mathsf{srs}, z, \mathsf{r}(z,1), \mathsf{r}(X,1))$$

$$[o_b]_1 = \mathsf{Op}(\mathsf{srs}, yz, \mathsf{r}(yz,1), \mathsf{r}(X,1))$$
$$[o_t]_1 = \mathsf{Op}(\mathsf{srs}, z, \mathsf{t}(z,y), \mathsf{t}(X,y))$$

along with evaluations $a' = \mathsf{r}(z,1), b' = \mathsf{r}(y,z), t' = \mathsf{t}(z,y)$. Then it engages in the signature of correct computation playing the role of the helper, i.e. it commits to $\mathsf{s}(X,y)$ and sends the commitment $[s]_1$, commitment opening

$$[o_s]_1 = \mathsf{Op}(\mathsf{srs}, z, \mathsf{s}(z,y), \mathsf{s}(X,y)),$$

and $s' = \mathsf{s}(z,y)$. Then it obtains a challenge u from the verifier.

Round 4 In the next round the prover computes $[c]_1 \leftarrow \mathsf{Com}(\mathsf{srs}, \mathsf{d}, \mathsf{s}(u,Y))$ and computes commitments' openings

$$[w]_1 = \mathsf{Op}(\mathsf{srs}, u, \mathsf{s}(u,y), \mathsf{s}(X,y)),$$
$$[q_y]_1 = \mathsf{Op}(\mathsf{srs}, y, \mathsf{s}(u,y), \mathsf{s}(u,Y)),$$

and returns $[w]_1, [q_y]_1, s = \mathsf{s}(u,y)$. Eventually the prover gets the last challenge from the verifier—z'.

Round 5 In the final round, P computes opening $[q_{z'}]_1 = \mathsf{Op}(\mathsf{srs}, z', \mathsf{s}(u,z'), \mathsf{s}(u,X))$ and outputs $[q_{z'}]_1$.

Sonic *verifier* $\mathsf{V}(\mathbf{R}, \mathsf{srs}, \mathsf{x}, \pi)$. The verifier in Sonic runs as subroutines the verifier for the polynomial commitment. That is it sets $t' = a'(b' + s') - \mathsf{k}(y)$ and checks the following:

$$\begin{aligned}
&\mathbf{PC_S}.\mathsf{V}(\mathsf{srs}, \mathsf{n}, [r]_1, z, a', [o_a]_1),\\
&\mathbf{PC_S}.\mathsf{V}(\mathsf{srs}, \mathsf{n}, [r]_1, yz, b', [o_b]_1),\\
&\mathbf{PC_S}.\mathsf{V}(\mathsf{srs}, \mathsf{d}, [t]_1, z, t', [o_t]_1),\\
&\mathbf{PC_S}.\mathsf{V}(\mathsf{srs}, \mathsf{d}, [s]_1, z, s', [o_s]_1),
\end{aligned}
\qquad
\begin{aligned}
&\mathbf{PC_S}.\mathsf{V}(\mathsf{srs}, \mathsf{d}, [s]_1, u, s, [w]_1),\\
&\mathbf{PC_S}.\mathsf{V}(\mathsf{srs}, \mathsf{d}, [c]_1, y, s, [q_y]_1),\\
&\mathbf{PC_S}.\mathsf{V}(\mathsf{srs}, \mathsf{d}, [c]_1, z', \mathsf{s}(u,z'), [q_{z'}]_1),
\end{aligned}$$

and accepts the proof iff all the checks holds. Note that the value $\mathsf{s}(u,z')$ that is recomputed by the verifier uses separate challenges u and z'. This enables the batching of many proof and the outsourcing of this part of the proof to an untrusted helper.

D.2 Unique opening property of $\mathbf{PC_S}$

Lemma 11. $\mathbf{PC_S}$ *has the unique opening property in the AGM.*

Proof. Let $z \in \mathbb{F}_p$ be the attribute the polynomial is evaluated at, $[c]_1 \in \mathbb{G}$ be the commitment, $s \in \mathbb{F}_p$ the evaluation value, and $o \in \mathbb{G}$ be the commitment opening. We need to show that for every PPT adversary $\mathcal{A}$ probability

$$\Pr\left[\begin{array}{l|l} \mathsf{Vf}(\mathsf{srs}, [c]_1, z, s, [o]_1) = 1, & \mathsf{srs} \leftarrow \mathsf{KGen}(1^\lambda, \mathsf{max}),\\ \mathsf{Vf}(\mathsf{srs}, [c]_1, z, \tilde{s}, [\tilde{o}]_1) = 1 & ([c]_1, z, s, \tilde{s}, [o]_1, [\tilde{o}]_1) \leftarrow \mathcal{A}(\mathsf{srs}) \end{array}\right]$$

is at most negligible.

As noted in [29, Lemma 2.2] it is enough to upper bound the probability of the adversary succeeding using the idealised verification equation—which considers equality between polynomials—instead of the real verification equation—which considers equality of the polynomials' evaluations.

For a polynomial f, its degree upper bound max, evaluation point z, evaluation result s, and opening $[o(X)]_1$ the idealised check verifies that

$$\alpha(X^{\mathsf{d}-\mathsf{max}} f(X) \cdot X^{-\mathsf{d}+\mathsf{max}} - s) \equiv \alpha \cdot o(X)(X - z)\,, \tag{15}$$

what is equivalent to

$$f(X) - s \equiv o(X)(X - z)\,. \tag{16}$$

Since $o(X)(X-z) \in \mathbb{F}_p[X]$ then from the uniqueness of polynomial composition, there is only one $o(X)$ that fulfils the equation above. □

D.3 Unique response property

The unique response property of $\mathbf{S}$ follows from the unique opening property of the used polynomial commitment scheme $\mathbf{PC}_{\mathbf{S}}$.

Lemma 12. *If a polynomial commitment scheme* $\mathbf{PC}_{\mathbf{S}}$ *is evaluation binding with parameter* $\varepsilon_{\mathsf{bind}}(\lambda)$ *and has unique openings property with parameter* $\varepsilon_{\mathsf{op}}(\lambda)$*, then* $\mathbf{S}$ *is* 1-ur *with parameter* $\varepsilon_{\mathsf{ur}}(\lambda) \leq \varepsilon_{\mathsf{bind}}(\lambda) + \varepsilon_{\mathsf{op}}(\lambda)$.

Proof. Let $\mathcal{A}$ be an adversary that breaks 1-ur-ness of $\mathbf{S}$. We consider two cases, depending on which round $\mathcal{A}$ is able to provide at least two different outputs such that the resulting transcripts are acceptable. For the first case we show that $\mathcal{A}$ can be used to break the evaluation binding property of $\mathbf{PC}_{\mathbf{S}}$, while for the second case we show that it can be used to break the unique opening property of $\mathbf{PC}_{\mathbf{S}}$.

The proof goes similarly to the proof of Lemma 4 thus we provide only draft of it here. In each Round i, for $i > 1$, the prover either commits to some well-defined polynomials (deterministically), evaluates these on randomly picked points, or shows that the evaluations were performed correctly. Obviously, for a committed polynomial p evaluated at point x only one value $y = \mathsf{p}(x)$ is correct. If the adversary was able to provide two different values y and $\tilde{y}$ that would be accepted as an evaluation of p at x then the $\mathbf{PC}_{\mathbf{S}}$'s evaluation binding would be broken. Alternatively, if $\mathcal{A}$ was able to provide two openings W and $\tilde{\mathsf{W}}$ for $y = \mathsf{p}(x)$ then the unique opening property would be broken. Hence the probability that $\mathcal{A}$ breaks 1-ur-property of $\mathbf{PC}_{\mathbf{S}}$ is upper-bounded by $\varepsilon_{\mathsf{bind}}(\lambda) + \varepsilon_{\mathsf{op}}(\lambda)$. □

D.4 Forking soundness

Lemma 13. $\mathbf{S}$ *is* $(\varepsilon_{\mathsf{f}}(\lambda), 2, \mathsf{n}+\mathsf{Q}+1)$*-forking sound against algebraic adversaries, with*

$$\varepsilon_{\mathsf{f}}(\lambda) \leq \varepsilon_{\mathsf{s}}(\lambda) + \varepsilon_{\mathsf{ldlog}}(\lambda)\,,$$

where $\varepsilon_{\mathsf{s}}(\lambda)$ *is a soundness error of the protocol, and* $\varepsilon_{\mathsf{ldlog}}(\lambda)$ *is security of* (d, d)-ldlog *assumption.*

Proof. The proof goes similarly to the proof of Lemma 5. Let $\mathcal{A}$ be an adversary that produces a $(1, \mathsf{n}+\mathsf{Q}+1, 1, 1)$-tree of acceptable transcripts T for a statement x. We consider two disjunctive events E and $\overline{\mathsf{E}}$. The first corresponds to a case when all transcripts in T are acceptable for the ideal verifier, i.e. $\mathbf{ve}_{\mathbf{x},\boldsymbol{\pi}}(X) = \mathbf{0}$. In that case we show an extractor $\mathsf{Ext}_{\mathsf{ss}}$ that from T extracts a valid witness w. The second, corresponds to a case when T contains a transcript that is acceptable by the real verifier but is not acceptable by the ideal verifier. In that case we show a reduction $\mathcal{R}_{\mathsf{ldlog}}$ that uses $\mathcal{A}$ to break the (d, d)-ldlog assumption.

When E happens: Since $\mathbf{S}$ is statistically sound regarding the ideal verifier, for an acceptable proof π for a statement x there exists a witness w such that $\mathbf{R}(\mathsf{x}, \mathsf{w})$ holds and the polynomial $\mathsf{r}(X, y)$ contains witness at its coefficients. Note that the polynomial $\mathsf{r}(X, y)$, which has witness elements at its coefficients, has degree at most $\mathsf{n}+\mathsf{Q}$ and since $\mathcal{A}$ answered correctly on $(\mathsf{n}+\mathsf{Q}+1)$ different challenges z (for the sake of concreteness let us call them $z_1, \ldots, z_{\mathsf{n}+\mathsf{Q}+1}$) then $(\mathsf{n}+\mathsf{Q}+1)$ evaluations $\mathsf{r}(z_1, y), \ldots, \mathsf{r}(z_{\mathsf{n}+\mathsf{Q}+1}, y)$ of $\mathsf{r}(X, y)$ are known. The extractor $\mathsf{Ext}_{\mathsf{ss}}$ interpolates $\mathsf{r}(X, y)$ and reveals the corresponding witness w.

When $\overline{\mathsf{E}}$ happens: Consider a transcript such that for some verification equation $\mathsf{ve}_{i,\mathsf{x},\pi}(X) \neq 0$, but $\mathsf{ve}_{i,\mathsf{x},\pi}(\chi) = 0$. Since the adversary is algebraic, all group elements included in the tree of transcripts are extended by their representation as a combination of the input $\mathbb{G}_1$ or $\mathbb{G}_2$-elements. Hence all coefficients of the verification equation polynomial $\mathsf{ve}_{i,\mathsf{x},\pi}(X)$ are known and $\mathcal{R}_{\mathsf{dlog}}$ can find its zero points. Since $\mathsf{ve}_{i,\mathsf{x},\pi}(\chi) = 0$, the targeted discrete log value χ is among them. □

D.5 Honest verifier zero-knowledge

Lemma 14. Sonic *is honest verifier zero-knowledge.*

Proof. The simulator proceeds as follows. In the first round, it picks randomly vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and sets

$$\boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b}. \tag{17}$$

Then it pick randomisers $c_{\mathsf{n}+1}, \ldots, c_{\mathsf{n}+4}$, honestly computes polynomials $\mathsf{r}(X, Y), \mathsf{r}'(X, Y), \mathsf{s}(X, Y)$ and $\mathsf{t}(X, Y)$ and concludes the first round as an honest prover would. Because of the randomisers the polynomial $\mathsf{r}(X, Y)$ computed by the simulator is indistinguishable from a polynomial provided by an honest user for an adversary that only learns $[r]_1, [t]_1, a', b'$ from the proof (the other proof elements are fixed by these elements and the public information, see Supp. Mat. D.1).

Next, Sim computes the first verifier's challenge y such that $\mathsf{t}(X, y)$ is a polynomial that has 0 as a coefficient next to X^0. I.e. $\mathsf{t}(0, y) = 0$. By the definition of $\mathsf{t}(X, Y)$, the coefficient next to X^0 in $\mathsf{t}(X, Y)$ equals

$$\mathsf{t}(0, Y) = \boldsymbol{a} \cdot \mathbf{u}(Y) + \boldsymbol{b} \cdot \mathbf{v}(Y) + \boldsymbol{c} \cdot \mathbf{w}(Y) + \sum_{i=1}^{\mathsf{n}} a_i b_i (Y^i + Y^{-i}) - \mathsf{k}(Y), \tag{18}$$

for public $\mathbf{u}(Y), \mathbf{v}(Y), \mathbf{w}(Y), \mathsf{k}(Y)$ as defined in Supp. Mat. D.1 (Vectors $\boldsymbol{u_q}, \boldsymbol{v_q}, \boldsymbol{w_q}$ are n-elements long and correspond to Q linear constrains of the

proof system. Field element k_q is the instance value). When the proven instance is correct, $\mathsf{t}(0, Y)$ is a zero polynomial. See [45] for details. Also, when Eq. (17) holds, that polynomial simplifies to

$$\mathsf{t}(0, Y) = \boldsymbol{a} \cdot \mathbf{u}(Y) + \boldsymbol{b} \cdot \mathbf{v}(Y) + \boldsymbol{c} \cdot \tilde{\mathbf{w}}(Y) - \mathsf{k}(Y), \tag{19}$$

where $\tilde{\mathbf{w}}(Y)$ is defined as

$$\tilde{\mathsf{w}}_\mathsf{i}(Y) = \sum_{q=1}^{\mathsf{Q}} Y^{q+\mathsf{n}} w_{q,i}\,.$$

Note that $\mathsf{t}(0, Y)$ is a "classical", i.e. non-Laurent, polynomial. Also, it is a polynomial of degree $\mathsf{Q} + \mathsf{n}$ with 0 coefficients for Y^i, for $i \in [0\,..\,\mathsf{n}]$. Also, since $\boldsymbol{a}, \boldsymbol{b}$ were picked at random, $\mathsf{t}(0, Y)/Y^{\mathsf{n}+1}$ is a degree-$(\mathsf{Q}-1)$ polynomial of random coefficients. Recall, that the view of the adversary is independent of $\boldsymbol{a}, \boldsymbol{b}$ because of randomisers $c_{\mathsf{n}+1}, \ldots, c_{\mathsf{n}+4}$.

The probability that a random degree-$(\mathsf{Q}-1)$ polynomial over $\mathbb{F}_p[Y]$ has a root is at least $1/(\mathsf{Q}-1)!$, see Lemma 10 for a proof of that bound. Since we assume that $\mathsf{Q} = \mathsf{poly}(\lambda)$, we can say that the polynomial $\mathsf{t}(0, Y)$ as computed by the simulator has roots with non-negligible probability. Furthermore, these roots can be found and the simulator picks fresh $\boldsymbol{a}, \boldsymbol{b}$ until $\mathsf{t}(0, Y)$ has a root. As the roots of a random polynomial are themselves random $\mathbb{F}_p$ elements, the challenge y picked by the simulator comes from the same distribution as if it was picked by an honest verifier.

The simulator continues building the transcript by honestly computing the prover's messages and by picking verifier's challenges at random. This and the fact that $\mathsf{t}(0, y) = 0$ guarantees that the transcript provided by the simulator is acceptable and comes from the same distribution as a transcript of an honest prover and verifier. □

Remark 2. As noted in [45], Sonic is statistically subversion-zero knowledge (Sub-ZK). As noted in [1], one way to achieve subversion zero knowledge is to utilise an extractor that extracts a SRS trapdoor from a SRS-generator. Unfortunately, a NIZK made subversion zero-knowledge by this approach cannot achieve perfect Sub-ZK as one has to count in the probability of extraction failure. However, with the simulation presented in Lemma 14, the trapdoor is not required for the simulator as it is able to simulate the execution of the protocol just by picking appropriate (honest) verifier's challenges. This result transfers to $\mathbf{S}_{\mathsf{FS}}$, where the simulator can program the random oracle to provide challenges that fits it.

D.6 From forking soundness and unique response property to forking simulation extractability of $\mathbf{S}_{\mathsf{FS}}$

Since Lemmas 12 and 13 hold, $\mathbf{S}$ is 1-ur and forking sound. We now make use of Theorem 2 and show that $\mathbf{S}_{\mathsf{FS}}$ is forking simulation-extractable as defined in Definition 2.

Corollary 3 (Forking simulation extractability of $\mathbf{S}_{\mathsf{FS}}$). *Assume that* $\mathbf{S}$ *is* 1-ur *with security* $\varepsilon_{\mathsf{ur}}(\lambda)$*, and forking-sound with security* $\varepsilon_{\mathsf{f}}(\lambda)$*. Let* $\mathcal{H}\colon \{0,1\}^* \to \{0,1\}^\lambda$ *be a random oracle. Let* $\mathcal{A}$ *be a* PPT *adversary that can make up to* q *random oracle queries and outputs an acceptable proof for* $\mathbf{S}_{\mathsf{FS}}$ *with probability at least* acc*. Then* $\mathbf{S}_{\mathsf{FS}}$ *is forking simulation-extractable with extraction error* $\eta = \varepsilon_{\mathsf{ur}}(\lambda)$*. The extraction probability* ext *is at least*

$$\mathsf{ext} \geq \frac{1}{q^{\mathsf{n}+\mathsf{Q}}}(\mathsf{acc} - \varepsilon_{\mathsf{ur}}(\lambda))^{\mathsf{n}+\mathsf{Q}+1} - \varepsilon(\lambda).$$

for some negligible $\varepsilon(\lambda)$*,* n *and* Q *being, respectively, the number of multiplicative and linear constrains of the system.*