# Non-Malleability of the Fiat–Shamir Transform Revisited for Multi-round SRS-Based Protocols

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**Abstract.** The Fiat–Shamir transformation turns public-coin (three round) sigma protocol into signature schemes, non-interactive proof systems, and signatures of knowledge (SoK). The security of the transformation relies on a powerful forking lemma that extracts the secret key or the witness, even in the presence of signing queries for signatures and simulation queries for prove systems and SoK, respectively. We extend this line of work and formally define simulation extractability for protocols in the random oracle model (ROM) which use a structured reference string (SRS). We then show sufficient conditions for compiling via the Fiat–Shamir transformation public-coin multi-round interactive protocol with SRS into simulation-extractable NIZK proof systems. We also consider the case that the SRS is updatable and define a strong simulation extractability notion that allows for simulated proofs with respect to past and present SRS. In the ROM, we obtain simulation-extractable and updatable NIZKs. Importantly, we show that thee popular zero knowledge SNARKs — Plonk, Sonic, and Marlin — are simulation extractable out-of-the-box. This also results in the first construction of update simulation-extractable SNARKs and succinct updatable SoK.

## 1 Introduction

Zero-knowledge proof systems that allow a prover to convince a verifier of a statement without revealing anything beyond the truth of the statement have applications in cryptography and theory of computation [8, 23, 28]. When restricted to computationally sound proofs, called *argument systems*, proof length can be shorter than the length of the witness [**?**]. Zero-knowledge Succinct Non-interactive ARguments of Knowledge (zkSNARKs) are zero-knowledge argument systems that additionally have a succinctness property – small proof sizes and fast verification. Since their introduction in [**?**], zk-SNARKs have been a powerful and versatile design tool for secure cryptographic protocols. They became particularly relevant for blockchain applications that demand short proofs and fast verification, such as privacy-preserving cryptocurrencies [9] in Zcash and scalable and private smart contracts in Ethereum[4].

The work of [26] proposed a preprocessing zk-SNARK for general NP statements phrased in the language of Quadratic Span Programs (QSP) and Quadratic Arithmetic Programs (QAP) for Boolean and arithmetic circuits respectively. This built on previous works of [**?**, 31, 39] and led to several works [10, 12, 13, 32, 40, 43] with very short proof sizes and fast verification. The line of work on pre-processing zkSNARKs has seen rapid progress with many works proposing significant improvements in efficiency of different parameters like proof size, verifier efficiency, complexity of setup etc. Most modern zkSNARK constructions follow a modular blueprint that involves the design of an information theoretic interactive protocol, e.g. an Interactive Oracle Proof (IOP), that is then compiled via cryptographic tools to obtain an interactive argument system. This is then turned into a zkSNARK using the Fiat-Shamir transformation in the Random Oracle Model (ROM). In particular, several schemes such as Sonic in [42], Plonk [25], Marlin [18] follow this approach where the information theoretic object is an algebraic variant of IOP, and the cryptographic primitive in the compiler is a polynomial commitment scheme (PC).

*Simulation extractability.* Most zkSNARKs are shown to satisfy a standard knowledge soundness property. Intuitively, this guarantees that a prover that creates a valid proof knew a valid witness. However, deployments of zkSNARKs in real-world applications require a stronger property – *simulation-extractability* (SE). This is because, in practice, an adversary against the zkSNARK has access to proofs provided by

[4] https://z.cash/,https://ethereum.org

other parties using the same zkSNARK. The definition of knowledge soundness ignores the ability of an adversary to see other valid proofs that may occur in real-world applications. For instance, in applications of zkSNARKs in privacy-preserving blockchains, proofs are posted on the chain for all blockchain-participants to see. Therefore, it is necessary for a zero-knowledge proof system to be *non-malleable*, that is, resilient against adversaries that additionally get to see proofs generated by different parties before trying to forge. Therefore, it is necessary for a zero-knowledge proof system to be *simulation-extractable*, that is, resilient against adversaries that additionally get to see proofs generated by different parties before trying to forge. This captures the more general scenario where an adversary against the zkSNARK has access to proofs provided by other parties as it is in applications of zkSNARKs in privacy-preserving blockchains, where proofs are posted on the chain for all participants in the network to verify.

*zkSNARKs in the updatable setting.* One of the downsides of efficient zkSNARKs like [19, 26, 31, 32, 39, 40, 43] is that they rely on a *trusted setup*, where there is a structured reference string (SRS) that is assumed to be generated by a trusted party. In practice, however, this assumption is not well founded; if the party that generates the SRS is not honest, then they can produce proofs of false statement. That is, if the trusted setup assumption does not hold, knowledge soundness breaks down. Groth et al [33] propose a setting to tackle this challenge which allows parties – provers and verifiers – to *update* the SRS, that is, take a current SRS and contribute to it randomness in a verifiable way to obtain a new SRS. The guarantee in this *updatable setting* is that knowledge soundness holds as long as one of the parties who updates the SRS is honest. The SRS is also *universal*, in that it does not depend on the relation to be proved but only an upper bound on the size of the statements. Although inefficient, as the SRS length is quadratic to the size of the statement, [33] set a new paradigm of universal updatable setting for designing zkSNARKs.

The first universal zkSNARK with updatable and linear size SRS was Sonic proposed by Maller et al. in [42]. Subsequently, Gabizon et al. designed Plonk [25] which currently is the most efficient updatable universal zkSNARK. Independently, Chiesa et al. [18] proposed Marlin with comparable efficiency to Plonk.

*The challenge of SE in the updatable setting.* The notion of simulation-extractability for zkSNARKs which is well motivated in practice has not been studied in this updatable setting. Consider the following scenario: an instance proof pair with respect to some SRS is available for public verification, $(\mathsf{srs}, x, \pi)$. Now, if there is a new purported proof $(\mathsf{srs}', x, \pi')$ with respect to an updated $\mathsf{srs}'$, we would like the guarantee that the prover must have known a witness corresponding to $x$, and therefore computed $\pi'$. Since everybody is allowed to update an SRS, the ability for an adversary to perform an update $\mathsf{srs}$ to $\mathsf{srs}'$, and "move" the proof $\pi$ from the old SRS to a proof $\pi'$ for the new SRS without knowing a witness clearly violates security. That is, even an adversary who knows the trapdoor for the update from $\mathsf{srs}$ to $\mathsf{srs}'$ should not be able to break SE as long as there was at least one honest update to $\mathsf{srs}$.

As it turns out, defining SE for updatable SRS zkSNARKs requires some care. Since the SRS is being continually updated, there are proofs with respect to *different* SRSs available for the adversary to see before attempting to forge a proof with respect to a current SRS. That is, each SRS in the update chain spawns a simulation oracle. Intuitively, the updatability of the SRS allows an adversarial prover to contribute to updating, and see proofs with respect to different updated SRSs before attempting to provide a proof for a false statement (potentially output a proof wrt a SRS that is different from the SRSs corresponding to all the simulated proofs seen). A definition of SE in the updatable setting should take into account this additional power of the adversary, which is not captured by existing definitions of SE. While generic lifting techniques/compilers [1, 38] can be applied to updatable SRS SNARKs to obtain SE, not only do they inevitably incur overheads and lead to efficiency loss, we contend that the standard definition of SE does not suffice in the updatable setting.

*Fiat–Shamir.* The Fiat–Shamir (FS) transform takes a public-coin interactive protocol and makes it interactive by hashing the current protocol transcript to compute the verifier's public coins. While in principle justifiable in the random oracle model (ROM) [7], it is theoretically unsound [29] and so only a heuristic that should be used with care. The FS transform is a now popular design tool in constructing zkSNARKs. In the updatable universal SRS setting, works like Sonic [42] Plonk [25], and Marlin [18]

are designed and proven secure as multi-round interactive protocols. Security is then only *conjectured* for their non-interactive variants by employing the FS transform.

We investigate the non-malleability properties of a class of zkSNARK protocols obtained by FS-compiling multi-round protocols in the updatable SRS setting and give a modular approach to analyze non-malleablilty of zkSNARKs.

### 1.1 Our Contributions

- *Updatable simulation extractability (USE).* We propose a definition of simulation extractability in the updatable SRS setting called USE, that captures the additional power the adversary gets by having access to updating the SRS and seeing proofs with respect to different SRSs.
- *General theorem for USE of FS-compiled interactive protocols.* We then show that a class of interactive proofs of knowledge that are honest-verifier zero-knowledge, have a unique response property in the updatable setting, and satisfy a property we define called forking soundness *are USE out-of-the box* in the random oracle model when the Fiat–Shamir transformation is applied to them. Informally, our notion of forking soundness is a variant of special soundness where the transcripts provided to the extractor are obtained through interaction with an honest verifier, and the extraction guarantee is computational instead of unconditional. Our extractor only needs oracle access to the adversary, it does not depend on the adversary's code, nor relies on knowledge assumptions.
- *USE for concrete zkSNARKs.* We then prove that the most efficient updatable SRS SNARKS – Plonk/Sonic/Marlin – satisfy the notions necessary to invoke our general theorem, thus showing that these zkSNARKs are updatable simulation extractable. Proving that these protocols satisfy the required properties is done in the algebraic group model (AGM).

### 1.2 Technical Overview

The proof of our general theorem for USE is, at a high level, along the lines of the proof of SE for FS-compiled sigma protocol from [21]. However, we need new and stronger notions as we consider proof systems that are richer than simple sigma protocols **[Hamid:** and moreover they are in the stronger updatable setting]. We discuss some of the technical challenges below.

Plonk [25] and Sonic [42] were originally presented as interactive proofs of knowledge that are made non-interactive by the Fiat–Shamir transform. In the following, we denote the underlying interactive protocols by $\mathsf{P}$ (for Plonk) and $\mathsf{S}$ (for Sonic) and the resulting non-interactive proof systems by $\mathsf{P}_{\mathsf{FS}}$ and $\mathsf{S}_{\mathsf{FS}}$, respectively.

**Forking soundness in the updatable setting.** Following [21], one would have to show that for the protocols we consider, a witness can be extracted from sufficiently many valid transcripts with a common prefix. However, many protocols do not meet the standard definition of special soundness for sigma protocols, that requires extraction of a witness from any two transcripts, with the same first message. We put forth a notion analogous to special soundness, that is more general and applicable to a wider class of protocols – protocols that are more than three rounds, and rely on an updatable SRS. For $\mathsf{P}$ and $\mathsf{S}$ that are not three move protocols, the definition needs to be adapted. Furthermore, the number of transcripts required for extraction is more than two. Concretely, $(3\mathsf{n} + 1)$—where $\mathsf{n}$ is the number of constraints in the proven circuit—for $\mathsf{P}$ and $(\mathsf{n} + \mathsf{Q} + 1)$—where $\mathsf{n}$ and $\mathsf{Q}$ are the numbers of multiplicative and linear constraints—for $\mathsf{S}$. Hence, we do not have a pair of transcripts, but a *tree of transcripts*.

In protocols that rely on SRS that come with a trapdoor, an adversary in possession of the trapdoor can produce multiple valid proof transcripts without knowing the witness and potentially for false statements. This is true even in the updatable setting, where there exists a trapdoor for any updated SRS. Recall that the standard special soundness definition requires witness extraction from *any* tree of acceptable transcripts that share a common root. This means that there are no such trees for false statements. We define a different, forking lemma-related, version of soundness that we call forking soundness. Forking soundness guarantees that it is possible to extract a witness from all (but negligibly many) trees of accepting transcripts produced by probabilistic polynomial time (PPT) adversaries, given that the trees are generated as interactions between a (possibly malicious) prover and an honest verifier. That is, if

extraction from such a tree fails, then we break an underlying computational assumption.[**Hamid:** The fact that the definition is in the updatable setting indicates that this should hold even against adversaries that contribute in the SRS generation.]

**Unique response property in the updatable setting.** Another property required to show USE is the unique response property [22] which says that for 3-messages sigma protocols, all but the first message sent by the prover are deterministic (intuitively, the prover can only employ fresh randomness in the first message of the protocol). We cannot use this definition as is since the protocols we consider have other rounds where the prover messages are randomized. In $\mathsf{P}_{\mathsf{FS}}$, both the first and the second prover's messages are randomized. Although $\mathsf{S}$ prover is deterministic after it picks its first message, the protocol has more than 3 rounds. We propose a generalisation of the definition which states that a protocol is $i$-ur if the prover is deterministic starting from its $(i+1)$-th message. For our proof it is sufficient that this property is met by $\mathsf{P}$ for $i = 2$. Since Sonic's prover is deterministic from the second message on, it is 1-ur.

**HVZK.** In order to invoke our main theorem on $\mathsf{P}_{\mathsf{FS}}$ and $\mathsf{S}_{\mathsf{FS}}$ to conclude USE, we also need to show that (interactive) $\mathsf{P}$ and $\mathsf{S}$ are HVZK in the standard model. Although both Sonic and Plonk are zero-knowledge, their simulators utilize trapdoors. However, for our reduction, we need simulators that rely only on reordering the messages and picking suitable verifier challenges, without knowing the SRS trapdoor. That is, any PPT party should be able to produce a simulated proof by its own in a trapdoor-less way. (Note that this property does not necessary break soundness of the protocol as the simulator is required only to produce a transcript and is not involved in a real conversation with a real verifier). We show simulators for $\mathsf{P}_{\mathsf{FS}}$ and $\mathsf{S}_{\mathsf{FS}}$ that rely only on the programmability of the RO, where programmability is only needed from some round $i$ onwards. [**Chaya:** revisit this. is HVZK for the interactive protocol? then why programming? it might be a good idea to elaborate on why a trapdoor-based simulator does not work in the reduction. I am not sure I have clarity on this.] Michal 16.09: maybe we should just define Trapdoor-less simulatable (TLS) protocols?

**Generalisation of the general forking lemma.** Consider an interactive 3-message special-sound protocol $\Psi$ and its non-interactive version $\Psi_{\mathsf{FS}}$ obtained by the Fiat–Shamir transform. The general forking lemma provides an instrumental lower bound for the probability of extracting a witness from an adversary who provides two proofs for the same statement that share the first message. Since $\mathsf{P}$ and $\mathsf{S}$ have more than 3 messages and are not special-sound, the forking lemma of Bellare and Neven [6], cannot be used directly. We propose a generalization that covers multi-message protocols where witness extraction requires more transcripts than merely two. Unfortunately, we also observe that the security gap grows with the number of transcripts and the probability that the extractor succeeds diminishes significantly; the security loss, albeit big, is polynomial.

Most modern zkSNARKs [16, 42] heavily rely on the Fiat–Shamir transform and thus potentially the forking lemma. First, an interactive protocol is proposed and its security and forking soundness analysed. Second, one uses an argument that the Fiat–Shamir transform can be used to get a protocol that is non-interactive and shares the same security properties.

We see our generalized forking lemma as contributing to a critical assessment of this approach. The analysis of the interactive protocol is not enough and one has to consider the security loss implied by the Fiat-Shamir transform for the target security notion. Thus one has to either rely on our generalisation of the forking lemma or disclose a transformation that does not suffer this loss. We note that the security loss may also apply when knowledge soundness is proven. That is the case for the original Sonic paper, whose security proof relies on so-called witness-extended emulation. The authors of Plonk and recent work on Sonic [27] work around this problem by proving knowledge soundness directly in the AGM.

### 1.3 Related Work

*Simulation extractability.* There are many results on simulation extractability for non-interactive zero-knowledge proofs (NIZKs). First, Groth [30] noticed that a (black-box) SE NIZK is universally-composable (UC) [17]. Then Dodis et al. [20] introduced a notion of (black-box) *true simulation extractability* and showed that no NIZK can be UC-secure if it does not have this property.

In the context of zkSNARKs, the first SE zkSNARK was proposed by Groth and Maller [34] and SE zkSNARK for QAP by Lipmaa [41]. Kosba's et al. [38] give a general transformation from a NIZK to a black-box SE NIZK. Although their transformation works for zkSNARKs as well, succinctness of the proof system is not preserved by the transformation. Recently, Abdolmaleki et al. [1] showed another transformation that obtains non-black-box simulation extractability but also preserves succinctness of the argument. The zkSNARK of [32] has been shown to be SE by introducing minor modifications to the construction and making stronger assumptions [2, 15]. Recently, [3] showed that the original Groth's proof system from [32] is weakly SE and randomizable. None of these results are for zkSNARKs in the updatable SRS setting.

*Forking lemma generalizations.* There are several task specific variants, e.g., [4, 5, 35], of the general forking lemma [6, 44] for analyzing the forking behavior of random-oracle based executions. In [14], Bootle et al. proposed a novel inner-product argument which security relies on, so-called, witness-extended emulation. To show that property, the authors proposed a new version of forking lemma, which gives a lower bound on probability that a tree finding algorithm is able to produce a tree of acceptable transcripts by rewinding a conversation between a (potentially malicious) prover and verifier.

Although the result in that paper is dubbed a "forking lemma" it differs from forking lemmas known from e.g. [6, 44]. First of all, the forking lemmas in these papers analyse the probability of building a tree of acceptable transcripts for Fiat–Shamir based non-interactive proof systems, while the protocol presented by Bootle et al. is intended to work for interactive proof systems.

Importantly, it is not obvious how the result of Bootle et al. can be used to show security of non-interactive protocols as it relies on interactive provers whose proving strategies are more limited than proving strategies of non-interactive provers. For example, if a challenge given by the verifier does not suit an interactive prover, it can only try to finish a proof with it or abort. On the other hand, a non-interactive prover has far wider scope of possible actions–when the protocol is non-interactive the prover may adapt its strategy based on the random oracle outputs. This is reminiscent of *state restoration* security [11, 36] which is also about the security loss incurred by FS transformation for knowledge soundness from witness extended emulation.

Here, we directly capture the state restoration capability of the prover in the forking lemma instead of defining an interactive game where the prover can rewind the verifier to an earlier state as is done in [**?**]. The work of [**?**] further shows that state restoration security gives tight security guarantees for the non-interactive versions of Bulletproof [16] and Sonic . Our work differs from [**?**] in the following ways. First, they focus on showing security of concrete proof systems, while we show a general theorem about the security of a wide class of protocols. Second, they only consider knowledge soundness, while we focus on the stronger notion of simulation extractability. Third, the proof of [**?**] is in the AGM which allows for online extraction, whereas we aim to minimize our reliance on the AGM. In particular, our main theorem does not rely on AGM and we tackle technical challenges arising from extraction by rewinding. However, note that we show that concrete protocols satisfy the preconditions of our main theorem in the AGM.

## 2 Preliminaries

### 2.1 Notations

Let $\mathsf{PPT}$ denote probabilistic polynomial-time and $\lambda \in \mathbb{N}$ be the security parameter. All adversaries are stateful. For an algorithm $\mathcal{A}$, let $\mathrm{im}(\mathcal{A})$ be the image of $\mathcal{A}$ (the set of valid outputs of $\mathcal{A}$), let $\mathsf{R}(\mathcal{A})$ denote the set of random tapes of correct length for $\mathcal{A}$ (assuming the given value of $\lambda$), and let $r \leftarrow\!\!\$\, \mathsf{R}(\mathcal{A})$ denote the random choice of the randomiser $r$ from $\mathsf{R}(\mathcal{A})$. We denote by $\mathsf{negl}(\lambda)$ ($\mathsf{poly}(\lambda)$) an arbitrary negligible (resp. polynomial) function.

Probability ensembles $X = \{X_\lambda\}_\lambda$ and $Y = \{Y_\lambda\}_\lambda$, for distributions $X_\lambda, Y_\lambda$, have *statistical distance* $\Delta$ equal $\epsilon(\lambda)$ if $\sum_{a \in \mathrm{Supp}(X_\lambda \cup Y_\lambda)} |\Pr[X_\lambda = a] - \Pr[Y_\lambda = a]| = \epsilon(\lambda)$. We write $X \approx_\lambda Y$ if $\Delta(X_\lambda, Y_\lambda) \leq \mathsf{negl}(\lambda)$. For values $a(\lambda)$ and $b(\lambda)$ we write $a(\lambda) \approx_\lambda b(\lambda)$ if $|a(\lambda) - b(\lambda)| \leq \mathsf{negl}(\lambda)$.

For a probability space $(\Omega, \mathcal{F}, \mu)$ and event $\mathsf{E} \in \mathcal{F}$ we denote by $\overline{\mathsf{E}}$ an event that is complementary to $\mathsf{E}$, i.e. $\overline{\mathsf{E}} = \Omega \setminus \mathsf{E}$.

Denote by $\mathcal{R} = \{\mathbf{R}\}$ a family of relations. We assume that if $\mathbf{R}$ comes with any auxiliary input, the later is benign. Directly from the description of $\mathbf{R}$ one learns security parameter $\lambda$ and description of the group $\mathbb{G}$, if the relation is a relation of group elements (as it usually is in case of zkSNARKs).

*Bilinear groups.* A bilinear group generator $\mathsf{Pgen}(1^\lambda)$ returns public parameters $\mathsf{p} = (p, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T, \hat{e}, [1]_1, [1]_2)$, where $\mathbb{G}_1$, $\mathbb{G}_2$, and $\mathbb{G}_T$ are additive cyclic groups of prime order $p = 2^{\Omega(\lambda)}$, $[1]_1, [1]_2$ are generators of $\mathbb{G}_1$, $\mathbb{G}_2$, resp., and $\hat{e} : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$ is a non-degenerate PPT-computable bilinear pairing. We assume the bilinear pairing to be Type-3, i.e., that there is no efficient isomorphism from $\mathbb{G}_1$ to $\mathbb{G}_2$ or from $\mathbb{G}_2$ to $\mathbb{G}_1$. We use the by now standard bracket notation, i.e., we write $[a]_\iota$ to denote $ag_\iota$ where $g_\iota$ is a fixed generator of $\mathbb{G}_\iota$. We denote $\hat{e}([a]_1, [b]_2)$ as $[a]_1 \bullet [b]_2$. Thus, $[a]_1 \bullet [b]_2 = [ab]_T$. We freely use the bracket notation with matrices, e.g., if $\boldsymbol{AB} = \boldsymbol{C}$ then $\boldsymbol{A}\,[\boldsymbol{B}]_\iota = [\boldsymbol{C}]_\iota$ and $[\boldsymbol{A}]_1 \bullet [\boldsymbol{B}]_2 = [\boldsymbol{C}]_T$. Since every algorithm $\mathcal{A}$ takes as input the public parameters we skip them when describing $\mathcal{A}$'s input. Similarly, we do not explicitly state that each protocol starts with generating these parameters by $\mathsf{Pgen}$.

**Lemma 1 (Difference lemma, [46, Lemma 1]).** *Let* $\mathsf{A}, \mathsf{B}, \mathsf{F}$ *be events defined in some probability space, and suppose that* $\mathsf{A} \wedge \overline{\mathsf{F}} \iff \mathsf{B} \wedge \overline{\mathsf{F}}$. *Then* $|\Pr[\mathsf{A}] - \Pr[\mathsf{B}]| \leq \Pr[\mathsf{F}]$.

### 2.2 Algebraic Group Model

The algebraic group model (AGM) introduced in [24] lies between the standard model and generic bilinear group model. In the AGM it is assumed that an adversary $\mathcal{A}$ can output a group element $[y] \in \mathbb{G}$ if $[y]$ has been computed by applying group operations to group elements given to $\mathcal{A}$ as input. It is further assumed, that $\mathcal{A}$ knows how to "build" $[y]$ from that elements. More precisely, the AGM requires that whenever $\mathcal{A}([\boldsymbol{x}])$ outputs a group element $[y]$ then it also outputs $\boldsymbol{c}$ such that $[y] = \boldsymbol{c}^\top \cdot [\boldsymbol{x}]$. Both Plonk and Sonic have been shown secure using the AGM. An adversary that works in the AGM is called *algebraic*.

### 2.3 Polynomial commitment

In the polynomial commitment scheme $\mathsf{PC} = (\mathsf{KGen}, \mathsf{Com}, \mathsf{Op}, \mathsf{Vf})$ the committer $\mathsf{C}$ can convince the receiver $\mathsf{R}$ that some polynomial $\mathsf{f}$ which $\mathsf{C}$ committed to evaluates to $s$ at some point $z$ chosen by $\mathsf{R}$. $\mathsf{PC}$'s subroutines are defined as follows

$\mathsf{KGen}(1^\lambda, \mathsf{max})$**:** The key generation algorithm $\mathsf{KGen}(1^\lambda, \mathsf{max})$ takes in a security parameter $1^\lambda$ and a parameter $\mathsf{max}$ which determines the maximal degree of the committed polynomial. It outputs a structured reference string $\mathsf{srs}$ (including a commitment key).

$\mathsf{Com}(\mathsf{srs}, \mathsf{f})$**:** The commitment algorithm $\mathsf{Com}(\mathsf{srs}, \mathsf{f})$ takes in $\mathsf{srs}$ and a polynomial $\mathsf{f}$ with maximum degree $\mathsf{max}$, and outputs a commitment $c$.

$\mathsf{Op}(\mathsf{srs}, z, s, \mathsf{f})$**:** The opening algorithm $\mathsf{Op}(\mathsf{srs}, z, s\mathsf{f})$ takes as input $\mathsf{srs}$, an evaluation point $z$, a value $s$ and the polynomial $\mathsf{f}$. It outputs an opening $o$.

$\mathsf{Vf}(\mathsf{srs}, c, z, s, o)$**:** The verification algorithm takes in $\mathsf{srs}$, a commitment $c$, an evaluation point $z$, a value $s$ and an opening $o$. It outputs 1 if $o$ is a valid opening for $(c, z, s)$ and 0 otherwise.

Plonk and Sonic use variants of the KZG polynomial commitment scheme [37]. We denote the first by $\mathsf{PC}_\mathsf{P}$ and the latter by $\mathsf{PC}_\mathsf{S}$. Due to page limit, we omit their presentation here and refer to **??** and **??** in the **??**. In this paper we use evaluation binding, commitment of knowledge, and, newly introduced, unique opening and hiding properties. Formal definitions of these could be find in **??**, here we briefly introduce them.

**Evaluation binding** intuitively, this property assures that no adversary could provide two valid openings for two different evaluations of the same commitment in the same point.

**Commitment of knowledge** when a commitment scheme is "of knowledge" then if an adversary produces a (valid) commitment $c$, which it can open, then it also knows the underlying polynomial $\mathsf{f}$ which commits to that value. [42] shows, using AGM, that $\mathsf{PC}_\mathsf{S}$ is a commitment of knowledge. The same reasoning could be used to show that property for $\mathsf{PC}_\mathsf{P}$.

**Unique opening** this property assures that there is only one valid opening for the committed polynomial and given evaluation point. This property is crucial in showing forking simulation-extractability of Plonk and Sonic. We show that the Plonk's and Sonic's polynomial commitment schemes satisfy this requirement in **??** and **??** respectively.

**Hiding** assures that no adversary is able to tell anything about the polynomial given only its commitment and bounded number of evaluations.

### 2.4 Zero-Knowledge Proof Systems

Let $\mathcal{R}(1^\lambda) = \{\mathbf{R}\}$ be a family of NP relations. Denote by $\mathcal{L}_\mathbf{R}$ the language determined by $\mathbf{R}$. Let $\mathsf{P}$ be a *prover* and $\mathsf{V}$ be the *verifier*, both PPT algorithms. We allow our proof system to have a setup, i.e. there is a $\mathsf{KGen}$ algorithm that takes as input the relation description $\mathbf{R}$ and outputs a common reference string $\mathsf{srs}$. We assume that the $\mathsf{srs}$ defines the relation and for universal prove systems, such as Plonk and Sonic, we treat both the reference string and the relation as universal.

We denote by $\langle \mathsf{P}(\mathsf{srs}, \mathsf{x}, \mathsf{w}), \mathsf{V}(\mathsf{srs}, \mathsf{x})\rangle$ a *transcript* (also called *proof*) $\pi$ of a conversation between $\mathsf{P}$ with input $(\mathsf{srs}, \mathsf{x}, \mathsf{w})$ and $\mathsf{V}$ with input $(\mathsf{srs}, \mathsf{x})$. We write $\langle \mathsf{P}(\mathsf{srs}, \mathsf{x}, \mathsf{w}), \mathsf{V}(\mathsf{srs}, \mathsf{x})\rangle = 1$ if in the end of the transcript the verifier $\mathsf{V}$ returns $1$ and say that $\mathsf{V}$ accepts it. For non-interactive proof systems we abuse notation and write $\mathsf{V}(\mathsf{srs}, \mathsf{x}, \pi) = 1$ to denote a fact that $\pi$ is accepted by the verifier.

A proof system $\Psi = (\mathsf{KGen}, \mathsf{P}, \mathsf{V}, \mathsf{Sim})$ for $\mathcal{R}$ is required to have three properties: completeness, soundness and zero knowledge, which are defined as follows:

*Completeness.* An interactive proof system $\Psi$ is *complete* if an honest prover always convinces an honest verifier, that is for all $\mathbf{R} \in \mathcal{R}(1^\lambda)$ and $(\mathsf{x}, \mathsf{w}) \in \mathbf{R}$

$$\Pr[\langle \mathsf{P}(\mathsf{srs}, \mathsf{x}, \mathsf{w}), \mathsf{V}(\mathsf{srs}, \mathsf{x})\rangle = 1 \mid \mathsf{srs} \leftarrow \mathsf{KGen}(\mathbf{R})] = 1\,.$$

*Soundness.* We say that $\Psi$ for $\mathcal{R}$ is *sound* if no PPT prover $\mathcal{A}$ can convince an honest verifier $\mathsf{V}$ to accept a proof for a false statement $\mathsf{x} \notin \mathcal{L}$. More precisely, for all $\mathbf{R} \in \mathcal{R}(1^\lambda)$

$$\Pr[\langle \mathcal{A}(\mathsf{srs}, \mathsf{x}), \mathsf{V}(\mathsf{srs}, \mathsf{x})\rangle = 1 \wedge \mathsf{x} \notin \mathcal{L}_\mathbf{R} \mid \mathsf{srs} \leftarrow \mathsf{KGen}(\mathbf{R}), \mathsf{x} \leftarrow \mathcal{A}(\mathsf{srs})] \le \mathsf{negl}(\lambda)\,;$$

Sometimes a stronger notion of soundness is required—except requiring that the verifier rejects proofs of statements outside the language, we request from the prover to know a witness corresponding to the proven statement. This property is called *knowledge soundness*.

*Zero knowledge.* We call a proof system $\Psi$ *zero-knowledge* if for any $\mathbf{R} \in \mathcal{R}(1^\lambda)$, and adversary $\mathcal{A}$ there exists a PPT simulator $\mathsf{Sim}$ such that for any $(\mathsf{x}, \mathsf{w}) \in \mathbf{R}$

$$\left\{\langle \mathsf{P}(\mathsf{srs}, \mathsf{x}, \mathsf{w}), \mathcal{A}(\mathsf{srs}, \mathsf{x}, \mathsf{w})\rangle \;\middle|\; \mathsf{srs} \leftarrow \mathsf{KGen}(\mathbf{R})\right\} \qquad \approx_\lambda \qquad \left\{\mathsf{Sim}^{\mathcal{A}}(\mathsf{srs}, \mathsf{x}) \;\middle|\; \mathsf{srs} \leftarrow \mathsf{KGen}(\mathbf{R})\right\}\,.$$

We call zero knowledge *perfect* if the distributions are equal and *computational* if they are indistinguishable for any PPT distinguisher.

Alternatively, zero-knowledge can be defined by allowing the simulator to use the trapdoor $\mathsf{td}$ that is generated along the $\mathsf{srs}$. In this paper we distinguish simulators that requires a trapdoor to simulate and those that do not. We call the former *SRS-simulators*. We say that a protocol is zero knowledge in the standard model if its simulator does not require the trapdoor.

In security reductions in this paper it is sometimes needed to produce simulated NIZK proofs without knowning the trapdoor, just by programming the random oracle. We call protocols which allow for such kind of simulation *trapdoor-less simulatable* (TLS). More precisely,

**Definition 1 (Trapdoor-less simulatable proof system).** *Let* $\Psi = (\mathsf{KGen}, \mathsf{P}, \mathsf{V}, \mathsf{Sim})$ *be a NIZK proof system and* $\mathcal{H}$ *a random oracle. Let* $\mathsf{Sim}$ *be ana pair of algorithms:* $\mathsf{Sim}_\mathcal{H}$ *that takes random oracle queries and answers them,* $\mathsf{Sim}_\mathsf{P}$ *that takes as input an SRS* $\mathsf{srs}$ *and instance* $\mathsf{x}$ *and outputs a proof* $\pi_{\mathsf{Sim}}$*. We call* $\Psi$ trapdoor-less simulatable *if for any adversary* $\mathcal{A}$*,* $\varepsilon_0 \approx \varepsilon_1$*, where*

$$\varepsilon_b = \Pr\left[\mathcal{A}^{\mathsf{O}_b}(\mathsf{srs}) = 0 \mid \mathsf{srs} \leftarrow\!\!\$\, \mathsf{KGen}(\lambda)\right] \tag{1}$$

*where* $\mathsf{O}_b$ *takes two types of adversary's queries:*

**random oracle calls:** *on $\mathcal{A}$'s query $x$, $\mathsf{O}_b$ responds with $\mathcal{H}(x)$ if $b = 0$, and with $y \leftarrow \mathsf{Sim}_{\mathcal{H}}(\mathsf{srs}, x)$, if $b = 1$.*

**proof calls:** *on $\mathcal{A}$'s query $\mathsf{x}, \mathsf{w}$ responds with a real proof $\pi_{\mathsf{P}} \leftarrow \mathsf{P}(\mathsf{srs}, \mathsf{x}, \mathsf{w})$ if $b = 0q$ or a simulated proof $\pi_{\mathsf{Sim}} \leftarrow \mathsf{Sim}(\mathsf{srs}, \mathsf{x})$ if $b = 1$.*

**Definition 2 ($k$-programmable ZK).** *Let $\Psi$ be a $(2\mu + 1)$-message ZK proof system and let $\Psi_{\mathsf{FS}}$ be its Fiat–Shamir variant. We say that $\Psi_{\mathsf{FS}}$ is $k$-programmable ZK if there exists a simulator $\mathsf{Sim}_{\mathsf{FS}}$ that*

1. *produces proofs indistinguishable from proofs output by an honest prover;*
2. $\mathsf{Sim}_{\mathsf{FS}}$ *programs the random oracle* only *for challenges from round $k$ to $\mu + 1$.*

We note that Plonk is 2-programmable ZK, Sonic is 1-programmable ZK, and Marlin is 1-programmable ZK. This follows directly from the proofs of their standard model zero-knowledge property in **??????**.

**Idealised verifier and verification equations** Let $(\mathsf{KGen}, \mathsf{P}, \mathsf{V})$ be a proof system. Observe that the $\mathsf{KGen}$ algorithm provides an SRS which can be interpreted as a set of group representation of polynomials evaluated at trapdoor elements. E.g. for a trapdoor $\chi$ the SRS contains $[\mathsf{p}_1(\chi), \ldots, \mathsf{p}_k(\chi)]_1$, for some polynomials $\mathsf{p}_1(X), \ldots, \mathsf{p}_k(X) \in \mathbb{F}_p[X]$. On the other hand, the verifier $\mathsf{V}$ accepts if a (possibly set of) verification equation $\mathsf{ve}_{\mathsf{x},\pi}$ (note that the verification equation changes relate to the instance $\mathsf{x}$ and proof $\pi$), which can also be interpreted as a polynomial in $\mathbb{F}_p[X]$ whose coefficients depend on messages sent by the prover, zeroes at $\chi$. Following [25] we call verifiers who checks that $\mathsf{ve}_{\mathsf{x},\pi}(\chi) = 0$ *real verifiers* as opposed to *ideal verifiers* who accepts only when $\mathsf{ve}_{\mathsf{x},\pi}(X) = 0$. That is, while a real verifier accepts when a polynomial *evaluates* to zero, an ideal verifier accepts only when the polynomial *is* zero.

Although ideal verifiers are impractical, they are very useful in our proofs. More precisely, we show that

1. the idealised verifier accepts an incorrect proof (what "incorrect" means depends on the situation) with at most negligible probability (and many cases—never);
2. when the real verifier accepts, but not the idealised one, then we show how to use a malicious $\mathsf{P}$ to break the underlying security assumption (in our case—a variant of dlog.)

Analogously, idealised verifier can also be defined for polynomial commitment scheme.

**Sigma protocols** A sigma protocol $\Sigma = (\mathsf{P}, \mathsf{V}, \mathsf{Sim})$ for a relation $\mathbf{R} \in \mathcal{R}(1^\lambda)$ is a special case of an interactive proof where a transcript consists of three messages $(a, b, z)$, where $b$ is a challenge provided by the verifier. Sigma protocols are honest verifier zero-knowledge in the standard model and specially-sound. That is, there exists an extractor $\mathsf{Ext}$ which given two accepting transcripts $(a, b, z)$, $(a, b', z')$ for a statement $\mathsf{x}$ can recreate the corresponding witness if $b \neq b'$. More formally:

*Special soundness.* A sigma protocol $\Sigma$ is *specially-sound* if for any adversary $\mathcal{A}$ the probability

$$\Pr\left[\begin{array}{l|l} \mathsf{V}(\mathbf{R}, \mathsf{x}, (a, b, z)) = \mathsf{V}(\mathbf{R}, \mathsf{x}, (a, b', z')) = 1 & (\mathsf{x}, (a, b, z), (a, b', z')) \leftarrow \mathcal{A}(\mathbf{R}), \\ \wedge\ b \neq b' \wedge \mathbf{R}(\mathsf{x}, \mathsf{w}) = 0 & \mathsf{w} \leftarrow \mathsf{Ext}(\mathbf{R}, \mathsf{x}, (a, b, z), (a, b', z')) \end{array}\right]$$

is upper-bounded by some negligible function $\mathsf{negl}(\lambda)$.

Another property that sigma protocols may have is a unique response property [22] which states that no PPT adversary can produce two accepting transcripts that differ only on the last element. More precisely, *Unique response property.* Let $\Sigma = (\mathsf{P}, \mathsf{V}, \mathsf{Sim})$ be a sigma-protocol for $\mathbf{R} \in \mathcal{R}(1^\lambda)$ with proofs of the form $(a, b, z)$. We say that $\Sigma$ has the unique response property if for all PPT algorithms $\mathcal{A}$, it holds that,:

$$\Pr[\mathsf{V}(\mathbf{R}, \mathsf{x}, (a, b, z)) = \mathsf{V}(\mathbf{R}, \mathsf{x}, (a, b, z')) = 1 \wedge z \neq z' \mid (\mathsf{x}, a, b, z, z') \leftarrow \mathcal{A}(\mathbf{R})] \leq \mathsf{negl}(\lambda)\,.$$

If this property holds even against unbounded adversaries, it is called *strict*, cf. [21]. Later on we call protocols that follows this notion *ur-protocols*. For the sake of completeness we note that many sigma protocols, like e.g. Schnorr's protocol [45], fulfil this property.

### 2.5 From interactive to non-interactive—the Fiat–Shamir transform

Consider a $(2\mu + 1)$-message, public-coin, honest verifier zero-knowledge interactive proof system $\Psi = (\mathsf{KGen}, \mathsf{P}, \mathsf{V}, \mathsf{Sim})$ for $\mathbf{R} \in \mathcal{R}(1^\lambda)$. Let $\pi$ be a proof performed by the prover $\mathsf{P}$ and verifier $\mathsf{V}$ compound of messages $(a_1, b_1, \ldots, a_\mu, b_\mu, a_{\mu+1})$, where $a_i$ comes from $\mathsf{P}$ and $b_i$ comes from $\mathsf{V}$. Denote by $\mathcal{H}$ a random oracle. Let $\Psi_{\mathsf{FS}} = (\mathsf{KGen}_{\mathsf{FS}}, \mathsf{P}_{\mathsf{FS}}, \mathsf{V}_{\mathsf{FS}}, \mathsf{Sim}_{\mathsf{FS}})$ be a proof system such that

- $\mathsf{KGen}_{\mathsf{FS}}$ behaves as $\mathsf{KGen}$.
- $\mathsf{P}_{\mathsf{FS}}$ behaves as $\mathsf{P}$ except after sending message $a_i$, $i \in [1\,..\,\mu]$, the prover does not wait for the message from the verifier but computes it locally setting $b_i = \mathcal{H}(\pi[0..i])$, where $\pi[0..j] = (\mathsf{x}, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j)$. (Importantly, $\pi[0..\mu+1] = (\mathsf{x}, \pi)$).
- $\mathsf{V}_{\mathsf{FS}}$ behaves as $\mathsf{V}$ but does not provide challenges to the prover's proof. Instead it computes the challenges locally as $\mathsf{P}_{\mathsf{FS}}$ does. Then it verifies the resulting transcript $\pi$ as the verifier $\mathsf{V}$ would.
- $\mathsf{Sim}_{\mathsf{FS}}$ behaves as $\mathsf{Sim}$, except when $\mathsf{Sim}$ picks challenge $b_i$ before computing message $\pi[0, i]$, $\mathsf{Sim}_{\mathsf{FS}}$ programs the random oracle to output $b_i$ on $\pi[0, i]$.

The Fiat–Shamir heuristic states that $\Psi_{\mathsf{FS}}$ is a zero-knowledge non-interactive proof system for $\mathbf{R} \in \mathcal{R}(1^\lambda)$.

### 2.6 Updatable SRS schemes

We recall the definition of an updatable SRS scheme from [33] which consists of the following algorithms.

- $(\mathsf{srs}, \rho) \leftarrow \mathsf{KGen}(1^\lambda)$ is a PPT algorithm that takes a security parameter $\lambda$ and outputs a SRS $\mathsf{srs}$, and correctness proof $\rho$.
- $(\mathsf{srs}', \rho') \leftarrow \mathsf{Upd}(1^\lambda, \mathsf{srs}, \{\rho_i\}_{i=1}^n)$ is a PPT algorithm that takes as input the security parameter $\lambda$, a SRS $\mathsf{srs}$, a list of update proofs and outputs an updated SRS together with a proof of correct update.
- $b \leftarrow \mathsf{VerifySRS}(1^\lambda, \mathsf{srs}, \{\rho_i\}_{i=1}^n)$ is a DPT algorithm that takes the security parameter $\lambda$, a SRS $\mathsf{srs}$, a list of update proofs, and outputs a bit indicating acceptance or not.

In the next section, we define security notions in the updatable setting. To this end, we define an SRS update oracle $\mathsf{UpdO}$ in Fig. 1 by which the adversary updates the SRS. We also define the simulation oracle $\mathsf{SimO}$ in Fig. 1 that is the simulator w.r.t. the SRS finalised by the adversary using $\mathsf{UpdO}$. This simulation oracle will be used in the SE definition.

[**Hamid:** TODO: add a paragraph here about the universality!]

## 3 Definitions and lemmas for multi-round SRS-based protocols

The result of Faust et al. [21] do not apply to our setting since the protocols we consider have an SRS, more than three messages, require more than just two transcripts for standard model extraction and are not special sound. Furthermore, the advrersary gets to constribute to the SRS in the updatable setting which makes the analysis more complicated. We thus adapt special soundness to forking soundness, and generalize the forking lemma and the unique response property to make them compatible with multi-round SRS-based protocols in the updatable setting.

### 3.1 Generalised forking lemma.

First of all, although dubbed "general", **??** is not general enough for our purpose as it is useful only for protocols where witness can be extracted from just two transcripts. To be able to extract a witness from, say, an execution of $\mathsf{P}$ we need to obtain at least $(3\mathsf{n} + 1)$ valid proofs, and $(\mathsf{n} + \mathsf{Q} + 1)$ for $\mathsf{S}$. Here we propose a generalisation of the general forking lemma that given probability of producing an accepting transcript, $\mathsf{acc}$, lower-bounds the probability of generating a *tree of accepting transcripts* $\mathsf{T}$, which allows to extract a witness.

**Definition 3 (Tree of accepting transcripts, cf. [14]).** *Consider a $(2\mu + 1)$-message interactive proof system $\Psi$. A $(n_1, \ldots, n_\mu)$-tree of accepting transcripts is a tree where each node on depth $i$, for $i \in [1\,..\,\mu + 1]$, is an $i$-th prover's message in an accepting transcript; edges between the nodes are labeled with verifier's challenges, such that no two edges on the same depth have the same label; and each node*

| $\mathsf{UpdO}(\texttt{intent}, \mathsf{srs}_n, \{\rho_i\}_{i=1}^n)$ | $\mathsf{SimO}(\mathsf{x}')$ |
|---|---|
| **if** $\mathsf{srs} \neq \bot$ : **return** $\bot$ | **if** $(\mathsf{srs} = \bot)$ : **return** $\bot$ |
| **if** $(\texttt{intent} = \texttt{setup})$ : | $\pi' \leftarrow$ [**Hamid** :$\mathsf{Sim}_{\mathsf{FS}}$]$(\mathsf{srs}, \mathsf{td}, \mathsf{x}')$ |
| $(\mathsf{srs}', \rho') \leftarrow \mathsf{KGen}(\mathbf{R})$ | $Q = Q \cup \{(\mathsf{x}', \pi')\}$ |
| $Q_{\mathsf{srs}} \leftarrow Q_{\mathsf{srs}} \cup \{(\mathsf{srs}', \rho')\}$ | **return** $\pi'$ |
| **return** $(\mathsf{srs}', \rho')$ | |
| **if** $(\texttt{intent} = \texttt{update})$ : | |
| $b \leftarrow \mathsf{VerifySRS}(1^\lambda, \mathsf{srs}_n, \{\rho_i\}_{i=1}^n)$ | |
| **if** $(b = 0)$ : **return** $\bot$ | |
| $(\mathsf{srs}', \rho') \leftarrow \mathsf{Upd}(1^\lambda, \mathsf{srs}_n, \{\rho_i\}_{i=1}^n)$ | |
| $Q_{\mathsf{srs}} \leftarrow Q_{\mathsf{srs}} \cup \{(\mathsf{srs}', \rho')\}$ | |
| **return** $(\mathsf{srs}', \rho')$ | |
| **if** $(\texttt{intent} = \texttt{final})$ : | |
| $b \leftarrow \mathsf{VerifySRS}(1^\lambda, \mathsf{srs}_n, \{\rho_i\}_{i=1}^n)$ | |
| **if** $(b = 0) \vee Q_{\mathsf{srs}}^{(2)} \cap \{\rho_i\}_i = \emptyset$ : **return** $\bot$ //$Q_{\mathsf{srs}} = (Q_{\mathsf{srs}}^{(1)}, Q_{\mathsf{srs}}^{(2)})$ s.t. $Q_{\mathsf{srs}}^{(2)}$ contains the update proofs in $Q_{\mathsf{srs}}$ | |
| $\mathsf{td} \leftarrow \mathsf{Ext}_{\mathsf{srs}}(\mathsf{srs}_n, , Q_{\mathsf{srs}}, \{\rho_i\}_{i=1}^n)$ | |
| $\mathsf{srs} \leftarrow \mathsf{srs}_n$, **return** srs | |
| **else return** $\bot$ | |

Fig. 1: The left oracle defines the notion of updatable SRS setup. The right oracle is the simulation oracle.

*on depth $i$ has $n_i - 1$ siblings and $n_{i+1}$ children. The tree consists of $N = \prod_{i=1}^{\mu} n_i$ branches, where $N$ is the number of accepting transcripts. We require $N = \mathsf{poly}(\lambda)$.*

**Lemma 2 (General forking lemma II).** *Fix $q \in \mathbb{Z}$ and set $H$ of size $h \geq m$. Let $\mathcal{Z}$ be a* PPT *algorithm that on input $y, h_1, \ldots, h_q$ returns $(i, s)$ where $i \in [0\,..\,q]$ and $s$ is called a side output. Denote by* IG *a randomised instance generator. We denote by* acc *the probability*

$$\Pr\left[i \neq 0 \;\middle|\; y \leftarrow \mathsf{IG};\ h_1, \ldots, h_q \leftarrow\!\$\, H;\ (i, s) \leftarrow \mathcal{Z}(y, h_1, \ldots, h_q)\right].$$

*Let $\mathsf{GF}_{\mathcal{Z}}^m$ denote the algorithm described in Fig. 2 then the probability* $\mathsf{frk} := \Pr\left[b = 1 \;\middle|\; y \leftarrow \mathsf{IG};\ h_1, \ldots, h_q \leftarrow\!\$\, H;\ (b, \boldsymbol{s}) \leftarrow \mathsf{GF}_{\mathcal{Z}}^m(y, h_1, \ldots, h_q)\right]$ *is at least*

$$\frac{\mathsf{acc}^m}{q^{m-1}} - \mathsf{acc} \cdot \left(1 - \frac{h!}{(h-m)! \cdot h^m}\right).$$

The proof goes similarly to [6, Lemma 1] with some modifications required by the fact that the protocol has more than 3 rounds and the number of transcripts required is larger. Due to page limit, the proof is presented in **??**.

To highlight importance of the generalised forking lemma we describe how we use it in our forking simulation-extractability proof. Let $\Psi$ be a forking sound proof system where for an instance $\mathsf{x}$ the corresponding witness can be extracted from an $(1, \ldots, 1, n_k, 1, \ldots, 1)$-tree of accepting transcripts. Let $\mathcal{A}$ be the simulation-extractability adversary that outputs an accepting proof with probability at least acc. (Although we use the same acc to denote probability of $\mathcal{Z}$ outputing a non-zero $i$ and the probability of $\mathcal{A}$ outputing an accepting proof, we claim that these probabilities are exactly the same by the way we define $\mathcal{Z}$.) Let $\mathcal{A}$ produce an accepting proof $\pi_{\mathcal{A}}$ for instance $\mathsf{x}_{\mathcal{A}}$; $r$ be $\mathcal{A}$'s randomness; $Q$ the list of queries submitted by $\mathcal{A}$ along with simulator Sim's answers; **[Hamid:** $Q_{\mathsf{srs}}$ be the list of proofs for SRS honest updates;] and $Q_{\mathcal{H}}$ be the list of all random oracle queries made by $\mathcal{A}$. All of these are given to the extractor Ext that internally runs the forking algorithm $\mathsf{GF}_{\mathcal{Z}}^{n_k}$. Algorithm $\mathcal{Z}$ takes $(\mathsf{srs}, \mathcal{A}, Q, r)$ as input $y$ and $Q_{\mathcal{H}}$

$\mathsf{GF}^m_{\mathcal{Z}}(y, h^1_1, \ldots, h^1_q)$

$\rho \leftarrow\$ \mathsf{R}(\mathcal{Z})$
$(i, s_1) \leftarrow \mathcal{Z}(y, h^1_1, \ldots, h^1_q; \rho)$
$i_1 \leftarrow i$
**if** $i = 0$ **return** $(0, \perp)$
**for** $j \in [2 \,..\, m]$
  $h^j_1, \ldots, h^j_{i-1} \leftarrow h^{j-1}_1, \ldots, h^{j-1}_{i-1}$
  $h^j_i, \ldots, h^j_q \leftarrow\$ H$
  $(i_j, s_j) \leftarrow \mathcal{Z}(y, h^j_1, \ldots, h^j_{i-1}, h^j_i, \ldots, h^j_q; \rho)$
  **if** $i_j = 0 \vee i_j \neq i$ **return** $(0, \perp)$
**if** $\exists (j, j') \in [1 \,..\, m]^2, j \neq j' : (h^j_i = h^{j'}_i)$ **return** $(0, \perp)$
**else return** $(1, s)$

Fig. 2: Generalised forking algorithm $\mathsf{GF}^m_{\mathcal{Z}}$

as input $h^1_1, \ldots, h^1_q$. (For the sake of completeness, we allow $\mathsf{GF}^{n_k}_{\mathcal{Z}}$ to pick $h^1_{l+1}, \ldots, h^1_q$ responses if $Q_{\mathcal{H}}$ has only $l < q$ elements.)

Next, $\mathcal{Z}$ internally runs $\mathcal{A}(\mathsf{srs}; r)$ **[Hamid:** $\mathcal{A}(1^\lambda; r)$**]** and responds to its random oracle and simulator queries by using $Q_{\mathcal{H}}$ and $Q$. Note that $\mathcal{A}$ makes the same queries as it did before it output $(\mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}})$ as it is run on the same random tape and with the same answers from the simulator and random oracle. Once $\mathcal{A}$ outputs $\pi_{\mathcal{A}}$, algorithm $\mathcal{Z}$ outputs $(i, \pi_{\mathcal{A}})$, where $i$ is the index of a random oracle query submitted by $\mathcal{A}$ to receive the challenge after the $k$-th message from the prover—a message where the tree of transcripts branches. Then, after the first run of $\mathcal{A}$ is done, the extractor runs $\mathcal{Z}$ again, but this time it provides fresh random oracle responses $h^2_i, \ldots, h^2_q$. Note that this is equivalent to rewinding $\mathcal{A}$ to a point just before $\mathcal{A}$ is about to ask its $i$-th random oracle query. The probability that the adversary produces an accepting transcript with the fresh random oracle responses is at least $\mathsf{acc}$. This continues until the required number of transcripts is obtained.

We note that in the original forking lemma, the forking algorithm $\mathsf{F}$, cf. **??**, gets only as input $y$ and elements $h^1_1, \ldots, h^1_q$ are randomly picked from $H$ internally by $\mathsf{F}$. However, assuming that $h^1_1, \ldots, h^1_q$ are random oracle responses, and thus random, makes the change only notational.

We also note that the general forking lemma proposed in Lemma 2 works for protocols with an extractor that can obtain the witness from a $(1, \ldots, 1, n_k, 1, \ldots, 1)$-tree of accepting transcripts. This limitation however does not affect the main result of this paper, i.e. showing that both Plonk and Sonic are forking simulation extractable.

### 3.2 Unique-response protocols

Another technical hurdle is the assumption of unique response property of the transformed sigma protocol required by Faust et al. The original Fischlin's formulation, although suitable for applications presented in [21, 22], does not suffice in our case. First, the property assumes that the protocol has three messages, with the second being the challenge from the verifier. That is not the case we consider here. Second, it is not entirely clear how to generalize the property. Should one require that after the first challenge from the verifier, the prover's responses are fixed? That does not work since the prover needs to answer differently on different verifier's challenges, as otherwise the protocol could have fewer rounds. Another problem is that the protocol could consist of a round other than the first one where the prover message is randomized. Unique response cannot hold in this case. Finally, the protocols we consider here are not in the standard model, but use an SRS what also complicates things considerably.

We walk around these obstacles by providing a generalised notion of the unique response property. More precisely, we say that a $(2\mu + 1)$-message protocol has *unique responses from* $i$, and call it an $i$-ur-protocol, if it follows the definition below:

**Definition 4** (*i*-ur**-protocol in the updatable setting).** *Let* $\Psi$ *be a* $(2\mu+1)$*-message public coin proof system* $\Psi = (\mathsf{KGen}, \mathsf{P}, \mathsf{V}, \mathsf{Sim})$. *Let* $\Psi_{\mathsf{FS}}$ *be* $\Psi$ *after the Fiat–Shamir transform and* $\mathcal{H}$ *the random oracle. Denote by* $a_1, \ldots, a_\mu, a_{\mu+1}$ *protocol messages output by the prover, We say that* $\Psi$ *has* unique responses from $i$ on *if for any* PPT *adversary* $\mathcal{A}$*:*

$$\Pr\left[\begin{matrix}\boldsymbol{a} \neq \boldsymbol{a'}, a_1, \ldots, a_i = a'_1, \ldots, a'_i, \\ \mathsf{V}^{\mathcal{H}}_{\mathsf{FS}}(\mathsf{srs}, \mathsf{x}, \boldsymbol{a}) = \mathsf{V}^{\mathcal{H}}_{\mathsf{FS}}(\mathsf{srs}, \mathsf{x}, \boldsymbol{a'}) = 1\end{matrix} \;\middle|\; \mathsf{x}, \boldsymbol{a} = (a_1, \ldots, a_{\mu+1}), \boldsymbol{a'} = (a'_1, \ldots, a'_{\mu+1}) \leftarrow \mathcal{A}^{\mathcal{H}, \mathsf{UpdO}}(1^\lambda)\right]$$

*is upper-bounded by some negligible function* $\mathsf{negl}(\lambda)$. *[***Hamid:** *TODO: rephrase this: Note that in the updatable setting,* srs *is the SRS that* $\mathcal{A}$ *finalised using the update oracle* UpdO*, defined in Fig. 1.]*

Intuitively, a protocol is $i$-ur if it is infeasible for a PPT adversary to produce a pair of acceptable and different proofs $\pi, \pi'$ that are the same on first $i$ messages. We note that the definition above is also meaningful for protocols without an SRS. Intuitively in that case srs is the empty string.

### 3.3 Forking soundness

Note that the special soundness property (as usually defined) holds for all—even computationally unbounded—adversaries. Unfortunately, since a simulation trapdoors for P and S exist, the protocols cannot be special sound in that regard. This is because an unbounded adversary can recover the trapdoor and build a number of simulated proofs for a fake statement. Hence, we provide a weaker, yet sufficient, definition of *forking soundness*. More precisely, we state that an adversary that is able to answer correctly multiple challenges either knows the witness or can be used to break some computational assumption. However, differently from the standard definition of special soundness, we do not require from the extractor to be able to extract the witness from *any* tree of acceptable transcripts. We require that the tree be produced honestly, that is, all challenges are picked randomly—exactly as an honest verifier would pick. Intuitively, the tree is as it would be generated by a GF algorithm from the generalized forking lemma.

**Definition 5** ($(\varepsilon(\lambda), k, n)$**-forking soundness in the updatable setting).** *Let* $\Psi = (\mathsf{KGen}, \mathsf{P}, \mathsf{V}, \mathsf{Sim})$ *be an* $(2\mu+1)$*-message proof system for a relation* $\mathbf{R}$.

*For any* PPT *adversary* $\mathcal{A}^{\mathsf{UpdO}, \mathcal{H}}(1^\lambda; r)$ *with access to oracles* UpdO *defined in Fig. 1, and random oracle* $\mathcal{H}$*, we consider the procedure* $\mathcal{Z}$ *that provided the transcript* $(\mathsf{srs}, \mathcal{A}, r, Q_H)$ *and* $h_1, \ldots, h_q$ *runs* $\mathcal{A}$ *by providing it with random oracle queries and update oracle queries.* $\mathcal{Z}$ *returns the index* $i$ *of the random oracle query made for challenge* $k$ *and the proof* $\mathcal{A}$ *returns.*

*Consider the algorithm* $\mathsf{GF}^n_{\mathcal{Z}}$ *that rewinds* $\mathcal{Z}$ *to produce a* $(1, \ldots, n, \ldots, 1)$*-tree of transcripts.*

*We say that* $\Psi$ *is* $(\varepsilon(\lambda), k, n)$*-forking if for any PPT adversary the probability that*

$$\Pr\left[\mathbf{R}(\mathsf{x}, \mathsf{w}) = 0 \;\middle|\; \begin{matrix} r \leftarrow\!\!\$\, \mathsf{R}(\mathcal{A}), (\mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}) \leftarrow \mathcal{A}^{\mathsf{UpdO}, \mathcal{H}}(1^\lambda; r), \\ (1, \mathsf{T}) \leftarrow \mathsf{GF}^m_{\mathcal{Z}}((\mathsf{srs}, \mathcal{A}, r, Q_H), Q_H), \mathsf{w} \leftarrow \mathsf{Ext}_{\mathsf{tree}}(\mathsf{T}) \end{matrix}\right] \leq \varepsilon(\lambda).$$

*Here,* srs *is the SRS that* $\mathcal{A}$ *finalised using the update oracle* UpdO*. List* $Q_{\mathcal{H}}$ *contains all* $\mathcal{A}$*'s queries to* $\mathcal{H}$ *and* $\mathcal{H}$*'s answers.*

[**Chaya:** why does GF get $Q_H$ twice?]

### 3.4 Forking simulation extractability

**Definition 6 (Forking simulation-extractable NIZK, [21]).** *Let* $\Psi_{\mathsf{FS}} = (\mathsf{KGen}_{\mathsf{FS}}, \mathsf{P}_{\mathsf{FS}}, \mathsf{V}_{\mathsf{FS}}, \mathsf{Sim}_{\mathsf{FS}})$ *be a HVZK proof system[***Hamid:** $\Psi_{\mathsf{FS}}$ *is the Fiat-Shamir variant of the underlying proof system. So maybe we mean the underlying proof system is HVZK?]. We say that* $\Psi_{\mathsf{FS}}$ *is* updatable forking simulation-extractable *with* extraction error $\nu$ *if for any* PPT *adversary* $\mathcal{A}$ *that is given oracle access to an updatable SRS setup* UpdO*, a simulation oracle* SimO *[***Hamid:** *(See Fig. 1)], and a random oracle* $\mathcal{H}$*, and produces an accepting transcript of* $\Psi$ *with probability* acc*, where*

$$\mathsf{acc} = \Pr\left[\mathsf{V}_{\mathsf{FS}}(\mathsf{srs}, \mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}) = 1 \wedge (\mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}) \notin Q \;\middle|\; r \leftarrow\!\!\$\, \mathsf{R}(\mathcal{A}), (\mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}) \leftarrow \mathcal{A}^{\mathsf{UpdO}, \mathsf{SimO}, \mathcal{H}}(1^\lambda; r)\right],$$

*there exists an extractor* $\mathsf{Ext}_{\mathsf{se}}$ *such that*

$$\mathsf{ext} = \Pr\left[\begin{array}{l|l} \mathsf{V}_{\mathsf{FS}}(\mathsf{srs}, \mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}) = 1 \wedge & r \leftarrow\!\!\$\, \mathsf{R}(\mathcal{A}), (\mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}) \leftarrow \mathcal{A}^{\mathsf{UpdO},\mathsf{SimO},\mathcal{H}}(1^{\lambda}; r) \\ (\mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}) \notin Q \wedge \mathbf{R}(\mathsf{x}_{\mathcal{A}}, \mathsf{w}_{\mathcal{A}}) = 1 & \mathsf{w}_{\mathcal{A}} \leftarrow \mathsf{Ext}_{\mathsf{se}}(\mathsf{srs}, \mathcal{A}, r, \mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}, Q, Q_{\mathcal{H}}, Q_{\mathsf{srs}}) \end{array}\right]$$

*is at at least*

$$\mathsf{ext} \geq \frac{1}{\mathsf{poly}(\lambda)}(\mathsf{acc} - \nu)^d - \varepsilon(\lambda)\,,$$

*for some polynomial* $\mathsf{poly}(\lambda)$*, constant* $d$ *and negligible* $\varepsilon(\lambda)$ *whenever* $\mathsf{acc} \geq \nu$. *[***Hamid:** *We should add this: Here,* srs *is the finalised SRS.] List* $Q$ *contains all* $(\mathsf{x}, \pi)$ *pairs where* $\mathsf{x}$ *is an instance provided to the simulator by the adversary and* $\pi$ *is the simulator's answer. List* $Q_{\mathcal{H}}$ *contains all* $\mathcal{A}$*'s queries to* $\mathcal{H}$ *and* $\mathcal{H}$*'s answers.*

## 4 Forking simulation-extractability—the general result

Equipped with definitional framework of Section 3 we are ready to present the main result of this paper—a proof of forking simulation extractability of Fiat-Shamir NIZK based on multi-round protocols.

The proofs go by game hopping. The games are controlled by an environment $\mathcal{E}$ that internally runs a simulation extractability adversary $\mathcal{A}$, provides it with access to a random oracle and simulator, and when necessary, rewinds it. The games differ by various breaking points, i.e. points where the environment decides to abort the game.

Denote by $\pi_{\mathcal{A}}, \pi_{\mathsf{Sim}}$ proofs returned by the adversary and the simulator respectively. We use $\pi[i]$ to denote prover's message in the $i$-th round of the proof (counting from 1), i.e. $(2i-1)$-th message exchanged in the protocol. $\pi[i].\mathsf{ch}$ denotes the challenge that is given to the prover after $\pi[i]$, and $\pi[i..j]$ to denote all messages of the proof including challenges between rounds $i$ and $j$, but not challenge $\pi[j].\mathsf{ch}$. When it is not explicitly stated, we denote the proven instance $\mathsf{x}$ by $\pi[0]$ (however, there is no following challenge $\pi[0].\mathsf{ch}$).

Without loss of generality, we assume that whenever the accepting proof contains a response to a challenge from a random oracle, then the adversary queried the oracle to get it. It is straightforward to transform any adversary that violates this condition into an adversary that makes these additional queries to the random oracle and wins with the same probability.

We conjecture that based on the recent results on state restoration soundness [27], which effectively allows to query the verifier multiple times on different overlapping transcripts, the $q^{\mu}$ loss could be avoided. However, this would reduce the class of protocols covered by our results.

**Theorem 1 (Forking simulation-extractable multi-message protocols).** *Let* $\Psi = (\mathsf{KGen}, \mathsf{P}, \mathsf{V}, \mathsf{Sim})$ *be an interactive* $(2\mu+1)$*-message proof system for* $\mathcal{R}(1^{\lambda})$ *that is honest verifier zero-knowledge in the standard model[***Hamid:** *Is this different from Trapdoor-less simulatable proof system (def. 1)?]*[5]*, has* $k$-ur *property with security* $\varepsilon_{\mathsf{ur}}(\lambda)$*, and is* $(\varepsilon_{\mathsf{s}}(\lambda), k, n)$*-forking sound. Let* $\mathcal{H}\colon \{0,1\}^* \to \{0,1\}^{\lambda}$ *be a random oracle. Then* $\Psi_{\mathsf{FS}}$ *is forking simulation-extractable with extraction error* $\varepsilon_{\mathsf{ur}}(\lambda)$ *against* PPT *algebraic adversaries that makes up to* $q$ *random oracle queries and returns an acceptable proof with probability at least* acc*. The extraction probability* ext *is at least* $\mathsf{ext} \geq \frac{1}{q^{n-1}}(\mathsf{acc} - \varepsilon_{\mathsf{ur}}(\lambda))^n - \varepsilon(\lambda)$ *, for some negligible* $\varepsilon(\lambda)$.

*Proof.* **Game** $\mathsf{G}_0$**:** This is simulation extraction game played between an adversary $\mathcal{A}$ who is given access to an oracle UpdO that defines an updatable SRS setup, a random oracle $\mathcal{H}$ and a simulation oracle SimO. There is also an extractor Ext that, from a proof $\pi_{\mathcal{A}}$ for instance $\mathsf{x}_{\mathcal{A}}$ output by the adversary and from transcripts of $\mathcal{A}$'s operations is tasked to extract a witness $\mathsf{w}_{\mathcal{A}}$ such that $\mathbf{R}(\mathsf{x}_{\mathcal{A}}, \mathsf{w}_{\mathcal{A}})$ holds. $\mathcal{A}$ wins if it manages to produce an acceptable proof and the extractor fails to reveal the corresponding witness. In the following game hops we upper-bound the probability that this happens. Note that srs is with respect to the finalised SRS with respect to which $\mathcal{A}$ is allowed to make simulation queries.

[5] Crucially, we require that one can provide an indistinguishable simulated proof without any additional knowledge, as e.g knowledge of a SRS trapdoor.

**Game** $\mathsf{G}_1$**:** This is identical to $\mathsf{G}_0$ except that now the game is aborted if there is $(\mathsf{x}_{\mathcal{A}}, \pi_{\mathsf{Sim}}) \in Q$ such that $\pi_{\mathsf{Sim}}[1..k] = \pi_{\mathcal{A}}[1..k]$. That is, the adversary in its final proof reuses at least $k$ messages from a simulated proof it saw before and the proof is acceptable. Denote that event by $\mathsf{Err}_{\mathsf{ur}}$.

**Game 0 to Game 1:** $\Pr[\mathsf{Err}_{\mathsf{ur}}] \leq \varepsilon_{\mathsf{ur}}(\lambda)$. The proof goes exactly as in **??**.

**Game** $\mathsf{G}_2$**:** Define an adversary $\mathcal{B}$ against forking soundness such that given access to oracles $\mathsf{UpdO}$ and $\mathcal{H}$, randomness $r_{\mathcal{B}}$ internally runs $\mathcal{A}^{\mathsf{UpdO},\mathsf{SimO},\mathcal{H}}(1^{\lambda}; r_{\mathcal{A}})$, where

1. $r_{\mathcal{B}}$ is split into two substrings $r_{\mathcal{A}}$ and $r_{\mathsf{Sim}}$;
2. $\mathcal{B}$ answers $\mathcal{A}$ update queries by asking the same query from its own update oracle. Also, when $\mathcal{A}$ finalised an SRS $\mathsf{srs}$, $\mathcal{B}$ does the same.
3. $\mathcal{B}$ answers $\mathcal{A}$ simulator queries itself by programming the random oracle locally. Coins from $r_{\mathsf{Sim}}$ are use to that.
4. Eventually $\mathcal{A}$ outputs instance $\mathsf{x}_{\mathcal{A}}$ and proof $\pi_{\mathcal{A}}$. The same instance and proof are output by $\mathcal{B}$.
5. $\mathcal{B}$ sets up (initially empty) list $Q$ where all simulated proofs that $\mathcal{A}$ asked for are kept.

In this game, the environment aborts also when it fails to build a $(1, \ldots, 1, n, 1, \ldots, 1)$-tree of accepting transcripts $\mathsf{T}$ by rewinding $\mathcal{B}$. Denote that event by $\mathsf{Err}_{\mathsf{frk}}$.

**Game 1 to Game 2:** Note that for every acceptable proof $\pi_{\mathcal{A}}$, we may assume that whenever $\mathcal{A}$ outputs in Round $k$ message $\pi_{\mathcal{A}}[k]$, then the $(\mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}[1..k])$ random oracle query was made by the adversary, not the simulator[6], i.e. there is no simulated proof $\pi_{\mathsf{Sim}}$ on $\mathsf{x}_{\mathsf{Sim}}$ such that $(\mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}}[1..k]) = (\mathsf{x}_{\mathsf{Sim}}, \pi_{\mathsf{Sim}}[1..k])$. Otherwise, the game would be already interrupted by the error event in Game $\mathsf{G}_1$. As previously, $|\Pr[\mathsf{G}_1] - \Pr[\mathsf{G}_2]| \leq \Pr[\mathsf{Err}_{\mathsf{frk}}]$ .

We describe our extractor $\mathsf{Ext}$ here. The extractor takes as input relation $\mathbf{R}$, SRS $\mathsf{srs}$, $\mathcal{B}$'s code, its randomness $r_{\mathcal{B}}$, the output instance $\mathsf{x}_{\mathcal{A}}$ and proof $\pi_{\mathcal{A}}$, and the list of random oracle queries and responses $Q_{\mathcal{H}}$. Then, $\mathsf{Ext}$ starts a forking algorithm $\mathsf{GF}^n_{\mathcal{Z}}(y, h_1, \ldots, h_q)$ for $y = (\mathsf{srs}, \mathcal{B}, r_{\mathcal{B}}, \mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}})$ where we set $h_1, \ldots, h_q$ to be the consecutive queries from list $Q_{\mathcal{H}}$. We run $\mathcal{B}$ internally in $\mathcal{Z}$.

To assure that in the first execution of $\mathcal{Z}$ the adversary $\mathcal{B}$ produces the same $(\mathsf{x}_{\mathcal{A}}, \pi_{\mathcal{A}})$ as in the extraction game, $\mathcal{Z}$ provides $\mathcal{B}$ with the same randomness $r_{\mathcal{B}}$ and answers queries to the random oracle with pre-recorded responses in $Q_{\mathcal{H}}$. Note, that since the view of the adversary when run inside $\mathcal{Z}$ is the same as its view with access to the real random oracle, it produces exactly the same output. After the first run, $\mathcal{Z}$ outputs the index $i$ of a random oracle query that was used by $\mathcal{B}$ to compute the challenge $\pi[k].\mathsf{ch} = \mathcal{H}(\pi_{\mathcal{A}}[0..k])$ it had to answer in the $(k+1)$-th round and adversary's transcript, denoted by $s_1$ in $\mathsf{GF}$'s description. If no such query took place $\mathcal{Z}$ outputs $i = 0$.

Then new random oracle responses are picked for queries indexed by $i, \ldots, q$ and the adversary is rewound to the point just prior to when it gets the response to RO query $\pi_{\mathcal{A}}[0..k]$. The adversary gets a random oracle response from a new set of responses $h_i^2, \ldots, h_q^2$. If the adversary requests a simulated proof after seeing $h_i^2$ then $\mathcal{Z}$ computes the simulated proof on its own. Eventually, $\mathcal{Z}$ outputs index $i'$ of a query that was used by the adversary to compute $\mathcal{H}(\pi_{\mathcal{A}}[0..k])$, and a new transcript $s_2$. $\mathcal{Z}$ is run $n$ times with different random oracle responses. If a tree $\mathsf{T}$ of $n$ transcripts is built then $\mathsf{Ext}$ runs internally the tree extractor $\mathsf{Ext}_{\mathsf{tree}}(\mathsf{T})$ and outputs what it returns.

We emphasize here the importance of the unique response property. If it does not hold then in some $j$-th execution of $\mathcal{Z}$ the adversary $\mathcal{A}$ (run internally in $\mathcal{B}$) could reuse a challenge that it learned from observing proofs in $Q$. In that case, $\mathcal{B}$ would output a proof that would make $\mathcal{Z}$ output $i = 0$, making the extractor fail. Fortunately, the case that the adversary breaks the unique response property has already been covered by the abort condition in $\mathsf{G}_1$.

Denote by $\widetilde{\mathsf{acc}}$ the probability that $\mathcal{A}$ outputs a proof that is accepted and does not break $k$-$\mathsf{ur}$-ness of $\Psi$. With the same probability an acceptable proof is returned by $\mathcal{B}$. Denote by $\widetilde{\mathsf{acc}}'$ the probability that algorithm $\mathcal{Z}$, defined in the lemma, produces an accepting proof with a fresh challenge after Round $k$. Given the discussion above, we can state that $\widetilde{\mathsf{acc}} = \widetilde{\mathsf{acc}}'$.

[6] [21] calls these queries *fresh*.

Next, from the generalised forking lemma, cf. Lemma 2, we get that

$$\Pr[\mathsf{Err}_{\mathsf{frk}}] \leq 1 - \widetilde{\mathsf{acc}} \cdot \left( \widetilde{\mathsf{acc}}^{n-1}/q^{n-1} + (2^\lambda)!/((2^\lambda - n)! \cdot (2^\lambda)^n) - 1 \right). \tag{2}$$

**Game** $\mathsf{G}_3$**:** This game is identical to $\mathsf{G}_2$ except that it aborts if $\mathsf{Ext}_{\mathsf{tree}}(\mathsf{T})$ run by $\mathsf{Ext}$ fails to extract the witness.

**Game 2 to Game 3:** Since $\Psi$ is forking-sound the probability that $\mathsf{Ext}_{\mathsf{tree}}(\mathsf{T})$ fails is upper-bounded by $\varepsilon_{\mathsf{f}}(\lambda)$.

Since Game $\mathsf{G}_3$ is aborted when it is impossible to extract the correct witness from $\mathsf{T}$ and $\mathcal{B}$ only passes proofs produced by $\mathcal{A}$, the adversary $\mathcal{A}$ cannot win. Thus, by the game-hopping argument,

$$|\Pr[\mathsf{G}_0] - \Pr[\mathsf{G}_4]| \leq 1 - \left( \frac{\widetilde{\mathsf{acc}}^n}{q^{n-1}} + \widetilde{\mathsf{acc}} \cdot \frac{(2^\lambda)!}{(2^\lambda - n)! \cdot (2^\lambda)^n} - \widetilde{\mathsf{acc}} \right) + \varepsilon_{\mathsf{ur}}(\lambda) + \varepsilon_{\mathsf{f}}(\lambda) .$$

Thus the probability that extractor $\mathsf{Ext}_{\mathsf{ss}}$ succeeds is at least

$$\frac{\widetilde{\mathsf{acc}}^n}{q^{n-1}} + \widetilde{\mathsf{acc}} \cdot \frac{(2^\lambda)!}{(2^\lambda - n)! \cdot (2^\lambda)^n} - \widetilde{\mathsf{acc}} - \varepsilon_{\mathsf{ur}}(\lambda) - \varepsilon_{\mathsf{f}}(\lambda) .$$

Since $\widetilde{\mathsf{acc}}$ is probability of $\mathcal{A}$ outputting acceptable transcript that does not break $k$-ur-ness of $\Psi$, then $\widetilde{\mathsf{acc}} \geq \mathsf{acc} - \varepsilon_{\mathsf{ur}}(\lambda)$, where $\mathsf{acc}$ is the probability of $\mathcal{A}$ outputting an acceptable proof as defined in **??**. It thus holds

$$\mathsf{ext} \geq \frac{(\mathsf{acc} - \varepsilon_{\mathsf{ur}}(\lambda))^n}{q^{n-1}} - \underbrace{(\mathsf{acc} - \varepsilon_{\mathsf{ur}}(\lambda)) \cdot \left( 1 - \frac{(2^\lambda)!}{(2^\lambda - n)! \cdot (2^\lambda)^n} \right) - \varepsilon_{\mathsf{ur}}(\lambda) - \varepsilon_{\mathsf{f}}(\lambda)}_{\varepsilon(\lambda)} . \tag{3}$$

Note that the part of Eq. (3) denoted by $\varepsilon(\lambda)$ is negligible as $\varepsilon_{\mathsf{ur}}(\lambda), \varepsilon_{\mathsf{f}}(\lambda)$ are negligible, and $\frac{(2^\lambda)!}{(2^\lambda - n)! \cdot (2^\lambda)^n} \geq \left((2^\lambda - n)/2^\lambda\right)^n$ is overwhelming. Thus,

$$\mathsf{ext} \geq q^{-(n-1)} (\mathsf{acc} - \varepsilon_{\mathsf{ur}}(\lambda))^n - \varepsilon(\lambda) .$$

and $\Psi_{\mathsf{FS}}$ is forking simulation extractable with extraction error $\varepsilon_{\mathsf{ur}}(\lambda)$. □

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