

# BP

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## 1 Model

The atomic unit of the factor graph is shown in Figure 1.

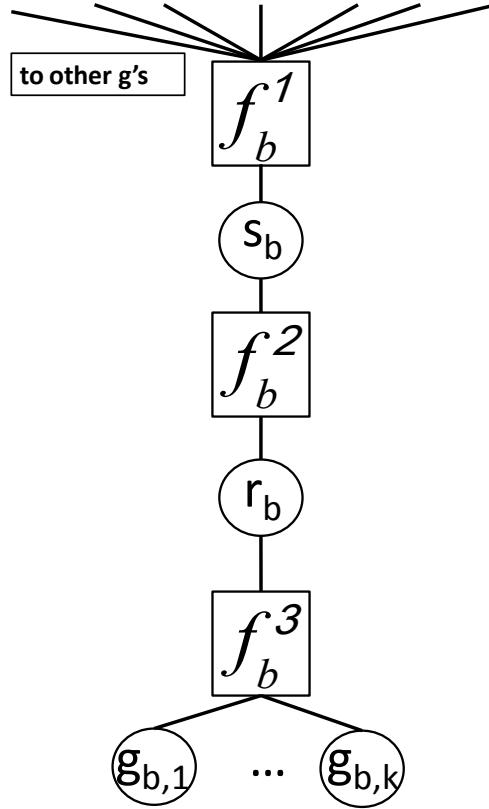


Figure 1: Factor graph

We require efficient computation of the messages.

## 2 Top-down messages

### 2.1 $\mu_{f_b^1 \rightarrow s_b}$

The general form of the message is given by

$$\mu_{f_b^1 \rightarrow s_b}(s_b) = \sum_{\vec{g}} f_b^1(s_b, \vec{g}) \prod_{g_{b',k}} \mu_{g_{b',k} \rightarrow f_b^1}(g_{b',k}) \quad (1)$$

We divide the cases into  $s_b = 0$ , and  $s_b = 1$ .

$$\mu_{f_b^1 \rightarrow s_b}(s_b = 0) = \sum_{\vec{g}} f_b^1(s_b = 0, \vec{g}) \prod_{g_{b',k}} \mu_{g_{b',k} \rightarrow f_b^1}(g_{b',k}) \quad (2)$$

$$= (1 - \epsilon) \prod_{g_{b',k}} \mu_{g_{b',k} \rightarrow f_b^1}(g_{b',k} = 0) \quad (3)$$

$$\mu_{f_b^1 \rightarrow s_b}(s_b = 1) = \sum_{\vec{g}} f_b^1(s_b = 1, \vec{g}) \prod_{g_{b',k}} \mu_{g_{b',k} \rightarrow f_b^1}(g_{b',k}) \quad (4)$$

$$= \epsilon \times \prod_{g_{b',k}} \mu_{g_{b',k} \rightarrow f_b^1}(g_{b',k} = 0) \quad (5)$$

$$+ \sum_{\vec{g}} \prod_{g_{b',k}} \mu_{g_{b',k} \rightarrow f_b^1}(g_{b',k}) - \prod_{g_{b',k}} \mu_{g_{b',k} \rightarrow f_b^1}(g_{b',k} = 0)$$

$$= \epsilon \times \prod_{g_{b',k}} \mu_{g_{b',k} \rightarrow f_b^1}(g_{b',k} = 0) \quad (6)$$

$$+ 1 - \prod_{g_{b',k}} \mu_{g_{b',k} \rightarrow f_b^1}(g_{b',k} = 0)$$

$$= \prod_{g_{b',k}} \mu_{g_{b',k} \rightarrow f_b^1}(g_{b',k} = 0)(\epsilon - 1) + 1 \quad (7)$$

$$= 1 - (1 - \epsilon) \prod_{g_{b',k}} \mu_{g_{b',k} \rightarrow f_b^1}(g_{b',k} = 0) \quad (8)$$

$$= 1 - \mu_{f_b^1 \rightarrow s_b}(s_b = 0) \quad (9)$$

where we have made use of the property that messages from a node/factor sum to 1

### 2.2 $\mu_{s_b \rightarrow f_b^2}$

if  $s_b$  has not been observed, this message is given by

$$\mu_{s_b \rightarrow f_b^2}(s_b) = \mu_{f_b^1 \rightarrow s_b}(s_b) \quad (10)$$

otherwise, it is a constant message that indicates its observed state.

### 2.3 $\mu_{f_b^2 \rightarrow r_b}$

$$\mu_{f_b^2 \rightarrow r_b}(r_b) = \sum_{s_b} f_b^2(s_b, r_b) \mu_{s_b \rightarrow f_b^2}(s_b) \quad (11)$$

$$= \sum_{s_b} P(r_b \mid s_b) \mu_{s_b \rightarrow f_b^2}(s_b) \quad (12)$$

$s_b$  only has two values, so this is easy to compute.

### 2.4 $\mu_{r_b \rightarrow f_b^3}$

$$\mu_{r_b \rightarrow f_b^3}(r_b) = \mu_{f_b^2 \rightarrow r_b}(r_b) \quad (13)$$

### 2.5 $\mu_{f_b^3 \rightarrow g_{b,k}}$

$$\mu_{f_b^3 \rightarrow g_{b,k}}(g_{b,k}) = \sum_{r_b} \sum_{k' \neq k} f_b^3(r_b, \vec{g}_b) \mu_{r_b \rightarrow f_b^3}(r_b) \prod_{k' \neq k} \mu_{g_{b,k} \rightarrow f_b^3}(g_{b,k'}) \quad (14)$$

$$= \sum_{r_b} \mu_{r_b \rightarrow f_b^3}(r_b) \sum_{k' \neq k} P(\vec{g}_b \mid r_b) \prod_{k' \neq k} \mu_{g_{b,k} \rightarrow f_b^3}(g_{b,k'}) \quad (15)$$

$$= \sum_{r_b} \mu_{r_b \rightarrow f_b^3}(r_b) \sum_{k' \neq k} P(\vec{g}_{b,k} \mid r_b) \prod_{k' \neq k} P(\vec{g}_{b,k'} \mid r_b) \mu_{g_{b,k} \rightarrow f_b^3}(g_{b,k'}) \quad (16)$$

$$= \sum_{r_b} \mu_{r_b \rightarrow f_b^3}(r_b) P(\vec{g}_{b,k} \mid r_b) \prod_{k' \neq k} \sum_{g_{b,k'}} P(\vec{g}_{b,k'} \mid r_b) \mu_{g_{b,k} \rightarrow f_b^3}(g_{b,k'}) \quad (17)$$

The number of values  $r_b$  can take should be small (no more than, say 10? 20?), so this should be possible to compute.

### 2.6 $\mu_{g_{b,k} \rightarrow f_b^1}$

$$\mu_{g_{b,k} \rightarrow f_b^1}(g_{b,k}) = \mu_{f_b^3 \rightarrow g_{b,k}}(g_{b,k}) \prod_{b' \neq b} \mu_{f_{b'}^1 \rightarrow g_{b,k}}(g_{b,k}) \quad (18)$$

## 3 Bottom-up messages

### 3.1 $\mu_{f_b^1 \rightarrow g_{b',k}}$

Recall that  $f_b^1$  connects to other bricks. We denote one such brick by  $b'$ .

Because  $f_b^1$  only cares about whether  $g_{b',k}$  points to  $b$ , we reduce the domain of  $g_{b',k}$  to 0, 1 which means “doesn’t point to  $b$ ” and “points to  $b$ ”, respectively. The messages can be “expanded” by  $g_{b',k}$  to its original domain when  $g_{b',k}$  needs to send a message to  $f_b^3$ .

The general form of this message is

$$\mu_{f_b^1 \rightarrow g_{b',k}}(g_{b,k'}) = \sum_{s_b} \sum_{g \neq g_{b',k}} f_b^1(s_b, \mathbf{g}) \mu_{s_b \rightarrow f_b^1}(s_b) \prod_{g \neq g_{b',k}} \mu_{g \rightarrow f_b^1}(g). \quad (19)$$

We divide the cases into  $g_{b',k} = 0$ , and  $g_{b',k} = 1$

$$\mu_{f_b^1 \rightarrow g_{b',k}}(g_{b',k} = 0) = \mu_{s_b \rightarrow f_b^1}(s_b = 0) \times (1 - \epsilon) \times \prod_{g \neq g_{b',k}} \mu_{g \rightarrow f_b^1}(g = 0) \quad (20)$$

$$\begin{aligned} & + \mu_{s_b \rightarrow f_b^1}(s_b = 1) \times \epsilon \times \prod_{g \neq g_{b',k}} \mu_{g \rightarrow f_b^1}(g = 0) \\ & + \mu_{s_b \rightarrow f_b^1}(s_b = 1) \left( \prod_{g \neq g_{b',k}} \sum_g \mu_{g \rightarrow f_b^1}(g) - \prod_{g \neq g_{b',k}} \mu_{g \rightarrow f_b^1}(g = 0) \right) \\ = & \mu_{s_b \rightarrow f_b^1}(s_b = 0) \times (1 - \epsilon) \times \prod_{g \neq g_{b',k}} \mu_{g \rightarrow f_b^1}(g = 0) \quad (21) \\ & + \epsilon \times \mu_{s_b \rightarrow f_b^1}(s_b = 1) \prod_{g \neq g_{b',k}} \mu_{g \rightarrow f_b^1}(g = 0) \\ & + \mu_{s_b \rightarrow f_b^1}(s_b = 1) \left( 1 - \prod_{g \neq g_{b',k}} \mu_{g \rightarrow f_b^1}(g = 0) \right) \end{aligned}$$

$$\begin{aligned} = & \mu_{s_b \rightarrow f_b^1}(s_b = 0) \times (1 - \epsilon) \times \prod_{g \neq g_{b',k}} \mu_{g \rightarrow f_b^1}(g = 0) \\ & + \mu_{s_b \rightarrow f_b^1}(s_b = 1) \left( 1 - \prod_{g \neq g_{b',k}} \mu_{g \rightarrow f_b^1}(g = 0) \right) (1 - \epsilon) \end{aligned} \quad (22)$$

where we have made use of the property that messages from a node/factor sum to 1.

$$\mu_{f_b^1 \rightarrow g_{b',k}}(g_{b',k} = 1) = \mu_{s_b \rightarrow f_b^1}(s_b = 1) \prod_{g \neq g_{b',k}} \sum_g \mu_{g \rightarrow f_b^1}(g) \quad (23)$$

$$= \mu_{s_b \rightarrow f_b^1}(s_b = 1) \quad (24)$$

### 3.2 $\mu_{g_{b,k} \rightarrow f_b^3}$

$$\mu_{g_{b,k} \rightarrow f_b^3}(g_{b,k}) = \prod_{b' \in V(b)} \mu_{f_{b'}^3 \rightarrow g_{b,k}}(g_{b,k}) \quad (25)$$

wher  $V(b)$  is the after set of brick  $b$ . Note that  $\mu_{f_{b'}^3 \rightarrow g_{b,k}}(g_{b,k})$  needs to be “expanded” to the proper domain of  $g_{b,k}$

### 3.3 $\mu_{f_b^3 \rightarrow r_b}$

$$\mu_{f_b^3 \rightarrow r_b}(r_b) = \sum_{\vec{g}} f_b^3(\vec{g}, r_b) \prod_k \mu_{g_{b,k} \rightarrow f_b^3}(g_{b,k}) \quad (26)$$

$$= \sum_{\vec{g}} \prod_k P(g_{b,k} \mid r_b) \prod_k \mu_{g_{b,k} \rightarrow f_b^3}(g_{b,k}) \quad (27)$$

$$= \prod_k \sum_{g_{b,k}} P(g_{b,k} \mid r_b) \mu_{g_{b,k} \rightarrow f_b^3}(g_{b,k}) \quad (28)$$

### 3.4 $\mu_{r_b \rightarrow f_b^2}$

$$\mu_{r_b \rightarrow f_b^2}(r_b) = \mu_{f_b^3 \rightarrow r_b}(r_b) \quad (29)$$

### 3.5 $\mu_{f_b^2 \rightarrow s_b}$

$$\mu_{f_b^2 \rightarrow s_b}(s_b) = \sum_{r_b} f_b^2(r_b, s_b) \mu_{r_b \rightarrow f_b^2}(r_b) \quad (30)$$

$$= \sum_{r_b} P(r_b \mid s_b) \mu_{r_b \rightarrow f_b^2}(r_b) \quad (31)$$

These messages should be trivial to compute, since the set of values  $r_b$  is small.

### 3.6 $\mu_{s_b \rightarrow f_b^1}$

$$\mu_{s_b \rightarrow f_b^1}(s_b) = \mu_{f_b^2 \rightarrow s_b}(s_b) \quad (32)$$