Dimensionality reduction

Machine Learning and Deep Learning

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November 15th, 2022

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Agenda

Principal Component Analysis

Eigenfaces

Principal Component Analysis

- Linear dimensionality reduction model
  - Subspace projection is linear
  - Reconstruction is linear
- Projects data in a new space subject to:
  - the direction exhibiting highest variance in feature space is projected on the first axis, the one exhibiting the second highest variance on the second axis, and so on.
  - axis of the new space are orthogonal (covariance is zero).

PCA: algorithm

- Arrange your data in a $n \times d$ matrix X, where n is the number of samples and d is data dimensionality
- Compute the mean μ (d-dimensional vector) of all samples

$\mu_{\text{PIXEL } 1,1} = \dfrac{\Sigma \text{ TUTTI PIXEL 1 IMG.}}{\text{N° PIXEL 1}}$

. . . .

$\mu_{\text{PIXEL } nn} = \text{''} \; |_{1=nn}$

- Compute convariance matrix

$$\Sigma = (X - \mu)^T (X - \mu)$$

→ SI ESTRAGGONO AUTOVETTORI ORDINATI DAI CORRISP. AUTOVALORI

- Pick the first m eigenvectors of Σ (ordered by decreasing eigenvalues), where m is the dimensionality you want your data to be projected to
- Arrange such eigenvectors in a $d \times m$ matrix E → MATRICE DI TRASFORMAZIONE
- Compute the projected samples as $P = X \cdot E$ (PROIEZIONE)

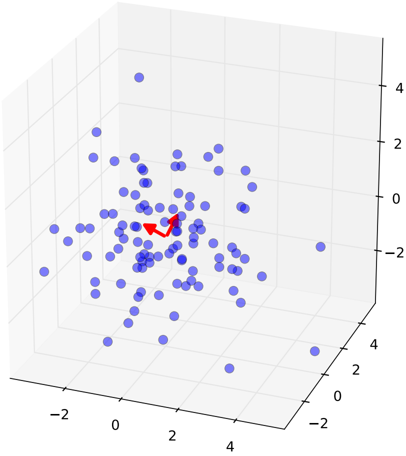
DIMENSIONE RIDOTTA

$P_{dim} = (n \times d)(d \times k) = n \times k$

- You can compute the reconstruction as $\tilde{X} = P \cdot E^T$ (RICOSTRUZIONE)

$\tilde{X}_{dim} = (n \times k)(d \times n) = n \times n$ (CI SARÀ PERDITA DI INFORMAZIONE)

PCA: plotting components



PCA: projecting and reconstructing (2D)

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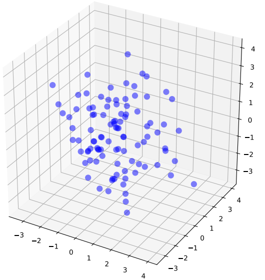
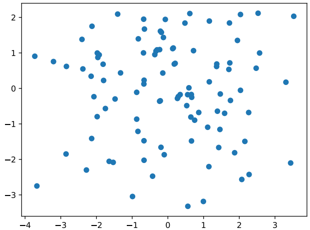
C'È UN MINIMO DI PERDITA.

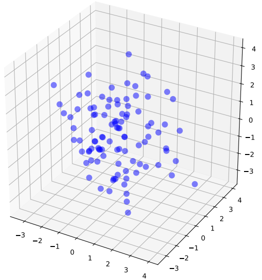
original data



projection



reconstruction

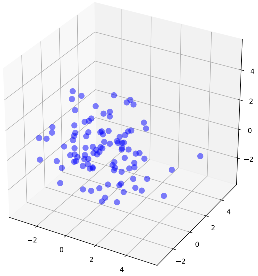


PCA: projecting and reconstructing (1D)

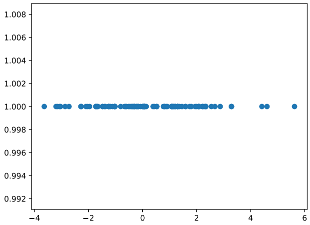
3D RIDUCO A CASO
1-DIM

PERDO MOLTA
INFORMAZIONE.

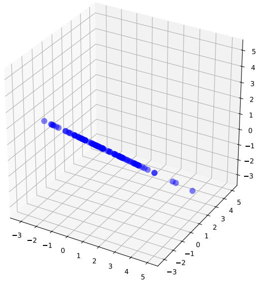
original data



projection



reconstruction

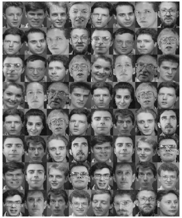


Eigenfaces

Eigenfaces

Famous algorithm for face recognition. Training is as simple as:

- load faces and annotations from the Olivetti dataset (datasets.get_faces_dataset takes care of loading and flattening images)



- Select a number of principal components and fit a PCA on training faces

To classify a test image:

- Project the image in the reduced spaces built in the training phase
- Perform **nearest neighbor classification**:
  - Roughly speaking, choose the class of the nearest training example (in the reduced space)

Eigenfaces: a magic trick to compute eigenvectors

Each Olivetti image is 112×92. Once flattened, is a vector of 10304 pixels: (TANTE FEATURES)

- The convariance matrix is 10304×10304 → ALTO COSTO DI COMPUTAZ.
- Computing eigenvectors and eigenvalues is a pain
- Instead, compute the covariance matrix of transposed X:

$$\Sigma = (X - \mu)(X - \mu)^T$$

(invece che DIM > n° features · n° ... spazio dim = n° el. × n° campi)

- Once selected the principal components $\tilde{E}$ of this weirdo space, you can compute the original eigenvectors just like:

$$E = X^T \cdot \tilde{E}$$

- Normalize the retrieved eigenvectors to have unit lenght:

$$E_i \leftarrow \frac{E_i}{\sqrt{\lambda_i}} \qquad i = 1, 2, \ldots, m$$

where λ_i is the eigenvalue corresponding to the eigenvector E_i

Perché autovalori Ei sono normalizzati rispetto la radice del corrispettivo autovalore.
Pertìchè Ei /sqrt(Gamma_i) é sempre <0 e <1 ?

In particolare la matrice di covarianza $\Sigma = \dim(d \times d)$, $d = n°$ features

stimare 10000 autovalori e autovettori è costoso.

Uso questo trick, invece di $\Sigma_{d\times d} = (n\times d)^T @ \, n\times d$

$= d\times n @ d\times n = d\times d$

Ottengo invece $\Sigma^T = \tilde{\Sigma}$ che è una $n\times n = 200 \times 200$ ~ questa scompo.

$$\tilde{S} = n\cdot d @ n\cdot d^T = n\cdot d @ d\cdot n = n\times n$$

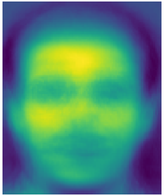
Poi da $\tilde{E} = \dim(n\times c)$, $c = n°$ dimensioni PCA.

$\tilde{S} \to B$ con $E = \underbrace{d\times n}_{X^T} @ \underbrace{n\times c}_{\tilde{E}} \to d\times c$

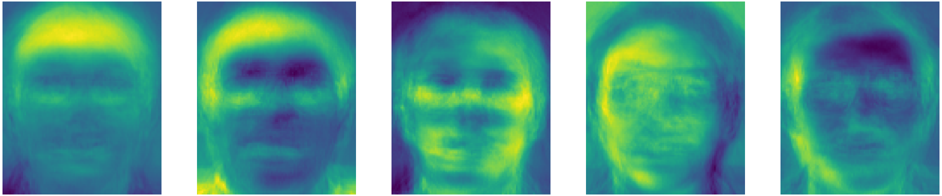
$$\Rightarrow P = n\times d @ d\times c = n\times c$$

Eigenfaces: some plots

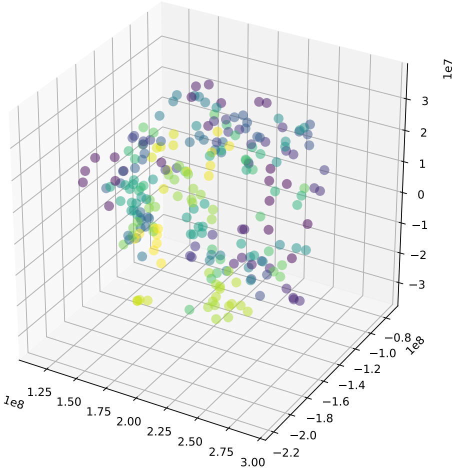
- Mean face:



- Eigenvectors:



Eigenfaces: face space



Eigenfaces: how many dimensions?