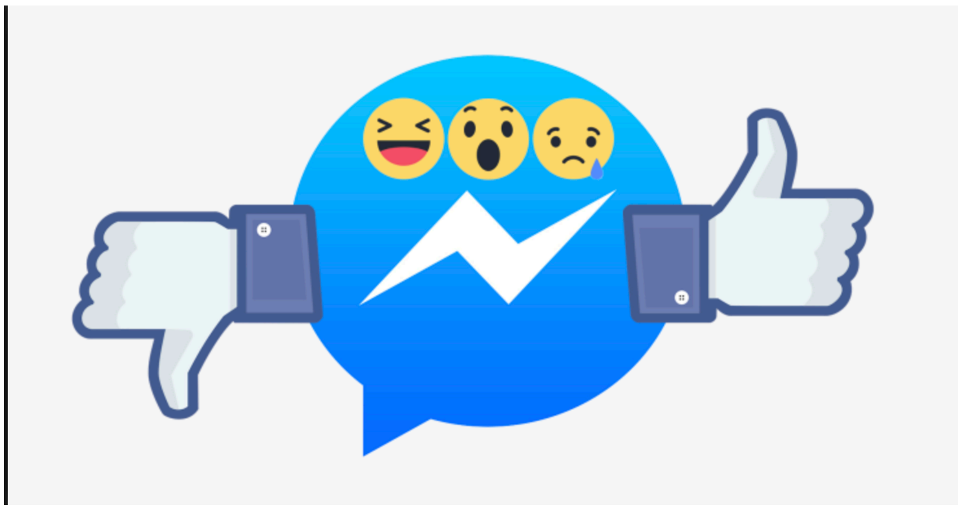
Network with Positive and Negative Ties



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Based on http://www.cs.uoi.gr/~tsap/teaching/2013-cs-l14/references.html

So far…

- Relationships have always positive connotations:
  - Friendship, collaboration, sharing of information
- However, there are also negative:
  - Controversy interactions between people or groups
  - Conflicts, Disapproval

How should we reason about the mix of positive and negative relationships that take place within a network?

Structural Balance

- Theory that involves taking a network and annotating its links with positive and negative signs
  - Positive links represent friendship
  - Negative links represent antagonism

- Understand the tension between these two forces

- Nice connection between local and global network properties:
  - Way in which local effects can have global consequences: Network as a whole!

Structural Balance

- Most basic model:
  - Set of people, everyone knows everyone else
  - Each edge is labeled with either + or -
    - +: friends
    - -: enemies
- Model makes the most sense:
  - For a group of people small enough to have this level of mutual awareness: classroom, small company
  - For international relations: nodes are countries and each country has an official diplomatic position toward every other.

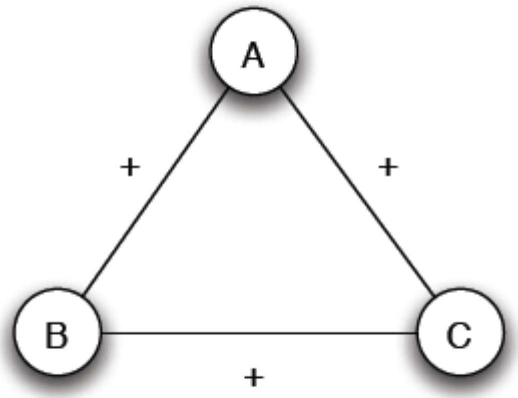
Structural Balance

- Principles are based on theories in social psychology dating back to the work of Heider in the 1940s
- Generalized and extended to the language of graphs with the work of Cartwright and Harary in the 1960s
  - Two people in isolation: + or –
  - Sets of three people: configurations

  of +s and –s are socially more plausible than others

Structural Balance

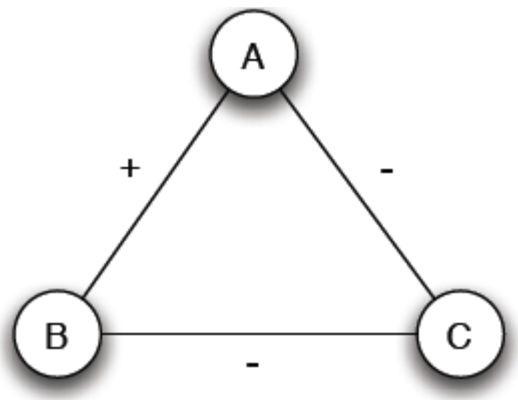
Case 1 - 3 +: it corresponds to three people who are mutual friends



Mutual friends

Structural Balance

Case 2 - 1 +,2 -: it means that two of the three are friends, and they have a mutual enemy in the thrid

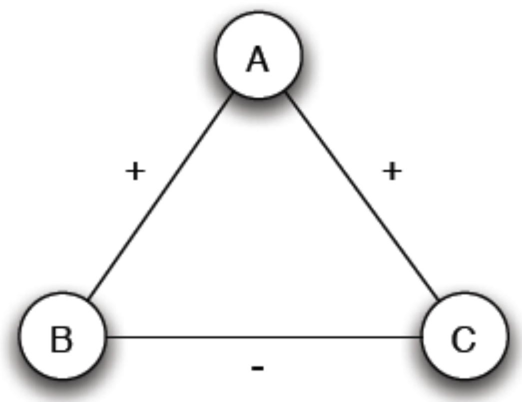


A and B are friends with a mutual enemy

Some cases introduce amount of psychological stress or instability

Structural Balance

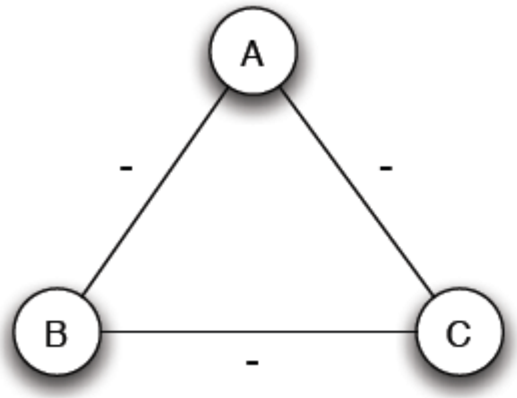
Case 3 - 2 +,1 -: it means A is friend with each of B and C, but B and C don't get along with each other. Implict forces pushing A to try to ge B and C to become friends; or else for A to side with of B or C against the other



A is friend with B and C, but B and C do not get well together

Structural Balance

Case 4 - 3 -: In this case, there would be forces motivating two of the three people to team up against the third (turning one of the three labels to a +)

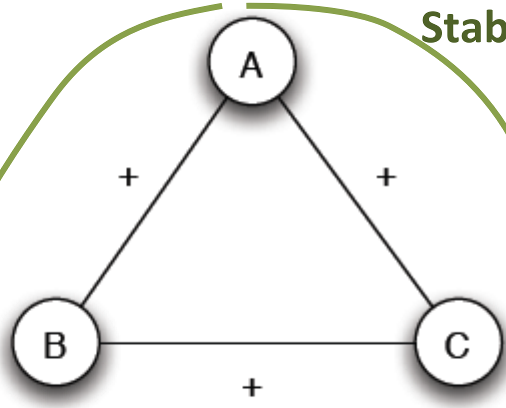


Mutual enemies

Structural Balance

Case 1: 3 +

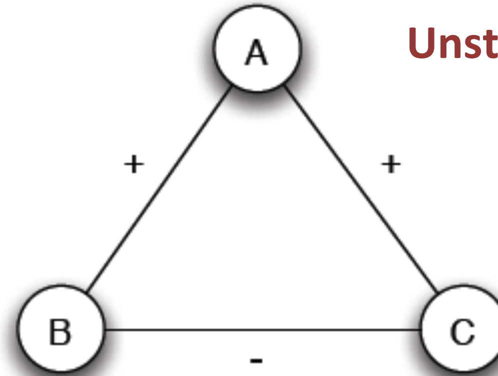
Stable or balanced



Mutual friends

Case 3: 2 +, 1 -

Unstable

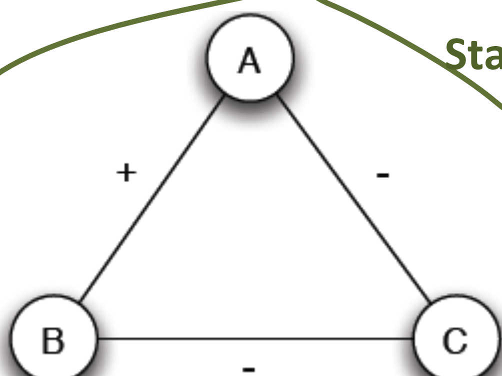


A is friend with B and C, but B and C do not get well together

Implicit force to make B and C friends (- => +) or turn one of the + to -

Case 2: 1 +, 2 -

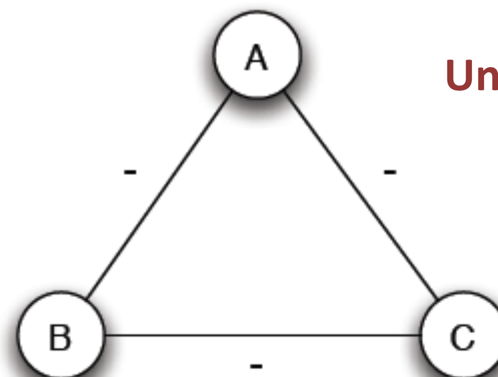
Stable or balanced



A and B are friends with a mutual enemy

Case 4: 3 -

Unstable



Mutual enemies

Forces to team up against the third (turn 1 – to +)

Structural Balance

- Unbalanced triangles are sources of stress or psychological dissonance
- People strive to minimize them in their personal relationships
- Unbalanced triangles will be less abundant in real social settings than balanced triangles

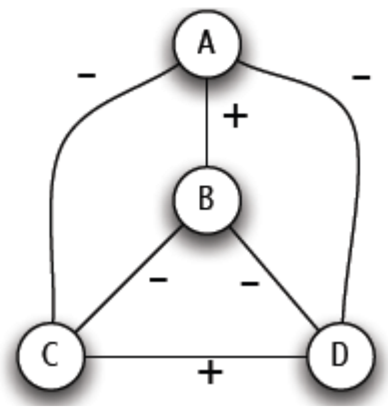
Structural Balance for Networks

- Definition that generalizes to complete graphs on an arbitrary number of nodes, with edges labeled by pluses and minuses

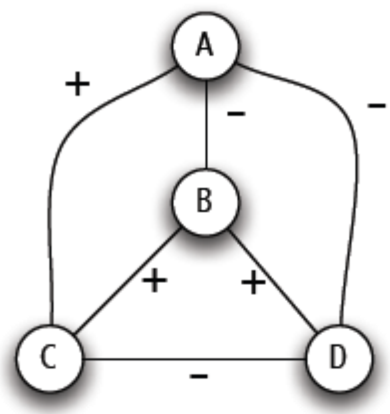
- A labeled complete graph is balanced if every one of its triangles is balanced

Structural Balance Property: For every set of three nodes, if we consider the three edges connecting them, either all three of these are labeled +, or else exactly one of them is labeled + (odd number of +)

Structural Balance for Networks



- Balanced: Each set of three nodes satisfies the Structural Balanced property



- Unbalanced: Among nodes A, B and C, there are exactly two edges labeled +, in violation of structural balance. Triangle B,C and D also violates the condition

Structural Balance for Networks

- Definition represents the limit of a social system that has eliminated all unbalanced triangles
- Extreme definition
- Propose a definition only requiring that at least some large percentage of all triangles were balanced, allowing a few triangles to be unbalanced (weaker structural balance)
- Strong definition is fundamental first step in thinking about the concept

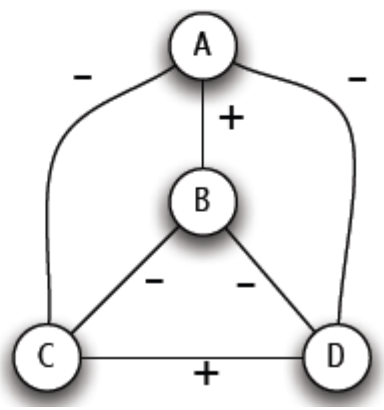
Structural Balance for Networks

What does a balanced network look like?

Simple conceptual description of what a balanced network looks like in general, instead of checking each single triangle!

Structural Balance for Networks

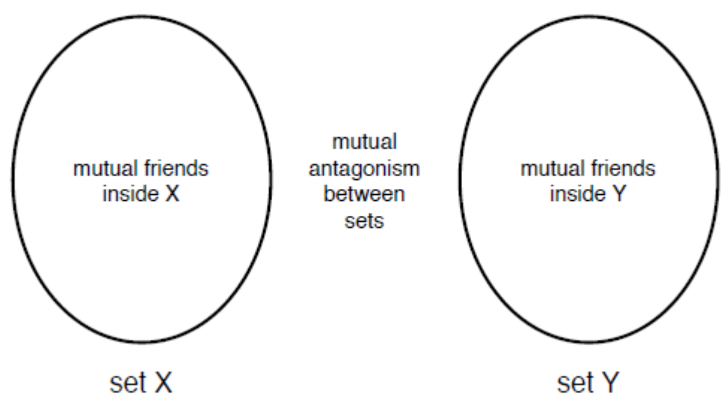
- Ways for a network to be balanced
  - Everyone likes each other: all triangles have three + labels



- More complicated way: two groups of friends (A and B, C and D), with negative relations between people in different groups

Structural Balance for Networks

- True in general
  - Suppose we have a complete graph in which nodes can be divided into two groups X and Y
  - Each pair in X likes each other, each pair in Y likes each other, and everyone in X is the enemy of everyone in Y
  - Such a network is balanced: a triangle contained entirely in one group or the other has three +, and a triangle with two people in one group and one in the other has exactly one + label



The Structure of Balanced Networks

Balance Theorem: If a labeled complete graph is balanced,

(a) all pairs of nodes are friends, or

(b) the nodes can be divided into two groups X and Y, such that every pair of nodes in X like each other, every pair of nodes in Y like each other, and every one in X is the enemy of every one in Y.

- Purely local property: structural balanced applied to only three nodes at a time
- Strong global property: either everyone gets along or the world is divided into two battling factions

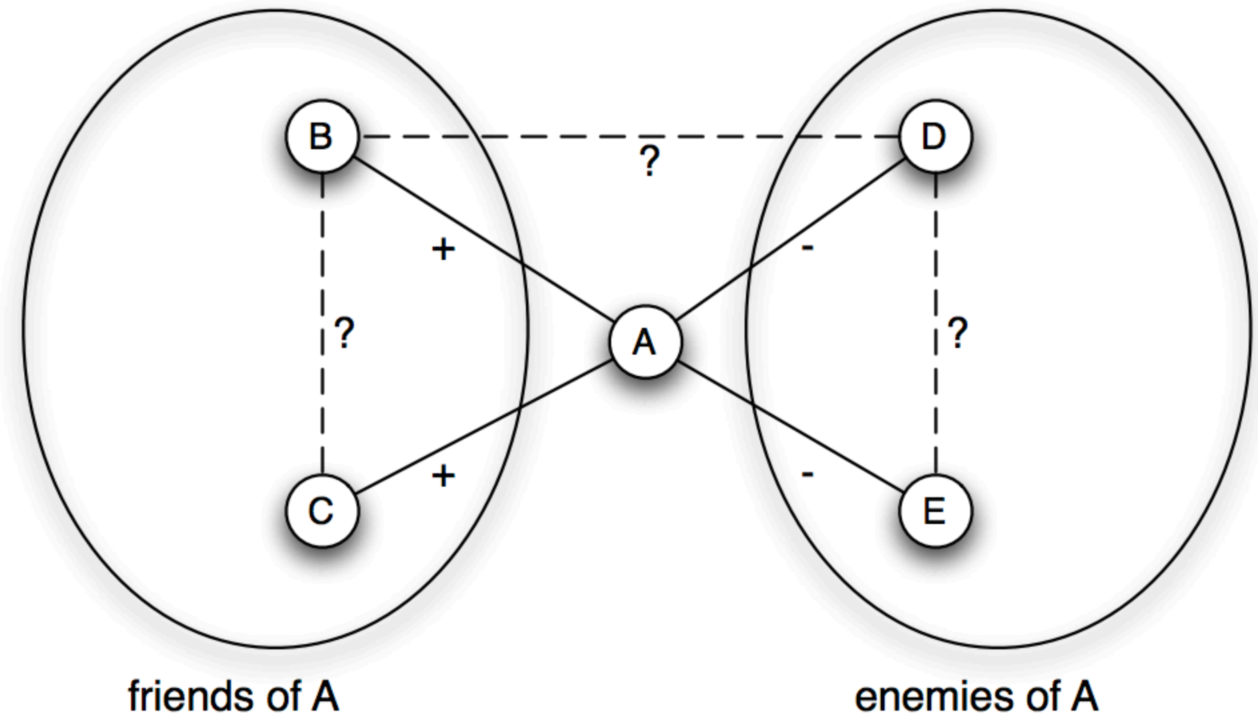
Proving the Balance Theorem

- Proof steps:
  - Suppose we have an arbitrary labeled complete graph
  - Assume it is balanced
  - Conclude either everyone is friend, or that there are sets X and Y, as previously described

Proving the Balance Theorem

- Pick any node in the network (A) and consider things from A's perspective:
  - Every node is either friend of A or enemy of A
  - Set X: A and all its friends
  - Set Y: All enemies of A
- Show for these two sets
  1. Every two nodes in X are friends
  2. Every two nodes in Y are friends
  3. Every node in X is enemy of every node in Y

Proving the Balance Theorem



Proving the Balance Theorem

- Condition 1
  - A is friend with every other node in X
  - How about two other nodes in X (B and C)?
    - A is friend with both B and C
    - If B and C were enemies, then A, B, and C would form a triangle two + labels: violation!
    - As network is balanced: B and C are friends
    - Every two nodes in X are friends

- Condition 2
  - Consider two nodes in Y: D and E are friends?
  - A is enemy with both D and E
  - If D and E were enemies of each other, the A, D and E would form a triangle with no + labels: violation!
  - As network is balanced: D and E are friends
  - Every two nodes in Y are friends

Proving the Balance Theorem

- Condition 3:
  - Consider a node in X (B) and Y (D): they are enemies?
  - A is friend with B and enemy with D
  - If B and D were friends, A, B, and D would form a triangle with two + labels: violation!
  - As network is balanced: B and D are enemies
  - Any node in X and any node in Y is a pair of enemies

Proving the Balance Theorem

By assuming only that the network is balanced, we checked conditions 1, 2, and 3.

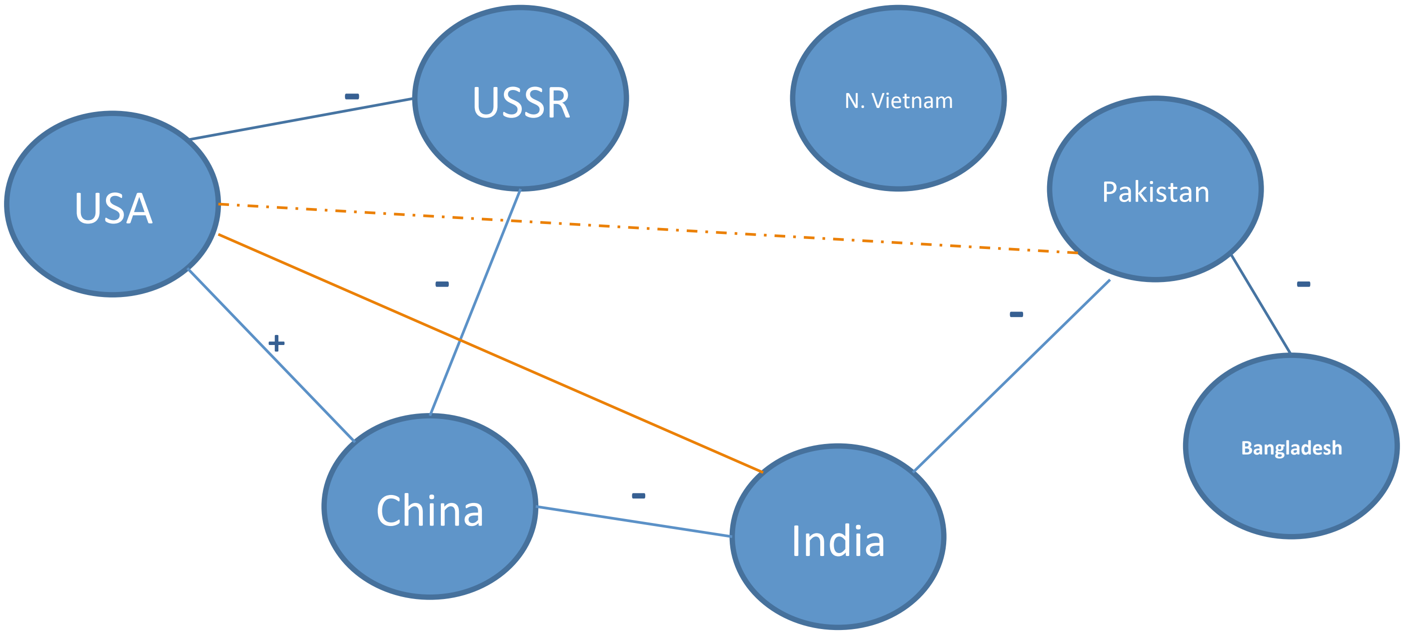
Applications of Structural Balance

- Growing into a large area of study
- Dynamics aspects of structural balance
  - How the set of friendships and antagonisms might evolve over time as the social network implicitly seeks out structural balance
- Two examples
  - International relations: nodes are different countries
  - Online social media sites: users can express positive or negative opinions about each other

International Relations

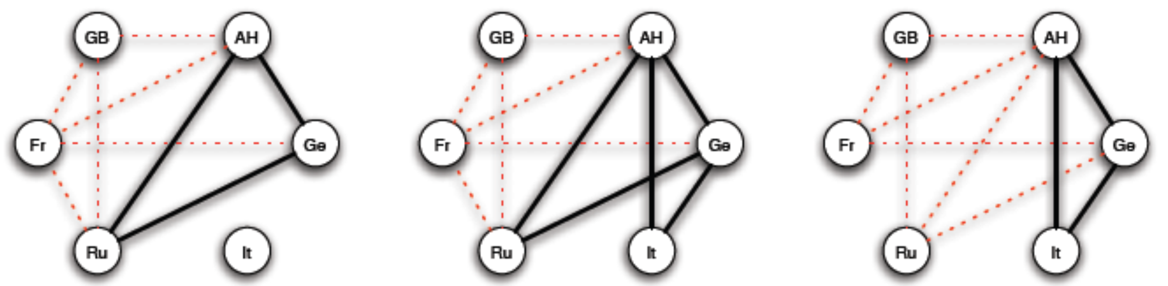
- Nodes are nations
  - Positive label : alliance
  - Negative label : animosity
- Research in political science: structural balance can sometimes provide an effective explanation for the behavior of nations during various international crises
  - Conflict over Bangladesh's separation from Pakistan in 1972 (Moore,1978)

International Relations

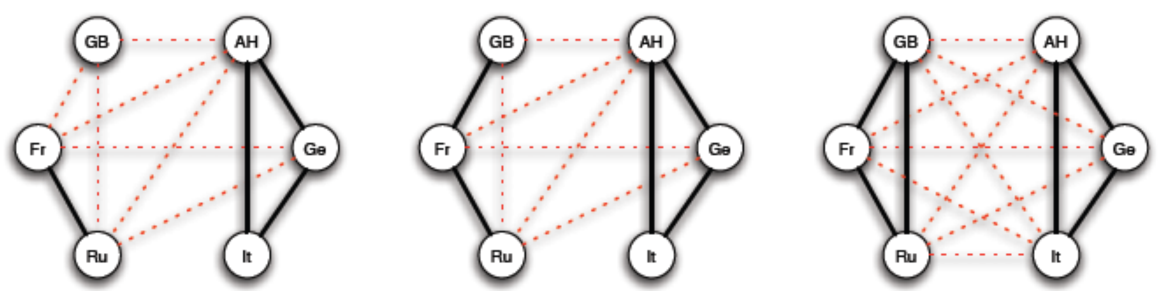


USA support to Pakistan?

International Relations (II)



(a) *Three Emperors' League 1872–81* (b) *Triple Alliance 1882* (c) *German-Russian Lapse 1890*



(d) *French-Russian Alliance 1891–94* (e) *Entente Cordiale 1904* (f) *British Russian Alliance 1907*

Figure 5.5: The evolution of alliances in Europe, 1872-1907 (the nations GB, Fr, Ru, It, Ge, and AH are Great Britain, France, Russia, Italy, Germany, and Austria-Hungary respectively). Solid dark edges indicate friendship while dotted red edges indicate enmity. Note how the network slides into a balanced labeling — and into World War I. This figure and example are from Antal, Krapivsky, and Redner [20].

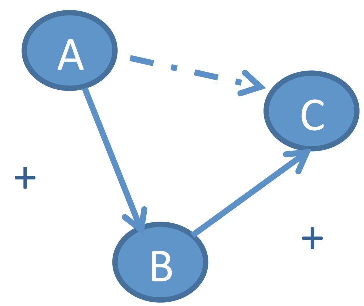
Trust, Distrust and Online Ratings

- Communities on the Web: source for network data with both positive and negative edges, where people can express positive or negative sentiments about each other
  - Slashdot: users can designate each other as a friend or a foe
  - Epinions: online product-rating, where a user can evaluate different products and also express trust or distrust of other users

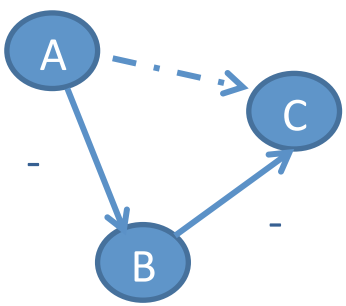
Trust, Distrust and Online Ratings

Evaluation of products and trust/distrust of other users

Directed Graphs



A trusts B, B trusts C, A ? C

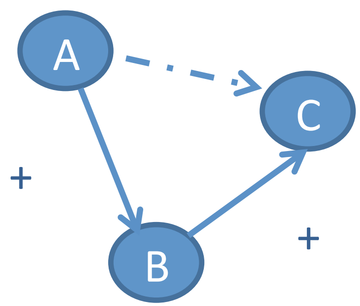


A distrusts B, B distrusts C, A ? C

Trust, Distrust and Online Ratings

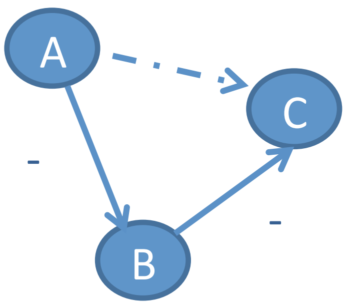
Evaluation of products and trust/distrust of other users

Directed Graphs



A trusts B, B trusts C, A ? C

It is natural to expect that A trusts C. Analogy with the all-positive (undirected) triangles of structural balance theory



A distrusts B, B distrusts C, A ? C
If distrust enemy relation, + (structural balance)
A distrusts means that A is better than B, -

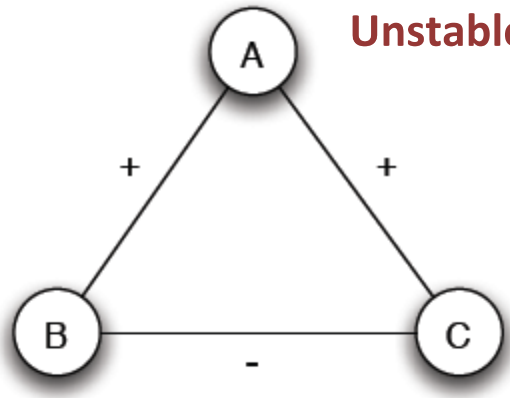
Depends on the application:
1) Rating political books (agreement or disagreement in the users' own political orientations)
2) Consumer rating electronics products (expertise)

Weaker Form of Structural Balance

- Alternate notions of structural balance

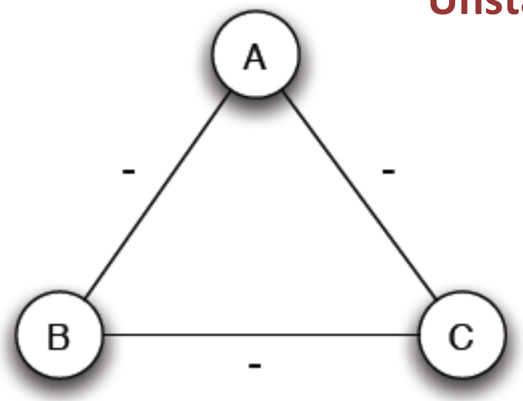
Case 3: 2 +, 1 -

Unstable



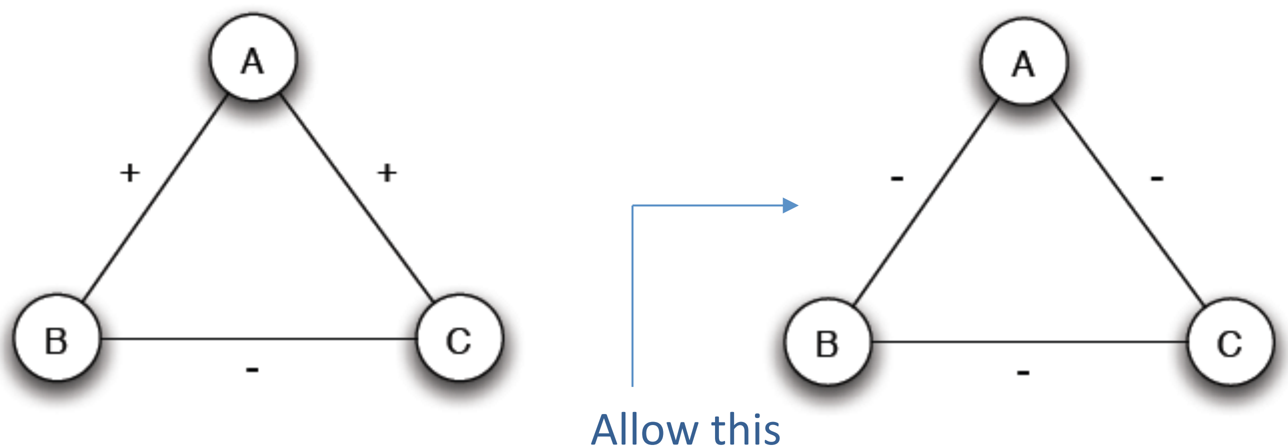
Case 4: 3 -

Unstable



Weaker Form of Structural Balance

- James Davis [1967]
  - Case with 2+ and 1- stronger than case 3-:
    - Friends of friends trying to reconcile their differences
    - Less force leading any two of three mutual enemies to become friendly
- What structural properties arise when we allow triangle with 3- to be present in the network?



Weaker Form of Structural Balance

- We will say that a complete graph, with each edge labeled by + or -, is weakly balanced if:

Weak Structural Balance Property: There is no set of three nodes such that the edges among them consist of exactly two positive edges and one negative edge

Weaker Form of Structural Balance

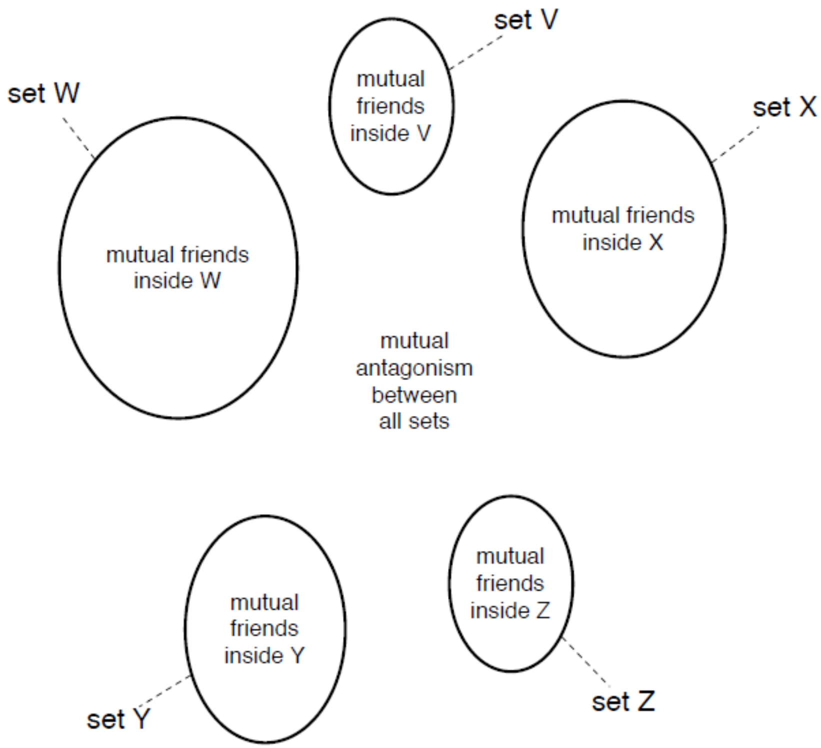
- Weak balance imposes less of a restriction, we should expect to see a broader range of possible structures for weakly balanced networks
- New kind of structure can arise

Weaker Form of Structural Balance

- Weak balance imposes less of a restriction, we should expect to see a broader range of possible structures for weakly balanced networks

- New kind of structure can arise

Weaker Form of Structural Balance



Weakly Balance Theorem: If a labeled complete graph is weakly balanced, its nodes can be divided into groups in such a way that every two nodes belonging to the same group are friends, and every two nodes belonging to different groups are enemies

Proving Weakly Balanced Theorem

- Proof steps:
  - Suppose we have an arbitrary labeled complete graph
  - Assume it is weakly balanced
  - Conclude that nodes are divided into groups of mutual friends, such that all relations between nodes in different groups are negative

Proving Weakly Balanced Theorem

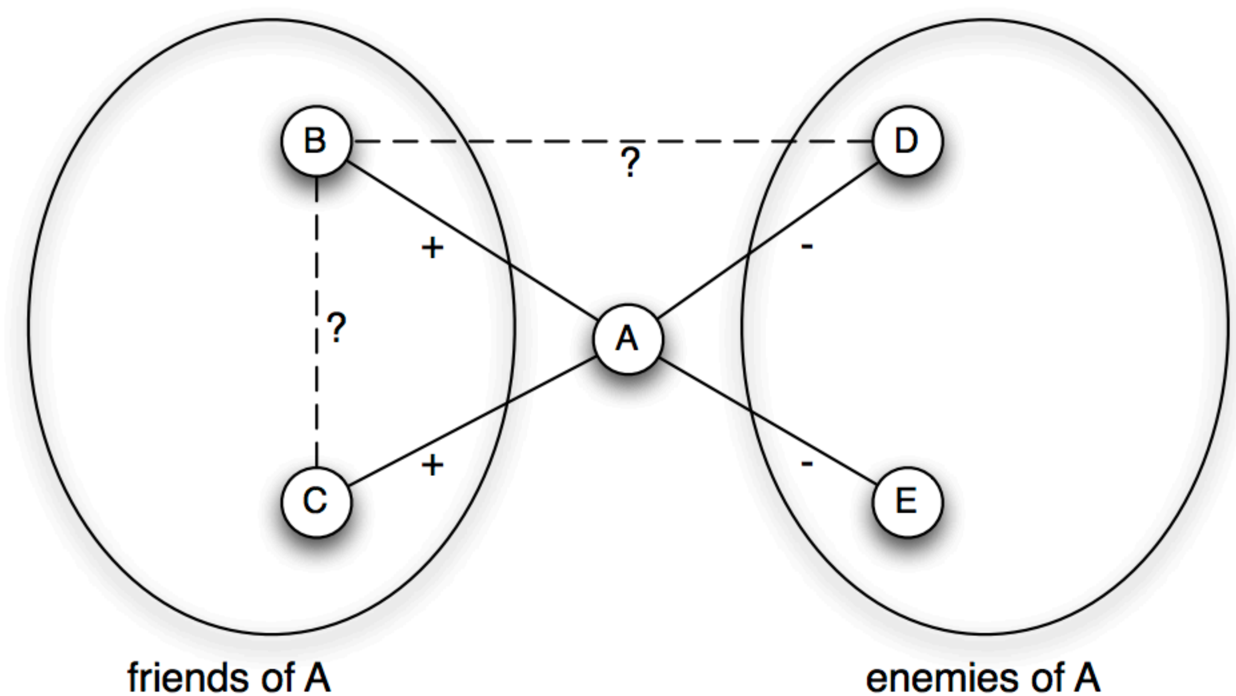
- Pick any node in the network (A) and consider things from A's perspective:

  - Every node is either friend of A or enemy of A
  - Set X: A and all its friends

- Show for these two sets

  1. All A's friends are friends with each other (group of mutual friends)
  2. A and all his friends are enemies with everyone else in the graph (the people in this group will be enemies with everyone in other groups, regardless of how we divide up the rest of the graph)

Proving Weakly Balanced Theorem



Proving Weakly Balanced Theorem

- Condition 1
  - Consider two nodes, B and C, who are both friends with A
  - If B and C were enemies of each other, then the triangle on nodes A,B and C would have two + labels: violation of weak structural balance!
  - B and C must be friends with each other

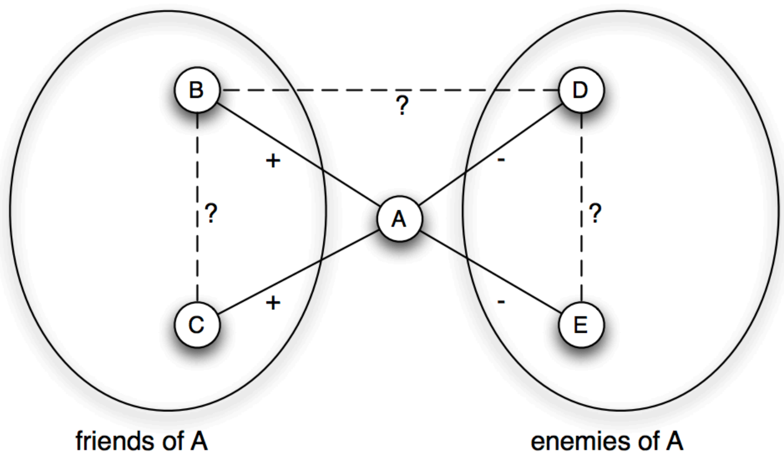
Proving Weakly Balanced Theorem

- Condition 2
  - A is enemies with all nodes in the graph outside X
  - How about an edge between a node B in X and a node D outside X?
  - If B and D are friends, then the triangle on nodes A, B , and D would have exactly two + labels: violation!
  - B and D must be enemies

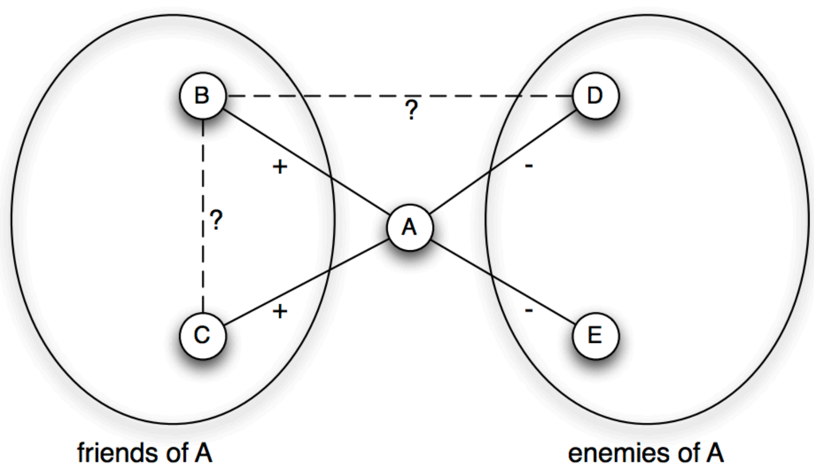
Proving Weakly Balanced Theorem

- Proprieties (i) and (ii) hold
  - Remove the set X: A and all his friends
  - Declare it to be the first group
  - Smaller complete graph that is still weakly balanced
  - Find a second group and proceed to remove groups until all the nodes have been assigned to a group
    - Each group consists of mutual friends (i)
    - Each group has only negative relations with everyone outside the group (ii)
    - Proved!

Small Differences



- Structural Balance



- Weakly Balanced

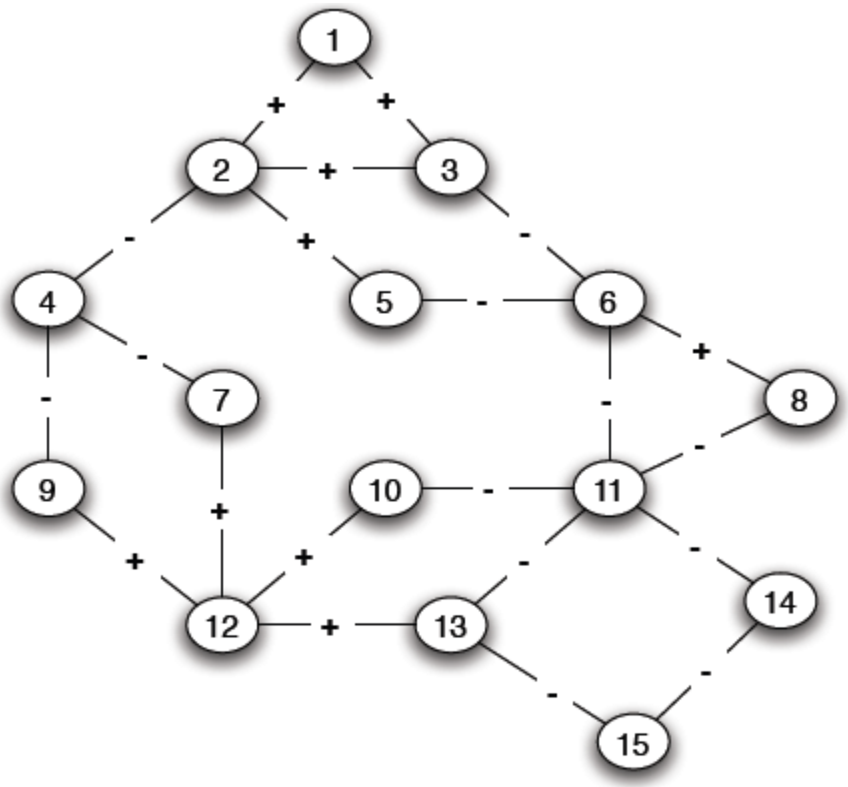
Generalizing the Definition of Structural Balance

- More general ways
  - It applies only to complete graphs. What if only some pairs of people know each other?

  - Balance Theorem implies a global division of the world into two factions, only applies to the case in which every triangle is balanced. Can we relax this requirement to say that, if most triangles are balanced, then the world can be approximately divided into to factions?

We shall use the original ("non-weak" definition of structural balance)

Arbitrary (Noncomplete) Networks



- Positive edge
- Negative edge
- Absence of an edge

Arbitrary (Noncomplete) Networks

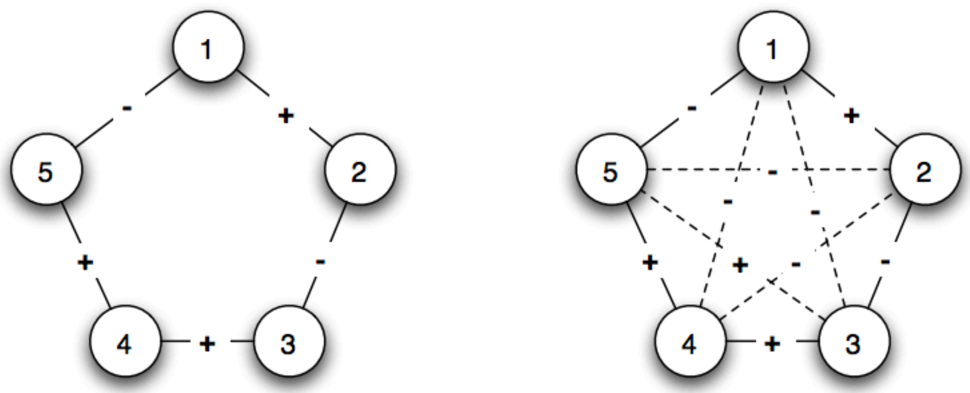
- Balance Theorem suggests that structural balance can be viewed in either of two equivalent ways:
  - Local view: condition on each triangle of the network
  - Global view: the world be divided into two mutually opposed sets of friends
- Each of these views suggests a way of defining structural balance for general signed graphs

Arbitrary (Noncomplete) Networks

- Option 1 (local view):
  - Treat as a problem of filling the missing values
    - Imagine all people in the group in fact do know and have an opinion on each other
    - Graph under consideration is not complete only because we failed to observe the relation between some pairs
    - We could say that the graph is balanced if it is possible to fill in all the missing labeled edges, resulting signed complete balanced graph
  - Noncomplete graph is balanced if it can be completed by adding edges to form a signed complete graph that is balanced

Arbitrary (Noncomplete) Networks

- Option 1 (local view)



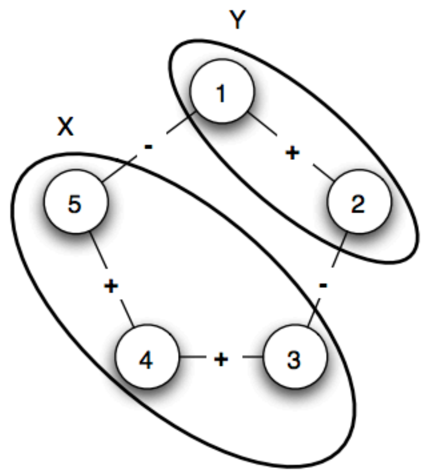
(a) *A graph with signed edges.* (b) *Filling in the missing edges to achieve balance.*

Arbitrary (Noncomplete) Networks

- Option 2 (global view):
  - Signed graph is balanced if it is possible to divide the nodes into two sets, X and Y, in which people in X are all mutual friends and also for Y; people in X are all enemies of people in Y

Arbitrary (Noncomplete) Networks

- Option 2 (global view):

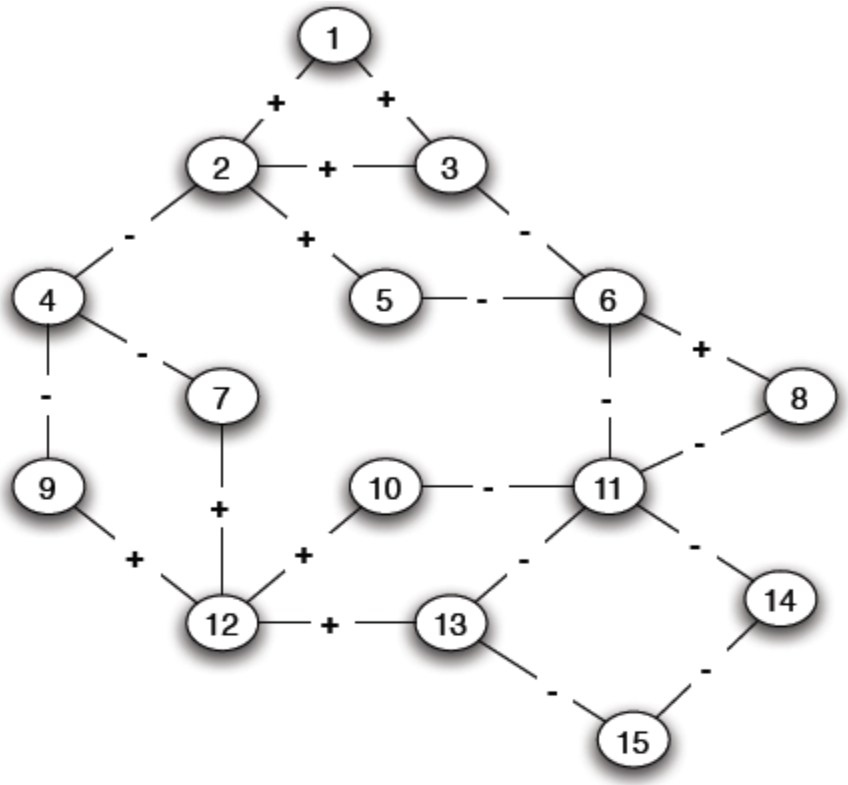


(c) *Dividing the graph into two sets.*

Characterizing Balance for General Networks

- Definitions themselves do not provide much insight into how to easily check whether a graph is balanced or not balanced
  - Lots of ways to choose signs for the missing edges
  - Lots of ways of splitting the nodes into sets X and Y

Characterizing Balance for General Networks

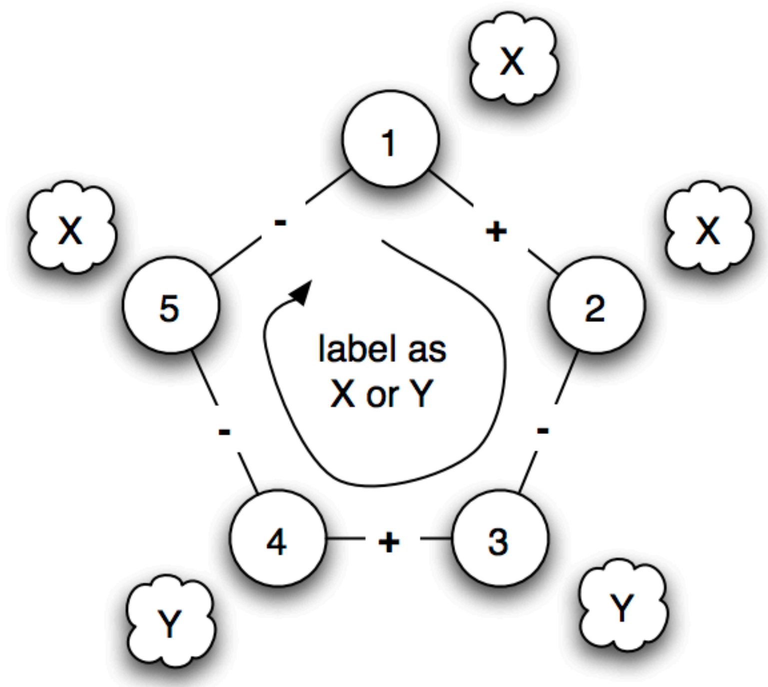


Not obvious from a quick inspection the graph is not balanced

Changing the edge connecting nodes 2 and 4 to be positive instead of negative: balanced graph!

Characterizing Balance for General Networks

What prevents a graph from being balanced?



- Start at node 1: try divide nodes into sets X and Y (node 1 belongs to X)
- Node 2 is friend of 1 - X
- Node 3 is enemy of 2 - Y
- Node 4 is friend of 3 - Y
- Node 5 is enemy of 4 - X
- Node 1 is enemy of 5 - should belong to Y, but we already assigned it to X!
- Hence, nodes can't be divided in sets X and Y - graph not balanced

Characterizing Balance for General Networks

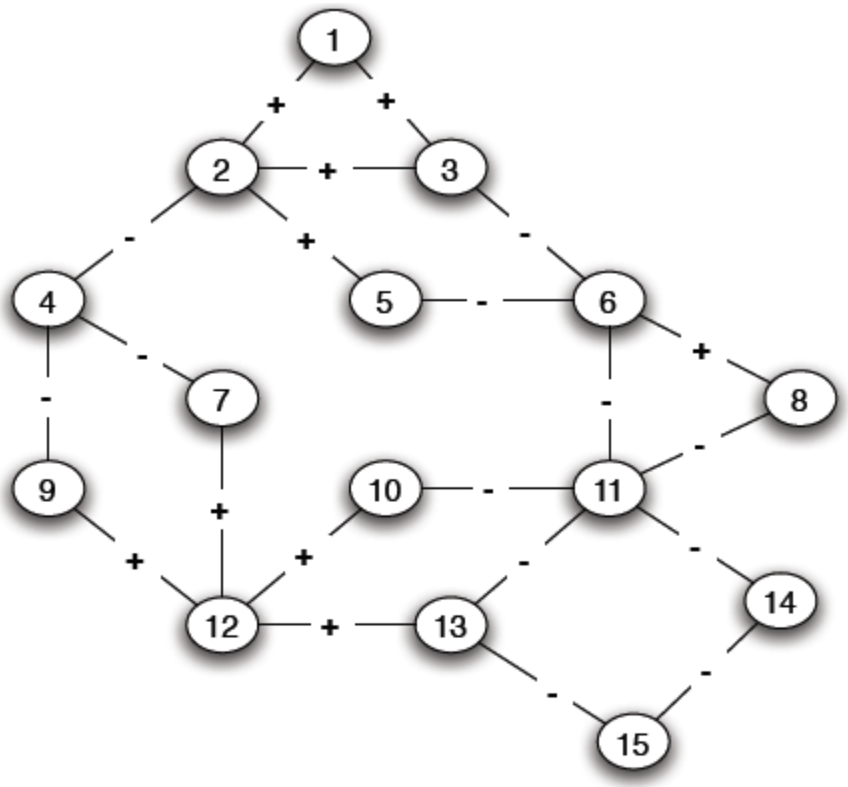
- Why not balanced?
  - We are walking around a cycle
  - Every time we crossed a negative edge we have to change the set into which we are putting nodes
  - Getting back around to node 1 requires crossing an odd number of negative edges
  - Original decision to put node 1 into X crashes with the eventual conclusion that node 1 ought to be in Y

Characterizing Balance for General Networks

- If the graph contains a cycle with an odd number of negative edges, then this implies that the graph is not balanced

- A cycle with an odd number of negative edges is thus a very simple-to-understand reason why a graph is not balanced

Characterizing Balance for General Networks



- Cycle: 2,3,6,11,13,12,9, and 4 Contains 5 negative edges

- Are there other more complex reasons why a graph is not balanced?

Characterizing Balance for General Networks

- Cycles with an odd number of negative edges are the only obstacles to balance

Claim: A signed graph is balanced, if and only if, it contains no cycles with an odd number of negative edges

- Proof by construction

Proving the Characterization

Find *a balanced division:* partition into sets X and Y, all edges in X and Y positive, crossing edges negative

Either succeeds or Stops with a cycle containing an odd number of - links

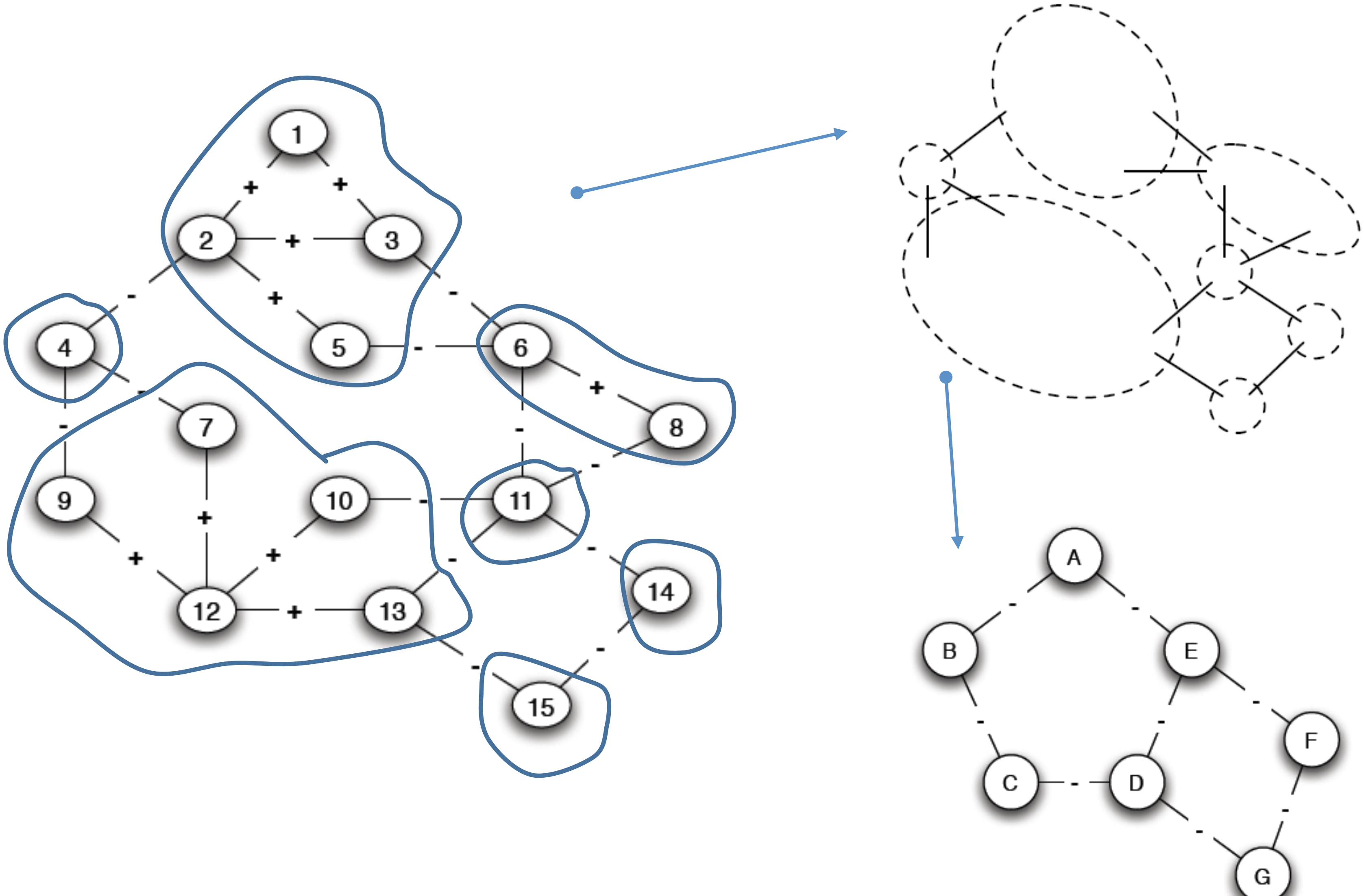
Two steps:

1. Convert the graph into a reduced one with only negative edges
2. Solve the problem in the reduced graph

Identifying Supernodes (Step 1)

- Whenever two nodes are connected by a positive edge, they must belong to the same set, X or Y
- What the connected components of the graph would be if we were to consider only positive edges?
  - Supernodes: each supernode is connected internally via positive edges, and the only edges going between two different supernodes are negative
  - If there were a positive edge linking two different supernodes, then we should have combined them together into a single supernodegraph is not balanced

Identifying Supernodes (Step 1)



Identifying Supernodes

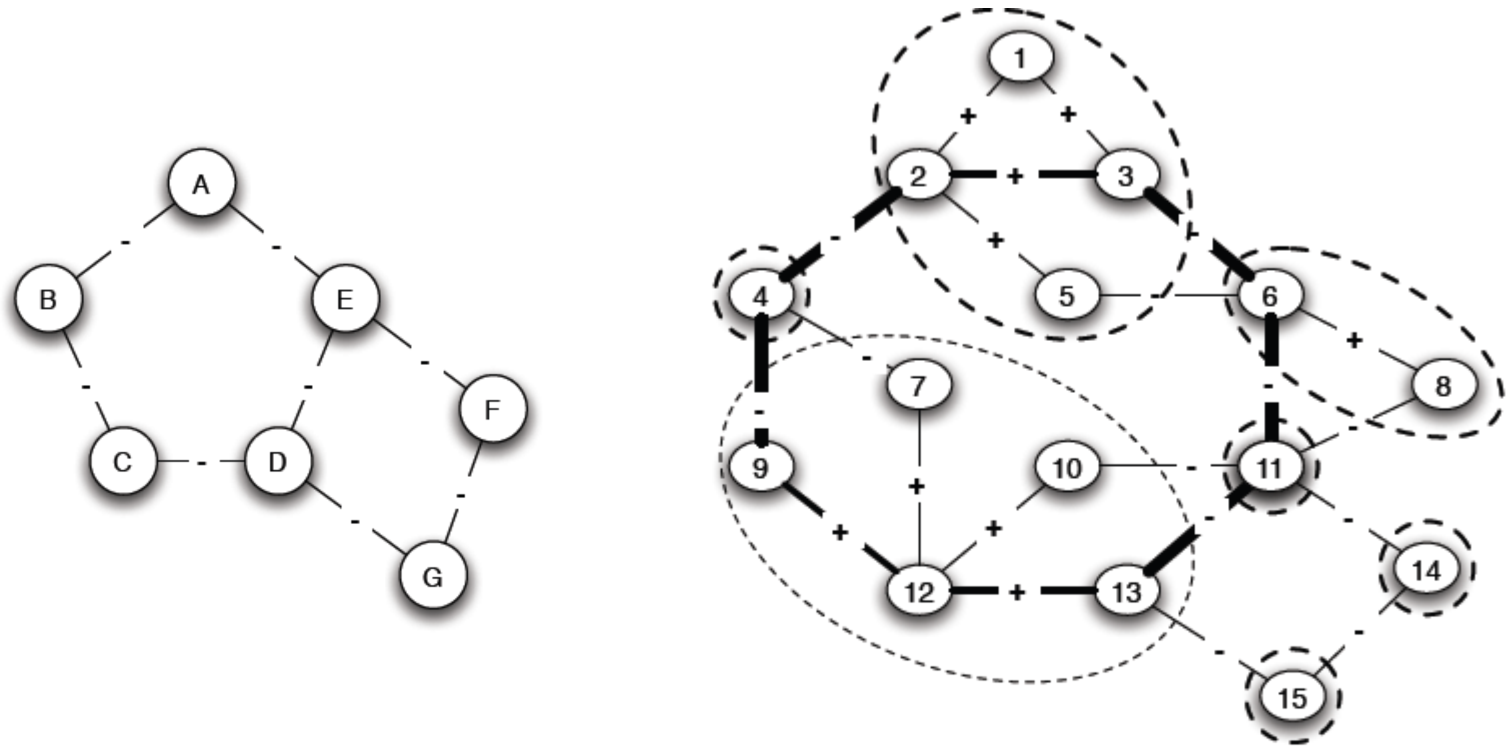
Now: Reduced graph, whose nodes are the supernodes of the original graph

- Step 2: Breadth-First Search of the Reduced Graph

Breadth-First Search of the Reduced Graph (Step 2)

- Only negative edges occur between supernodes
- Our reduced graph has only negative edges
- One of two possible outcomes:
  - Labeling each node in the reduced graph as either X or Y in such a way that every edge has endpoints with opposite labels
    - Balanced division of the original graph
  - To find a cycle in the reduced graph that has as odd number of edges
    - Convert this to a (potentially longer) cycle in the original graph
    - This path will contain an odd number of negative edges in the original graph

Breadth-First Search of the Reduced Graph (Step 2)



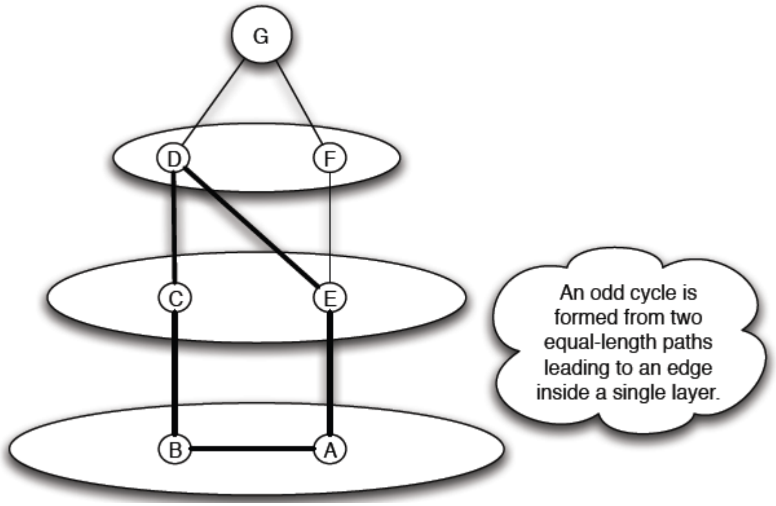
Breadth-First Search of the Reduced Graph (Step 2)

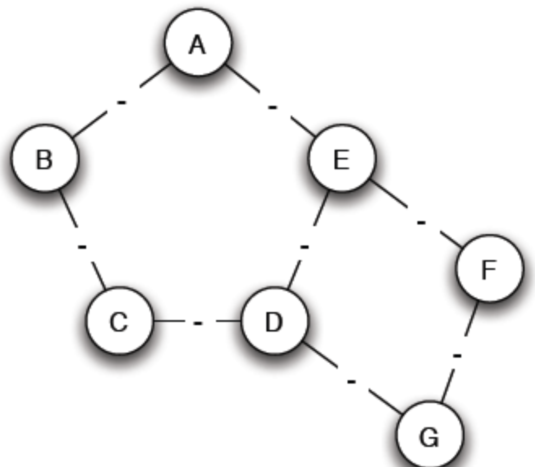
Determining whether the graph is bipartite

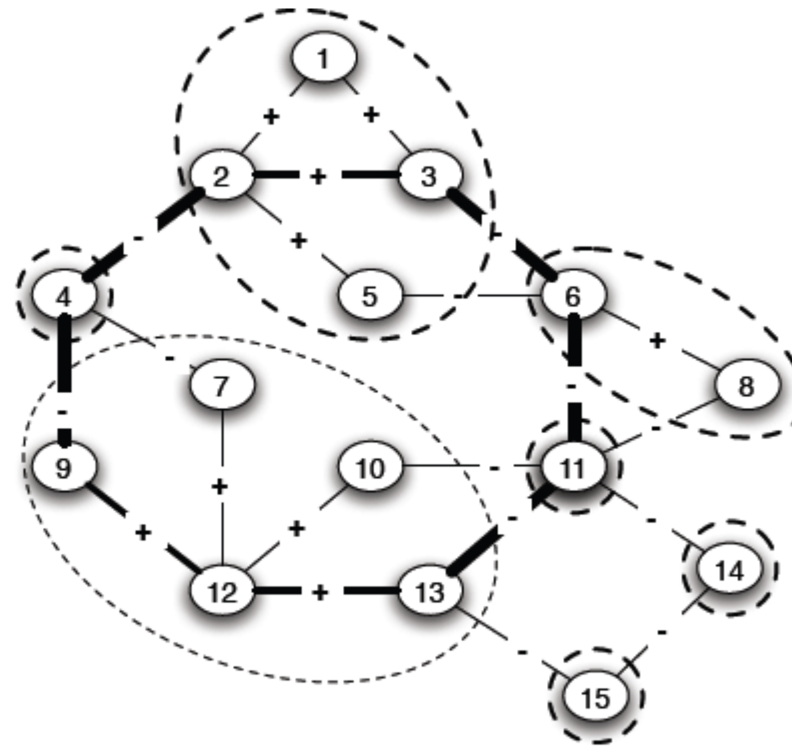
Use Breadth-First-Search (BFS)

- Start the search at any node and give alternating labels to the vertices visited during the search. *That is, give label X to the starting node, Y to all its neighbors, X to those neighbors' neighbors, and so on.*
- If at any step a node has (visited) neighbors with the same label as itself, then the graph is not bipartite *(cross-level edge)*
- If the search ends without such a situation occurring, then the graph is bipartite.

Breadth-First Search of the Reduced Graph (Step 2)







Approximately Balance Networks

a complete graph (or clique): every edge either + or -

Claim: If all triangles in a labeled complete graph are balanced, than either
(a) all pairs of nodes are friends or,
(b) the nodes can be divided into two groups X and Y, such that
  (i) every pair of nodes in X like each other,
  (ii) every pair of nodes in Y like each other, and
  (iii) every one in X is the enemy of every one in Y.

Not all, but most, triangles are balanced

Claim: If *at least 99.9%* of all triangles in a labeled compete graph are balanced, then either,
(a) There is a set consisting of *at least 90%* of the nodes in which *at least 90%* of all pairs are friends, or,
(b) the nodes can be divided into two groups X and Y, such that
  (i) *at least 90%* of the pairs in X like each other,
  (ii) *at least 90%* of the pairs in Y like each other, and
  (iii) *at least 90%* of the pairs with one end in X and one in Y are enemies

Approximately Balance Networks

Claim: If *at least 99.9%* of all triangles in a labeled complete graph are balanced, then either,

(a) There is a set consisting of *at least 90%* of the nodes in which *at least 90%* of all pairs are friends, or,

(b) the nodes can be divided into two groups X and Y, such that
   (i) *at least 90%* of the pairs in X like each other,
   (ii) *at least 90%* of the pairs in Y like each other, and
   (iii) *at least 90%* of the pairs with one end in X and one in Y are enemies

Claim: Let ε be any number, such that $0 \le \varepsilon < 1/8$, If at *least* $1 - \varepsilon$ in a labeled complete graph are balanced, then either

(a) There is a set consisting of *at least* $1-\delta$ of the nodes in which *at least* $1-\delta$ of all pairs are friends, or,

(b) the nodes can be divided into two groups X and Y, such that
   (i) *at least* $1-\delta$ of the pairs in X like each other,
   (ii) *at least* $1-\delta$ of the pairs in Y like each other, and
   (iii) *at least* $1-\delta$ of the pairs with one end in X and one in Y are enemies

$$\delta = \sqrt[3]{\varepsilon}$$