LECTURE 1: MODULAR PROOFS IN ISABELLE HOL

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Midlands Graduate School 2025

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COURSE OVERVIEW

A practical course on effective use of the Isabelle/HOL proof assistant in mathematics and programming languages

Lectures:

- **Introduction to Proof Assistants**
- **Formalising the basics in Isabelle/HOL**
- Introduction to Isar, more types, Locales and Type-classes
- Case studies:
  - Formalising Mathematics: Combinatorics & advanced locale reasoning patterns
  - Program Verification: Formalising semantics, program properties, and introducing modularity/abstraction.

Example Classes:

- Isabelle exercises based on the previous lecture
- Will be drawing from the existing Isabelle tutorials/Nipkow's Concrete Semantic Book, as well as custom exercises (e.g. for locales).

Acknowledgement: Slides partially inspired by slides/notes by Larry Paulson, Tobias Nipkow, Gerwin Klein, Clemens Ballarin, Georg Struth, Andrei Popescu (and many more who've come before me!)

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PRE-REQUISITE KNOWLEDGE

- No prior proof assistance is assumed:
  - If you've used Isabelle before, perhaps this will offer a new perspective/closer look at certain features
  - If you've used other proof assistants before, there'll be plenty of Isabelle specific concepts as well as more familiar ones.
  - We'll discuss topics that are both Isabelle specific and more general in the proof assistant landscape.
- What is assumed:
  - Some familiarity with functional programming
  - Basic logic, discrete maths, some semantics (for the last lecture).

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A DISCLAIMER ....

This course IS...

...unashamedly a course on the practical use of proof assistants and in particular, Isabelle/HOL

Main course goals:

- Be able to use Isabelle to start your own project/keep learning yourself.
- Understand the importance of modularity in formal proof and use important tools/advanced proof techniques in Isabelle/HOL to manage such modularity
- Understand the role proof assistants can play in several areas of foundations research

This course IS NOT:

- A type theory course
- A course on the details of all proof assistants (or for that matter, even all the details of Isabelle/HOL!).
- An introduction to a particular foundational concept which only uses Isabelle for exercises

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COURSE RESOURCES

- Documentation
  - See the course website for slides, notes, and exercises:
  - https://cledmonds.github.io/mgs2025/
  - Will be updated throughout this week!
- Other useful resources:
  - The official documentation (particularly prog-prove & locales tutorials): Comes with Isabelle distribution
  - Tobias Nipkow's Concrete Semantics Book: http://concrete-semantics.org/
  - Machine Logic Blog: Interesting exploration of Isabelle and history by Larry Paulson - https://lawrencecpaulson.github.io/

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LECTURE 1 OVERVIEW

- Introduction to Proof Assistants
  - History, major developments, motivation
- Introduction to Isabelle/HOL
- A fast-paced "tour" through key basic concepts
  - The editors
  - Some logical proofs
  - Functions, datatypes, tactics.
  - More examples!
  - Isabelle Infrastructure: AFP, automation, search, etc
  - Summary of other advanced features

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INTRODUCTION TO PROOF ASSISTANTS

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PROOF ASSISTANTS

- Interactive proof assistants allow us to prove theorems in a logical formalism:
  - With precise definitions of concepts
  - A formal deductive system
  - And (hopefully) automated tools
- We can create hierarchies of definitions and proofs
  - Specifications of components and properties
  - Proofs that designs meet their requirements.
- Interactive = “guided” by a human user to produce a formalisation or mechanisation.

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WHY FORMALISE?

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WHY FORMALISE?

A very simple example ....

Are the proofs below correct? Are they valid theorems to begin with?

$(P \to Q), (Q \to R) \vdash R$

1.	$(P \to Q)$	hyp
2.	$(Q \to R)$	hyp
3.	P	hyp
4.	Q	$(\to E)$, 1, 3
5.	R	$(\to E)$, 2, 4
6.	$P \to R$	$(\to I)$ 3-5
7.	R	$(\to E)$ 6,3

$\forall x \exists y P(x, y) \vdash \exists x \forall y P(x, y)$

1.	$\forall x \exists y P(x, y)$	hyp
2.	$\exists y P(a, y)$	$(\forall E)$ 1
3.	$P(a, b)$	$(\exists E)$ 2
4.	$\forall x P(x, b)$	$(\forall I)$ 3
5.	$\exists y \forall x P(x, y)$	$(\exists I)$ 4

$(P \land Q) \to R \vdash P \to (Q \to R)$

1.	$(P \land Q) \to R$	hyp
2.	P	hyp
3.	Q	hyp
4.	$P \land Q$	$(\land E_l)$ 2, 3
5.	R	$(\to E)$ 1, 4
6.	$Q \to R$	$(\to I)$ 3-5
7.	$P \to Q \to R$	$(\to I)$ 2-6

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WHY FORMALISE?

A very simple example ....

$(P \to Q), (Q \to R) \vdash R$

1.	$(P \to Q)$	hyp
2.	$(Q \to R)$	hyp
3.	P	hyp
4.	Q	$(\to E)$, 1, 3
5.	R	$(\to E)$, 2, 4
6.	$P \to R$	$(\to I)$ 3-5
7.	R	$(\to E)$ 6,3

NOT A THEOREM! $(\to E)$ at 7

$\forall x \exists y P(x, y) \vdash \exists x \forall y P(x, y)$

1.	$\forall x \exists y P(x, y)$	hyp
2.	$\exists y P(a, y)$	$(\forall E)$ 1
3.	$P(a, b)$	$(\exists E)$ 2
4.	$\forall x P(x, b)$	$(\forall I)$ 3
5.	$\exists y \forall x P(x, y)$	$(\exists I)$ 4

NOT A THEOREM! $(\exists E)$ at 3

$(P \wedge Q) \to R \vdash P \to (Q \to R)$

1.	$(P \wedge Q) \to R$	hyp
2.	P	hyp
3.	Q	hyp
4.	$P \wedge Q$	$(\wedge E_l)$ 2, 3
5.	R	$(\to E)$ 1, 4
6.	$Q \to R$	$(\to I)$ 3-5
7.	$P \to Q \to R$	$(\to I)$ 2-6

PROOF ERROR: $(\wedge I)$ at 4

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WHY FORMALISE?

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Quasi-projectivity of moduli spaces of polarized varieties
Pages 597-639 from Volume 159 (2004), Issue 2 *by Georg Schumacher, Hajime Tsuji*

Abstract

By means of analytic methods the quasi-projectivity of the moduli space of algebraically polarized varieties with a not necessarily reduced complex structure is proven including the case of nonuniruled polarized varieties.

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Non-quasi-projective moduli spaces
Pages 1077-1096 from Volume 164 (2006), Issue 3 *by János Kollár*

Abstract

We show that every smooth toric variety (and many other algebraic spaces as well) can be realized as a moduli space for smooth, projective, polarized varieties. Some of these are not quasi-projective. This contradicts a recent paper (Quasi-projectivity of moduli spaces of polarized varieties, *Ann. of Math.* **159** (2004) 597–639.).

[1] The result of Problem 11 contradicts the results announced by Levy [1963b]. Unfortunately, the construction presented there cannot be completed.
[2] The transfer to ZF was also claimed by Marek [1966] but the outlined method appears to be unsatisfactory and has not been published.
[3] A contradicting result was announced and later withdrawn by Truss [1970].
[4] The example in Problem 22 is a counterexample to another condition of Mostowski, who conjectured its sufficiency and singled out this example as a test case.
[5] The independence result contradicts the claim of Felgner [1969] that the Cofinality Principle implies the Axiom of Choice. An error has been found by Morris (see Felgner's corrections to [1969]).

*Footnotes on page 118 of Jech's *The Axiom of Choice (1973)*

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WHY FORMALISE?

- To validate complex proofs
- To reveal hidden assumptions & proof steps
- To create central libraries of verified mathematical/CS knowledge
- To benefit from advances in automation and technology

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PROOF ASSISTANT COMPONENTS

User Interface

Proof Libraries

Automation Tools

Notational Support

Basic Proof Language

Theory Management

Core Logical Formalism

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SOME HISTORY

- **Automath** (de Bruijn, 1968): The first! Novel type theory. Formalised the construction of the reals.
- **Mizar** (Trybulec, 1973): Set theory with "soft typing". Structured formal language
- **Rocq (Coq)** (Coquand and Huet et al, 1984): Dependent type theory.
- **HOL [Light]** (Gorden, 1988, Harrison, 1992): Simple type theory/Higher-order logic. First to verify real analysis.
- **Isabelle[HOL]** (Paulson, 1986)**:** Isabelle is a generic proof assistant. Its main instance is simple type theory/higher order logic.
- **Agda** (Coquand, 1999, Ulf, 2007): A dependently typed functional programming language, that is also a proof assistant. Based on Intuitionistic type theory.
- **Lean** (de Moura et al, 2015)**:** Dependent type theory. Has a strong community for formalised maths.
- **And many more …**

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THE ISABELLE PROOF ASSISTANT

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THE ISABELLE PROOF ASSISTANT

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ISABELLE OVERVIEW

- Simple type theory/HOL
- Sledgehammer – automated proof search.
- Counter-example generators
- Search tools: Query Search, Find Facts, SErAPIS
- The Isar structured proof language
- Jedit/VS Codium IDE
- Extensive existing libraries in Maths & Computer Science (AFP)
- Additional features: Code generation, documentation generation …

```
theorem  assumes "prime p"  shows "sqrt p ∉ ℚ"
proof
  from ‹prime p› have p: "1 < p" by (simp add: prime_def)
  assume "sqrt p ∈ ℚ"
  then obtain m n :: nat where
      n: "n ≠ 0" and sqrt_rat: "¦sqrt p¦ = m / n"
    and "coprime m n" by (rule Rats_abs_nat_div_natE)
  have eq: "m² = p * n²"
  proof -
    from n and sqrt_rat have "m = ¦sqrt p¦ * n" by simp
    then show "m² = p * n²"
      by (metis abs_of_nat of_nat_eq_iff of_nat_mult power2_eq_square real_sqrt_abs2 rea
  qed
  have "p dvd m ∧ p dvd n"
  proof
    from eq have "p dvd m²" ..
    with ‹prime p› show "p dvd m" by (rule prime_dvd_power_nat)
    then obtain k where "m = p * k" ..
    with eq have "p * n² = p² * k²" by (auto simp add: power2_eq_square ac_simps)
    with ‹prime p› show "p dvd n"
      by (metis dvd_triv_left nat_mult_dvd_cancel1 power2_eq_square prime_dvd_power_nat
  qed
  then have "p dvd gcd m n" by simp
  with ‹coprime m n› have "p = 1" by simp
  with p show False by simp
qed
```

sledgehammer proofs

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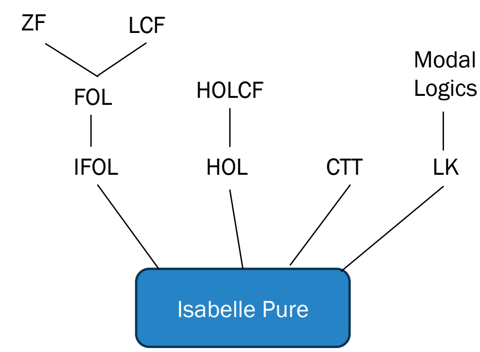
ISABELLES FAMILY OF LOGICS

- Isabelle is a *generic* theorem prover
- Overtime, several different logics have been developed – Isabelle/HOL is by far the most widely used.



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ISABELLE/HOL FOUNDATIONS

- Isabelle/HOL is based on a Higher-Order logic (i.e. simple type theory)
  - First order logic extended with functions and sets.
  - Extended to also incorporate rank-1 polymorphism (we'll get to type classes later!).
  - ML-style functional programming.
- Often introduced as HOL
- Variation of Gordon's HOL (also led to the logic behind HOL4/HOL Light)

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BASIC TYPES / TERMS / FUNCTIONS

$$\begin{array}{rcl} \tau & ::= & (\tau) \\ & | & bool \mid nat \mid int \mid \ldots \\ & | & 'a \mid 'b \mid \ldots \\ & | & \tau \Rightarrow \tau \\ & | & \tau \times \tau \\ & | & \tau\ list \\ & | & \tau\ set \\ & | & \ldots \end{array}$$

- Base types
- Type variables
- Function types
- Pairs
- Lists
- Sets
- User defined types

-Postfix types have precedence over function types (i.e. $'a \Rightarrow 'b\ list$ means $'a \Rightarrow ('b\ list)$)

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TERMS

$$\begin{array}{rcl} t & ::= & (t) \\ & | & a \\ & | & t\ t \\ & | & \lambda x.\ t \\ & | & \ldots \end{array}$$

Terms (follow the typed λ calculus)

- Constants, c and Variables, x
- Function applications $t\ u$
- Abstractions $\lambda x\ .\ t$
- Lots of syntactic sugar

- i.e. The language of terms is a simply type $\lambda -$ calculus, noting Isabelle performs β-reduction ($(\lambda x.\ t)\ u$ to $t[u/x]$) automatically.
- Terms must be **well-typed** ($t\ :: \tau$)
- Isabelle automatically computers the type of each variable in a term (type inference), except for overloaded functions where type annotations can be useful.

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ISABELLE'S META LOGIC

- **Implication:** $\Longrightarrow$
  - For separating premises and conclusions of theorems
- **Equality** $\equiv$
  - For definitions
- **Universal Quantifier** $\bigwedge$
  - For binding local variables

Do not use inside HOL formula!

Logically the same meaning, but differences is usability/automation

NB: The Metalogic, has itself been formalised! https://www.isa-afp.org/entries/Metalogic_ProofChecker.html

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EDITORS

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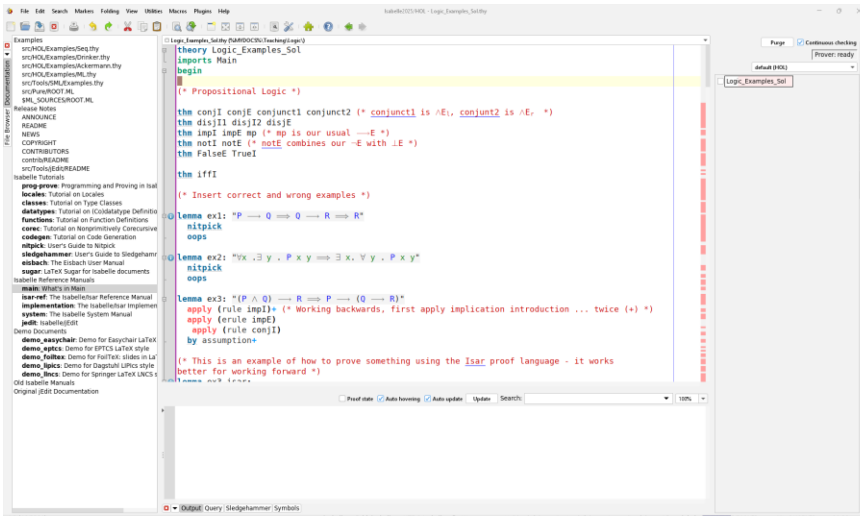
ISABELLE JEDIT



Includes the most customised support for Isabelle developments

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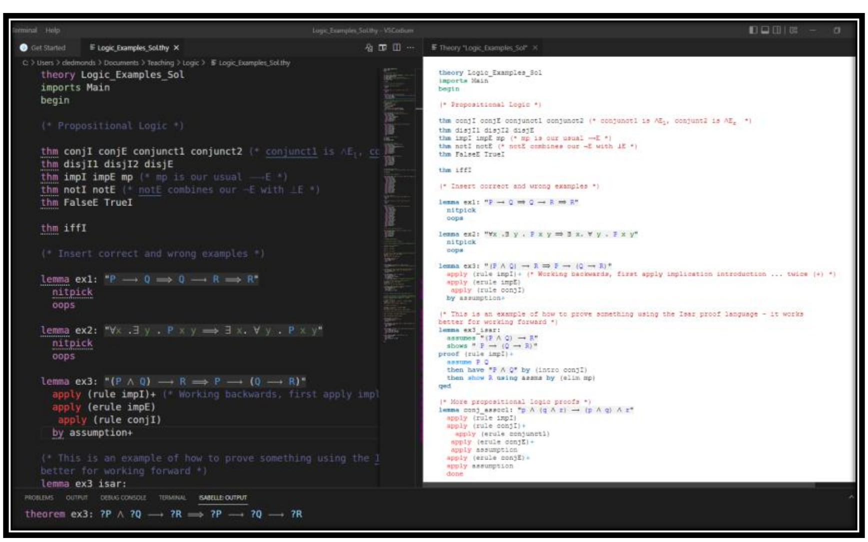
ISABELLE VSCODE



New VSCode based editor

- Must use instance in the Isabelle download
- Start via:

  "isabelle vscode"
- Nice html preview
- Many less Isabelle features than jedit
- Don't use the old VSCode extension

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INTRODUCTION BY EXAMPLE

1. BOOLEAN LOGIC AND FUNCTIONS

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FUNCTIONS/DATATYPES

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DATATYPES

- Functional style datatypes
- Generates lots of useful facts/properties:
  - distinctness and injectivity (applied automatically).
  - Induction (needs to be applied)

```
datatype 'a mylist = Nill | Consl 'a " 'a mylist"

thm mylist.induct
thm mylist.case
```

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FUNCTIONS & DEFINITIONS

- All Functions must be total!
- Fun – termination proved automatically (most things we'll deal with),

```
fun app :: "'a mylist ⇒ 'a mylist ⇒ 'a mylist" where
"app Nill ys = ys" |
"app (Consl x xs) ys = Consl x (app xs ys)"
```

- Function – user supplied termination proof.
- Definition: non-recursive definitions

```
definition prime :: "nat ⇒ bool" where
"prime p = (1 < p ∧ (∀ m. m dvd p ⟶ m = 1 ∨ m = p))"
```

- Recursive functions have more built in facts that are useful in proofs than a definition.

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TACTICS

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AUTO VS SIMP

Auto

- `auto` applies `simp` rules + all obvious logical steps, e.g.:
  - Splitting conjunctive goals and disjunctive assumptions
  - Performing obvious quantifier removal
- It operates on *all* subgoals
- Designated intro and elimination rules included in this

Simp

- `Simp` performs *rewriting* (along with simple arithmetic simplification)
- It only operates on the *first* subgoal
- Some facts are included in the simplifier
- Other facts are often useful, e.g. for arithmetic, consider trying the following:
  - `algebra_simps`
  - `field_simps`
  - `divide_simps`

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MORE REWRITING

- Simp rules work left to right, i.e. at each step transform the LHS into the RHS
- Isabelle enables you to add rules to the simplifier by declaring them as such
- Rewrite rules can be conditional (and are applied if the conditions can themselves be recursively proved via simplification)
- But! We need to be careful to avoid loops.
  - The following pair of "simp" rules would cause issues:
    $$f(x) = h(g(x)), g(x) = f(x+2)$$
  - Permutative rewrite rules (e.g. $x + y = y + x$) are applied but only if they make the term "lexicographically smaller"

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VARIATIONS ON SIMP/AUTO

- Add a fact (once-off) to be used for simplification: `simp add: app_assoc`
- Omit a fact (once-off) from simplification: `simp del: rev_rev`
- Don't simplify the assumptions: `simp (no_asm_simp)`
- Ignore the assumptions: `simp (no_asm)`
- Simplify all the subgoals: `simp_all`
- Add rewriting rules/introduction rules etc to auto: `auto simp add: … intro: …`
- You can combine many of these!

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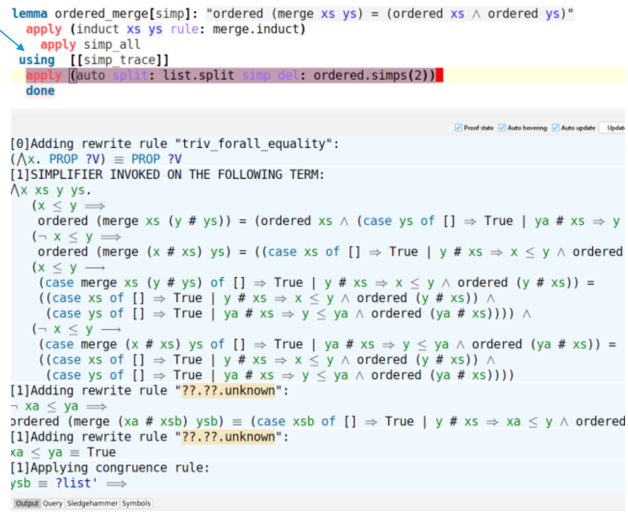
SIMP TRACE

- Insert: `using [[simp_trace]]` (inline proof) or `declare [[simp_trace]]` (theory wide)



```
lemma ordered_merge[simp]: "ordered (merge xs ys) = (ordered xs ∧ ordered ys)"
  apply (induct xs ys rule: merge.induct)
    apply simp_all
 using  [[simp_trace]]
  apply (auto split: list.split simp del: ordered.simps(2))
  done

[0]Adding rewrite rule "triv_forall_equality":
(⋀x. PROP ?V) ≡ PROP ?V
[1]SIMPLIFIER INVOKED ON THE FOLLOWING TERM:
⋀x xs y ys.
   (x ≤ y ⟹
    ordered (merge xs (y # ys)) = (ordered xs ∧ (case ys of [] ⇒ True | ya # xs ⇒ y
   (¬ x ≤ y ⟹
    ordered (merge (x # xs) ys) = ((case xs of [] ⇒ True | y # xs ⇒ x ≤ y ∧ ordered
   (x ≤ y ⟶
    (case merge xs (y # ys) of [] ⇒ True | y # xs ⇒ x ≤ y ∧ ordered (y # xs)) =
    ((case xs of [] ⇒ True | y # xs ⇒ x ≤ y ∧ ordered (y # xs)) ∧
     (case ys of [] ⇒ True | ya # xs ⇒ y ≤ ya ∧ ordered (ya # xs)))) ∧
   (¬ x ≤ y ⟶
    (case merge (x # xs) ys of [] ⇒ True | ya # xs ⇒ y ≤ ya ∧ ordered (ya # xs)) =
    ((case xs of [] ⇒ True | y # xs ⇒ x ≤ y ∧ ordered (y # xs)) ∧
     (case ys of [] ⇒ True | ya # xs ⇒ y ≤ ya ∧ ordered (ya # xs))))
[1]Adding rewrite rule "??.??.unknown":
¬ xa ≤ ya ⟹
ordered (merge (xa # xsb) ysb) ≡ (case xsb of [] ⇒ True | y # xs ⇒ xa ≤ y ∧ ordered
[1]Adding rewrite rule "??.??.unknown":
xa ≤ ya ≡ True
[1]Applying congruence rule:
ysb ≡ ?list' ⟹
```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MORE TACTICS

- Basic tactics such as `rule, erule, assumption, intro, elim`, used in conjunction with a *known fact*
- These can often be *combined* with auto/simp (like other variations of simp)
- We also have other automated tactics:
  - `force, fastforce`
  - `blast`: uses intro + elimination rules with powerful search heuristics (not simplification/arithmetic reasoning) and won't terminate if it doesn't work
  - Arithmetic tactics: `arith, linarith`
  - Use of tactics like "`metis`" and "`smt`" often indicate use of sledgehammer
- Other good tactics for starting a proof (less powerful, but safer): `safe, clarify, standard`
- And many more tactics: `cases, split` …
- Tactics can be combined e.g. by `(induction) (blast | fastforce)+` applies induction then repeatedly shows the subgoals using either blast or fastforce

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INDUCTION

- Inductive tactics are well-developed with many options for application.
- The `induction` tactic tries to figure out what to do automatically:

```
lemma app_assoc: "app (app xs ys) zs = app xs (app ys zs)"
  apply (induction xs)
   apply auto
  done
```

- Sometimes it can't, and we need to be more specific

```
lemma "itlen xs n = size xs + n"
  apply (induct xs arbitrary: n rule: list.induct)
   apply auto
  done
```

Specify n should be universally quantified in induction

Specify induction rule to use (unnecessary in this case)

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USEFUL FEATURES

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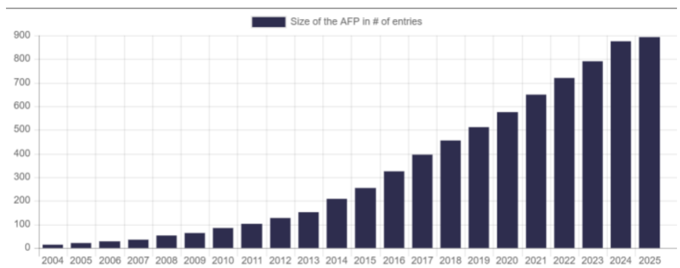
THE ISABELLE AFP

- A significant archive of (refereed) formalised mathematics and computer science concepts.
  - More of an “archive” than a constantly modified “library”
- https://www.isa-afp.org/
- It can be easily imported into a local instance of Isabelle by adding it as a component, see here: https://www.isa-afp.org/help/
- Over 4.5 million lines of code across 894 entries – and still growing!



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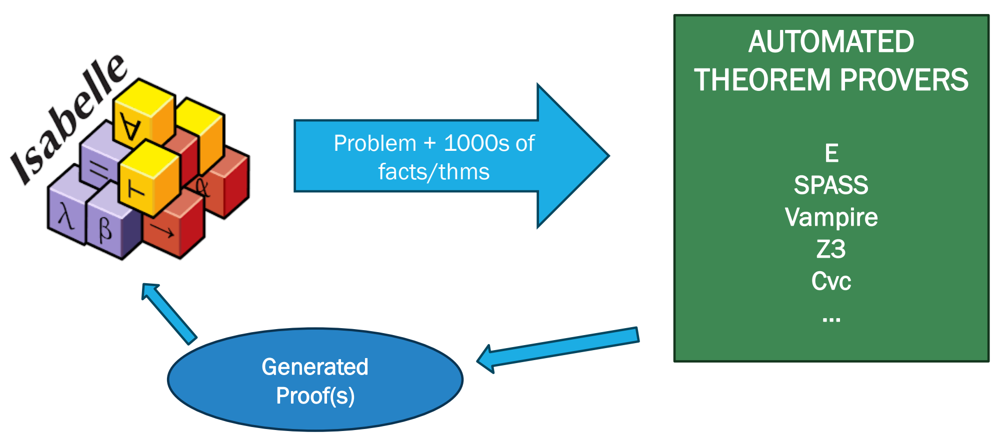
SLEDGEHAMMER



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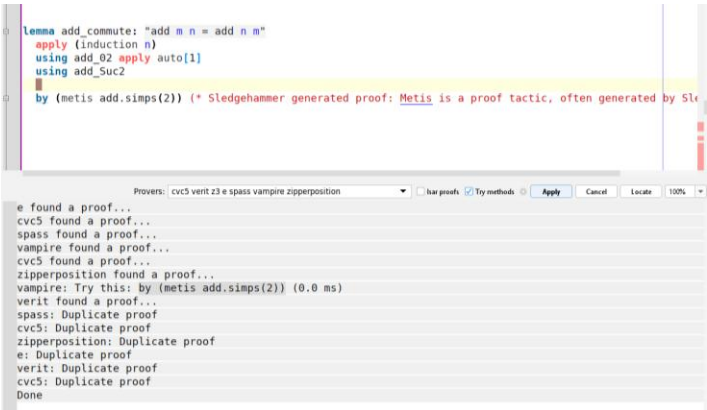
SLEDGEHAMMER



- Simplify the goal and break down into pieces
- Sledgehammer doesn't prove the goal, but returns a "proof" which is a call to metis, smt, blast, auto etc…
- Translations are not sound, hence sledgehammer provided proof *may not work* when inserted.
- Generated proofs can be ugly/messy – there are usually cleaner ways!

- For more history: https://lawrencecpaulson.github.io/2022/04/13/Sledgehammer.html
- For a more technical overview: https://www.cl.cam.ac.uk/~lp15/papers/Automation/paar.pdf (or many of Jasmin Blanchette's papers for more recent work).

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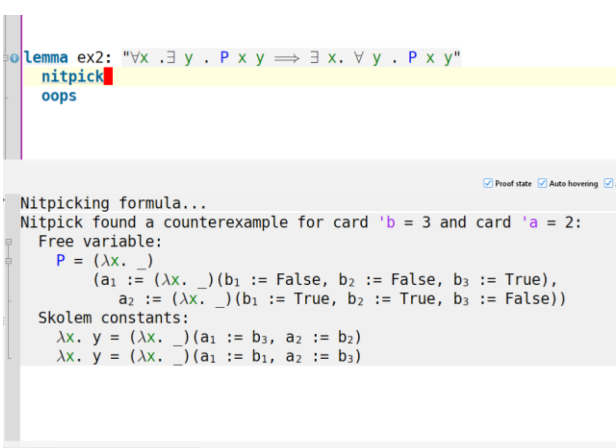
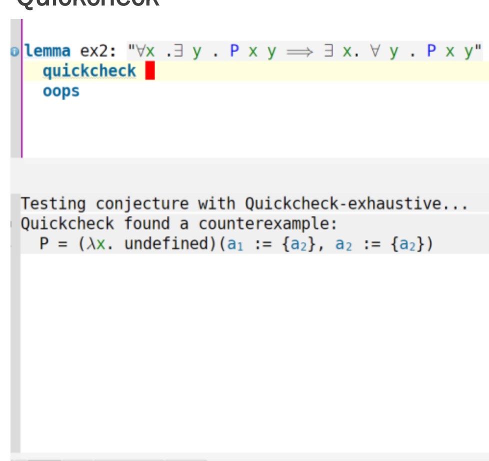
COUNTER EXAMPLE

Nitpick



Quickcheck



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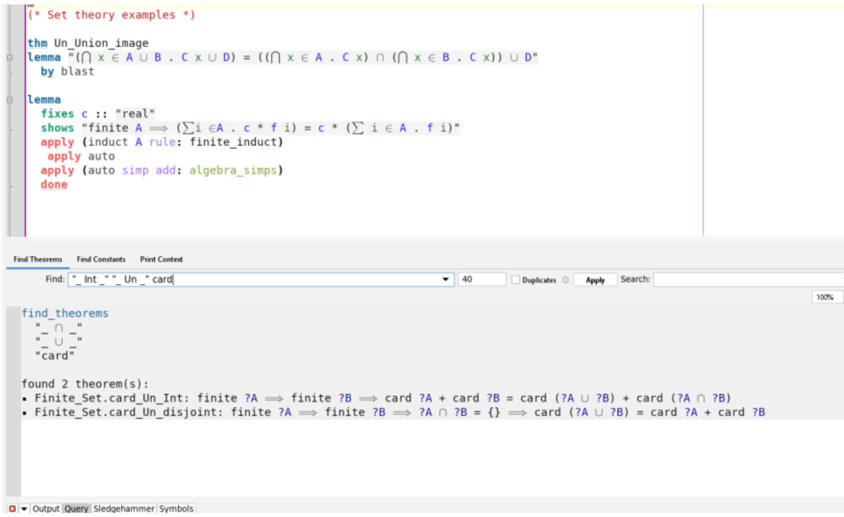
SEARCH: QUERY



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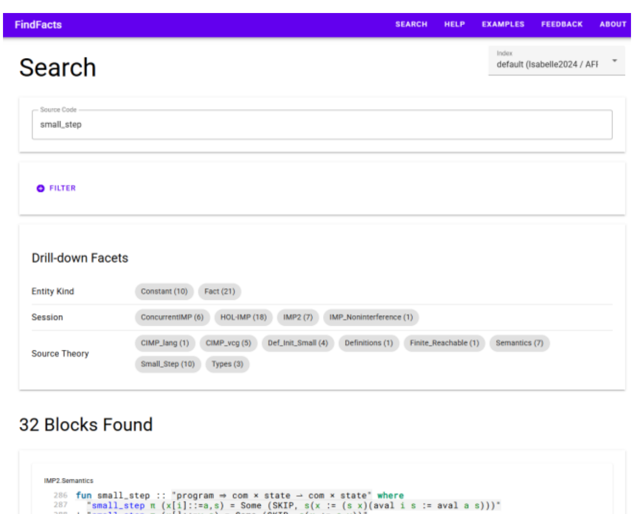
SEARCH: FINDFACTS



https://search.isabelle.in.tum.de/

OR

Local Database with Isabelle2025

isabelle find_facts_server -p 8080 -o find_facts_database_name=isabelle

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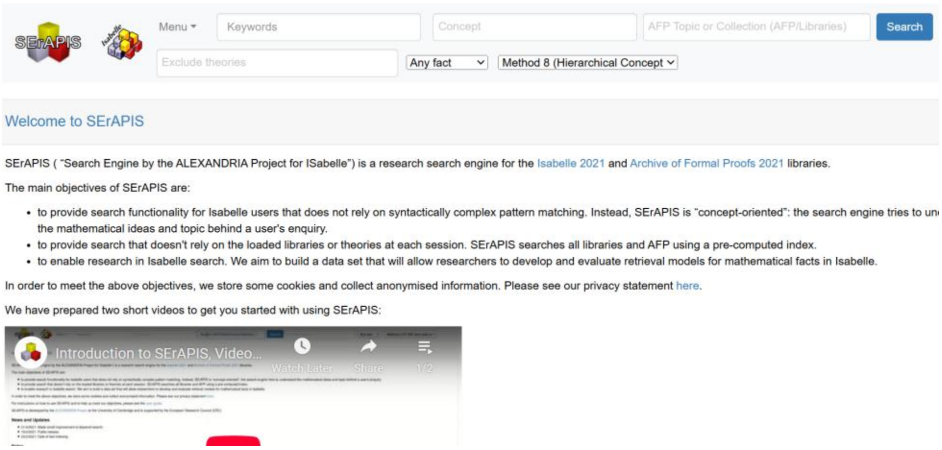
SEARCH: SERAPIS



https://behemoth.cl.cam.ac.uk/search/
Note: Last AFP Index was in 2021

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OTHER COOL FEATURES

- Code Generation
- Document Preparation
- Lifting and Transfer
- Eisbach => Proof Method language
- **Polymorphism (Type classes) and a powerful module system (Locales)**

TOMORROW

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NEXT TIME…

- Example Class:
  - Get started with Isabelle: Logic and function proofs
  - Test out sledgehammer for yourself
  - Try out different tactics
  - Gain familiarity with Isabelle tools
- Next Lecture
  - Starting on modularity!
  - Finish off your “tour” overview of Isabelle with the Isar proof language and more advanced types
  - Introducing type classes and locales
- To come… more advanced case studies in mathematics and program verification!

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