LECTURE 4: SEMANTICS AND ABSTRACTION

MODULAR PROOFS IN ISABELLE/HOL

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COURSE OVERVIEW

A practical course on the effective use of the Isabelle/HOL proof assistant in mathematics and programming languages

Lectures:

- Introduction to Proof Assistants
- Formalising the basics in Isabelle/HOL
- Introduction to Isar, more types, Locales and Type-classes
- Case studies:
  - Formalising Mathematics: combinatorics & advanced locale reasoning patterns
  - **Semantics, Abstraction, PL: Formalising semantics, program properties, and introducing modularity/abstraction.**

Example Classes:

- Isabelle exercises based on the previous lecture
- Will be drawing from the existing Isabelle tutorials/Nipkow's Concrete Semantic Book, as well as custom exercises (e.g. for locales).

LECTURE 4 OVERVIEW

Modular proofs = an engineering-like approach to formalisation.

Yesterday: mathematical formalisations/case-study

TODAY:

- Program verification and proof assistants
- Review: operational semantics
- Formalising semantics and working with basic properties
- Examples of locales/modularity in program verification
  - Refinement
  - Abstract reasoning
- Proof assistants in the wider-research landscape.

PROGRAM VERIFICATION & PROOF ASSISTANTS

SOME WELL-KNOWN PROGRAM VERIFICATION EXAMPLES

The development of several proof assistants was (and continues to be) motivated by program verification in many cases.

Some historical/long running applications

- Intel HOL-light (Floating Point verification): https://www.cl.cam.ac.uk/~jrh13/papers/sfm.pdf
- Sel4 (Isabelle): first formally verified operating system https://sel4.systems/About/

Currently

- Increasingly seen in industry (proof assistants are no longer just the domain of research!).
- Increasing interest in widely used frameworks (that help with modularity!) specific to program verification e.g. Iris in Rocq. https://iris-project.org/

SEMANTICS REVIEW

SEMANTICS INTRODUCTION

- We'll consider *operational* semantics, which can be given inductively:
  - Specifying syntax
  - Expression evaluation
  - Command Execution
- Typically, properties are proven using induction.
- We won't consider type checking in this lecture due to time, but also easy to do!

LET'S CONSIDER A BASIC SMALL-STEP SEMANTICS

- In the "Concrete Semantics" textbook (Nipkow & Klein, 2014), a basic "IMP" language is introduced. We'll use this as our initial case study today:

$$com \ ::= \text{SKIP} \mid string \ ::= aexp \mid com \,;;\, com \mid \text{IF } bexp \text{ THEN } com \text{ ELSE } com \mid \text{WHILE } bexp \text{ DO } com$$

$$\frac{}{(x ::= a,\ s) \to (SKIP,\ s(x := aval\ a\ s))}\ Assign$$

$$\frac{}{(SKIP;;\ c_2,\ s) \to (c_2,\ s)}\ Seq1 \qquad \frac{(c_1,\ s) \to (c_1',\ s')}{(c_1;;\ c_2,\ s) \to (c_1';;\ c_2,\ s')}\ Seq2$$

$$\frac{bval\ b\ s}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \to (c_1,\ s)}\ IfTrue$$

$$\frac{\neg\ bval\ b\ s}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \to (c_2,\ s)}\ IfFalse$$

$$\frac{}{(WHILE\ b\ DO\ c,\ s) \to (IF\ b\ THEN\ c;;\ WHILE\ b\ DO\ c\ ELSE\ SKIP,\ s)}\ While$$

AND HERE'S AN EQUIVALENT BIG STEP SEMANTICS

- In the "Concrete Semantics" textbook (Nipkow & Klein, 2014), a basic "IMP" language is introduced. We'll use this as our initial case study today:

$$com\ ::= \text{SKIP} \mid string\ ::= aexp \mid com\ ;;\ com \mid \text{IF}\ bexp\ \text{THEN}\ com\ \text{ELSE}\ com \mid \text{WHILE}\ bexp\ \text{DO}\ com$$

$$\frac{}{(SKIP,\ s) \Rightarrow s}\ \textit{Skip} \qquad \frac{}{(x ::= a,\ s) \Rightarrow s(x := aval\ a\ s)}\ \textit{Assign}$$

$$\frac{(c_1,\ s_1) \Rightarrow s_2 \qquad (c_2,\ s_2) \Rightarrow s_3}{(c_1;;\ c_2,\ s_1) \Rightarrow s_3}\ \textit{Seq}$$

$$\frac{bval\ b\ s \qquad (c_1,\ s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t}\ \textit{IfTrue}$$

$$\frac{\neg\ bval\ b\ s \qquad (c_2,\ s) \Rightarrow t}{(IF\ b\ THEN\ c_1\ ELSE\ c_2,\ s) \Rightarrow t}\ \textit{IfFalse}$$

$$\frac{\neg\ bval\ b\ s}{(WHILE\ b\ DO\ c,\ s) \Rightarrow s}\ \textit{WhileFalse}$$

$$\frac{bval\ b\ s_1 \qquad (c,\ s_1) \Rightarrow s_2 \qquad (WHILE\ b\ DO\ c,\ s_2) \Rightarrow s_3}{(WHILE\ b\ DO\ c,\ s_1) \Rightarrow s_3}\ \textit{WhileTrue}$$

SEMANTICS IN ISABELLE

DATATYPES

- This is a basic command datatype in Isabelle (from the IMP language) with custom syntax

```
datatype
  com = SKIP
      | Assign vname aexp        (‹_ ::= _› [1000, 61] 61)
      | Seq    com   com         (‹_;;/ _›  [60, 61] 60)
      | If     bexp com com      (‹(IF _/ THEN _/ ELSE _)›  [0, 0, 61] 61)
      | While  bexp com          (‹(WHILE _/ DO _)›  [0, 61] 61)
```

- We could either define elements of **com** abstractly or concretely:

Abstract

```
type_synonym aexp = "state ⇒ val"
type_synonym bexp = "state ⇒ bool"
```

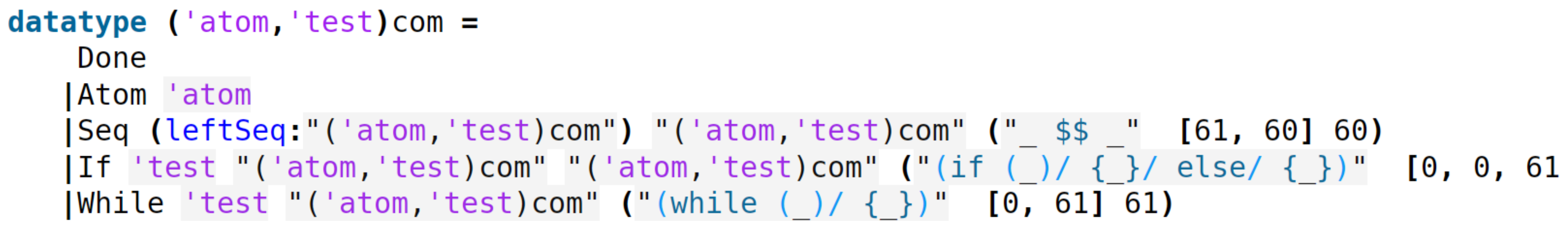
Concrete

```
datatype bexp = Bc bool | Not bexp |
       And bexp bexp | Less aexp aexp

datatype aexp = N int | V vname | Plus aexp aexp
```

DATATYPES

- Say for example we wanted to also make our **com** definition more abstract.
- As Isabelle's datatypes allow for parameterisation, it is quite easy to do this!
- In the example below, parametrised **com** with two additional type parameters instead of using a more concrete *aexp* and *bexp,* noting *Assign* has also been generalised to an Atomic command.



```
datatype ('atom,'test)com =
    Done
   |Atom 'atom
   |Seq (leftSeq:"('atom,'test)com") "('atom,'test)com" ("_ $$ _"  [61, 60] 60)
   |If 'test "('atom,'test)com" "('atom,'test)com" ("(if (_)/ {_}/ else/ {_})"  [0, 0, 61
   |While 'test "('atom,'test)com" ("(while (_)/ {_})"  [0, 61] 61)
```

ASIDE: INDUCTIVE SETS AND PREDICATES

Inductive Set Approach

```
inductive_set
Evens :: "nat set"
where
zeroin: "0 ∈ Evens"
| EvensI: "n ∈ Evens ⟹ Suc (Suc n) ∈ Evens"
```

Inductive Predicate

```
inductive evn :: "nat ⇒ bool" where
zero: "evn 0" |
step: "evn n ⟹ evn (Suc (Suc n))"
```

- After functions and datatypes, inductive definitions are one of the more valuable basic features of Isabelle
- They generate numerous useful facts (induct rules, cases etc).
- For semantics, we typically use the predicate style (as we also are often dealing with quite complex “triples”).

SMALL STEP DEFINITION

- Like other definitions, inductive definitions allow us to specify special syntax.

```
inductive
  small_step :: "com * state ⇒ com * state ⇒ bool" (infix ‹→› 55)
where
Assign:  "(x ::= a, s) → (SKIP, s(x := aval a s))" |

Seq1:    "(SKIP;;c₂,s) → (c₂,s)" |
Seq2:    "(c₁,s) → (c₁',s') ⟹ (c₁;;c₂,s) → (c₁';;c₂,s')" |

IfTrue:  "bval b s ⟹ (IF b THEN c₁ ELSE c₂,s) → (c₁,s)" |
IfFalse: "¬bval b s ⟹ (IF b THEN c₁ ELSE c₂,s) → (c₂,s)" |

While:   "(WHILE b DO c,s) →
            (IF b THEN c;; WHILE b DO c ELSE SKIP,s)"
```

AUTOMATION AND RULE INVERSION

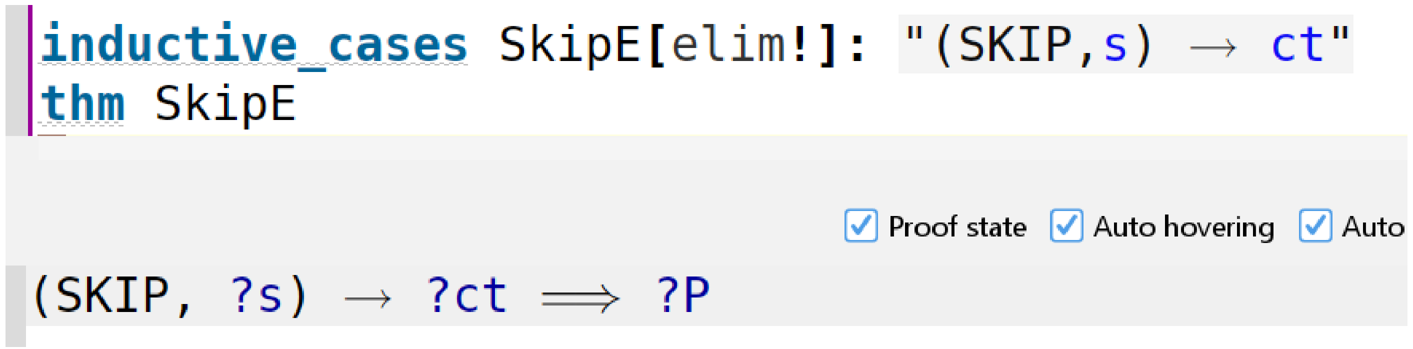
- We can make induction rules more useful by “reformatting it”, such as splitting into pairs:

```
lemmas small_step_induct = small_step.induct[split_format(complete)]
```

- This adds the automatically generated “introduction” rules to the simp/intro sets (so tactics like **auto** will automatically use them).

```
declare small_step.intros[simp,intro]
```

- Rule inversion: We can also use “inductive cases” to get our rule inversion facts of our semantics for free!



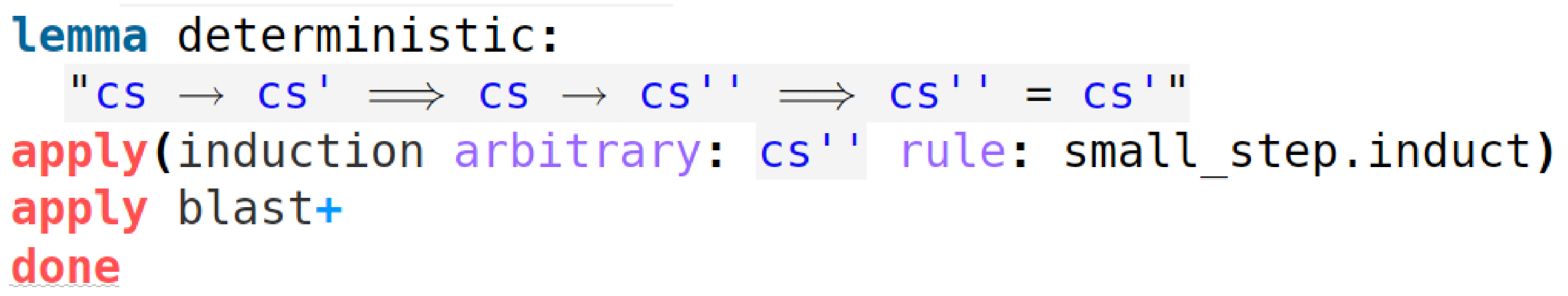
```
inductive_cases SkipE[elim!]: "(SKIP,s) → ct"
thm SkipE
```

✓ Proof state ✓ Auto hovering ✓ Auto

```
(SKIP, ?s) → ?ct ⟹ ?P
```

A SAMPLE PROOF: DETERMINISTIC

- We can use our inductive rules easily as normal, and for simple facts the proofs can be very fast!



```
lemma deterministic:
  "cs → cs' ⟹ cs → cs'' ⟹ cs'' = cs'"
apply(induction arbitrary: cs'' rule: small_step.induct)
apply blast+
done
```

- DEMO! More of the IMP theory

BUT WHAT ABOUT MODULARITY?

INTRODUCING A BASIC LOCALE

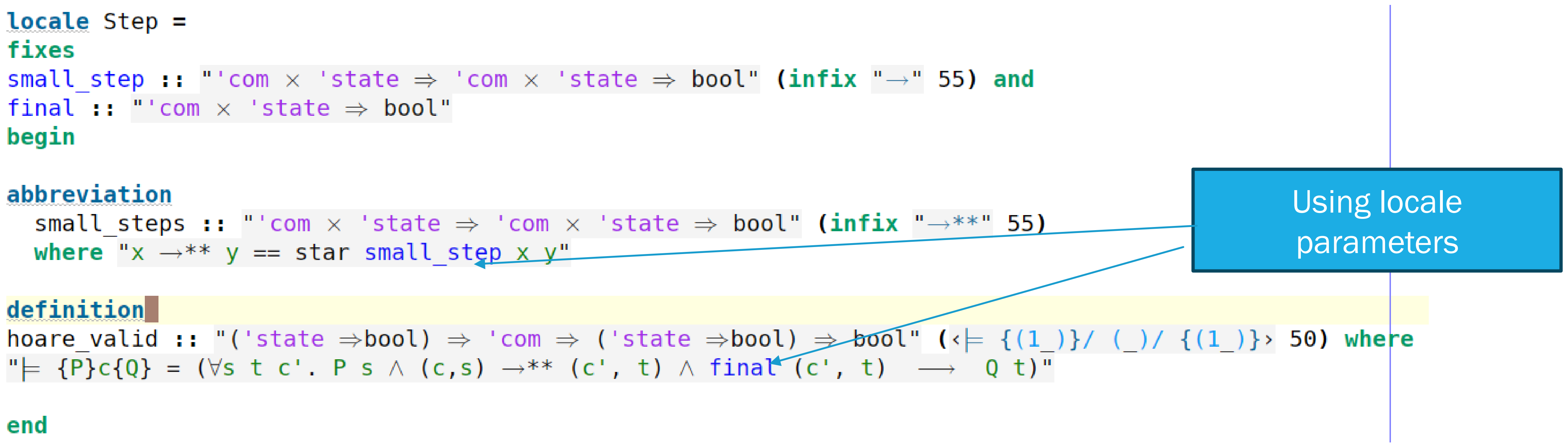
```
locale Step =
fixes
small_step :: "'com × 'state ⇒ 'com × 'state ⇒ bool" (infix ":→" 60) and
final :: "'com × 'state ⇒ bool"
begin

abbreviation
  small_steps :: "'com × 'state ⇒ 'com × 'state ⇒ bool" (infix "→**" 55)
  where "x →** y == star small_step x y"
```

- A basic locale which represents a context that defines a “small step” semantics with a “final” operator (i.e. representing a program terminating)
- Other useful definitions and properties can now be defined/proven locally.

DEFINING ABSTRACT PROPERTIES

- Consider introducing a Hoare logic.
- We can abstractly define if a Hoare triple is valid without needing to know anything about the semantics is valid. And therefore other useful lemmas on this definition!



```
locale Step =
fixes
small_step :: "'com × 'state ⇒ 'com × 'state ⇒ bool" (infix "→" 55) and
final :: "'com × 'state ⇒ bool"
begin

abbreviation
  small_steps :: "'com × 'state ⇒ 'com × 'state ⇒ bool" (infix "→**" 55)
  where "x →** y == star small_step x y"

definition
hoare_valid :: "('state ⇒bool) ⇒ 'com ⇒ ('state ⇒bool) ⇒ bool" (‹⊨ {(1_)}/ (_)/ {(1_)}› 50) where
"⊨ {P}c{Q} = (∀s t c'. P s ∧ (c,s) →** (c', t) ∧ final (c', t)  ⟶  Q t)"

end
```

INHERITANCE

- We can use locales in all the same way we used them for mathematics, including inheritance.
- For example, we may want to abstractly reason on a semantics with a deterministic characteristic.

```
locale Step_Deterministic = Step +
  assumes determ: "((cs :→ cs') ∧ (cs :→ cs'')) ⟶ (cs'' = cs')"
```

INTERPRETING THE LOCALE (CONCRETE)

- Interpreting the **Step** locale with our concrete small-step semantics from earlier is trivial, as there are no assumptions!

```
interpretation Step small_step final .
```

- Similarly, we can use our deterministic lemma from earlier to establish an interpretation for our abstract deterministic locale

```
interpretation det: Step_Deterministic small_step final
  apply (unfold_locales)
  using deterministic by blast
```

OR “REFINING” THE LOCALE

- As is typical of program verification, we often want to gradually refine our specification, rather than jump straight to a concrete definition.
- This is an example of a locale which contains a concrete inductive definition of the semantics, that assumes the existence of evaluation functions for arithmetic and Boolean expressions

```
locale SeqParWhileLang =
fixes evalA :: "'atom ⇒ 'state ⇒ 'state"
and evalT :: "'test ⇒ 'state ⇒ bool"
and Skip :: 'atom and Not :: "'test ⇒ 'test"
assumes evalA_Skip_id[simp,intro!]: "evalA Skip = id"
and evalT_Not[simp]: "⋀s. evalT (Not t) s = (¬ evalT t s)"
```

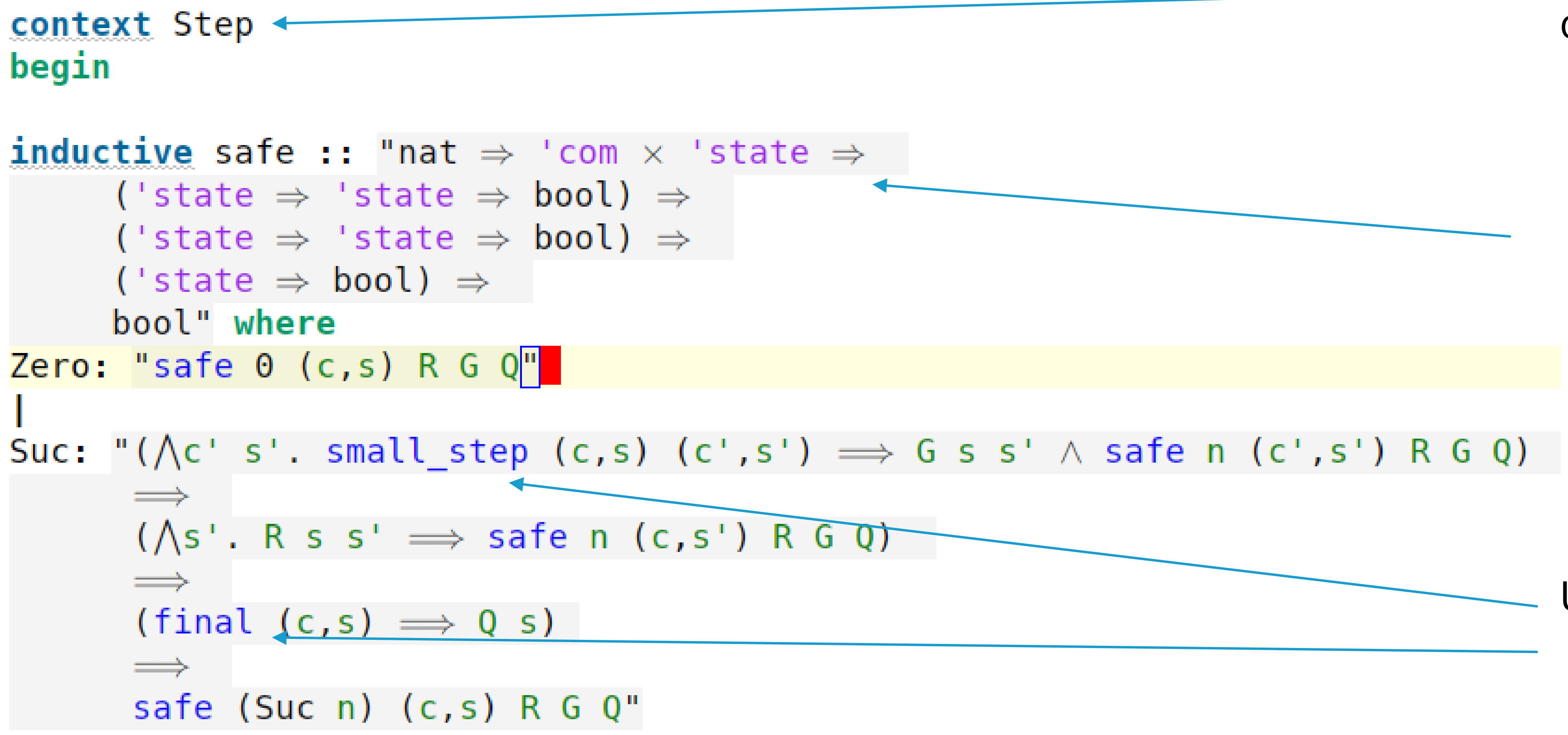
- We use **sublocale** to establish the relationship

```
sublocale SeqParWhileLang < Step where
small_step = small_step and final = final .
```

MORE ADVANCED CASE STUDIES

EXAMPLE 1: MORE ABSTRACT PROPERTIES

- In some recent joint work (w/ A. Popescu & J. Wright), we needed to abstractly reason on safety for Rely-Guarantee reasoning, and could then show our theorem held for any small step semantics, as well as interpret it for practical use.



```
context Step
begin

inductive safe :: "nat ⇒ 'com × 'state ⇒
     ('state ⇒ 'state ⇒ bool) ⇒
     ('state ⇒ 'state ⇒ bool) ⇒
     ('state ⇒ bool) ⇒
     bool" where
Zero: "safe 0 (c,s) R G Q"
|
Suc: "(⋀c' s'. small_step (c,s) (c',s') ⟹ G s s' ∧ safe n (c',s') R G Q)
       ⟹
       (⋀s'. R s s' ⟹ safe n (c,s') R G Q)
       ⟹
       (final (c,s) ⟹ Q s)
       ⟹
       safe (Suc n) (c,s) R G Q"
```

Open abstract context

New local definition of desired property

Using locale parameters

EXAMPLE 2: MODELLING ATTACKER LEVELS

- Information-flow security investigates if any information can leak from “high valued” variables to “low security” variables through the execution of a program
- Relative security is a new concept that focuses on checking if an enhanced (e.g. optimized) system, is secure with respect to the original (“vanilla”) system (i.e. if any leaks occur, they already occurred in the basic version).
- We can model the idea of “leaks” in different ways, depending on how “abstract” a property we want to reason on.
- For further details:
  - See the original conference paper here (B. Dongol, M. Griffin, A. Popescu, J. Wright, 2024): https://andreipopescu.uk/pdf/relative_security_CSF_2024.pdf
  - The AFP Entry here: https://www.isa-afp.org/entries/Relative_Security.html

EXAMPLE 2: SETTING UP A TRANSITION SYSTEM

- The base transition system locale

```
locale Transition_System =
fixes istate :: "'state ⇒ bool"
  and validTrans :: "'trans ⇒ bool"
  and srcOf :: "'trans ⇒ 'state"
  and tgtOf :: "'trans ⇒ 'state"
```

- A transition system that includes the definition of finality as an assumption

```
locale System_Mod =
Simple_Transition_System istate validTrans
for istate :: "'state ⇒ bool"
and validTrans :: "'state × 'state  ⇒ bool"
+
fixes final :: "'state ⇒ bool"
assumes final_def: "final s1 ⟷ (∀s2. ¬ validTrans (s1,s2))"
```

EXAMPLE 2: MODELLING ATTACKER LEVELS

- Leakage Model: Assumes the existence of some function describing leaks

```
locale Leakage_Mod = System_Mod istate validTrans final
  for istate :: "'state ⇒ bool" and validTrans :: "'state × 'state ⇒ bool"
    and final :: "'state ⇒ bool"
+
fixes lleakVia :: "'state llist ⇒ 'state llist ⇒ 'leak ⇒ bool"
```

- Attacker Model: Specifies the leak via function using more precise functions on secrets, attackers and observers

```
locale Attacker_Mod = System_Mod istate validTrans final
  for istate :: "'state ⇒ bool" and validTrans :: "'state × 'state ⇒ bool"
and final :: "'state ⇒ bool"
+
fixes S :: "'state llist ⇒ 'secret llist"
and A :: "'state ltrace ⇒ 'act llist"
and O :: "'state ltrace ⇒ 'obs llist"
begin

fun lleakVia :: "'state llist ⇒ 'state llist ⇒ 'secret llist × 'secret llist ⇒ bool"
where
"lleakVia tr tr' (sl,sl') = (S tr = sl ∧ S tr' = sl' ∧ A tr = A tr' ∧ O tr ≠ O tr')"
```

EXAMPLE 2: MODELLING ATTACKER LEVELS

- Relative Security – uses two instances of attacker models.

```
locale Rel_Sec =
  Van: Statewise_Attacker_Mod istateV validTransV finalV isSecV getSecV isIntV getIntV
+
  Opt: Statewise_Attacker_Mod istateO validTransO finalO isSecO getSecO isIntO getIntO
  for validTransV :: "'stateV × 'stateV ⇒ bool"
  and istateV :: "'stateV ⇒ bool" and finalV :: "'stateV ⇒ bool"
  and isSecV :: "'stateV ⇒ bool" and getSecV :: "'stateV ⇒ 'secret"
  and isIntV :: "'stateV ⇒ bool" and getIntV :: "'stateV ⇒ 'actV × 'obsV"
  (* NB: we have the same notion of secret, but everything else can be different  *)
  and validTransO :: "'stateO × 'stateO ⇒ bool"
  and istateO :: "'stateO ⇒ bool" and finalO :: "'stateO ⇒ bool"
  and isSecO :: "'stateO ⇒ bool" and getSecO :: "'stateO ⇒ 'secret"
  and isIntO :: "'stateO ⇒ bool" and getIntO :: "'stateO ⇒ 'actO × 'obsO"
  and corrState :: "'stateV ⇒ 'stateO ⇒ bool"
```

- We restate all the parameters using **for** to keep our custom type names

RELATIVE SECURITY FINAL LOCALE INFRASTRUCTURE

$$\begin{tikzcd}
 & & \text{Transition\_System} \arrow[d] & & \\
 & & \text{Simple\_Transition\_System} \arrow[d] & & \\
 & & \text{System\_Mod} \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
\text{Leakage\_Mod\_fin} \arrow[r, dashed] \arrow[d, bend right, "Van"'] \arrow[d, bend left, "Opt"] & \text{Attacker\_Mod\_fin} \arrow[r, dashed] \arrow[d, bend right, "Van"'] \arrow[d, bend left, "Opt"] & \text{Statewise\_Attacker\_Mod} \arrow[d, bend right, "Van"'] \arrow[d, bend left, "Opt"] & \text{Attacker\_Mod} \arrow[l, dashed] \arrow[d, bend right, "Van"'] \arrow[d, bend left, "Opt"] & \text{Leakage\_Mod} \arrow[l, dashed] \arrow[d, bend right, "Van"'] \arrow[d, bend left, "Opt"] \\
\text{Relative\_Security''\_fin} \arrow[r, dashed] & \text{Relative\_Security'\_fin} \arrow[r, dashed, bend right] & \text{Rel\_Sec} & \text{Relative\_Security'} \arrow[l, dashed, bend left] & \text{Relative\_Security''} \arrow[l, dashed]
\end{tikzcd}$$

CONCLUSION

Any feedback/questions/thoughts? Feel free to get in touch at:

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We've covered

- A fast-paced introduction to the basics of Isabelle/HOL!
- An in-depth discussion of type classes and locales, including advanced reasoning patterns on locales.
- An introduction to reasoning on semantics in Isabelle/HOL
- Research case studies: formalized combinatorics, relative security, refinement!

And along the way:

- Some history (proof assistants, formalised maths, verification)
- Insight and links to current research in proof assistants/formal verification

CONCLUSION

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Your Challenge:

- Try out formalising your own work in Isabelle (or any other proof assistant out there).
- Keep "software engineering" principles in mind:
  - Verification is only half the goal
  - Modular, reusable, and maintainable formal proof libraries can go much further!

More Resources:

- To continue the work we started on semantics today, see Nipkow and Klein's book: http://concrete-semantics.org
- CPP/ITP are good starting points for formalisation focused research.
- See more links at start of lecture 1!