

Verifying Tail Modulo Cons using Relational Separation Logic

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Verifying Tail Modulo Cons using Relational Separation Logic

Program transformation implemented in the
OCAML compiler by Frédéric Bour, Basile
Clément & Gabriel Scherer.

Verifying Tail Modulo Cons using Relational Separation Logic

Formalize the transformation and its soundness.

Verifying Tail Modulo Cons using Relational Separation Logic

Prove soundness using an adequate IRIS binary
logical relation à la SIMULIRIS.

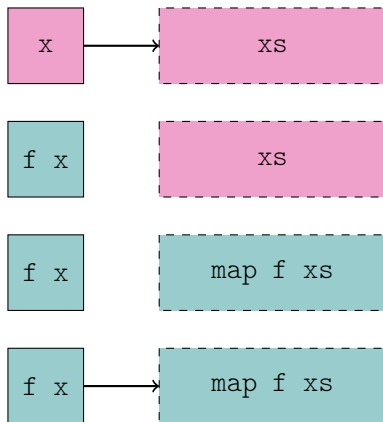
The map problem: natural implementation

```
let rec map f xs =  
  match xs with  
  | [] →  
    []  
  | x :: xs →  
    let y = f x in  
    y :: map f xs
```

```
# List.init 250_000 (fun _ → ())  
|> map Fun.id  
|> ignore  
;;
```

Stack overflow during evaluation (looping recursion?).

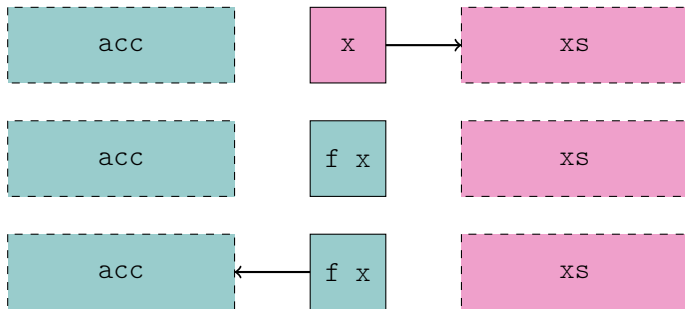
The map problem: natural implementation



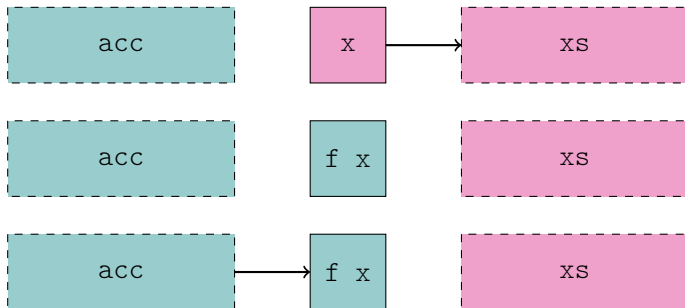
The map problem: APS implementation

```
let rec map ys f xs =  
  match xs with  
  | [] →  
    List.rev ys  
  | x :: xs →  
    let y = f x in  
    map (y :: ys) f xs  
let map xs =  
  map [] f xs  
  
# List.init 250_000 (fun _ → ())  
|> map Fun.id  
|> ignore  
;;  
- : unit = ()
```

The map problem: APS implementation



The map problem: DPS implementation



The map problem: DPS implementation

```
let rec map_dps dst f xs =      let map f xs =
  match xs with
  | [] →
    set_field dst 1 []
  | x :: xs →
    let y = f x in
    let dst' = y :: [] in
    set_field dst 1 dst' ;
    map_dps dst' f xs

# List.init 250_000 (fun _ → ())
|> map Fun.id
|> ignore
;;
- : unit = ()
```

The map problem: TMC

```
let[@tail_mod_cons] rec map f xs =  
  match xs with  
  | [] →  
    []  
  | x :: xs →  
    let y = f x in  
    y :: map f xs
```

```
# List.init 250_000 (fun _ → ())  
|> map Fun.id  
|> ignore  
;;  
- : unit = ()
```

DATA LANG: syntax

Index	\ni	i	$::=$	$0 \mid 1 \mid 2$
Tag	\ni	t		
\mathbb{B}	\ni	b		
\mathbb{L}	\ni	ℓ		
\mathbb{F}	\ni	f		
\mathbb{X}	\ni	x, y		
Val	\ni	v, w	$::=$	$() \mid i \mid t \mid b \mid \ell \mid @f$
Expr	\ni	e	$::=$	$v \mid x \mid \mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2 \mid e_1 \ \overline{e_2}$ $\mid e_1 = e_2 \mid \mathbf{if} \ e_0 \ \mathbf{then} \ e_1 \ \mathbf{else} \ e_2$ $\mid \{t, e_1, e_2\}$ $\mid e_1.(e_2) \mid e_1.(e_2) \leftarrow e_3$
Def	\ni	d	$::=$	$\mathbf{rec} \ \overline{x} = e$
Prog	\ni	p	$::=$	$\mathbb{F} \xrightarrow{\text{fn}} \text{Def}$
State	\ni	σ	$::=$	$\mathbb{L} \xrightarrow{\text{fn}} \text{Val}$
Config	\ni	ρ	$::=$	$\text{Expr} \times \text{State}$

DATA LANG: map

```
map := rec f xs =  
  match xs with  
  | [] →  
    []  
  | x :: xs →  
    let y = f x in  
    y :: @map f xs
```

TMC transformation

$$e_s \xrightarrow[\text{dir}]{\xi} e_t$$

$$d_s \xrightarrow[\text{dir}]{\xi} d_t$$

$$(e_{dst}, e_{idx}, e_s) \xrightarrow[\text{dps}]{\xi} e_t$$

$$d_s \xrightarrow[\text{dps}]{\xi} d_t$$

$$p_s \rightsquigarrow p_t$$

TMC transformation: map

```
map := rec f xs =  
  match xs with  
  | [] →  
    []  
  | x :: xs →  
    let y = f x in  
    let dst = y :: ■ in  
    @map_dps dst 2 f xs ;  
    dst
```

```
map_dps := rec dst idx f xs =  
  match xs with  
  | [] →  
    dst.(idx) ← []  
  | x :: xs →  
    let y = f x in  
    let dst' = y :: ■ in  
    dst.(idx) ← dst' ;  
    @map_dps dst' 2 f xs
```

Transformation soundness

$p_s \rightsquigarrow p_t$ program p_s transforms into program p_t

\Downarrow

$p_s \sqsupseteq p_t$ program p_t refines program p_s
(termination-preserving refinement)

Transformation soundness

$p_s \rightsquigarrow p_t$ program p_s transforms into program p_t



$p_s \gtrsim p_t$ program p_t simulates program p_s
(*relational separation logic*, SIMULIRIS)



$p_s \sqsupseteq p_t$ program p_t refines program p_s
(termination-preserving refinement)

Specification in separation logic

$$\frac{\{\text{???}\}}{\frac{\text{@map } v_s \gtrsim \text{@map } v_t}{\{\text{???}\}}}$$

$$\frac{\{\text{???}\}}{\frac{\text{@map } v_s \gtrsim \text{@map_dps } \ell \ i \ v_t}{\{\text{???}\}}}$$

Direct transformation

$$\frac{\frac{\{v_s \approx v_t\}}{\text{@map } v_s \gtrsim \text{@map } v_t}}{\{w_s, w_t. w_s \approx w_t\}}$$

RELDIR (SIMULIRIS)

$$f \in \text{dom}(p_s)$$

$$v_s \approx v_t$$

$$\frac{\forall w_s, w_t. w_s \approx w_t \rightarrow \Phi(w_s, w_t)}{\text{@f } v_s \gtrsim \text{@f } v_t [\Phi]}$$

DPS transformation

$$\frac{\frac{\{v_s \approx v_t * (\ell + i) \mapsto_t \blacksquare\}}{\textcircled{\text{map}} \textcolor{violet}{v}_s \gtrsim \textcircled{\text{map_dps}} \textcolor{blue}{\ell} \textcolor{violet}{i} \textcolor{violet}{v}_t}}{\{\langle \rangle, w_t. \exists w_t. w_s \approx w_t * (\ell + i) \mapsto_t w_t\}}$$

RELDPS

$$\frac{\begin{array}{c} \xi[f] = f_{dps} \\ \overline{v_s} \approx \overline{v_t} \\ \textcolor{blue}{\ell} \mapsto_t \overline{v} \\ \forall w_s, w_t. w_s \approx w_t \multimap \textcolor{blue}{\ell} \mapsto_t \overline{v}[i \mapsto w_t] \multimap \Phi(w_s, \langle \rangle) \end{array}}{\textcircled{f} \overline{v_s} \gtrsim \textcircled{f_{dps}} \textcolor{blue}{\ell} \textcolor{violet}{i} \overline{v_t} [\Phi]}$$

RELPROTOCOL

$$\frac{\textcolor{red}{X}(e_s, e_t, \textcolor{blue}{\Psi}) \quad \forall e'_s, e'_t. \textcolor{blue}{\Psi}(e'_s, e'_t) \multimap e'_s \gtrsim e'_t \langle \textcolor{red}{X} \rangle [\Phi]}{e_s \gtrsim e_t \langle \textcolor{red}{X} \rangle [\Phi]}$$

Proof sketch

$$f_s \approx f_t$$

$$xs_s \approx xs_t$$

$$@map\ f_s\ xs_s$$

$$\gtrsim$$

$$@map\ f_t\ xs_t$$

Proof sketch

$$\begin{array}{c}
 f_s \approx f_t \qquad \qquad \qquad xs_s \approx xs_t \\
 \boxed{\text{REL PURE}} \\
 \frac{e_s \xrightarrow[\text{pure}]{p_s} e'_s \quad e_t \xrightarrow[\text{pure}]{p_t} e'_t \quad e'_s \gtrsim e'_t [\Phi]}{e_s \gtrsim e_t [\Phi]} \\
 @map \ f_s \ xs_s \qquad \qquad \qquad \gtrsim \qquad \qquad \qquad @map \ f_t \ xs_t
 \end{array}$$

Proof sketch

$$f_s \approx f_t$$

$$xs_s \approx xs_t$$

```
match xs_s with
| [] →
  []
| x :: xs' →
  let y = f_s x in
  y :: @map f_s xs'
```

$$\gtrsim$$

```
match xs_t with
| [] →
  []
| x :: xs' →
  let y = f_t x in
  let dst = y :: ■ in
  @map_dps dst 2 f_t xs' ;
  dst
```

Proof sketch

RELMATCH

$$e_{s0} \gtrsim e_{t0} \{ \approx \}$$

$$e_{s1} \gtrsim e_{t1} [\Phi]$$

$$\frac{\forall x_s, x_t, xs_s, xs_t. x_s \approx x_t \multimap xs_s \approx xs_t \multimap e_{s2} \gtrsim e_{t2} [\Phi]}{\text{match } e_{s0} \text{ with } \begin{array}{l} | [] \rightarrow e_{s1} \\ | x_s :: xs_s \rightarrow e_{s2} \end{array} \gtrsim \text{match } e_{t0} \text{ with } \begin{array}{l} | [] \rightarrow e_{t1} \\ | x_t :: xs_t \rightarrow e_{t2} \end{array} [\Phi]}$$

y :: @map f_s xs'

@map_dps dst 2 f_t xs' ;
dst

Proof sketch

$$f_s \approx f_t$$

$$xs_s \approx xs_t$$

[]

\wedge

[]

Proof sketch

$$f_s \approx f_t$$

$$xs_s \approx xs_t$$

$$x_s \approx x_t$$

$$xs'_s \approx xs'_t$$

```
let y = f_s x_s in  
y :: @map f_s xs'_s
```

$$\gtrsim$$

```
let y = f_t x_t in  
let dst = y :: ■ in  
@map_dps dst 2 f_t xs'_t ;  
dst
```

Proof sketch

$$f_s \approx f_t \qquad x_{s_s} \approx x_{s_t}$$

$$x_s \approx x_t \qquad x_{s'} \approx x_{s'_t}$$

$$\frac{\text{RELVCALLSIMILAR} \quad f_s \approx f_t \quad x_s \approx x_t}{f_s \ x_s \gtrsim f_t \ x_t \ \{\approx\}}$$

```
let y = f_s x_s in
y :: @map f_s xs'_s
```

 \gtrsim

```
let dst = y :: ■ in
@map_dps dst 2 f_t xs'_t ;
dst
```

Proof sketch

$$f_s \approx f_t$$

$$xs_s \approx xs_t$$

$$x_s \approx x_t$$

$$xs'_s \approx xs'_t$$

$$y_s \approx y_t$$

```
let y = y_s in  
y :: @map f_s xs'_s
```

$$\gtrsim$$

```
let y = y_t in  
let dst = y :: ■ in  
@map_dps dst 2 f_t xs'_t ;  
dst
```

Proof sketch

$$f_s \approx f_t$$

$$xs_s \approx xs_t$$

$$x_s \approx x_t$$

$$xs'_s \approx xs'_t$$

$$y_s \approx y_t$$

$$y_s \quad :: \quad @map \quad f_s \quad xs'_s \quad \quad \quad \geq \quad \quad \quad \mathbf{let} \quad dst = y_t \quad :: \quad \blacksquare \quad \mathbf{in} \\ @map_dps \quad dst \quad 2 \quad f_t \quad xs'_t \quad ; \\ dst$$

Proof sketch

$$f_s \approx f_t$$

$$xs_s \approx xs_t$$

$$x_s \approx x_t$$

$$xs'_s \approx xs'_t$$

REL TGT CONS

$$\frac{\forall \ell. \ell \mapsto_t (\text{CONS}, v_1, v_2) \multimap e_s \gtrsim \ell [\Phi]}{e_s \gtrsim v_1 :: v_2 [\Phi]}$$

$$e_s \gtrsim v_1 :: v_2 [\Phi]$$

$y_s :: \text{@map } f_s \text{ } xs'_s$

\gtrsim

```
let dst = y_t :: ■ in
@map_dps dst 2 f_t xs'_t ;
dst
```

Proof sketch

$$f_s \approx f_t$$

$$xs_s \approx xs_t$$

$$x_s \approx x_t$$

$$xs'_s \approx xs'_t$$

$$y_s \approx y_t$$

$$\ell_t \mapsto_t (\text{CONS}, y_t, \blacksquare)$$

$$y_s :: @map\ f_s\ xs'_s \quad \geq \quad \begin{array}{l} \text{let } dst = \ell_t \text{ in} \\ @map_dps\ dst\ 2\ f_t\ xs'_t\ ; \\ dst \end{array}$$

Proof sketch

$$f_s \approx f_t$$

$$xs_s \approx xs_t$$

$$x_s \approx x_t$$

$$xs'_s \approx xs'_t$$

RELGTGPURE

$$\frac{e_t \xrightarrow[pure]{p_t} e'_t \quad e_s \gtrsim e'_t [\Phi]}{e_s \gtrsim e_t [\Phi]}$$

$y_s :: @map \text{ f}_s \text{ xs}'_s$

\gtrsim

```
let dst = l_t in
@map_dps dst 2 f_t xs'_t ;
dst
```


Proof sketch

$$f_s \approx f_t$$

$$xs_s \approx xs_t$$

$$x_s \approx x_t$$

$$xs'_s \approx xs'_t$$

$$y_s \approx y_t$$

$$\ell_t \mapsto_t (\text{CONS}, y_t, \blacksquare)$$

$$y_s :: @\text{map} \ f_s \ xs'_s \quad \geq \quad @\text{map_dps} \ \ell_t \ 2 \ f_t \ xs'_t \ ;$$

Proof sketch

$$f_s \approx f_t$$

$$xs_s \approx xs_t$$

RELDPS2

$$\xi[f] = f_{dps}$$

$$\overline{v_s} \approx \overline{v_t}$$

$$\ell \mapsto_t (t, v_1, v_2)$$

$$\forall w_s, w_t. w_s \approx w_t \multimap \ell \mapsto_t (t, v_1, w_t) \multimap \Phi(w_s, ())$$

$$\textcircled{f} \overline{v_s} \gtrsim \textcircled{f_{dps}} \ell \textcircled{2} \overline{v_t} [\Phi]$$

$$y_s :: y_{s_s}$$

$$\gtrsim$$

$$\textcircled{\text{map_dps}} \ell_t \textcircled{2} f_t xs'_t ;$$

$$\ell_t$$

Proof sketch

$$f_s \approx f_t$$

$$xS_s \approx xS_t$$

$$x_s \approx x_t$$

$$xs'_s \approx xs'_t$$

$$y_s \approx y_t$$

$$ys_s \approx ys_t$$

$$\ell_t \mapsto_t (\text{CONS}, y_t, y_{s_t})$$

$$y_s \quad :: \quad yS_s$$

$\wedge 2$

$$(\cdot) ; \ell_t$$

Proof sketch

$$f_s \approx f_t$$

$$xs_s \approx xs_t$$

$$x_s \approx x_t$$

$$xs'_s \approx xs'_t$$

$$y_s \approx y_t$$

$$ys_s \approx ys_t$$

$$\ell_t \mapsto_t (\text{CONS}, y_t, ys_t)$$

$$y_s \quad :: \quad ys_s$$

$$\succeq$$

$$\ell_t$$

Proof sketch

$$f_s \approx f_t$$

$$xs_s \approx xs_t$$

$$x_s \approx x_t$$

$$xs'_s \approx xs'_t$$

RELSRCCONS

$$\frac{\forall \ell. \ell \mapsto_s (\text{CONS}, v_1, v_2) \rightarrow * \ell \gtrsim e_t [\Phi]}{v_1 :: v_2 \gtrsim e_t [\Phi]}$$

 ℓ_s \gtrsim ℓ_t

Proof sketch

$$f_s \approx f_t \qquad xs_s \approx xs_t$$

$$x_s \approx x_t \qquad xs'_s \approx xs'_t$$

$$y_s \approx y_t$$

$$ys_s \approx ys_t$$

$$\ell_t \mapsto_t (\text{CONS}, y_t, ys_t)$$

$$\ell_s \mapsto_s (\text{CONS}, y_s, ys_s)$$

 ℓ_s \geq ℓ_t

Proof sketch

$$f_s \approx f_t$$

$$xs_s \approx xs_t$$

RELBIIINSERT

$$\ell_s \mapsto_s \overline{v_s}$$

$$\ell_t \mapsto_t \overline{v_t}$$

$$\overline{v_s} \approx \overline{v_t}$$

$$\ell_s \approx \ell_t \multimap e_s \gtrsim e_t [\Phi]$$

$$\hline e_s \gtrsim e_t [\Phi]$$

 ℓ_s \gtrsim ℓ_t

Proof sketch

$$f_s \approx f_t$$

$$xs_s \approx xs_t$$

$$x_s \approx x_t$$

$$xs'_s \approx xs'_t$$

$$y_s \approx y_t$$

$$ys_s \approx ys_t$$

$$\ell_s \approx \ell_t$$

 ℓ_s \succeq ℓ_t

Thank you for your attention!